

# **Constructibility of Regular Polygons**

**Iowa State University**

**MATH 599**

**Creative Component**

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## Constructibility of Regular Polygons

3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 48, 51, 60, 64, 68,  
80, 85, 96, 102, 120, 128, 136, 160, 170, 192, 204, 240, 255, 256, 257, 272

The above list of 38 numbers appears as the concluding statement of the greatest work written by one of the elite mathematicians of all time. Carl Friedrich Gauss ended his masterpiece *Disquisitiones Arithmeticae* with this sequence. It represents the numbers of sides (under 300) of regular polygons that can be constructed by traditional straight-edge and compass techniques. While the idea of constructing regular polygons had been studied by the early Greeks, along with many others, it was Gauss that proved with finality which regular polygons could be constructed, and which ones could not. In this paper, I will discuss what was known on this topic in the time of Euclid, and relate it to what Gauss was able to prove in what could be described as the discovery in which he took the most pride. ([mathcircle.berkeley.edu](http://mathcircle.berkeley.edu))

The construction of regular polygons is a topic that has been studied for many years. Going back to the early Greek mathematicians, we can see that by using specific tools with well defined rules for their use, the techniques to construct certain regular polygons were widely known. It was the Greek mathematician Euclid who chronicled what was known at that time in what was regarded as the greatest compilation on the subject of Geometry for over two millennia, *Euclid's Elements*.

While very few of the specific details of his life are known, there is evidence that Euclid must have lived during the reign of Ptolemy I (323 – 283 BC). He was active in

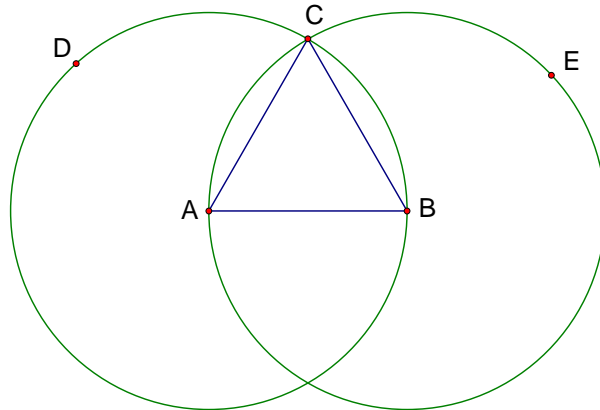
the great library in Alexandria, and could have possibly studied at Plato's Academy in Greece. However, the exact dates of his birth and death, and the manner of his death are not known. (Stillwell, p35) The *Dictionary of Scientific Biography* contains the following passage regarding Euclid: (Euclid, p xx)

Euclid's fame rests preeminently upon the *Elements*, which he wrote in thirteen books and which has exercised an influence upon the human mind greater than that of any other work except the Bible. For this reason he became known in antiquity as **Ὁ Στοιχειωτής**, "The Writer of the Elements", and sometimes simply as **Ὁ Γεωμέτρης**, "The Geometer". There had been *Elements* written before Euclid – notably by Hippocrates, Leo, and Theudius of Magnesia – but Euclid's work superseded them so completely that they are now known only from Eudemus's references as preserved by Proclus.

*Euclid's Elements* remained the primary textbook on Geometry for hundreds of years, until men like Saccheri, Lobachevsky, and Bolyai (and arguably Gauss himself) began to question the choices that Euclid had made regarding his postulates. This would be the beginning of what is now referred to as Non-Euclidean Geometry. However, Euclidean Geometry is still what is taught in many secondary schools as an introduction to axiomatic systems and formal proof. (wikipedia.org)

In the *Elements*, Euclid addresses the topic of construction of regular polygons. Euclid describes techniques for constructing several specific polygons, including steps for an equilateral triangle, a square, a regular pentagon, and a regular hexagon. For instance, the following process is Euclid's instruction for the construction of an equiangular triangle (Book I, Proposition 1):

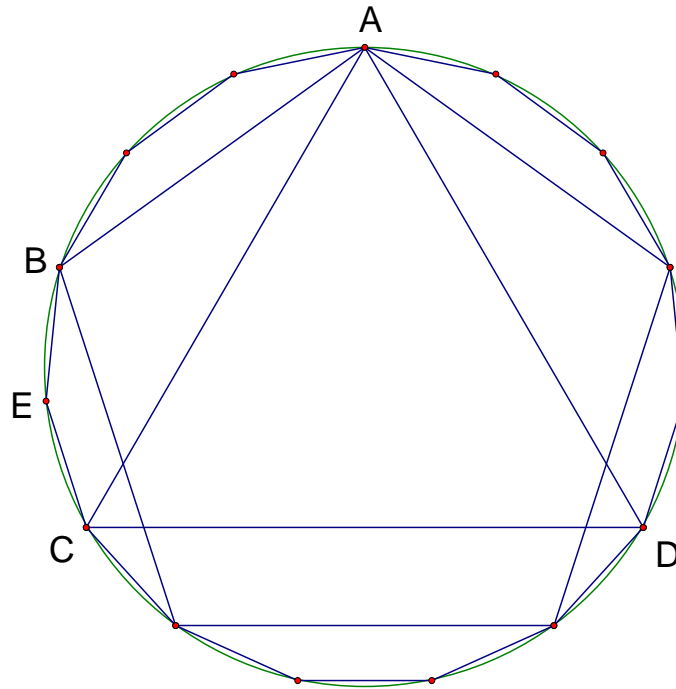
**To construct an equilateral triangle on a given finite straight-line:**



Let AB be the given finite straight-line. So it is required to construct an equilateral triangle on the straight-line AB. Let the circle BCD with center A and radius AB have been drawn, and again let the circle ACE with center B and radius BA have been drawn. And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another, to the points A and B (respectively). And since the point A is the center of the circle CDB, AC is equal to AB. Again, since the point B is the center of the circle CAE, BC is equal to BA. But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another. Thus, CA is also equal to CB. Thus, the three (straight-lines) CA, AB, and BC are equal to one another. Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB. (Which is) the very thing it was required to do. (Euclid, p3)

Euclid also demonstrates how to construct a regular fifteen-sided polygon. By constructing an equilateral triangle and a regular pentagon so that they are inscribed in the same circle, and sharing one vertex, the two shapes divide the circle into thirds and fifths, respectively. Using the following diagram, we can see that of the fifteen equal segments that the circle must contain, three of them are contained in arc AB, and five of them are contained in arc AC. Therefore, two of them are contained in the remainder, arc

