Black Holes

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Contents

Conventions

18 Jan

We set $c = G = 1$; our metric signature is $(- + ++)$. (1)
Spacetime coordinates are labelled by Latin letters: $x^a,\ a = 0, 1, 2, 3$
The Riemann tensor is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

that is in components

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \ldots$$

1 Mathematical Preliminaries

1.1 Differential Forms

Let $\mathcal{M}$ be a differentiable manifold.

Definition 1.1 *A* **differentiable p-form** *on $\mathcal{M}$ is an antisymmetric $(0, p)$ tensor field.*

Remark A zero-form is a function; a one-form is a covector field.

Given a p-form X and a q-form Y we define a $(p + q)$-form $X \wedge Y$ by (in components)

$$(X \wedge Y)_{a_1 \ldots a_p b_1 \ldots b_q} = \frac{(p+q)!}{p!q!} X_{[a_1 \ldots a_p} Y_{b_1 \ldots b_q]}. \tag{1.1}$$

The wedge product has the following properties

a) $X \wedge Y = (-1)^{pq} Y \wedge X$,

b) $X \wedge X = 0$ if p is odd,

c) $X \wedge (Y \wedge Z) = (X \wedge Y) \wedge Z$.

We can form a basis for the set of p-forms using coordinate differentials dx^a. Any p-form can be written as a linear combination of

$$dx^{a_1} \wedge dx^{a_2} \wedge \ldots \wedge dx^{a_p}$$

as

$$X = \frac{1}{p!} X_{a_1 \ldots a_p} dx^{a_1} \wedge dx^{a_2} \wedge \ldots \wedge dx^{a_p}. \tag{1.2}$$

We define the **exterior derivative** of a given p-form X as the $(p + 1)$-form dX given by

$$(dX)_{a_1 \ldots a_{p+1}} = (p + 1) \partial_{[a_1} X_{a_2 \ldots a_{p+1}]}. \tag{1.3}$$

Exercise: Check that this is a tensor.

The exterior derivative has the following properties

a) $d(dX) = 0$, d is nilpotent,

b) $d(X \wedge Y) = dX \wedge Y + (-1)^p X \wedge dY$,

c) For $h : \mathcal{M} \to \mathcal{N}$, $d(h^* X) = h^* dX$.

If $dX = 0$ everywhere, then X is **closed**, if there is a form Y such that $X = dY$ everywhere, then X is **exact**. Any exact form is closed; the converse is true locally, i.e. if X is closed, then for any point p there is a neighbourhood of p in which there exists a Y such that $X = dY$. (Poincaré Lemma)

1.2 Frobenius' Theorem

Assume that we have a family of hypersurfaces in $\mathcal{M}$ specified by equations of the form $f(x) =$ constant, where f is a smooth function with $df \neq 0$ everywhere. Then df is normal to these hypersurfaces, since for any tangent vector T

$$< df, T >= T(f) = T \cdot \partial f = 0.$$

The most general one-form normal to these hypersurfaces is

$$n = g\, df, \tag{1.4}$$

where g is any function that is non-zero everywhere. Then $dn = dg \wedge df$ and hence

$$n \wedge dn = g\, df \wedge dg \wedge df = 0. \tag{1.5}$$

Conversely, if $n \wedge dn = 0$, then locally there exist functions f, g such that $n = g\, df$. Hence n is normal to hypersurfaces $n =$ constant., i.e.

$$n \wedge dn = 0 \quad \Leftrightarrow \quad n \text{ hypersurface-orthogonal}.$$

This is Frobenius' Theorem.

1.3 Killing Vector Fields

Definition 1.2 *An* **isometry** *of a (pseudo-)Riemannian manifold* $(\mathcal{M}, g)$ *is a diffeomorphism* $\varphi : \mathcal{M} \to \mathcal{M}$ *such that*

$$\varphi^* g_{\varphi(p)} = g_p \quad \forall\, p \in \mathcal{M}. \tag{1.6}$$

Exercise: Show that the set of all isometries forms a group.

Now assume that $(\mathcal{M}, g)$ admits a one-parameter group of isometries φ_λ, obeying

$$\varphi_{\lambda_1} \circ \varphi_{\lambda_2} = \varphi_{\lambda_1 + \lambda_2}. \tag{1.7}$$

The map $\lambda \mapsto \varphi_\lambda(p)$ defines a curve through p for each p in $\mathcal{M}$. Define a vector field k by defining $k|_p$ to be tangent to the curve at p. The integral curve of k through any point p is then just the map $\lambda \mapsto \varphi_\lambda(p)$, and hence

$$(\mathcal{L}_k g)_p = \lim_{\lambda \to 0} \left[\frac{\varphi_\lambda^* g_{\varphi(p)} - g_p}{\lambda} \right] = 0, \tag{1.8}$$

since φ_λ is an isometry. This is the defining property of a **Killing vector field**. So a one-parameter group of isometries defines a Killing vector field.
Conversely, given a Killing vector field k we define $\varphi_\lambda(p)$ to be the point parameter distance λ along the integral curve of k through p. If these curves are complete (extendible to $\lambda = \pm\infty$), this defines a one-parameter family of isometries.

So continuous isometries correspond to Killing vector fields.
In components,

$$\mathcal{L}_k g = 0 \quad \Leftrightarrow \quad \underline{\nabla_a k_b + \nabla_b k_a = 0}. \tag{1.9}$$

This is **Killing's equation**.

1.3.1 Conservation Laws

According to Noether's theorem, symmetries correspond to conservation laws. We have just seen that in general relativity, Killing vector fields represent symmetries, so there are conserved quantities. Let k be a Killing vector field.

(i) Let U^a be tangent to an affinely parametrized geodesic (parameter λ), i.e. $U \cdot \nabla U^a = 0$. Then

$$\frac{d}{d\lambda}(k \cdot U) = U \cdot \nabla(k \cdot U) = U^a \nabla_a (k_b U^b) = U^a U^b \nabla_a k_b + U^a k_b \nabla_a U^b = 0, \tag{1.10}$$

since the first term is a symmetric tensor contracted with an antisymmetric tensor and the second term vanishes because of $U \cdot \nabla U^a = 0$. Hence $k \cdot U$ is conserved along the geodesic.

(ii) Consider $J^a = T^{ab} k_b$, where T^{ab} is the energy-momentum tensor. Then

$$\nabla_a J^a = \nabla_a (T^{ab} k_b) = k_b (\nabla_a T^{ab}) + T^{ab} \nabla_a k_b = 0, \tag{1.11}$$

where we have used the conservation equation $\nabla_a T^{ab}$ and Killing's equation.
J^a is a conserved current.

1.3.2 Adapted Coordinates

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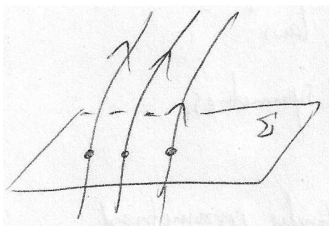
Pick a hypersurface Σ that intersects each integral curve of k once. Let x^i be coordinates on Σ; assign coordinates (λ, x^i) to the point parameter distance λ along the integral curve of k starting at the point on Σ with coordinates x^i.



In these coordinates, $k = \frac{\partial}{\partial \lambda}$ and Killing's equation reduces to

$$\mathcal{L}_k g_{ab} = \frac{\partial g_{ab}}{\partial \lambda} = 0. \tag{1.12}$$

Hence each Killing vector k corresponds to a coordinate λ such that the metric g_{ab} is independent of λ.

1.3.3 Time Independence

Definition 1.3 *A spacetime* $(\mathcal{M}, g)$ *is* **stationary** *if it admits a timelike Killing vector field.*

If we use adapted coordinates (t, x^i) so that $k = \frac{\partial}{\partial t}$, then g_{ab} is independent of t:

$$ds^2 = g_{00}(x^k)dt^2 + 2g_{0i}(x^k)dt\,dx^i + g_{ij}(x^k)dx^i\,dx^j, \quad g_{00} < 0. \tag{1.13}$$

Definition 1.4 $(\mathcal{M}, g)$ *is* **static** *if it admits a hypersurface-orthogonal timelike Killing vector field.*

(Each static spacetime is stationary.)
For a static spacetime, choose some surface Σ orthogonal to k and introduce adapted coordinates (t, x^i) as above. In these coordinates, Σ is the surface $t = 0$. Hence dt is normal to Σ, and k must be proportional to dt at $t = 0$.
From the line element (1.13), we see that the covector dual to k has the form

$$k = g_{00}(x^k)dt + g_{0i}(x^k)dx^i. \tag{1.14}$$

We conclude that

$$\underline{g_{0i}(x^k) = 0.} \tag{1.15}$$

So in adapted coordinates (t, x^i), a static metric takes the form

$$ds^2 = g_{00}(x^k)dt^2 + g_{ij}(x^k)dx^i\,dx^j, \quad g_{00} < 0. \tag{1.16}$$

<u>Note</u> A static metric admits a discrete time reversal isometry $t \to -t$.
In this sense, "static" is equivalent to "time-independent and time-reversal invariant".

2 The Schwarzschild Solution

2.1 Spherical Symmetry

The standard metric on a unit two-sphere is

$$ds^2 = d\theta^2 + \sin^2\theta\,d\varphi^2. \tag{2.1}$$

This has an $SO(3)$ isometry group.

Definition 2.1 *A spacetime is* **spherically symmetric** *if its isometry group has an* $SO(3)$ *subgroup, and the orbits of* $SO(3)$ *are two-spheres. (The orbit is the set of point resulting from the action of* $SO(3)$ *on a given point.)*

The spacetime metric induces a metric on each two-sphere. By $SO(3)$ symmetry, it has to be proportional to (2.1). Let A be the area of a two-sphere and define a function r by

$$r = \sqrt{\frac{A}{4\pi}}.$$

Then the metric on the two-sphere takes the form

$$ds^2 = r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right). \tag{2.2}$$

<u>Remark</u> There exist spacetimes with $SO(3)$ symmetry with three-dimensional orbits, e.g. Taub-NUT (see [1]).

2.1.1 Birkhoff's Theorem

Theorem 2.2 *The unique spherically symmetric solution of the vacuum Einstein equations is the* **Schwarzschild solution***, which in Schwarzschild coordinates has the metric*

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 \, d\Omega^2. \tag{2.3}$$

Here $d\Omega^2$ is the metric on S^2 and $M > 0$. The proof can be found in [1].

Note

(i) The metric is well-behaved for $r > 2M$ (exterior Schwarzschild) and also for $0 < r < 2M$ (interior Schwarzschild). $r = 2M$ is called the **Schwarzschild radius**.

(ii) As $r \to \infty$ the metric approaches Minkowski metric, the exterior solution is "asymptotically flat". At large r, it is well-approximated by the linarised solution for a localised object of mass M.

(iii) $\frac{\partial}{\partial t}$ is Killing and hypersurface-orthogonal, so the exterior Schwarzschild solution is static. ("vacuum plus spherical symmetry gives static")

Since the sun is nearly spherically symmetric, the metric outside the sun is approximately the (exterior) Schwarzschild solution. The radius of the sun is about $7 \cdot 10^5$ km, but its Schwarzschild radius is only about 3 km and so well inside the sun, where the Schwarzschild solution does not apply anyway.

Do we have to worry about what happens at $r = 2M$? Yes! Stars like our sun are supported against gravitational collapse by pressure generated by nuclear reactions.
As the star uses up its nuclear "fuel", it will cool and shrink. Can some non-thermal source of pressure balance gravity? Yes, but only up to a maximum mass of about two solar masses. (see later)
More massive stars must either shed mass (e.g. supernova) or must undergo complete gravitational collapse to a black hole.

2.2 Spherically Symmetric Pressure-Free Collapse

Consider a spherically symmetric star made of "dust", i.e. a pressure-free fluid.
From Birkhoff's theorem, the metric outside the star is the Schwarzschild solution. By continuity, this will also be valid at the star's surface. Let the surface of the star be at $r = R(t)$ in Schwarzschild coordinates.
Because of zero pressure and spherical symmetry, particles on the star's surface will follow timelike radial geodesics of the Schwarzschild geometry. 23 Jan
We write the 4-velocity as (3)

$$U^a = \left(\frac{dt}{d\tau}, \frac{dR}{d\tau}, 0, 0\right).$$

Since $k = \frac{\partial}{\partial t}$ is Killing, the quantity $k \cdot U = -\epsilon$ is constant along geodesics. Explicitly

$$\left(1 - \frac{2M}{R}\right) \frac{dt}{d\tau} = \epsilon. \tag{2.4}$$

ϵ is the energy per unit mass of the particle (see example sheet). Since τ is proper time, we also have $U^2 = -1$, so

$$-1 = -\left(1 - \frac{2M}{R}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2M}{R}\right)^{-1}\left(\frac{dR}{d\tau}\right)^2$$

$$1 = \left(\left(1 - \frac{2M}{R}\right) - \left(1 - \frac{2M}{R}\right)^{-1}\dot{R}^2\right)\left(\frac{dt}{d\tau}\right)^2, \quad \dot{R} = \frac{dR}{dt}. \tag{2.5}$$

Substitute (2.4)

$$1 = \left(\left(1 - \frac{2M}{R}\right) - \left(1 - \frac{2M}{R}\right)^{-1}\dot{R}^2\right)\left(1 - \frac{2M}{R}\right)^{-2}\epsilon^2. \tag{2.6}$$

We can rewrite this as

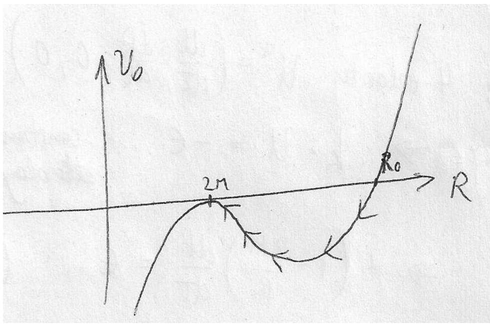
$$\frac{1}{2}\dot{R}^2 + V_0(R) = 0, \tag{2.7}$$

where

$$V_0(R) = -\frac{1}{2\epsilon^2}\left(1 - \frac{2M}{R}\right)^2\left(\frac{2M}{R} - 1 + \epsilon^2\right).$$

Assume the star is initially at rest ($\dot{R} = 0$) at $R = R_0 > 2M$. This fixes ϵ^2 as

$$\epsilon^2 = 1 - \frac{2M}{R_0} < 1, \quad R_0 = \frac{2M}{1 - \epsilon^2}. \tag{2.8}$$



R decreases with $R \to 2M$ as $t \to \infty$, so it takes an infinite Schwarzschild time to reach $r = 2M$. But what about proper time? From (2.4), we have

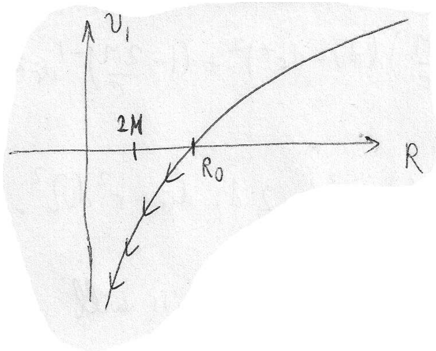
$$\frac{dR}{dt} = \frac{dR}{d\tau}\frac{d\tau}{dt} = \frac{1}{\epsilon}\left(1 - \frac{2M}{R}\right)\frac{dR}{d\tau} \tag{2.9}$$

and (2.7) becomes

$$\frac{1}{2}\left(\frac{dR}{d\tau}\right)^2 + V_1(R) = 0, \tag{2.10}$$

where

$$V_1(R) = -M\left(\frac{1}{R} - \frac{1}{R_0}\right).$$



The surface of the star will fall through $R = 2M$ in finite proper time. This suggests investigating the spacetime near $r = 2M$ using coordinates adapted to infalling particles.
It is simplest to use massless particles. For a radial null geodesic $ds^2 = 0$ and $d\Omega^2 = 0$ and we introduce a new radial coordinate by

$$dt^2 = \left(1 - \frac{2M}{r}\right)^{-2} dr^2 \equiv (dr^*)^2, \tag{2.11}$$

where $r^* = r + 2M \log |\frac{r-2M}{2M}|$. r^* is called the "Regge-Wheeler radial coordinate" or "tortoise coordinate". As r ranges from $2M$ to ∞, r^* ranges from $-\infty$ to ∞.
Radial null geodesics are described by $dt = \pm dr^*$ and hence

$$t = \pm r^* + \text{constant} \qquad \text{is an} \begin{cases} \text{outgoing} \\ \text{ingoing} \end{cases} \text{radial null geodesic.} \tag{2.12}$$

Let $v = t + r^*$ (constant on ingoing null geodesics) be a new time coordinate and rewrite the metric in **ingoing Eddington-Finkelstein coordinates** (v, r, θ, φ)

$$ds^2 = -\left(1 - \frac{2M}{r}\right)(dv - dr^*)^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dv\,dr + r^2\,d\Omega^2. \tag{2.13}$$

Initally we had $r > 2M$ but this form of the metric is well behaved for all $r > 0$. (i.e. the metric and its inverse are smooth) Hence we can analytically continue through $r = 2M$ to a new region $r < 2M$.
The bad behaviour at $r = 2M$ is a "coordinate singularity"; the coordinate chart, and not spacetime, is badly behaved.

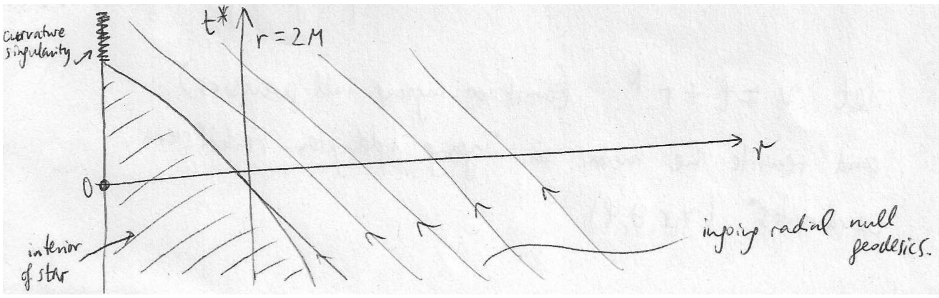
However, $r = 0$ is a physical **curvature singularity** since

$$R_{abcd} R^{abcd} \propto \frac{M^2}{r^6} \to \infty \quad \text{as } r \to 0,$$

so tidal forces are infinite there.

Summary
The star collapses through $r = 2M$ and forms a curvature singularity in finite proper time. (Exercise)

We can draw a Finkelstein diagram by plotting $t^* \equiv v - r$ against r.



Note

(i) For $r > 2M$,

$$\left(\frac{\partial}{\partial t}\right)_{SS} = \left(\frac{\partial}{\partial v}\right)_{EF}$$

(Exercise) is the static Killing vector field, which becomes null at $r = 2M$, and spacelike in $r < 2M$.

The solution is still spherically symmetric in $r < 2M$, hence it must be described by the interior Schwarzschild solution. Explicitly, let $t' = v - r^*$ for $r < 2M$. Then the metric in coordinates (t', r, θ, φ) is interior Schwarzschild. (Exercise)

(ii) In ingoing Eddington-Finkelstein coordinates, $-\frac{\partial}{\partial r}$ is tangent to ingoing radial null geodesics, hence it is future directed.

Consider any future-directed non-spacelike curve (in the interior) $x^a(\lambda)$ with tangent $U^a = \frac{dx^a}{d\lambda}$. Then

$$\left(-\frac{\partial}{\partial r}\right) \cdot U \leq 0, \quad \frac{dv}{d\lambda} \geq 0.$$

So v is non-decreasing. It follows from $U^2 \leq 0$ that

$$-2\frac{dv}{d\lambda}\frac{dr}{d\lambda} = -U^2 + \left(\frac{2M}{r} - 1\right)\left(\frac{dv}{d\lambda}\right)^2 + r^2\left(\frac{d\Omega}{d\lambda}\right)^2 \geq 0, \tag{2.14}$$

so we have

$$\frac{dv}{d\lambda}\frac{dr}{d\lambda} \leq 0$$

with equality only for radial null geodesics with $r \equiv 2M$ or $v =$ constant (ingoing null geodesics). In the interior $(r < 2M)$ r decreases along any future-directed causal curve.

Hence no such curve connects a point in $r \leq 2M$ to a point in $r > 2M$, in particular to $r = \infty$. This is the defining property of a **black hole**.

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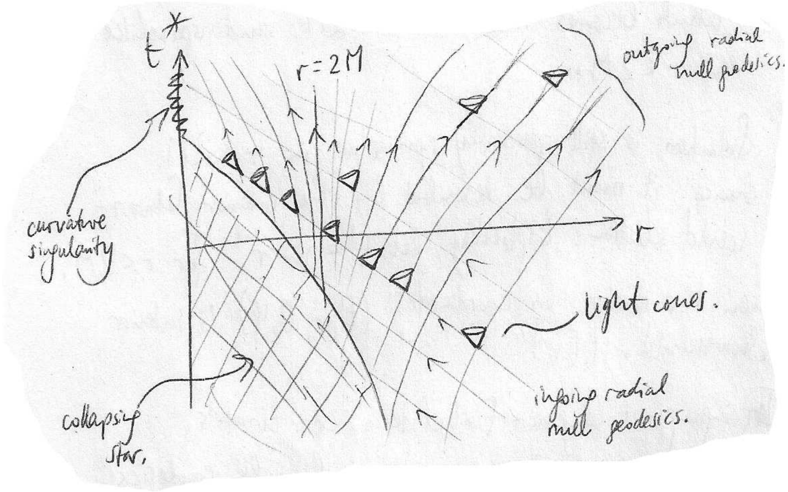


Exercise Show that "outgoing" radial null geodesics are given by

$$v - 2\left(r + 2M\log\left|\frac{r-2M}{2M}\right|\right) = \text{constant} \quad \text{or r} \equiv 2\text{M}. \tag{2.15}$$

Light cones "tip over" towards the singularity. An external observer never sees the star fall through $r = 2M$, it just fades from view.

2.3 White Holes

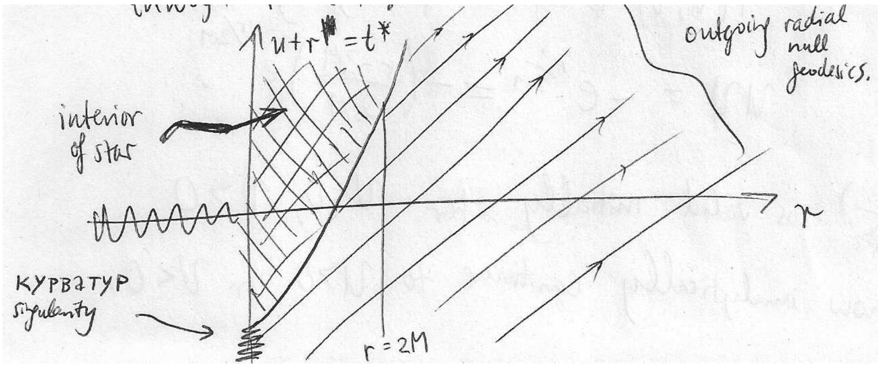
For a black hole, $r = 2M$ acts as a one-way membrane. But Einstein's equations are time-reversal invariant!
We introduce a coordinate $u = t - r^*$ in $r > 2M$, which is constant on outgoing radial null geodesics, and rewrite the Schwarzschild metric in **outgoing Eddington-Finkelstein coordinates** (u, r, θ, φ)

$$ds^2 = -\left(1 - \frac{2M}{r}\right)du^2 - 2du\,dr + r^2\,d\Omega^2. \tag{2.16}$$

This is initially defined for $r > 2M$ but can be analytically continued to $r < 2M$.
This region must be interior Schwarzschild but it is not the same as the $r < 2M$ region in ingoing Eddington-Finkelstein coordinates.

Exercise Show that once can argue as before, using the fact that $\frac{\partial}{\partial r}$ is tangent to outgoing radial null geodesics, to show $\frac{du}{d\lambda} \geq 0$ and $\frac{dr}{d\lambda} \geq 0$ in $r < 2M$ along any future-directed causal curve.

A star intially with $r < 2M$ must expand out through $r = 2M$!



The region $r < 2M$ in outgoing Eddington-Finkelstein coordinates is a **white hole**; the time-reverse of a black hole.
Both are allowed by general relativity, but white holes require special initial conditions (at the singularity), whereas black holes do not, so only black holes occur in nature.

