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Metriplectic Algebra for Dissipative Fluids in Lagrangian Formulation

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Abstract: The dynamics of dissipative fluids in Eulerian variables may be derived from an algebra of Leibniz brackets of observables, the *metriplectic algebra*, that extends the Poisson algebra of the frictionless limit of the system via a symmetric semidefinite component, encoding dissipative forces. The metriplectic algebra includes the conserved total Hamiltonian H, generating the non-dissipative part of dynamics, and the entropy S of those microscopic degrees of freedom draining energy irreversibly, which generates dissipation. This S is a *Casimir invariant* of the Poisson algebra to which the metriplectic algebra reduces in the frictionless limit. The role of S is as paramount as that of H, but this fact may be underestimated in the Eulerian formulation because S is not the only Casimir of the symplectic non-canonical part of the algebra. Instead, when the dynamics of the non-ideal fluid is written through the parcel variables of the Lagrangian formulation, the fact that entropy is symplectically invariant clearly appears to be related to its dependence on the microscopic degrees of freedom of the fluid, that are themselves in involution with the position and momentum of the parcel.

1. Introduction

The history of Theoretical Physics is, to a certain extent, the discovery of symmetries of physical laws, allowing to bypass the necessity of solving the equations of motion explicitly and gaining deep insights about the essence of first principles themselves.

The highest achievements of this simplification process are the least action principles [1,2], with the Feynman path integral [3] as their most recent descendant, and the study of invariances [4], the Hamiltonian formalism [2,5] and the Hamilton-Jacobi theory [5−7]. In the context of Hamiltonian mechanics, the dynamics of physical systems appears in the form of *algebra of Poisson brackets* [8], composing together the physical observables to both represent the motion of the system and the symmetry properties of its dynamics. An element of the Poisson algebra, namely the Hamiltonian, is the observable that generates the time evolution of any other quantity through the brackets.

This route to the algebrization of dynamics also leads to Dirac's formulation of Quantum Mechanics [9], according to which the algebra of quantum observables is simply a commutation algebra of operators, isomorphic to the Poisson algebra of the respective classical ones. A further progress along this path is the introduction of non-canonical symplectic tensors [10−12], yielding the concept of Lie-Poisson bracket and rendering Casimir invariant quantities important in dynamics. The inclusion of constrained systems in the Hamiltonian framework [13] is another crucial step, allowing for the quantization of gauge systems (including potentially the "fundamental forces of nature", which are indeed gauge forces).

Almost all the benefits of the development just mentioned are, generally speaking, restricted to the Physics of non-dissipative systems: in Lagrangian and Hamiltonian mechanics, as well as in the context of action principles [5,8], only systems undergoing "conservative forces" are treated, while no form of "dissipation" is considered in the fundamental quantum laws, intended as the basic principles of Physics [14]. When dissipative quantum systems are referred to, one typically considers open systems in interaction with some "environment" only partially observed [15], and these are not regarded as "fundamental" (one should however mention the "line of thought", expressed in [16], for instance, in which dissipation is included in the fundamental laws of Quantum Mechanics).

Dissipation, *i.e.*, the presence of *interactions rendering the dynamics non-Hamiltonian and non-time-reversible*, however appears to be omnipresent in the known universe (especially in the world of complex systems [17], from space plasmas [18] to planetary systems, to life [19]). Dissipation is not only an annoying obstacle to the ambitious abstract dream of finding a Hamiltonian version of as many systems as possible, it rather appears as a constitutive factor of the world [20]: the attempt to bring it under the framework of algebrized dynamics is henceforth expected to give deep insights in our understanding of nature.

Enz demonstrated in [21] that, in order for a system to have *decreasing energy*, *non-time-reversible dynamics* and *phase volume shrinking with time* (three conditions defining *dissipative* dynamics), it is necessary to add to the antisymmetric symplectic product of Hamiltonian systems, a symmetric semidefinite part. Such a formalism was originally introduced as *mixed-bracket formulation* by Enz and Turski, in [22−24], in the kinetic theory of the phase transitions, particularly I order transitions. This mixed-bracket formulation was used in the early 1970s to formulate the dynamics of viscous and

heat transporting fluids [25]; then, it was later used extensively in the theory of magnetic systems [26−28], and relativistic plasma [29].

Once a dissipative system is algebrized through the use of mixed-brackets, its quantization may be tried, as described in [21] and [25]: the symplectic part is mapped into the usual quantum commutation algebra, while an anticommutator algebra possibly quantizes the semimetric part giving rise to dissipation [25]. Extensive reviews of mixed-bracket approach are found in [30] and [31].

In the mixed-bracket representation, the system evolution is still generated by the Hamiltonian *H* that may be either conserved or not, due to the semidefinition of the symmetric part of the bracket. When *H* is not conserved, however, the formalism does not keep track of the dissipated energy, as the system does not include that part of the environment giving rise to dissipation and draining energy (e.g., a thermal bath, microscopic degrees of freedom or radiation dissipated). A deeper insight about irreversibility of the process may, however, be gained by closing the system with the degrees of freedom that are draining energy. In order to do this, one has to extend the Hamiltonian and include a new observable, namely an *entropy*.

The dynamics of energetically closed systems relaxing to asymptotic equilibria due to dissipation has been described in by Morrison in [32] via the so-called *metriplectic formalism*, and this is the approach adopted here. The non-dissipative limit of the system is Hamiltonian, so that there exists some function *H* and an algebra of Poisson brackets that describes the system in the absence of dissipation. When dissipation is turned on, the Hamiltonian is still constant during the motion, but friction drives the system to an asymptotic equilibrium: this is done introducing the semimetric bracket, and an observable representing the entropy *S* of the closure of the system.

All the foregoing fruitful attempts to put dissipative systems on the way to algebrization of Physics is represented in [33,34] in terms of Leibniz *algebræ*.

