A Note on the Role of Idempotents in the Extended Heisenberg Algebra.

B. J. Hiley.

Theoretical Physics Research Unit, Birkbeck, Malet Street, London WC1E 7HX.

b.hiley@bbk.ac.uk

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Abstract.

Motivated by a process explanation of quantum phenomena, we explore an algebraic representation-free approach to the quantum formalism. We show that the Schrödinger equation can be written in a representation-free way provided the appropriate algebra contains idempotents. To fit into the general scheme, the nilpotent Heisenberg algebra must be extended to include suitable idempotents. We show that the extended boson algebra, which includes the projector to the vacuum, can be used to generate these new idempotents using the Heisenberg and metaplectic groups. We briefly discuss some of the consequences of this approach.

1. Introduction

In this note I want to re-examine the Dirac algebraic approach to quantum mechanics. Like Dirac (1956) I feel the representation free approach to quantum mechanics offers greater possibilities. For reasons which have to do with my speculations about the nature of quantum processes, I want to make the algebraic structure primary and work for as long as possible in this algebraic structure. If you like, I want to take up the suggestion of Haag (1992) (Local Quantum Theory) "The intrinsic structure of the [quantum field] theory is fully characterised by the algebraic relations in the net of abstract algebras (as opposed to their representative algebras on a Hilbert space.) The divorce of the basic description of the theory from Hilbert space brings a tremendous additional freedom, that is it allows the introduction of thermodynamics at the quantum level." Dirac (1956) expressed a similar sentiment, but did not mention thermodynamics. He also claimed the approach offered a more general way of dealing with situations where it was not possible to represent results in Hilbert space. His concern was that there did not seem to be any readily available interpretation to explain this approach. I have claimed elsewhere (Hiley

2000) that a generalised notion of process which is basic to the implicate order begins to provide an explanation. I will not justify this proposal here.

Since the unfolding process described in Hiley (2000) takes the form of an inner automorphism, I start with the Heisenberg picture and consider the time development operator to be a one-parameter automorphism of a general element of the appropriate algebra in the usual way. Thus an $M(t)$ such that

$$M(t)\colon\ A \rightarrow A \qquad\qquad A \quad \mathsf{A}$$

such that $A(t) = M(t)^{-1}A(0)M(t)$ with $M(t) = \exp[iHt]$

Here H of course is the Hamiltonian. The infinitesimal transformation gives the Heisenberg equation of motion

$$i\frac{dA}{dt} = [A, H] \tag{1}$$

We then compare this with the corresponding classical equation of motion with the Poisson brackets replacing the commutator. Thus

$$\frac{dA}{dt} = \{A, H\} + \frac{\partial A}{\partial t} \tag{2}$$

where we have included the explicit time dependence. if we also allow for the explicit time dependence in the Heisenberg equation of motion, then (1) becomes

$$i\frac{dA}{dt} = [A, H] + i\frac{\partial A}{\partial t} \tag{3}$$

Physicists usually derive this equation by first starting from the Schrödinger picture. Because of my starting point, I want to reverse this procedure and arrive at the Schrödinger equation by starting from the algebra

We know that in the usual approach the Hamilton acts on a vector in Hilbert space. But we want to remain in the algebra without first going to an abstract Hilbert space. Therefore we need to look for a natural vector space in the algebra. In fact there is such a natural vector space in the algebra, namely, a left ideal, I_L[1]. There is also exists a dual vector space, the right ideal, I_R. Unfortunately if we act from the right on an element of the left ideal we will in general leave the left ideal. Thus we cannot put an element from a left ideal into equation (3) and remain in that left ideal. However if we form a two-sided ideal such that any element belonging to it can be written in the form $A = B\varepsilon C$, with $B\varepsilon \quad I_L$, and $\varepsilon C \quad I_R$. Here $\varepsilon = \varepsilon^2$ is an appropriate idempotent. Such an element can be put into equation (3) and it will remain in the two-sided ideal.

