

Classical dynamics with curl forces, and motion driven by time-dependent flux

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Abstract

For position-dependent forces whose curl is non-zero ('curl forces'), there is no associated scalar potential and therefore no obvious Hamiltonian or Lagrangean and, except in special cases, no obvious conserved quantities. Nevertheless, the motion is nondissipative (measure-preserving in position and velocity). In a class of planar motions, some of which are exactly solvable, the curl force is directed azimuthally with a magnitude varying with radius, and the orbits are usually spirals. If the curl is concentrated at the origin (for example, the curl force could be an electric field generated by a changing localized magnetic flux, as in the betatron), a Hamiltonian does exist but violates the rotational symmetry of the force. In this case, reminiscent of the Aharonov–Bohm effect, the spiralling is extraordinarily slow.

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1. Introduction

For a particle moving classically under a force $F(r)$ depending only on position—that is, in the absence of velocity-dependent forces, e.g., dissipative or magnetic—the trajectory $\mathbf{r}(t)$ is the solution of Newton's equation:

$$\ddot{\mathbf{r}} = F(r). \quad (1.1)$$

If the force is derivable from a scalar potential, that is if $F(r) = -\nabla U(r)$, the dynamics can be generated by a Lagrangean, or a Hamiltonian of the familiar type $H(r, p) = \frac{1}{2}p^2 + U(r)$. But if no such potential exists—if the force has a non-zero curl, i.e.

$$\nabla \times F(r) \neq 0, \quad (1.2)$$

then there is no Hamiltonian or corresponding Lagrangean, at least of the usual kind. Nevertheless, in the 'phase space' of position r and velocity $v = \dot{r}$, the motion is volume-preserving, that is,

$$\nabla_r \cdot \dot{r} + \nabla_v \cdot \dot{v} = \nabla_r \cdot v + \nabla_v \cdot F(r) = 0, \quad (1.3)$$

irrespective of the nonexistence of a potential $U(r)$. We note that unconventional Lagrangeans can be found in some particular cases [1]; and we do not consider empty reformulations in which the phase space is doubled, enabling any evolution equation to be expressed in Hamiltonian form.

Our aim here is to explore, in an elementary way, motion determined by (1.1) with forces satisfying (1.2), which we will refer to as ‘curl forces’. As we will see, such nondissipative yet non-Hamiltonian dynamics has interesting features. For example, Noether’s theorem [2] does not apply, so there can be symmetries not associated directly with a conservation law and, conversely, conserved quantities not associated with any symmetry.

Dynamics driven by curl forces has been studied by mechanical engineers and applied mathematicians, for example in the context of whirling shafts [3, 4], and pendulums with rotational dissipation [5, 6], but less often by physicists [7–9]. Most of these studies concentrate on the simplest linear cases (for an exception, see [10]), and emphasize the sometimes unexpected effects of curl forces on stability [11–18]. In the literature, curl forces have been variously referred to as ‘follower forces’ [19–21], ‘circulatory forces’ [17, 22], ‘pseudo-gyroscopic forces’ [18] and ‘positional nonconservative forces’ [23].

The existence of curl forces is a matter of some controversy [24–27], centred on the question of whether it is physically possible for them to act alone, that is in the absence of additional forces not of curl type. This can be illustrated by attempting to reformulate the dynamics in terms of an electric curl force, with a time-dependent vector potential

$$A(r, t) = tF(r). \quad (1.4)$$

This would suggest describing the dynamics by the Hamiltonian

$$H(r, p) = \frac{1}{2}(p - A(r, t))^2. \quad (1.5)$$

But the attempt fails, because H generates an additional magnetic force, so the total force acting on the particle is

$$F_{\text{total}} = \frac{\partial A(r, t)}{\partial t} + v \times B(r, t) = F(r) + v \times B(r), \quad (1.6)$$

where

$$B(r, t) = \nabla \times A(r, t) = t\nabla \times F(r). \quad (1.7)$$

The unwanted magnetic force is non-zero precisely in the situation we are interested in, namely forces satisfying (1.2). But it is the basis of the betatron [28], in which electrons are accelerated (albeit relativistically) by the electric curl forces generated by a time-dependent magnetic field, with the unavoidable associated magnetic force tailored to stabilise the motion by maintaining circular orbits. In other cases that have been described, the non-curl forces have been dissipative [4, 5, 15].

In section 2 we present a class of models for particles moving under curl forces alone. We cannot find closed-form solutions for general motions, but the dynamics can be reduced to the Emden–Fowler equation [29, 30], for which a particular exact solution exists for a wide class of cases, described in section 3. Conserved quantities are known for two particular cases (section 4). The solutions in section 3 fail for three special cases of particular interest, and we consider them separately. Section 5 describes a case for which the Emden–Fowler equation reduces to a triviality. Section 6 describes the simple linear case, which can be solved exactly and indeed represented by infinitely many Hamiltonians, albeit not of the usual type. Section 7 describes another case that can be solved exactly, by transformation into linear form. The most interesting case (section 8) corresponds to particle motion determined by a time-dependent magnetic flux that is localized, so the only force is azimuthal electric. In this case there is a Hamiltonian but it has a peculiar feature: a classical evocation of the Aharonov–Bohm effect [31, 32]. And the orbits display pathologically slow windings.

2. Curl force models

In perhaps the simplest class of models of the type we are interested in, the particle moves in the plane (x, y) under the action of a curl force directed azimuthally, with a magnitude depending on distance r from the origin:

$$F(r) = f(r)e_\theta. \quad (2.1)$$

Since there is no radial force, the radial evolution $r(t)$ is determined by centrifugal acceleration alone:

$$\ddot{r}(t) = r(t)\dot{\theta}(t)^2. \quad (2.2)$$

This is never negative, so the particle will almost always eventually recede from the origin. It is worth emphasizing that there are azimuthal forces which are conventionally derived from a potential, i.e. not of the curl type (2.1); as has been noted [10], an example is the potential $U(r) = x/(x+y)$, which generates $F(r) = re_\theta/(x+y)^2$.