Quite a few dynamical systems have been reformulated as metriplectic: in [32] the kinetic Vlasov-Poisson approximation of a collisional plasma was described as Hamiltonian in the collisionless limit, while collisional terms are shown to arise from a metric bracket. In [35], a non-ideal fluid described in Eulerian variables (EV) is presented as a non-canonical Hamiltonian system, with the addition of a metric bracket providing the dissipative terms due to the finite viscosity and thermal conductivity. In [36], the non-canonical Hamiltonian dynamics of a free rigid body throughout the space of its angular velocity is enriched by a metric contribution allowing the rotator to relax down to asymptotic equilibria (corresponding to spinning around one of its inertial axes).

In [37], there appeared a specific formulation turning the Boltzmann equation into algebraic dynamics, which violates the antisymmetric-symmetric duality of conservative-dissipative components. The formulation of metriplectic dynamics given in [32,35,36] has been refined in the work of Grmela and Ottinger (see [38] and [39]), extended to more complex system contexts, and renamed as GENERIC (General Equation for the Nonequilibrium Reversible-Irreversible Coupling).

A general review of Poisson and metric brackets to describe energetically isolated or non-isolated systems (referred to as complete and incomplete) may be found in [40]. In his PhD thesis [41], Fish applies the results of [32,35,36] examining metriplectic systems of various types under the point of view of manifold properties, and also giving interesting examples from applied physics and biophysics. In [42] the unity of all those approaches is recognized.

The metriplectic system describing neutral fluids in [35] has been generalized to non-ideal magneto-hydrodynamics in [43], while examples of how to algebrize simple mechanical systems with friction are provided in [44].

In the present paper, *the Lagrangian Formulation (LF) of the metriplectic algebra for a viscous fluid* is constructed. The symplectic part of the metriplectic system is taken from [35,45,46], while the metric part is an original contribution presented here for the first time as far as the author is aware of, by mapping the metric bracket in EV to its expression in parcel variables.

Even if rather interesting from the viewpoint of mathematical completeness, still the translation of the Eulerian metriplectic algebra to the Lagrangian one can be questioned as to whether it is worth the effort in physical terms, due to its very complicated appearance. Instead, it should be underlined that the symmetry-related role of the fluid entropy appears much clearer in the Lagrangian algebra than in the Eulerian one, not to mention that whenever the use of LF is preferred to that of the Eulerian Formulation (EF), the expressions found here will be applied.

The fluid entropy has zero Poisson bracket with any other quantity in both formulations, but the expression of the symplectic product in Lagrangian variables (LV) makes it clear that *S is not a Casimir invariant due to the parcel relabeling symmetry* (that allows the fluid to possess an Eulerian representation at all), but simply because it encodes degrees of freedom in involution with the parcel position and momentum (and that enter parcel dynamics only through dissipation).

The paper is organized as follows.

In Section 2 the general framework of metriplectic complete systems is sketched, while in Section 3 we discuss briefly the role of Casimir invariants of the theory with respect to algebra reduction and dissipative processes.

Section 4 is dedicated to the key result of this paper: the LF is constructed for viscous fluids, and their metriplectic algebra is formulated in the material variables. With this result in mind, a speculation on the nature of the fluid entropy as a Casimir invariant of the theory is presented in Section 5.

Conclusions are reported in Section 6, where possible applications and future developments of the present research are also sketched.

2. Metriplectic Complete Systems

In this § we essentially follow the line of reasoning of [32,35,36], and resume those results.

Consider an energetically closed system with dissipation, and describe its state as a point ψ moving in a suitable phase space **V**. Also, refer to its algebra of observables O as a subset of $C^\infty(\mathbf{V}, \mathbf{R})$.

According to the metriplectic scheme, its dynamics $\dot{\psi}$ will be expressed as the sum of a non-dissipative part $\dot{\psi}_{\text{non-diss}} = \{\psi, H\}$, generated by the Hamiltonian $H \in O$ through a Poisson bracket structure, and the dissipative part $\dot{\psi}_{\text{diss}} = \lambda(\psi, S)$, where (.,.) is a symmetric semidefinite Leibniz bracket

$$(A, B) = (B, A), \quad (A, A) \leq 0 \quad \forall \quad A, B \in O,$$

referred to as metric bracket. λ is a negative constant parameter (making physical sense only in the correspondence with an asymptotic equilibrium [43]).

The generator S of the dissipative dynamics $\dot{\psi}_{\text{diss}}$ has zero Poisson bracket with any other observable depending on ψ

$$\{S, A\} = 0 \quad \forall \quad A \in O,\tag{1}$$

while the metric bracket $(., .)$ must have H among its null modes

$$(H, A) = 0 \quad \forall \quad A \in O.\tag{2}$$

If the evolution of the system works as

$$\dot{\psi} = \dot{\psi}_{\text{non-diss}} + \dot{\psi}_{\text{diss}} = \{\psi, H\} + \lambda(\psi, S),\tag{3}$$

then any quantity $\Phi \in O$ depending on ψ evolves according to the same rule

$$\dot{\Phi}(\psi) = \{\Phi(\psi), H\} + \lambda(\Phi(\psi), S).\tag{4}$$

Due to the conditions (1) and (2), this general rule also implies

$$\dot{H} = 0, \quad \dot{S} = \lambda(S, S) \geq 0:\tag{5}$$

the first of these equations means that H is constant because it is not altered by dissipation, that just redistributes energy but does not destroy nor creates it; the second condition in (5) states that S asymptotically and monotonically grows during the motion, as a Lyapunov quantity is expected to do in the correspondence with an asymptotic stable state [47]. This also implies that the energy redistribution takes place irreversibly.

The conditions (1) and (2), together with the properties of $\{., .\}$ and $(., .)$ as Leibniz brackets [33], allow for the definition of a total metriplectic generator $F = H + \lambda S$ so that, provided the new bracket

$$\langle\langle A, B \rangle\rangle = \{A, B\} + (A, B)\tag{6}$$

is defined, one may simply state

$$\dot{\psi} = \langle\langle \psi, F \rangle\rangle,$$

and $\dot{\Phi} = \langle\langle \Phi, F \rangle\rangle$ for any observable Φ. The new Leibniz structure defined in (6) is the metriplectic bracket, while the metriplectic generator F is sometimes referred to as *free energy*.

