Kähler and Symplectic Manifolds Quotient Constructions

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Introduction

This dissertation concerns the geometry of complex and symplectic manifolds, and Kähler geometry, which unifies both. The original aim of the project was to understand the quotient construction for Kähler manifolds, which uses differential geometry, and the quotient construction for complex algebraic manifolds, which uses Geometric Invariant Theory, and to explain how these two constructions are related.

In the event, this goal proved too ambitious to tackle fully in the time and space available. Instead, we have concentrated mostly on the differential-geometric side of the story, discussing complex manifolds, almost complex structures and integrability; Hermitian, Kähler and hyperkähler metrics; examples and conditions for a manifold to be complex or almost complex; symplectic manifolds, and examples including toric symplectic manifolds; and quotient constructions for symplectic and Kähler manifolds. We do discuss some algebraic geometry topics, namely projective complex manifolds, Chow's Theorem and the Kodaira Embedding Theorem, and GIT quotients, although these are covered briefly.

We now summarize the contents of each chapter.

Chapter 1

This first chapter is a self-contained discussion about fundamental concepts in complex geometry. We analyze different definitions of *complex manifold* and discuss the different compatible structures that they can carry. We also study Kähler manifolds, which provide one of the most important types of structures in complex geometry. And as a fundamental example of a Kähler manifold we analyze the projective space $\mathbb{CP}^n$, whose complex submanifolds are the subject of *(projective) complex algebraic geometry*. This leads us to a fundamental theorem in geometry, the Kodaira Embedding Theorem, that explains which complex compact manifolds can be embedded in projective space, and hence are objects of study in projective algebraic geometry. Lastly, we discuss holonomy groups as a way to understand and classify different geometric structures on Riemannian manifolds.

Chapter 2

In this chapter we explain some of the fundamental concepts of symplectic geometry. We aim at understanding what the symplectic structure on a manifold, i.e., a non-degenerate closed 2-form, means both topologically and geometrically. Topologically, because we are interested in the topological conditions that ensure that a symplectic form exists on a manifold. Geometrically, because by Darboux's Theorem, symplectic manifolds are all locally isomorphic. Although all Kähler manifolds are symplectic, there are many examples of symplectic manifolds with an integrable complex structure which are not Kähler. Here we describe the first example that was constructed. Lastly, we discuss of one of the most important tools used to study symplectic manifolds, J-holomorphic curves.

Chapter 3

This last chapter is dedicated to the study of quotient constructions for different geometric structures. We construct explicit examples of quotients and discuss the differences and relationship between the different approaches. We begin by studying symplectic group actions and moment maps and learn how to construct quotients of symplectic manifolds using the Marsden-Weinstein-Meyer reduction. We will see how the resulting quotient can be understood in terms of combinatorial data when the group is a maximal torus. We discuss Kähler quotients as a similar construction to the symplectic case and, after this, we consider the conditions under which the Kähler quotient will be equivalent to the GIT quotient in algebraic geometry. Lastly, we hyperkähler quotient as a generalization of the Kähler quotient construction, closely related to symplectic reduction.

Contents

Chapter 1

Complex Geometry

In this chapter we discuss different structures on complex manifolds and how they are compatible. We also discuss the conditions under which a complex manifold is projective and lies then in the area of study of algebraic geometry and which are Kähler. We can see these relations as follows

$$\text{Complex Geometry} \supset \text{Kähler Geometry} \supset \text{Projective Geometry}.$$

We will also study hyperkähler manifolds as a generalization of Kähler geometry and how the different types of geometric structures can be classified by the different holonomy groups of each manifold. It is important to remark that complex geometry represents a small part of the geometry of manifolds carrying an endomorphism J of the tangent bundle such that $J^2 = -id$. Only those manifolds which have an *integrable* almost complex structure will be studied using the techniques of complex geometry.

The main references for this chapter are [4–9, 15, 18–20, 22, 23, 32, 36].

1.1 Complex Manifolds

Definition 1.1. *Let M be a real manifold of dimension $2n$. A complex chart on M is a pair $(\mathcal{U}, \psi)$, where $\mathcal{U}$ is open in M and $\psi : \mathcal{U} \to \mathbb{C}^n$ is a diffeomorphism between $\mathcal{U}$ and some open set in $\mathbb{C}^n$. Equivalently, ψ gives a set of complex coordinates $(z_1, \ldots, z_n)$ on $\mathcal{U}$. If $(\mathcal{U}_1, \psi_1)$ and $(\mathcal{U}_2, \psi_2)$ are two complex charts, then the transition function is $\psi_{12} : \psi_1(\mathcal{U}_1 \cap \mathcal{U}_2) \to \psi_2(\mathcal{U}_1 \cap \mathcal{U}_2)$, given by $\psi_{12} = \psi_2 \circ \psi_1^{-1}$. We say M is a complex manifold if it has an atlas $\{(\mathcal{U}_i, \psi_i) : i \in I\}$ of complex charts $(\mathcal{U}, \psi)$, such that all the transition functions are biholomorphic as maps from open subsets of $\mathbb{C}^n$ to open subsets of $\mathbb{C}^n$.*

Example 1.2 (Complex Manifold)**.** *Here we give three different examples of complex manifolds.*

(i) *Let us consider the manifold $M = \mathbb{C}^n$. $(\mathcal{U}, \psi) = (\mathbb{C}^n, id_{\mathbb{C}^n})$ is a chart on $\mathbb{C}^n$ and $\{(\mathbb{C}^n, id_{\mathbb{C}^n})\}$ is an atlas on $\mathbb{C}^n$.*

(ii) *Let us consider the manifold given by the projective space $\mathbb{CP}^n$.*

Let $\mathcal{U}_k = \{[z_0 : z_1 : \cdots : z_n] \in \mathbb{CP}^n : z_k \neq 0\}$ for each $k = 0, 1, \ldots n$. Then $\mathcal{U}_k$ is open in $\mathbb{CP}^n$.

We can define a map $\psi_k : \mathcal{U}_k \longrightarrow \mathbb{C}^n$ by

$$\psi_k\left([z_0 : \cdots : z_{k-1} : z_k : z_{k+1} : \cdots : z_n]\right) = \left(\frac{z_0}{z_k}, \ldots, \frac{z_{k-1}}{z_k}, \frac{z_{k+1}}{z_k}, \ldots, \frac{z_n}{z_k}\right).$$

Let us write two maps $\psi_i : \mathcal{U}_i \longrightarrow \mathbb{C}^n$ and $\psi_j : \mathcal{U}_j \longrightarrow \mathbb{C}^n$ as

$$\psi_i\left([z_0 : \cdots : z_{i-1} : z_i : z_{i+1} : \cdots : z_n]\right) \longmapsto \left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i}\right) = (v_1, \ldots, v_n)$$

and

$$\psi_j\left([z_0 : \cdots : z_{j-1} : z_j : z_{j+1} : \cdots : z_n]\right) \longmapsto \left(\frac{z_0}{z_j}, \ldots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \ldots, \frac{z_n}{z_j}\right) = (w_1, \ldots, w_n)$$

for $0 \leq i < j \leq n$.

The transition function $\psi_{ij} = \psi_j \circ \psi_i^{-1}$ maps $\{(v_1, \ldots, v_n) \in \mathbb{C}^n : v_i \neq 0\}$ to $\{(w_1, \ldots, w_n) \in \mathbb{C}^n : w_{j+1} \neq 0\}$ by

$$\begin{aligned}
\psi_j \circ \psi_i^{-1}(v_1, \ldots, v_n) &= (w_1, \ldots, w_n) \\
&= \left(\tfrac{z_0/z_i}{z_j/z_i}, \ldots, \tfrac{z_{i-1}}{z_j}, \tfrac{z_i}{z_j}, \tfrac{z_{i+1}}{z_j}, \ldots, \tfrac{z_{j-1}}{z_j}, \tfrac{z_{j+1}}{z_j}, \ldots, \tfrac{z_n}{z_j}\right) \\
&= \left(\tfrac{v_1}{v_j}, \ldots, \tfrac{v_i}{v_j}, \tfrac{1}{v_j}, \tfrac{v_{i+1}}{v_j}, \ldots, \tfrac{v_{j-1}}{v_j}, \tfrac{v_{j+1}}{v_j}, \ldots, \tfrac{v_n}{v_j}\right).
\end{aligned}$$

This is a biholomorphic mapping. So $(\mathcal{U}_i, \psi_i)$, $(\mathcal{U}_j, \psi_j)$ are compatible and $\{(\mathcal{U}_k, \psi_k) \mid k = 0, \ldots, n\}$ is a complex chart on $\mathbb{CP}^n$. The charts $(\mathcal{U}_k, \psi_k)$ for $k = 0, 1, \ldots, n$ form a holomorphic atlas for $\mathbb{CP}^n$ and thus $\mathbb{CP}^n$ is a complex manifold.

(iii) *Let $n > 1$, and $a \in \mathbb{C}^*$, $0 < |a| < 1$. Let $\mathbb{Z}$ act on $\mathbb{C}^n \backslash \{0\}$ by holomorphic transformations $(z_1, \ldots, z_n) \mapsto (a^k z_1, \ldots, a^k z_n)$ with $k \in \mathbb{Z}$. Then the quotient $(\mathbb{C}^n \backslash 0)/\mathbb{Z}$ is diffeomorphic to $S^1 \times S^{2n-1}$. Hence $S^1 \times S^{2n-1}$ is a complex manifold. This example is due to Hopf and is named the Hopf manifold after him. Calabi and Eckmann [8] extended this result to show that the product of odd dimensional spheres are complex manifolds.*

We now explain an alternative, differential geometric way to define complex manifolds. We no longer deal with holomorphic coordinates and charts, but prefer to think of vector fields on a manifold, tensors and tangent spaces. This new point of view, widens the range of manifolds that can be studied using geometric methods to those manifolds that do not fit into Definition 1.1. We will see that, under certain conditions, given by Theorem 1.7, both definitions are equivalent.

Definition 1.3. *Let V be a vector space. A **complex structure** on V is a linear map:*

$$J : V \longrightarrow V \quad \text{with } J^2 = -id.$$

Definition 1.4. *If M is a real $2n$-dimensional manifold, and we can define a smooth field of complex structures as given in Definition 1.3 on the tangent bundles $J : T_pM \to T_pM$ where $J^2 = -id_{T_pM}$ and we make the underlying vector space into a complex vector space by setting $(a + ib) \cdot v = a \cdot v + b \cdot J(v)$, for $a, b \in \mathbb{R}$ and $v \in TM$, then J is an almost complex structure on the manifold and (M, J) is an almost complex manifold.*

A complex manifold M as given in Definition 1.1 will also admit further structure on its tangent bundle. Let M be a real $2n$-dimensional manifold. If we complexify the tangent bundle TM (isomorphic to $\mathbb{R}^{2n}$) we obtain the bundle

$$\begin{array}{c} TM \otimes_{\mathbb{R}} \mathbb{C} \\ \downarrow \\ M \end{array}$$

with fiber $(TM \otimes \mathbb{C})_p = T_pM \otimes \mathbb{C}$ at $p \in M$. The complex vector space $T_pM \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $\mathbb{C}^{2n}$.