BC. By bisecting arc BC, the side of the fifteen-sided polygon is established. (Euclid, p98)



By following the techniques outlined in *Euclid's Elements*, it is possible to construct regular polygons with the following numbers of sides: 3, 4, 5, and 15. By bisection, polygons of sides 6, 8, 10, 12, 16, 20, etc. can also be constructed. “This is where things stood until March 29, 1796, when at age eighteen Carl Friedrich Gauss (1777-1855) created a ruler and compass construction for a regular 17-gon.” (Martin, p18)

Carl Friedrich Gauss was a German mathematician and physicist who is regarded as one of the top mathematical minds of all time. He was the only son of lower-class uneducated parents, but at a young age demonstrated a remarkable ability for mathematical calculation. Well-known anecdotes of him teaching himself to read at age two, correcting his father's arithmetic at age three, or adding the numbers 1 to 100

mentally at the age of seven have been used to illustrate his genius at a young age. These stories have been cast into some doubt, and the details are subject to embellishment by the years of retelling. However, looking at the results that he was able to prove in mathematics throughout the course of his life, it is not disputed that his understanding of mathematics was extraordinary. (americanscientist.org)

Gauss was only twenty-one years old when he finished *Disquisitiones Arithmeticae*, although he did not publish it for three more years. The first several sections of the book deal with a systematic structuring of number theory, which became the groundwork for that area of study that is still used to this day. The last section of the book deals with construction of regular polygons, and outlines that the construction of a regular 17-gon would be possible. He proved that there were certain prime numbers that dictated which polygons could or could not be constructed using the rules of the straightedge and compass. (Stillwell, p335)

The primes that Gauss referred to fit the form  $2^{2^n} + 1$ , where  $n$  is an integer. They are called Fermat primes, discovered by the French mathematician Pierre de Fermat. Fermat claimed that all numbers of this particular form must be prime. However, at this time, there are only five known Fermat primes; when  $n = 0$  ( $P = 3$ ),  $n = 1$  ( $P = 5$ ),  $n = 2$  ( $P = 17$ ),  $n = 3$  ( $P = 257$ ), and  $n = 4$  ( $P = 65,537$ ). When  $n = 5$ ,  $2^{2^5} + 1$  equals 4,294,967,297. This number is not prime, as demonstrated by Leonhard Euler in 1732. He found that it could be factored into the product  $641 \cdot 6,700,417$ . (Gauss, p 459) No other Fermat primes have been discovered, nor is it known if there are no more to be found. It is known that for values as large as  $n = 32$ ,  $2^{2^n} + 1$  is composite, but there has

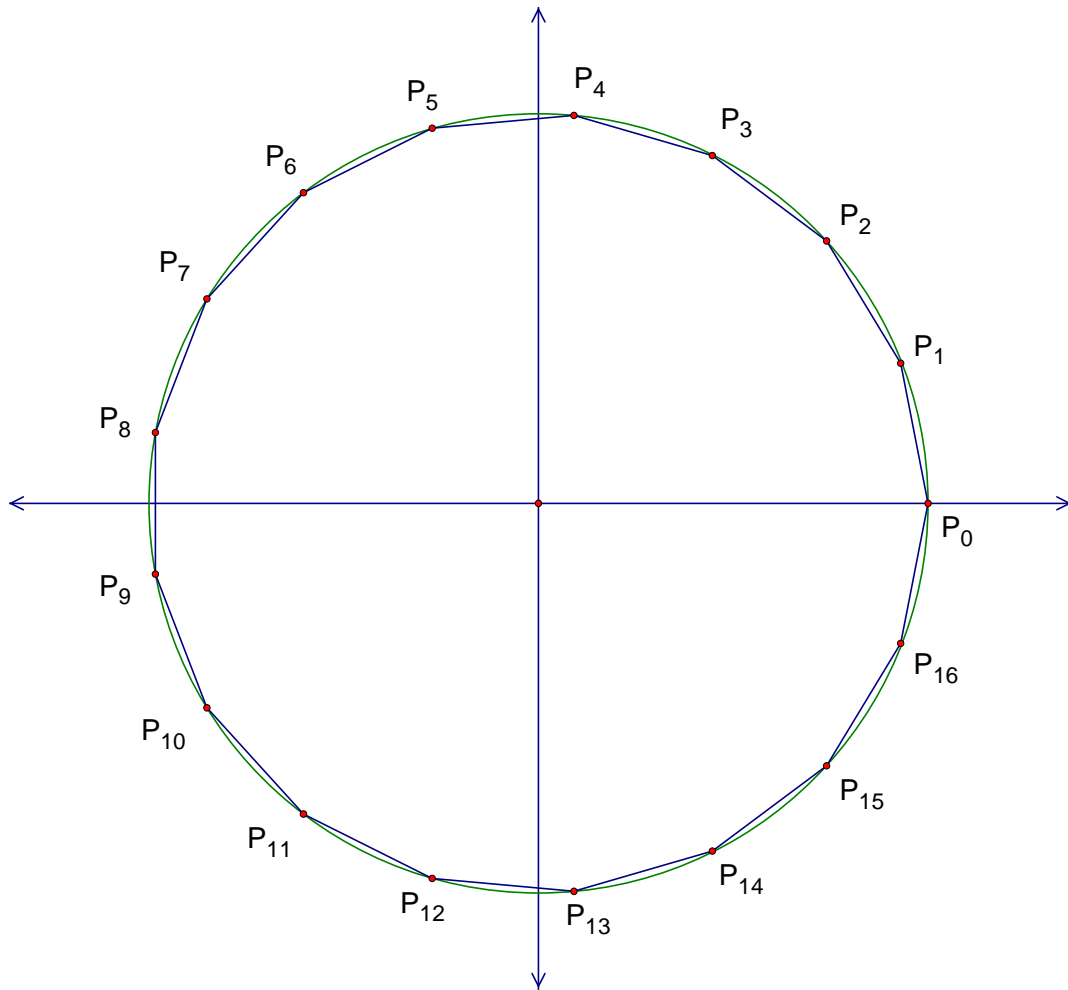
not been a successful proof that there are not any more Fermat primes either.