The fundamental ingredients of the metriplectic formalism, presented here as a refined and complete framework, *i.e.*, *algebrization of dynamics*, the *antisymmetric-symmetric dualism* of $\{A, B\}$ and (A, B) respectively, and the fact that (A, B) has *degeneracy with respect to the gradient of conserved quantities* (e.g., $(A, H) = 0$ for any A), have appeared one after the other in the literature, even before the papers [32,35,36]. In particular, one may be reminded by [48] and [49] of the algebrization process (not to mention the historical path [1–15]), while [10] discussed the issue of symmetric brackets for dissipation. Enz stressed the necessity of a symmetric component of the dynamical algebra in order to have dissipation in [21].

3. Casimir Invariants

The condition (1) attributes to the metric generator S, whatever it is physically, the algebraic character of Casimir invariant (CI) of the Poisson bracket $\{., .\}$, as was recognized in [25].

Now, the metric generator may be a CI for one of the two following reasons.

Either, the symplectic bracket $\{.,.\}$ includes derivatives with respect to the variables on which S depends too, and nevertheless admits a non-trivial kernel to which S belongs; this case, we will refer to as C1, is typical for Poisson *algebræ* $(A_{\text{red}}, \{.,.\}_{\text{red}})$ obtained by reducing some Poisson algebra $(A, \{.,.\})$ to the algebra of all the observables invariant under a certain group of transformations G: if $g \in A$ is a symplectic generator of those transformations, clearly $\{\Phi, g\}_{\text{red}} = 0$ for any element $\Phi \in A_{\text{red}}$, and g is a CI for the bracket $\{.,.\}_{\text{red}}$.

Alternatively, no derivatives with respect to the variables forming S appear at all in the definition of the Poisson bracket, so that S belongs to the kernel of it as does any variable outside the system; this other case, referred to as C2, is that of a Poisson algebra $(A_0, \{.,.\}_0)$ describing a system of variables ψ_0 in interaction with some environment, an effective description of which is given via a variable z external to the system: then, any $C(z)$ is trivially a CI of $\{.,.\}_0$, since the latter depends only on derivatives with respect to ψ_0 but does not involve any derivative in z. A metriplectic system describing the relaxation of "macroscopic" variables ψ_0 due to the interaction with some *microscopic degrees of freedom* (µDoF) may be conceived by defining a metric bracket "driven" by $C(z)$ and acting on the "total" state $\psi \stackrel{\text{def}}{=} (\psi_0, z)$, where z is a coarse grained description of the µDoF. Throughout the literature mentioned in our Introduction above, one meets examples of both kinds C1 and C2.

The free dumped rotator presented in [36], and revisited in [41], is easily recognized to be a C1 case: the square angular momentum is such a CI when the phase space of the rotator is reduced from the 6-dimensional space of angles and their canonical momenta to the $\mathbf{R}^3$ of angular velocities; in this case, the Casimir quantity is the square angular momentum $\left|\vec{L}\right|^2$, that depends on the same vector $\vec{L}$ (the angular momentum) appearing in the symplectic part.

Systems with dissipative constants regarded as control parameters depending on an external variable are properly C2 cases (e.g., the Lodka-Volterra, Lorentz and Van Der-Pole systems in [41], or the elementary mechanics dissipative systems reported in [44], where the external variable is the state of a thermal bath).

One should underline that the distinction between C1 and C2 Casimir just stressed is, in a sense, matter of choice of coordinates in the phase space **V**, since, as some S is in involution with any other quantity, it must be possible to find a description of **V** with $\{.,.\}$ completely independent of the arguments of S: in this sense, *all the metriplectic systems might be cast into the C2 format*. This is indeed the case for the rotator mentioned in [36] and [41]: the $\mathbf{V} = \mathbf{R}^3$ space of vectors $\vec{L}$ may be described by spherical coordinates $(L, \varphi_L, \vartheta_L)$, with $L = \sqrt{\left|\vec{L}\right|^2}$, and no derivative $\frac{\partial}{\partial L}$ appears in $\{.,.\}$, as in the C2 type systems: this reads the original algebra of the rotator as *reducible to a C2 one*. Reviewing this line of thought of the fluid dynamical systems is postponed to § 6.

Last but not least, deciding whether the Boltzmann entropy playing the role of the metric generator for the Vlasov-Poisson collisional plasma is a C1 or C2 quantity deserves a deeper investigation, involving the fact that Vlasov-Poisson equation results from the truncation of a hierarchy of equations involving many-particle variables [50], the symplectic limit of which has been studied in [51].

The origin of being a CI for the entropy of a viscous fluid is investigated here, writing its metriplectic algebra explicitly in LV.

4. Lagrangian Formulation for Viscous Fluids

In [34], the viscous fluid equation is described in the Eulerian Formalism (EF), via the fields mass density $\rho(\vec{x}, t)$, velocity $\vec{v}(\vec{x}, t)$ and mass-specific entropy density $\sigma(\vec{x}, t)$. In the non-dissipative limit the dynamics takes a non-canonical Hamiltonian form: the Poisson bracket between two any functionals $\Phi[\rho, \vec{v}, \sigma]$ and $\Psi[\rho, \vec{v}, \sigma]$ is defined as

$$\{\Phi, \Psi\}_E = -\int_{\mathbf{R}^3} d^3x \left[\frac{\delta\Phi}{\delta\rho} \partial_\alpha \left(\frac{\delta\Psi}{\delta v_\alpha} \right) + \frac{\delta\Psi}{\delta\rho} \partial_\alpha \left(\frac{\delta\Phi}{\delta v_\alpha} \right) - \frac{1}{\rho} \frac{\delta\Psi}{\delta v_\alpha} \varepsilon_{\alpha\gamma\beta} \varepsilon^{\beta\delta\eta} \frac{\delta\Psi}{\delta v_\gamma} \partial_\delta v_\eta + \right.$$
$$\left. + \frac{1}{\rho} \partial_\alpha \sigma \left(\frac{\delta\Phi}{\delta\rho} \frac{\delta\Psi}{\delta v_\alpha} - \frac{\delta\Psi}{\delta\rho} \frac{\delta\Phi}{\delta v_\alpha} \right) \right]. \tag{7}$$

Greek indices are used for the *SO*(3)-vector components of $\vec{v}$ and of the position $\vec{x}$ in the space, and a summation convention holds, so that scalar products in $\mathbf{R}^3$ read $\vec{v} \cdot \vec{w} = v^\alpha w_\alpha$. The symbol $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$ is used for spatial gradients.

