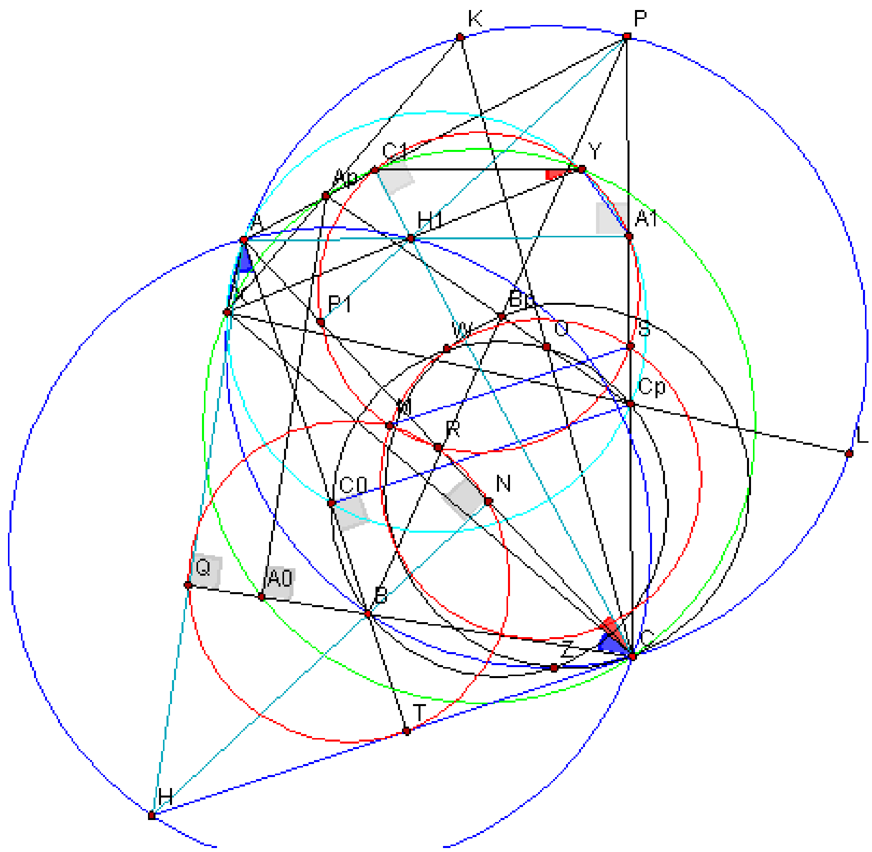
Problem: Let ABC be a triangle, let a line l through its circumcenter O, let a point P lie on the circumcircle of ΔABC. Let AP, BP, CP meet l at A_P, B_P, C_P, respectively. Denote A_0, B_0, C_0 projections of A_P, B_P, C_P to BC, CA, AB, respectively. Then A_0, B_0, C_0 are collinear and the line $\overline{A_0B_0C_0}$ bisects the line segment joining the orthocenter of ΔBAC and P.

Proof:

Lemma 1 (well-known): The two circles with diameter AC_P, CA_P intersects at two points X, Y, one of them (say X) lies on (O), the second, (say Y), lies on the nine-point circle of ΔPAC.

Proof:

Let CO cut (O) again at K; KA_P cut (O) at X; XC_P cut (O) again at L, then by Pascal in hexagon $LAPCKX$ we get A, O, L collinear. Hence X lies on the circle with diameter CA_P as well as the circle with diameter AC_P. Let the circle with diameter AC_P cut PC at A_1, the circle with diameter CA_P cut PA at C_1, then AA_1, CC_1 are two altitudes of ΔPAC. If they intersect at H_1 then H_1 obviously lies on the radical axis of the two spoken circles, thus H_1 lies on XY. Notice that $\angle C_1YA_1 = \angle C_1YX + \angle XYA_1 = \angle C_1CX + \angle XAH_1 = 360^\circ - \angle AXC - \angle AH_1C = (180^\circ - \angle AXC) + (180^\circ - \angle AH_1C) = \angle APC + \angle APC = 2\angle APC$. This means that if I is the midpoint of PH_1 then C_1IYA_1 cyclic, or Y lies on the nine-point circle of ΔPAC.



Lemma 2: A_0C_0 passes through Y.

Proof: Notice that $\angle XYC_0 = \angle XAC_0 = \angle XCA_0 = \angle XYA_0$. This implies that A_0, C_0, Y are collinear.

Let AQ, CT the altitudes of ΔABX with H the orthocenter. Easy to get A, H_1, C, H cyclic and the circle they lie on is the mirror of the circle O over AC. If R is the midpoint of AC and M midpoint of PH then there is no problem to see that R, M are the two common points of the nine-point circle of ΔPAC and the nine-point circle of ΔBAC. By another words, M is the midpoint of PH. So if S is the midpoint of PC then $MS \parallel C_0C_P$.

Now we are going to show that M lies on the line YC_0A_0. Notice that $\angle XYM = \angle XYA_1 - \angle MYA_1 = \angle XAA_1(180° - \angle MSA_1) = \angle XAA_1(180° - \angle C_0C_PA_1) = \angle XAA_1 - \angle C_0AA_1 = \angle XYC_0$. This means that Y, M, C_0 are collinear or M lies on the line YC_0A_0 as desired.

Next, let the circle with diameter CB_P cuts the circle with diameter BC_P intersect at W, Y. By lemma 1 Z lies on (O) and W lies on the nine-point circle of ΔPBC. There is no problem to see that the nine-point circle of ΔPAC

goes through S, M. By lemma 2 we have C_0, B_0, W collinear. As $\angle WMS = \angle WCS = \angle WC_0C_P$, but as $MS \parallel C_0C_P$, then W, M, C_0 are collinear. Hence all the points A_0, C_0, M, Y, B_0, W lie on a line d, this line bisects the line segment joining P and the orthocenter of ΔBAC (that is PH).

Of course then the line l goes through P, then$X, C_P, A_P \equiv P$; $P_1 \equiv Y$, d is the Simson line of ΔABC.