Proposition 1.5. *If M has an almost complex structure J, then J induces a splitting of the complexified tangent and cotangent bundles $TM \otimes_{\mathbb{R}} \mathbb{C}$ and $T^*M \otimes_{\mathbb{R}} \mathbb{C}$ such that we can decompose the complex tangent bundle as*

$$T_pM \otimes_{\mathbb{R}} \mathbb{C} \cong T_p^{1,0}M \oplus T_p^{0,1}M,$$

where $T^{1,0}M = \{v \in TM \otimes \mathbb{C} \mid Jv = iv\}$ and $T^{0,1}M = \{v \in TM \otimes \mathbb{C} \mid Jv = -iv\}$ are eigenspaces for the eigenvalues i and $-i$ of J respectively.

Definition 1.6. *Suppose M is a $2n$-dimensional manifold and J an almost complex structure on M. Let $f : M \to \mathbb{C}$ be a smooth function and write $f = u + iv$. We call f a **holomorphic function** or J-holomorphic if $du = Jdv$, i.e., $df \circ J = idf$.*

Note that the splitting of the complexified tangent bundles of a manifold M produced by the almost complex structure J does not guarantee the existence of an atlas of complex charts $(\mathcal{U}, \psi)$, such that all the transition functions as in Definition 1.1 are holomorphic. Indeed the conditions set in Definition 1.6 will only be satisfied under certain quite restrictive conditions.

Theorem 1.7 (Newlander-Nirenberg)**.** *The almost complex structure J gives each tangent space T_pM the structure of a complex vector space. A necessary and sufficient condition for there to exist a holomorphic chart around each point of M, is the vanishing of the* ***Nijenhuis tensor*** *N_J of J, where we write $N_J(v, w) = [v, w] + J([Jv, w] + [v, Jw]) - [Jv, Jw]$ for all smooth vector fields v, w on M.*

The Nijenhuis tensor N_J represents an obstruction to the existence of holomorphic functions on M (see Joyce 5.1, [23]). The equations that a function f must satisfy in order to be holomorphic on M, as given in Definition 1.6, are called the Cauchy-Riemann equations. For $n = 1$, the manifold always admits holomorphic coordinates, but for $n > 1$ the Cauchy-Riemann equations are overdetermined, and N_J is an obstruction to the existence of holomorphic functions on M. If $N_J \neq 0$, then there can still exist some holomorphic functions on M, but not enough to construct holomorphic coordinates. Thus, an almost complex manifold only admits many, i.e., enough, holomorphic functions if the Nijenhuis tensor N_J vanishes.

The Newlander-Nirenberg Theorem 1.7 implies a new definition of **complex manifold** as a $2n$-dimensional manifold with an almost complex structure J such that $N_J \equiv 0$. This definition is equivalent to Definition 1.1.

If we can define a structure J on a real $2n$-dimensional manifold M such that the Nijenhuis tensor N_J vanishes, then the Newlander-Nirenberg Theorem 1.7 ensures that there exist locally holomorphic coordinates $(z_1, \ldots, z_n)$ on M so that M is a topological space X with an atlas of charts $(\mathcal{U}_i, \psi_i)$ defined on it such that the transition functions ψ_{ij} are holomorphic as defined in Definition 1.1. In this case, we define the complex manifold (M, J) as an almost complex manifold with integrable complex structure J, i.e, $N_J = 0$. This integrability condition is also given by the fact that the Lie bracket of two $(1, 0)$ vector fields described in Proposition 1.5 should be also of type $(1, 0)$.

1.2 Compatible Structures

We have discussed two different definitions of complex manifold. The first definition depends on the possibility of defining holomorphic functions on a manifold, while the second relies on the integrability of the almost complex structure J defined on a manifold M, which ensures a vanishing tensor N_J. Thus a more general idea of **almost complex manifold** follows, where none of the almost complex structures on the manifold M are integrable and we therefore no longer have holomorphic functions on M. In this section we define further structures on manifolds, and we explain how they are compatible with one another.

1.2.1 Metrics

A metric on a manifold is given by a tensor that assigns to each point $p \in M$ an inner product on the tangent space T_pM. That is, a metric g on a manifold M determines an inner product on each tangent space T_pM, so that $\langle X, Y \rangle := g(X, Y)$ for $X, Y \in T_pM$. Depending on how we define $g(X, Y)$ we can define different metrics on M.

Definition 1.8. *A **Riemannian metric** g on M determines an inner product on each tangent space T_pM defined as $\langle X, Y \rangle := g(X, Y)$ for $X, Y \in T_pM$ depending smoothly on $p \in M$ such that $g(X, Y)$ is a positive definite bilinear symmmetric differential form on M. A **Riemannian manifold** (M, g) is a differentiable manifold, equipped with a Riemannian metric.*

We now analyze the conditions under which a Riemannian metric is compatible with an almost complex structure on a manifold (M, J).

Definition 1.9. *If a Riemannian metric g on an almost complex manifold (M, J) is compatible with the complex structure J on (M, J) so that $\langle JX, JY \rangle = \langle X, Y \rangle$, that is, $g(X, Y) = g(JX, JY)$ for all $p \in M$ and $X, Y \in T_pM$, then g is called a **Hermitian metric** on (M, J).*

Proposition 1.10. *An almost complex manifold M can always be equipped with a Hermitian metric.*

Proof. If g is any Riemannian metric, then

$$h(X, Y) = \frac{1}{2}\left(g(X, Y) + g(JX, JY)\right)$$

is a Hermitian metric. □

1.2.2 Differential Forms

We have approached complex and almost complex manifolds in terms of the almost complex structure $J : T_pM \to T_pM$, the role of its integrability as explained in Theorem 1.7, and the possibility of defining a tensor g, a metric, that is compatible with J. Now we analyze manifolds by considering the space of differential forms on their tangent bundles. Important references for the material discussed here are [7, 9, 15, 23, 36].

de Rham Cohomology

There are several different ways to define the cohomology of topological spaces, for instance, singular cohomology, Čech cohomology, Alexander-Spanier cohomology. If the topological space is sufficiently nice (for instance paracompact and Hausdorff), then the corresponding cohomology groups are all isomorphic. Here we study the de Rham cohomology of a smooth n-manifold M.

Definition 1.11. *Let $C^\infty\left(\Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C}\right)$ denote the space of smooth complex differential forms of degree k on M, then we can write the exterior differential* d *as*

$$\mathrm{d} : C^\infty\left(\Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C}\right) \longrightarrow C^\infty\left(\Lambda^{k+1} T^*M \otimes_{\mathbb{R}} \mathbb{C}\right), \tag{1.1}$$

where T^*M is the cotangent bundle, $\Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C}$ is a complex vector bundle over M and $C^\infty\left(\Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C}\right)$ is the space of smooth sections of $\Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C}$. The exterior differential d *satisfies the Leibniz rule and* $\mathrm{d} \circ \mathrm{d} = 0$.

Since $\mathrm{d} \circ \mathrm{d} = 0$, the chain of operators

$$0 \xrightarrow{\mathrm{d}} C^\infty\left(\Lambda^0 T^*M \otimes_{\mathbb{R}} \mathbb{C}\right) \xrightarrow{\mathrm{d}} C^\infty\left(\Lambda^1 T^*M \otimes_{\mathbb{R}} \mathbb{C}\right) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} C^\infty\left(\Lambda^n T^*M \otimes_{\mathbb{R}} \mathbb{C}\right) \xrightarrow{\mathrm{d}} 0 \tag{1.2}$$

forms a complex and we may find the corresponding cohomology groups.

Definition 1.12. *The kernel of* d *are the closed forms and the image of* d *are the exact forms.*

Definition 1.13 (de Rham Cohomology Groups)**.** *For $k = 0, \ldots, n$ we define the* ***kth de Rham cohomology group*** *of M by*

$$H^k_{dR}(M; \mathbb{C}) = \frac{\mathrm{Ker}(\mathrm{d} : C^\infty\left(\Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C}\right) \longrightarrow C^\infty\left(\Lambda^{k+1} T^*M \otimes_{\mathbb{R}} \mathbb{C}\right))}{\mathrm{Im}(\mathrm{d} : C^\infty\left(\Lambda^{k-1} T^*M \otimes_{\mathbb{R}} \mathbb{C}\right) \longrightarrow C^\infty\left(\Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C}\right))}.$$

Remark 1.14. *The de Rham cohomology of a smooth manifold M, $H^k_{dR}(M;\mathbb{C})$ is isomorphic to the cohomology $H^k(M;\mathbb{C})$ of M as topological space.*

If we endow an oriented n-manifold M with a Riemannian metric g, then we obtain a *volume form* dV_g on M, which can be used to integrate functions on M.

The Hodge star $*$ is an isomorphism of vector bundles $*: \Lambda^k T^*M \to \Lambda^{n-k}T^*M$, which is defined as follows. For α and β k-forms, $*\beta$ is the unique $(n-k)$-form that satisfies the equation $\alpha \wedge (*\beta) = (\alpha, \beta) dV_g$ for all k-forms α on M.

Definition 1.15. *Let (M, g) be a Riemannian manifold, then we can define the corresponding adjoint differential operator* d^* *where*

$$\mathrm{d}^* : C^\infty\left(\Lambda^{k+1}T^*M \otimes_{\mathbb{R}} \mathbb{C}\right) \longrightarrow C^\infty\left(\Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C}\right) \tag{1.3}$$

by

$$\mathrm{d}^*\alpha = (-1)^{kn+n+1} * \mathrm{d}(*\alpha).$$

Let α be a k-form and β be a (k+1)-form. Then $\langle \mathrm{d}\alpha, \beta\rangle_{L^2} = \int_M (\mathrm{d}\alpha, \beta) dV_g = \int_M \mathrm{d}\alpha \wedge *\beta dV_g = \int_M \mathrm{d}\alpha \wedge *\beta - \mathrm{d}(\alpha \wedge *\beta) = \int_M \mathrm{d}\alpha \wedge *\beta - \mathrm{d}\alpha \wedge *\beta \pm \alpha \wedge \mathrm{d}(*\beta) = \int_M (\alpha, *\mathrm{d}(\alpha\beta)) dV_g = \langle \alpha, \mathrm{d}^*\beta\rangle_{L^2}$.