The Hamiltonian functional of the system reads

$$H[\rho, \vec{v}, \sigma] = \int_{\mathbf{R}^3} d^3x \left[\frac{\rho v^2}{2} + \rho U(\rho, \sigma) + \rho\phi \right], \tag{8}$$

where $\rho U d^3x$ is the amount of internal energy attributed to the infinitesimal volume d^3x around the position $\vec{x}$. ϕ is an external potential.

H generates the motion of any observable $\Phi[\rho, \vec{v}, \sigma]$ as $\dot{\Phi} = \{\Phi, H\}_E$ thanks to the Poisson bracket (7): the non-dissipative Navier-Stokes equations hence follow

$$\begin{cases} \partial_t v_\alpha = -v_\beta \partial^\beta v_\alpha - \frac{1}{\rho} \partial_\alpha p - \partial_\alpha \phi, \\ \partial_t \rho = -\partial^\beta (\rho v_\beta), \\ \partial_t \sigma = -v_\beta \partial^\beta \sigma, \end{cases}$$

where *p* is the pressure.

Let then viscosity and thermal conductivity be finite.

Let the viscosity tensor be of the form $\Sigma^{\alpha\beta} = \Lambda^{\alpha\beta\gamma\delta} \partial_\gamma v_\delta$, with Λ constant (this Λ is formed by Kronecker tensors and physical "constitutive" constants, see [43]); let the heat flux $\vec{I}$ be related to local temperature *T* as $I_\alpha = -\kappa \partial_\alpha T$: then, the symplectic algebra (7) must be completed by the metric bracket [35]

$$(\Phi, \Psi)_E = \frac{1}{\lambda} \int_{\mathbf{R}^3} d^3x \left\{ T\Lambda_{\alpha\beta\gamma\delta} \left[\partial^\alpha \left(\frac{1}{\rho} \frac{\delta\Phi}{\delta v_\beta} \right) - \frac{1}{\rho T} \partial^\alpha v^\beta \frac{\delta\Phi}{\delta\sigma} \right] \left[\partial^\gamma \left(\frac{1}{\rho} \frac{\delta\Psi}{\delta v_\delta} \right) - \frac{1}{\rho T} \partial^\gamma v^\delta \frac{\delta\Psi}{\delta\sigma} \right] + \right.$$
$$\left. + \kappa T^2 \partial^\alpha \left(\frac{1}{\rho T} \frac{\delta\Phi}{\delta\sigma} \right) \partial_\alpha \left(\frac{1}{\rho T} \frac{\delta\Psi}{\delta\sigma} \right) \right\} \tag{9}$$

so that, given the total entropy of the fluid as

$$S[\rho, \sigma] = \int_{\mathbf{R}^3} \rho\sigma d^3x,$$

the dynamics reads

$$\dot{\Phi} = \{\Phi, H\}_E + \lambda(\Phi, S)_E.$$

This gives rise to the equations of motion:

$$\begin{cases} \partial_t v_\alpha = -v_\beta \partial^\beta v_\alpha - \frac{1}{\rho} \partial_\alpha p - \partial_\alpha \phi + \frac{1}{\rho} \partial^\kappa \left(\Lambda_{\kappa\alpha\beta\gamma} \partial^\beta v^\gamma \right), \\ \partial_t \rho = -\partial^\beta (\rho v_\beta), \\ \partial_t \sigma = -v_\beta \partial^\beta \sigma + \frac{1}{\rho T} \Lambda_{\kappa\alpha\beta\gamma} \partial^\kappa v^\alpha \partial^\beta v^\gamma + \frac{\kappa}{\rho T} \partial^2 T. \end{cases} \tag{10}$$

The symbol $\partial^2 = \partial^\beta \partial_\beta$ has been used.

In the LF, the fluid is subdivided into material parcels labeled by a continuous three-index $\vec{a}$, and the motion and evolution of each $\vec{a}$-th parcel is followed [52]. As far as its motion throughout the space is concerned, the $\vec{a}$-th fluid parcel is described at time t by its position $\vec{\zeta}(\vec{a}, t)$ and its momentum $\vec{\pi}(\vec{a}, t)$ (in order to give a more concrete sense to the label $\vec{a}$, the choice

$$\vec{a} = \vec{\zeta}(\vec{a}, 0) \tag{11}$$

can be made). Being that the parcel is not purely pointlike, these will be understood as the position and momentum of the center-of-mass of the parcel. Since the parcel is a system of $\mathbf{O}(10^{23})$ microscopic particles, it must be equipped also by some variable describing those μDoF: its mass-specific entropy density $s(\vec{a}, t)$ is given this role [53]. The fact that the μDoF of the $\vec{a}$-th parcel are all encoded in the thermodynamical variable $s(\vec{a}, t)$ suggests that they are treated statistically: in a sense, the metriplectic formalism is the algebrization of a stochastic dynamics in which what remains of the probabilistic noise is the equilibrium thermodynamics of it [23,24,44].

In the optics of [54,55], s encodes *the thermodynamics of the relative variables*.