The azimuthal force (2.1) corresponds to a torque, giving the particle angular acceleration. Therefore the angular momentum $J = r^2\dot{\theta}$ is not conserved, even though the force (2.1) is rotationally symmetric: Noether's theorem does not apply because there is no corresponding Lagrangian or Hamiltonian. The J evolution is governed by

$$r(t)^2\ddot{\theta}(t) + 2r(t)\dot{r}(t)\dot{\theta}(t) = \frac{d}{dt}[r(t)^2\dot{\theta}(t)] \equiv \dot{J}(t) = r(t)f(r(t)). \quad (2.3)$$

Thus if $f(r)$ is positive, J never decreases. Often this leads to the particle winding round the origin forever, i.e. θ increasing without bound. When combined with the radial motion governed by (2.2), this implies that the orbits are spirals. However, we shall see that if $f(r)$ decreases fast enough the azimuthal motion can freeze; and section 8 will deal with a marginal case where the winding is perpetual but pathologically slow.

There is no obvious Hamiltonian for general $f(r)$. Nevertheless, the dynamics is separable, in at least two ways. The first starts by writing (2.2) as

$$J(t) = \sqrt{r(t)^3 \ddot{r}(t)}, \quad (2.4)$$

and then differentiating again. This leads to a third-order equation for $r(t)$ alone:

$$r + 3\dot{r}\ddot{r} - 2f(r)\sqrt{r\ddot{r}} = 0. \quad (2.5)$$

The initial conditions required to define a trajectory are

$$r(0) \equiv r_0, \quad \dot{r}(0) \equiv v_0, \quad \ddot{r}(0) = r_0\dot{\theta}(0)^2 = \frac{J(0)^2}{r_0^3}. \quad (2.6)$$

The second way is more useful and will play a major part in the following. It starts by writing the radial equation (2.2) in terms of angular momentum:

$$\ddot{r} = \frac{J^2}{r^3}. \quad (2.7)$$

Next, J is regarded as a new independent variable, and a new dependent variable is defined as the integrated torque:

$$T(J) \equiv \int_{r_0}^{r(J)} dr r f(r), \quad (2.8)$$

in which we will usually take $r_0 = 0$. It now follows from (2.3) that

$$\frac{dT}{dJ} \equiv T' = \dot{r}, \quad (2.9)$$

leading to the equation for $T(J)$:

$$T''(J) = \frac{J^2}{r(J)^4 f(r(J))}. \quad (2.10)$$

When this can be solved, $r(J)$ can be determined from (2.8), and its time dependence follows from the torque equation (2.3), i.e. by inverting

$$t = \int_{J_0}^{J(r)} \frac{dJ}{r(J)f(r(J))}. \quad (2.11)$$

Then the angular evolution can be found from

$$\theta(t) = \int_0^t dt \frac{J(t)}{r(t)^2}. \quad (2.12)$$

We will concentrate on a subclass of curl forces of the type (2.1), in which $f(r)$ is a power law:

$$f(r) = r^\mu. \quad (2.13)$$

We do not include a constant coefficient, because this can be removed by scaling time in (1.1). There is an additional scaling under which (1.1) is invariant, namely replacing r by ρ , defined by

$$\rho(t) = R r(R^{\mu-1}/2t). \quad (2.14)$$

This means we can eliminate one degree of initial-condition freedom by taking $r_0 = 1$, and we will frequently assume this in what follows.

Using the second separation described above, we find the variable T in (2.8) as

$$T = \frac{r^{\mu+2}}{\mu+2}. \quad (2.15)$$

Now the T evolution (2.10) can be conveniently expressed as the Emden–Fowler equation [29]

$$T''(J) = A J^n T^m, \quad (2.16)$$

in which

$$n = 2, \quad m = -\frac{\mu+4}{\mu+2}, \quad A = (\mu+2)^{-(\mu+4)/(\mu+2)}. \quad (2.17)$$

The exact closed-form solution of the Emden–Fowler equation is not known for general μ , so we cannot solve the dynamics explicitly, and nor can we find any conserved quantities that are functions of $r(t)$ and $v(t)$. However, special solutions are known for a class of μ , and these will be described in section 3. For some special μ , conserved quantities are known and will be described in section 4. In three special cases (sections 5–7) general solutions are known. And in section 8 we give an extended discussion of what is perhaps the most interesting case, namely $\mu = -1$, where a Hamiltonian does exist but is not single-valued.

Before proceeding, we mention a class of curl forces complementary to (2.1), in which the force is radial but can vary with azimuth. Thus

$$F(r) = g(r, \theta) \mathbf{e}_r, \quad (2.18)$$

in which $g(r, \theta)$ is a 2π -periodic function of θ . Because there is no torque, angular momentum is conserved, even though the force is not rotationally symmetric (except for the familiar central forces, in which g is independent of θ). Again, Noether's theorem does not apply. Motion under these non-symmetric central forces deserves further study, but we do not consider them further here.

We acknowledge that our inability to find exact solutions and conserved quantities (functions of r and v only) for general curl forces of the type (2.1) does not imply that none exist. If any such unconventional invariants are found, the question of their physical meaning will arise.

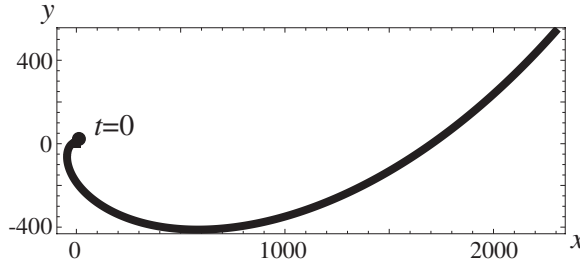


Figure 1. Orbit (3.1) (for curl force with exponent $\mu = 0$) for times $0 \leq t \leq 100$.