Thus d^* has the formal properties of the adjoint of d. As $\mathrm{d}^2 = 0$ we find that $(\mathrm{d}^*)^2 = 0$, so that the corresponding chain of operators form a complex, similar to the expression given in Equation (1.2), of which we can compute the cohomology groups as

$$H^k_{dR}(M, \mathbb{R}) \cong \frac{\operatorname{Ker}\left(\mathrm{d}^* : C^\infty\left(\Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C}\right) \longrightarrow C^\infty\left(\Lambda^{k-1}T^*M \otimes_{\mathbb{R}} \mathbb{C}\right)\right)}{\operatorname{Im}\left(\mathrm{d}^* : C^\infty\left(\Lambda^{k+1}T^*M \otimes_{\mathbb{R}} \mathbb{C}\right) \longrightarrow C^\infty\left(\Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C}\right)\right)}.$$

Theorem 1.16. *For (M, g) a compact Riemannian manifold, we write the* ***Laplacian*** *as*

$$\Delta_{\mathrm{d}} = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d} \tag{1.4}$$

and we have

$$\operatorname{Ker}(\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}) = \mathcal{H}^k$$

where the elements of $\mathcal{H}^k$ are called the harmonic k-forms.

Lemma 1.17. *For (M, g) a compact manifold, we have $\Delta\alpha = 0 \iff \mathrm{d}\alpha = 0$ and $\mathrm{d}^*\alpha = 0$.*

Corollary 1.18. *On a compact Riemannian manifold, every harmonic function is constant.*

Theorem 1.19 (Hodge's Theorem)**.** *Let (M, g) be a compact, oriented Riemanninan manifold. Then every de Rham cohomology class on M contains a unique harmonic representative, and $\mathcal{H}^k \cong H^k_{dR}(M, \mathbb{R})$.*

Dolbeault Cohomology

Given an almost complex structure J on a manifold M, the decomposition of the complex tangent bundle as $TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$ given in Proposition 1.5 induces a similar decomposition on the bundles of complex differential forms:

$$\left(\Lambda^k T^* M\right) \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{\substack{p,q \\ p+q=k}} \Lambda^{p,q} M,$$

where $\Lambda^{p,q}M$ is the bundle $\Lambda^p {T^*}^{1,0} M \otimes_{\mathbb{C}} \Lambda^q {T^*}^{0,1} M$. A section of $\Lambda^{p,q}M$ is called a (p,q)-form which is a complex-valued differential form which can be expressed in local holomorphic coordinates $(z_1, \dots, z_n)$ as

$$\sum_{\substack{1 \le i_1 < i_2 < \dots < i_p \le n \\ 1 \le j_1 < j_2 < \dots < j_q \le n}} f_{i_1 \dots i_p, j_1 \cdots j_q} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}.$$

The exterior differential operator d in Equation (1.1) splits informally as

$$\mathrm{d}\alpha^{p,q} = (N_J \cdot \alpha)^{p+2,q-1} + \partial \alpha^{p+1,q} + \bar{\partial}\alpha^{p,q+1} + (N_J \cdot \alpha)^{p+2,q-1}, \tag{1.5}$$

where ∂ and $\bar{\partial}$ are operators such that

$$\partial : C^\infty\left(\Lambda^{p,q}M\right) \longrightarrow C^\infty\left(\Lambda^{p+1,q}M\right) \tag{1.6}$$

$$\bar{\partial} : C^\infty\left(\Lambda^{p,q}M\right) \longrightarrow C^\infty\left(\Lambda^{p,q+1}M\right), \tag{1.7}$$

and N_J is the Nijenhuis tensor as given in Theorem 1.7.

If the almost complex structure J defined on the manifold M is integrable, i.e, M is a complex manifold and $N_J \equiv 0$, then we can rewrite Equation (1.5) as

$$\mathrm{d}\alpha^{p,q} = \partial\alpha^{p+1,q} + \bar{\partial}\alpha^{p,q+1}, \tag{1.8}$$

which we rewrite more easily as

$$\mathrm{d} = \partial + \bar{\partial}. \tag{1.9}$$

Proposition 1.20. *If M is a complex manifold then $\partial \circ \partial = 0$, $\bar{\partial} \circ \bar{\partial} = 0$ and $\partial \circ \bar{\partial} + \bar{\partial} \circ \partial = 0$.*

Proof. We have $\mathrm{d}^2 = 0$ and since $\mathrm{d} = \partial + \bar{\partial}$ we can write

$$\mathrm{d}^2 = (\partial + \bar{\partial})(\partial + \bar{\partial}) = \partial^2 + [\partial \circ \bar{\partial} + \bar{\partial} \circ \partial] + \bar{\partial}^2 = 0.$$

We have that ∂^2 maps (p,q)-forms to $(p+2,q)$-forms, $\partial\circ\bar{\partial}+\bar{\partial}\circ\partial$ maps (p,q)-forms to $(p+1,q+1)$-forms and $\bar{\partial}^2$ maps (p,q)-forms to $(p,q+2)$-forms respectively. Thus each component vanishes separately, and we have

$$\partial^2=\partial\circ\bar{\partial}+\bar{\partial}\circ\partial=\bar{\partial}^2=0.$$

□

Since $\bar{\partial}^2=0$ for each $p=0,\dots,n$, the chain of operators

$$0\xrightarrow{\bar{\partial}}C^\infty\left(\Lambda^{p,0}M\right)\xrightarrow{\bar{\partial}}C^\infty\left(\Lambda^{p,1}M\right)\xrightarrow{\bar{\partial}}\cdots\xrightarrow{\bar{\partial}}C^\infty\left(\Lambda^{p,n}M\right)\tag{1.10}$$

forms a complex and we may find the corresponding cohomology groups.

Definition 1.21 (Dolbeault Cohomology Groups)**.** *For $p,q=0,\dots,n$ we define the Dolbeault cohomology groups of M by*

$$H^{p,q}_{\bar{\partial}}(M;\mathbb{C})=\frac{\operatorname{Ker}\left(\bar{\partial}:C^\infty\left(\Lambda^{p,q}M\right)\longrightarrow C^\infty\left(\Lambda^{p,q+1}M\right)\right)}{\operatorname{Im}\left(\bar{\partial}:C^\infty\left(\Lambda^{p-1,q}M\right)\longrightarrow C^\infty\left(\Lambda^{p,q}M\right)\right)}.$$

If we impose the extra condition given in Definition 1.15 that the complex manifold (M,J) carries a Hermitian metric g, then the coadjoint operator to d given by d^* as defined in Equation (1.3) splits similarly as

$$\mathrm{d}^*=\partial^*+\bar{\partial}^*,\tag{1.11}$$

where we define formal adjoints ∂^* and $\bar{\partial}^*$ in a similar way as ∂ and $\bar{\partial}$ in Equations (1.6) and (1.7) by

$$\partial^*:C^\infty\left(\Lambda^{p+1,q}M\right)\longrightarrow C^\infty\left(\Lambda^{p,q}M\right)\quad\text{and}\quad\bar{\partial}^*:C^\infty\left(\Lambda^{p,q+1}M\right)\longrightarrow C^\infty\left(\Lambda^{p,q}M\right).$$

The corresponding Laplacian operators are

$$\Delta_\partial=\partial\partial^*+\partial^*\partial\tag{1.12}$$

and

$$\Delta_{\bar{\partial}}=\bar{\partial}\bar{\partial}^*+\bar{\partial}^*\bar{\partial}.\tag{1.13}$$

Compatibility

Since any almost complex manifold M can always be equipped with a Hermitian metric as we saw in Definition 1.9, we have that to each Hermitian metric g we can always associate a 2-form ω. The 2-form ω and the complex structure J are compatible if $\omega(JX, JY) = \omega(X, Y)$ and $\omega(X, JX) > 0$ for $X, Y \in T_pM$ at each point $p \in M$ and $X \neq 0$.

Lemma 1.22. *ω, g and J are compatible if and only if $g(X, Y) = \omega(X, JY)$ where g is a Riemannian metric. In this case, any two of g, J or ω determine the third so*

$$\omega(X, Y) = g(X, JY) \quad \text{and} \quad J(X) = \tilde{g}^{-1}(\tilde{\omega}(X))$$

where

$$\tilde{\omega} : TM \longrightarrow T^*M, \quad \tilde{g} : TM \longrightarrow T^*M$$

are the linear isomorphisms induced by the bilinear forms ω and g.

1.2.3 Connections and Curvature

Let (M, g) be a manifold and $E \to M$ a vector bundle on M. Then a **connection** on E is a map

$$\nabla : C^\infty(E) \longrightarrow C^\infty(E \otimes T^*M), \tag{1.14}$$

such that the Leibniz rule given by

$$\nabla(fe) = e \otimes df + f\nabla e,$$

holds for $e \in C^\infty(E)$ where $C^\infty(E)$ is the space of smooth sections of E, T^*M is the cotangent bundle of M, and f is a smooth function $f : M \to \mathbb{R}$.

Theorem 1.23 (Fundamental Theorem of Riemannian Geometry)**.** *Let (M, g) be a Riemannian manifold. Then (M, g) always has a unique preferred connection ∇ on TM called the Levi-Civita connection which satisfies $\nabla g = 0$, and is torsion-free, i.e., the torsion tensor $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ vanishes and we have $\nabla_X Y - \nabla_Y X = [X, Y]$ for X and Y vector fields.*

The **curvature** of a connection measures how close a connection is to being flat. The curvature of the Levi-Civita connection, is denoted by $R(\nabla)$ and is a tensor R^a_{bcd}.

- The Ricci curvature of a Riemannian metric g is $R_{ab} = R^c_{acb}$.
- The scalar curvature of g is $s = g^{ab}R_{ab}$.

Definition 1.24. *A Riemannian metric g is Ricci-flat if $R_{ab} = 0$.*

1.3 Kähler Manifolds

Definition 1.25. *Let (M, J) be a complex manifold. A Hermitian metric g on M is called Kähler if $\mathrm{d}\omega = 0$, where $\omega(X, Y) = g(X, JY)$ is the associated 2-form. The closed 2-form ω is called the Kähler form of g. Then, if g is a Kähler metric on a complex manifold (M, J, g) with Kähler form ω, (M, J, g) is a Kähler manifold. The condition on the 2-form ω given by $\mathrm{d}\omega = 0$ is called the Kähler condition.*

Proposition 1.26. *Let (M, J, g) be a complex manifold with Hermitian metric g. Then the following conditions are equivalent:*

(i) $\nabla J = 0$,

(ii) $\nabla \omega = 0$,

(iii) $\mathrm{d}\omega = 0$,

where g is a Hermitian metric on M with respect to J, ω is the corresponding Hermitian form and ∇ is the Levi-Civita connection.