The field configuration $(\vec{\zeta}, \vec{\pi}, s)$ represents the state of the fluid in LF; let us indicate its functional phase space as $\mathbf{V}_L$. In LF the hypothesis of parcel identity conservation is made: this means that at every time t the map $\vec{a} \mapsto \vec{\zeta}(\vec{a}, t)$ is a diffeomorphism from the space initially occupied by the continuum $\mathbf{D}_0$ and the one it occupies at time t, $\mathbf{D}(t) \subseteq \mathbf{R}^3$. If its Jacobian matrix $J_i^\mu = \frac{\partial \zeta^\mu}{\partial a^i}$ is defined, with the volume expansion factor $\mathbf{J} = \det J$, then the measure of the infinitesimal volume $d^3\zeta(\vec{a}, t)$ of the $\vec{a}$-th parcel at time t is related to its initial volume d^3a by the law $d^3\zeta = \mathbf{J}d^3a$. Also, these

diffeomorphisms show a (semi)-group property with respect to the evolution parameter *t*:
$$\vec{\zeta}(\vec{a},t_1+t_2)=\vec{\zeta}\left(\vec{\zeta}(\vec{a},t_1),t_2\right).$$

Vector components of $\vec{\zeta}$ and $\vec{\pi}$ are labeled by Greek indices, as $\vec{v}$ and $\vec{x}$ in the EF, while Latin indices label the components of $\vec{a}$ (even if $\vec{\zeta}$ and $\vec{a}$ belong to the same physical space, as the possible assumption (11) shows, we prefer to use different indices for components of dynamical variables and of the label $\vec{a}$).

The Hamiltonian (8) is easily re-written in the LF as

$$H\left[\vec{\zeta},\vec{\pi},s\right]=\int_{\mathbf{D_0}} d^3a\left[\frac{\pi^2}{2\rho_0}+\rho_0 U\left(\frac{\rho_0}{\mathbf{J}},s\right)+\rho_0\phi\left(\vec{\zeta}\right)\right]. \tag{12}$$

$\rho_0(\vec{a})$ is the initial mass density of the $\vec{a}$-th parcel. The mass-specific internal energy density U depends on the density of the parcel, that reads $\rho=\frac{\rho_0}{\mathbf{J}}$ because of mass conservation [56], and on its entropy. The dynamics of the nondissipative limit in LV is governed by an *apparently canonical Poisson bracket*, reading:

$$\{\Phi,\Psi\}_L=\int_{\mathbf{D_0}} d^3a\left[\frac{\delta\Phi}{\delta\zeta^\alpha(\vec{a})}\frac{\delta\Psi}{\delta\pi_\alpha(\vec{a})}-\frac{\delta\Psi}{\delta\zeta^\alpha(\vec{a})}\frac{\delta\Phi}{\delta\pi_\alpha(\vec{a})}\right] \tag{13}$$

(the expression "apparently canonical" will be commented soon). For any physical observable Φ one has simply $\dot{\Phi}=\{\Phi,H\}_L$, giving rise to the equations of motion:

$$\begin{cases}\dot{\zeta}_\alpha=\pi_\alpha,\\[2mm]\dot{\pi}_\alpha=-\rho_0\dfrac{\partial\phi}{\partial\zeta^\alpha}+A_\alpha^i\dfrac{\partial}{\partial a^i}\left(\rho_0\dfrac{\partial U}{\partial\mathbf{J}}\right),\quad A_\alpha^i=\dfrac{\varepsilon_{\alpha\kappa\lambda}\varepsilon^{imn}}{2}\dfrac{\partial\zeta^\kappa}{\partial a^m}\dfrac{\partial\zeta^\lambda}{\partial a^n},\\[2mm]\dot{s}=0\end{cases} \tag{14}$$

(in Equation (14) the "dot" means "time derivative along the motion of the parcel", also called *Lagrangian*, or *material*, derivative).

In order to complete the algebraic dynamics of the non-ideal fluid in LF, the metric part must be produced. The first step is to consider that the equations of motion to be reproduced are the translation of the system (10) in parcel variables:

$$\begin{cases}\dot{\zeta}_\alpha=\pi_\alpha,\\[2mm]\dot{\pi}_\alpha=-\rho_0\dfrac{\partial\phi}{\partial\zeta^\alpha}+A_\alpha^i\dfrac{\partial}{\partial a^i}\left(\rho_0\dfrac{\partial U}{\partial\mathbf{J}}\right)+\Lambda_{\alpha\beta\gamma\delta}\mathbf{J}\nabla^\beta\nabla^\gamma\left(\dfrac{\pi^\delta}{\rho_0}\right),\\[2mm]\dot{s}=\dfrac{\mathbf{J}}{\rho_0 T}\Lambda_{\alpha\beta\gamma\delta}\nabla^\alpha\left(\dfrac{\pi^\beta}{\rho_0}\right)\nabla^\gamma\left(\dfrac{\pi^\delta}{\rho_0}\right)+\dfrac{\kappa\mathbf{J}}{\rho_0 T}\nabla^\alpha\nabla_\alpha T.\end{cases} \tag{15}$$

The definition of A_α^i was already given in (14). The operator ∇_μ is the derivative with respect to ζ^μ intended as the differential operator $\nabla_\mu=\frac{\partial a_i}{\partial\zeta^\mu}\frac{\partial}{\partial a_i}$, and it acts on $\vec{a}$-dependent fields through the chain

rule; the operator ∇_μ reads $\nabla_\mu = (J^{-1})_{\mu i}(\partial\vec{\zeta})\frac{\partial}{\partial a_i}$ in terms of the Jacobian $J(\partial\vec{\zeta})$. In (15) T represents the temperature of the $\vec{a}$-th parcel.

The metric bracket $(.,.)_L$ is obtained by requiring that it reproduces the Equation (15) via the prescription

$$\dot{\Phi} = \{\Phi, H\}_L + \lambda(\Phi, S)_L :$$

in order to obtain it explicitly, one may consider $(\Phi, \Psi)_E$ in (9) and reason on the relationship between the parcel variables and the Eulerian fields

$$\begin{cases} \rho(\vec{x}, t) = \int_{\mathbf{D}_0} d^3a\, \rho_0(\vec{a})\mathbf{J}(\partial\vec{\zeta}(\vec{a}, t))\delta^3(\vec{\zeta}(\vec{a}, t) - \vec{x}), \\ \vec{v}(\vec{x}, t) = \int_{\mathbf{D}_0} d^3a\, \frac{\vec{\pi}(\vec{a}, t)}{\rho_0(\vec{a})}\delta^3(\vec{\zeta}(\vec{a}, t) - \vec{x}), \\ \sigma(\vec{x}, t) = \int_{\mathbf{D}_0} d^3a\, s(\vec{a}, t)\delta^3(\vec{\zeta}(\vec{a}, t) - \vec{x}). \end{cases} \tag{16}$$