We must restrict the choice of A further and assume dA/dt = 0. Our reason for introducing this restriction will become clear from equation (10). We these assumptions we write equation (3) in the form

$$i\frac{\partial A}{\partial t} = [H, A] \tag{4}$$

Since $A = B\varepsilon C$ equation (4) becomes

$$i\left(\frac{\partial B}{\partial t}\right)\varepsilon C + iB\varepsilon\left(\frac{\partial C}{\partial t}\right) = HB\varepsilon C - B\varepsilon CH.$$

Multiplying this equation from the left by $B^\dagger$ and from the right by $C^\dagger$, and assuming $B^\dagger B = CC^\dagger = 1$, we find

$$i\,B^\dagger\left(\frac{\partial B}{\partial t}\right)\varepsilon - B^\dagger HB\varepsilon = -\varepsilon i\left(\frac{\partial C}{\partial t}\right)C^\dagger - \varepsilon CHC^\dagger.$$

Since $B\varepsilon$ is any element of the left ideal εC any element of a right ideal, we can write

$$i\left(\frac{\partial B}{\partial t}\right) = HB \tag{5}$$

and

$$-i\left(\frac{\partial C}{\partial t}\right) = CH \tag{6}$$

We shall see immediately that equation (5) and equation (6) have the same *form* as the Schrödinger equation and its conjugate counterpart. However it should be noted that here

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[1] Recall that a left ideal is defined by $I_L = \{ A \quad A \mid KA \quad I_L \mid \quad K \quad A\}$. While the right ideal is defined by $I_R = \{A \quad A \mid AK \quad I_R \mid \quad K \quad A\}$.

$B\varepsilon$ and εC are elements of the algebra and not elements of a Hilbert space. Because of this we will call $B\varepsilon$ and εC 'wave operators'.

Thus we see that the algebraic content of the Schrödinger equation is equivalent to the one-parameter transformation on ideals in the algebra

$$M(t)\text{:}\ \ B\varepsilon \to B\varepsilon \qquad B\varepsilon \quad I_{\mathrm{L}} \quad \text{such that} \qquad B(t)\varepsilon = M(t)^{-1}B(0)\varepsilon.$$

While the conjugate equation is equivalent to

$$M(t)\text{:}\ \ \varepsilon C \to \varepsilon C \qquad \varepsilon C \quad I_{\mathrm{R}} \qquad \text{such that} \qquad \varepsilon C(t) = \varepsilon C(0)M(t).$$

It should be noted in coming to this conclusion, we are using equation (3) and noting that, since $\mathrm{d}A/\mathrm{d}t = 0$, and B and C are independent, $\mathrm{d}B/\mathrm{d}t = \mathrm{d}C/\mathrm{d}t = 0$.

We now need to show that equations (5) and (6) become the standard Schrödinger equation and its conjugate. In order to do this we need to make a closer connection with the Dirac notation. For this reason we will write A = B   C.[2] Further we will assume B   = $B(X, t)$ I_{L} [3]. Then in order to project this element onto a Hilbert space, we follow Dirac (1947) and define

$$\langle x|B\rangle = B(x,t)$$

which is now the usual wave function, conventionally written as $\Psi(x, t)$[4]. It is easy to show that equation (4) becomes the Schrödinger equation

$$i\frac{\partial\Psi(x,t)}{\partial t} = \int H(x,x')\Psi(x',t)dx' \tag{7}$$

When $H(x',x)$ is diagonal, we can write this in the more usual form

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[2] NB. The symbol   (without the line ê) is NOT the ordinary ket, ê . Dirac calls this the standard ket. For further details see Dirac (1947, p. 79-83.) We re-introduced this notation to emphasise that B in an element of the left ideal, which, we emphasise again, belongs to a vector sub-space of the algebra itself. The usual ket êa is an element of an abstract vector space, which is labelled by an eigenvalue of some appropriate observable.

[3] We use upper case letters for elements of the algebra and lower case for the eigenvalues of these elements.