Proof sketch. Since ω is a 2-form, $\mathrm{d}\omega$ is a 3-form and hence locally determined by $\binom{2n}{3}$ functions. We also find that $\nabla\omega \in C^\infty\left(\Lambda^2 T^*M \otimes T^*M\right)$ is locally determined by $\binom{2n}{2}\cdot 2n$ functions. Hence we deduce that $\nabla\omega$ lies in a bigger vector space than $d\omega$. In fact, $\nabla\omega \cong \mathrm{d}\omega \oplus N_J$, where N_J is the Nijenhuis tensor. Now since we have defined (M, g, J) to have an integrable complex structure, then $N_J = 0$ so that $\nabla\omega = d\omega$. So $\mathrm{d}\omega = 0$ iff $\nabla\omega = 0$, i.e., (ii) and (iii) are equivalent. (i) and (ii) are equivalent as $\nabla g = 0$. $\square$

We now give examples of how to construct a Kähler metric on a complex manifold. Note that a Kähler metric is a Riemannian metric on a complex manifold that is compatible with J is a natural way.

Example 1.27 (Complex Manifold with Kähler Metric)**.** *The Euclidean metric g on $\mathbb{C}^n$ is given by*

$$\begin{aligned} g &= \tfrac{1}{2}\textstyle\sum_{k=1}^n (dz_1 \otimes d\bar{z}_k + d\bar{z}_k \otimes dz_k) = \sum_{k=1}^n (dx_k^2 + dy_k^2) \\ \omega &= \tfrac{i}{2}\textstyle\sum_{k=1}^n dz_k \wedge d\bar{z}_k = \sum_{k=1}^n dx_k \wedge dy_k. \end{aligned}$$

The 2-form $\omega = \frac{i}{2}\sum dz_i \wedge d\bar{z}_i$ is the Kähler form of the Kähler metric g on $\mathbb{C}^n$.

Example 1.28 (Fubini-Study metric)**.** *The complex projective space* $\mathbb{CP}^n$ *can be made into a Kähler manifold by defining an appropriate metric. This metric is called the* ***Fubini-Study metric****.*

Firstly, we define the projection

$$\begin{array}{rccc} \pi : & \mathbb{C}^{n+1}\backslash\{0\} & \longrightarrow & \mathbb{CP}^n \\ & (z_0, \dots, z_n) & \longmapsto & [z_0 : z_1 : \cdots : z_n]. \end{array}$$

We can show there exists a unique 2-form ω_{FS} *on* $\mathbb{CP}^n$ *such that*

$$\begin{array}{rcl} \pi^*(\omega_{FS}) & = & i\partial\bar{\partial}\log|Z|^2 \\ & = & i\sum_{j,k=1}^n \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log|Z|^2 dz_j d\bar{z}_k. \end{array}$$

where $|Z|^2 = |z_0|^2 + \cdots + |z_n|^2$. *Then* ω_{FS} *is the Kähler form of a Kähler metric* g_{FS} *on* $\mathbb{CP}^n$ *called the Fubini-Study metric.*

It is important to remark that the Kähler metric is inherited by the complex submanifolds of a Kähler manifold.

Proposition 1.29. *If* S *is a complex submanifold of a Kähler manifold* M*, then the restriction of* g *to* S *is also Kähler. Any complex submanifold of a Kähler manifold is a Kähler manifold.*

Proof. If $S \subset M$ is a manifold and ω is the associated $(1,1)$-form of a Kähler metric on M, we find that the associated $(1,1)$-form of the induced metric on S is the restriction to S of ω. Thus if M is Kähler then S is Kähler. □

Remark 1.30. *The condition given in Proposition 1.29 is very strong since we require the submanifold to be complex.*

As we have seen, the conditions for a manifold to admit a Kähler form are quite restrictive. Indeed, we need the complex structure to be integrable and furthermore, the 2-form ω has to be non-degenerate and closed, i.e. $\mathrm{d}\omega = 0$. These conditions provide the following results which follow from the results in Subsection 1.2.2 but are not true in general for non-Kähler manifolds.

Definition 1.31 (Lefschetz operators)**.** *For* (M, J, ω) *a Kähler manifold we can define two further operators called the* ***Lefschetz operator*** L *and the* ***dual Lefschetz operator*** $\bigwedge$ *by*

$$L : C^\infty\left(\Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C}\right) \longrightarrow C^\infty\left(\Lambda^{k+2} T^*M \otimes_{\mathbb{R}} \mathbb{C}\right) \tag{1.15}$$

and

$$\bigwedge : C^\infty\left(\Lambda^{k+2} T^*M \otimes_{\mathbb{R}} \mathbb{C}\right) \longrightarrow C^\infty\left(\Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C}\right) \tag{1.16}$$

by $L(\alpha) = \alpha \wedge \omega, \quad \text{and } \bigwedge = L^*.$

The operators d, d^*, ∂, ∂^*, $\bar{\partial}$, $\bar{\partial}^*$, L and $\bigwedge$ are related by the *Kähler identities* as given in [36, Chapter 6] and are essential for developing Hodge Theory for Kähler manifolds.

Proposition 1.32. *The operators* d, d^*, ∂, ∂^*, $\bar{\partial}$, $\bar{\partial}^*$, L *and* $\bigwedge$ *satisfy the identities*

$$[\bigwedge, \bar{\partial}] = -i\partial^*, \quad [\bigwedge, \partial] = i\bar{\partial}^*. \tag{1.17}$$

These identities are used in proving the following results, also a Kähler identity, that will be important to study the cohomology of Kähler manifolds.

Proposition 1.33. *For (M, J, ω) a Kähler manifold, the Laplacian splits as*

$$\Delta_{\mathrm{d}} = \Delta_{\partial} + \Delta_{\bar{\partial}}.$$

Proof. Using the definitions of the coadjoint operator d^* as given in Definition 1.15 and Equations (1.9) and (1.11) we find

$$\mathrm{dd}^* + \mathrm{d}^*\mathrm{d} = (\partial + \bar{\partial})(\partial + \bar{\partial})^* + (\partial + \bar{\partial})^*(\partial + \bar{\partial}) = (\partial\partial^* + \partial^*\partial) + (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}),$$

since by Equation 1.17 we have that $\bar{\partial}\partial^* = -\partial^*\bar{\partial}$ and $\partial\bar{\partial}^* = -\bar{\partial}^*\partial$. Then we write,

$$\Delta_{\mathrm{d}} = \Delta_{\partial} + \Delta_{\bar{\partial}}.$$

where Δ_{d}, Δ_{∂} and $\Delta_{\bar{\partial}}$ are Laplacians associated to the operators d, ∂ and $\bar{\partial}$ as given in Equations (1.4), (1.12) and (1.13) respectively. □

Proposition 1.34. *For (M, J, ω) a Kähler manifold we find the following relations between Laplacians,*

$$\Delta_{\partial} = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta_{\mathrm{d}}.$$

Proof. For a proof see Voisin [36, Theorem 6.7, p.141]. □

Theorem 1.35 (Hard Lefschetz Condition)**.** *The linear map*

$$L_{[\omega]^r} : H^{n-r}(M; \mathbb{C}) \longrightarrow H^{n+r}(M; \mathbb{C}), \quad L_{[\omega]^r}([x]) = [\omega]^r \cup [x]$$

is an isomorphism. In this case we call ω a Lefschetz element.

Theorem 1.36 (Hodge Decomposition)**.** *Let (M, J, ω) be a compact Kähler manifold. Every Dolbeault cohomology class on (M, J, ω) has a unique harmonic representative, i.e.,*

$$\mathcal{H}^{p,q} \cong \mathcal{H}^{p,q}_{\bar{\partial}}(M; \mathbb{C})$$

and the spaces $\mathcal{H}^{p,q}$ are finite-dimensional. Hence, we have the following isomorphism:

$$H^k_{dR}(M; \mathbb{C}) \cong \mathcal{H}^k \cong \bigoplus_{\substack{p,q \\ p+q=k}} \mathcal{H}^{p,q} \cong \bigoplus_{\substack{p,q \\ p+q=k}} H^{p,q}_{\bar{\partial}}(M; \mathbb{C})$$

where $H^k_{dR}(M; \mathbb{C})$ are the de Rham cohomology groups as given in Definition 1.13 and $H^{p,q}_{\bar{\partial}}(M; \mathbb{C})$ are the Dolbeault cohomology groups as given in Definition 1.21 and the isormorphism

$$H^k_{dR}(M; \mathbb{C}) \cong \bigoplus_{\substack{p,q \\ p+q=k}} H^{p,q}_{\bar{\partial}}(M; \mathbb{C})$$

is called the ***Hodge decomposition****.*

As a consequence of the Hodge Decomposition Theorem 1.36 we find the following properties of the Betti numbers of Kähler manifolds.

Proposition 1.37. *Let M be a compact complex manifold on which there exists a Kähler metric. Then*

- *Even Betti numbers $\beta_{2q}(M)$ are positive*
- *Odd Betti numbers $\beta_{2q+1}(M)$ are even*

Proposition 1.38 (Gompf [13])**.** *A closed complex surface S is Kähler iff $b_1(S)$ is even. Every simply connected compact complex surface is Kähler.*

The restrictive conditions that guarantee that a manifold is Kähler, imply that there are many non-Kähler complex manifolds.

Example 1.39 (Complex Manifold without Kähler Metric)**.** *We have shown before in Example 1.2 that the $2n$-dimensional manifold $M = S^{2n-1} \times S^1$ for $n > 1$ is a complex manifold. Now $[\omega] \in H^2(M)$, if there existed a Kähler metric with Kähler form ω then $n! vol(M) = \int_M \omega^n > 0$. But $\int_M \omega^n = [\omega]^n \cdot [M] = 0$ as $[\omega] \in H^2(S^{2n-1} \times S^1) = 0$. Thus, by Proposition 1.37 M is not a Kähler manifold. Indeed, by Künneth Theorem the Betti numbers of M are $b^k(M) = 1$ for $k = 0, 1, 2n-1, 2n$ and $b^k(M) = 0$ otherwise.*

The Kähler Cone

Let (M, J) be a fixed compact complex manifold. We find

$$H^2(M;\mathbb{R}) \subset H^2(M;\mathbb{C})$$

and

$$H^2(M;\mathbb{R}) \cap H^{1,1}(M,\mathbb{C}) \subset H^2(M;\mathbb{C})$$

if M admits a Kähler metric.

If ω is a Kähler form on (M, J), then $[\omega] \in H^2(M;\mathbb{R}) \cap H^{1,1}(M;\mathbb{C})$ and we define

$$\mathcal{K} = \{[\omega] : \omega \text{ is a Kähler form}\}$$

to be the Kähler cone.