(mathworld.wolfram.com)

Constructing regular polygons is possible when the numbers that dictate the location of the vertices of the polygon are “constructible numbers”. The idea of constructible numbers goes back to the Greeks once again. There were three classic construction problems whose solutions were not known to the Greeks: trisecting the angle, squaring the circle, and doubling the cube. It has been proven that these constructions are indeed not possible using the rules of the Greeks. For constructions to be possible, it was shown that the values required must be the result of performing addition, subtraction, multiplication, division, or taking square roots on integers. For instance, it can be shown that trisecting a 60 degree angle would require the construction of the solutions to an irreducible third degree polynomial. To show this, use the trigonometric identity of  $\cos 3x = 4 \cos^3 x - 3 \cos x$ . If  $x = 20$ , this gives us that  $\cos 60 = 4 \cos^3 20 - 3 \cos 20$ , which is equivalent to the equation  $\frac{1}{2} = 4y^3 - 3y$ , where  $y = \cos 20$ . This cubic polynomial is an irreducible cubic equation, so its solutions would require at the minimum the construction of cube roots, not square roots. This also implies that it would not be possible to construct the regular nonagon, which could be constructed by trisecting the 60 degree angles of an equilateral triangle. (mathforum.org)

For the construction of the 17-gon, Gauss used the fact that its vertices can be represented as the seventeenth roots of unity, or the complex solutions to the equation  $x^{17} - 1 = 0$ , plotted on the complex plane. What Gauss was able to show was that the solutions of this polynomial could be found by solving a series of nested quadratic equations, where the solutions of one determine the coefficients of the next.

While the following derivation does not follow Gauss' work identically, it does show how the value for  $\cos \frac{2\pi}{17}$  can be derived.



In the above diagram, a sketch of the regular 17-gon is shown. If the circle has center at the origin and radius of one, the coordinates of the vertices correspond to the seventeenth roots of unity. To find the locations for these 17 points, we must solve the following equation:

$$x^{17} - 1 = 0$$



By de Moivre's theorem, we know the seventeenth roots of unity can be represented by the following expression:

$$P_k = \left( \cos \frac{2\pi k}{17} + i \sin \frac{2\pi k}{17} \right), \text{ where } k = 0, 1, 2, \dots, 16.$$

From the symmetry of the above figure, we know that the cosine values for  $P_1$  and  $P_{16}$  are equal, the cosines of  $P_2$  and  $P_{15}$  are the same,  $P_3$  and  $P_{14}$ , etc. Let  $x_1$  represent the cosine value for  $P_1$ , in other words, if  $\theta = \frac{2\pi}{17}$ ,  $x_1 = \cos \theta$ . In the same way,  $x_2 = \cos 2\theta$ , etc.

We know that the sum of the roots of unity is zero,  $x_0 + x_1 + x_2 + \dots + x_{16} = 0$ ,

or because of the duplicate values and the fact that  $x_0 = 1$ , we find that

$$x_1 + x_2 + x_3 + \dots + x_8 = -\frac{1}{2}$$

We will use the following identity for the cosine function repeatedly:

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = (\cos \alpha \cos \beta - \sin \alpha \sin \beta) + (\cos \alpha \cos \beta + \sin \alpha \sin \beta) = 2 \cos \alpha \cos \beta$$

Since  $x_1 = \cos \theta$  and  $x_4 = \cos 4\theta$ , and since  $x_4 = x_{13}$ , the above identity gives us

$$x_1 + x_4 = 2x_6 \cdot x_7.$$

In much the same way,

$$x_3 + x_5 = 2x_1 \cdot x_4$$

$$x_2 + x_8 = 2x_3 \cdot x_5$$

$$x_6 + x_7 = 2x_2 \cdot x_8$$

Let  $2A = x_1 + x_4 = 2x_6 \cdot x_7$

$$2B = x_3 + x_5 = 2x_1 \cdot x_4$$

$$2C = x_2 + x_8 = 2x_3 \cdot x_5$$

$$2D = x_6 + x_7 = 2x_2 \cdot x_8$$

This means that  $A \cdot C = \frac{(x_1 + x_4)(x_2 + x_8)}{4} = \frac{x_1x_2 + x_2x_4 + x_4x_8 + x_1x_8}{4}$ , but  $x_1x_2 = x_1 + x_3$

by the cosine identity given before. In the same way,

$$x_2x_4 = x_2 + x_6$$

$$x_4x_8 = x_4 + x_{12} = x_4 + x_5$$

$$x_1x_8 = x_7 + x_9 = x_7 + x_8.$$

Therefore,  $A \cdot C = \frac{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8}{4} = -\frac{1}{16}$

In the same way,  $B \cdot D = -\frac{1}{16}$ .

Also,

$$A \cdot B = \frac{(x_1 + x_4)(x_3 + x_5)}{4} = \frac{x_1x_3 + x_3x_4 + x_4x_5 + x_1x_5}{4} = \frac{2x_1 + x_2 + 2x_4 + x_6 + x_7 + x_8}{8} = -\frac{1}{16} + \frac{A}{4} - \frac{B}{4}$$

or

$$16AB = -1 + 4A - 4B$$

$$16BC = -1 + 4B - 4C$$

$$16CD = -1 + 4C - 4D$$

$$16DA = -1 + 4D - 4A$$

Define  $2X = A + C$  and  $2Y = B + D$ .

This means that  $X + Y = -\frac{1}{8}$ , since  $A + B + C + D = -\frac{1}{4}$

$$4XY = (A + C)(B + D) = AB + BC + CD + DA =$$

$$\frac{-1+4A-4B}{16} + \frac{-1+4B-4C}{16} + \frac{-1+4C-4D}{16} + \frac{-1+4D-4A}{16} = -\frac{1}{4}$$

$$\text{so } XY = -\frac{1}{16}$$

If  $X + Y = -\frac{1}{8}$  and  $XY = -\frac{1}{16}$ , this means that  $X$  and  $Y$  are roots of the following

quadratic equation:

$$R^2 + \frac{1}{8}R - \frac{1}{16} = 0 \text{ or } 16R^2 + 2R - 1 = 0$$

$$\text{by the quadratic formula, } R = \frac{-2 \pm \sqrt{2^2 - 4(16)(-1)}}{2(16)} = \frac{-2 \pm \sqrt{68}}{32} = \frac{-1 \pm \sqrt{17}}{16}$$

$$\text{Since } 2X = A + C = x_6x_7 + x_3x_5 > 0, X = \frac{-1 + \sqrt{17}}{16} \text{ and } Y = \frac{-1 - \sqrt{17}}{16}$$

$$\text{This gives us } A + C = \frac{-1 + \sqrt{17}}{8} \text{ and } AC = -\frac{1}{16},$$

so  $A$  and  $C$  are roots to the following quadratic equation:

$$S^2 - \frac{-1 + \sqrt{17}}{8}S - \frac{1}{16} = 0 \text{ or } 16S^2 + (2 - 2\sqrt{17})S - 1 = 0$$