2.4 Kruskal-Szekeres Coordinates

The region $r > 2M$ is covered by both ingoing and outgoing Eddington-Finkelstein coordinates. We can write the Schwarzschild metric in coordinates (u, v, θ, φ) in the region $r > 2M$:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du\, dv + r^2\, d\Omega^2, \tag{2.17}$$

where $r = r(u, v)$ is now an implicitly defined function of u and v. Let us introduce new coordinates

$$U = -\exp\left(-\frac{u}{4M}\right), \quad V = \exp\left(\frac{v}{4M}\right), \tag{2.18}$$

so that

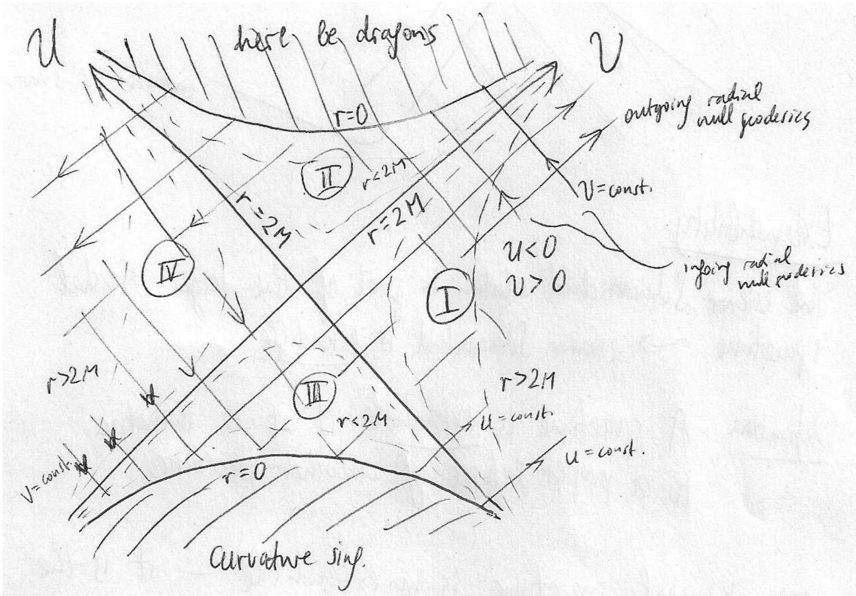
$$ds^2 = -\frac{32M^3}{r} e^{-\frac{r}{2M}} dU\, dV + r^2\, d\Omega^2, \tag{2.19}$$

where $r(U, V)$ is defined by

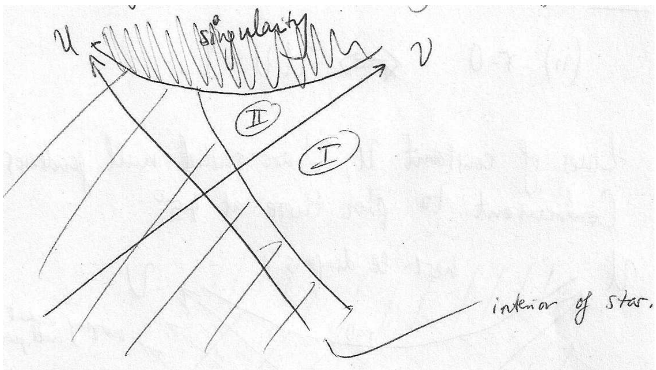
$$UV = -\exp\left(\frac{r^*}{2M}\right) = -\frac{r - 2M}{2M} e^{\frac{r}{2M}}. \tag{2.20}$$

(2.19) is valid initially for $U < 0,\ V > 0$ but can be analytically continued to positive U or negative V.
Note $r = 2M$ corresponds to $U = 0$ or $V = 0$; $r = 0$ corresponds to $UV = 1$.

Lines of constant U, V are radial null geodesics. It is convenient to plot these at 45 degrees.



Regions I and II are covered by ingoing Eddington-Finkelstein coordinates; regions I and III are covered by outgoing Eddington-Finkelstein coordinates. Region IV is a new asymptotically flat region.
The isometry $(U, V) \to (-U, -V)$ interchanges regions I and IV and II and III.
Only regions I and II are relevant in gravitational collapse, the other regions are covered up by the interior of a collapsing star.



2.4.1 Extendibility

The exterior Schwarzschild solution is part of the larger Kruskal spacetime; it is extendible.

Definition 2.3 *A spacetime is* **extendible** *if it is isometric to a proper subset of another spacetime.*

The Kruskal spacetime is **inextendible** - it is the "maximal analytic extension" of the Schwarzschild solution.

2.4.2 Singularities

The metric is **singular** if it or its inverse is not smooth somewhere.
A **coordinate singularity** can be eliminated by a change of coordinates (e.g. $r = 2M$ in Schwarzschild). These are unphysical.
Curvature singularities where some scalar built from $R^a{}_{bcd}$ diverges, <u>are</u> physical.
Not all physical singularities are curvature singularities, e.g. a conical singularity: Consider the metric

$$ds^2 = dr^2 + \lambda^2 r^2\, d\varphi^2, \quad \lambda \neq 1, \tag{2.21}$$

where φ is an angular coordinate that is periodically identified with period 2π. If $r \neq 0$ then this is flat (to see this, let $\varphi' = \lambda\varphi$), hence $r = 0$ can not be a curvature singularity. But if we consider the circle given by $r = \epsilon$,

$$\frac{\text{Circumference}}{\text{Radius}} = \frac{2\pi\lambda\epsilon}{\epsilon} = 2\pi\lambda \not\to 2\pi \qquad \text{as } \epsilon \to 0.$$

The geometry is <u>not</u> locally flat at $r = 0$, hence it is not regular there; the point $r = 0$ can not be part of the manifold.

Physical singularities are not "places" as they do not belong to the spacetime manifold. But in a singular manifold some geodesics will "end" at a singularity. This motivates the following definition. 28 Jan (5)

Definition 2.4 *A spacetime is* **non-singular** *if all geodesics are complete (i.e. can be extended to arbitrary values of the affine parameter).*

<u>Note</u>

(i) The Kruskal spacetime is singular as some geodesics reach $r = 0$ in finite affine parameter.

(ii) An extendible spacetime is trivially singular, so usually assume the spacetime is inextendible.

(iii) The **singularity theorems** of Penrose and Hawking prove that geodesic incompleteness is a generic feature of gravitational collapse, not just a special property of spherically symmetric collapse.

2.4.3 Time Translation in Kruskal

<u>Exercise</u> Show that, in Kruskal coordinates, $k = \frac{\partial}{\partial t}$ is

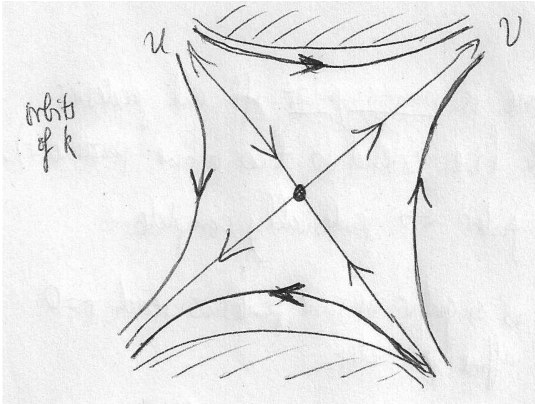
$$k = \frac{1}{4M}\left(v\frac{\partial}{\partial v} - u\frac{\partial}{\partial u}\right). \tag{2.22}$$

<u>Note</u>

(i) $k^2 = -\left(1 - \frac{2M}{r}\right)$, hence k is $\begin{cases} \text{timelike for } r > 2M \text{ (regions I, IV)}, \\ \text{spacelike in regions II, III}, \\ \text{null on } r = 2M \text{ (i.e. } U = 0 \text{ or } V = 0). \end{cases}$

(ii) $\{U = 0\}$ and $\{V = 0\}$ are fixed sets of k.

(iii) $k = 0$ on the "bifurcation two-sphere" $U = V = 0$.



2.5 Relativistic Stars

A star can be supported against gravitational collapse by a non-thermal source of pressure:
White dwarf/neutron star, supported by degeneracy pressure of electrons/neutrons.
But stars supported by cold matter have a maximum mass.

A general static spherically symmetric metric has the form

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2\, d\Omega^2. \tag{2.23}$$

The metric outside the star is the Schwarzschild metric. We model matter inside the star as a perfect fluid,

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab}. \tag{2.24}$$

If we assume the fluid is at rest, $u = \frac{1}{\sqrt{A}}\frac{\partial}{\partial t}$ (so that $u^2 = -1$). Because of spherical symmetry, $p = p(r)$ and $\rho = \rho(r)$ are functions of r only. We assume that

(i) Cold matter has some equation of state $p = p(\rho)$ with $p > 0,\ \rho > 0$ and $\frac{dp}{d\rho} > 0$ (needed for stability).

(ii) The equation of state is known up to nuclear density ($\sim 3 \cdot 10^{14}$ gcm^{-3}).

If you then solve the Einstein equations, you will discover that solutions can only exist for $M < M_c \approx 2$ solar masses.

3 The Event Horizon

3.1 Null Hypersurfaces

Definition 3.1 *A* **null hypersurface** *is a hypersurface whose normal is null.*

For instance, consider a surface $r =$ constant in the Schwarzschild solution using ingoing Eddington-Finkelstein coordinates. The normal is $n = dr$ and hence

$$n^a = g^{ab}n_b = g^{ar} = \left(\frac{\partial}{\partial v}\right)^a + \left(1 - \frac{2M}{r}\right)\left(\frac{\partial}{\partial r}\right)^a, \quad n^2 = n^a n_a = 1 - \frac{2M}{r}. \tag{3.1}$$

Hence this is a null hypersurface if $r = 2M$. Note that

$$n^a|_{r=2M} = \left(\frac{\partial}{\partial v}\right)^a.$$

Now let N be a general null hypersurface with normal n. A tangent vector t to N obeys $t \cdot n = 0$, but since $n \cdot n = 0$, n is both a normal and a tangent vector.
Hence $n^a = \frac{dx^a}{d\lambda}$ for some null curve $x^a(\lambda)$ in N.

Proposition 3.2 *The curves $x^a(\lambda)$ are geodesics.*

Proof n is hypersurface-orthogonal, hence $n = h\,df$ for some functions h and f (i.e. N belongs to a family $f(x) =$ constant). Then compute

$$n \cdot \nabla n^a = n \cdot \nabla(g^{ab}h\partial_b f) = g^{ab}\partial_b f\, n \cdot \nabla h + g^{ab}hn^c\nabla_c\partial_b f = \frac{1}{h}n^a\, n \cdot \nabla h + g^{ab}hn^c\nabla_b\partial_c f$$

$$= n^a\, n \cdot \partial \log|h| + g^{ab}hn^c\nabla_b\left(\frac{1}{h}n_c\right) = n^a\, n \cdot \partial \log|h| + g^{ab}hn^2\partial_b\frac{1}{h} + g^{ab}n^c\nabla_b n_c,$$

$$n \cdot \nabla n^a = n^a\, n \cdot \partial \log|h| - n^2\partial^a \log|h| + \frac{1}{2}\partial^a(n^2). \tag{3.2}$$

Now $n^2 = 0$ on N, and $t \cdot \partial(n^2) = 0$ for any t tangent to N, i.e. $\partial_a(n^2)$ is proportional to n_a. All terms on the right-hand side of (3.2) are proportional to n^a on N, so $n \cdot \nabla^a|_N \propto n^a$ and $x^a(\lambda)$ is a geodesic, q.e.d.

We can choose h so that $n \cdot \nabla n^a|_N = 0$, i.e. λ is an affine parameter. We can assume this henceforth.

Definition 3.3 *The null geodesics $x^a(\lambda)$ with affine parameter λ for which the tangent vectors $\frac{dx^a}{d\lambda}$ are normal to a null hypersurface N are called the* **generators** *of N.*

For example take the surface $U = 0$ in the Kruskal spacetime. The normal has the form $n = h\,dU$ and hence

$$n^a = -\frac{r}{16M^2}e^{\frac{r}{2M}}h\left(\frac{\partial}{\partial V}\right)^a.$$

Here $n^2 = n^a n_a = 0$ everywhere, not just on N. All $U =$ constant surfaces are null. From (3.2), $n \cdot \nabla n^a = 0$ for $h =$ constant. We choose

$$h = -\frac{8M^2}{e} \quad \Rightarrow \; n^a|_N = \left(\frac{\partial}{\partial V}\right)^a.$$

V is an affine parameter for the null geodesic generators of $\{U = 0\}$.

3.1.1 Killing Horizons and Surface Gravity

Definition 3.4 *A null hypersurface N is a* **Killing horizon** *of a Killing vector field ξ if $\xi|_N$ is normal to N.*

Let n be a normal to N such that $n \cdot \nabla n^a = 0$ on N, $\xi = h\, n$ for some h and so

$$\underline{\xi \cdot \nabla \xi^a|_N = \kappa \xi^a}, \tag{3.3}$$

where $\kappa = \xi \cdot \partial \log |h|$ is called the **surface gravity** of the Killing horizon.

Exercise Show that (3.3) can be written as

$$-\frac{1}{2}\partial^a(\xi^2)|_N = \kappa \xi^a.$$

Example In the Kruskal spacetime, define $N^+ = \{U = 0\},\ N^- = \{V = 0\}$. 30 Jan
$\xi = k$ is the stationary Killing vector field (6)

$$k = \begin{cases} \frac{1}{4M} V \frac{\partial}{\partial V} & \text{on } N^+, \\ -\frac{1}{4M} U \frac{\partial}{\partial U} & \text{on } N^-. \end{cases}$$

$k = h\, n$ where

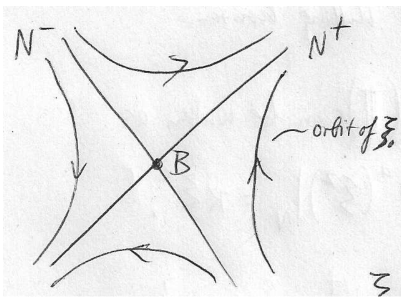
$$h = \begin{cases} \frac{1}{4M} V & \text{on } N^+, \\ -\frac{1}{4M} U & \text{on } N^-. \end{cases}, \quad n = \begin{cases} \frac{\partial}{\partial V} & \text{on } N^+, \\ \frac{\partial}{\partial U} & \text{on } N^-. \end{cases}$$

n is normal to $N^\pm$, hence $N^\pm$ is a Killing horizon of k. Since $n \cdot \nabla n^a = 0$ (above), the surface gravity is

$$h = k \cdot \partial \log |h| = \begin{cases} \frac{1}{4M} & \text{on } N^+, \\ -\frac{1}{4M} & \text{on } N^-. \end{cases}. \tag{3.4}$$

Note $\kappa =$ constant on $N^\pm$ is a special case of the Zeroth Law of Black Hole Mechanics. (see later)

$N = N^+ \cup N^-$ is called a "bifurcate Killing horizon" with bifurcation two-sphere $B = N^+ \cap N^- = \{U = V = 0\}$; $\xi = 0$ on B.



3.1.2 Normalisation of κ

If N is a Killing horizon of ξ with surface gravity κ, then N is also a Killing horizon of $c\xi$ with surface gravity $c\kappa$, hence κ depends on the normalisation of ξ.
In asymptotically flat spacetimes, we can normalise ξ at infinity, e.g. for time translations choose $k^2 = -1$ at infinity. This fixes k up to a sign; we can fix the sign by requiring k to be future-directed at infinity.

3.2 Rindler Spacetime

We start from the Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2\, d\Omega^2. \tag{3.5}$$

Now let $r - 2M = \frac{x^2}{8M}$, so that

$$1 - \frac{2M}{r} = \frac{\kappa^2 x^2}{1 + \kappa^2 x^2} \approx \kappa^2 x^2 \text{ near } x = 0, \quad \kappa = \frac{1}{4M}.$$

The metric near $x = 0$ is

$$ds^2 = -\kappa^2 x^2 dt^2 + dx^2 + (2M)^2 d\Omega^2 + \text{ subleading terms.}$$

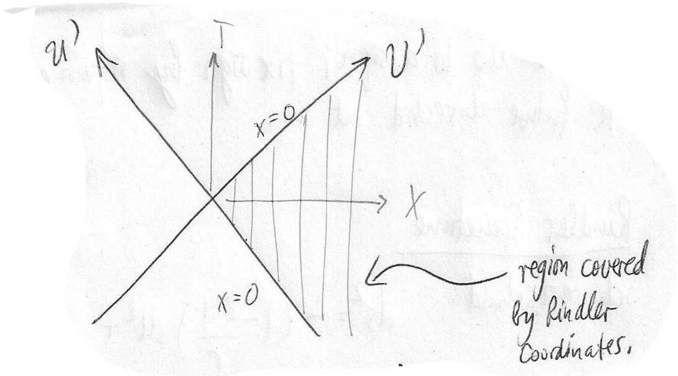
We can thus learn about the Schwarzschild spacetime near $r = 2M$ by studying the two-dimensional **Rindler spacetime** given by

$$ds^2 = -\kappa^2 x^2 dt^2 + dx^2, \quad x > 0. \tag{3.6}$$

This has a coordinate singularity at $x = 0$. After the coordinate transformation $U' = -xe^{-\kappa t},\ V' = xe^{\kappa t}$ the metric becomes

$$ds^2 = -dU'\, dV' = -dT^2 + dX^2, \tag{3.7}$$

where $U' = T - X,\ V' = T + X$ (so $U' \leq 0,\ V' \geq 0$).
This shows that the Rindler spacetime is flat and covers the region $U' \leq 0,\ V' \geq 0$ of Minkowski spacetime (corresponding to region I of the Kruskal spacetime).



$\{U' = 0\}$ and $\{V = 0\}$ form a bifurcate Killing horizon of $k = \frac{\partial}{\partial t}$ with surface gravity $\pm\kappa$ (Exercise).
Note $k^2 = -\kappa^2 x^2 \to -\infty$ as $x \to \infty$, there is no natural way to normalise κ in the Rindler spacetime.

3.2.1 Acceleration Horizons

Consider orbits of a stationary Killing vector field k; the normalised four-velocity is $u = \frac{k}{\sqrt{-k^2}}$. The proper acceleration is given by

$$a^a = u \cdot \nabla u^a = \frac{k \cdot \nabla k^a}{-k^2} + \frac{k^a}{2(-k^2)^2} k \cdot \partial(k^2).$$

By using $k^b\nabla_b k^a = -k^b\nabla^a k_b = -\frac{1}{2}\partial^a(k^2)$ and $k\cdot\partial(k^2) = 2k^a k^b \nabla_a k_b = 0$ (since k is Killing) we can rewrite this as

$$\underline{a^a = -\frac{1}{2}\frac{\partial^a(k^2)}{(-k^2)} = \frac{1}{2}\partial^a \log(-k^2).} \tag{3.8}$$

Examples

(i) Rindler spacetime

$$a = \frac{1}{2}d\log(\kappa^2 x^2) = \frac{dx}{x},$$

and $|a| = \sqrt{g^{ab}a_a a_b} = \frac{1}{x}$. Orbits of k have $x =$ constant, hence they correspond to curves of constant acceleration.

(ii) Schwarzschild spacetime

$$a = \frac{1}{2}d\log\left(1-\frac{2M}{r}\right) = \frac{M}{r^2\left(1-\frac{2M}{r}\right)}dr,$$

$$|a| = \sqrt{g^{ab}a_a a_b} = \sqrt{\frac{M^2}{r^4\left(1-\frac{2M}{r}\right)}} = \frac{M}{r^2\sqrt{1-\frac{2M}{r}}}.$$

Orbits of k have $r =$ constant, hence constant acceleration.

In both of these cases $|a|\to\infty$ at the Killing horizon. It is an "acceleration horizon" (a surface in a static spacetime where the norm of the proper acceleration of orbits of k diverges).

3.2.2 Interpretation of κ

In a static, asymptotically flat spacetime, consider a static particle P of unit mass (i.e. following an orbit of k) held at rest by a massless, inelastic string, whose other end is held by an observer at infinity.
Let F be the force (tension) in the string measured at infinity. Then $F\to\kappa$ as we consider orbits closer and closer to a Killing horizon.
So κ is the force required at infinity to hold a unit mass particle at rest (i.e. on an orbit of k) near the Killing horizon.
Proof (for Schwarzschild) Example Sheet 2.

Later we will see that $\frac{\kappa}{2\pi}$ is the Hawking temperature of the black hole.

3.3 Conformal Compactification

Given a spacetime $(\mathcal{M}, g)$ consider a new metric

$$\tilde{g} \equiv \Omega^2 g, \quad \Omega(x) > 0 \text{ in } \mathcal{M}.$$

g and $\tilde{g}$ have the same light cones, i.e. the same causal structure.
Choose Ω so that "points at infinity" with respect to g are at finite affine parameter with respect to $\tilde{g}$. For this we need $\Omega(x)\to 0$ "at infinity with respect to g". We then define "infinity" to be x

such that $\Omega(x) = 0$. Such points do not belong to $\mathcal{M}$, but we can add them to $\mathcal{M}$ to give a new manifold $(\tilde{\mathcal{M}}, \tilde{g})$, a "conformal compactification" of $(\mathcal{M}, g)$.

Examples

(i) Minkowski spacetime

$$ds^2 = -dt^2 + dr^2 + r^2\, d\Omega^2$$

Introduce light cone coordinates $u = t - r,\ v = t + r$, so that $-\infty < u \leq v < \infty$ and the metric becomes

$$ds^2 = -du\, dv + \frac{1}{4}(u - v)^2 d\Omega^2. \tag{3.9}$$

Now let $u = \tan p,\ v = \tan q$, so that $-\frac{\pi}{2} < p \leq q < \frac{\pi}{2}$ and the metric is

$$ds^2 = \frac{1}{(2\cos p \cos q)^2}\left(-4dp\, dq + \sin^2(p - q) d\Omega^2\right). \tag{3.10}$$

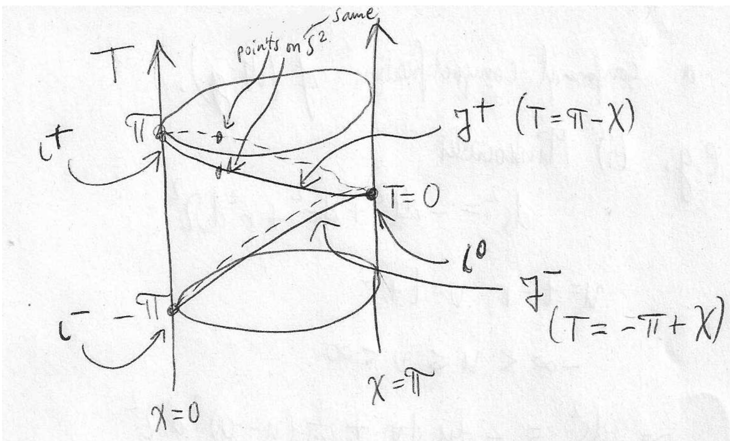
Infinity is $|p| \to \frac{\pi}{2}$ or $|q| \to \frac{\pi}{2}$.

1 Feb (7)

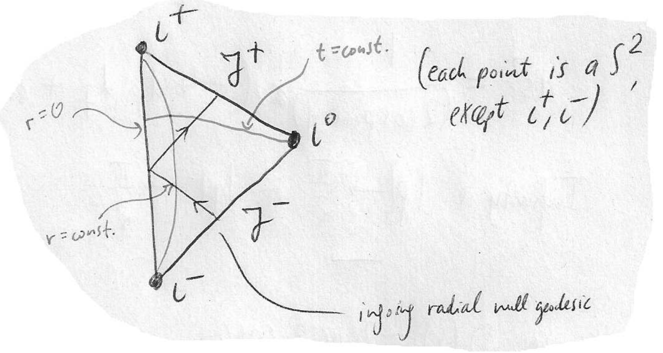
We choose the conformal factor to be $\Omega(x) = 2\cos p \cos q$:

$$d\tilde{s}^2 = -4dp\, dq + \sin^2(p - q) d\Omega^2 = -dT^2 + d\chi^2 + \sin^2 \chi\, d\Omega^2 \tag{3.11}$$

where $T = q + p,\ \chi = q - p$ and now $T \in (-\pi, \pi)$ and $\chi \in (0, \pi)$.
Minkowski spacetime is conformal to a region of the Einstein static universe $\mathbb{R} \times S^3$. We "add in points at infinity", which are the boundary of this region.



Suppressing 2-spheres (keeping angular coordinates fixed) we obtain the **Penrose diagram**

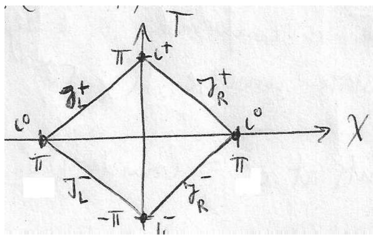


Radial null curves are at 45 degrees.

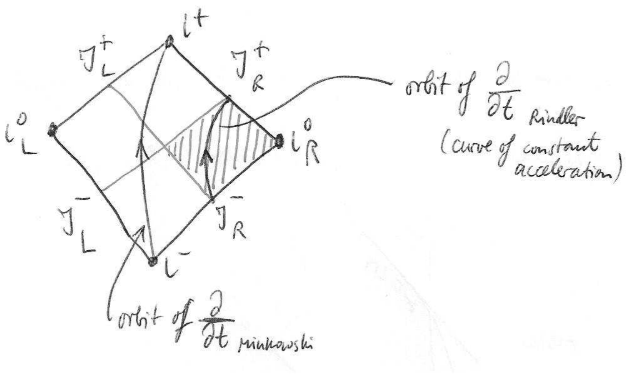
(ii) Two-dimensional Minkowski spacetime

$$ds^2 = -dt^2 + dr^2,$$

where now $-\infty < r < \infty$. Do the same as before, i.e. introduce coordinates u, v and p, q; the conformally related metric will be (3.11) but now the ranges of the coordinates are $T \in (-\pi, \pi)$, $\chi \in (-\pi, \pi)$.



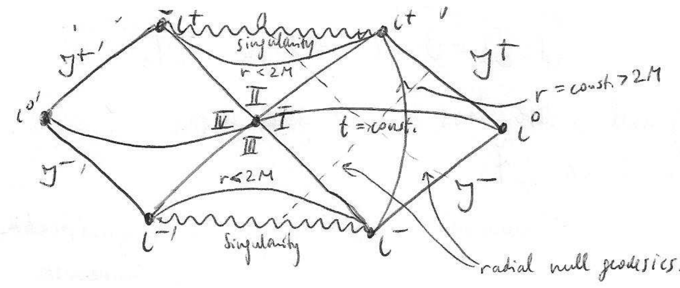
In two dimensions, there are left and right infinities.
Rindler spacetime is the $u < 0,\ v > 0$ part of two-dimensional Minkowski spacetime.



(iii) Kruskal spacetime
Use (u, v) coordinates in region I and let $u = \tan p,\ v = \tan q$, then $-\frac{\pi}{2} < p < q < \frac{\pi}{2}$ and we use the same conformal factor as before:

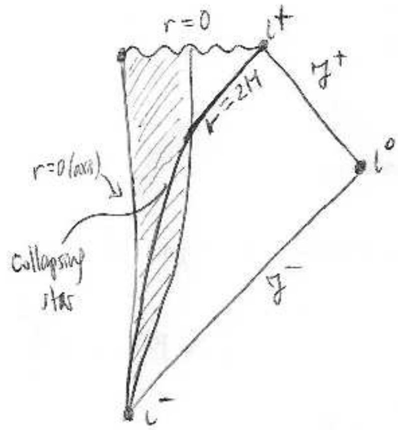
$$d\tilde{s}^2 = (2\cos p \cos q)^2 ds^2 = -4\left(1 - \frac{2M}{r}\right) dp\, dq + \left(\frac{r}{r_*}\right)^2 \sin^2(q - p) d\Omega^2. \qquad (3.12)$$

This approaches the metric of compatified Minkowski space, so we can add $\mathcal{I}^{\pm}, \imath^0$ as before. Near $r = 2M$ we use Kruskal coordinates to go through the horizon; again we suppress two-spheres to get the Penrose diagram.