Proposition 1.40. *$\mathcal{K}$ is an open, convex cone in $H^2(M;\mathbb{R}) \cap H^{1,1}(M;\mathbb{C})$.*

Proof. Let ω be a Kähler form and η any closed real $(1,1)$-form. Then $\omega + \epsilon\eta$ is a Kähler form if ϵ is small enough $\left(\epsilon \cdot \frac{1}{2}||\eta||_{C^0} < \frac{1}{2}\right)$. So $\mathcal{K}$ is **open** in $H^2(M;\mathbb{R}) \cap H^{1,1}(M;\mathbb{C})$. Now, let $[\omega_1], [\omega_2] \in \mathcal{K}$, then $t[\omega_1] + (1-t)[\omega_2] \in \mathcal{K}$ for all $t \in [0,1]$. This follows as $t\omega_1 + (1-t)\omega_2$ is also a Kähler form. So $\mathcal{K}$ is **convex**. Since $t\omega$ is also Kähler for all $t > 0$, $\mathcal{K}$ is a **cone**. $\square$

The Kähler cone can sometimes be described explicitly.

Example 1.41. *Let $(M, \omega) = \mathbb{CP}^1 \times \mathbb{CP}^1$. Then the corresponding Kähler cone is given by $\mathcal{K} = \{(x,y) : x, y > 0\} \subset \mathbb{R}^2 = H^2(\mathbb{CP}^1 \times \mathbb{CP}^1;\mathbb{R})$.*

1.3.1 Complex Projective Varieties

We have seen that $\mathbb{CP}^n$ admits a Kähler metric, the Fubini-Study metric. So a very important type of Kähler manifolds is given by complex submanifolds of $\mathbb{CP}^n$. Moreover, a compact complex manifold is called *projective* if it is isomorphic to a complex submanifold of $\mathbb{CP}^n$, for some n.

Proposition 1.42. *Since $\mathbb{CP}^n$ is a Kähler manifold, any complex submanifold of $\mathbb{CP}^n$ will also be Kähler. These submanifolds of $\mathbb{CP}^n$ are defined as complex algebraic varieties. Any compact complex manifold that can be embedded in $\mathbb{CP}^n$ is Kähler.*

Theorem 1.43 (Chow)**.** *All compact complex submanifolds of $\mathbb{CP}^n$ are non-singular algebraic varieties. i.e., defined by zeros of polynomials.*

Example 1.44 (Non algebraic submanifolds of $\mathbb{CP}^2$)**.** *Let $M = \{[1, z, \exp(z)] : z \in \mathbb{C}\} \subset \mathbb{CP}^2$. M is a non-closed complex submanifold of $\mathbb{CP}^2$ and it is not algebraic since it is not locally defined by zeros of polynomials. Note that* exp *is a transcendental function.*

Example 1.45 (Rational Normal Curve of degree 3, Twisted cubic)**.** *We define a holomorphic map*

$$\begin{array}{llcl} f: & \mathbb{CP}^1 & \longrightarrow & \mathbb{CP}^3, \\ f: & [x:y] & \longmapsto & [x^3 : x^2y : xy^2 : y^3]. \end{array}$$

The image of $\mathbb{CP}^1$ under f, that we denote by M, is a complex submanifold of $\mathbb{CP}^3$ and is isomorphic as a complex manifold to $\mathbb{CP}^1$.

We can define M as a subset of $\mathbb{CP}^3$ as

$$M = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \ : \ z_0z_3 - z_1z_2 = 0, \ z_0z_2 - z_1^2 = 0, \ z_1z_3 - z_2^2 = 0\},$$

which is the intersection of three conics.

Corollary 1.46. *If a complex manifold does not admit a Kähler form, then it cannot be projective.*

Example 1.47. *Hopf manifolds $S^1 \times S^{2n-1}$ are not Kähler, hence they fail to be projective.*

However, there are Kähler manifolds which cannot be embedded in $\mathbb{CP}^n$ and hence cannot be studied using algebraic geometry.

Example 1.48. *Here we give two important examples of Kähler manifolds which are non-projective.*

- *Let Λ be a lattice in $\mathbb{C}^2$, so that $\Lambda \cong \mathbb{Z}^4 = \{a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 : a_i \in \mathbb{Z}\}$. Then $\mathbb{C}^2/\Lambda$ is a complex torus. For a generic choice of $\{v_1, \ldots, v_4\}$, $\mathbb{C}^2/\Lambda$ will not be projective.*

- *$K3$ surfaces are compact complex surfaces with $c_1 = 0$ and $b_1 = 0$. There is a 20 dimensional family $\mathcal{M}_{K3}$ of $K3$ surfaces up to isomorphism. All $K3$ surfaces admit Kähler metrics. Inside $\mathcal{M}_{K3}$ are many 19-dimensional families of* ***projective*** *$K3$ surfaces, but generic $K3$ surfaces are not projective.*

The conditions under which a manifold can be embedded in $\mathbb{CP}^n$ are given in Kodaira embedding theorem.

1.3.2 The Kodaira Embedding Theorem

A line bundle $\mathcal{L}$ over a compact complex manifold M is called *very ample* if $\mathcal{L}$ has no base points in M, and the map $\iota_{\mathcal{L}} : M \to \mathbb{CP}^n$ is an embedding of M in $\mathbb{CP}^n$. Also, $\mathcal{L}$ is called *ample* if $\mathcal{L}^k$ is very ample for some $k > 0$.

A line bundle $\mathcal{L} \to M$ is called **positive** if its Chern class $c_1(\mathcal{L})$ can be represented as a de Rham cohomology class by a closed $(1,1)$-form which is positive ($\omega(v, Jv) > 0$).

Theorem 1.49. *A line bundle $\mathcal{L}$ is ample iff it is positive.*

Corollary 1.50. *A compact complex manifold with positive line bundle is a projective variety.*

Theorem 1.51 (The Kodaira Embedding Theorem)**.** *A compact complex manifold M can be embedded in $\mathbb{CP}^n$ if and only if it has a closed, positive $(1,1)$-form ω whose cohomology class $[\omega]$ is rational. A metric whose $(1,1)$-form is rational is a Hodge metric.*

Corollary 1.52. *A compact Kähler manifold with a Hodge metric (integral cohomology class) can be embedded in $\mathbb{CP}^n$. Hodge manifolds with integral cohomology class can be embedded in $\mathbb{CP}^n$. M is embeddable in $\mathbb{CP}^n$ iff $[\omega]$ is rational.*

The Kodaira Embedding Theorem says that $\mathcal{L}$ is ample if and only if it is positive. Therefore, a compact complex manifold with a positive line bundle is a projective variety, follows from Chow's Theorem 1.43 [23, p. 98]. M^n admits a holomorphic embedding into $\mathbb{CP}^n$ for some $n \geq m$ if M has a positive line bundle.

Corollary 1.53. *Let (M, J, g) be a compact Kähler manifold with $h^{2.0}(M) = 0$. Then (M, J, g) is projective.*

Proof. Let (M, J, g) be a compact Kähler manifold and $h^{2,0}(M) = h^{0,2}(M) = 0$ so that $H^{1,1}(M) = H^2(M; \mathbb{C})$ and $H^{1,1}(M) \cap H^2(M; \mathbb{R}) = H^2(M; \mathbb{R})$. The Kähler cone $\mathcal{K}_M$ of M is a nonempty open subset of $H^{1,1}(M) \cap H^2(M; \mathbb{R})$, so that $\mathcal{K}_M$ is open in $H^2(M; \mathbb{R})$. But $H^2(M; \mathbb{Q})$ is dense in $H^2(M; \mathbb{R})$, so $\mathcal{K}_M \cap H^2(M; \mathbb{Q})$ is nonempty. By The Kodaira Embedding Theorem 1.51 there exists $\alpha \in \mathcal{K}_M \cap H^2(M; \mathbb{Q})$ such that $k\alpha \in K \cap H^2(M, \mathbb{Z})$ for $k > 1$ an integer. Then $k\alpha \in \mathcal{K}_M$, as $\mathcal{K}_M$ is a cone, so there exists a closed positive $(1,1)$-form β on M such that $[\beta] = k\alpha$.

Then there exists a holomorphic line bundle $\mathcal{L}$ over M with $c_1(\mathcal{L}) = k\alpha$. Since $c_1(\mathcal{L})$ is represented by a positive $(1,1)$-form β, the Kodaira Embedding Theorem shows that $\mathcal{L}$ is ample. $\mathcal{L}$ is an holomorphic line bundle and $c_1(\mathcal{L}) = k\alpha$ is ample. Hence M is projective. $\square$

1.3.3 Hyperkähler Manifolds

Hyperkähler manifolds are the geometric analogue of the algebra of quaternions, which constitutes an extension of the algebra of complex numbers.

The algebra $\mathbb{H}$ of quaternions is generated by the symbols i, j, k satisfying the identities

$$\begin{aligned} &i^2 = j^2 = k^2 = -1 \\ &ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \end{aligned} \tag{1.18}$$

Then a quaternion x is written as $x = a_0 + a_1 i + a_2 j + a_3 k$ for $a_0, a_1, a_2, a_3 \in \mathbb{R}$.

Definition 1.54 (Hyperkähler Manifold)**.** *A hyperkähler manifold is a $4m$-dimensional Riemannian manifold (M, g) where we have defined three orthogonal complex structures I, J and K on the tangent bundle which are covariant constant with respect to the Levi-Civita connection and satisfy the quaternion algebra identities, given in Equation* (1.18)*.*

Choosing any of I, J and K and ignoring the other two, so that we consider a single automorphism of the tangent bundle, we see that M is a Kähler manifold.

For $a_1, a_2, a_3 \in \mathbb{R}$ we find that $a_1 I + a_2 J + a_3 K$ is also a complex structure, so that g is Kähler with respect to a whole 2-sphere of complex structures. Also, a hyperkähler manifold admits three compatible Kähler forms ω_I, ω_J, ω_K compatible with the complex structures I, J and K respectively. If we fix on the complex structure defined by I then ω_I is the $(1,1)$-form associated to the Kähler metric, while $\omega_{\mathbb{C}} = \omega_J + i\omega_K$ is a closed holomorphic 2-form [1].

Example 1.55. *The quaternionic space $\mathbb{H}^m$ with standard metric is a hyperkähler manifold.*

The different structure of the quaternionic algebra and the algebra of complex numbers, also accounts for important differences between Kähler and hyperkähler manifolds. A Kähler metric on a complex manifold can be modified to another one by adding a Hermitian form $i\partial\bar{\partial}f$ for an arbitrary small C^∞ function f, which ensures that the space of Kähler metric is infinite dimensional [22]. This is not the case for hyperkähler metrics.