By the quadratic formula,

$$S = \frac{-(2-2\sqrt{17}) \pm \sqrt{(2-2\sqrt{17})^2 - 4(16)(-1)}}{2(16)} = \frac{-2+2\sqrt{17} \pm \sqrt{4-8\sqrt{17}+68+64}}{32}$$

$$= \frac{-2+2\sqrt{17} \pm \sqrt{136-8\sqrt{17}}}{32} = \frac{-1+\sqrt{17} \pm \sqrt{34-2\sqrt{17}}}{16}$$

Since  $A = x_6 x_7 > 0$ ,  $A = \frac{-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}}{16}$  and  $C = \frac{-1+\sqrt{17}-\sqrt{34-2\sqrt{17}}}{16}$

Also,  $B + D = \frac{-1-\sqrt{17}}{8}$  and  $BD = -\frac{1}{16}$ ,

so B and D are roots to the following quadratic equation:

$$T^2 - \frac{-1-\sqrt{17}}{8}T - \frac{1}{16} = 0 \text{ or } 16T^2 + (2+2\sqrt{17})T - 1 = 0$$

By the quadratic formula,

$$T = \frac{-(2+2\sqrt{17}) \pm \sqrt{(2+2\sqrt{17})^2 - 4(16)(-1)}}{2(16)} = \frac{-2-2\sqrt{17} \pm \sqrt{4+8\sqrt{17}+68+64}}{32}$$

$$= \frac{-2-2\sqrt{17} \pm \sqrt{136+8\sqrt{17}}}{32} = \frac{-1-\sqrt{17} \pm \sqrt{34+2\sqrt{17}}}{16}$$

Since  $B = x_1 x_4 > 0$ ,  $B = \frac{-1-\sqrt{17}+\sqrt{34+2\sqrt{17}}}{16}$ , and  $D = \frac{-1-\sqrt{17}-\sqrt{34+2\sqrt{17}}}{16}$

Finally, since  $x_1 + x_4 = 2A$ , and  $x_1 x_4 = B$ ,

$x_1$  and  $x_4$  are roots to the following quadratic equation:

$$U^2 - 2AU + B = 0, \text{ or } U^2 - 2\frac{-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}}{16}U + \frac{-1-\sqrt{17}+\sqrt{34+2\sqrt{17}}}{16} = 0$$

Simplifying,

$$16U^2 + (2 - 2\sqrt{17} - 2\sqrt{34 - 2\sqrt{17}})U - 1 - \sqrt{17} + \sqrt{34 + 2\sqrt{17}} = 0$$

By the quadratic formula:

$$U = \frac{-(2 - 2\sqrt{17} - 2\sqrt{34 - 2\sqrt{17}}) \pm \sqrt{(2 - 2\sqrt{17} - 2\sqrt{34 - 2\sqrt{17}})^2 - 4(16)(-1 - \sqrt{17} + \sqrt{34 + 2\sqrt{17}})}}{2(16)}$$

$$U = \frac{-2 + 2\sqrt{17} + 2\sqrt{34 - 2\sqrt{17}} \pm \sqrt{272 + 48\sqrt{17} - 64\sqrt{34 + 2\sqrt{17}} + 8\sqrt{578 - 34\sqrt{17}} - 8\sqrt{34 - 2\sqrt{17}}}}{32}$$

$$U = \frac{-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} \pm \sqrt{68 + 12\sqrt{17} - 16\sqrt{34 + 2\sqrt{17}} + 2\sqrt{578 - 34\sqrt{17}} - 2\sqrt{34 - 2\sqrt{17}}}}{16}$$

$$U = \frac{-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} \pm 2\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}}{16}$$

Since  $x_1 > x_4$ ,  $x_1 = \frac{-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + 2\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}}{16}$

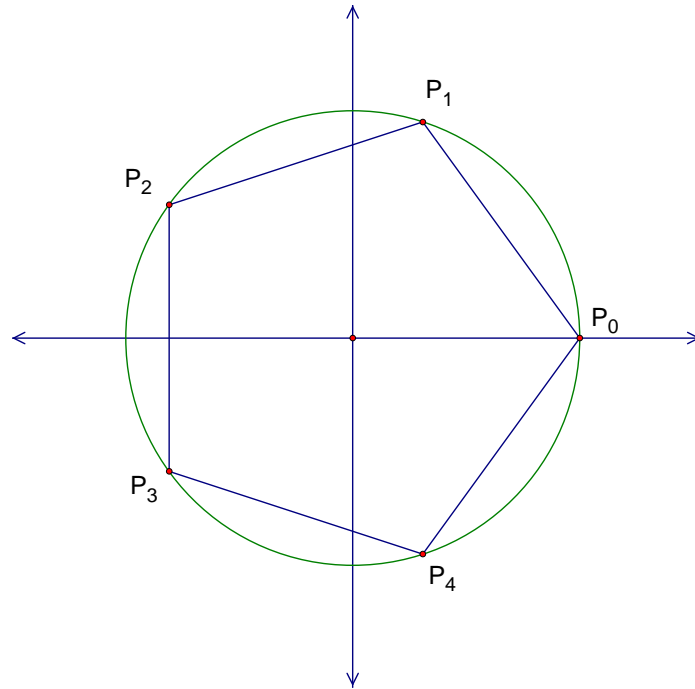
or  $\cos\left(\frac{2\pi}{17}\right) = \frac{-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + 2\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}}{16}$

All other values for  $x_i$  can be found in a similar manner. ([www.nn.ij4u.or.jp](http://www.nn.ij4u.or.jp)),

([www.iisc.ernet.in](http://www.iisc.ernet.in)), (Gauss, p 458)

After finally completing the previous derivation, I thought it might be interesting to try to apply the same technique to the regular pentagon. I expected since the 17-gon required solving three levels of quadratics, the pentagon would only require two. I was surprised to find the solution by solving only one quadratic equation. I had seen other methods for

deriving an exact value expression for  $\cos(2\pi/5)$ , but I had never seen the following technique before.