The surfaces of constant r all intersect at $\imath^{\pm}$, including $r = 0$, so $\imath^{\pm}$ is singular and can not be added to the spacetime (but we draw them anyway).
We can choose Ω so that $r = 0$ is a straight line.

(iv) Spherically symmetric gravitational collapse



3.3.1 Asymptotic Flatness

Definition 3.5 *A spacetime $(\mathcal{M}, g)$ is* **asymptotically simple** *if there exists a manifold with boundary $(\tilde{\mathcal{M}}, \tilde{g})$ such that*

(a) $\mathcal{M} = int\ \tilde{\mathcal{M}}$, so that $\tilde{\mathcal{M}} = \mathcal{M} \cup \partial\tilde{\mathcal{M}}$,

(b) $\tilde{g} = \Omega^2 g$ for some function $\Omega(x)$ with $\Omega(x) > 0$ on $\mathcal{M}$ and $\Omega = 0$, $d\Omega \neq 0$ on $\partial\tilde{\mathcal{M}}$,

(c) every null geodesic has a past and future endpoint on $\partial\tilde{\mathcal{M}}$.

Minkowski spacetime is asymptotically simple because we can take $\tilde{\mathcal{M}}$ to be the conformal compactification of Minkowski spacetime. There is a technical subtlety; a manifold with boundary can not have corners, so we must delete $\imath^{\pm}$ and $\imath^0$ from $\tilde{\mathcal{M}}$, hence $\partial\tilde{\mathcal{M}} = \mathcal{I}^+ \cup \mathcal{I}^-$.
This definition is quite restrictive as condition (c) excludes black holes. We would like to have a notion of asymptotic flatness that includes black holes.

Definition 3.6 *A spacetime* $(\mathcal{M}, g)$ *is* **weakly asymptotically simple** *if there exists an open set* $U \subset \mathcal{M}$ *that is isometric to a neighbourhood of* $\partial\tilde{\mathcal{M}}$ *of some asymptotically simple spacetime.*

Exercise Prove that the Kruskal spacetime is weakly asymptotically simple (Example sheet 2).

Definition 3.7 *A spacetime* $(\mathcal{M}, g)$ *is* **asymptotically flat** *if it is weakly asymptotically simple and* **asymptotically empty**, *i.e.* $R_{ab} = 0$ *in a neighbourhood of* $\partial\tilde{\mathcal{M}}$.

This definition needs to be modified in the presence of an electromagnetic field when $R_{ab} = 0$ only on $\partial\tilde{\mathcal{M}}$.
An asymptotically flat spacetime has the same structure for $\mathcal{I}^{\pm}, i^0$ as Minkowski spacetime.

3.4 The Event Horizon

Definition 3.8 *The* **causal past** *of* $U \subset \mathcal{M}$ *is*

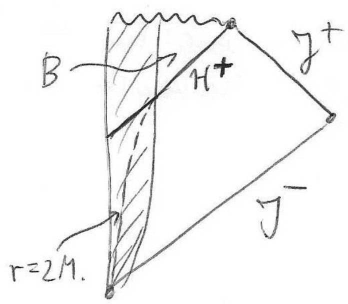
$$\mathcal{J}^-(U) = \{p \in \mathcal{M} : \text{ there exists a future-directed causal curve from } p \text{ to some } q \in U\}.\}$$

Similarly, we define the **causal future** $\mathcal{J}^+(U)$.

Definition 3.9 *The* **black hole region** *of an asymptotically flat spacetime* $(\mathcal{M}, g)$ *is* $\mathcal{B} = \mathcal{M} \backslash \mathcal{J}^-(\mathcal{I}^+)$, *i.e.*

$$\mathcal{B} = \{p \in M : \text{ there is no future-directed causal curve from } p \text{ to } \mathcal{I}^+\}.$$

Definition 3.10 *The* **future event horizon** $\mathcal{H}^+$ *is the boundary of the black hole region* $\mathcal{B}$ *in* $\mathcal{M}$, *i.e. the boundary of* $\mathcal{J}^-(\mathcal{I}^+)$ *in* $\mathcal{M}$.



Properties 4 Feb (8)

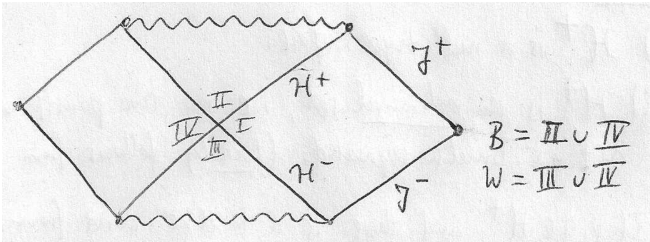
(i) $\mathcal{H}^+$ is a null hypersurface.

(ii) $\mathcal{H}^+$ is an **achronal** set, i.e. no two points in $\mathcal{H}^+$ are timelike separated. (this follows locally from (i))
To see this, assume $p, q \in \mathcal{H}^+$ and there is a timelike curve from p to q. We could continuously deform it to a timelike curve from p' inside the horizon ($p' \notin \mathcal{J}^-(\mathcal{I}^+)$) to q' outside the horizon ($q' \in \mathcal{J}^-(\mathcal{I}^+)$). But then $p' \in \mathcal{J}^-(q')$ and this gives the contradiction $p' \in \mathcal{J}^-(\mathcal{I}^+)$, as $\mathcal{J}^-(q') \subset \mathcal{J}^-(\mathcal{I}^+)$.

(iii) Null geodesic generators of $\mathcal{H}^+$ can have "past endpoints", in the sense that the continuation of a generator into the past may leave $\mathcal{H}^+$.

(iv) **Theorem 3.11** *(Penrose) The generators of $\mathcal{H}^+$ can not have future endpoints.*

This means that geodesics can enter the horizon $\mathcal{H}^+$ but can not leave it.
The time reverse of this statement is that null geodesics can leave, but not enter the past event horizon $\mathcal{H}^-$, which is the boundary of the white hole region $\mathcal{W} = \mathcal{M} \backslash \mathcal{J}^+(\mathcal{J}^-)$.



In spherically symmetric gravitational collapse, the past event horizon $\mathcal{H}^-$ is empty.

Locating $\mathcal{H}^+$ requires knowledge of the full spacetime - it can not be determined locally. But if we wait until the black hole settles down to equilibrium (i.e. a stationary spacetime) then

Theorem 3.12 *(Hawking)*
In a stationary, asymptotically flat spacetime, $\mathcal{H}^+$ is a Killing horizon (but not necessarily of $\frac{\partial}{\partial t}$).

3.4.1 Predictability

Definition 3.13 *(i) A* **partial Cauchy surface** *Σ for a spacetime $(\mathcal{M}, g)$ is a hypersurface which no causal curve intersects more than once.*

(ii) A causal curve is **past inextendible** *if it has no past endpoint in $\mathcal{M}$.*

(iii) The **future domain of dependence** *of Σ is*

$$\mathcal{D}^+(\Sigma) = \{p \in M : \textit{ every past inextendible causal curve through } p \textit{ intersects } \Sigma\}.$$

The significance of $\mathcal{D}^+(\Sigma)$ is that solutions of hyperbolic PDEs (e.g. Klein-Gordon/Maxwell equations) in $\mathcal{D}^+(\Sigma)$ are fully determined by initial data in Σ.
The **past domain of dependence** $\mathcal{D}^-(\Sigma)$ is defined analogously.

Definition 3.14 *A partial Cauchy surface Σ is a* **Cauchy surface** *for $\mathcal{M}$ if*

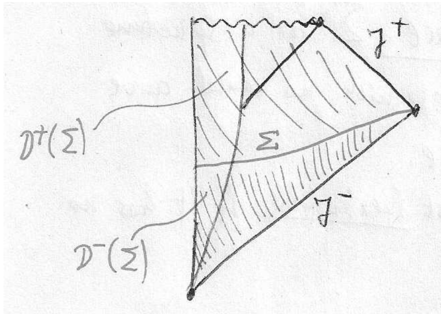
$$\mathcal{M} = \mathcal{D}^+(\Sigma) \cup \mathcal{D}^-(\Sigma).$$

$\mathcal{M}$ is **globally hyperbolic** *if it admits a Cauchy surface.*

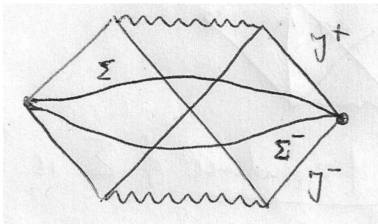
If $\mathcal{M}$ is globally hyperbolic, physics can be predicted to the future or (retrodicted to the) past from data prescribed on Σ.

Examples of globally hyperbolic spacetimes

(i) Spherically symmetric collapse



(ii) Kruskal spacetime



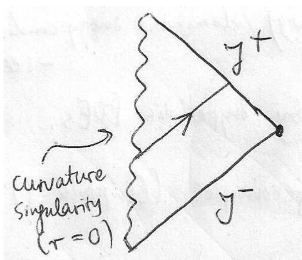
If $\mathcal{M}$ is not globally hyperbolic, then $\mathcal{D}^+(\Sigma)$ (or $\mathcal{D}^-(\Sigma)$) will have a future (past) boundary in $\mathcal{M}$ - this is a null hypersurface called the future (past) **Cauchy horizon** of Σ.

3.4.2 Cosmic Censorship

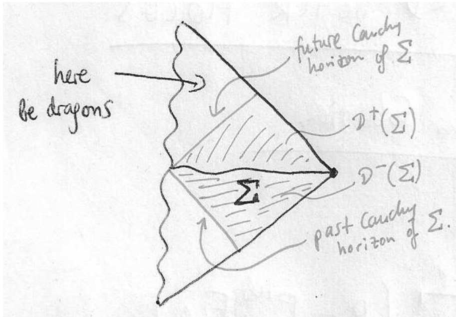
In spherically symmetric gravitational collapse, the singularity at $r = 0$ is hidden in the sense that no signal from it can reach $\mathcal{I}^+$.
This is not true for the Kruskal spacetime: A signal from the white hole singularity <u>can</u> reach $\mathcal{I}^+$; it is a **naked singularity**.
Another example of a naked singularity is the Schwarzschild spacetime with $M < 0$

$$ds^2 = -\left(1 + \frac{2|M|}{r}\right) dt^2 + \frac{dr^2}{1 + \frac{2|M|}{r}} + r^2 d\Omega^2.$$

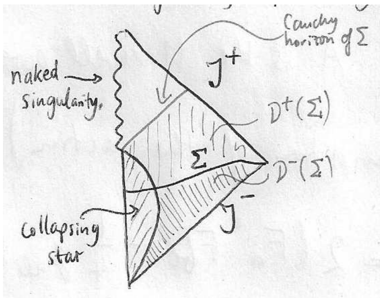
(This solves the vacuum Einstein equations.) Its Penrose diagram is



The spacetime is <u>not</u> globally hyperbolic.



Could a naked singularity form by gravitational collapse?



If this happened, predictability would be lost.
Considerable evidence (e.g. numerical simulations) suggests this does not happen.

Proposition 3.15 *(Cosmic Censorship Conjecture, Penrose)*
Naked singularities do not form in the evolution of generic initial data describing physically reasonable matter on a non-singular, asymptotically flat, spacelike surface.

"Physically reasonable" means that the matter has positive energy/satisfies the dominant energy condition (see later) and that it is described by hyperbolic PDEs.
One can construct non-generic counterexamples to cosmic censorship.

4 Charged Black Holes

4.1 The Reissner-Nordström Solution

6 Feb (9)

We start from the Einstein-Maxwell action

$$S = \frac{1}{16\pi} \int d^4x \, \sqrt{-g} \left(R - F^{ab} F_{ab} \right) , \tag{4.1}$$

where $F = dA$ and A is the Maxwell potential.
By varying the action one obtains the Einstein equations

$$R_{ab} - \frac{1}{2} R g_{ab} = 2 \left(F_a{}^c F_{bc} - \frac{1}{4} g_{ab} F^{cd} F_{cd} \right) , \tag{4.2}$$

variation with respect to F_{ab} gives Maxwell's equations

$$\nabla^a F_{ab} = 0, \tag{4.3}$$

together with $dF = 0$ (from $F = dA$).
The unique spherically symmetric solution of the Einstein-Maxwell equations is the **Reissner-Nordström solution**

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{e^2}{r^2}} + r^2 d\Omega^2, \tag{4.4}$$

with

$$A = -\frac{Q}{r} dt - P \cos\theta \, d\varphi, \;\; e = \sqrt{Q^2 + P^2}. \tag{4.5}$$

Q and P are the electric and magnetic charges, respectively. (Of course there is no evidence that magnetic charges exist in nature but they are allowed by the equations.)
Q is defined so that the Coulomb force between two particles in flat space is $\frac{Q_1 Q_2}{r^2}$ (geometric units of charge).

The Reissner-Nordström solution is asymptotically flat (as $r \to \infty$).

Let $\Delta = r^2 - 2Mr + e^2 = (r - r_+)(r - r_-)$, where

$$r_\pm = M \pm \sqrt{M^2 - e^2}.$$

In this notation, the metric is

$$ds^2 = -\frac{\Delta}{r^2} dt^2 + \frac{r^2}{\Delta} dr^2 + r^2 \, d\Omega^2. \tag{4.6}$$

There are three distinct cases:

(i) $M < e$, and so $\Delta > 0$ for all $r > 0$.
There is no horizon and only a naked curvature singularity at $r = 0$. This case is very similar to the Schwarzschild spacetime with $M < 0$, and is excluded by cosmic censorship.
For a charged dust ball with $M < e$, electromagnetic repulsion overcomes gravitational attraction; there is no collapse.

(ii) $M = e$, see later

(iii) $M > e$, Δ has simple zeros at $r = r_\pm > 0$.

These singularities are coordinate singularities, and we will remove them in a similar fashion as before. Start in $r > r_+$ and define $dr^* = \frac{r^2}{\Delta} dr$, i.e.

$$r^* = r + \frac{1}{2\kappa_+} \log \left| \frac{r - r_+}{r_+} \right| + \frac{1}{2\kappa_-} \log \left| \frac{r - r_-}{r_-} \right| + \text{constant}, \tag{4.7}$$

where

$$\kappa_\pm = \frac{r_\pm - r_\mp}{2 r_\pm^2}.$$

Let $v = t + r^*, \;\; u = t - r^*$. In ingoing Eddington-Finkelstein coordinates (v, r, θ, φ), the Reissner-Nordström metric is

$$ds^2 = -\frac{\Delta}{r^2} dv^2 + 2 dv \, dr + r^2 \, d\Omega^2. \tag{4.8}$$

So we can analytically continue to $0 < r < r_+$. Surfaces $r =$ constant have normal $n = dr$; this is null when $g^{rr} = 0$, i.e. when $\frac{\Delta}{r^2} = 0$.
Hence $r = r_\pm$ are null hypersurfaces which we will denote by $\mathcal{S}_\pm$.

Proposition 4.1 *$\mathcal{S}_\pm$ are Killing horizons of $k = \frac{\partial}{\partial v}$ (the extension of $\frac{\partial}{\partial t}$) with surface gravities $\kappa_\pm$.*

Proof
$n = dr$ is normal to $\mathcal{S}_\pm$. The covector corresponding to k is

$$k_a = g_{ab}k^b = g_{av} = -\frac{\Delta}{r^2}(dv)_a + (dr)_a. \tag{4.9}$$

On $\mathcal{S}_\pm$, $k = dr$, hence k is normal to $\mathcal{S}_\pm$ and $\mathcal{S}_\pm$ are Killing horizons of k. Also

$$k \cdot \nabla k_a = -\frac{1}{2}\partial_a(k^2) = -\frac{1}{2}\partial_a\left(-\frac{\Delta}{r^2}\right) = \frac{1}{2}\left(\frac{\Delta}{r^2}\right)'(dr)_a = \frac{1}{2}\left(\frac{\Delta}{r^2}\right)' k_a|_{\mathcal{S}_\pm}. \tag{4.10}$$

Hence the surface gravities are

$$\kappa = \frac{1}{2}\left(\frac{\Delta}{r^2}\right)'|_{\mathcal{S}_\pm} = \frac{M}{r^2} - \frac{e^2}{r^3}|_{\mathcal{S}_\pm} = \kappa_\pm, \tag{4.11}$$

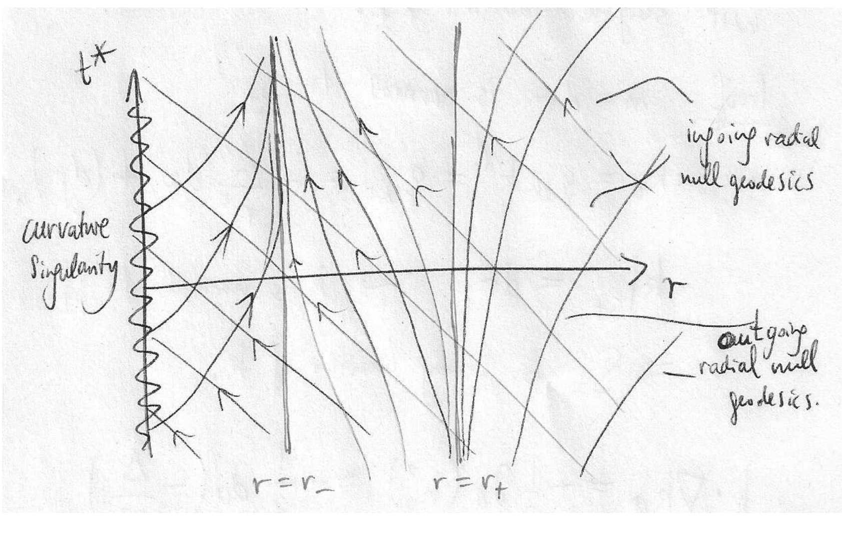
q.e.d.

Note that $k = \frac{\partial}{\partial t}$ in static coordinates, so $k^2 \to -1$ as $r \to \infty$. Hence $\kappa_\pm$ are correctly normalised.

k is spacelike for $r_- < r < r_+$ and timelike for $0 < r < r_-$.

Exercise

(a) Show that the Finkelstein diagram for the Reissner-Nordström solution with $M > e$ is



(b) Show that r decreases along any future-directed causal curve in $r_- < r < r_+$.

No point with $r < r_+$ can send a signal to infinity, so this region corresponds to a black hole.
In outgoing Eddington-Finkelstein coordinates, the Reissner-Nordström metric is

$$ds^2 = -\frac{\Delta}{r^2} du^2 - 2du\, dr + r^2\, d\Omega^2. \tag{4.12}$$

Hence the region $r \leq r_+$ is a white hole.
To construct the maximal analytic extension, we use Kruskal-type coordinates:

$$U^\pm = -e^{-\kappa_\pm u},\ V^\pm = \pm e^{\kappa_\pm v}.$$

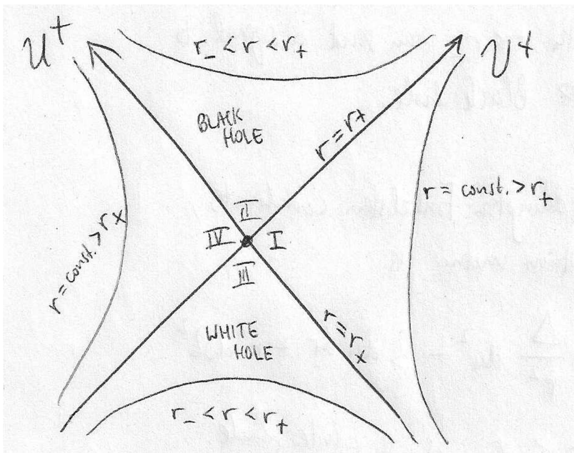
We start in $r > r_+$ and use (U^+, V^+); the metric becomes

$$ds^2 = -\frac{r_+ r_-}{\kappa_+^2} \frac{e^{-2\kappa_+ r}}{r^2} \left(\frac{r - r_-}{r_-}\right)^{1+\frac{\kappa_+}{|\kappa_-|}} dU^+\, dV^+ + r^2 d\Omega^2, \tag{4.13}$$

where $r(U^+, V^+)$ is defined implicitly by

$$U^+ V^+ = -e^{2\kappa_+ r} \left(\frac{r - r_+}{r_+}\right) \left(\frac{r_-}{r - r_-}\right)^{\frac{\kappa_+}{|\kappa_-|}}. \tag{4.14}$$

Initially, we have $U^+ < 0,\ V^+ > 0$, but can now analytically continue to $U^+ > 0$ or $V^+ < 0$.



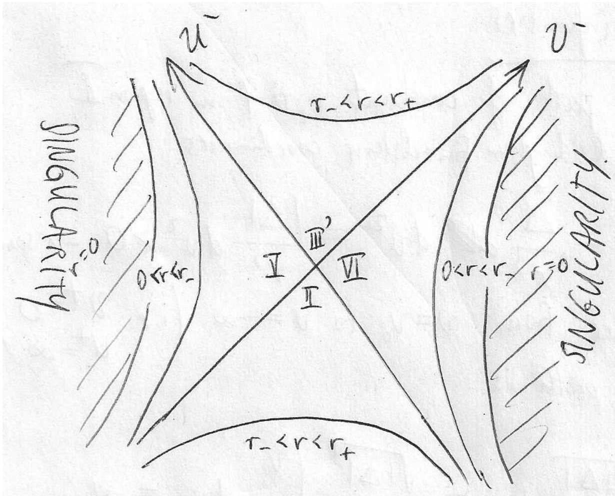
We have a bifurcate Killing horizon and a bifurcation two-sphere at $U^+ = V^+ = 0$.
Note that $r(U^+, V^+) > r_-$, so these coordinates do not cover $r \leq r_-$.
In region II, we can use ingoing Eddington-Finkelstein coordinates (v, r, θ, φ). We can then define $t' = v - r^*$ to obtain the metric in region II in Reissner-Nordström coordinates (t', r, θ, φ).
Alternatively let $u = t' - r^* = v - 2r^*$ and then define U^-, V^- as above; then $U^- < 0,\ V^- < 0$ in region II. The metric becomes

$$ds^2 = -\frac{r_+ r_-}{\kappa_-^2} \frac{e^{2|\kappa_-| r}}{r^2} \left(\frac{r_+ - r}{r_+}\right)^{1+\frac{|\kappa_-|}{\kappa_+}} dU^-\, dV^- + r^2 d\Omega^2, \tag{4.15}$$

where $r(U^-, V^-)$ is defined implicitly by

$$U^-V^- = e^{-2|\kappa_-|r}\left(\frac{r-r_-}{r_-}\right)\left(\frac{r_+}{r_+-r}\right)^{\frac{|\kappa_-|}{\kappa_+}}. \tag{4.16}$$

We can now analytically continue to $U^- > 0$ or $V^- > 0$. The Kruskal diagram is



The regions V and VI contain curvature singularities at $r = 0$, i.e. $U^-V^- = -1$.

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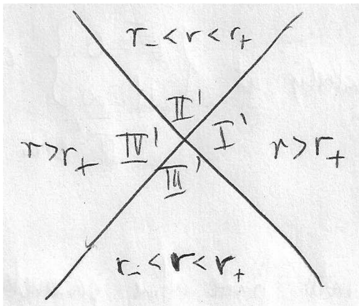
We know region II is connected to I, III and IV to the past, so by time-reversal invariance we expect (10)
that III' is connected to isometric regions I', II', and IV' to the future:



Proof: We can introduce new Kruskal coordinates U'', V'' in III' and proceed as before.
I' and IV' are new asymptotically flat regions. We can keep going and get infinitely many regions.

4.1.1 Internal Infinities

Consider a path of constant r, θ and φ in region II in ingoing Eddington-Finkelstein coordinates.

$$ds^2 = -\frac{\Delta}{r^2}dv^2 = \frac{|\Delta|}{r^2}dv^2 > 0,$$

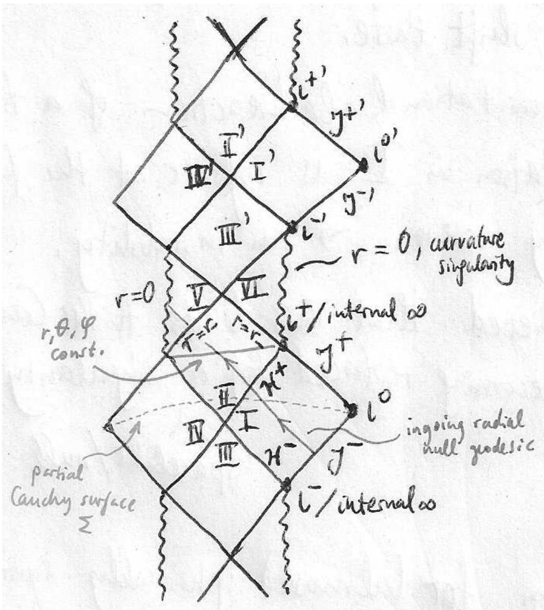
so the path is spacelike.
The proper distance from $v = v_0$ to $v = -\infty$ (i.e. $V^+ = 0$ and $U^+ = \infty$) along <u>this</u> path is

$$\int_{-\infty}^{v_0} \frac{\sqrt{|\Delta|}}{r}dv = \frac{\sqrt{|\Delta|}}{r}\int_{-\infty}^{v_0} dv = \infty. \tag{4.17}$$

There exists an "internal" spacelike infinity in region II. (One can still reach $V^+ = 0$ in finite affine parameter along a timelike/null path, so $V^+ = 0$ is part of the spacetime.) We can do the same thing using outgoing Eddington-Finkelstein coordinates in region II, there are timelike infinities at $U^+ = 0,\ V^+ = \infty$, similarly at $V^- = 0,\ U^- = \infty$ or $U^- = 0,\ V^- = \infty$ in region III etc.