Proposition 1.56. *If there exists a hyperkähler metric on a compact complex manifold, then up to isometry there is only a finite dimensional space of them.*

Proposition 1.57. *An irreducible hyperkähler metric is uniquely determined (up to a constant scale factor) by its family of complex structures.*

This result is false for Kähler metrics, where there are many Kähler metrics on a fixed complex manifold.

The existence of three complex structures I, J, K on a hyperkähler manifold also accounts for further important features. The holomorphic volume form $\omega_{\mathbb{C}}^m$ must for a hyperkähler manifold give a covariant constant trivialization of the canonical line bundle. The curvature of this bundle for any Kähler metric is the Ricci form, and so a hyperkähler metric has in particular vanishing Ricci tensor, see Hitchin [22].

Hyperkähler manifolds are obviously Kähler. Moreover, they are also Ricci-flat.

Note however that not all Kähler and Ricci-flat manifolds are hyperkähler, since the extra structure given by the three Kähler forms compatible with the three complex structures may not exist. In this case, the manifold is a Calabi-Yau manifold.

Example 1.58. *While it is very easy to construct examples of Kähler manifolds, e.g. closed projective varieties, it is not so easy to construct examples of hyperkähler manifolds. Important examples are the following:*

- $K3$ *surfaces are hyperkähler 4-manifolds and constitute one of the fundamental examples.*
- *In higher dimensions, the Hilbert scheme of zero cycles on a* $K3$ *surface or a 2-dimensional complex torus yields a natural class of hyperkähler metrics, see Hitchin [22].*
- *ALE spaces which are examples of non-compact hyperkähler 4-manifolds [26].*

Hyperkähler manifolds may be constructed via twistor theory and as hyperkähler quotients [22].

1.4 Holonomy Groups and Classification

Definition 1.59. *Let M be a manifold, E a vector bundle over M and ∇^E a connection on E. Let us fix a point $x \in M$. We call γ a* ***loop based at*** *x if $\gamma : [0,1] \to M$ is a (piecewise)smooth path with $\gamma(0) = \gamma(1) = x$. The parallel transport map $P_\gamma : E_x \to E_x$ is an invertible linear map so P_γ lies in the group of invertible linear transformations of E_x. The set of linear transformations arising from parallel transport along closed loops is a group called the* ***holonomy group*** *$Hol(\nabla^E) = \{P_\gamma : \gamma \text{ is a loop based at } x\}$.*

Holonomy groups classify types of geometric structures and encode this information about the manifold into purely algebraic terms. Note that holonomy groups will provide information about our manifolds being Kähler or not, and about its curvature.

Proposition 1.60. *Since parallel transport preserves the Riemann metric g the holonomy group of the Levi-Civita connection is contained in $O(n)$. If the manifold M is orientable, then its holonomy group lies in $SO(n)$.*

Since the holonomy group of a Riemannian manifold (M, g) is a subgroup of $O(n)$, we are interested in describing which subgroups of $SO(n)$ are holonomy groups. Under some simplifying assumptions, Berger proved that there are only 7 different possibilities. It took more time to prove that there actually existed manifolds with these holonomy groups.

Theorem 1.61 (Berger). *Let M be a simply-connected manifold of dimension n and g an irreducible, nonsymmetric Riemannian metric on M. Then $Hol(g)$ is exactly one of the following:*

(i) $SO(n)$.

(ii) $n = 2m$, $Hol(g) = U(m) \subset SO(2m)$ *for* $m \geq 2$. *Represents Kähler manifolds.*

(iii) $n = 2m$, $Hol(g) = SU(m) \subset SO(2m)$ *for* $m \geq 2$. *Represents Kähler and Ricci-flat manifolds, i.e., Calabi-Yau manifolds.*

(iv) $n = 4m$, $Hol(g) = Sp(m) \subset SO(4m)$ *for* ≥ 2. *Represents hyperkähler manifolds. (Kähler and Ricci-flat manifolds with m triples of compatible complex structures J_1, J_2, J_3).*

(v) $n = 4m$, $Hol(g) = Sp(m)Sp(1) \subset SO(4m)$ *for* $m \geq 2$. *Represents quaternionic Kähler manifolds, which are neither Kähler nor Ricci-flat. (However, if the curvature of the manifold is positive, its twistor space is Kähler).*

(vi) $n = 7$, $Hol(g) = G_2 \subset SO(7)$.

(vii) $n = 8$, $Hol(g) = Spin(7) \subset SO(8)$.

We can think of the groups $O(n)$, $U(m)$, $Sp(m)Sp(1)$ and $Spin(7)$ as automorphisms of $\mathbb{R}^n$, $\mathbb{C}^m$, $\mathbb{H}^m$ and $\mathbb{O}$ respectively with corresponding "determinant 1"subgroups given by $SO(n)$, $SU(m)$, $Sp(m)$ and G_2.

The following result follows from Proposition 1.26:

Proposition 1.62. *Let (M, J, g) be a Kähler manifold with g a Kähler metric. Then the holonomy group is $Hol(g) \subseteq U(m)$.*

Proof. Since the holonomy group $Hol(g)$ preserves constant tensors and we have $\nabla g = \nabla J = \nabla \omega = 0$ where ∇ is the Levi-Civita connection, we deduce that g, J, ω are constant tensors. Hence $Hol(g) \subseteq \{A \in GL(2, \mathbb{R}) : A \text{ preserves } g_0, \omega_0, J_0 \in \mathbb{R}^{2m}\} = U(m)$. $\square$

We have shown that if $Hol(g) \subseteq U(m)$, then (M, g) is a Kähler manifold. If we impose the extra condition that the metric is Ricci-flat, and (M, g) is simply-connected, we restrict the holonomy group $U(m)$ to its subgroup $SU(m)$. These groups represent Calabi-Yau manifolds which are indeed Kähler manifolds.

We can define a further restriction on the holonomy of Kähler manifolds given by the relation $Sp(m) \subseteq SU(2m) \subset U(2m)$. Note that $Sp(m) = O(4m) \cap GL(m, \mathbb{H})$. This relation represents the fact that parallel translation preserves three complex structures J_1, J_2 and J_3 instead of only one, so the holonomy group is contained in $O(4m)$ and $GL(m, \mathbb{H})$ corresponds to hyperkähler manifolds as were discussed in Subsection 1.3.3. As $Sp(m) \subseteq SU(2m)$, it follows that all hyperkähler manifolds are Kähler and Ricci-flat.

The holonomy groups $Sp(m)$ correspond to hyperkähler manifolds which are Kähler and Ricci-flat and carry triples of complex structures and corresponding compatible metric instead of single complex structures as is the case by a Kähler manifold, see Definition 1.25. This further geometric structure given by J_1, J_2 and J_3 which need not be present in Calabi-Yau manifolds accounts for the restriction from $U(2n)$, corresponding to Kähler and Ricci-flat manifolds, to $Sp(n)$, which encodes the existence of triples of complex structures.

As explained in Proposition 1.37 for Kähler manifolds, hyperkähler metrics also have consequences in the homological features of the manifold.

Remark 1.63. *If $H^{2,0}(M, \mathbb{C}) = \mathbb{C}$, M can be hyperkähler. Moreover we find the following:*

- $\dim_{\mathbb{C}} M = n$ *and M is a Calabi Yau manifold, then*

$$H^{p,0}(M) = \begin{cases} \mathbb{C} & p = 0, n \\ 0 & \text{otherwise} \end{cases}$$

- $\dim_{\mathbb{C}} M = 2k$ *and M is hyperkähler manifold, then*

$$H^{p,0}(M) = \begin{cases} \mathbb{C} & p = 0, 2, 4, \ldots, 2k \\ 0 & \text{otherwise} \end{cases}$$

We now discuss the three simplifying conditions used by Berger in his classification, namely:

(i) M is simply connected,

(ii) g is irreducible,

(iii) M is non-symmetric.

To study ***simply-connectedness***, we have to consider that the group $Hol(g)$ described in Definition 1.59 has an identity element which is contained in $Hol^0(g)$.

Definition 1.64. *Let (M, g) be a Riemannian manifold. The set of linear transformations arising from parallel transport only along loops which are homotopic to the identity, is a subgroup of the whole holonomy group called the restricted holonomy group of M and denoted by $Hol^0(g)$. $Hol^0(g)$ is the connected component of the identity element of $Hol(g)$, i.e. $Hol^0(g)$ contains the identity of $Hol(g)$.*

$Hol^0(g)$ plays an important role in describing the holonomy groups of not simply-connected manifolds. If (M, g) is not simply-connected, then we consider $Hol^0(g)$, where $Hol^0(g) \subset Hol(g)$ is a normal subgroup of $Hol(g)$ and the quotient $Hol(g)/Hol^0(g)$ is a discrete subgroup with a natural surjective morphism $\pi_1(M) \twoheadrightarrow Hol(g)/Hol^0(g)$.

If $(\tilde{M}, \tilde{g})$ denotes the universal Riemannian covering of (M, g), then $Hol(\tilde{g}) = Hol^0(\tilde{g}) = Hol^0(g)$, where $\tilde{g}$ is the pull-back metric on the universal cover $\tilde{M}$ of M, with $\pi_1(\tilde{M}) = 1$.

Proposition 1.65. *For M a simply-connected manifold, E a vector bundle over M with fibre $\mathbb{R}^k$ and ∇^E a connection on E, the holonomy $Hol(\nabla^E)$ is a connected Lie subgroup of $GL(k, \mathbb{R})$.*

If g is ***locally reducible*** then $Hol^0(g)$ is product of holonomy groups in lower dimensions.

Proposition 1.66. *Let (M_1, g_1), (M_2, g_2) be Riemannian manifolds. Then the product metric $g_1 \times g_2$ has holonomy $Hol(g_1 \times g_2) = Hol(g_1) \times Hol(g_2)$.*

Lastly, ***symmetric*** Riemannian spaces were classified by Élie Cartan using his classification of irreducible representations of Lie groups.

Example 1.67. *In Example 1.27 we studied $\mathbb{CP}^n$ as a complex manifold with a Kähler metric, the Fubini-Study metric. $\mathbb{CP}^n$ endowed with the Fubini-Study metric is a symmetric space.*

Chapter 2

Symplectic Manifolds

Symplectic geometry is a cousin of complex geometry. Just as all complex manifolds are locally trivial (that is, locally biholomorphic to $\mathbb{C}^n$), so by Darboux' Theorem, all symplectic manifolds are locally trivial (that is, locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$). This means that all nontrivial symplectic and complex geometry concerns global properties of symplectic and complex manifolds, in contrast to Riemannian geometry or Kähler geometry, for instance, in which one has local invariants such as curvature.