[4] Not all elements of a left ideal produce state functions that are physically meaningful. We will not discuss these restriction here (See Ballentine 1990).

$$i\frac{\partial\Psi(x,t)}{\partial t} = H(x)\Psi(x,t) \qquad (8)$$

The conjugate equation can be derived by assuming $C = C(X, t)$ and now multiplying from the right by the usual position ket êx ,we find

$$\langle C(X,t)|x\rangle = C^*(x,t) = \Psi^*(x,t).$$

So that equation (5) becomes

$$-i\frac{\partial\Psi^*(x,t)}{\partial t} = \int\Psi^*(x',t)H(x',x)dx' = H(x)\Psi^*(x,t) \qquad (9)$$

In this way we make contact with the usual approach.

There is one more relationship worth pointing out here to clarify the notation we are using. In the x-representation, $A = B \quad C$ becomes

$$\langle x|A|x'\rangle = \langle x|C \quad B|x'\rangle = \Psi^*(x,t)\Psi(x',t) = \rho(x,x',t). \qquad (10)$$

$\rho(x,x',t)$ is immediately recognised as the density matrix in the x-representation. Thus a sub-class of these the two-sided ideals are playing the role of density operators. It is now clear why we imposed the assumption dA/dt = 0. It is introduced to ensure the density operator will satisfiy Liouville's theorem. Further equation (4) is immediately recognised as the Liouville equation.

By assumption that $A = \Psi \quad \Psi$ in the above, we have restricted ourselves to pure state density operators. It is straightforward to generalise this procedure to mixed states but we will not carry that out here.

From the mathematical point of view the status of the symbols , , and can be considered as unsatisfactory. Dirac simply introduced them to distinguish between elements that could be operated on from the left, , and right, . He did not consider the role of . As we will show the symbol plays the role of an idempotent. An idempotent can be used to define left and right ideals. To see this, consider an algebra with a non-trivial idempotent, $\varepsilon = \varepsilon^2$. We can then construct a left ideal in the following manner,

$$I_{L\varepsilon} = \{A \quad A: A = B\varepsilon, \quad B \quad A\} \qquad (11)$$

Also the right ideal can be defined through

$$I_{\mathrm{R}\varepsilon} = \{A \quad A\text{: A} = \varepsilon B, \quad B \quad A\} \tag{12}$$

Note that since $I_{\mathrm{R}\varepsilon} : I_{\mathrm{L}\varepsilon} \rightarrow \mathrm{C}$,

$$B(X,t) \cdot B(X,t) = \int \langle B(X,t) | x \quad x | B(X,t) \rangle dx$$

Thus in an algebra with an idempotent, the idempotent can be used to define a state. Furthermore if A contains a non-trivial idempotent ε, we can identify ε with so that $A = B \; C = B\varepsilon C$[5]. Thus is part of the algebra. The reason why Dirac was forced to bring in this symbol was because the Heisenberg algebra, being nilpotent has no non-trivial idempotents. We will show how to remedy this in the next section.

Before doing this, however, I would like to summarize the position we have reached so far. From the point of view we are adopting in this note, writing the Schrödinger equation in the form (5) has the advantage that it is independent of the representation we ultimately choose. What we will also do is to assume that equations (5) and (6) are independent. This then allows us to form two new equations, the first by subtracting the two equations and the second by adding the two equations.

Subtracting the two equations gives us back the Liouville equation (4). As is well known this gives us an equation for the conservation of probability.