The Eulerian field is the value taken by the corresponding Lagrangian quantity attributed to the parcel that, at that given time, transits at that given point in the space: hence, one should understand $\rho_0(\vec{a})\mathbf{J}(\partial\vec{\zeta}(\vec{a}, t))$ in (9) in the place of $\rho(\vec{x}, t)$, $\frac{\vec{\pi}(\vec{a}, t)}{\rho_0(\vec{a})}$ in the place of $\vec{v}(\vec{x}, t)$ and $s(\vec{a}, t)$ in the place of $\sigma(\vec{x}, t)$, provided that the label $\vec{a}$ is chosen so that $\vec{\zeta}(\vec{a}, t) = \vec{x}$. The integral over $\mathbf{R}^3$ in d^3x is replaced by an integral over $\mathbf{D}_0$ in $d^3\zeta = \mathbf{J}d^3a$.

Special care must be used to treat the relationship between the functional derivative with respect to any Eulerian field $\psi_E(\vec{x})$ and that with respect to the corresponding Lagrangian variable $\psi_L(\vec{a})$. These operations are in fact defined via Frechet derivatives

$$\begin{cases} \dfrac{\delta\Phi}{\delta\psi_E(\vec{x})} = \lim_{\varepsilon\to 0}\dfrac{d}{d\varepsilon}\Phi[\psi_E(\vec{y}) + \varepsilon\delta^3(\vec{y} - \vec{x})], \\ \dfrac{\delta\Phi}{\delta\psi_L(\vec{a})} = \lim_{\varepsilon\to 0}\dfrac{d}{d\varepsilon}\Phi[\psi_L(\vec{b}) + \varepsilon\delta^3(\vec{b} - \vec{a})] \end{cases}$$

so that, even if ψ_E and ψ_L may be identified with each other, still the distributions $\delta^3(\vec{b} - \vec{a})$ and $\delta^3(\vec{y} - \vec{x})$, here to be understood as $\delta^3(\vec{\zeta}(\vec{b}) - \vec{\zeta}(\vec{a}))$, do not exactly coincide: $\delta^3(\vec{b} - \vec{a}) = \mathbf{J}\delta^3(\vec{\zeta}(\vec{b}) - \vec{\zeta}(\vec{a}))$. As a result, one may write:

$$\frac{\delta\Phi}{\delta\psi_E(\vec{\zeta}(\vec{a}))} = \frac{1}{\mathbf{J}(\partial\vec{\zeta}(\vec{a}))}\frac{\delta\Phi}{\delta\psi_L(\vec{a})}.$$

All in all, the metric bracket for a viscous fluid in LF reproducing Equations (15) with $S[s]$ as a metric generator, reads:

$$\left(\Phi,\Psi\right)_L = \frac{1}{\lambda}\int_{\mathbf{D}_0}\mathbf{J}d^3a\left\{T\Lambda_{\alpha\beta\gamma\delta}\left[\nabla^\alpha\left(\frac{\delta\Phi}{\delta\pi_\beta}\right)-\frac{1}{\rho_0 T}\nabla^\alpha\left(\frac{\pi^\beta}{\rho_0}\right)\frac{\delta\Phi}{\delta s}\right]\left[\nabla^\gamma\left(\frac{\delta\Psi}{\delta\pi_\delta}\right)-\frac{1}{\rho_0 T}\nabla^\gamma\left(\frac{\pi\delta}{\rho_0}\right)\frac{\delta\Psi}{\delta\sigma}\right]+ \right.$$
$$\left. +\kappa T^2\nabla^\alpha\left(\frac{1}{\rho_0 T}\frac{\delta\Phi}{\delta s}\right)\nabla_\alpha\left(\frac{1}{\rho_0 T}\frac{\delta\Psi}{\delta s}\right)\right\}. \tag{17}$$

The bracket (17) is easily shown to exhibit all the necessary properties for it to be a metric bracket: it is thoroughly symmetric in the $\Phi\leftrightarrow\Psi$ exchange, while about semidefiniteness one may note that

$$\left(\Phi,\Psi\right)_L = \left(\Phi,\Psi\right)_E$$

holds, provided the correct "dictionary" is used, so that one may count of the fact that $\left(\Phi,\Psi\right)_L$ inherits all the good properties from those demonstrated for $\left(\Phi,\Psi\right)_E$ in [35,43] and references therein.

With the finding (17) we have the complete metriplectic algebra of viscous fluid dynamics in the LF, which can be reported as:

$$\begin{cases} \dot{\Phi} = \left\langle\!\left\langle\Phi,F\right\rangle\!\right\rangle_L, \\ \left\langle\!\left\langle\Phi,\Psi\right\rangle\!\right\rangle_L = \left\{\Phi,\Psi\right\}_L + \left(\Phi,\Psi\right)_L, \\ F = H + \lambda S, \\ H = \int_{\mathbf{D}_0}d^3a\left[\frac{\pi^2}{2\rho_0}+\rho_0 U\left(\frac{\rho_0}{\mathbf{J}},s\right)+\rho_0\phi(\vec{\zeta})\right], \quad S = \int_{\mathbf{D}_0}d^3a\rho_0 s. \end{cases}$$

As anticipated before, the advantage of looking at the metriplectic fluid dynamics in the LF, instead of in the EF, is that a certain subtlety about entropy is clarified, that has to do with the question of it to be a CI of the theory. This clarification comes in Section 5.

5. Entropy and the Casimir Invariant Condition

Back to what was described in the Introduction, we speculate here on the entropy of fluids [35,43], that appears as in-between the "two ways of being a Casimir" C1 and C2.