The study of the global structure of symplectic manifolds is the subject of symplectic topology. Here we are interested in the topological conditions for the existence a symplectic structure of a manifold, and what these topological conditions mean geometrically. The main references for this chapter are [2, 9, 12–14, 17, 28, 31, 33, 35].

2.1 Symplectic Structure

Definition 2.1. *Let M be a $2n$-dimensional manifold. A 2-form is* ***nondegenerate*** *if for every nonzero vector $X \in T_pM$ for $p \in M$ there exists a vector $Y \in T_pM$ such that $\omega|_p(X, Y) \neq 0$.*

Definition 2.2. *A manifold M equipped with a non-degenerate closed 2-form ω is called a* ***symplectic manifold*** *(M, ω), and ω is called a symplectic form.*

Theorem 2.3 (Darboux)**.** *Let (M^{2n}, ω) be a symplectic manifold, and let p be a point in (M, ω). Then around each point $p \in (M, \omega)$ there is a local coordinate system $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ such that on $\mathcal{U}$*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Such a chart $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ is called a Darboux chart.

Example 2.4. *Let $M = \mathbb{C}^{2n}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. Then*

$$\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$$

is a symplectic structure.

Proposition 2.5. *Since the symplectic form ω on a manifold (M, ω) is closed and non-degenerate, i.e., $\omega^n \neq 0$, we have that symplectic manifolds are orientable.*

Proof. Since the symplectic form ω is closed, it represents a cohomology class

$$\alpha = [\omega] \in H^2(M; \mathbb{R})$$

If (M, ω) is closed, then the cohomology class $a^n \in H^{2n}(M; \mathbb{R})$ is represented by the volume form $\omega^n \in C^\infty\left(\Lambda^{2n} T^* M\right)$ and the integral of this form over M does not vanish. □

Example 2.6 (Cotangent bundle)**.** *The cotangent bundle T^*M of any manifold M^k is a symplectic manifold.*

The cotangent bundle T^*M is the vector bundle whose sections are 1-forms on M, so T^*M carries a tautological 1-form $\beta \in C^\infty\left(\Lambda^1 T^* M\right)$ given by $\beta|_{(x,\alpha)} = \pi^*(\alpha)$ where we have a map

$$\begin{array}{rccc} \pi: & T^*M & \longrightarrow & M \\ & (x, \alpha) & \longmapsto & x, \end{array}$$

and we write $\omega = -\mathrm{d}\beta$.

In standard local coordinates (x, y) where $x \in \mathbb{R}^n$ is the coordinate on M and $y \in \mathbb{R}^n$ is the coordinate on the fibre T_xM, the canonical 1-form and its differential are given by the formulae

$$\beta = y dx, \quad \omega = dx \wedge dy.$$

Example 2.7. *Let (M, J, g) be a Kähler manifold, with Kähler form ω. Since the Hermitian metric g on M induces a positive definite bilinear symmetric differential form $g(X, Y)$ on T_pM, where $X, Y \in T_pM$, we deduce that the 2-form ω is non-degenerate. Since $\mathrm{d}\omega = 0$, we find that ω is closed and non-degenerate and hence symplectic.*

Proposition 2.8. *Any coadjoint orbit $\mathbb{O} \subset \mathfrak{g}^*$ of a Lie group G has a natural symplectic structure, [10].*

The symplectic structure on a manifold is closely related to the almost complex structure J we described in Section 1.1, see Definition 1.4. All symplectic manifolds admit an almost complex structure.

Definition 2.9. *An almost complex structure J is* ***compatible*** *with the 2-form ω if for every non-zero tangent vector u we have $\omega(u, Ju) > 0$ and $\omega(u, Jv) = -\omega(Ju, v)$.*

Differential forms also have a cohomological interpretation, and indeed we will use the cohomological features of the manifold M to understand which forms, and consequently, which metrics we can define on the manifold.

Proposition 2.10. *If M is a compact manifold with $H^2(M, \mathbb{R}) = 0$, then the manifold M does not admit a symplectic structure.*

Proof. Suppose ω is a symplectic structure on M. Then $[\omega]^n = vol(M)n! > 0$ in $H^{2n}(M; \mathbb{R}) \cong \mathbb{R}$. Hence $[\omega] \neq 0$ in $H^2(M; \mathbb{R})$, contradicting $H^2(M; \mathbb{R}) = 0$. □

Example 2.11. *We have shown in Example 1.2 that the $2n$-dimensional manifold $M = S^{2n-1} \times S^1$ is a complex manifold, i.e. it admits a complex structure. Now, we have $H^2(M, \mathbb{R}) = 0$ for $n > 1$, so we can deduce that M does not admit a symplectic structure. Note that we proved in Example 1.39 that M carries no Kähler form.*

The condition given in Proposition 2.10 is very strong and there exist complex manifolds with $H^2(M, \mathbb{R}) \neq 0$ which do not admit a symplectic structure either.

Example 2.12.

- *Let $M = S^1 \times S^3 \# \overline{\mathbb{CP}^2}$. Then we find $\beta_1 = 1$ and $\beta_2 = 1$ with $b_2^+ = 0$ so M does not carry any symplectic structure.*
- *The 8 dimensional complex manifold $M = S^2 \times S^3 \times S^3$ has $H^2(M; \mathbb{R}) = \mathbb{R}$ but $H^4(M; \mathbb{R}) = 0$. If ω is a symplectic form then $[\omega]^4 = vol(M)4! > 0$ as in the proof of Proposition 2.10. But $[\omega]^2 = 0$ as $H^4(M; \mathbb{R}) = 0$, so $[\omega]^4 = 0$ which is a contradiction. So M admits no symplectic forms even though $H^2(M; \mathbb{R}) \neq 0$.*

Symplectic forms are closely related to almost complex structures. Indeed, every symplectic manifold admits an almost complex structure which is compatible as described in Definition 2.9. The converse is true for open manifolds.

Theorem 2.13 (Gromov)**.** *Every* ***open*** *almost complex manifold admits a symplectic structure, see [28].*

Thus, we are interested in studying under what conditions a closed manifold M admits a symplectic structure. Note that if M does not admit an almost complex structure, it will not admit a symplectic structure either.

2.2 Almost Complex Structures and Symplectic Manifolds

A manifold may fail to admit holomorphic coordinates and still carry almost complex structures J on its tangent bundle. This obstruction to the existence of holomorphic functions is due to the non-integrability of the structure J and is represented by the Nijenhuis tensor $N_J(v, w)$ as in Theorem 1.7. However, not all real $2n$-dimensional manifolds admit an almost complex structure. Whether we can define an almost complex structure on a manifold M or not will depend on its topological features.

In the case of 4-dimensional manifolds, the topological conditions that must hold in order to define a structure J on M and which guarantee the existence of such an structure on M were set out by Ehresmann and Wu in the following theorem.

Theorem 2.14 (Ehresmann, Wu [13])**.** *Let M be a compact, connected, differentiable, oriented 4-manifold, and let $c_1(TM) \in H^2(M, \mathbb{Z})$. There exists on M an almost complex structure with first Chern class $c_1(TM)$ if and only if $c_1(TM) \equiv w_2(M)$ $\pmod 2$ and $c_1^2[M] = 3\sigma(M) + 2\chi(M)$, where $w_2(M)$ is the second Stiefel-Whitney class, $c_1^2[M]$ is the Chern number $c_1c_1[M]$, $\sigma(M)$ the signature and $\chi(M)$ the Euler-Poincaré characteristic of M.*

Remark 2.15. *For any compact oriented $2n$-manifold M, we have $c_n[TM] = \chi(M)$.*

The following formula provides conditions on the Chern numbers $c_1^2[TS]$ and $c_2[TS]$ for a surface S to admit an almost complex structure.

Theorem 2.16 (Noether's Formula)**.** *For an almost complex 4-manifold S the integer $c_1^2[TS] + c_2[TS] \equiv 0 \pmod{12}$. Also, $1 - b_1(S) + b_2^+(S)$ is even.*

Example 2.17. *We showed in Example 1.28 that $\mathbb{CP}^n$ is a Kähler manifold. We now check that Wu's formula holds for $M = \mathbb{CP}^2$ verifying that it admits an almost complex structure.*

Proof. We can show that $c_1c_1[\mathbb{CP}^2] = \binom{3}{1}\binom{3}{1} = 9$ see [29]. The Euler characteristic is $\chi(\mathbb{CP}^2) = 3$, the signature is $\sigma(\mathbb{CP}^2) = 1$ so that $c_1^2[\mathbb{CP}^2] = 3 \times 1 + 2 \times 3 = 9$ which is the case. Thus Wu's formula holds. Indeed, Noether's formula holds since $9 + 3 \equiv 0 \pmod{12}$. □

Example 2.18. $\mathbb{CP}^2\#\mathbb{CP}^2$ *does not admit an almost complex structure.*

Proof. We compute the Euler characteristic to be $\chi(\mathbb{CP}^2\#\mathbb{CP}^2) = \chi(\mathbb{CP}^2)+\chi(\mathbb{CP}^2)-\chi(S^4) = 2\times 3 - 2 = 4$. The signature is given by the trace of intersection form $Q_{\mathbb{CP}^2\#\mathbb{CP}^2} = Q_{\mathbb{CP}^2} \oplus Q_{\mathbb{CP}^2} = \mathbb{I}_{2\times 2} = 2$. Now, if we could define an almost complex structure J on $M = \mathbb{CP}^2\#\mathbb{CP}^2$, then we could find $c_1c_1[M]$ such that $c_1^2[M] = c_1c_1[M] = 2\chi + 3\sigma = 14$, but $14 + 4 \not\equiv 0 \pmod{12}$ so by Noether's formula we conclude that $M = \mathbb{CP}^2\#\mathbb{CP}^2$ does not admit an almost complex structure. $\square$

Note that $H^2(\mathbb{CP}^2\#\mathbb{CP}^2) = \mathbb{Z}\oplus\mathbb{Z}$ which is non-zero, and we can see that a non-zero 2-cohomology class does not guarantee that the manifold admits a symplectic structure.

Example 2.19. $\#n\mathbb{CP}^2$ *admits an almost complex structure iff n is odd.*

Proof. The Euler characteristic is $\chi(\#n\mathbb{CP}^2) = n+2$ and the signature is $\sigma(\#n\mathbb{CP}^2) = n$. Hence if we could define an almost complex structure J on M, then we could find $c_1[TM] \in H_2(M;\mathbb{Z})$ with $c_1^2[M] = 2(n+2)+3n = 5n+4$ and $c_2[M] = n+2$. But $(5n+4)+(n+2) = 6n+6 \not\equiv 0 \pmod{12}$ for n even, so by Noether's formula we deduce that n odd is a *necessary* condition for M to admit almost complex structures.