The second equation gives us a new equation. If we write $B = Re^{iS}$ This is can be written in the form

$$\rho_R \frac{\partial S}{\partial t} + \frac{1}{2}[\rho, H]_+ = 0 \tag{13}$$

where $\rho_R = R^2$. This equation describes the time development of the phase operator and if the energy is well defined then the equation is an expression for the conservation of energy. It is in fact the quantum generalisation of the Hamiltonian-Jacobi equation. This

[5] If there are more than one idempotent in the algebra, this identification is not unique. As we will show in the next two sections, we can generate many equivalent idempotents in the extended enveloping Heisenberg algebra. In most cases in physics we do not need to identify these idempotents and this is why the Dirac notation is so useful to physicists.

can be seen by multiplying from the left by xê and from the right by êx which produces the equation

$$\frac{\partial S(x)}{\partial t} + H(x) + Q(x) = 0 \tag{14}$$

where $Q(x) = -\frac{1}{2m}\frac{\partial^2 R(x)}{\partial^2 x}\frac{1}{R(x)}$ is the quantum potential. If we regard $\hat{H} = H + Q$ as being the effective Hamiltonian, then the classical canonical formalism gives

$$\dot{x} = \frac{\partial \hat{H}}{\partial p} \qquad \dot{p} = -\frac{\partial \hat{H}}{\partial x} \tag{15}$$

The first equation in equation (14) is just the 'guidance condition' usually written in the form

$$\dot{x} = \frac{1}{m}\,\mathrm{Im}\frac{\partial \psi}{\partial x}\frac{1}{\psi} \tag{16}$$

Thus the representation-free formalism enables us to obtain a different insight into the Bohm interpretation. Some further implications of this equation are discussed in detail in Brown and Hiley (2000).

2. The Nilpotent Heisenberg Algebra.

Clearly I would like to apply the above techniques to the Heisenberg algebra, but as we have already remarked the Heisenberg algebra is a nilpotent algebra of degree three under the product [*A*, *B*]. As a consequence of a well-known theorem, nilpotent algebras do not contain any non-trivial idempotents and if we have no non-trivial idempotents, we cannot construct non-trivial left ideals.

To remedy the absence of such an idempotent in the Heisenberg algebra, Frescura and Hiley (1984) followed a suggestion made by Schönberg (1958) who showed that it was possible to generalise the (enveloping) Heisenberg algebra by adding to it a new fundamental idempotent. Unfortunately his papers were long and the methods far from clear. In this paper we simplify his ideas and bring out the full implications of his method using techniques that will be recognised by physicists.

We begin by noting that the Heisenberg algebra, HA, is isomorphic to the boson algebra, BA, defined through creation and annihilation operators satisfying the commutation relations $[a, a^\dagger] = 1$. (For simplicity we will consider one pair of operators only.) This isomorphism can be realised through the Bargmann transformation, which formally establishes the well-known relations

$$a = Q + iP, \quad \text{and} \quad a^\dagger = Q - iP,$$

What physicists actually work with in physics are the respective enveloping algebras, EHA and EBA. Clearly the EBA is also nilpotent. However this has not caused any problems for physicists because they have introduced the projector onto the vacuum state, V, which satisfies the conditions

$$aV = 0, \qquad V a^\dagger = 0 \qquad \text{and} \qquad V^2 = V \tag{17}$$

The algebra generated by $\{1, a, a^\dagger, V\}$ is no longer nilpotent because $[a, V] = Va$ and $[a^\dagger, V] = a^\dagger V$ so that higher order commutators do not vanish. We will call this algebra the extended enveloping boson algebra, EEBA. [6]

All of this is very straightforward once we realise that V in the algebra is playing the role that the vacuum projector, $|0\rangle\langle 0|$, in the Dirac notation. Here $|0\rangle$ is the vacuum state. We can write an element in the form

$$\sum_{m,n} c_{mn} V^{mn} = \sum_{m,n} c_{mn} (a^\dagger)^m |0\rangle\langle 0| a^n = \sum_{m,n} c_{mn} |m\rangle\langle n|,$$

which, once again, is easily recognised as the density operator, $\rho(m,n)$. This emphasises that it is the density operator that lies at the heart of the algebraic description. If we require a pure state then consider the special case with $\rho^2 = \rho$.