In the same way as with the 17-gon, let  $x_1 = \cos \theta$ ,  $x_2 = \cos 2\theta$ , etc, where  $\theta = \frac{2\pi}{5}$ .

Again,  $x_0 + x_1 + x_2 + x_3 + x_4 = 0$ ,  $x_1 = x_4$ , and  $x_2 = x_3$  so  $x_1 + x_2 = -\frac{1}{2}$ .

By the same cosine identity used before,  $x_1 + x_2 = 2 \cdot x_1 \cdot x_2$ .

Since  $x_1 + x_2 = -\frac{1}{2}$ ,  $x_1 \cdot x_2 = -\frac{1}{4}$ .

Therefore,  $x_1$  and  $x_2$  are roots to the following quadratic equation:

$$x^2 + \frac{1}{2}x - \frac{1}{4} = 0, \text{ or } 4x^2 + 2x - 1 = 0$$

By the quadratic formula,

$$x = \frac{-2 \pm \sqrt{2^2 - 4(4)(-1)}}{2(4)} = \frac{-2 \pm \sqrt{20}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

Since  $x_1 > x_2$ ,  $x_1 = \frac{-1 + \sqrt{5}}{4}$  and  $x_2 = \frac{-1 - \sqrt{5}}{4}$

This gives us that  $\cos\left(\frac{2\pi}{5}\right) = \cos\left(\frac{8\pi}{5}\right) = \frac{-1 + \sqrt{5}}{4}$

and  $\cos\left(\frac{4\pi}{5}\right) = \cos\left(\frac{6\pi}{5}\right) = \frac{-1 - \sqrt{5}}{4}$

While the expression  $\cos\frac{2\pi}{17}$  seems to be daunting, it is comprised only of sums, differences, products, quotients, and square roots, making it a constructible number. Gauss was able to prove that the 17-gon was constructible by deriving this expression. While Gauss proved that this polygon could be constructed, he did not provide instructions as to how to complete the construction itself. The construction of the 17-gon was first developed by Johannes Erchinger a few years after Gauss published *Disquisitiones Arithmeticae*. Reportedly, Gauss was so pleased with this particular discovery that he requested that his gravestone contain a regular 17-gon. The mason who was to cut the stone declined, claiming that the image would just appear to be a circle. However, there is a monument in Braunschweig that is a statue of Gauss where the base contains a 17-pointed star. ([mathcircle.berkeley.edu](http://mathcircle.berkeley.edu))

Gauss demonstrated that if one less than the number of sides of the polygon to be constructed had a prime factorization of  $2^n$ , the corresponding equation could be represented as solely quadratics. If not, the equation would require higher degree

polynomials, as in the case of the 19-gon. Since 18's prime factorization is  $2 \cdot 3^2$ , this meant that the solutions to the equation could be found by one quadratic, and two irreducible cubic equations. The numbers, which Gauss calculated to ten place accuracy, would not be constructible numbers, so the 19-gon cannot be constructed by straightedge and compass techniques. (Gauss, p 429)

Gauss went on to show that if the polygon had a prime number of sides, and one less than the number of sides were a power of two, the corresponding polynomial could be represented by only quadratic equations, and if the number of quadratics were a power of two, then the reduction process using nested quadratics described earlier would lead to a constructible solution. This meant that the number of sides would have to be one more than two raised to a power of two, or  $2^{2^n} + 1$ , which is a Fermat prime. (Gauss, p 458)

Gauss concluded that for a regular polygon to be constructible, it had to have a number of sides whose odd factors were only distinct Fermat primes. (Gauss, p 458) For instance, the regular 255-gon can be constructed, since its prime factorization,  $3 \cdot 5 \cdot 17$ , is made up of distinct Fermat primes. The 256-gon is constructible, since it has no odd factors. The 257-gon is also constructible, since it is a Fermat prime. The 258-gon is not constructible, since its prime factorization,  $2 \cdot 3 \cdot 43$ , contains a prime that is not a Fermat prime.

Gauss, in *Disquisitiones Arithmeticae*, explained that this condition would guarantee that one could construct a regular polygon. He also stated, but did not prove, that this would also be a necessary requirement for all possible constructible regular polygons. It was several years later, in 1837, that Pierre Wantzel proved that this was necessary. It is noteworthy that in the same paper, Wantzel also proved that doubling the



cube and trisecting any angle were impossible. Through this, the criterion was finally established for which regular polygons can be constructed by straightedge and compass techniques. (Martin, p46)

“The following are the 38 values of N below 300:

3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 48, 51, 60, 64, 68, 80, 85  
96, 102, 120, 128, 136, 160, 170, 192, 204, 240, 255, 256, 257, 272.” (Gauss, p460)

There remains one question, however. If these are the only regular polygons that can be constructed using typical straightedge and compass construction, what if we change the rules slightly? What if we allow ourselves to use a “marked” ruler in place of our straightedge, for instance? What would that allow us to construct? Would we be able to construct other regular polygons that we were not able to otherwise?

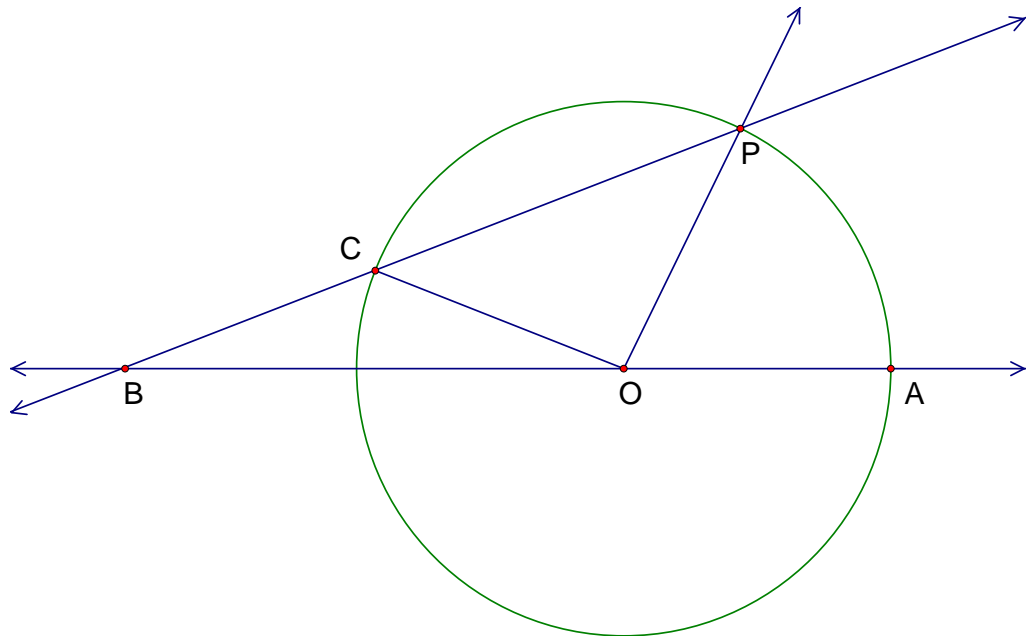
A “marked” ruler is simply a straightedge with a pair of markings on the edge that are a set distance apart. (Conway & Guy, p194) While this does not seem to enhance the use of the tool, it does have a significant impact on what can and cannot be done. To begin this investigation, again we must return to the early Greeks.