3. Special solutions for a range of exponents μ

From Emden–Fowler solution tables [29, 30], or directly from (2.2), (2.3) and (2.13), we can find the following special solutions, in which the particle emerges from the origin (i.e. the initial condition is not $r(0) = 1$) with infinite angular velocity and then recedes while spiralling ever more slowly:

$$\left. \begin{aligned} r(t) &= \left(\frac{(1-\mu)^2}{3+\mu} \right)^{1/(1-\mu)} \frac{t^{2/(1-\mu)}}{(2(1+\mu))^{1/(2(1-\mu))}} \\ \theta(t) &= \frac{\sqrt{2(1+\mu)}}{1-\mu} \log t \end{aligned} \right\} \quad (-1 < \mu < 1). \quad (3.1)$$

Figure 1 illustrates the case $\mu = 0$ (i.e. $m = -2$ in (2.17)), for which the orbit is

$$r(t) = \frac{t^2}{3\sqrt{2}}, \quad \theta(t) = \sqrt{2} \log t. \quad (3.2)$$

The solution (3.1) fails when μ takes the limiting values 1 or -1 and also $\mu = -3$, and sections 6–8 will be devoted to these cases ($\mu = 1$ and $\mu = -1$ have been cited, without elaboration, [33] as examples of non-potential forces).

4. Conserved quantities for $\mu = -3/2$ and $\mu = -5/3$

For Emden–Fowler equations with $n = 2$, the following functions of r and J (or equivalently r and $\dot{\theta}$) can be extracted from tabulated solutions [29] and easily confirmed to be conserved under the dynamics

$$\ddot{r} = \frac{J^2}{r^3}, \quad \dot{J} = r f(r) = r^{\mu+1} \quad (4.1)$$

for two special cases.

In the first, $\mu = -3/2$, i.e. $m = -5$, and the conserved quantity is

$$C_{-3/2} = (J\dot{r} - 2\sqrt{r})^2 + \frac{J^4}{r^2}. \quad (4.2)$$

In the second, $\mu = -5/3$, i.e. $m = -7$, the conserved quantity is

$$C_{-5/3} = J\dot{r}^2 - 3r^{1/3}\dot{r} + \frac{J^3}{r^2}. \quad (4.3)$$

Neither has any obvious physical interpretation.

5. Solvable case $\mu = -4$

This corresponds to $m = 0$, so the Emden–Fowler equation takes the trivial form

$$T''(J) = J^2. \quad (5.1)$$

The particular solution

$$T(J) = \int_1^{r(J)} dr r f(r) = \int_1^{r(J)} \frac{dr}{r^3} = \frac{1}{2} \left(1 - \frac{1}{r^2}\right) = \frac{1}{12} J^4 \quad (5.2)$$

(incorporating (2.8)) connects r with J and will correspond to the dynamics in which the particle starts from rest at $r = 1$.

We can find the shape $r(\theta)$ of the trajectory from (2.12), transforming the integration variable from t to r and using (2.9):

$$\theta = \int_1^r \frac{dr J^2}{\dot{r} r} = \int_1^r \frac{dr J^2}{T'(J) r} = 3 \int_1^r \frac{dr J^2}{J^2 r} = \sqrt{\frac{3}{2}} \int_1^r \frac{dr J^2}{r \sqrt{r^2 - 1}} = \sqrt{\frac{3}{2}} \sec^{-1} r. \quad (5.3)$$

Thus

$$r(\theta) = \sec\left(\theta \sqrt{\frac{2}{3}}\right), \quad (5.4)$$

implying that the orbit departs from $r = 1$ in the y direction and recedes towards infinity in the asymptotic direction $\theta_{\max} = \frac{\pi}{2} \sqrt{\frac{3}{2}} \approx 110.23^\circ$.

To find the time dependence of r , we use (2.11), to get

$$\begin{aligned} t = \int_0^{J(r)} dJ r(J)^3 &= \int_1^r dr \frac{dJ}{dr} r^3 = \left(\frac{3}{8}\right)^{1/4} \int_1^r \frac{dr r^{3/2}}{(r^2 - 1)^{3/4}} \\ &= \left(\frac{3}{8}\right)^{1/4} \left[\sqrt{r(r^2 - 1)}^{1/4} + \frac{1}{\sqrt{2}} \left(K\left(\frac{1}{\sqrt{2}}\right) - F\left(\cos^{-1}(1 - 1/r^2)^{1/4}, \frac{1}{\sqrt{2}}\right) \right) \right], \end{aligned} \quad (5.5)$$

where K and F are elliptic integrals, in the notation of [34]. This shows how the radial coordinate increases from $r = 1$ at $t = 0$ and grows linearly as $t \rightarrow \infty$.

6. Solvable linear curl force ($\mu = 1$)

For this case, $\mu = -5/3$. The corresponding Emden–Fowler equation is exactly solvable [29], but it is easier to note that the force determined by (2.13) is $F(r) = r e_\theta$, so (1.1) is a linear equation, that can be written

$$\ddot{x} = -y, \quad \ddot{y} = x. \quad (6.1)$$

The exact solution is very simple. A convenient way starts by defining

$$\zeta \equiv x + iy, \quad (6.2)$$

so (6.1) can be written as

$$\ddot{\zeta}(t) = i\zeta(t), \quad (6.3)$$

whose general solution is

$$\begin{aligned} \zeta(t) &= A \exp(t\sqrt{i}) + B \exp(-t\sqrt{i}) \\ &= A \exp\left(\frac{t}{\sqrt{2}}\right) \exp\left(\frac{it}{\sqrt{2}}\right) + B \exp\left(-\frac{t}{\sqrt{2}}\right) \exp\left(-\frac{it}{\sqrt{2}}\right). \end{aligned} \quad (6.4)$$

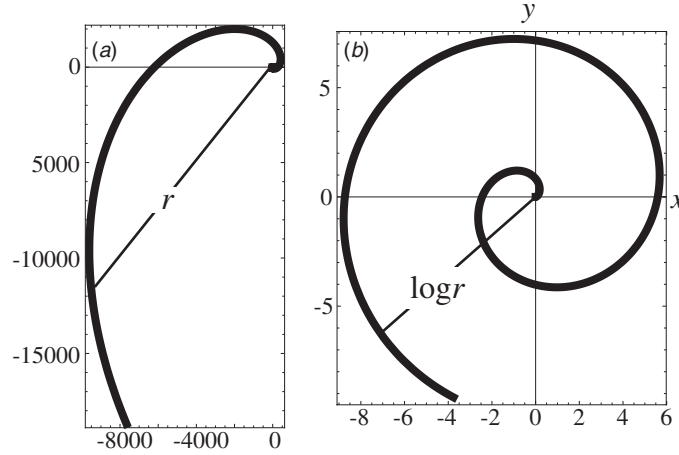


Figure 2. (a) Orbit (6.6) under linear rotational curl force (exponent $\mu = 1$); (b) as (a) but with radius logarithmically scaled, revealing the inner windings of the orbit.