Notice, we could have written $B(X,t) \quad = B\hat{e}$ in analogy with $F(a^\dagger,t) \;= F(a^\dagger,t)\hat{e}0$ in which case we see the standard ket plays a role in the Heisenberg algebra that the vacuum state plays in the boson algebra. Furthermore, as we have already remarked, $\quad = \varepsilon$, plays the role of an idempotent, the projector onto the standard ket, which again is analogous to

[6] It is interesting to note that V is the limit point $V = \lim_{\beta \to \infty} \exp[-\beta a^\dagger a]$

V. (Frescura and Hiley (1984) have already obtained this result.) Since ε is an idempotent equation (11) tells us that $B = A\varepsilon$, A $\mathcal{A}$, generates the left ideal.

In the structure we have outlined so far, the idempotent ε is not well defined since we not know the product rule for ε with P. We need an extra defining relation and the simplest rule would be $P\varepsilon = 0$. If we assume $\varepsilon^\dagger = \varepsilon$, then $\varepsilon P = 0$ and ε projects the EEHA onto the polynomial sub-algebra $F(X)$.

By symmetry we could also extend the EHA by introducing a different idempotent Π such that, $\Pi^2 = \Pi$ and $X\Pi = 0$. Again if $\Pi^\dagger = \Pi$ then $\Pi X = 0$, thus Π projects the EEHA onto the polynomial sub-algebra $\Phi(P)$.

We could also extend the EHA by including both idempotents and make some assumption about their product, for example, $\varepsilon\Pi = \Pi\varepsilon$. Such an assumption is arbitrary and, as we will show shortly, is incorrect. What we now show is how these idempotents can be derived from the EEBA, which includes V. We will see that it is the presence of V that enables these new idempotents to be generated.

3. Relations between the idempotents.

3.1 The Heisenberg Group.

So far we have provided no way to systematically generate the idempotents that we require. To show we can do this we must consider the structure of EHA a little more carefully.

The EHA contains two groups of inner automorphisms in which we are interested. The first is the Heisenberg group generated by linear elements of the algebra. The general element of this group can be written as

$$H(a,b,c) = \exp[\alpha X + \beta P + \varepsilon Z]$$

where Z . Details of the structure of the Heisenberg group will be found in Guillemin and Sternberg, (1984).

To illustrate the method we choose a specific element of this group

$$\exp[\beta P - \varepsilon X)] = \exp[\alpha a^\dagger - \alpha\ a] = D(a)$$

where α, β, ε   C. Some will immediately recognise that $D(a)$ generates coherent states, states that are well known in quantum optics. We can also use this operator to generate a new idempotent, the projector onto the coherent states, in the following manner

$$\Omega = |\alpha\rangle\langle\alpha| = D(a)VD^{-1}(a)$$

Umezawa has an interesting way of looking at the coherent state ê α. He regards it as the α-vacuum state, which is a superposition of states with many particles and can be thought of a condensation of many particles. This statement becomes more transparent by noting that

$$|\alpha\rangle = \exp[-\tfrac{1}{2}|\alpha|^2]\exp[-\alpha a^\dagger]\,|0\rangle$$

As the properties of the coherent states are well known and we will not discuss them further here, particularly as these are not the idempotents that we need.

3.2 The Metaplectic Group.

The second group of inner automorphisms contained in the EHA that is of importance to our discussion can most easily be seen by first noting that the EHA is isomorphic to the symplectic Clifford algebra. (See Crumeyrolle 1990.) This algebra is analogous to the usual orthogonal Clifford algebra, which is known to contain the spin group formally known as the Clifford group. The Clifford group is generated by bilinear combinations of the generating elements of the algebra. The corresponding spin group generated by bilinear combinations of the symplectic Clifford algebra is known as the metaplectic group which double covers the symplectic group. We expect this metaplectic group to give rise to some symplectic spinor properties.