4.2 Penrose Diagram

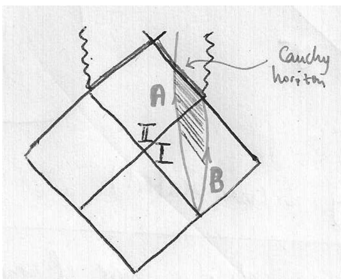
We bring points at infinity into finite affine parameter using a conformal transformation; the Penrose diagram extends infinitely into the future and past.



Note

(i) The singularity is timelike.

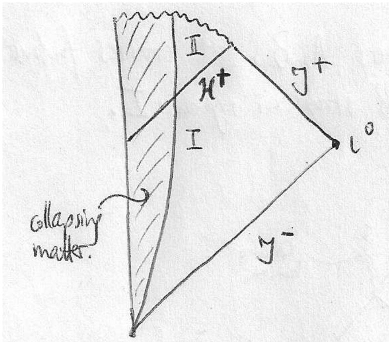
(ii) The spacetime is not globally hyperbolic. A partial Cauchy surface Σ has future/past Cauchy horizons at $r = r_-$.
Consider two observers A and B. A crosses the future Cauchy horizon, B stays in region I.



Assume that B sends light signals to A at proper time intervals of one second. B sends infinitely many signals, A receives them all in finite proper time as he crosses the Cauchy horizon. Therefore signals from region I undergo an infinite blue shift there.

The gravitational backreaction of a tiny perturbation in I is infinite at the future Cauchy horizon, this is an instability. It is believed that this leads to the Cauchy horizon becoming replaced by a spacelike/null singularity.

The Penrose diagram for (almost) spherically symmetric gravitational collapse of a charged ball of dust (with $M > e$) to form a Reissner-Nordström black hole is the following:



The structure of the singularity is complicated.

<u>Note</u>
In the real world, black holes have $e \ll M$ as

(i) large imbalances of charge do not occur in nature, so collapsing matter will be almost neutral;

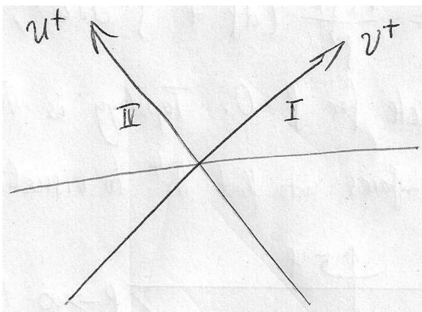
(ii) a charged black hole preferentially attracts particles of opposite charge.

4.3 Geometry of $t =$ constant Surfaces

In region I,

$$\frac{U^+}{V^+} = -e^{-2\kappa_+ t}$$

Hence surfaces of $t =$ constant have $U^+ = V^+ \cdot$ constant and are straight lines through the origin of the Kruskal diagram.



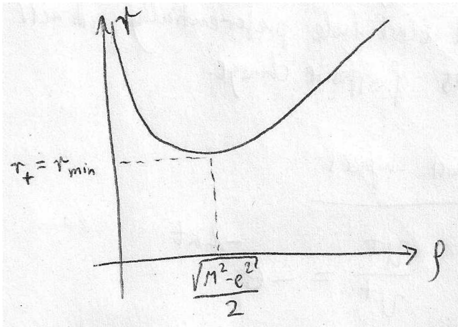
These extend naturally into region IV.
Let $r = \rho + M + \frac{M^2 - e^2}{4\rho}$; the Reissner-Nordström metric in **isotropic coordinates** $(t, \rho, \theta, \varphi)$ is

$$ds^2 = -\frac{\Delta}{r(\rho)^2} dt^2 + \frac{r(\rho)^2}{\rho^2} \left(d\rho^2 + \rho^2 d\Omega^2\right), \tag{4.18}$$

where

$$\Delta = \left(\rho - \frac{M^2 - e^2}{4\rho}\right)^2 .$$

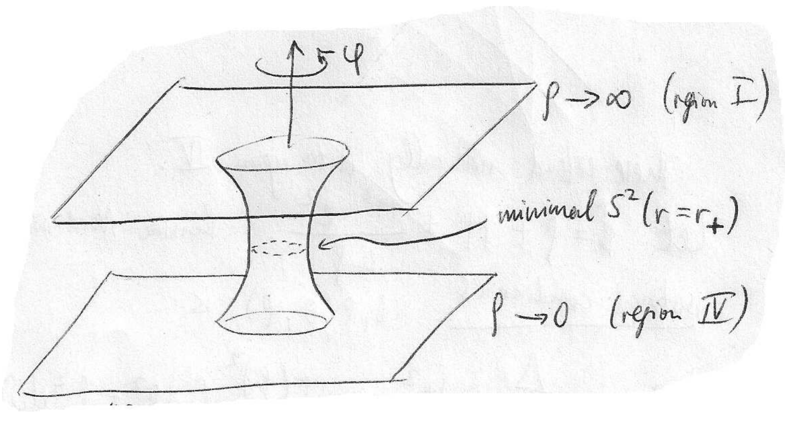
In these coordinates surfaces of constant t are manifestly conformally flat.
We can plot r as a function of ρ:



$\rho > \frac{\sqrt{M^2-e^2}}{2}$ in region I and $\rho < \frac{\sqrt{M^2-e^2}}{2}$ in region IV.
$\rho \to \frac{M^2-e^2}{4\rho}$ is an isometry that interchanges regions I and IV.
The metric of $t =$ constant surfaces is

$$ds^2 = \frac{r(\rho)^2}{\rho^2}\left(d\rho^2 + \rho^2 d\Omega^2\right), \tag{4.19}$$

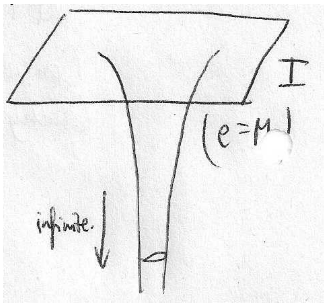
they are geodesically complete for $\rho > 0$. The topology of these surfaces is $\mathbb{R} \times S^2$.
We can embed these surfaces into flat $\mathbb{R}^4$ to visualize the geometry; suppressing the coordinate θ, the picture looks as follows



Each surface is an "Einstein-Rosen bridge" connecting the two asymptotically flat regions.
The proper distance from $r = R$ to $r = r_+$ along a curve of constant t, θ, φ is

$$\int_{r_+}^{R} \frac{dr}{\sqrt{(1-\frac{r_+}{r})(1-\frac{r_-}{r})}} \to \infty \qquad \text{as } r_+ - r_- \to 0, \ \text{ i.e. as } e \to M.$$

In this limit the Einstein-Rosen bridge becomes an infinite throat:



4.4 Extremal Reissner-Nordström Solution

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Let $e = M$ in (4.4):

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \frac{dr^2}{\left(1 - \frac{M}{r}\right)^2} + r^2 d\Omega^2. \tag{4.20}$$

We start in the region $r > M$ and introduce

$$r^* = r + 2M \log\left|\frac{r - M}{M}\right|$$

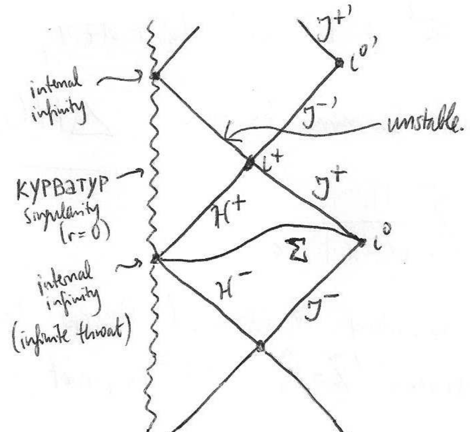
and ingoing Eddington-Finkelstein coordinates (v, r, θ, φ); the metric becomes

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dv^2 + 2dv\,dr + r^2 d\Omega^2. \tag{4.21}$$

As before, we can now analytically continue to the region $r < M$ and include all positive values for r.

Exercise Show that $r = M$ is a **degenerate** Killing horizon (i.e. $\kappa = 0$) of $\frac{\partial}{\partial v}$.

The Penrose diagram is



As before, $\mathcal{H}^+$ is a Cauchy horizon for Σ.

To understand the geometry near the horizon, let $r = M(1 + \lambda)$. To leading order to λ,

$$ds^2 \sim -\lambda^2 dt^2 + M^2 \frac{d\lambda^2}{\lambda^2} + M^2 d\Omega^2. \tag{4.22}$$

This is the **Robinson-Bertotti** solution. The metric is a product metric $adS_2 \times S^2$.

4.5 Multiple Black Hole Solutions

The extremal Reissner-Nordström solution is in isotropic coordinates (4.18)

$$ds^2 = -\frac{1}{H^2}dt^2 + H^2(d\rho^2 + \rho^2 d\Omega^2), \quad H = 1 + \frac{M}{\rho}. \tag{4.23}$$

This is a special case of the Majumdar-Papapetrou solution

$$ds^2 = -\frac{1}{H^2}dt^2 + H^2(dx^2 + dy^2 + dz^2), \tag{4.24}$$

where $H(x, y, z)$ is a general harmonic function:

$$\Delta H(x, y, z) = 0.$$

Choosing

$$H = 1 + \sum_{i=1}^{N} \frac{M_i}{|\vec{x} - \vec{x}_i|} \tag{4.25}$$

gives a solution describing N extremal Reissner-Nordström black holes at positions $\vec{x} = \vec{x}_i$ (these are actually two-spheres, not points).
The solution exists only for the extremal case because $M_i = Q_i$ for all black holes, and there is an exact cancellation of gravitational attraction and electrostatic repulsion.

5 Rotating Black Holes

5.1 Spacetime Symmetries

We need to weaken the definition of "stationary" to cover rotating black holes.

Definition 5.1 *An asymptotically flat spacetime is* **stationary** *if it admits a Killing vector field k that is timelike in a neighbourhood of $\mathcal{I}^\pm$. It is* **static** *if it is stationary and k is hypersurface-orthogonal.*

As an example, Kruskal is static (even though k is spacelike in regions II and III).

Definition 5.2 *An asymptotically flat spacetime is* **stationary and axisymmetric** *if*

(i) it is stationary,

(ii) it admits a Killing vector field m that is spacelike near $\mathcal{I}^\pm$,

(iii) m generates a one-parameter group of isometries isomorphic to $U(1)$,

(iv) $[k, m] = 0$.

For an stationary and axisymmetric spacetime, we can choose adapted coordinates so that $k = \frac{\partial}{\partial t}$, $m = \frac{\partial}{\partial \varphi}$ and φ is identified with $\varphi + 2\pi$, e.g. for Minkowski spacetime

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

5.2 Uniqueness Theorems

Birkhoff's theorem states that

"spherical symmetry + vacuum $\Rightarrow$ static."

The converse is untrue; a static vacuum spacetime need not be spherically symmetric, e.g. outside a cubic-shaped star.

But if the spacetime describes only a black hole, then

Theorem 5.3 *(Israel [2], Bunting/Masood [3], ...)*
If $(\mathcal{M}, g)$ is a static, asymptotically flat vacuum spacetime, non-singular on, and outside, an event horizon, then $(\mathcal{M}, g)$ is Schwarzschild.

<u>Note</u>

(i) This proves that static vacuum multi-black holes do not exist.

(ii) The Einstein-Maxwell generalisation is: $(\mathcal{M}, g)$ must be Reissner-Nordström <u>or</u> Majumdar-Papapetrou.

What about stationary black holes?

Theorem 5.4 *(Hawking [4], Wald 1992)*
If $(\mathcal{M}, g)$ is a stationary, non-static, asymptotically flat solution of the Einstein-Maxwell equations that is non-singular on, and outside, an event horizon, then

(i) $(\mathcal{M}, g)$ is axisymmetric,

(ii) the event horizon is a Killing horizon of $\xi = k + \Omega_H m$ for some constant $\Omega_H \neq 0$.

So for black holes,

"stationary $\Rightarrow$ axisymmetric."

Theorem 5.5 *(Carter [5], Robinson [6])*
If $(\mathcal{M}, g)$ is an asymptotically flat, stationary axisymmetric vacuum spacetime, non-singular on, and outside a connected event horizon, then $(\mathcal{M}, g)$ is a member of the two-parameter **Kerr family** *[7] of solutions. The parameters are mass M and angular momentum J.*

<u>Note</u>

(i) We expect the final state of gravitational collapse to be stationary. The initial state can be arbitrarily complicated - so there are arbitrarily many independent multipole moments of the gravitational field. But the final state has only two independent multipole moments.
All information about the initial state <u>except for</u> M, J is radiated away during collapse.

(ii) The Einstein-Maxwell generalisation is: $(\mathcal{M}, g)$ belongs to the **Kerr-Newman** family [8] of solutions with four paramters (M, J, Q, P).

5.3 The Kerr-Newman Solution

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In Boyer-Lindqvist coordinates, the Kerr-Newman solution is

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 - 2a\sin^2\theta\frac{r^2+a^2-\Delta}{\Sigma}dt\,d\varphi + \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta\,d\varphi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\,d\theta^2, \tag{5.1}$$

with

$$A = \frac{-Qr(dt - a\sin^2\theta\,d\varphi) + P\cos\theta(a\,dt - (r^2+a^2)d\varphi)}{\Sigma} \tag{5.2}$$

and

$$\Sigma = r^2 + a^2\cos^2\theta,\ \Delta = r^2 - 2Mr + a^2 + e^2,\ e = \sqrt{Q^2+P^2}. \tag{5.3}$$

We will see later that $aM = J$, where J is the total angular momentum.
Note

(i) This is asymptotically flat as $r \to \infty$.

(ii) If $a = 0$, the metric reduces to the Reissner-Nordström solution.

(iii) The transformation $\varphi \to -\varphi$ has the same effect as $a \to -a$, so without loss of generality we can assume $a \geq 0$.

(iv) The metric has a discrete isometry $t \to -t,\ \varphi \to -\varphi$.

5.4 The Kerr Solution

We set $Q = P = 0$ in the Kerr-Newman solution to get the Kerr solution, then

$$\Sigma = r^2 + a^2\cos^2\theta,\ \Delta = r^2 - 2Mr + a^2. \tag{5.4}$$

There is no known way of matching this onto a non-vacuum metric describing a stellar interior; so there is no reason to expect the metric outside a rotating star to be exactly the Kerr solution.
In Boyer-Lindqvist coordinates, the metric is singular at

(a) $\theta = 0, \pi$ (axis of symmetry: coordinate singularity),

(b) $\Delta = 0$.

We can write $\Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-)$, where

$$r_\pm = M \pm \sqrt{M^2 - a^2}. \tag{5.5}$$

As usual, there are three different cases depending on the magnitudes of M and a:

Case (i) $M^2 < a^2,\ \Delta > 0$ for all r.
The quantity $R_{abcd}R^{abcd}$ diverges at $\Sigma = 0$, i.e. at $r = 0$ and $\theta = \frac{\pi}{2}$. This is a curvature singularity.

The understand the geometry here, transform to **Kerr-Schild** coordinates $(\tilde{t}, x, y, z)$

$$x + iy = (r + ia)\sin\theta \exp\left(i\int(d\varphi + \frac{a}{\Delta}dr)\right),\ z = r\cos\theta,\ \tilde{t} = \int\left(dt - \frac{r^2+a^2}{\Delta}dr\right) - r.$$

Then $r = r(x, y, z)$ is given implicitly by

$$r^4 - (x^2 + y^2 + z^2 - a^2)r^2 - a^2 z^2 = 0. \tag{5.6}$$

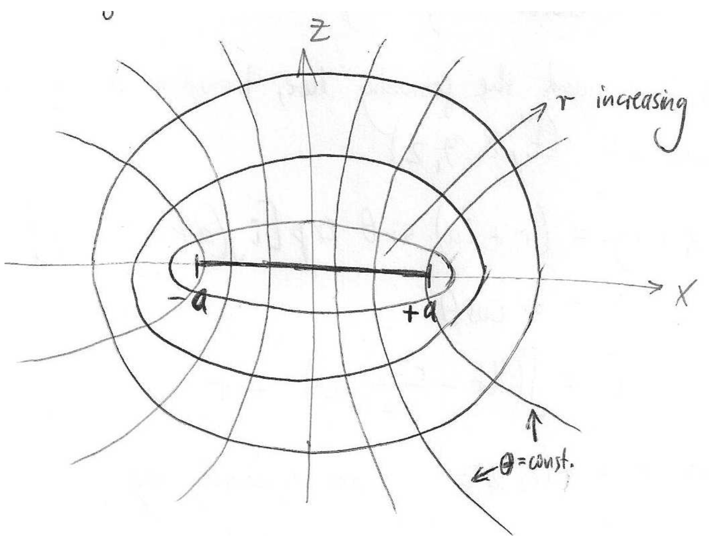
This defines r up to a sign, we can choose r to be positive everywhere.
The metric is

$$ds^2 = -d\tilde{t}^2 + dx^2 + dy^2 + dz^2 + \frac{2Mr^3}{r^4 + a^2 z^2}\left(\frac{r(x\,dx + y\,dy) - a(x\,dy - y\,dx)}{r^2 + a^2} + \frac{z\,dz}{r} + d\tilde{t}\right)^2. \tag{5.7}$$

<u>Note</u>
This metric is non-singular at $x = y = 0$ (i.e. $\theta = 0, \pi$). If $M = 0$, then the Kerr metric is flat.
Since

$$x^2 + y^2 + \left(\frac{a^2 + r^2}{r^2}\right) z^2 = r^2 + a^2,$$

surfaces of constant $r, \tilde{t}$ are confocal ellipsoids which degenerate at $r = 0$ to a disc $z = 0,\ x^2 + y^2 \le a^2$.

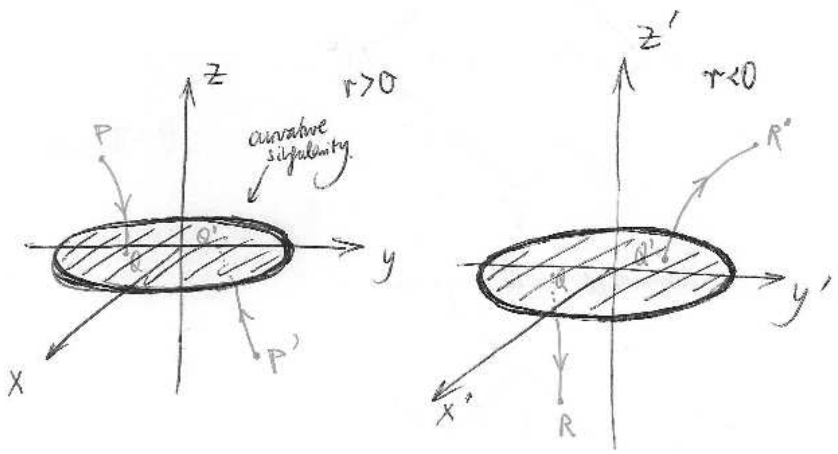


The singularity at $r = 0,\ \theta = \frac{\pi}{2}$ corresponds to $z = 0$ and $x^2 + y^2 = a^2$, i.e. a "ring" that bounds the disc. For $x^2 + y^2 < a^2$ and $z \downarrow 0$

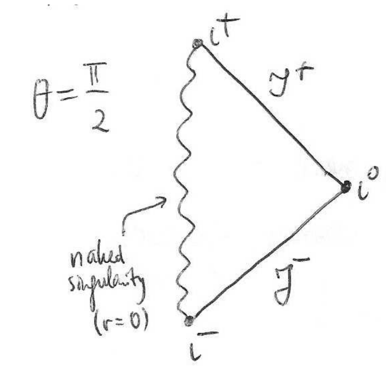
$$r \approx \frac{az}{\sqrt{a^2 - x^2 - y^2}}.$$

For the metric to be analytic, we need r to be analytic; hence r becomes negative when z becomes negative.

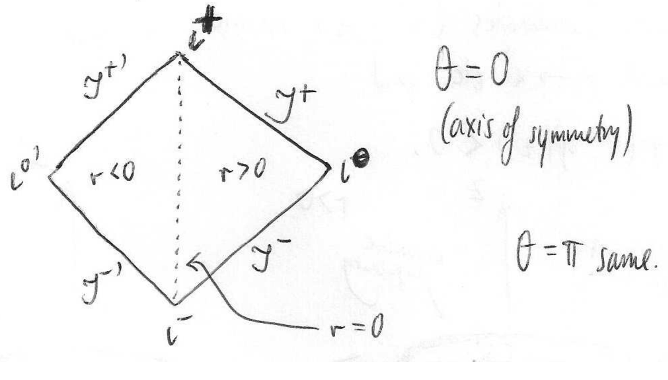
But we know that $r > 0$ for $z < 0$!
Resolution: Crossing the disc leads to a new region with coordinates (x', y', z'), with metric as before with $x \to x'$ etc. and $r(x', y', z') < 0$.



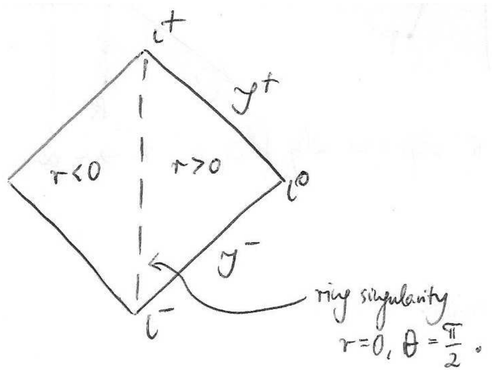
PQR, $P'Q'R'$ are smooth curves.
The region $r < 0$ is asymptotically flat as $r \to -\infty$.
To understand the causal structure, note that $\theta = 0, \frac{\pi}{2}, \pi$ are **totally geodesic** submanifolds (i.e. a geodesic initially tangent to the submanifold remains tangent). Hence we can draw Penrose diagrams for these submanifolds:



(each points represents a circle $0 \leq \varphi \leq 2\pi$),



We can summarize both diagrams as



This spacetime is unphysical because of cosmic censorship, and for another reason: Consider the axial Killing vector field m, then

$$m^2 = g_{\varphi\varphi} = a^2 \sin^2\theta \left(1 + \frac{r^2}{a^2}\right) + \frac{Ma^2}{r} \cdot \frac{2\sin^4\theta}{1 + \frac{a^2}{r^2}\cos^2\theta} \tag{5.8}$$

For $r = a\delta,\ \theta = \frac{\pi}{2} + \delta$ with $|\delta| \ll 1$,

$$m^2 = \frac{Ma}{\delta} + a^2 + O(\delta) < 0$$

for small negative δ.
Hence m is timelike near the ring singularity in the $r < 0$ region, but its orbits are closed; these orbits are **closed timelike curves** (CTCs).
One can show that there exist CTCs through any point of this spacetime. 15 Feb

Case (ii) $M^2 > a^2$ (13)

Here $r = r_\pm$ are coordinate singularities. Define **Kerr coordinates** (v, r, θ, χ) by

$$dv = dt + \frac{r^2 + a^2}{\Delta} dr,\ d\chi = d\varphi + \frac{a}{\Delta} dr. \tag{5.9}$$

Then the Killing vectors are $k = \frac{\partial}{\partial v}$ and $m = \frac{\partial}{\partial \chi}$ and χ is periodically identified with period 2π. The metric becomes

$$\begin{aligned} ds^2 = &-\frac{\Delta - a^2\sin^2\theta}{\Sigma} dv^2 + 2dv\,dr - 2a\frac{\sin^2\theta(r^2 + a^2 - \Delta)}{\Sigma} dv\,d\chi - 2a\sin^2\theta\,d\chi\,dr \\ &+ \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \sin^2\theta\,d\chi^2 + \Sigma\,d\theta^2 \end{aligned} \tag{5.10}$$

Starting in $r > r_+$, we can analytically continue through $r = r_\pm$.

Proposition 5.6 *$r = r_\pm$ is a Killing horizon of the Killing vector field*

$$\xi_\pm = k + \Omega_H m, \quad \Omega_H = \frac{a}{r_\pm^2 + a^2}$$

with surface gravity

$$\kappa_\pm = \frac{r_\pm - r_\mp}{2(r_\pm^2 + a^2)}.$$

Proof

Exercise; follow the following steps

(i) Determine $(\xi_\pm)_a$, show that

$$(\xi_\pm)_a dx^a|_{r=r_\pm} \propto dr,$$

hence $\xi_\pm$ is normal to the surface $r = r_\pm$.

(ii) Calculate $(\xi_\pm)^2$ and show that

$$(\xi_\pm)^2|_{r=r_\pm} = 0.$$

Hence $r = r_\pm$ is a null hypersurface and a Killing horizon of $\xi_\pm$.

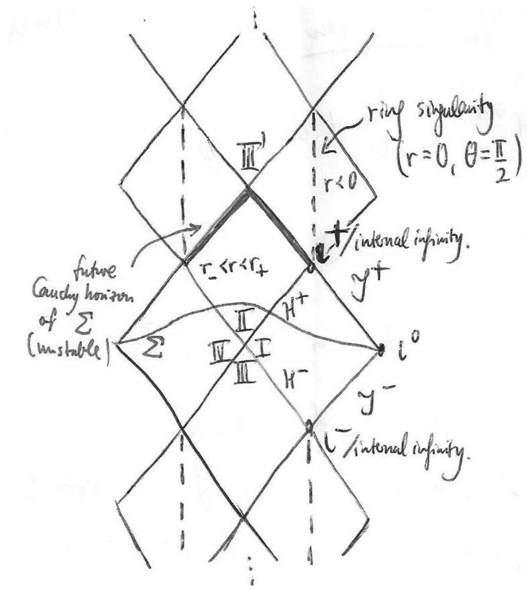
(iii) Determine κ using

$$-\frac{1}{2}\partial_a(\xi_\pm^2)|_{r=r_\pm} = \kappa_\pm(\xi_\pm)_a|_{r=r_\pm}.$$

We can use his to find Kruskal-Szekeres-like coordinates covering four regions around a bifurcation two-sphere of each horizon.

