Fugit (options)

INTRODUCTION

The terminology of fugit refers to the risk neutral expected time to exercise an American option. Invented by Mark Garman, while professor at Berkeley, in the context of a binomial tree for American option, this concept has been since extended to other numerical methods for American options and in particular to American Monte Carlo.

The concept of fugit is quite relevant for convertible bonds, equity linked convertible notes, Bermudan option and any. putable or callable exotic coupon notes. It provides an estimate of when the option would be exercised. This is a useful indication for the determination of the maturity to use to hedge American or Bermudan products with European options.

In the case of a tree or a pde, the fugit is easily computed when discounting the cashflow backward. We initially start in the last slice of the tree or pde with an estimated time equal to the final date. When computing backward the cashflow, one checks whether the option is exercised or not. If so, the time of exercise at this node becomes precisely the current date. Otherwise, the estimated exercise time is computed as the probability weighted average of the exercise time of the previous slice. The estimated exercise time or fugit is given when reaching the last slice.

In the case of an American Monte Carlo, the computation of the fugit implies to first determine the exercise boundary and then using previous or new sampled paths (according to the method) compute the expected exercise time.

EXERCISE BOUNDARY PROBLEM

More importantly the concept of fugit for American and Bermudan (also referred to as Mid-Atlantic options) provides valuable information about the exercise boundary. The pricing of American or Bermudan options can be in fact reformulated in terms of dynamic programming and the Bellman principle. Simply speaking, the Bellman equation states that the optimal exercise strategy for an American option is to compare the early exercise value with the present value of all futures cash flow (wait for a further exercise) and take the strategy with greater value.

$$V(t, S_t) = \max\{F(t, S_t), PV_t[V(t + dt, S_t + dS_t)]\} \quad (1.1),$$

where S_t is the underlying asset, $V(t, S_t)$ is the value of the American option, $F(t, S_t)$ the value in case of an early exercise and $PV_t[V(t + dt, S_t + dS_t)]$ is the present value of the strategy consisting in waiting for the next exercise period. The dynamic programming formulation is generally used by American Monte Carlo techniques, together with the fact that one can generally approximate the early exercise decision by a function of the state variable S_t.

A straightforward American Monte method consists in maximizing the exercise boundary for a given simulation. Obviously, this leads to a lower bound for the American option value.

The other formulation of the problem used by trees and finite difference uses the dual formulation in term of Partial Differential Inequality. The relationship between the two approaches lies in the Feyman Kac formulations that translate this probabilistic problem into a Partial Differential Inequality problem. In the case for instance of local volatility models, this leads to

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma(t,S)^2 S^2 \frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV\leq 0 \qquad (1.2),$$

with the early exercise condition $V(t,S_t)\geq F(t,S_t)$ (1.3),

and the final boundary condition $V(T,S_T)\geq F(T,S_T)$ (1.4).

Compared with standard European Option pricing, the boundary condition are not well known, hence the name free boundary problem for it. The usual discretisation based on general theta scheme leads to use a Crank Nicolson method ($\theta=1/2$ [1]) on the logarithm of the underlying asset (guaratees that the equation is a real hyperbollic one). Denoting by $V_{i,j}$ the discretised function where the first variable i stands for time whereas the second one j for the space variable, one obtains the following results:

[1] Although a Douglas scheme is more accurate and very similar to the Crank Nicolson scheme, the banking industry seems to prefer Crank Nicolson algorithm. Recent developments in terms of Alternative Direction Methods have been popularised by local volatility and jumps and stochastic volatility and jumps models.

$$\frac{V_{i+1,j}-V_{i,j}}{\Delta T}$$
$$+(1-\theta)\left(r(i)-\frac{\sigma(i+1,j)^2}{2}\right)\frac{V_{i+1,j+1}-V_{i+1,j-1}}{2\Delta S}+\theta\left(r(i)-\frac{\sigma(i,j)^2}{2}\right)\frac{V_{i,j+1}-V_{i,j-1}}{2\Delta S}$$
$$+(1-\theta)\frac{\sigma^2(i+1,j)}{2}\left(\frac{V_{i+1,j+1}-2V_{i+1,j}+V_{i+1,j-1}}{\Delta S^2}\right)+\theta\frac{\sigma^2(i,j)}{2}\left(\frac{V_{i,j+1}-2V_{i,j}+V_{i,j-1}}{\Delta S^2}\right)$$
$$=r\frac{V_{i+1,j}-V_{i,j}}{2}$$

wich can be regroup to lead to the traditionnal triband diffusion matrix solved with the linear LU decomposition. The boundary conditions can take various forms (see boundary condition for options) like Dirichlet Van Neuman and second order derivatives.

CASE STUDY: BERMUDAN SWAPTION

Consider the case of a 10 into 20 year Bermudan swaption with exercise date every year starting at year 5 and onwards. Like for any other Bermudan or American structure, the question of the optimal exercise is quite crucial both for the seller of the option (for her hedging strategy) and the option holder (monetise the option value). Although, the party long the option often would like to sell the option to cash in the option value, it is rarely easy to do so as these products are OTC and barely trade cheaply. The party long the option would have no choice but to exercise the option if she thinks that this is ideal timing for the option. In contrast to exercising the option, selling the option could have provided more value as the option's value is the combined result of the early exercise strategy and the other wait and see strategy.

Unfortunately, this theoretical value is very hard to cash in. The Bermudan swaption market is a big business market since many corporate are interested in flexible swaptions. On the liability side, receiver Bermudan swaptions have been massively sold to issurance companies for their liabilities problems as they offer them protection against interest rate rise.

Bermudan swaption can be seen as an option on the most expensive European one plus the right to switch. Usually (in common examples), when computing the fugit,one finds a maturity quite close to the one of the most expensive European option. However, for Bermudan options that do not have a clear most expensive European option, the fugit can significantly differ from the maturity of the most expensive European option. Hedging only with the most expensive European option can lead to serious timing problem as the hedge may be inactivated before the exercise of the Bermudan option or vice versa.

Entry category: options

Scope: Option pricing, American option.

Related articles: American-style option pricing, Greeks

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