The generators of the metaplectic group can be written in the following form

$$M = \exp[\alpha X^2 + \beta P^2 + \varepsilon(XP + PX)].$$

Such elements are not unfamiliar in quantum optics and form the basis of the description of squeezed states. These states are generated by operators of the form

$$M_s = \exp[\tfrac{i}{2}(a^2 - a^{\dagger 2})]$$

I will not discuss the properties of squeezed states here. I merely introduce them to show that elements of the metaplectic group have physical significance. What I now want to do is to show how elements of the metaplectic group enable us to generate the idempotents we need from the algebraic equivalent of the vacuum projector, V.

We find

$$\varepsilon = \sqrt{N}\exp[X^2/2]V\exp[X^2/2] \qquad \text{and} \qquad \Pi = \sqrt{N'}\exp[P^2/2]V\exp[P^2/2]$$

where N and N' are suitable constants to ensure $\varepsilon^2 = \varepsilon$ and $\Pi^2 = \Pi$. It is a straightforward but tedious task to show that $P\varepsilon = \varepsilon P = 0$ and $X\Pi = \Pi X = 0$. Furthermore we can show that

$$\varepsilon\Pi = N_1\exp[X^2/2]V\exp[P^2/2]$$

and

$$\Pi\varepsilon = N_2\exp[P^2/2]V\exp[X^2/2].$$

We will show below that the expressions on the right hand side of these last two equations are essentially the idempotent E that we have used elsewhere in Frescura and Hiley (1984) and in Hiley and Monk (1993). We now call the conjugate $E^\dagger = \Delta$ and will show that Δ is the algebraic equivalent of the Dirac delta function $\delta(x)$.

4. More Idempotents.

Consider the two idempotents defined by

$$E = \sqrt{N}\exp[X^2/2]V\exp[P^2/2] \quad \text{and} \quad \Delta = \sqrt{N'}\exp[P^2/2]V\exp[X^2/2].$$

Where N and N' are again some numbers introduced to ensure that E and Δ are idempotent. For convenience we will omit writing these numbers from the discussions below since they do not alter the results.

We first start by recalling an old suggestion made by Weyl (1930). He noticed that in quantum mechanics we are interested in ray representations so that we can consider iQ and iP as infinitesimal unitary rotations of the ray field. The Heisenberg commutation rules show that these ray rotations are commutative. Weyl then suggested that we should explore the properties of Abelian groups of unitary rotations in the ray field of an n-dimensional space. This suggestion has been taken up by Schwinger (1960). Davies et al

(1982), and Monk and Hiley (1993) explored this structure, which was called the discrete Weyl algebra, and showed how to construct a discrete space with interesting properties.

This algebra is generated by {1, A, B} subject to

$$AB = \omega BA \qquad A^n = 1, \qquad B^n = 1, \qquad \text{with } \omega = \exp[2\pi i/n]$$

and where n is an integer.

Weyl shows that we can write $A = \exp[i\xi P]$, and $B = \exp[i\eta X]$. In the limit as $n \to \infty$ this algebra approaches the Heisenberg algebra. However the discrete Weyl algebra is not nilpotent and therefore has idempotents that are not difficult to construct from elements of the algebra.

In Hiley and Monk (1993) we show how these idempotents can be constructed. One such idempotent can be written in the form

$$\varepsilon_{00} = \frac{1}{n}\sum_{\beta} \exp[i\beta X] \quad \to \quad \frac{1}{2\pi}\int d\beta \exp[i\beta X] \quad \to \quad \delta(x) \tag{18}$$

We have followed the limiting process described by Weyl. Thus our idempotent ε_{00} is playing the role of a point at the origin of the co-ordinates. The limiting process suggests that we need to extend the Heisenberg algebra by adding the idempotent, Δ, which is to play the role of the algebraic equivalent of the Dirac delta function at the origin. Thus our EEHA is generated by {1, X, P, Δ}. We will show that Δ is exactly the idempotent defined above and we will evaluate the multiplication rules with other elements of the algebra.