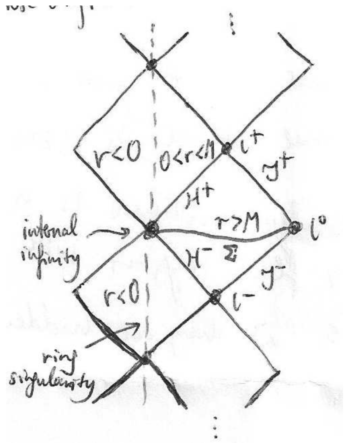
The region $r < r_-$ has a ring singularity at $r = 0, \theta = \frac{\pi}{2}$. It has the same structure as in case (i). Therefore we can analytically continue to a new asymptotically flat region with $r < 0$. (We will still have CTCs but they are hidden behind the $r = r_-$ horizon.)

Again, the submanifolds $\theta = 0, \frac{\pi}{2}, \pi$ are totally geodesic; we can determine the Penrose diagram as before.



Case (iii) $M^2 = a^2$, "extremal Kerr solution"

Here $r = M$ is a degenerate Killing horizon ($\kappa = 0$) of $\xi = k + \Omega_H m$, $\Omega_H = \frac{1}{2M}$. The Penrose diagram is



$\mathcal{H}^+$ is a future Cauchy horizon for Σ.

5.5 Angular Velocity of the Horizon

We previously defined the Killing vector field

$$\xi = k + \Omega_H m = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \varphi}, \quad \Omega_H = \frac{a}{r_+^2 + a^2}$$

in Boyer-Lindqvist coordinates. Since

$$\xi^a \partial_a(\varphi - \Omega_H t) = 0, \tag{5.11}$$

$\varphi = \Omega_H t +$ constant on orbits of ξ.
Note that $\varphi =$ constant on orbits of k, and so particles moving on orbits of ξ rotate with angular velocity Ω_H relative to a stationary observer (someone on an orbit of k), e.g. an inertial observer at infinity.

$\mathcal{H}^+$ is a Killing horizon of ξ, hence the null geodesic generators of $\mathcal{H}^+$ rotate with angular velocity Ω_G, i.e. the black hole has angular velocity Ω_H (relative to a stationary observer).

5.6 The Ergosphere

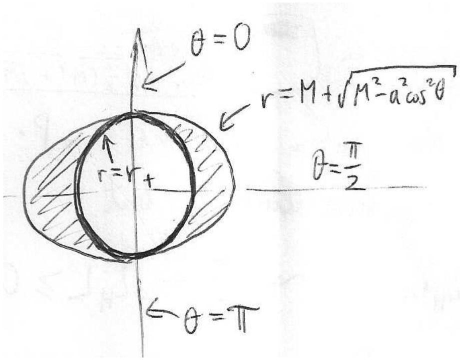
The Killing vector field k has

$$k^2 = g_{tt} = -\frac{\Delta - a^2 \sin^2\theta}{\Sigma} = -\left(1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}\right), \tag{5.12}$$

so k is timelike if and only if $r^2 + a^2\cos^2\theta - 2Mr > 0$, i.e. if and only if (in $r > 0$)

$$r > M + \sqrt{M^2 - a^2\cos^2\theta}. \tag{5.13}$$

The region $r_+ < r < M + \sqrt{M^2 - a^2\cos^2\theta}$, in which k is spacelike, is called the **ergosphere**.



Observers cannot remain stationary in the ergosphere - they must rotate in the same direction as the horizon.

The Penrose Process

Consider a particle with 4-momentum $P^a = (\text{mass}\cdot U^a)$ that approaches a Kerr black hole along a geodesic. Its energy mesaured at infinity is $E = -k \cdot P$. Suppose that the particle decays inside the ergosphere into two particles with 4-momenta P_1^a, P_2^a.
By 4-momentum conservation, we must have

$$P^a = P_1^a + P_2^a \quad \Rightarrow \; E = E_1 + E_2. \tag{5.14}$$

$E_1 = -k \cdot P_1$, but since k is spacelike in the ergosphere we can have $E_1 < 0$.
The second particle has $E_2 = E - E_1 > E$ for $E_1 < 0$. One can show that the first particle can always fall into the black hole (and the second can escape to infinity). Then the mass of the black hole decreases, energy is <u>extracted</u> from the black hole!

Limits to Energy Extraction

A particle crossing the horizon $\mathcal{H}^+$ has $-P \cdot \xi \geq 0$ (as P and ξ are both future-directed), but

$$\xi = k + \Omega_H m \quad \Rightarrow \; E - \Omega_H L \geq 0,$$

where $L = m \cdot P$ is the angular momentum of the particle. Hence $L \leq \frac{E}{\Omega_H}$; so if for particle one E is negative, so is L and the Penrose process also decreases the angular momentum of the black hole.
In the Penrose process, $\delta M = E$, $\delta J = L$ and so

$$\delta J \leq \frac{1}{\Omega_H}\delta M = 2M\frac{M^2 + \sqrt{M^4 - J^2}}{J}\delta M. \tag{5.15}$$

<u>Exercise</u> Show that this equivalent to $\delta M_{irr} \geq 0$, where the **irreducible mass** is

$$M_{irr} = \sqrt{\frac{1}{2}(M^2 + \sqrt{M^4 - J^2})}. \tag{5.16}$$

Inverting gives

$$M = M_{irr}^2 + \frac{J^2}{4M_{irr}^2} \geq M_{irr}^2. \tag{5.17}$$

Hence one can not reduce M below the inital value of M_{irr} via the Penrose process. This limits the amount of energy that can be extracted via the Penrose process.

18 Feb (14)

Exercise Show that $A = 16\pi M_{irr}^2$ is the "area of the event horizon", i.e. the area of the intersection of $\mathcal{H}^+$ with a partial Cauchy surface (such as $v =$ constant in Kerr coordinates), and hence that $\delta A \geq 0$ in the Penrose process. This is a special case of the Second Law of Black Hole Mechanics.

6 Mass, Charge and Angular Momentum

6.1 More on Differential Forms

Remark

Any two n-forms on an n-dimensional manifold are proportional. If X and X' are two such n-forms, then $X = fX'$ where f is a function on the manifold.

Definition 6.1 *An n-manifold is* **orientable** *if it admits a smooth nowhere vanishing n-form ε. Two such* **orientations** *$\varepsilon, \varepsilon'$ are* **equivalent** *if $\varepsilon' = f(x)\varepsilon$ for some everywhere positive function f. (An orientable manifold has two inequivalent orientations, given by $\pm\varepsilon$.)*

Definition 6.2 *A coordinate chart (x^a) is* **right-handed** *with respect to ε if*

$$\varepsilon = h(x)dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \tag{6.1}$$

for an everywhere positive function h.
The **volume form** *is defined by $\varepsilon_{12\ldots n} = \sqrt{|g|}$ in any coordinate chart.*

Note that

$$\varepsilon^{12\ldots n} = \pm\frac{1}{\sqrt{|g|}}$$

with a positive sign for Riemannian and a negative sign for Lorentzian signature.
Also remember the relation

$$\varepsilon^{a_1\ldots a_p c_{p+1}\ldots c_n}\varepsilon_{b_1\ldots b_p c_{p+1}\ldots c_n} = \pm p!(n-p)!\delta^{a_1}{}_{[b_1}\ldots\delta^{a_p}{}_{b_p]}. \tag{6.2}$$

Definition 6.3 *The* **Hodge dual** *of a p-form X is the $(n-p)$-form $(*X)$ defined by*

$$(*X)_{a_1\ldots a_{n-p}} = \frac{1}{p!}\varepsilon_{a_1\ldots a_{n-p}b_1\ldots b_p}X^{b_1\ldots b_p}. \tag{6.3}$$

Properties

(i) $*(*X) = \pm(-1)^{p(n-p)}X$,

(ii) $(*d*X)_{a_1\ldots a_{p-1}} = \pm(-1)^{p(n-p)}\nabla^b X_{a_1\ldots a_{p-1}b}$, hence J_a is conserved if and only if $d*J = 0$.

Example Maxwell's equations, $\nabla^a F_{ab} = -4\pi j_b$, where j_b is the current density 4-vector, and $\nabla_{[a}F_{bc]} = 0$ can be written as

$$d*F = -4\pi *j, \quad dF = 0. \tag{6.4}$$

(Then locally, there exists an A such that $F = dA$.)

6.1.1 Integration of Forms

Let $\mathcal{M}$ be an oriented manifold, $U \subset \mathcal{M}$ a region covered by a right-handed coordinate chart $\phi = (x^a)$, and X an n-form.

Definition 6.4 *The integral of X over $\mathcal{M}$ is defined by*

$$\int_{\mathcal{M}} X = \int_{\phi(\mathcal{M})} dx^1 dx^2 \dots dx^n X_{12\dots n}. \tag{6.5}$$

Exercise Check that this is coordinate-invariant.

In case there exists no global chart, we divide $\mathcal{M}$ into topologically trivial regions U_i such that $\mathcal{M} = \bigcup_i U_i$ and $U_i \cap U_j = \emptyset$ for $i \neq j$. Then we can define

$$\int_{\mathcal{M}} X = \sum_i \int_{U_i} X. \tag{6.6}$$

The volume of the manifold $\mathcal{M}$ is

$$V = \int_{\mathcal{M}} \varepsilon.$$

If $f : \mathcal{M} \to \mathbb{R}$ is a function we also define

$$\int_{\mathcal{M}} f = \int_{M} f\varepsilon.$$

Now let Σ be an m-dimensional oriented submanifold of $\mathcal{M}$ and X an n-form. If (ξ^i) are coordinates on Σ $(1 \leq i \leq m)$, we can specify Σ by $x^a = x^a(\xi^i)$, i.e. a map $\varphi : \Sigma \to \mathcal{M}$. Then the pull-back $\varphi^* X$ is an m-form on Σ. We then define the integral of X over Σ to be

$$\int_{\Sigma} X = \int_{\Sigma} \varphi^* X. \tag{6.7}$$

Note If Y is an $(n-1)$-form, then

$$\int_{\Sigma} dY = \int_{\Sigma} \varphi^*(dY) = \int_{\Sigma} d(\varphi^* Y).$$

6.1.2 Stokes' Theorem

states that

$$\int_{\Sigma} dX = \int_{\partial\Sigma} X, \tag{6.8}$$

where Σ is an n-dimensional oriented manifold with boundary $\partial\Sigma$ and X is an $(n-1)$-form.
A "manifold with boundary" is defined in the same way as a manifold except that the coordinate charts are maps $\Sigma \to \frac{1}{2}\mathbb{R}^n = \{(x^1, \dots, x^n) : x^1 \leq 0\}$. The boundary $\partial\Sigma$ is the set of points in Σ mapped to $\{x^1 = 0\}$. The orientation of $\partial\Sigma$ is fixed by saying that $(x^2, \dots, x^n)$ is a right-handed coordinate chart on $\partial\Sigma$ if $(x^1, \dots, x^n)$ is right-handed on Σ.

6.2 Charges in Curved Spacetime

Let Σ be a spacelike hypersurface with boundary $\partial\Sigma$. The total electric charge on Σ is

$$Q = -\int_\Sigma *j,$$

where the orientation of Σ is enduced from $\mathcal{J}^-(\Sigma)$; e.g. consider Minkowski space with volume form $dt \wedge dx \wedge dy \wedge dz$ and a hypersurface $t =$ constant which has charge $Q = \int dx\, dy\, dz\, (*j)_{123}$. Using Maxwell's equations,

$$Q = \frac{1}{4\pi}\int_\Sigma d * F = \frac{1}{4\pi}\int_{\partial\Sigma} *F. \tag{6.9}$$

In an asymptotically flat spacetime, we choose Σ to have a single boundary of topology S^2 at spatial infinity (e.g. at $\imath^0$). Then

$$Q = \frac{1}{4\pi}\int_{S^2_\infty} *F.$$

In Minkowski space, we can choose the volume form $r^2 \sin\theta\, dt \wedge dr \wedge d\theta \wedge d\varphi$ to define the orientation. Let Σ be the hypersurface $t = 0$, it is the boundary of the region $\{t \leq 0\}$. Then $dr \wedge d\theta \wedge d\varphi$ defines the orientation of Σ. Similarly, let $S^2_R = \{t = 0, r = R\}$ be the boundary of the region $\{r \leq R\}$ in Σ; then an orientation on S^2_R is defined by $d\theta \wedge d\varphi$.
Consider an electric potential $A = -\frac{Q}{r}dt$, then $F = -\frac{Q}{r^2}dt \wedge dr$,

$$(*F)_{\theta\varphi} = \varepsilon_{\theta\varphi tr}F^{tr} = r^2 \sin\theta \frac{Q}{r^2} = Q\sin\theta \quad \Rightarrow \frac{1}{4\pi}\int_{S^2_R} *F = \frac{1}{4\pi}\int d\theta\, d\varphi\, Q\sin\theta = Q.$$

Our definition of electric charge agrees with the electric charge in the electrostatic potential.

Definition 6.5 *In an asymptotically flat spacetime, the total electric and magnetic charges are*

$$Q = \frac{1}{4\pi}\int_{S^2_\infty} *F, \quad P = \frac{1}{4\pi}\int_{S^2_\infty} F, \tag{6.10}$$

where S^2_∞ is a two-sphere that approaches $\imath^0$.

<u>Exercise</u> Show that these agree with Q and P in the Reissner-Nordström solution.

<u>Note</u>
These charges are conserved, hence we can consider different spheres S^2_∞, $S^{2\prime}_\infty$ at different points of infinity, i.e. S^2_∞ at spatial infinity and $S^{2\prime}_\infty$ at future timelike infinity. Then

$$Q' - Q = \frac{1}{4\pi}\int_{S^{2\prime}_\infty} *F - \frac{1}{4\pi}\int_{S^2_\infty} *F = \frac{1}{4\pi}\int_C d * F,$$

where C is a cylinder which connects S^2_∞ and $S^{2\prime}_\infty$. If no charge flows through $\mathcal{I}^+$,

$$Q' - Q = -\int_C *j = 0.$$

Similarly $P' = P$.

6.2.1 Komar Integrals

20 Feb (15)

If $(\mathcal{M}, g)$ is stationary then there exists a conserved energy-momentum current $J_a = -T_{ab}k^b$. So we can define the total energy of matter on a spacelike hypersurface Σ by

$$E(\Sigma) = -\int_\Sigma *J. \tag{6.11}$$

This is conserved: If Σ and Σ' bound the spacetime region R, then

$$E(\Sigma') - E(\Sigma) = -\int_{\partial R} *J = -\int_R d*J = 0. \tag{6.12}$$

If $*J = dX$ for some two-form X, then we could convert $E(\Sigma)$ into an integral over $\partial\Sigma$ (as for Q and P) and thereby define energy for any asymptotically flat spacetime. We cannot do this for J; but consider

$$(*d*dk)_a = -\nabla^b(dk)_{ab} = -\nabla^b\nabla_a k_b + \nabla^b\nabla_b k_a = 2\nabla^b\nabla_b k_a,$$

since k is a Killing vector field.

<u>Exercise</u> Show that a Killing vector field k obeys

$$\nabla_a\nabla_b k^c = R^c{}_{bad}k^d. \tag{6.13}$$

We use this to rewrite

$$(*d*dk)_a = -2R_{ab}k^b = 8\pi J'_a$$

where

$$J'_a \equiv -2\left(T_{ab} - \frac{1}{2}Tg_{ab}\right)k^b$$

by the Einstein equations, so

$$d*dk = 8\pi * J'$$

and $*J'$ is exact and hence

$$-\int_\Sigma *J' = -\frac{1}{8\pi}\int_\Sigma d*dk = -\frac{1}{8\pi}\int_{\partial\Sigma} *dk.$$

Definition 6.6 *In a stationary, asymptotically flat spacetime, the* **Komar mass** *(or energy) is*

$$M_{Komar} = -\frac{1}{8\pi}\int_{S^2_\infty} *dk, \tag{6.14}$$

where S_2^∞ approaches spacelike infinity $\imath^0$.

<u>Note</u>

(i) M_{Komar} is conserved, as we showed previously for Q and P.

(ii) M_{Komar} is the total energy of spacetime, not just the energy from matter. For instance, for the Schwarzschild spacetime, $M_{Komar} = M$ (<u>Exercise</u>).

(iii) One can use a similar defintion for any Killing vector field, e.g.

Definition 6.7 *In an axisymmetric, asymptotically flat spacetime, the* **Komar angular momentum** *is*

$$J_{Komar} = \frac{1}{16\pi}\int_{S^2_\infty} *dm. \tag{6.15}$$

6.2.2 ADM Energy

How do we define total energy in a non-stationary spacetime?
Consider a weak field, so that we can write the metric as

$$g_{ab} = \eta_{ab} + h_{ab}$$

with Cartesian coordinates $x^a = (x^0, x^i)$ and $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$. Then we define

$$\bar{h}_{ab} = h_{ab} - \frac{1}{2} h \eta_{ab}, \quad h = \eta^{ab} h_{ab}$$

and fix the gauge freedom corresponding to infinitesimal coordinate transformations by imposing deDonder gauge

$$\partial^a \bar{h}_{ab} = 0. \tag{6.16}$$

We then obtain the linearised Einstein equations

$$\Box \bar{h}_{ab} \equiv (-\partial_0^2 + \partial_i \partial_i) \bar{h}_{ab} = -16\pi T_{ab}. \tag{6.17}$$

Here the spatial index i is being summed over when it appears twice. Setting $a = 0, b = 0$ we get

$$16\pi T_{00} = \partial_0^2 \bar{h}_{00} - \partial_i \partial_i \bar{h}_{00} = \partial_0(\partial_i \bar{h}_{i0}) - \partial_i \partial_i \bar{h}_{00} = \partial_i \partial_j \bar{h}_{ij} - \partial_i \partial_i \bar{h}_{00}$$

$$= \partial_i \partial_j \left(h_{ij} - \frac{1}{2} h \eta_{ij} \right) - \partial_i \partial_i \left(h_{00} + \frac{1}{2} h \right) = \partial_i \partial_j h_{ij} - \partial_i \partial_i h_{00} - \partial_i \partial_i h = \partial_i \partial_j h_{ij} - \partial_i \partial_i h_{jj} = \partial_i (\partial_j h_{ij} - \partial_i h_{jj}),$$

where we repeatedly used the gauge condition.
Now let Σ be a surface of constant x^0. Then the total energy of the matter on Σ is

$$E = \int_\Sigma d^3x \; T_{00} = \frac{1}{16\pi} \int_\Sigma d^3x \; \partial_i (\partial_j h_{ij} - \partial_i h_{jj}) = \frac{1}{16\pi} \lim_{r \to \infty} \int_{S_r^2} dA \; n^i (\partial_j h_{ij} - \partial_i h_{jj}), \tag{6.18}$$

where S_r^2 is a sphere with radius r, i.e. all points with $x^i x^i = r^2$, at constant x^0, and n^i is the unit outward normal to S_r^2 in Σ.
We can now extend this to any (not necessarily weak field) asymptotically flat spacetime.

Definition 6.8 *Let Σ be an asymptotically flat spacelike hypersurface in an asymptotically flat spacetime. Let x^i be asymptotically Cartesian coordinates on Σ, i.e. the metric on Σ is $\delta_{ij} + h_{ij}$, where $h_{ij} = O(\frac{1}{r})$ (wirh $r = \sqrt{x^i x^i}$). Then the* **ADM energy** *is defined by (6.18).*

Note

(i) The integrand of (6.18) is not gauge-invariant. Under an infinitesimal coordinate transformation,

$$h_{ij} \to h_{ij} + \partial_{(i} \xi_{j)}.$$

But the integral is gauge-invariant.

(ii) One can also define an **ADM momentum** P^i_{ADM} tangent to Σ at i^0. If Σ is chosen so that $P^i_{ADM} = 0$, then E_{ADM} is called the **ADM mass**.

(iii) In a stationary spacetime, choosing Σ to be asymptotically orthogonal to k means that

$$P^i_{ADM} = 0, \quad E_{ADM} = E_{Komar}.$$

6.3 Energy Conditions

These are statements how "physical" energy-momentum tensors should behave. The **dominant energy condition** states that

$$-T^a{}_b V^b$$

is a future-directed timelike/null vector (or zero) for all future-directed timelike/null vectors V^a. ("All observes measure a speed of energy flow less than or equal to the speed of light.")
All physically reasonable matter obeys this, e.g. a massless scalar field with

$$T_{ab} = \partial_a \Phi \partial_b \Phi - \frac{1}{2} g_{ab} (\partial \Phi)^2. \tag{6.19}$$

Let

$$j^a = -T^a{}_b V^b = -(V \cdot \partial \Phi) \partial^a \Phi + \frac{1}{2} (\partial \Phi)^2 V^a,$$

then

$$j^2 = \frac{1}{4} V^2 \left((\partial \Phi)^2 \right)^2 \leq 0$$

if $V^2 \leq 0$, and hence j is timelike or null. Furthermore,

$$V \cdot j = -(V \cdot \partial \Phi)^2 + \frac{1}{2} V^2 (\partial \Phi)^2 = -\frac{1}{2} (V \cdot \partial \Phi)^2 + \frac{1}{2} V^2 \left(\partial \Phi - \frac{V \cdot \partial \Phi}{V^2} V \right)^2$$

The first term is clearly negative definite, and from

$$V \cdot \left(\partial \Phi - \frac{V \cdot \partial \Phi}{V^2} V \right) = 0$$

it follows that the bracket is spacelike, null or zero, and hence its square is greater than or equal to zero. Hence $V \cdot j \leq 0$ and j is future-directed.

The **weak energy condition** states that

$$T_{ab} V^a V^b \geq 0 \tag{6.20}$$

for any non-spacelike vector V. ("All observers measure non-negative energy density.")
The **null energy condition** states that

$$T_{ab} V^a V^b \geq 0 \tag{6.21}$$

for any null vector V. The dominant energy condition implies the weak energy condition, the weak energy condition implies the null energy condition.
The **strong energy condition** states that

$$\left(T_{ab} - \frac{1}{2} T g_{ab} \right) V^a V^b \geq 0 \quad \Leftrightarrow \quad R_{ab} V^a V^b \geq 0 \tag{6.22}$$

for any non-spacelike vector V^a. ("Gravity is attractive.")
<u>Note</u>
The strong energy condition does <u>not</u> imply the dominant or weak energy conditions.

It occurs in singularity theorems of general relativity. The dominant energy condition is most important physically.
A positive cosmological constant obeys the dominant, but violates the strong energy condition. 22 Feb

In Newtonian theory, gravitational energy is negative. Can the ADM mass be negative ? The answer is no: (16)

Theorem 6.9 *(Schoen, Yau, Witten [9])*
The ADM mass of an asymptotically flat spacetime satisfying Einstein's equations is non-negative, and vanishes only in Minkowski spacetime, provided that

(i) there exists a non-singular Cauchy surface (this excludes Schwarzschild spacetime with $M < 0$),

(ii) matter obeys the dominant energy condition.

7 Black Hole Mechanics

7.1 Geodesic Congruences

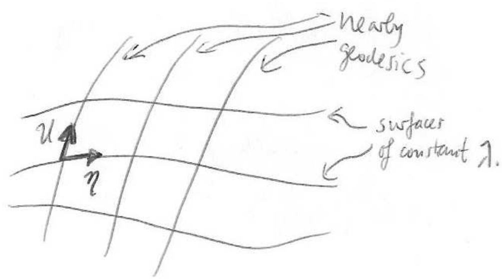
Definition 7.1 *Let U be an open subset of $\mathcal{M}$. A* **congruence** *in U is a family of curves such that through each point $p \in U$ there passes exactly one curve. It is a* **geodesic congruence** *if these curves are geodesics.*

We can specify a congruence by $x^a = x^a(\lambda, y^i)$, where λ is an affine parameter, and y^i labels the geodesics. The tangent

$$U^a = \left(\frac{\partial x^a}{\partial \lambda}\right)_y$$

obeys the geodesic equation $U \cdot \nabla U^a = 0$.

Definition 7.2 *A* **displacement vector field** *for a geodesic congruence with tangent U is a vector field η, nowhere parallel to U, that obeys $\mathcal{L}_U \eta = [U, \eta] = 0$.*



Hence by definition,

$$U \cdot \nabla \eta - \eta \cdot \nabla U = 0 \tag{7.1}$$

or with $B^a{}_b = \nabla_b U^a$,

$$U \cdot \nabla \eta^a = B^a{}_b \eta^b.$$

$B^a{}_b$ measures the "geodesic deviation" - the failure of η to be parallelly propagated.
Let Σ be a spacelike hypersurface that intersects each geodesic in the congruence exactly once. Let λ be the affine parameter distance from Σ along each geodesic. Now

$$\frac{d}{d\lambda}(U \cdot \eta) = U \cdot \nabla(U \cdot \eta) = U_a \left(U \cdot \nabla \eta^a\right) = U_a B^a{}_b \eta^b = 0 \tag{7.2}$$

because

$$U_a B^a{}_b = \frac{1}{2}\partial_b(U^2) = 0$$

as we can fix $U^2 = \pm 1$ or 0. Then $U \cdot \eta =$ constant along geodesics, and the component of η in the direction of U exhibits boring behaviour, so we restrict attention to displacement vectors satisfying $U \cdot \eta = 0$ on Σ, which is then satisfied for all λ.
If U is null there is still the freedom to shift η by a term proportional to U, i.e. $\eta' = \eta + f(x)U$ is also a displacement vector orthogonal to U provided $U \cdot \nabla f = 0$. We fix this freedom by introducing a vector N obeying $N^2 = 0$ and $U \cdot N = -1$ on Σ and extend N off Σ by parallel propagation:

$$U \cdot \nabla N = 0.$$

Then $N^2 = 0$ and $U \cdot N = -1$ are satisfied everywhere. We cannot insist that η be orthogonal to N everywhere since $U \cdot \nabla(\eta \cdot N) \neq 0$.
We define a two-dimensional subspace $T_\perp$ of the tangent space by $P\eta = \eta$ where

$$P^a{}_b = \delta^a{}_b + N^a U_b + U^a N_b.$$

P is a projection which projects to the subspace orthogonal to N and U.

Proposition 7.3 *For $\eta \in T_\perp$,*

$$U \cdot \nabla \eta^a = \hat{B}^a{}_b \eta^b, \tag{7.3}$$

where $\hat{B}^a{}_b = P^a{}_c B^c{}_d P^d{}_b$.