This is the superposition of two spiral motions with uniform angular velocity: one anticlockwise, and spiralling exponentially fast outwards, and the other clockwise and spiralling exponentially fast inwards.

For the particle starting from rest on the x axis, the initial conditions are

$$A = B = \frac{1}{2}, \quad \text{i.e. } x(0) = 1, y(0) = \dot{x}(0) = \dot{y}(0) = 0, \quad (6.5)$$

and the orbit is simply

$$\zeta(t) = \cosh(t\sqrt{i}), \quad (6.6)$$

as illustrated in figure 2.

The evolution equation (6.3) possesses the obvious complex conserved quantity

$$\dot{\zeta}(t)^2 - i\zeta(t)^2 = C_1 + iC_2. \quad (6.7)$$

Therefore C_1 and C_2 , namely

$$\begin{aligned} v_x^2 - v_y^2 + 2xy &= C_1, \quad 2v_x v_y - x^2 + y^2 = C_2, \quad \text{i.e.} \\ C_1 &= (\dot{r}^2 - r^2 \dot{\theta}^2) \sin 2\theta - (r - 2\dot{r}\dot{\theta}) \cos 2\theta \\ C_2 &= (\dot{r}^2 - r^2 \dot{\theta}^2) \cos 2\theta + (r - 2\dot{r}\dot{\theta}) \sin 2\theta \end{aligned} \quad (6.8)$$

are real invariants, as is any combination of them. Associated with C_1 and C_2 are the following Hamiltonians (suggested by manipulations in [1]):

$$\begin{aligned} H_1(r, p) &= \frac{1}{2} (p_x^2 - p_y^2) + xy \\ H_2(r, p) &= p_x p_y + \frac{1}{2} (y^2 - x^2). \end{aligned} \quad (6.9)$$

Either will generate the evolution (6.1), but neither reflects the rotational symmetry of the force $F(r) = re_\theta$, and C_1 and C_2 are not related to this symmetry.

7. Solvable inverse cube curl force ($\mu = -3$)

For this case, $m = 1$ and the Emden–Fowler equation (2.16) is linear:

$$T''(J) = J^2 T(J). \quad (7.1)$$

The solution [34] is conveniently written, incorporating (2.8) with $r_0 = \infty$, as

$$T(J) = - \int_r^\infty \frac{dr}{r^2} = -\frac{1}{r} = -\sqrt{J} \left(A J_{1/4} \left(\frac{1}{2} J^2 \right) + B Y_{1/4} \left(\frac{1}{2} J^2 \right) \right), \quad (7.2)$$

in which $J_{1/4}$ and $Y_{1/4}$ (not italics, and not to be confused with J) denote Bessel functions, and A and B are constants.

To find the spatial track we use (2.12), which for this case gives

$$\theta = \int_0^J \frac{dJ}{J} \frac{J}{r^2} = \int_0^J dJ J = \frac{1}{2} J^2. \quad (7.3)$$

Thus, from (7.2), the track is

$$r(\theta) = \frac{1}{\theta^{1/4} (A J_{1/4}(\theta) + B Y_{1/4}(\theta))}. \quad (7.4)$$

And for the time-dependence, (2.11) gives

$$\begin{aligned} t(\theta) &= \int_0^\theta \frac{dJ(\theta')}{d\theta'} r(\theta')^2 = \int_0^\theta \frac{d\theta' r^2(\theta')}{\sqrt{2\theta'}} \\ &= \frac{\pi}{2\sqrt{2}} \left(\frac{A Y_{1/4}(\theta) - B J_{1/4}(\theta)}{A J_{1/4}(\theta) + B Y_{1/4}(\theta)} - \frac{A}{B(A^2 + B^2)} \right). \end{aligned} \quad (7.5)$$

This is the general solution. It will suffice to consider the particular case $A = 0$, $B = -1$, for which

$$r(\theta) = -\frac{1}{\theta^{1/4} Y_{1/4}(\theta)}, \quad t(\theta) = -\frac{\pi}{2\sqrt{2}} \frac{J_{1/4}(\theta)}{Y_{1/4}(\theta)}. \quad (7.6)$$

This corresponds to the initial conditions

$$\begin{aligned} x(0) &= \frac{\pi}{2^{1/4} \Gamma(\frac{1}{4})} = 0.7287, \dots, \quad y(0) = 0, \\ \dot{x}(0) &= v_0 = -\frac{\Gamma(-\frac{1}{4})}{2^{5/4} \pi} = 0.6560, \dots, \quad \dot{y}(0) = 0. \end{aligned} \quad (7.7)$$

Figure 3 shows the trajectory. The particle's azimuth increases from $\theta = 0$ and freezes asymptotically at θ_{\max} , given by the first zero of $Y_{1/4}$:

$$Y_{1/4}(\theta_{\max}) = 0, \quad \text{i.e. } \theta_{\max} = 1.241\,66, \dots, \approx 71.142^\circ. \quad (7.8)$$

8. Localized time-dependent flux (curl force with $\mu = -1$)

Here, $m = -3$ and the Emden–Fowler does not appear to be solvable in closed form. In this most interesting case, the force, namely

$$F(r) = \frac{e_\theta}{r} = \frac{x e_y - y e_x}{r^2}, \quad (8.1)$$

is irrotational everywhere except at the origin, where the curl is localized:

$$\nabla \times F(r) = 2\pi \delta(r) e_z. \quad (8.2)$$

Therefore by excluding the origin it is possible to define a scalar potential, namely $U(r) = -\theta$, and thus a Hamiltonian representing a conserved energy:

$$H(r, p) = \frac{1}{2} p^2 - \theta = \frac{1}{2} \left(p_r^2 + \frac{J^2}{r^2} \right) - \theta = E = \text{constant}. \quad (8.3)$$

Excluding the origin carries a price: the plane is no longer simply connected, so the domain of the angular coordinate is the entire real line $-\infty < \theta < +\infty$. Therefore H does