Before doing this, we will recall in more detail how the ideas discussed in Hiley and Monk (1993) are related to Weyl's discussions. With the idempotent defined in equation (10), we can translate to a new point through the relation $\varepsilon_{\alpha\alpha} = A^{-\alpha}\varepsilon_{00}A^{\alpha}$. If we act only on one side with $A^{-\alpha}$, this will corresponds to the first expression in the Weyl equation (15.2), $A^s : x'_k = x_{k+s}$, which in the limit gives $\psi(x) \to \psi(x - \alpha)$. Thus $A^{-\alpha}$ produces a translation through α. Our equation (3.6), namely, $B^{\beta}\varepsilon_{\alpha 0} = \omega^{\alpha-\beta}\varepsilon_{\alpha 0}$ corresponds to the second part of the Weyl equation (15.2), viz: B^t: $x'_k = \varepsilon^{kt}x_k$, which in the limit corresponds to $\psi(x) \to \exp[i\beta x].\psi(x)$. Thus we see how the Schrödinger representation actually emerges from our algebraic structure as a limiting feature.

In order to identify the algebraic structure of Δ we have to show that $A^\beta = \exp[i\beta P]$ translates Δ to Δ_β such that

$$\Delta_\beta = A^{-\beta}\Delta A^\beta = \exp[-i\beta P]\Delta\exp[i\beta P]$$

and if we let $X\Delta = 0$ then

$$X\Delta_\beta = X\exp[-i\beta P]\Delta\exp[i\beta P] = \exp[-i\beta P](X+\beta)\Delta\exp[i\beta P] = \beta\Delta_\beta$$

If we write $\Delta = \exp[P^2/2]V\exp[X^2/2]$ then $X\Delta = 0$ so we can label $\Delta = \Delta_0$. Then

$$\Delta_x = \exp[-ixP]\Delta_0\exp[ixP]$$

Thus we have generated a continuum of delta functions producing a straight line. Thus our conditions for Δ are

$$X\Delta = 0, \qquad \Delta P = 0, \qquad \text{and} \qquad \Delta^2 = \Delta.$$

Then the $E = \Delta^\dagger$ introduced by Frescura and Hiley (1984) satisfies

$$EX = 0, \qquad PE = 0, \qquad \text{and} \qquad E^2 = E.$$

This completes our identification of the idempotents that we have used in our previous work.

5. Conclusions.

We have shown that if we extend the boson algebra by incorporating the limit point V, which physicists identify with the projection onto the vacuum, we use V to generate idempotents to extend the Heisenberg algebra to EEHA. We have shown these idempotents are generated by using elements of the Heisenberg and the metaplectic groups. The consequence of this is that EEHA with its new set of idempotents is isomorphic to EEHA just as EBA is isomorphic to EHA.

Within the EEHA we can now discuss the Heisenberg equation and Schrödinger equation in a representation-free manner. We then have the possibility of approaching quantum mechanics in a more general way. As we show in Brown and Hiley (2000) this leads us to see the Schrödinger equation as describing simultaneously both the conservation of energy and the conservation of probability. These equations can also be extended to the relativistic Dirac equation.

The conservation of energy equation can also be interpreted as giving phase information which leads to a very simple account of gauge effects. This approach also enables us to see the Bohm interpretation in a new light. It shows that the non-commutative structure enables us to construct "shadow phase spaces". The recent work of de Gosson (2001) has shown how these ideas are related to the work on Lagrangian quantisation. This shows the key role the metaplectic group is playing in quantum mechanics.

As a further example of the role of the metaplectic group, Fernandes and Hiley (2000) have shown how the symplectic spinor, which is central to the symplectic Clifford algebra can account for the Guoy phase in optics and the discontinuous change of phase when light passes through a focal point.

In this paper we have also re-examined how the discrete Weyl algebra can be extended to the continuum limit. The novelty of our approach is that the points of the continuum are carried in the algebra itself. This is done by constructing the special idempotent, Δ, which in the algebra plays the role of the Dirac delta function. This additional element has enabled us to complete the discussion of how the discrete Weyl algebra approaches the continuum limit.

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