Proof

$$U \cdot \nabla \eta^a = U \cdot \nabla(P^a{}_b \eta^b) = P^a{}_b U \cdot \nabla \eta^b = P^a{}_b B^b{}_c \eta^c = P^a{}_b B^b{}_c P^c{}_d \eta^d, \tag{7.4}$$

since U and N are parallelly propagated and $\eta \in T_\perp$. In the last equality we use

$$P^c{}_d \eta^d = \eta^c + U^c N_d \eta^d,$$

but also $B^b{}_c U^c = 0$ as $U \cdot \nabla U = 0$, q.e.d.

Now we decompose the rank two matrix $\hat{B}$ into irreducible parts:

$$\hat{B}^a{}_b = \frac{1}{2}\theta\, P^a{}_b + \hat{\sigma}^a{}_b + \hat{\omega}^a{}_b, \tag{7.5}$$

where

$$\theta = \hat{B}^a{}_a, \quad \hat{\sigma}_{ab} = \hat{B}_{(ab)} - \frac{1}{2}P_{ab}\theta, \quad \hat{\omega}_{ab} = \hat{B}_{[ab]}$$

are called **expansion, shear** and **twist** respectively.

Exercise Show that $\theta = g^{ab}B_{ab} = \nabla \cdot U$, independent of the choice of N.

Lemma 7.4

$$U_{[a}\hat{B}_{bc]} = U_{[a}B_{bc]}. \tag{7.6}$$

Proof

$$\hat{B}^a{}_b = B^a{}_b + U^a N_c B^c{}_b + U_b B^a{}_c N^c + U^a U_b N_c B^c{}_d N^d, \tag{7.7}$$

using $U \cdot B = B \cdot U = 0$, and the result follows.

Proposition 7.5 *U is hypersurface-orthogonal if and only if $\hat{\omega} = 0$.*

Proof

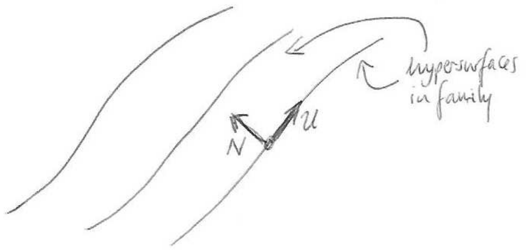
$$U_{[a}\hat{\omega}_{bc]} = U_{[a}\hat{B}_{bc]} = U_{[a}B_{bc]} = -\frac{1}{6}(U \wedge dU)_{abc};$$

so if $\hat{\omega} = 0$, then U is hypersurface-orthogonal from Frobenius' theorem.
If $U \wedge dU = 0$, then

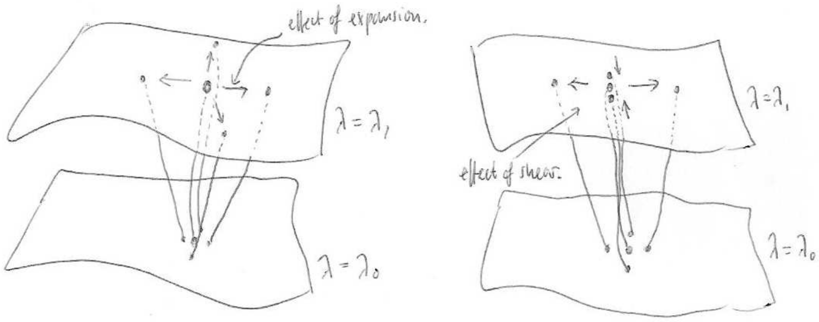
$$0 = U_{[a}\hat{\omega}_{bc]} = \frac{1}{3}\left(U_a\hat{\omega}_{bc} + U_b\hat{\omega}_{ca} + U_c\hat{\omega}_{ab}\right)$$

and contracting this with N^a gives $\hat{\omega}_{bc} = 0$ since $N \cdot \hat{\omega} = 0$, q.e.d.

If $\hat{\omega} = 0$, the family of these hypersurfaces is parametrised by the displacement along N:



It is easiest to visualise θ and $\hat{\sigma}$ in this case: Consider a hypersurface N in the family at two values for λ.



Now consider two infinitesimal displacement vectors η_1 and η_2. Then they span a two-dimensional surface element, whose area A is given by

$$A^2 = \frac{1}{2}(\eta_1 \wedge \eta_2)^2$$

(Since η_i are infintesimal, one can work in normal coordinates, and flat-space arguments can be used.) But then we have

$$A^2 = \frac{1}{2}(\hat{\eta}_1 \wedge \hat{\eta}_2)^2$$

with $\hat{\eta}_i = P\eta_i$, since η_i and $\hat{\eta}_i$ differ by a multiple of U and $U \cdot \eta_i = U^2 = 0$. Then

$$\frac{d}{d\lambda}(A^2) = U \cdot \nabla(A^2) = (\hat{\eta}_1 \wedge \hat{\eta}_2)_{ab} U \cdot \nabla(\hat{\eta}_1 \wedge \hat{\eta}_2)^{ab} = (\hat{\eta}_1 \wedge \hat{\eta}_2)_{ab} \left[(\hat{B}\hat{\eta}_1) \wedge \hat{\eta}_2 + \hat{\eta}_1 \wedge \hat{B}\hat{\eta}_2\right]^{ab}. \quad (7.8)$$

<u>Exercise</u> Let ξ^i be the dual one-form to $\hat{\eta}_i$, i.e. $\xi^i(\hat{\eta}_j) = \delta^i{}_j$, $\xi^i(U) = \xi^i(N) = 0$. Show that

(i) $(\hat{B}\hat{\eta}_1) \wedge \hat{\eta}_2 = \xi^1(\hat{B}\hat{\eta}_1)\hat{\eta}_1 \wedge \hat{\eta}_2$, $\hat{\eta}_1 \wedge \hat{B}\hat{\eta}_2 = \xi^2(\hat{B}\hat{\eta}_2)\hat{\eta}_1 \wedge \hat{\eta}_2$;

(ii) $\hat{\eta}_i^a \xi_b^i = P^a{}_b$.

Hence

$$\frac{d}{d\lambda}(A^2) = \theta(\hat{\eta}_1 \wedge \hat{\eta}_2)^2 = 2\theta A^2, \quad (7.9)$$

and

$$\frac{dA}{d\lambda} = \theta\, A. \quad (7.10)$$

So θ measures the rate of increase of the area element. For $\theta > 0$, geodesics diverge; for $\theta < 0$, geodesics converge.

7.1.1 Raychaudhuri's Equation

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governs the expansion along null geodesics:

$$\frac{d\theta}{d\lambda} = U \cdot \nabla(B^a{}_b P^b{}_a) = P^b{}_a U \cdot \nabla B^a{}_b = P^b{}_a U^c \nabla_c \nabla_b U^a = P^b{}_a U^c (\nabla_b \nabla_c U^a + R^a{}_{dcb} U^b)$$

$$= P^b{}_a(\nabla_b(U^c \nabla_c U^a) - (\nabla_b U^c)\nabla_c U^a) + P^b{}_a R^a{}_{dcb} U^c U^d,$$

the first term vanishes because U is tangent to a geodesic, and it is an exercise that

$$\frac{d\theta}{d\lambda} = -P^b{}_a B^c{}_b B^a{}_c - R_{cd} U^c U^d = -\hat{B}^c{}_a \hat{B}^a{}_c - R_{ab} U^a U^b$$

and hence

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}^{ab}\hat{\sigma}_{ab} + \hat{\omega}^{ab}\hat{\omega}_{ab} - R_{ab} U^a U^b. \quad (7.11)$$

This is Raychaudhuri's equation.

Proposition 7.6 *The expansion of the null geodesic generators of a null hypersurface obeys*

$$\frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2 \quad (7.12)$$

if the Einstein equations and the null energy condition are satisfied.

Proof
The generators are hypersurface-orthogonal, hence $\hat{\omega} = 0$. As the metric in $T_\perp$ is positive definite, we have $\hat{\sigma}^2 \geq 0$. Since U is null,

$$R_{ab}U^aU^b = 8\pi T_{ab}U^aU^b \geq 0$$

by the null energy condition. Then the result follows from Raychaudhuri's equation, q.e.d.

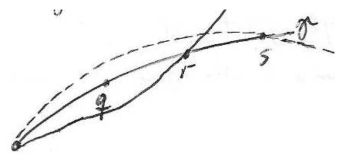
Corollary 7.7 *If $\theta = \theta_0 < 0$ at a point p on a null generator γ of a null hypersurface, then $\theta \to -\infty$ along γ within affine parameter $\frac{2}{|\theta_0|}$.*

Proof
Let $\lambda = 0$ at p. Then

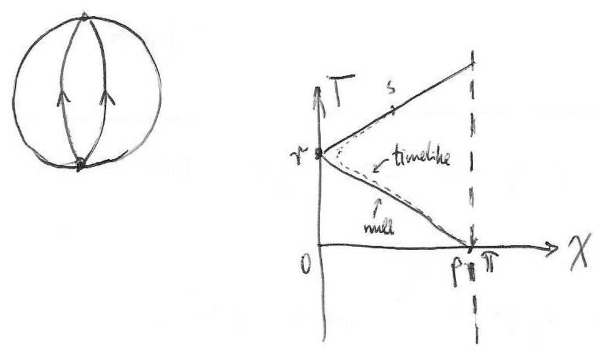
$$\frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2 \quad \Rightarrow \quad \frac{d(\theta^{-1})}{d\lambda} \geq \frac{1}{2} \quad \Rightarrow \quad \frac{1}{\theta} - \frac{1}{\theta_0} \geq \frac{1}{2}\lambda \quad \Rightarrow \theta \leq \frac{\theta_0}{1 + \frac{1}{2}\lambda\theta_0}.$$

If $\theta_0 < 0$, the right-hand side goes to $-\infty$ at $\lambda = \frac{2}{|\theta_0|}$, q.e.d.

The significance of this statement is the following:
Consider the congruence of all null geodesics through p; this congruence is singular at p. If $\theta = \theta_0 < 0$ at a point q on a geodesic γ, then $\theta \to -\infty$ at some later point r. r is called a **conjugate point** to p.
Roughly speaking, infinitesimally nearby null geodesics from p intersect at r. One can show that γ can be infinitesimally deformed to a timelike curve between p and any point s beyond r along γ.



As an example, consider $\mathbb{R} \times S^3$ with $p = \{\chi = \pi, T = 0\}$, the south pole at $T = 0$. Geodesics from the south pole refocus at the north pole. These are conjugate points.



Proposition 7.8 *If $\mathcal{N}$ is a Killing horizon, then $\theta = \hat{\sigma} = \hat{\omega} = 0$.*

Proof

Let ξ be the Killing vector field, then write $\xi = h\cdot U$ so that U is tangent to null geodesic generators. Then U is hypersurface-orthogonal, so $\hat{\omega} = 0$. With $U = h^{-1}\xi$ we have

$$B^a{}_b = \nabla_b U^a = \partial_b(h^{-1})\xi^a + h^{-1}\nabla_b \xi^a,$$

so that, since ξ is Killing,

$$B_{(ab)} = \partial_{(a}(h^{-1})\xi_{b)}, \quad \hat{B}_{(ab)} = P_a{}^c B_{(cd)} P^d{}_a = 0 \quad (\xi \propto U), \quad \theta = \hat{\sigma} = 0.$$

Corollary 7.9 *For a Killing horizon $\mathcal{N}$ of ξ, $R_{ab}\xi^a\xi^b = 0$ on $\mathcal{N}$.*

Proof

$\theta = 0$ and so $\frac{d\theta}{d\lambda} = 0$ on $\mathcal{N}$; then use Raychaudhuri's equation, q.e.d.

7.2 Zeroth Law of Black Hole Mechanics

Theorem 7.10 *(Zeroth Law of Black Hole Mechanics)*
κ is constant on each connected component of the future event horizon of a stationary black hole spacetime satisfying the dominant energy condition.

Proof

By Hawking's theorem (3.12), which applies to "physical" matter satisfying the dominant energy condition, $\mathcal{H}^+$ is a Killing horizon.
Let ξ be a Killing vector field normal to $\mathcal{H}^+$, then

$$0 = R_{ab}\xi^a\xi^b|_{\mathcal{H}^+} = 8\pi T_{ab}\xi^a\xi^b|_{\mathcal{H}^+} = -8\pi J\cdot\xi|_{\mathcal{H}^+}, \tag{7.13}$$

where $J_a = -T_{ab}\xi^b$ and we used that $\xi^2 = 0$. But J and ξ are both future-directed timelike or null (by dominant energy), so if $J\cdot\xi|_{\mathcal{H}^+} = 0$, then $J \propto \xi$ on $\mathcal{H}^+$. Then $\xi \wedge J = 0$, so

$$0 = \xi_{[a}J_{b]}|_{\mathcal{H}^+} = -\xi_{[a}T_{b]}{}^c\xi_c|_{\mathcal{H}^+} = -\frac{1}{8\pi}\xi_{[a}R_{b]}{}^c\xi_c|_{\mathcal{H}^+} = \frac{1}{8\pi}\xi_{[a}\partial_{b]}\kappa|_{\mathcal{H}^+}, \tag{7.14}$$

where the last equality is shown on example sheet 3. It follows that $\partial_a\kappa$ is proportional to ξ_a on $\mathcal{H}^+$, so $t\cdot\partial\kappa = 0$ for any t tangent to $\mathcal{H}^+$, and $\kappa = $ constant on $\mathcal{H}^+$, q.e.d.

7.3 Second Law of Black Hole Mechanics

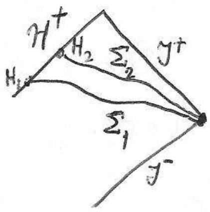
Definition 7.11 *An asymptotically flat spacetime $(\mathcal{M}, g)$ is* **strongly asymptotically predictable** *if there exists some globally hyperbolic open set $U \subset \mathcal{M}$ such that*

$$\mathcal{J}^-(\mathcal{I}^+) \cup \mathcal{H}^+ \subset U,$$

i.e. if physics is predictable from a Cauchy surface for U oustide the black hole and on $\mathcal{H}^+$.

Theorem 7.12 *(Second Law of Black Hole Mechanics, Hawking Area Theorem)*
Let $(\mathcal{M}, g)$ be a strongly asymptotically predictable spacetime which obeys the null energy condition. Let Σ_1, Σ_2 be spacelike Cauchy surfaces for U, where $\Sigma_2 \subset \mathcal{J}^+(\Sigma_1)$. Then for $H_i = \Sigma_i \cap \mathcal{H}^+$,

$$\textit{area}(H_2) \geq \textit{area}(H_1). \tag{7.15}$$

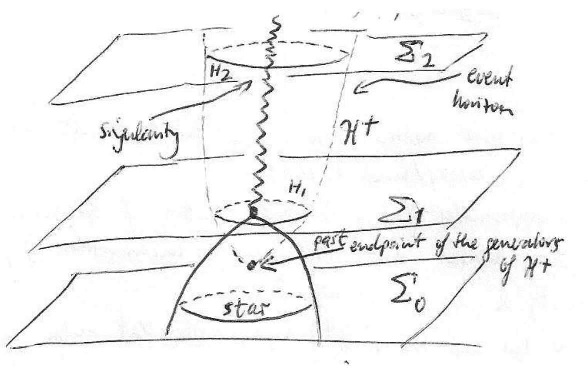


Proof
First show that $\theta \geq 0$ on $\mathcal{H}^+$; so assume that $\theta < 0$ at a point $p \in \mathcal{H}^+$.
Let γ be a generator of $\mathcal{H}^+$ through p, let q be slightly to the future of p along γ. Then by continuity, $\theta < 0$ at q. Hence there exists a point r conjugate to p on γ (assuming that γ is complete - one can relax this condition).
We can deform the curve to a timelike curve from p to s, where s is slightly beyond r along γ. But $\mathcal{H}^+$ is achronal, so no two points can be timelike separated, which is a contradiction. Hence $\theta \geq 0$ on $\mathcal{H}^+$.
Now let $p \in H_1$. A null geodesic generator γ of $\mathcal{H}^+$ through p can not leave $\mathcal{H}^+$, so must intersect H_2 (as Σ_2 is a Cauchy surface). This defines a map $\varphi : H_1 \to H_2$. Since $\theta \geq 0$ on $\mathcal{H}^+$,

$$\text{area}(H_2) \geq \text{area}(\varphi(H_1)) \geq \text{area}(H_1),$$

q.e.d.

As an example consider the formation of a black hole in spherically symmetric collapse. The Finkelstein diagram is

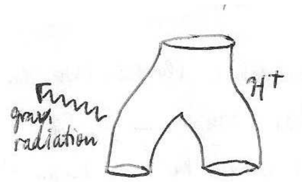


Here,

$$\text{area}(H_2) = 16\pi M^2 \geq \text{area}(H_1).$$

7.3.1 Consequences of the Second Law

(i) There are limits to the efficiency of mass/energy conversion in black hole collisions, e.g. for two coalescing black holes 25 Feb (18)



Assume that the inital black holes are well separated, so we can approximate them as Schwarzschild black holes. The final black hole is a Schwarzschild solution.
The inital mass is $M_1 + M_2$, and by the area theorem

$$A_3 \geq A_1 + A_2 = 16\pi(M_1^2 + M_2^2),$$

but $A_3 = 16\pi M_3^2$ and so $M_3^2 \geq M_1^2 + M_2^2$. The energy radiated is then $E = M_1 + M_2 - M_3$, and the efficiency

$$\eta = \frac{E}{M_1 + M_2} = \frac{M_1 + M_2 - M_3}{M_1 + M_2} \leq 1 - \frac{\sqrt{M_1^2 + M_2^2}}{M_1 + M_2} \leq 1 - \frac{1}{\sqrt{2}}.$$

The radiated energy could be used to do work; then the area theorem limits useful energy that can be extracted from a black hole (compare this to the second law of thermodynamics!).

(ii) Black holes can not bifurcate; assume Schwarzschild black holes and consider a process $M_1 \to M_2 + M_3$. Then from the area theorem,

$$M_1 \leq \sqrt{M_2^2 + M_3^2} < M_2 + M_3.$$

This violates energy conservation! (This statement holds in general.)

7.4 First Law of Black Hole Mechanics

Let $M \to M + \delta M,\ J \to J + \delta J$ in the Kerr solution. Then

$$\frac{\kappa \delta A}{8\pi} = \delta M - \Omega_H \delta J. \tag{7.16}$$

(Proof: Exercise.)
This corresponds to a particular metric perturbation. But the result holds for <u>any</u> metric perturbation [10]:
Consider a stationary, asymptotically flat black hole solution of the vacuum Einstein equations, with bifurcate Killing horizon, mass M, angular momentum J, horizon area A, surface gravity κ and angular velocity Ω_H.
Let δg be a non-singular (on and outside $\mathcal{H}^+$) asymptotically flat metric perturbation, satisfying the linearised Einstein equations. Then the perturbed spacetime has (ADM) mass $M + \delta M$, angular momentum $J + \delta J$, and horizon area (measured at the bifuraction two-sphere) $A + \delta A$ satisfying (7.16).

<u>Note</u>

(i) By the uniqueness theorems, the inital black hole is a Kerr solution. But the theorem generalises to include matter fields, and applies even when the exact black hole solution is not known.

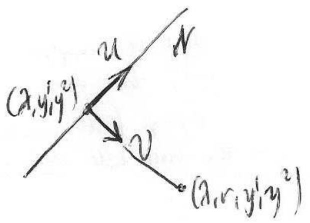
(ii) It was first proved for stationary, axisymmetric perturbations [11].

(iii) For charged black holes, the right-hand side has an additional term $-\Phi_H \delta Q$, where Φ_H is the electric potential difference between the event horizon $\mathcal{H}^+$ and $\imath^0$.

Alternative formulation:

Theorem 7.13 *(Hawking, Hartle [12])*
Consider a stationary, asymptotically flat black hole solution of the vacuum Einstein equations, with bifurcate Killing horizon. Assume a small amount of matter, carrying energy δM and angular momentum δJ crosses $\mathcal{H}^+$ and the black hole eventually becomes stationary. Then the area of $\mathcal{H}^+$ increases by δA, given by (7.16).

Proof
Matter has an energy-momentum four-vector $J_a = -T_{ab}k^b$, and an angular momentum four-vector $L_a = T_{ab}m^b$. T_{ab} is assumed small.
Let $\mathcal{N}$ be a portion of $\mathcal{H}^+$ to the future of the bifurcation sphere B.



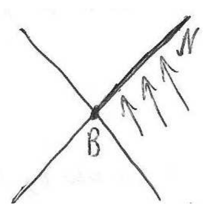
Then (see example sheet 3)

$$\delta M = -\int_{\mathcal{N}} *J, \quad \delta J = -\int_{\mathcal{N}} *L. \tag{7.17}$$

Let U be tangent to generators of $\mathcal{H}^+$ with $\lambda = 0$ on B. Let (y^1, y^2) be coordinates on B; then assign coordinates (λ, y^1, y^2) to a point affine parameter distance λ along a generator through a point on B with coordinates (y^1, y^2). Note that

$$U \cdot \frac{\partial}{\partial y^i} = 0,$$

as $\frac{\partial}{\partial y^i}$ are tangent to $\mathcal{H}^+$.
Let V be any vector field obeying $V^2 = 0$, $U \cdot V = 1$ and $V \cdot \frac{\partial}{\partial y^i} = 0$ on $\mathcal{H}^+$. Assign coordinates (λ, r, y^1, y^2) to a point parameter distance r along an integral curve of V through the point (λ, y^1, y^2) on $\mathcal{H}^+$:



Then $V = \frac{\partial}{\partial r}$, $U = \frac{\partial}{\partial \lambda}$ and $\mathcal{H}^+$ is at $r = 0$. On $\mathcal{H}^+$

$$ds^2 = 2d\lambda\, dr + h_{ij}(\lambda, y)dy^i dy^j.$$

We can order (y^1, y^2) so that

$$\varepsilon|_{\mathcal{H}^+} = \sqrt{h}\, d\lambda \wedge dr \wedge dy^1 \wedge dy^2.$$

Then the orientation of $\mathcal{N}$ is $d\lambda \wedge dy^1 \wedge dy^2$ and hence

$$(*J)_{\lambda 12}|_{\mathcal{N}} = \sqrt{h}J^r|_{\mathcal{N}} = \sqrt{h}J_\lambda|_{\mathcal{N}} = \sqrt{h}U \cdot J|_{\mathcal{N}}, \quad \delta M = -\int_{\mathcal{N}} d\lambda\, d^2y\, \sqrt{h}\, U \cdot J \tag{7.18}$$

and similarly

$$\delta J = -\int_{\mathcal{N}} d\lambda\, d^2y\, \sqrt{h}\, U \cdot L. \tag{7.19}$$

To first order, we can treat the background as fixed in these equations. Then $\mathcal{N}$ is a Killing horizon of $\xi = k + \Omega_H m$ and $\xi|_{\mathcal{N}} = fU|_{\mathcal{N}}$ for a function f obeying $\xi \cdot \partial \log |f||_{\mathcal{N}} = \kappa$.
Hence $U \cdot \partial f|_{\mathcal{N}} = \kappa$, and $f = \kappa\lambda + f_0(y)$.
But $\xi = 0$ on B, i.e. at $\lambda = 0$, and hence $f_0(y) \equiv 0$. So $\xi = \kappa\lambda U$ on $\mathcal{N}$, and

$$\begin{aligned}\delta M = -\int_{\mathcal{N}} d\lambda\, d^2y\, \sqrt{h}U \cdot J &= \int_{\mathcal{N}} d\lambda\, d^2y\, \sqrt{h}T_{ab}U^a k^b = \int_{\mathcal{N}} d\lambda\, d^2y\, \sqrt{h}T_{ab}U^a(\xi^b - \Omega_H m^b) \\ &= \int_{\mathcal{N}} d\lambda\, d^2y\, \sqrt{h}T_{ab}U^a U^b \kappa\lambda - \Omega_H \int_{\mathcal{N}} d\lambda\, d^2y\, \sqrt{h}U \cdot L,\end{aligned}$$

and

$$\delta M - \Omega_H \delta J = \int_{\mathcal{N}} d\lambda\, d^2y\, \sqrt{h}T_{ab}U^a U^b \kappa\lambda = -\int_{\mathcal{N}} d\lambda\, d^2y\, \sqrt{h}\, \kappa\lambda \frac{d\theta}{d\lambda} \tag{7.20}$$

to first order by Raychaudhuri's equation (remember $\theta = \hat{\sigma} = \hat{\omega} = 0$, neglect second order terms). Then by the zeroth law,

$$\delta M - \Omega_H \delta J = -\kappa \int_{\mathcal{N}} d\lambda\, d^2y\, \sqrt{h}\lambda \frac{d\theta}{d\lambda} = -\kappa \int d^2y \left\{ [\sqrt{h}\lambda\theta]_0^\infty - \int_0^\infty d\lambda\, \sqrt{h}\, \theta \right\}. \tag{7.21}$$

But $\sqrt{h}$ is an area element, $\frac{d\sqrt{h}}{d\lambda} = \theta\sqrt{h}$. As the black hole becomes stationary, $\sqrt{h}$ approaches a finite limit as $\lambda \to \infty$ and hence $\theta = o(\frac{1}{\lambda})$ as $\lambda \to \infty$. Hence the surface term in the integral vanishes and

$$\delta M - \Omega_H \delta J = \kappa \int d^2y \int_0^\infty d\lambda\, \frac{d\sqrt{h}}{d\lambda} = \kappa \int d^2y\, [\sqrt{h}]_0^\infty = \kappa\delta A, \tag{7.22}$$

q.e.d.

Remark
The first version is the most commonly encountered version but it compares two different spacetimes. The second version seems more physical, but does not refer to M, J defined at ∞. (Indeed $\delta M_{ADM} = 0$ in any asymptotically flat spacetime - it is conserved!)
The consistency of the two versions supports the idea that the black hole does settle down to a stationary state.