The use of the marked straightedge was not unknown to the Greeks in the time of Euclid. These types of constructions were referred to as “neusis” constructions, meaning “verging”. In fact, there had been individuals who were able to solve the unsolvable problems that had perplexed many of the mathematicians of the time. These solutions were looked upon by the Greeks as secondary techniques for construction, and less

rigorous than the traditional constructions. One individual who did spend some time working with these neusis constructions was Archimedes. (Conway & Guy, p195)

Archimedes is regarded as one of the greatest mathematicians of all time. He was born in the city of Syracuse, Sicily in the year 287 BC. He lived for 75 years, and is renowned for his work in mathematics and physics. He is credited with discoveries in the areas of geometry, physics, logic, and even calculus. Although much information about his life has been lost, he is known to have greatly assisted in the defense of the city of Syracuse against the Roman forces in the Second Punic War. Many of his discoveries were the result of designing machines to defend the city. (wikipedia.com)

Among his many contributions to the fields of mathematics and physics, Archimedes is credited with a neusis technique for trisecting an angle. His technique is demonstrated in the diagram below:



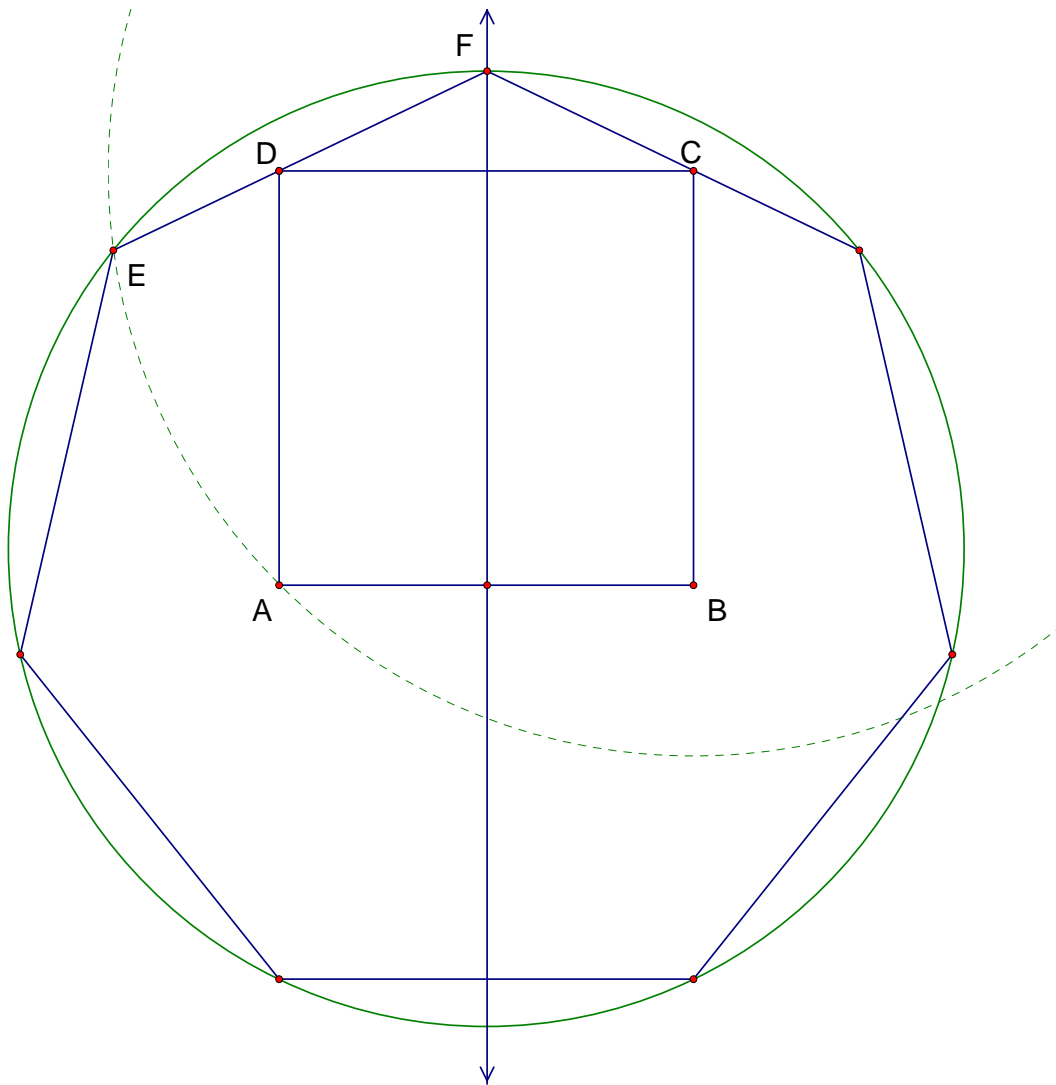
To trisect angle POA, first draw a circle with a radius equal to the space between markings on the ruler. Then, with the ruler aligned on point P, adjust the markings to lie on line OA and on circle O, forming angle PBA. We must now show that the measure of PBA is one third the measure of POA. Segments BC and OC must be congruent, making triangle BOC isosceles, with angle PBA congruent to angle COB. Set their measures equal to  $x$ . Since angle PCO is exterior to triangle BOC, its measure is equal to  $2x$ . Since triangle COP is also isosceles, angle COP's measure must be equal to  $180 - 4x$ . Angles BOC, COP, and POA add up to 180, so  $x + 180 - 4x + m\angle POA = 180$ . This gives us that  $m\angle POA = 3x$ , which is what we needed to show. (Conway & Guy, p195)

So using a marked ruler allows us to solve one of the impossible construction problems from antiquity. Granted, we are bending the rules for construction, so this is not a true solution, as far as how the early Greeks would have viewed it. The Greeks were aware of constructions using the marked ruler, but looked at solutions such as these as inferior to straightedge and compass techniques. (wikipedia.com) However, looking back to our original question, this does have implications for a more general question regarding construction of regular polygons. If the use of a marked ruler is allowed, it is immediate that the construction of the regular nonagon is possible by simply trisecting the central angles of an equilateral triangle. In fact, it would be possible to trisect the central angles of any previously constructible regular polygon, so polygons with 18, 27, 36, 45, ... would now be possible.