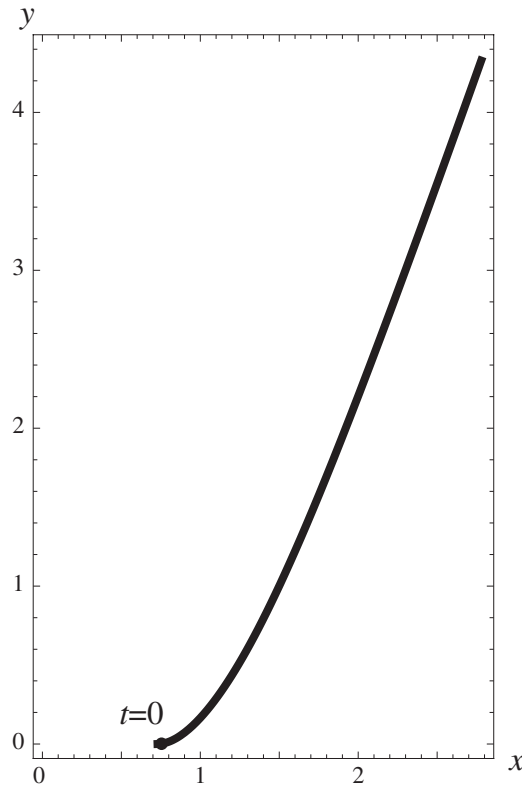


Figure 3. Orbit (7.6) under inverse-cube curl force (exponent $\mu = -3$) for times $0 \leq t \leq 4.3$.

not reflect the periodicity of the force (8.1). This phenomenon is well known in classical Hamiltonian and quantum physics—for example, the scalar potential of a uniform electric force, or the vector potential of a uniform magnetic field, are not themselves uniform and so violate the translation symmetry of the fields (or the periodicity of a crystal to which the field is applied). Here, the situation is worse because (8.3) is not single-valued: the same point can be represented by values of θ differing by multiples of 2π . H could be made single-valued by eliminating θ by the transformation (1.4) and (1.5), but then the time symmetry would be violated and energy would no longer be conserved.

As in our other examples, angular momentum J is not conserved with the Hamiltonian (8.3), but in this special case J evolves in a very simple way. From (2.3), the torque \dot{J} has the constant value unity, so J itself increases linearly:

$$J(t) = r^2(t) \dot{\theta}(t) = t + J_0. \quad (8.4)$$

(With the transformation (1.4) and (1.5), rotational symmetry would be restored and the canonical angular momentum conserved, and J would differ from the kinetic angular momentum precisely by the term t in (8.4).)

We can set $J_0 = 0$ by suitably defining the origin of time. If in addition we choose an orientation of coordinates such that $\theta(0) = 0$, the general motion under the force (8.1) corresponds to the particle released from $x_0 = 1$, $y_0 = 0$ with velocity v_0 along the x axis. Thus the energy is

$$E = \frac{1}{2} v_0^2, \quad (8.5)$$

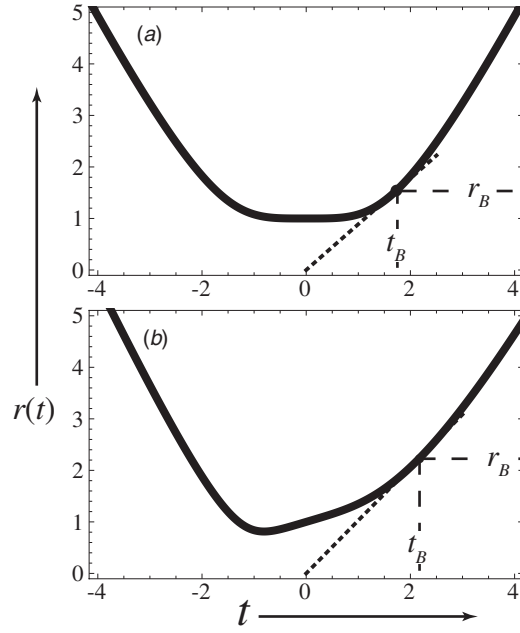


Figure 4. Evolution of radial coordinate for particle driven by time-dependent flux, calculated by solving (8.7) for (a) $v_0 = 0$ (particle at rest at $t = 0$), (b) $v_0 = 0.3$. The tangent construction (dotted lines) is explained after (8.13), and corresponds to (a) $t_B = 1.6226$ and $r_B = z_{Bt_B} = 1.4423$, (b) $t_B = 2.1338$ and $r_B = z_{Bt_B} = 2.1941$.

and $\theta(t)$ can be determined from the Hamiltonian if $r(t)$ is known:

$$\theta(t) = \frac{1}{2} \left(\dot{r}^2(t) - v_0^2 + \frac{t^2}{r^2(t)} \right). \quad (8.6)$$

The radial coordinate is now determined from (2.2) by

$$r^3(t) \ddot{r}(t) = t^2. \quad (8.7)$$

This is simply the Emden–Fowler equation (2.16), because for $\mu = -1$ we have $J(t) = t$ from (8.4), and, from (2.8), $T = r$. It is easy to solve (8.7) numerically; figure 4 shows the solutions for two different values of v_0 . But it is instructive to understand the evolution analytically for short and long times. The symmetry

$$r(-t, v_0) = r(t, -v_0) \quad (8.8)$$

means that if we consider general v_0 we need study only $t > 0$.

8.1. Short times

After some inspection, it becomes clear that the general power series solution of (8.7) has the form

$$r(t) = 1 + v_0 t + \sum_{n=4}^{\infty} b_n t^n. \quad (8.9)$$

The coefficients are easy to calculate recursively; table 1 shows the first few.

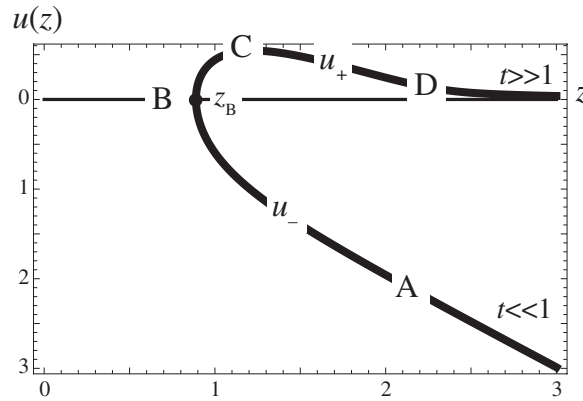


Figure 5. Double-valued solution for $u(z)$ corresponding to figure 4(a), obtained by solving (8.11) for $z_B = 0.888\ 87$, corresponding to $v_0 = 0$; the behaviour for other values of v_0 is qualitatively identical.