8 Quantum Field Theory in Curved Spacetime

8.1 Quantization of the Free Scalar Field

29 Feb (19)

Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime, with Cauchy surface Σ. Let $\mathcal{S}$ be the space of complex solutions to the Klein-Gordon equation

$$(\nabla^2 - M^2)\Phi = 0. \tag{8.1}$$

A point in $\mathcal{S}$ is uniquely determined by initial data on Σ. $\mathcal{S}$ has a natural Hermitian form: If $\alpha, \beta \in \mathcal{S}$,

$$(\alpha, \beta) = -\int_\Sigma *j(\alpha, \beta), \tag{8.2}$$

where $j(\alpha, \beta) = -i(\bar{\alpha} d\beta - \beta d\bar{\alpha})$. Since

$$*d * j(\alpha, \beta) = \nabla^a j_a = -i(\bar{\alpha}\nabla^2\beta - \beta\nabla^2\bar{\alpha}) = 0, \tag{8.3}$$

j is conserved and (α, β) does not depend on the choice of Σ. The Hermitian form has the following properties:

- $(\alpha, \beta) = \overline{(\beta, \alpha)}$, so $(\cdot, \cdot)$ is Hermitian.
- $(\alpha, \beta) = -(\bar{\beta}, \bar{\alpha})$, so in particular $(\alpha, \alpha) = -(\bar{\alpha}, \bar{\alpha})$, and $(\cdot, \cdot)$ is <u>not</u> positive definite.
- $(\cdot, \cdot)$ is non-degenerate; if $(\alpha, \beta) = 0$ for all β, then $\alpha = 0$.

We can choose a basis $\{\psi_i, \bar{\psi}_i\}$ for $\mathcal{S}$, so that $(\cdot, \cdot)$ has matrix form

$$\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix},$$

i.e.

$$(\psi_i, \psi_j) = -(\bar{\psi}_i, \bar{\psi}_j) = \delta_{ij}; \quad (\psi_i, \bar{\psi}_j) = (\bar{\psi}_i, \psi_j) = 0. \tag{8.4}$$

A <u>real</u> solution of the Klein-Gordon equation can be expanded as

$$\Phi(x) = \sum_i (a_i \psi_i(x) + \bar{a}_i \bar{\psi}_i(x)),$$

where a_i are constants. After quantization, Φ and a_i become operators and one has

$$\Phi(x) = \sum_i (a_i \psi_i(x) + a_i^\dagger \bar{\psi}_i(x)). \tag{8.5}$$

We impose

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i^\dagger, a_j^\dagger] = \delta_{ij} \tag{8.6}$$

(we set $\hbar = 1$). These operators act in a Hilbert space $\mathcal{H}$ defined to be the Fock space built from a vacuum state $|0\rangle$ obeying

$$a_i|0\rangle = 0 \quad \forall\, i, \quad \langle 0|0\rangle = 1, \tag{8.7}$$

i.e. $\mathcal{H}$ has basis $\{|0\rangle, a_i^\dagger|0\rangle, a_i^\dagger a_j^\dagger|0\rangle, \ldots\}$. Note that $\langle\cdot|\cdot\rangle$ is positive definite on $\mathcal{H}$. We interpret $a_i^\dagger|0\rangle$ as one-particle states, $a_i^\dagger a_j^\dagger|0\rangle$ as two-particle states, etc.
This construction depends on the choice of basis $\{\psi, \bar{\psi}\}$ for $\mathcal{S}$. In flat spacetime, we choose ψ_i to be positive frequency modes, i.e.

$$k \cdot \partial\psi_i = -i\omega_i\psi_i, \tag{8.8}$$

where $k = \frac{\partial}{\partial t}$ is the timelike Killing vector field and $\omega_i > 0$.
But in a general speactime, there is no invariant way of defining "positive frequency". A second basis $\{\psi_i', \bar{\psi}_i'\}$ can be related to the first by a **Bogoliubov transformation**

$$\psi_i' = \sum_j (A_{ij}\psi_j + B_{ij}\bar{\psi}_j), \quad \bar{\psi}_i' = \sum_j (\bar{A}_{ij}\bar{\psi}_j + \bar{B}_{ij}\psi_j), \tag{8.9}$$

which obeys (8.4) if and only if

$$\sum_k (\bar{A}_{ik}A_{jk} - \bar{B}_{ik}B_{jk}) = \delta_{ij}, \quad \sum_k (A_{ik}B_{jk} - B_{ik}A_{jk}) = 0, \tag{8.10}$$

i.e.

$$AA^\dagger - BB^\dagger = \mathbf{1}, \quad AB^T - BA^T = 0. \tag{8.11}$$

A_{ij} and B_{ij} are called **Bogoliubov coefficients**. Inverting this gives

$$\psi_i = \sum_j (A'_{ij}\psi'_j + B'_{ij}\bar{\psi}'_j),$$

where $A' = A^\dagger,\ B' = -B^T$. (Exercise)
These must obey the same conditions as A and B, i.e.

$$A^\dagger A - B^T\bar{B} = \mathbf{1}, \quad A^\dagger B - B^T\bar{A} = 0. \tag{8.12}$$

These are not implied by (8.11), they are additional constraints required for the change of basis to be invertible.
We can quantize using the second basis:

$$\Phi(x) = \sum_i (a_i'\psi_i'(x) + (a_i')^\dagger\bar{\psi}_i'(x)).$$

Exercise Show that

$$a_j = \sum_i (a_i' A_{ij} + (a_i')^\dagger \bar{B}_{ij}). \tag{8.13}$$

Hence the vacuum state $|0'\rangle$ defined by $a_i'|0'\rangle$ for all i is not the same as $|0\rangle$ in general. Single-particle states will also disagree; hence the notion of "particle" is basis-dependent!
In a stationary spacetime, there is a preferred choice of basis $\{u_i, \bar{u}_i\}$ where u_i are positive frequency eigenfunctions of $\mathcal{L}_k$, i.e. $\mathcal{L}_k u_i = -i\omega_i u_i$, with $\omega_i > 0$.

Note

(i) k is a Killing vector, so $\mathcal{L}_k$ commutes with $(\nabla^2 - m^2)$ and $\mathcal{L}_k$ maps $\mathcal{S}$ to $\mathcal{S}$.

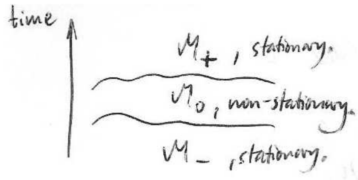
(ii) $\mathcal{L}_k$ is anti-Hermitian (Exercise), so it can be diagonalised with imaginary eigenvalues.

(iii) Eigenfunctions with distinct eigenvalues will be orthogonal, so $(u_i, \bar{u}_j) = 0$. We can normalise eigenfunctions so that $(u_i, u_j) = \delta_{ij}$.

(iv) The basis is unique up to relabelling the u_i which is a Bogoliubov transformation with $B = 0$ (and unitary A), so then $|0'\rangle = |0\rangle$.

8.1.1 Particle Production in a Non-stationary Spacetime

Consider a "sandwich" spacetime $\mathcal{M} = \mathcal{M}_- \cup \mathcal{M}_0 \cup M_+$, of the following form:



Let $\{u_i^\pm\}$ denote positive frequency modes in $\mathcal{M}_\pm$. We can extend these to the whole spacetime by solving the Klein-Gordon equation.
Then $\{u_i^\pm\}$ will <u>not</u> necessarily be positive frequency modes in $\mathcal{M}_\mp$. The sets $\{u_i^+\}$ and $\{u_i^-\}$ will be related by a Bogoliubov transformation:

$$u_i^- = \sum_j (A_{ij} u_j^\dagger + B_{ij} \bar{u}_j^\dagger).$$

So if

$$\Phi(x) = \sum_i (a_i^- u_i^- + (a_i^-)^\dagger \bar{u}_i^-) = \sum_i (a_i^+ u_i^+ + (a_i^+)^\dagger \bar{u}_i^+), \tag{8.14}$$

where $a_j^+ = \sum_i (a_i^- A_{ij} + (a_i^-)^\dagger \bar{B}_{ij})$, the vacua in $\mathcal{M}_\pm$ are $|0_\pm\rangle$ obeying $a_i^\pm |0_\pm\rangle$ for all i.
We assume that no particles are present in $\mathcal{M}_-$, so the state is $|0_-\rangle$.
The particle number operator for the ith mode in $\mathcal{M}_+$ is $N_i^+ = (a_i^+)^\dagger a_i^+$, then the expected number of particles in the jth mode in $\mathcal{M}_+$ is

$$\begin{aligned} \langle 0_- | N_j^\dagger | 0_- \rangle = \langle 0_- | (a_j^+)^\dagger a_j^+ | 0_- \rangle = \sum_{i,k} \langle 0_- | a_i^- B_{ij} (a_k^-)^\dagger \bar{B}_{kj} | 0_- \rangle = \sum_{i,k} B_{ij} \bar{B}_{kj} \langle 0 - | a_i^- (a_k^-)^\dagger | 0 - \rangle \\ = \sum_{i,k} \delta_{ik} B_{ij} \bar{B}_{kj} = \sum_i B_{ij} \bar{B}_{ij} = (B^\dagger B)_{jj}. \end{aligned} \tag{8.15}$$

The expected total number of particles in $\mathcal{M}_+$ is $\mathrm{tr}(B^\dagger B)$. This vanishes if and only if $B = 0$.

8.2 The Unruh Effect

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Consider two-dimensional Minkowski spacetime with line element (20)

$$ds^2 = -dT^2 + dX^2.$$

The massless Klein-Gordon equation is

$$\left(-\frac{\partial^2}{\partial T^2}+\frac{\partial^2}{\partial X^2}\right)\Phi=0.$$

Positive frequency modes are

$$u_k=\frac{1}{\sqrt{4\pi\omega}}e^{-i(\omega T-kX)},\quad \omega=|k|.$$

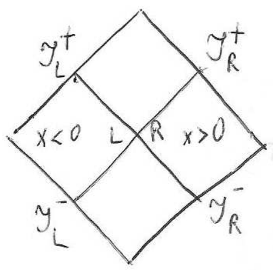
If we introduce coordinates $U=T-X,\ V=T+X$, then positive frequency modes are

$$u_k=\begin{cases}\frac{1}{\sqrt{4\pi\omega}}e^{-i\omega U} & \text{if } k>0 \text{ (right-movers)},\\ \frac{1}{\sqrt{4\pi\omega}}e^{-i\omega V} & \text{if } k<0 \text{ (left-movers)}.\end{cases} \tag{8.16}$$

<u>Remark</u>
Positive frequency modes are analytic in the lower half of the complex T-plane. In fact, if $f(T,X)$ is a superposition of positive frequency modes, then $f(T,X)$ is analytic in the lower-half T-plane $\{\text{Im } T\leq 0\}$.
From (8.16), a function is a superposition of positive frequency modes if and only if it is analytic in the lower half U <u>and</u> V planes (Im $U,V\leq 0$).

Now let $U=-xe^{-\kappa t}, V=xe^{\kappa t}$, then we obtain Rindler spacetime with line element

$$ds^2=-\kappa^2x^2dt^2+dx^2.$$



$x>0$ and $x<0$ are distinct Rindler regions. The vector

$$\frac{\partial}{\partial t}=\kappa\left(V\frac{\partial}{\partial V}-U\frac{\partial}{\partial U}\right)$$

is future directed in R, where $U<0$ and $V>0$, and past-directed in L, where $U>0$ and $V<0$.
Let $x=\pm e^{\kappa y}$, then

$$ds^2=\kappa^2e^{2\kappa y}(-dt^2+dy^2).$$

<u>Exercise</u> Show that the massless Klein-Gordon equation in two dimensions is <u>conformally invariant</u>, i.e. if $\tilde{g}=\Omega^2 g$ with $\Omega(x)>0$ then

$$\tilde{\nabla}^2\Phi=0\quad\Leftrightarrow\quad\nabla^2\Phi=0. \tag{8.17}$$

So the Klein-Gordon equation in L and R is

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial y^2}\right)\Phi=0$$

and positive frequency modes (with respect to $\frac{\partial}{\partial t}$) in R are

$$^{R}u_K = \frac{1}{\sqrt{4\pi\sigma}}e^{-i(\sigma t - Ky)}, \quad \sigma = |K|.$$

We can extend this into L by setting $^{R}u_K = 0$ in L. In (U, V) coordinates,

$$^{R}u_K = \begin{cases} \begin{cases} \frac{1}{\sqrt{4\pi\sigma}} \exp\left(\frac{i\sigma}{\kappa}\log(-U)\right), & U < 0 \\ 0, & U > 0 \end{cases} & \text{if } K > 0 \text{ (right-movers)}, \\ \begin{cases} 0, & V < 0 \\ \frac{1}{\sqrt{4\pi\sigma}} \exp\left(-\frac{i\sigma}{\kappa}\log V\right), & V > 0 \end{cases} & \text{if } K < 0 \text{ (left-movers)}. \end{cases} \tag{8.18}$$

These modes are well-defined on all of Minkowski space.
In L, $\frac{\partial}{\partial t}$ is past-directed, so the positive frequency modes are

$$^{L}u_K = \frac{1}{\sqrt{4\pi\sigma}}e^{i(\sigma t - Ky)}, \quad \sigma = |K|.$$

Wave fronts have $y = \text{sgn}(K)t$, t decreases to the future; so if $K > 0$, then y is decreasing to the future and x is increasing (right-mover), $K < 0$ is a left-mover.
We can extend this into R by setting $^{L}u_K = 0$ in R; then in (U, V) coordinates,

$$^{L}u_K = \begin{cases} \begin{cases} 0, & U < 0 \\ \frac{1}{\sqrt{4\pi\sigma}} \exp\left(-\frac{i\sigma}{\kappa}\log U\right), & U > 0 \end{cases} & \text{if } K > 0 \text{ (right-movers)}, \\ \begin{cases} \frac{1}{\sqrt{4\pi\sigma}} \exp\left(\frac{i\sigma}{\kappa}\log(-V)\right), & V < 0 \\ 0, & V > 0 \end{cases} & \text{if } K < 0 \text{ (left-movers)}. \end{cases} \tag{8.19}$$

The set $\{^{R}u_K, ^{L}u_K, ^{R}\bar{u}_K, ^{L}\bar{u}_K\}$ is a basis for $\mathcal{S}$ in full Minkowski spacetime. We can expand

$$\Phi = \sum_K (a_K^R\, ^{R}u_K(x) + a_K^L\, ^{L}u_K(x) + (a_K^R)^\dagger\, ^{R}\bar{u}_K(x) + (a_K^L)^\dagger\, ^{L}\bar{u}_K(x)). \tag{8.20}$$

The **Rindler vacuum state** is defined by

$$a_K^R|0_{Rin}\rangle = a_K^L|0_{Rin}\rangle = 0 \quad \forall\, K. \tag{8.21}$$

The functions $^{R}u_K, ^{L}u_K$ are <u>not</u> analytic in the lower half U and V planes. Hence they are <u>not</u> superpositions of positive frequency Minkowski modes. We have $B_{ij} \neq 0$ and $|0_{Rin}\rangle \neq |0_{Mink}\rangle$.
Let $K > 0$ and consider

$$^{L}\bar{u}_K = \frac{1}{\sqrt{4\pi\sigma}} \exp\left(\frac{i\sigma}{\kappa}\log U\right) = \frac{1}{\sqrt{4\pi\sigma}} \exp\left(\frac{i\sigma}{\kappa}(\log(-U) - i\pi)\right) \quad \text{for } U > 0.$$

We take the branch cut for the logarithm along the negative imaginary axis, so that

$$\log z = \log|z| + i\arg z, \quad \arg z \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

Then, for positive U,

$$^{L}\bar{u}_K = \frac{1}{\sqrt{4\pi\sigma}} e^{\frac{\pi\sigma}{\kappa}} \exp\left(\frac{i\sigma}{\kappa}\log(-U)\right)$$

and hence

$$ {}^{R}u_K + e^{-\frac{\pi\sigma}{\kappa}}\, {}^{L}\bar{u}_K = \frac{1}{\sqrt{4\pi\sigma}} \exp\left(\frac{i\sigma}{\kappa}\log(-U)\right) \text{ for } \underline{\text{all}}\ U. \tag{8.22} $$

Because of the choice of the branch cut, this is analytic in the lower-half U plane, and so is a superposition of positive-frequency Minkowski modes. Similarly, if $K < 0$,

$$ {}^{R}u_K + e^{-\frac{\pi\sigma}{\kappa}}\, {}^{L}\bar{u}_K = \frac{1}{\sqrt{4\pi\sigma}} e^{-\frac{\pi\sigma}{\kappa}} \exp\left(\frac{i\sigma}{\kappa}\log(-V)\right) \text{ for } \underline{\text{all}}\ V, \tag{8.23} $$

and

$$ {}^{L}u_K + e^{-\frac{\pi\sigma}{\kappa}}\, {}^{R}\bar{u}_K = \begin{cases} \frac{1}{\sqrt{4\pi\sigma}} e^{-\frac{\pi\sigma}{\kappa}} \exp\left(-\frac{i\sigma}{\kappa}\log(-U)\right), & K > 0, \\ \frac{1}{\sqrt{4\pi\sigma}} \exp\left(\frac{i\sigma}{\kappa}\log(-V)\right), & K < 0. \end{cases} \tag{8.24} $$

These are all analytic in the lower-half U, V plane, and hence are superpositions of positive frequency Minkowski modes. Now

$$ v_K^{(1)} = \lambda_K^{(1)} \left(e^{\frac{\pi\sigma}{2\kappa}}\, {}^{R}u_K + e^{-\frac{\pi\sigma}{2\kappa}}\, {}^{L}\bar{u}_K\right), \quad v_K^{(2)} = \lambda_K^{(2)} \left(e^{\frac{\pi\sigma}{2\kappa}}\, {}^{L}u_K + e^{-\frac{\pi\sigma}{2\kappa}}\, {}^{R}\bar{u}_K\right) \tag{8.25} $$

gives a basis of modes that are superpositions of positive frequency Minkowski modes. Hence the Bogoliubov transformation relating $\{v_K^{(i)}, \bar{v}_K^{(i)}\}$ to $\{u_K, \bar{u}_K\}$ has $B = 0$. These bases define the same vaccum, $|0_{Mink}\rangle$.

We can normalise these so that $(v_K^{(1)}, v_{K'}^{(1)}) = \delta_{KK'}$ etc., this is satisfied by

$$ \lambda_K^{(1)} = \lambda_K^{(2)} = \left(2\sinh\frac{\pi\sigma}{\kappa}\right)^{-\frac{1}{2}}. $$

Then expand:

$$ \Phi = \sum_K \left(b_K^{(1)} v_K^{(1)} + b_K^{(2)} v_K^{(2)} + (b_K^{(1)})^\dagger \bar{v}_K^{(1)} + (b_K^{(2)})^\dagger \bar{v}_K^{(2)}\right), \tag{8.26} $$

where $b_K^{(i)}|0_{Mink}\rangle = 0$ for all K and $i = 1, 2$, so

$$ \Phi = \sum_K \left(2\sinh\frac{\pi\sigma}{\kappa}\right)^{-\frac{1}{2}} \left({}^{R}u_K \left(e^{\frac{\pi\sigma}{2\kappa}} b_K^{(1)} + e^{-\frac{\pi\sigma}{2\kappa}} (b_K^{(2)})^\dagger\right) + {}^{L}u_K \left(e^{\frac{\pi\sigma}{2\kappa}} b_K^{(2)} + e^{-\frac{\pi\sigma}{2\kappa}} (b_K^{(1)})^\dagger\right) + \text{h.c.}\right), \tag{8.27} $$

hence

$$ a_K^R = \left(2\sinh\frac{\pi\sigma}{\kappa}\right)^{-\frac{1}{2}} \left(e^{\frac{\pi\sigma}{2\kappa}} b_K^{(1)} + e^{-\frac{\pi\sigma}{2\kappa}} (b_K^{(2)})^\dagger\right), \quad a_K^L = \left(2\sinh\frac{\pi\sigma}{\kappa}\right)^{-\frac{1}{2}} \left(e^{\frac{\pi\sigma}{2\kappa}} b_K^{(2)} + e^{-\frac{\pi\sigma}{2\kappa}} (b_K^{(1)})^\dagger\right). \tag{8.28} $$

The number operator for particles with Rindler momentum K in R is $N_K^R = (a_K^R)^\dagger a_K^R$, the expected particle number for the Kth mode in Minkowski vacuum would be

$$ \langle 0_{Mink}|N_K^R|0_{Mink}\rangle = \frac{1}{2\sinh\frac{\pi\sigma}{\kappa}} \langle 0_{Mink}|e^{-\frac{\pi\sigma}{2\kappa}} b_K^{(2)} e^{-\frac{\pi\sigma}{2\kappa}} (b_K^{(2)})^\dagger|0_{Mink}\rangle = \frac{1}{2\sinh\frac{\pi\sigma}{\kappa}} e^{-\frac{\pi\sigma}{\kappa}} = \frac{1}{e^{\frac{2\pi\sigma}{\kappa}} - 1}. \tag{8.29} $$

This gives the Planck spectrum of black-body radiation at temperature

$$ T = \frac{\kappa}{2\pi}, \tag{8.30} $$

where κ is the surface gravity. 5 Mar
This is temperature defined with respect to infinity: Let E_∞ be the energy defined with respect to k ("energy measured at infinity" in an asymptotically flat spacetime), e.g. for a particle with 4-momentum P, $E_\infty = -k \cdot P$. (21)
An observer on an orbit of k has 4-velocity

$$U^a = \frac{k^a}{\sqrt{-k^2}}.$$

The energy measured locally by the observer is

$$E_{loc} = -U \cdot P = \frac{1}{\sqrt{-k^2}} E_\infty,$$

i.e. E_∞ is redshifted with respect to E_{loc}. But temperature and energy scale in the same way, so

$$T_{loc} = \frac{1}{\sqrt{-k^2}} T_\infty,$$

this is the **Tolman law**. For the scalar field in Rindler spacetime, $\sigma = \sqrt{-k^2} \cdot \sigma_{loc}$ and so $\frac{\sigma}{T} = \frac{\sigma_{loc}}{T_{loc}}$, so

$$T_{loc} = \frac{1}{\sqrt{-k^2}} \frac{\kappa}{2\pi} = \frac{1}{\kappa x} \frac{\kappa}{2\pi} = \frac{1}{2\pi x} = \frac{a}{2\pi}, \tag{8.31}$$

where a is the magnitude of proper acceleration. Hence an observer on an orbit of $\frac{\partial}{\partial t}$ in Rindler sees $|0_{Mink}\rangle$ as a thermal bath at temperature $\frac{a}{2\pi}$. This is the **Unruh effect**.
Putting units back in.

$$T_{loc} = \left(\frac{a}{10^{19} ms^{-2}} \right) K,$$

so this is a very small effect!
We can define Fock spaces $\mathcal{H}_L, \mathcal{H}_R$ for L and R starting from $|0_{Rin}\rangle$:

$$\mathcal{H}_R = \bigotimes_K \mathcal{H}_K^R,$$

where $\mathcal{H}_K^R$ is the Fock space defined by a_K^R.
Let $|n_{K;R}\rangle$ be the normalised n_K-particle state in $\mathcal{H}_K^R$. You can show that

$$|0_{Mink}\rangle = N \prod_K \left\{ \sum_{n_K=0}^{\infty} e^{-\frac{\pi n_K \sigma_K}{\kappa}} |n_{K;L}\rangle |n_{K;R}\rangle \right\} \tag{8.32}$$

is a representation of $|0_{Mink}\rangle$ in $\mathcal{H}_L \otimes \mathcal{H}_R$ (N is a normalisation factor), i.e. $|0_{Mink}\rangle$ is an <u>entangled state</u> in $\mathcal{H}_L \otimes \mathcal{H}_R$ (not of the form $|\psi_L\rangle|\psi_R\rangle$). If an operator $\mathcal{O}$ acts on $\mathcal{H}_R$,

$$\langle O_{Mink}|\mathcal{O}|O_{Mink}\rangle = \operatorname{tr}_{\mathcal{H}_R}(\rho \mathcal{O}), \tag{8.33}$$

where

$$\rho = N \prod_K \left\{ \sum_{n_K=0}^{\infty} e^{-\frac{\pi n_K \sigma_K}{\kappa}} |n_{K;R}\rangle\langle n_{K;R}| \right\}. \tag{8.34}$$

Hence the restriction of $|0_{Mink}\rangle$ to $\mathcal{H}_R$ gives a (thermal) **density matrix**, not a pure state.
<u>Remarks</u>

(i) We discussed massless scalars. But the Unruh effect is general and holds even for interacting fields.

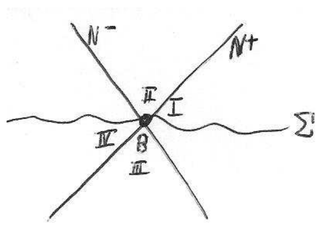
(ii) T_{ab} has a divergence in QFT. One can define a **renormalised stress tensor** T_{ab}^{ren} so that $\langle 0_{Mink}|T_{ab}^{ren}|0_{Mink}\rangle = 0$. This is observer-independent. One finds that $\langle 0_{Rin}|T_{ab}^{ren}|0_{Rin}\rangle \neq 0$ and is singular on the Killing horizon.

8.2.1 Unruh Effect in Curved Spacetime

Note In Minkowski spacetime, a mode is positive frequency with respect to $\frac{\partial}{\partial t}$ if and only if

$$\begin{cases} \text{its restriction to } V = 0 \text{ is positive frequency with respect to } \frac{\partial}{\partial U} \\ \underline{\text{and}} \text{ its restriction to } U = 0 \text{ is positive frequency with respect to } \frac{\partial}{\partial V}. \end{cases}$$

Let $(\mathcal{M}, g)$ be globally hyperbolic, with bifurcate Killing horizon, and a Cauchy surface Σ passing through the bifurcation sphere B.