We are not done with neusis constructions that incorporate angle trisection, however. We now are able to construct almost all regular polygons with single-digit numbers of sides. The only one remaining is the heptagon. Francois Viète was a French

mathematician who predated Gauss by two hundred years, and he was able to formulate a neusis construction of the regular heptagon. (Hartshorne, p264) There are some sources that credit Archimedes with a technique for construction of the regular heptagon, but there are questions as to their authenticity. (Martin, p136) Viète's discovery is impressive, since he did this without the benefit of modern algebra. Here is a neusis construction of a regular heptagon which is similar to Viète's construction:

(wikipedia.com)

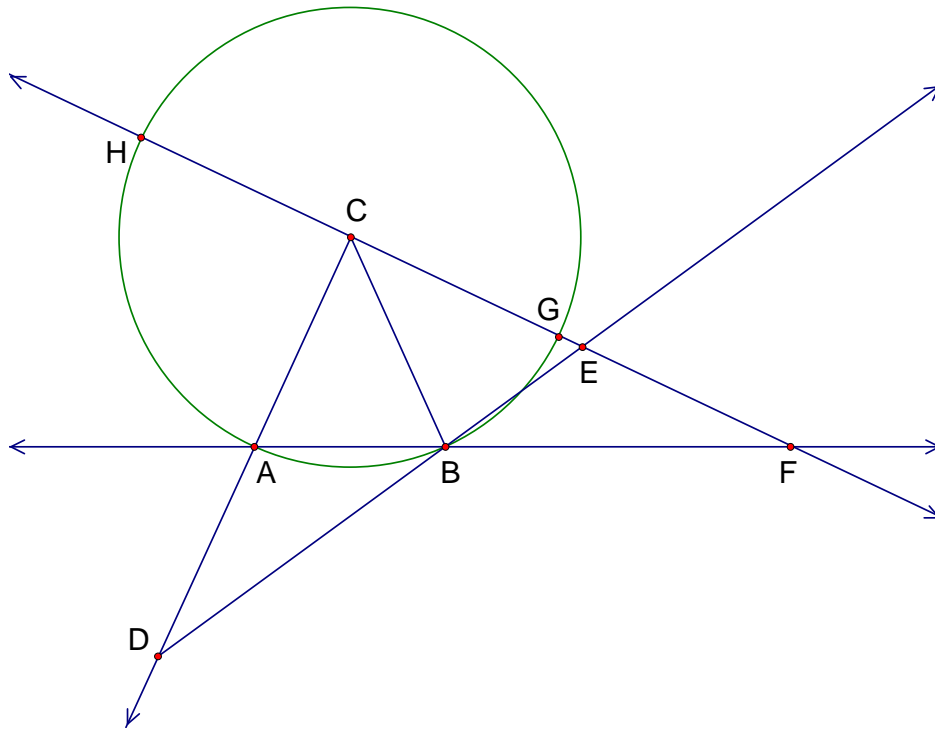


Begin by constructing square ABCD, with side length equal to the distance between the markings on the ruler. Draw circle with center at C and radius AC. Construct the perpendicular bisector of side AB. With the ruler lying on point D, adjust so that the marks lay on circle C and on the perpendicular bisector of AB (points E and F in the diagram). This gives one side of the heptagon. Find the center of the heptagon using a perpendicular bisector of EF, and draw the circumscribed circle, and use the ruler markings to locate the rest of the vertices.

Should it be surprising that using a technique that allows us to trisect angles should also allow us to construct a regular heptagon? Let's go back to Gauss briefly. His technique of solving nested quadratics to show that the angle for a 17-gon was constructible relied on the fact that one could construct solutions to quadratic equations using straightedge and compass. If the polynomial could be reduced to solely quadratic equations, then the polygon would be constructible. If the polynomial included, for instance, irreducible cubic equations, then the solutions were not constructible numbers. The heptagon, with  $n = 7$ , gives us  $n - 1 = 6$ , which is equal to  $2 * 3$ . This implies that the polynomial would reduce to a quadratic and a cubic. But since the heptagon is constructible by marked ruler and compass, the solutions to the cubic equation must be constructible as well. This implies that cube roots may be constructible numbers using a marked ruler and compass, implying that another of the impossible constructions from antiquity may be solvable using a marked ruler, namely the doubling of a cube.

To demonstrate that cube roots are constructible, we must show that given some number  $a$  and the unit length of 1, that the cube root of  $a$  can be constructed. Begin the construction with segment AB with length  $a$ . Using the unit length, determine a length  $b$

such that  $b$  is of the form  $2^{3k-1}$ , and  $b > a$ . Construct the isosceles triangle  $ABC$  with  $AC = BC = b$ . Extend  $CA$  to  $D$  such that  $AD = b$  as well. Draw line  $DB$ . Using the marked ruler, place the ruler on point  $C$ , and find point  $E$  on  $DB$  and  $F$  on  $AB$  such that  $EF = b$ , as shown in the diagram below.