Table 1. Coefficients in the series (8.9).

n	4	5	6	7	8
b_n	$-\frac{1}{12}$	$-\frac{3}{20}v_0$	$\frac{1}{5}v_0^2$	$-\frac{5}{21}v_0^3$	$\frac{1}{224}(-1 + 60v_0^4)$

Numerical calculation of many coefficients suggests that they decay exponentially with increasing n , so the series appears to be convergent with a finite radius of convergence. In any case it is useless for the numerical computation of $r(t)$ beyond $t \sim 1$ and we do not consider this series further.

8.2. Long times

It is possible to study the large t asymptotics of (8.7) directly, but it is interesting and instructive to transform to a first-order equation, involving the new independent variable z and dependent function $u(z)$, defined by

$$z \equiv \frac{r(t)}{t}, \quad u(z) \equiv \dot{r}(t) - \frac{r(t)}{t} = \dot{r}(t) - z = t\dot{z} \quad (8.10)$$

(this is a slight modification of a transformation suggested to us by Professor Clara Nucci). The transformed equation—also deceptively simple—is

$$u(z)u'(z) + u(z) = \frac{1}{z^3}. \quad (8.11)$$

Once the solution is known, $r(t)$ can be reconstructed from any given time t^* using the relations

$$t(z) = t^* \exp\left(\int_{z^*}^z \frac{dz}{u(z)}\right), \quad r(z) = zt(z), \quad z^* = \frac{r^*}{t^*}, \quad (8.12)$$

to which we will return later.

Figure 5 shows a numerically computed solution of (8.11), indicating that the function $u(z)$ is double-valued, with positive- and negative-valued branches $u_{\pm}(z)$. These are connected where $u = 0$, at $z = z_B$, corresponding, according to (8.10), by

$$\dot{r}(t_B) = \frac{r_B}{t_B}. \quad (8.13)$$

Table 2. Coefficients in the large z expansion (8.20) for $u_+(z)$, calculated from (8.21).

n	0	1	2	3	4	5	6
C_n	1	3	30	483	10314	268686	8167068

On the graph of $r(t)$ (figure 4) this is the point where the tangent to the curve passes through the origin. Solution of (8.11) for $u \ll 1$ gives the local behaviour

$$u_{\pm}(z) \approx \pm \sqrt{\frac{2}{z_B^3}} (z - z_B) \quad (z \approx z_B). \quad (8.14)$$

Significant regions of the graph of $u(z)$ are denoted A, B, C, D on figure 5. The regions A and D both correspond to $z \rightarrow \infty$, with A representing $t \ll 1$, and region D representing $t \gg 1$. To get the large t asymptotics of $r(t)$, in a way that incorporates the initial velocity v_0 , it is therefore necessary to connect region A to region D.

In region A, where $1/z^3$ can be neglected in (8.11), the asymptotic solution is

$$u_-(z) \rightarrow v_0 - z \quad (z \rightarrow \infty). \quad (8.15)$$

In region B we have the solution (8.14). In region C, which we will not consider further but include for completeness, the defining conditions are

$$u'_+(z_C) = 0 \rightarrow u_+(z_C) = \frac{1}{z_C^3} \rightarrow \dot{r}(t_C) = \frac{r_C}{t_C} + \frac{1}{z_C^3}. \quad (8.16)$$

This leaves the asymptotic region D of u_+ . In the leading order, the term in (8.11) involving u' can be neglected, giving

$$u_+(z) \sim \frac{1}{z^3}. \quad (8.17)$$

From (8.12) and (8.6), this gives

$$r(t) \sim t (4 \log t)^{1/4}, \quad \theta(t) \sim \sqrt{\log t}. \quad (8.18)$$

We will see that these are only rough approximations. But they do indicate that despite the decay of $f(r)$ the orbit spirals perpetually, with the windings slowing dramatically for long times: the time for n turns around the origin is approximately

$$t_n \sim t_{n0} = \exp(4\pi^2 n^2). \quad (8.19)$$

To go further, we write the formal large z series solution of (8.11) as

$$u_+(z) = \frac{1}{z^3} \sum_{n=0}^{\infty} \frac{C_n}{z^{4n}}, \quad C_0 = 1. \quad (8.20)$$

The coefficients are determined by

$$C_n = \sum_{m=1}^n C_{m-1} C_{n-m} (4m-1). \quad (8.21)$$

Table 2 shows the first few coefficients. Although we will need only the first two terms, the high orders are interesting and are discussed in the [appendix](#). Note that the coefficients are independent of the initial velocity v_0 . It is likely that the contribution of v_0 is exponentially small and not captured by the formal series, and some preliminary numerical simulations support this; but such refined exponential asymptotics is unnecessary because, as will now be shown, the contribution of v_0 to the asymptotics of $r(t)$ is much larger.

Now we describe the connection between region A and region D. For this we use (8.12) twice: for u_+ and u_- . Between A and B, we subtract the large z asymptotics, by writing

$$\frac{1}{u_-(z)} = \frac{1}{v_0 - z} + \Delta_-(z). \quad (8.22)$$

Then (8.12) gives

$$\begin{aligned} t(z) &= t_B \exp \left(\int_{z_B}^z dz \left(\frac{1}{v_0 - z} + \Delta_-(z) \right) \right) \\ &= t_B \frac{(z_B - v_0)}{(z - v_0)} \exp \left(\int_{z_B}^z dz \Delta_-(z) \right), \end{aligned} \quad (8.23)$$

and hence

$$r(z) = z t(z) = t_B z \frac{(z_B - v_0)}{(z - v_0)} \exp \left(\int_{z_B}^z dz \Delta_-(z) \right). \quad (8.24)$$

In the limit $z \rightarrow \infty$, and using $r_0 = 1$, this enables $t_B(v_0)$ to be identified as

$$t_B(v_0) = \frac{1}{z_B(v_0) - v_0} \exp \left(- \int_{z_B(v_0)}^{\infty} dz \Delta_-(z) \right), \quad (8.25)$$

where $z_B(v_0)$ can be identified as the place where the solution with the asymptotic form (8.15) is zero. The integral converges because of the subtraction in (8.23).