Assume the Killing vector field k is timelike in I and IV, and future-directed in I (e.g. Kruskal spacetime). Then we can define "positive frequency" modes with respect to k in I and IV, as for Rindler, and a vacuum state analogous to $|0_{Rin}\rangle$. But we can also define a global notion of "positive frequency". Let $\{{}^{U}_{V}\}$ be the affine parameter distance from B along future-directed generators of $\{{}^{N^-}_{N^+}\}$. We define a mode to be positive frequency if and only if its restriction to $N^-(N^+)$ is a positive frequency function of $U(V)$. We can now define a vacuum state analogous to $|0_{Mink}\rangle$. Then the restriction of this state to region I is a thermal state at temperature $T = \frac{\kappa}{2\pi}$, and an observer on an orbit of k measures a temperature

$$T_{loc} = \frac{\kappa}{2\pi\sqrt{-k^2}},$$

e.g. in Kruskal spacetime.
The "global" vacuum state is called the **Hartle-Hawking state** $|0_{HH}\rangle$. The vacuum state defined with respect to k in I and IV is the **Boulware state** $|0_{Boulware}\rangle$.
$\langle 0_{Boulware}|T_{ab}^{ren}|0_{Boulware}\rangle$ is singular on $\mathcal{H}^{\pm}$; $\langle 0_{HH}|T_{ab}^{ren}|0_{HH}\rangle$ is regular (and non-zero). In $|0_{HH}\rangle$, an inertial observer at infinity experiences thermal radiation at the **Hawking temperature**

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{8\pi M}. \tag{8.35}$$

This is not a derivation of Hawking radiation because

(i) $|0_{HH}\rangle$ describes an outgoing flux of thermal radiation at $\mathcal{I}^+$, but $|0_{HH}\rangle$ is time-reversal invariant and hence we also have incoming thermal radiation from $\mathcal{I}^-$. This is unphysical.

(ii) Bifurcate Killing horizons do not form in gravitational collapse, i.e. there are no regions III and IV.

(iii) $|0_{HH}\rangle$ does not exist for Kerr black holes because the Killing vector field ξ normal to $\mathcal{H}^{\pm}$ is spacelike far from the black hole. (particles following orbits of ξ are corotating with the black hole and would have to travel faster than light far from the black hole).

8.3 Scalar Field in Schwarzschild Spacetime

In Schwarzschild coordinates, consider a scalar field model of the form

$$\Phi_{\omega l\tilde{m}} = \frac{1}{r}e^{-i\omega t}R_l(r)Y_{l\tilde{m}}(\theta,\varphi) \quad (\omega > 0),$$

where $Y_{l\tilde{m}}$ is a spherical harmonic. These modes define the Boulware vacuum in regions I and IV.
Exercise Show that the Klein-Gordon equation reduces to the radial equation

$$-\frac{d^2}{dr^{*2}}R_l + V_l(r^*)R_l = \omega^2 R_l \tag{8.36}$$

(cf. time-independent Schrödinger equation), where

$$V_l(r^*) = \left(1 - \frac{2M}{r}\right)\left(m^2 + \frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right).$$

Behaviour of solutions:

(i) As $r^* \to \infty$ (so $r \to \infty$), $V_l \sim m^2$ so $R_l \sim e^{\pm ikr^*}$, where $\omega^2 = k^2 + m^2$. We choose positive sign for outgoing and negative sign for ingoing waves and set $k \geq 0$. For $m = 0$, this means that $\Phi \sim e^{-i\omega u}$ or $e^{-i\omega v}$.

(ii) As $r^* \to -\infty$ (so $r \to 2M$), $V_l \sim 0$ so $R_l \sim e^{\pm i\omega r^*}$. Then $e^{-i\omega t}e^{-i\omega r_*} = e^{-i\omega v}$ is a mode propagating into $\mathcal{H}^+$ and $e^{-i\omega t}e^{+i\omega r_*} = e^{-i\omega u}$ is a mode propagating into $\mathcal{H}^-$.

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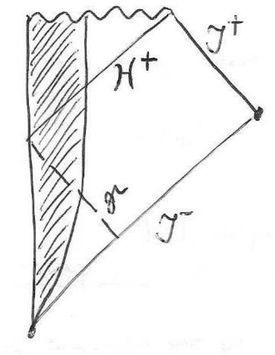
Note that $u \to \infty$ on $\mathcal{H}^+$ and so $e^{-i\omega u}$ is not regular on $\mathcal{H}^+$. We can superpose such modes to build a purely outgoing wave packet vanishing on $\mathcal{H}^+$ (where $u \to \infty$), but individual modes $e^{-i\omega u}$ are not regular on $\mathcal{H}^+$ (similarly for ingoing modes on $\mathcal{H}^-$).
We focus on the massless case ($m = 0$); then the above results suggest that, for every wave packet,

$$\Phi \sim f_{\pm}(u) + g_{\pm}(v) \quad \text{as } t \to \pm\infty.$$

Hence Φ is determined by its values on $\mathcal{H}^-$ and $\mathcal{I}^-$ (fixing f_-, g_-), or on $\mathcal{H}^+$ and $\mathcal{I}^+$ (fixing f_+, g_+).

8.4 Hawking Radiation

Consider a massless scalar Φ in the spacetime of spherically symmetric gravitational collapse.



Note that the spacetime <u>is</u> time-dependent (inside matter).
In the far past, all modes are specified by initial data on $\mathcal{I}^-$. Choose a basis $\{f_i, \bar{f}_i\}$ where $i = \{\omega, l, \tilde{m}\}$ and $\omega > 0$. Then

$$f_i \sim e^{-i\omega v} \qquad \text{near } \mathcal{I}^-.$$

Expand

$$\Phi = \sum_i (a_i f_i(x) + a_i^\dagger \bar{f}_i(x)). \tag{8.37}$$

Define the vacuum state $|0\rangle$ to contain no particles at $\mathcal{I}^-$:

$$a_i|0\rangle \quad \forall\, i.$$

At late times, waves scatter to $\mathcal{I}^+$ or fall through $\mathcal{H}^+$, so modes are specified by final data on $\mathcal{I}^+ \cup \mathcal{H}^+$. Use an orthonormal basis $\{p_i, \bar{p}_i, q_i, \bar{q}_i\}$, where

$$\begin{cases} \{p_i, \bar{p}_i\} \text{ is a basis for solutions outgoing at } \mathcal{I}^+, \text{ zero on } \mathcal{H}^+; \\ \{q_i, \bar{q}_i\} \text{ is a basis for solutions outgoing at } \mathcal{H}^+, \text{ zero on } \mathcal{I}^+ \end{cases}.$$

Note hat the individual $p_i(q_i)$ need not vanish on $\mathcal{H}^+(\mathcal{I}^+)$.
Near $\mathcal{I}^+$, we have a notion of positive frequency (with respect to k). So choose $p_i \sim e^{-i\omega u}$, $\omega > 0$, near $\mathcal{I}^+$. The "future" basis is related to the "past" basis by a Bogoliubov transformation:

$$p_i = \sum_j (A_{ij} f_j + B_{ij} \bar{f}_j), \quad q_i = \sum_j (C_{ij} f_j + D_{ij} \bar{f}_j). \tag{8.38}$$

Expand

$$\Phi = \sum_i (b_i p_i(x) + b_i^\dagger \bar{p}_i(x) + c_i q_i(x) + c_i^\dagger \bar{q}_i(x)). \tag{8.39}$$

<u>Exercise</u> Show that

$$b_i = \sum_j (\bar{A}_{ij} a_j - \bar{B}_{ij} a_j^\dagger). \tag{8.40}$$

The expected number of outgoing particles in the ith mode at $\mathcal{I}^+$ is

$$\langle 0| b_i^\dagger b_i |0\rangle = (BB^\dagger)_{ii}.$$

We need to determine B.
Let γ be a generator of $\mathcal{H}^+$ extended to the past. Without loss of generality γ intersects $\mathcal{I}^-$ at

$v = 0$.
Ingoing null geodesics reach $\mathcal{I}^+$ if $v < 0$ and hit the singularity if $v > 0$.

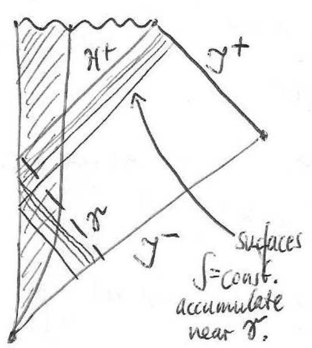
Consider "evolving" p_i to the past, starting at $\mathcal{I}^+$. Outside matter, near $\mathcal{H}^+$, p_i will be a superposition of $e^{-i\omega v}$ and $e^{-i\omega u}$. The latter mode has an "infinite oscillation" singularity at $\mathcal{H}^+$:

$$e^{-i\omega u} = e^{iS}, \quad S = \frac{\omega}{\kappa}\log(-U).$$

Consider a geodesic crossing $\mathcal{H}^+$:

$$\frac{dS}{d\lambda} = \frac{\omega\alpha}{kU} \quad \text{with } \alpha = \frac{dU}{d\lambda} \neq 0,$$

where λ is an affine parameter, and so $\frac{dS}{d\lambda}$ diverges at $U = 0$.
The phase of p_i is rapidly varying near $\mathcal{H}^+$, and hence everywhere along γ (as propagation through matter gives finite redshift).



The **geometric optics approximation** is valid: If $\Phi = Ae^{iS}$ and $|S| \gg 1$ with S rapidly varying (compared with $\log A$), then

$$\nabla^2\Phi = 0 \quad \Leftrightarrow \quad (\nabla S)^2 = 0. \tag{8.41}$$

That is, surfaces of constant S are <u>null</u> near γ.
$S = \infty$ on γ, so we work with $\frac{1}{S}$ instead. Let U be tangent to generators of surfaces $\frac{1}{S} = \text{constant}$.

$$U \cdot \nabla U^a = 0, \quad U = h\, dS^{-1}\ (h \neq 0) \tag{8.42}$$

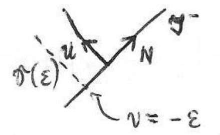
Note that U is tangent to γ. Let N be a future-directed null vector parallelly propagated along γ, then $U \cdot \nabla N^a = 0$, with $U \cdot N \neq 0$.
For any $p \in \gamma$, consider a point p' affine parameter distance $-\epsilon$ along the null geodesic through p with tangent N. (assume ϵ is small) Such points form a curve $\gamma(\epsilon)$.
On $\gamma(\epsilon)$,

$$S^{-1} = -\epsilon N \cdot \partial S^{-1}\big|_{\gamma} \tag{8.43}$$

as $S^{-1} = 0$ on γ.
<u>Exercise</u> Show that

$$U \cdot \nabla(N \cdot \partial S^{-1})\big|_{\gamma} = 0. \tag{8.44}$$

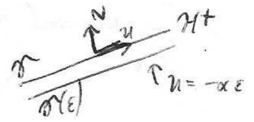
Hence, $N \cdot \partial S^{-1} = \text{constant}$ on γ, S^{-1} is constant on $\gamma(\epsilon)$, so S is constant on $\gamma(\epsilon)$.



Choose $N = \frac{\partial}{\partial v}$ at $\mathcal{I}^-$. $\gamma(\epsilon)$ intersects $\mathcal{I}^-$ at $v = -\epsilon$. Outside matter, near $\mathcal{H}^+$, $S = \frac{\omega}{\kappa}\log(-U)$, and

$$\frac{dU}{d\lambda}|_{\mathcal{H}^+} = \alpha \quad \Rightarrow U = -\alpha\epsilon \quad \text{on } \gamma(\epsilon), \tag{8.45}$$

where λ is the affine parameter along a geodesic with tangent N.



Hence, $S = \frac{\omega}{\kappa}\log(\alpha\epsilon)$ on $\gamma(\epsilon)$ and $S = \frac{\omega}{\kappa}\log(-\alpha v)$ on I^-. So on $\mathcal{I}^-$,

$$p_i = \begin{cases} 0, & v > 0, \\ A_i e^{i\frac{\omega}{\kappa}\log(-\alpha v)}, & v < 0 \ (|v| \text{ small}). \end{cases} \tag{8.46}$$

This is not analytic in the lower-half v-plane, and so is <u>not</u> a superposition of positive frequency modes f_i. 10 Mar
We want to write p_i in terms of $f_j \sim e^{-i\omega' v}$ and $\bar{f}_j \sim e^{i\omega' v}$. So Fourier transform: (23)

$$\tilde{p}_i(\omega') = \int_{-\infty}^{\infty} dv\, e^{i\omega' v} p_i(v). \tag{8.47}$$

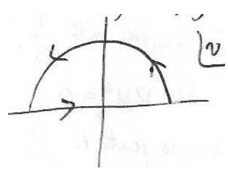
For large $|\omega'|$, $e^{i\omega' v}$ oscillates rapidly, so there are cancellations except near $v = 0$:

$$\tilde{p}_i(\omega') \approx \int_{-\infty}^{0} dv\, e^{i\omega' v} A_i e^{i\frac{\omega}{\kappa}\log(-\alpha v)}. \tag{8.48}$$

Assume that $\omega' > 0$, then

$$\tilde{p}_i(-\omega') \approx \int_{-\infty}^{0} dv\, e^{-i\omega' v} A_i e^{i\frac{\omega}{\kappa}\log(-\alpha v)} = \int_{0}^{\infty} dv\, e^{i\omega' v} A_i e^{i\frac{\omega}{\kappa}(\log(-\alpha v) + i\pi)} = e^{-\frac{\omega\pi}{\kappa}} \int_{0}^{\infty} dv\, e^{i\omega' v} A_i e^{i\frac{\omega}{\kappa}\log(-\alpha v)}. \tag{8.49}$$

We evaluate the integral in the complex v-plane, and we take the branch cut for $\log z$ along the positive imaginary axis, so that $\arg z \in (-\frac{3\pi}{2}, \frac{\pi}{2})$. Then the integrand is analytic in the upper-half v-plane and decays exponentially for Im $v > 0$.



Hence

$$\int_{-\infty}^{\infty} dv\; e^{i\omega' v} A_i e^{i\frac{\omega}{\kappa}\log(-\alpha v)} = 0.$$

It follows that

$$\tilde{p}_i(-\omega') = -e^{-\frac{\omega\pi}{\kappa}} \int_{-\infty}^{0} dv\; e^{i\omega' v} A_i e^{i\frac{\omega}{\kappa}\log(-\alpha v)} = -e^{-\frac{\omega\pi}{\kappa}} \tilde{p}_i(\omega') \tag{8.50}$$

for large ω', and so

$$B_{ij} = -e^{-\frac{\pi\omega_i}{\kappa}} A_{ij} \tag{8.51}$$

for large ω_i. Now we use the condition $AA^\dagger - BB^\dagger = \mathbf{1}$ for Bogoliubov transformations to get

$$\delta_{ij} = \sum_k \left(A_{ik}\bar{A}_{jk} - B_{ik}\bar{B}_{jk}\right) = \sum_k \left(e^{\frac{\pi(\omega_i+\omega_j)}{\kappa}} - 1\right) B_{ik}\bar{B}_{jk} \tag{8.52}$$

if low frequency modes on $\mathcal{I}^-$ are negligible, and hence

$$(BB^\dagger)_{ij} = \frac{\delta_{ij}}{e^{\frac{\pi(\omega_i+\omega_j)}{\kappa}} - 1}, \quad (BB^\dagger)_{ii} = \frac{1}{e^{\frac{2\pi\omega_i}{\kappa}} - 1}. \tag{8.53}$$

This is a thermal spectrum at the Hawking temperature

$$T_H = \frac{\kappa}{2\pi}. \tag{8.54}$$

Why did we neglect low frequency modes on $\mathcal{I}^-$? We are interested in late times on $\mathcal{I}^+$ ("long after the formation of the black hole"). We should really consider a basis on $\mathcal{I}^+$ of wave packets localised around time u_0 with width Δu, we want to consider large u_0. Evolving back to $\mathcal{I}^-$ gives a wave packet centred on v_0 with width Δv, where

$$-u_0 = \frac{1}{\kappa}\log(-\alpha v_0) \quad \Rightarrow \; -\alpha v_0 = e^{-\kappa u_0},$$

and v_0 is small. Consider

$$-(u_0 + \Delta u) = \frac{1}{\kappa}\log(-\alpha(v_0+\Delta v)), \quad \Delta u = \frac{1}{\kappa}\log\left(\frac{v_0}{v_0+\Delta v}\right) = -\frac{1}{\kappa}\log\left(1 + \frac{\Delta v}{v_0}\right) \simeq \frac{\Delta v}{\kappa|v_0|},$$

Then $\Delta v \simeq \kappa|v_0|\Delta u \ll \Delta u$. So at $\mathcal{I}^-$, the wave packet has very narrow spread around small v_0, so it mainly involves high frequency modes on $\mathcal{I}^-$.

Note

(i) Our calculation overlooked the fact that part of the mode p_i will be scattered back to $\mathcal{I}^-$ by the static Schwarzschild metric outside matter. We should really write

$$p_i = p_i^{(1)} + p_i^{(2)},$$

where $p_i^{(1)}$ is scattered by the static metric and $p_i^{(2)}$ propagates through matter. We have analyzed $p_i^{(2)}$. $p_i^{(1)}$ gives a contribution to $A_{ij} \propto A_i\delta_{ij}$ and a vanishing contribution to B_{ij}. A more careful analysis gives

$$(BB^\dagger)_{ii} = \frac{\Gamma_i}{e^{\frac{2\pi\omega_i}{\kappa}} - 1}, \tag{8.55}$$

where Γ_i is the fraction of the mode that enetres a white hole if the collapsing star were absent. This is also the fraction of the mode f_i that enters the black hole (by time-reversal symmetry of Kruskal spacetime), and so this is the absorption cross-section of f_i.
This is precisely the formula obeyed by a perfect black body at temperature $\frac{\kappa}{2\pi}$. Including all constants, we get

$$T = \frac{\kappa}{2\pi} = \frac{\hbar c^3}{8\pi GM} = 6 \cdot 10^{-8} \left(\frac{M_{Sun}}{M}\right) \kappa. \tag{8.56}$$

(ii) One can generalise the result to non-spherically symmetric collapse.

(iii) T decreases with M, and so the heat capacity of a black hole is <u>negative</u>.

(iv) In the Kerr generalisation, if we consider a mode $\propto e^{-i\omega t}e^{i\tilde{m}\varphi}$, just replace ω_i by $\omega_i - \tilde{m}\Omega_H$. This means that the black hole preferentially emits modes with $\tilde{m}\Omega_H > 0$, and so loses angular momentum.

(v) The result generalises to any free field.

(vi) By Stefan's law, the black hole loses energy:

$$\frac{dE}{dt} \sim -\sigma T^4 A$$

if we approximate Γ_i by treating the black hole as an absorbing sphere of area A in Minkowski space. But $E = M,\ A \propto M^2$ and $T \propto \frac{1}{M}$ and so

$$\frac{dM}{dt} \propto -\frac{1}{M^2}$$

and the black hole **evaporates away** in time $\tau \propto M^3 \sim 10^{71} \left(\frac{M}{M_{Sun}}\right)^3$ seconds.
This neglects backreaction but should be a good approximation, at least until the mass of the black hole approaches the Planck mass.

8.5 Black Hole Thermodynamics

As we have established that black holes radiate with temperature

$$T_H = \frac{\kappa}{2\pi}, \tag{8.57}$$

the zeroth law of black hole mechanics is now the same as the zeroth law of thermodynamics (the temperature is constant throughout a body in thermal equilibrium). The first law can be rewritten as

$$dE = T_H dS_{BH} + \Omega_H dJ, \tag{8.58}$$

where

$$S_{BH} = \frac{A}{4}. \tag{8.59}$$

This is the same as the first law of thermodynamics if we interpret S_{BH} as the (Bekenstein-Hawking) **entropy** of a black hole. Restoring units,

$$S_{BH} = \frac{c^3 A}{4G\hbar}.$$

The second law of black hole mechanics states that S_{BH} can not decrease classically. But quantum mechanically, S_{BH} can decrease by Hawking radiation.
However, this thermal radiation has entropy, and the total entropy

$$S = S_{radiation} + S_{BH}$$

does not decrease. This is a special case of the **generalised second law** (Bekenstein):

$$S = S_{matter} + S_{BH}$$

is non-decreasing in any physical process.

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A solar mass black hole has entropy $S_{BH} \sim 10^{77}$; the entropy of the sun is $\sim 10^{58}$. The entropy of the Universe would be much higher if all mass was in the form of black holes, so our Universe is very special, i.e. a very low entropy state (Penrose).
The derivation of $S_{BH} = \frac{A}{4}$ treated the gravitational field classically. But statistical physics suggests that a black hole has $N \sim e^{\frac{A}{4}}$ microstates, what are these? One needs quantum gravity to answer this.
A statistical derivation of S_{BH} is a major goal of quantum gravity research. One can do this in string theory for supersymmetric black holes ($M = |Q|$) [13] and, very recently, for extremal but non-supersymmetric black holes, e.g. extremal Kerr black holes [14].

8.6 Euclidean Methods

Consider a massive scalar in Minkowski space

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

Now do a Wick rotation $t = -i\tau$ with real τ, then the Euclidean metric is

$$ds^2 = d\tau^2 + dx^2 + dy^2 + dz^2. \tag{8.60}$$

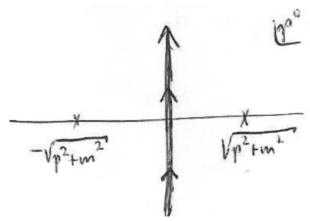
Let $x_E^a = (\tau, x, y, z)$, let $G(x_E)$ be a Green's function for $-\nabla^2 + m^2$:

$$(-\nabla^2 + m^2) G(x_E) = \delta^4(x_E) \quad \Rightarrow \; G(x_E) = \int \frac{d^4 p_E}{(2\pi)^4} \frac{e^{ip_E \cdot x_E}}{p_E^2 + m^2}, \tag{8.61}$$

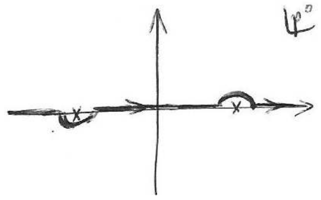
we can analytically continue this to Lorentzian signature. Let $\tau = it$, with t real, then

$$G(x) = i \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ip\cdot x}}{p^2 + m^2}, \tag{8.62}$$

where $p_E^0 = ip^0$ and $p \cdot x$ and p^2 are calculated with a Lorentzian metric. Here the p^0 contour of integration is along the imaginary axis,



We can rotate it to the real axis:



Then

$$G(x) = \Delta_F(x) = \langle 0|T(\varphi(x)\varphi(0))|0\rangle \tag{8.63}$$

is the **Feynman propagator**.
Now do the same for Schwarzschild spacetime; let $t = -i\tau$, then the metric is

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2. \tag{8.64}$$

Let $r = 2M + \frac{x^2}{8M}$ (cf. Rindler); then near $x = 0$ ($r = 2M$) the metric is

$$ds^2 = \kappa^2 x^2 d\tau^2 + dx^2 + (2M)^2 d\Omega^2 + \text{ subleading terms.} \tag{8.65}$$

We need to identify $\tau \sim \tau + \frac{2\pi}{\kappa}$ to avoid a conical singularity at $x = 0$. Then the "Euclidean Schwarzschild" spacetime has topology $\mathbb{R}^2 \times S^2$.
The Green's function $G(x_E)$ will then be periodic in τ with period $\beta = \frac{2\pi}{\kappa}$. This is characteristic of a **thermal correlation function** at temperature $T = \frac{1}{\beta} = \frac{\kappa}{2\pi}$:
Let $Z = \text{tr } e^{-\beta H}$, where H is the Hamiltonian, and

$$G_T(x_E, y_E) = \frac{1}{Z}\text{tr}\left[e^{-\beta H} T(\varphi(x_E)\varphi(y_E))\right]$$

be a thermal correlation function. Assume $\tau_1 < \tau_2 < \tau_1 + \beta$, then (suppressing spatial coordinates in G_T)

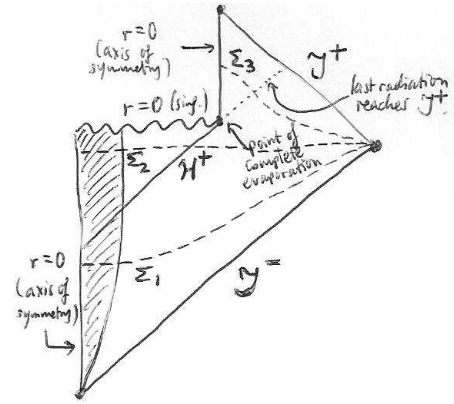
$$\begin{aligned} G_T(\tau_1 + \beta, \tau_2) &= \frac{1}{Z}\text{tr}\left[e^{-\beta H}\varphi(\tau_1 + \beta)\varphi(\tau_2)\right] = \frac{1}{Z}\text{tr}\left[e^{-\beta H} e^{\beta H}\varphi(\tau_1) e^{-\beta H}\varphi(\tau_2)\right] \\ &= \frac{1}{Z}\text{tr}\left[\varphi(\tau_1) e^{-\beta H}\varphi(\tau_2)\right] = \frac{1}{Z}\text{tr}\left[e^{-\beta H} T(\varphi(\tau_2)\varphi(\tau_1))\right] = G_T(\tau_1, \tau_2), \end{aligned} \tag{8.66}$$

where we used the fact that H generates time translations, and here $\Delta t = -i\beta$, and the cyclicity of the trace.
For Schwarzschild spacetime, $G(x_E)$ describes a correlation function for the field at temperature $\frac{\kappa}{2\pi}$. This suggests the existence of a thermal equilibirium state at this temperature (the Hartle-Hawking state). This would not be possible if the black hole did not have the same temperature!

This argument applies even to <u>interacting fields</u> [15].

8.7 The Black Hole Information Paradox

If the black hole evaporates away complete, the Penrose diagram is



Assume the field Φ is in a definite quantum state on Σ_1 (i.e. a pure state, not a density matrix). The state on Σ_2 is also pure but appears mixed to an observer outside the black hole (i.e. such an observer would use a density matrix). The state on Σ_3 can only be described by a density matrix. So a pure state on Σ_1 evolves to a mixed state on Σ_3. This violates unitary time evolution of quantum mechanics! Information appears to be lost in black hole evaporation.

The gauge/gravity correspondence of string theory [16] defines quantum gravity in asymptotically anti-de Sitter spacetimes in terms of a conventional QFT defined on an Einstein static universe in one dimension less (the conformal boundary of adS).

Hence the true evolution is unitary.

But the information paradox still exists in anti-de Sitter space. So what goes wrong with the argument for information loss?

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