Let  $BF = y$  and  $CE = x$ . Notice that triangle  $ACF$  has transversal  $BE$ . By Menelaus's

Theorem,  $\frac{AB}{BF} \cdot \frac{FE}{EC} \cdot \frac{CD}{DA} = 1$ , or  $\frac{a}{y} \cdot \frac{b}{x} \cdot \frac{2b}{b} = 1$ , which gives us  $xy = 2ab$ . Because  $FA$

and  $FH$  are both secants to the circle,  $FA \cdot FB = FH \cdot FG$ , or  $(a + y)y = (x + 2b)x$ . By eliminating  $x$  from the two equations, we get  $y^3 = 4ab^2$ . By substituting  $2^{3k-1}$  for  $b$ , this

gives us  $y^3 = 2^{6k} \cdot a$ , or  $\sqrt[3]{a} = \frac{y}{2^{2k}}$ . This means that to find the cube root of  $a$ , simply

bisect  $y$  a total of  $2k$  times. Therefore, cube roots of any number are constructible by marked ruler construction. (Hartshorne, p262)

By allowing the use of a marked ruler, or any tool that would be able to trisect an angle, we have seen that many more regular polygons can be constructed. But are there others in addition to the ones mentioned already? This question leads us, in much the same way that Gauss was led to Fermat primes, to another type of prime number, the Pierpont prime. Pierpont primes, identified by James Pierpont, are primes of the form  $2^a 3^b + 1$ , where  $a$  and  $b$  are non-negative integers. (Martin, p140) The set of Fermat primes are of course included in the Pierpont primes, taking  $b = 0$ . The Pierpont primes are also much more prevalent than the Fermat primes. The following table shows the first several Pierpont numbers, highlighting which ones are prime:

		b							
$2^a 3^b + 1$		0	1	2	3	4	5	6	7
a	0	2	4	10	28	82	244	730	2188
	1	3	7	19	55	163	487	1459	4375
	2	5	13	37	109	325	973	2917	8749
	3	9	25	73	217	649	1945	5833	17497
	4	17	49	145	433	1297	3889	11665	34993
	5	33	97	289	865	2593	7777	23329	69985
	6	65	193	577	1729	5185	15553	46657	139969
	7	129	385	1153	3457	10369	31105	93313	279937

By the same reasoning that Gauss had used for the Fermat primes, the Pierpont primes represent constructible polygons, since the polynomial for the roots of unity will be able to be represented by a nesting of quadratic and cubic equations. If a regular polygon has a number of sides that is a product of a power of two, a power of three, and distinct Pierpont primes, or is of the form  $2^a 3^b p_1 p_2 \dots p_i$ , where the  $p_i$ 's are distinct Pierpont primes, it can be constructed using a compass and a marked ruler. This gives us a much larger list of constructible regular polygons: (Martin, p140)

This is the limit of what we are able to accomplish using a marked ruler. There are still gaps in the list. For instance, we are not able to construct the 11-gon, the 22-gon, 23-gon, or the 25-gon. These all involve polynomials that would lead to irreducible 5<sup>th</sup> degree polynomials. If we wanted to press on, it can be shown that a tool that would allow us to divide an angle into 5 equal parts, or an angle quintisector, would be useful in performing these constructions. Martin demonstrates that the use of a combination of construction tools called tomahawks, which when used by itself is an angle trisector, can be used to quintisect angles. This would allow us to construct the 11-gon, 22-gon, 23-gon, and the 25-gon. (Martin, p141)

In fact, if the polynomial involves an irreducible  $n^{\text{th}}$  degree polynomial, we would have to be able to divide an angle into  $n$  equal parts to be able to construct the regular polygon. (Martin, p141) However, keeping in mind that if we can divide an angle into as many equal parts as we like, any regular polygon may be constructed by simply dividing 360 degrees into whatever central angle we would desire, and produce the regular polygon this way. At some point, we need to draw the line, no pun intended, and define what constitutes a legitimate construction technique. The Greeks made the limit the straightedge and compass. Others have made use of the compass and marked ruler. There are other types of constructions that involve only the compass, or only the ruler. With the increasing availability of computers equipped with software specifically designed for geometry construction, students are able to perform many other constructions that do not follow what the Greeks would regard as “true” constructions. These types of constructions are interesting, and a good topic for further study. However, concerning the topic of construction of regular polygons, starting with what the Greeks



could do with only straightedge and compass, continuing with what Gauss was able to add, and finishing with what other construction techniques allow us to accomplish, I believe that we have completed this investigation. The following table shows the possible numbers of sides up to 100. Highlighted in yellow are the numbers of sides of regular polygons that the early Greeks would have been able to construct. Highlighted in green are the polygons that were added with Gauss's work. Highlighted in blue are polygons that can be constructed using a marked ruler. The remaining numbers represent polygons that would require the ability to divide an angle into more than three equal parts.

2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28
29	30	31	32	33	34	35	36	37
38	39	40	41	42	43	44	45	46
47	48	49	50	51	52	53	54	55
56	57	58	59	60	61	62	63	64
65	66	67	68	69	70	71	72	73
74	75	76	77	78	79	80	81	82
83	84	85	86	87	88	89	90	91
92	93	94	95	96	97	98	99	100

## Bibliography:

1. Euclid, *Euclid's Elements*, transl. T.L. Heath, Santa Fe: Green Lion Press 2002.
2. Gauss, Carl Friedrich, *Disquisitiones Arithmeticae*, transl. Arthur A. Clarke, New York: Springer-Verlag New York Berlin Heidelberg Tokyo 1986.
3. Martin, George E., *Geometric Constructions*, New York: Springer-Verlag New York, Inc. 1998.
4. Stillwell, John, *Mathematics and Its History*, New York: Springer-Verlag New York, Inc. 1989.
5. Hartshorne, Robin, *Geometry: Euclid and Beyond*, New York: Springer 2000.
6. Conway, John H. and Guy, Richard K., *The Book of Numbers*, New York: Springer Science+Business Media, Inc. 2006.
7. [http://en.wikipedia.org/wiki/Carl\\_Friedrich\\_Gauss](http://en.wikipedia.org/wiki/Carl_Friedrich_Gauss)
8. <http://en.wikipedia.org/wiki/Archimedes>
9. <http://en.wikipedia.org/wiki/Neusis>
10. <http://en.wikipedia.org/wiki/Image:Neusis-heptagon.png>
11. <http://mathworld.wolfram.com/FermatNumber.html>
12. <http://mathworld.wolfram.com/Heptadecagon.html>
13. <http://mathforum.org/library/drmath/view/51706.html>
14. <http://mathcircle.berkeley.edu/BMC6/ps0506/Heptadecagon.pdf>
15. <http://www.iisc.ernet.in/academy/resonance/Jun1997/pdf/June1997p60-67.pdf>
16. <http://www.nn.ij4u.or.jp/~hsat/engl/heptadecagon.htm#fn1>
17. <http://www.americanscientist.org/template/AssetDetail/assetid/50686?&print=yes>