We can connect B with D directly (it is unnecessary to consider C separately), by proceeding similarly, with the subtraction (from the first two terms of (8.20)) in this case being

$$\frac{1}{u_+(z)} = z^3 - \frac{3}{z} + \Delta_+(z). \quad (8.26)$$

Using (8.12) again gives

$$t = t_B \left(\frac{z_B}{z} \right)^3 \exp \left(\frac{1}{4} (z^4 - z_B^4) + \int_{z_B}^z dz \Delta_+(z) \right). \quad (8.27)$$

For $t \gg 1$ this is

$$t \approx \frac{1}{z^3} \exp \left(\frac{1}{4} z^4 + K_B(v_0) \right), \quad (8.28)$$

in which

$$K_B(v_0) = -\frac{1}{4} z_B^4 + \log(z_B^3 t_B) + \int_{z_B}^{\infty} dz \Delta_+(z), \quad (8.29)$$

and again the integral converges. Thus

$$4 \log t = z^4 - 12 \log z + 4K_B(v_0) + \dots \quad (8.30)$$

and, by inversion,

$$z^4 = 4 \log t + 3 \log(4 \log t) - 4K_B(v_0) + \dots \quad (8.31)$$

giving finally the desired more sophisticated asymptotic form for $r(t)$:

$$r(t) = t (4 \log t + 3 \log(4 \log t) - 4K_B(v_0))^{1/4} + \dots \quad (8.32)$$

Knowing the asymptotics of $r(t)$, the asymptotics of the azimuth $\theta(t)$ can be determined from (8.6).

The procedure based on (8.29) for getting the connection constant $K_B(v_0)$, incorporating the initial velocity into the final asymptotics, is admittedly numerical, and it would be desirable to get an analytical form for it. But now that the form (8.32) has been established, the constant

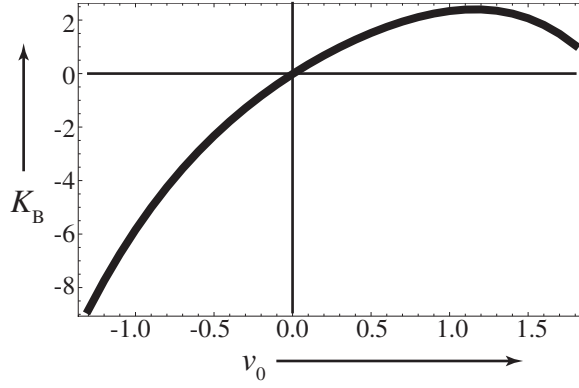


Figure 6. The connection constant K_B defined by (8.29), in the long-time asymptotics (8.32) of the radial coordinate, calculated from (8.33) as a function of the initial speed v_0 .

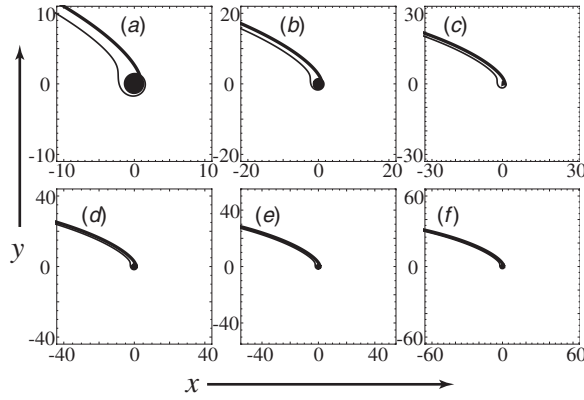


Figure 7. Orbits driven by time-dependent flux with $v_0 = 0$, for $0 \leq t \leq t_{\max}$, where (a) $t_{\max} = 10$, (b) $t_{\max} = 20$, (c) $t_{\max} = 30$, (d) $t_{\max} = 40$, (e) $t_{\max} = 50$, (f) $t_{\max} = 50$. Thick curves calculated numerically from (1.1) and (8.1); thin curves from asymptotic approximation (8.32) and (8.6) with $K_B = 0$.

can be alternatively and conveniently determined directly from a numerical solution of (8.7), as

$$K_B = \lim_{t \rightarrow \infty} \left(-\frac{r(t)^4}{4t^4} + \log t + \frac{3}{4} \log(4 \log t) \right). \quad (8.33)$$

Figure 6 is a graph of $K_B(v_0)$, calculated in this way. The limiting value for large t is attained rapidly, except near $v_0 = 0$. The value for $K_B(0)$ decreases slowly as t increases, leading us to conjecture that $K_B(0) = 0$; an analytical proof or refutation would be desirable.

Figures 7 and 8 show the trajectories computed numerically from (1.1) with the force (8.1), compared with the asymptotic trajectories calculated from (8.32) and (8.6). From figure 7, it is clear that as t increases the asymptotics converges rapidly onto the numerically computed solution. And from figure 8, for much larger values of t , it is clear that the spiralling is extremely slow. Table 3 shows the extraordinarily long times taken for the first few turns, that is for θ to reach $2\pi n$, starting from $\theta_0 = 0$; evidently the more sophisticated approximation based on (8.32) and (8.6) is a vast improvement.