On the one hand, this S clearly encodes information on the µDoF of the continuum, while the fluid velocity describes a macroscopic point of view of the system, as it happens in the C2 case. On the other hand, Morrison and Padhye had algebraic reasons to show, in [45,46], that this S belongs to a family of quantities conserved via a "C1 mechanism" (out of the reduction of the algebra (13) to the set A_E of quantities $\Theta\left[\vec{\zeta},\vec{\pi},s\right]$ that become the physical quantities in the EF, and are invariant under parcel relabeling transformations (RT) [56]). Examining the LF of the fluid, with the RT more clearly readable, the opinion of the author here has become that the viscous fluid may be considered on the same foot as those mentioned in [41] and [44], classifying its S in the C2 case.

The RTs, on which the LF to EF reduction is based, are smooth invertible maps $\vec{a}\mapsto\vec{a}'$ that leave the Hamiltonian (12) and the Eulerian fields (16) unchanged. The quantities, acting as symplectic generators of RTs via the bracket $\{.,.\}_L$, must belong to one of either the following families of functionals:

$$
\begin{cases}
C_1[\vec{\zeta},\vec{\pi},s] = \int\limits_{\mathbf{D}_0} d^3 a\, \eta(\vec{a}) Q_s(\vec{a}), \quad Q_s(\vec{a}) = \varepsilon^{ijk}\, \dfrac{\partial \pi^\alpha}{\partial a^i}\, \dfrac{\partial \zeta_\alpha}{\partial a^j}\, \dfrac{\partial s}{\partial a^k}, \\[2mm]
C_2[s] = \int\limits_{\mathbf{D}_0} d^3 a\, W(\vec{a}) s(\vec{a}),
\end{cases}
\tag{18}
$$

where $\eta(\vec{a})$ and $W(\vec{a})$ are arbitrary functions. The quantity $Q_s(\vec{a})$ is referred to as potential vorticity, while the entropy of the fluid is an example of $C_2[s]$, with $W(\vec{a}) = \rho_0(\vec{a})$. Both C_1 and C_2 are in involution with any quantity in A_E, so that if the reduction with respect to the symmetry they generate is performed, they do become CI. The point is that, due to the fact that no derivative with respect to s appears in $\{.,.\}_L$, the quantities $C_2[s]$ are "already CI" in the symplectic algebra of the Lagrangian Formulation. Instead, a non-trivial set exists of LF functionals $\Xi[\vec{\zeta},\vec{\pi},s]$ so that $\{\Xi, C_1\}_L \neq 0$, with C_1 given in (18): this is the set of the quantities that can be constructed in the LF *but have not a corresponding Eulerian quantity*, because they are not RT-invariant [45,46,56]. Moreover, the Poisson bracket $\{.,.\}_L$ has been indicated as "apparently canonical" because, even if the derivatives $\frac{\delta}{\delta \zeta^\alpha(\vec{a})}$ and $\frac{\delta}{\delta \pi_\alpha(\vec{a})}$ in (13) appear just like they would be expected to in canonical brackets, still s has no involvement in it, but is part of $\mathbf{V}_L$. This means that the symplectic operator, giving rise to $\{.,.\}_L$, is degenerate on $\mathbf{V}_L$, and the bracket is not "properly" canonical. It admits a nontrivial null space, the set of the quantities $C_2[s]$ in (18) of anything depending on s only: we could visualize this by expressing the matrix related to $\{.,.\}_L$ as

$$
\{\Phi,\Psi\}_L = (\partial_\psi \Phi)^{\mathrm{T}} \cdot Z \cdot \partial_\psi \Psi, \quad Z = \begin{pmatrix} 0 & \mathbf{1}_3 & 0 \\ -\mathbf{1}_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
$$

at each label $\vec{a}$, being $\psi = (\vec{\zeta}, \vec{\pi}, s)$, while one also has $\mathbf{V}_L = \mathbf{R}^6 \oplus \mathbf{R}$ at each $\vec{a}$, being $\mathbf{R}^6$ that of the canonical variables $(\vec{\zeta}, \vec{\pi})$ and $\mathbf{R}$ that of s.

The physical difference between S and any C_1 is that S includes only the μDoF responsible for dissipation, while the C_1s mix them with the parcel's center-of-mass variables $(\vec{\zeta}, \vec{\pi})$. In few words, only S is expected to play the driving role in dissipation processes.

The physical difference of roles for C_1 and C_2 in (18) persists in the metriplectic algebra of the fluid in the EF. In terms of Eulerian fields those quantities appear as follows

$$
\begin{cases}
C_{1E}[\rho,\vec{v},\sigma] = \int\limits_{\mathbf{D}} d^3 x\, \rho c_1(Q_{\sigma E}), \quad Q_{\sigma E} = \dfrac{1}{\rho}\, \vec{\partial}\sigma \cdot (\vec{\partial} \times \vec{v}), \\[2mm]
C_{2E}[\rho,\sigma] = \int\limits_{\mathbf{D}} d^3 x\, \rho c_2(\sigma)
\end{cases}
\tag{19}
$$

(use has been made of the symbol $\vec{\partial} = \frac{\partial}{\partial \vec{x}}$): clearly, all the quantities C_{1E} or C_{2E} in (19) satisfy the prescription (1), so one could be tempted to generalize the expression of the free energy as

$$
\mathbf{F} = H + \lambda_1 C_{1E} + \lambda_2 C_{2E};
\tag{20}
$$

the point is whether this gives rise to any sensible dynamics through the metriplectic algebra $\langle\langle .,. \rangle\rangle_E = \{.,.\}_E + (.,.)_E$; for sure, *as long as the metric bracket (9) is used*, the equations of motion (10) are produced only choosing $\lambda_1 = 0$ and $C_{2E} = S$, so that *entropy seems to play a role that no other CI plays*: assuming (10), the evolution of any C_{kE} may be expressed as $\dot{C}_{kE} = \lambda(C_{kE}, S)_E$, for $k = 1, 2$. The Casimir C_{1E} instead does not generate any dynamics. Whether **F** in (20) may be "useful to dynamics" with symmetric brackets other than that in (9) remains an open question, also because it is not clear yet how to define the metric bracket "from First Principles" (about this: Turski and his co-authors showed how to construct metriplectic systems out of Lie-Poisson *algebræ* via Cartan metrics and group theory reasoning [25−28,25], but this does not seem to be strictly the case here).

6. Conclusions

A viscous fluid with suitable border conditions relaxes to an asymptotic equilibrium due to the presence of dissipation, while it can be written in a Hamiltonian form in its frictionless limit. This is a perfect system to be put in a metriplectic form according to the formalism in [32,44] and references therein.

Fluids may be represented in EF or in LF, and the metriplectic framework for the EF was already known [35]. Here, the metriplectic algebra in the LF is obtained, adopting the parcel variables as in [52] to describe the fluid in a metriplectic form: the resulting picture is clearer than the one in EF.

The position of the center-of-mass of the $\vec{a}$-th parcel $\vec{\zeta}(\vec{a})$ and its momentum $\vec{\pi}(\vec{a})$ undergo the dissipative interaction with the μDoF of the nearby parcels, encoded in the entropy of nearby parcels (of course, $\vec{\zeta}(\vec{a})$ and $\vec{\pi}(\vec{a})$ cannot interact directly with the μDoF of their own parcel, since no internal force can alter the motion of the center-of-mass [57]). The novel result in this work is the expression (17) of the metric bracket in parcel variables, through which the metric generator of dissipation, namely the fluid entropy S, makes viscosity act.

The pure Hamiltonian limit of the metriplectic system would actively involve only the variables $\vec{\zeta}(\vec{a})$ and $\vec{\pi}(\vec{a})$, as demonstrated by the expression (13), in which no derivative appears with respect to fields encoding the μDoF. This renders the fluid entropy S a Casimir invariant "of C2 type": the degrees of freedom encoded in S act as "external variables" with respect to the field configuration, which would be sufficient to describe the ideal fluid in LF, *i.e.*, the Poisson algebra based on $\vec{\zeta}(\vec{a})$ and $\vec{\pi}(\vec{a})$. Hence, the metric generation of dissipation in this case shows the same mechanism as presented in [44] and in Chapter 8 of [41].

Despite the Lagrangian Formulation leads to equations of motion that are more complicated than the ones in EF, stating the dynamics of a viscous fluid in parcel variables appears crucial in order to describe more transparently mesoscopic coherent structures of matter [18].

In their EF, fluids (and plasmas) appear to be often characterized by modes representing local subsets of the continuum in which the parcels move with macroscopic scale correlations (e.g., in vortices or current structures); collective variables describing such field configurations will probably be better described by adopting parcel variables $(\vec{\zeta}, \vec{\pi}, s)$, because long range correlation are likely to form well defined patterns "in the $\vec{a}$-space" rather than "in the $\vec{x}$-space", since the "$\vec{a}$-space" is the set $\mathbf{D}_0$ of parcels' identities, where it is possible to keep track of which parcel has interacted with

which other one, and hence developed correlation at mesoscopic scales. Forthcoming studies will investigate the application of what obtained here to the LF of vortices [58,59], while a contact with the tetrad formalism, describing parcels of various scales [60−64], will be made. Moreover, the LF of an MHD collisional plasma may be constructed, as an extension of the present study to electromagnetic degrees of freedom.

An important point made here is about the distinction between Casimir C1 depending on variables with respect to which derivatives appear in the Poisson bracket, and C2 depending on degrees of freedom that are left untouched by the symplectic bracket. Here the fluid entropy has been clarified to be a C2 type Casimir because it encodes the µDoF of the parcel, while $\{.,.\}_L$ only acts on the parcel's center-of-mass. According to the line of thought that reinterprets the C2 as C1 systems written suitably, exposed in § 3 for the rigid rotator, those cases reported in [41] are expected to be obtainable from "big" symplectic *algebæ* $(A_{tot}, \{.,.\}_{tot})$ (including the degrees of freedom in $(A_0, \{.,.\}_0)$ and those describing microscopically the external forcing), that may be reduced to the given $(A_0, \{.,.\}_0)$. Realizing this explicitly for the systems undergoing dissipation due to a thermal bath of µDoF would mean to recognize *which symmetries of the "big" algebra* $(A_{tot}, \{.,.\}_{tot})$ *are encoded in what finishes to be interpreted as Boltzmann's entropy of the thermal bath*. In the case of the fluid parcel discussed here, a first step is to go from the variables $(\vec{x}_i, \vec{p}_i)$ of the $i = 1,..., N$ microscopic particles forming the $\vec{a}$-th parcel, to the six variables $(\vec{\zeta}(\vec{a}), \vec{\pi}(\vec{a}))$ plus the $6(N-1)$ ones relative-to-the-center-of-mass, collected as some $X_{\vec{a}} \in \mathbf{R}^{6(N-1)}$. These $X_{\vec{a}}$ are the proper µDoF entering the internal energy $U\left(\frac{\rho_0}{\mathbf{J}}, s\right)$ in (12) (clearly, $6N - 8$ components of $X_{\vec{a}}$ have disappeared without leaving tracks in the reduction $X_{\vec{a}} \mapsto (\mathbf{J}, s)$, as the system presented in (12) only tracks the volume of the parcel: this is properly *statistical coarse graining*). The mathematical work recognising the C1→C2 reduction would then be finding functions $s = s(X_{\vec{a}})$ not entering the dynamics of the parcel in the non-dissipative limit and playing the same role as the coordinate L of the free rotator to be reduced to C2. The present work has clarified that the Casimir built with the only entropy density $s(\vec{a})$ represents symmetries with respect to which the chain of reductions

$$\left(\vec{x}_i, \vec{p}_i, \{.,.\}_{6N}\right) \mapsto \left(\vec{\zeta}(\vec{a}), \vec{\pi}(\vec{a}), \{.,.\}_L\right) \oplus \left(X_{\vec{a}}, \{.,.\}_{6(N-1)}\right) \mapsto ... \mapsto \left(\vec{\zeta}(\vec{a}), \vec{\pi}(\vec{a}), \mathbf{J}, s, \{.,.\}_L\right) \qquad (21)$$

has already operated.

The author suggests that pursuing an investigation capable of filling in the gap "…" in (21) will give a deep insight into the problem of the origin of irreversibility *tout court*.

Conflicts of Interest

The authors declare no conflict of interest.

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