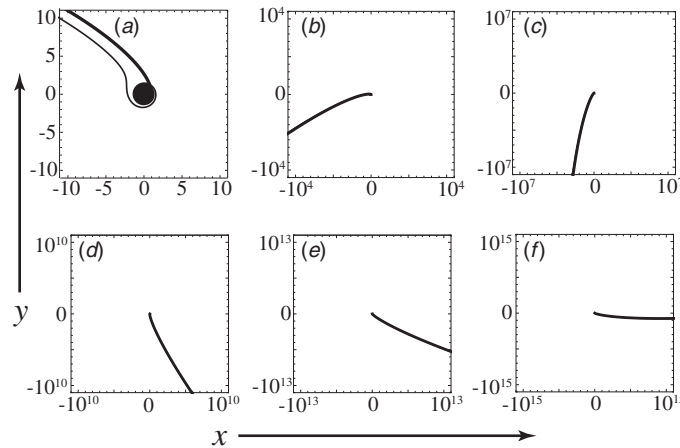


Figure 8. As figure 7, with (a) $t_{\max} = 10$, (b) $t_{\max} = 10^4$, (c) $t_{\max} = 10^7$, (d) $t_{\max} = 10^{10}$, (e) $t_{\max} = 10^{13}$, (f) $t_{\max} = 10^{15}$, showing very slow spiralling.

Table 3. Times t_n for n turns, compared with the crude approximation t_{n0} (equation (8.19)) and the more sophisticated approximate turn times t_{napp} obtained from (8.32) and (8.6).

n	1	2	3
t_n	7.138×10^{14}	6.746×10^{65}	1.946×10^{151}
t_{n0}	1.40×10^{17}	3.81×10^{68}	2.03×10^{154}
t_{napp}	7.475×10^{14}	6.894×10^{65}	1.977×10^{151}

9. Concluding remarks

It seems clear that motion under curl forces is a rich source of largely unexplored mathematical physics. Is it also physics that could be probed experimentally, at least in principle? We are not sure. The question concerns particle dynamics under the action of curl forces alone: if other forces (e.g. magnetic or dissipative) act as well, then physical implementation is unproblematic, as the betatron and many engineering examples cited in section 1 demonstrate. Of the cases we have considered, the only one that is obviously realizable is that considered in section 8: the electric force from a localized time-dependent flux, for example in a betatron without confining magnetic field. But in this case the force is curl-free in the region in which the particle moves, i.e. the plane with the origin removed. Nevertheless, the strange Hamiltonian (8.3), involving the angle θ which is not single-valued (a curious evocation of the Aharonov–Bohm effect as we remarked earlier), is interesting, as are the pathologically slow asymptotic windings and the fact that there seems to be no conserved quantity other than the energy.

There are several possible generalizations and extensions of this work. One is to explore curl forces of the form (2.18), where the force is radial, so angular momentum is conserved although there is apparently no Hamiltonian. Another is to study motion under fully three-dimensional curl forces.

A third direction involves nondissipative ‘magnetic’ forces $B(r)$ that are velocity-dependent but not divergenceless; Newtonian dynamics can be defined by a ‘Lorentz’ force $F = v \times B(r)$, but there is no corresponding vector potential $A(r)$ satisfying $B(r) = \nabla \times A(r)$ and so no Hamiltonian, even though such forces conserve the kinetic energy $\frac{1}{2}v^2$. One such situation, analogous to the localized time-dependent flux in section 8, is classical motion in

the field of a magnetic monopole [35]. In this case, which arises naturally in the dynamics of a slow system coupled to a fast one [36] (and was the system whose contemplation led to the present study), the divergence is zero everywhere except at the origin, where the monopole is located, so, as is well known [37, 38], A can be defined locally but not globally.

Finally, there is the question of whether quantum mechanics can be defined for curl forces. We do not know the answer in general, but remark that in the case of localized time-dependent flux the motion could perhaps be quantized on the multiply-connected space of the plane minus the origin, via the Hamiltonian (5.5).

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Appendix. Asymptotics of coefficients determined recursively by (8.21)

With the transformation

$$C_n \equiv 4^n D_n, \quad (\text{A.1})$$

and conflating the terms m and $n-m$, the recurrence relation (5.21) takes the more symmetrical form

$$D_n = \left(n + \frac{1}{2}\right) \sum_{m=1}^{\text{int}(n/2)} D_{m-1} D_{n-m}. \quad (\text{A.2})$$

The large n ansatz

$$D_n = G(n + \alpha)! \quad (\text{A.3})$$

leads to

$$(n + \alpha)! = \left(n + \frac{1}{2}\right) \left[(n + \alpha - 1)! + \frac{3}{4} (n + \alpha - 2)! \cdots\right], \quad (\text{A.4})$$

so

$$\begin{aligned} 1 &= \left(n + \frac{1}{2}\right) \left[\frac{1}{(n + \alpha)} + \frac{3}{4(n + \alpha)(n + \alpha - 1)} + \cdots\right] \\ &= 1 + \frac{1}{n} \left(\frac{5}{4} - \alpha\right) + \cdots \end{aligned} \quad (\text{A.5})$$

leading to the identification $\alpha = 5/4$. The constant G can be evaluated numerically:

$$G = \lim_{n \rightarrow \infty} \frac{C_n}{4^n (n + \frac{5}{4})!} = 0.352 \dots \quad (\text{A.6})$$

Thus the coefficients in (8.20) are

$$C_n \approx G 4^n \left(n + \frac{5}{4}\right)! \quad (n \gg 1), \quad (\text{A.7})$$

indicating that the series is factorially divergent and with all terms positive (i.e. on the Stokes line of its variable z). The terms decrease until $n \sim z^4/4$ and then increase rapidly. Truncation near the least term gives

$$u(z) \approx \frac{1}{z^3} \left(\sum_{n=0}^{\text{int}(z^4/4)-1} \frac{C_n}{z^{4n}} + G \sum_{n=\text{int}(z^4/4)}^{\infty} \left(\frac{4}{z^4}\right)^n \left(n + \frac{5}{4}\right)! \right). \quad (\text{A.8})$$

The divergent tail is in standard form [39] and can be estimated by Borel summation [40] as

$$\sum_{n=\text{int}(z^4/4)}^{\infty} \left(\frac{4}{z^4}\right)^n \left(n + \frac{5}{4}\right)! \approx \sqrt{2\pi} \left(\frac{z^2}{4}\right)^{3/4} \left(\frac{z^2}{4} - \text{int}\left[\frac{z^2}{4}\right] - \frac{11}{12}\right) \exp\left(-\frac{z^4}{4}\right). \quad (\text{A.9})$$

This is a precise version of the familiar rule [39, 41, 42] that when an asymptotic expansion whose terms are all positive is truncated near its least term, the remainder is of the same order as the least term, which is exponentially small.

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