Now, the quadratic form is

$$Q = a_1^2 + \cdots + a_n^2,$$

so writing $c_1[TM] = (a_1,\ldots,a_n)$ in $H_2(M;\mathbb{Z}) \equiv \mathbb{Z}^n$, we want to find odd integers such that

$$\sum_{i=1}^{n} a_i^2 = 2(n+2) + 3n = 5n+4. \tag{2.1}$$

If $(a_1,\ldots,a_n)$ is a solution for Equation (2.1), then

$$\sum_{i=1}^{n} a_i^2 + 1 + 9 = (5n+4) + 10 = 5\,(n+2) + 4,$$

and $(a_1,\ldots,a_n,1,3)$ is a solution for $n+2$. Thus we find that n odd is a *sufficient* condition for $\#n\mathbb{CP}^2$ to admit an almost complex structure. $\square$

Example 2.20. *We saw that $M = \mathbb{CP}^2$ and $N = \#3\mathbb{CP}^2$ both admit an almost complex structure. We can check that $M = \mathbb{CP}^2$ is a Kähler manifold and hence symplectic. $N = \#3\mathbb{CP}^2$ however, fails to inherit a symplectic structure, see Gompf [13, Prop 10.1.13].*

Proposition 2.21. *The 4-sphere S^4 does not admit an almost complex structure.*

Proof. Let J be an almost complex structure on S^4. Now, the first Chern class is trivial since $c_1(TS^4, J) \in H^2(S^4; \mathbb{Z}) = 0$, thus it follows that $c_1c_1[S^4] = 0$. We also find that the Euler characteristic of S^4 is $\chi(S^4) = 2$ and the signature is $\sigma(S^4) = 0$ (since all spheres are boundaries and hence have $\sigma(S^n) = 0$). But $3 \times 0 + 2 \times 2 \neq 0$ so we conclude that there exists no almost complex structure on S^4. $\square$

Corollary 2.22. *The 4-sphere S^4 is not symplectic.*

Proposition 2.23. *If M_1 and M_2 are oriented 4-manifolds with almost complex structures, then $W = M_1 \# M_2$ has no almost complex structure with this orientation and hence no symplectic structure.*

Proof. Except in degrees 0 and 4, the Betti numbers of the connected sum are the sums for the Betti numbers of the components. Hence,

$$\begin{aligned} 1 - b_1(M_1 \# M_2) + b_+(M_1 \# M_2) &= 1 - b_1(M_1) - b_1(M_2) + b_+(M_1) + b_+(M_2) \\ &= 1 - b_1(M_1) + b_+(M_1) - b_1(M_2) + b_+(M_2). \end{aligned}$$

But $1 - b_1(M_1) + b_+(M_1)$ and $1 - b_1(M_2) + b_+(M_2)$ are even so $-b_1(M_2) + b_+(M_2)$ is odd. Hence $1 - b_1(M_1 \# M_2) + b_+(M_1 \# M_2)$ is odd, so $M_1 \# M_2$ does not have an almost complex structure. $\square$

Example 2.24. *However, if we consider two manifolds with almost complex structure but reversed orientation, we find that their connected sum can admit an almost complex structure. Let us consider $M_1 = \mathbb{T}^4$ and $M_2 = \mathbb{CP}^2$, we see that $M_1 \# M_2 = \mathbb{T}^4 \# \overline{\mathbb{CP}^2}$ admits an almost complex structure.*

Similar topological conditions have been proved for manifolds of higher dimensions, see Heaps [21].

Example 2.25. *The 6-sphere S^6 does admit an almost complex structure.*

Proof. $\mathbb{R}^7 = Im\mathbb{O} = \{a + be, \mathrm{Re}\ a = 0\}$ carries a vector product $\times$ which is bilinear and skew symmetric given by $u \times v = Im(uv)$.

Let us define

$$J_x(y) = x \times y.$$

If $y \perp x$, we have $J_x(y) = x \times y = xy + \langle x, y \rangle = xy$.

We now want to prove that J_x maps T_xS^6 to itself and that $J_x^2 = -Id$. Let $x = a + be$ be a unit imaginary octonion ($\bar{a} = -a$) and let $y = c + de$ be any octonion. Then we can compute $x(xy) = -y$.

If $y \in T_xS^6$, i.e. y is imaginary and $y \perp x$, then $\langle x, xy \rangle = -Re(x(xy)) = Re(y) = 0$ so J_x maps T_xS^6 to itself and $J_x^2 = -Id$. □

Theorem 2.26. *If M_1 and M_2 are 6-manifolds with almost complex structures, then $W = M_1 \# M_2$ has also an almost complex structure.*

Proof. This result follows from a theorem by Audin [2]. □

Example 2.27. *S^6 has an almost complex structure but no symplectic form since $H^2(S^6;\mathbb{R}) = 0$.*

Remark 2.28 ($M^6 \# N^6$)**.** *Although the connected sum $M \# N$ of two almost complex 6-manifolds also carries an almost complex structure, it is an **open question** whether the connected sum of two symplectic 6-manifolds can be also symplectic [28].*

Example 2.29. *The product of even dimensional spheres $S^{2p} \times S^{2q}$ admits an almost complex structure iff $(p, q) = (1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (3, 3)$. See Datta [11] for the proof. However, for p or $q > 1$, the manifold is not symplectic.*

Remark 2.30. *If M is symplectic, noncompact and connected, then there is a symplectic structure on $M \times S^{2n}$ for each n. See Theorem 2.13.*

Proposition 2.31. *In dimension other than $2n = 2$ or $2n = 6$, the connected sum $M_1 \# M_2$ of compact almost complex manifolds does not admit an almost complex structure.*

However, Geiges [12] proves that for $M_1, \ldots, M_k$ connected almost complex manifolds of dimension $2n$ for $n \geq 2$ the connected sum

$$W = M_1 \# \ldots \# M_k \# (k + 1)(S^2 \times S^{2n-2}) \tag{2.2}$$

also admits an almost complex structure. Note that W is the connected sum of an odd number of manifolds.

In dimension 8, that is $n = 4$, we have that $S^2 \times S^6$ admits an almost complex structure so that the manifold given by Equation (2.2) is the connected sum of an odd number of almost complex manifolds. Hence we deduce that the connected sum of $2k + 1$ almost complex 8-manifolds admits an almost complex structure if there are at least $k + 1$ $S^2 \times S^6$ summands.

Remark 2.32 (Connected sum of an odd number of 8-manifolds)**.** *We can ask if the connected sum of an odd number of almost complex 8-manifolds will admit an almost complex structure even if we do not have the summands $S^2 \times S^6$.*

In general, it is quite difficult to decide which closed almost complex manifolds are also symplectic. Halic [17] in the 6-dimensional case and Pasquotto [31] in the 8-dimensional case prove that the sets of Chern numbers realized by almost complex manifolds can also be realized by symplectic manifolds in that dimension. This however, only guarantees the existence of symplectic manifolds for any choice of Chern numbers in dimension 6 and 8 and provides no tool to decide whether a manifold is symplectic.

Proposition 2.33. *If ω_1, ω_2 are Kähler forms on a compact manifold (M, J) and $[\omega_1] = [\omega_2]$ in $H^2(M; \mathbb{R})$ then $(M, \omega_1) \cong (M, \omega_2)$ are symplectomorphic.*

Proof. Let $(M, t\omega_1 + (1-t)\omega_2)$ be a family of symplectic manifolds and let $t \in [0, 1]$. There is no change in cohomology class, hence they are locally Hamiltonian isotopic. □

2.3 Symplectic manifolds without Kähler structure

There are many important examples of complex manifolds which are not symplectic, e.g. Example 1.39. However, we are interested in understanding and constructing examples of manifolds which do admit a non-degenerate closed 2-form, i.e. they carry a symplectic form, but no Kähler structure. This is also known as the Weinstein-Thurston problem.

Important examples of symplectic manifolds are given by *nilmanifolds* and manifolds constructed from them.

Definition 2.34 (Nilmanifold [34])**.** *A nilmanifold is a compact homogeneous space of the form N/Γ where N is a simply connected nilpotent Lie group and Γ is a discrete co-compact subgroup in N.*

Example 2.35 (Nilmanifold)**.** *We give several important examples of nilmanifolds*

- *The torus $\mathbb{R}^n/\mathbb{Z}^n$.*

- *The quotient $M = \mathbf{U}(n, \mathbb{R})/\mathbf{U}(n, \mathbb{Z})$ is a nilmanifold where $\mathbf{U}(n, \mathbb{R})$ is the nilpotent Lie group of upper triangular matrices having 1's along the diagonal and $\mathbf{U}(n, \mathbb{Z})$ is the subgroup of $\mathbf{U}(n, \mathbb{R})$ with integer entries. In the special case $n = 3$, the quotient manifold $M = \mathbf{U}(3, \mathbb{R})/\mathbf{U}(3, \mathbb{Z})$ is called the* ***Heisenberg nilmanifold****.*

Proposition 2.36 (Tralle and Oprea [34]). *All 4-dimensional nilmanifolds admit symplectic structures.*

However, there are nilmanifolds which are not symplectic.

Example 2.37. *The 6-dimensional nilmanifold* $\mathbf{U}(4,\mathbb{R})/\mathbf{U}(4,\mathbb{Z})$ *is not symplectic, see [34, p.54. Example 1.8.].*

Theorem 2.38. *A symplectic nilmanifold* M *of Lefschetz type, see Theorem 1.35 is diffeomorphic to a torus.*

Corollary 2.39. *Let* $M = N/\Gamma$ *be a compact nilmanifold. If* M *is a Kähler manifold, then it is diffeomorphic to a torus.*

The first example of a symplectic non-Kähler manifold was constructed in 1976 by Thurston, and is named the Kodaira-Thurston manifold (KT) after him.

Example 2.40 (Kodaira-Thurston [33]). *This manifold is the first example of a complex symplectic manifold admitting no Kähler structure.*

We can construct it in different ways:

(i) *Geometrically, the Kodaira-Thurston manifold is obtained by taking the product of the Heisenberg nilmanifold* $M = (\mathbf{U}(3,\mathbb{R})/\mathbf{U}(3,\mathbb{Z}))$ *and the circle so that we write* $KT = (\mathbf{U}(3,\mathbb{R})/\mathbf{U}(3,\mathbb{Z})) \times S^1$.

(ii) KT *is also the quotient* $\mathbb{R}^4/\Gamma$ *where* Γ *is the affine group of transformations generated by*

$$\begin{array}{rcccl}
\gamma_1 & := & (x_1, x_2, y_1, y_2) & \longmapsto & (x_1, x_2 + 1, y_1, y_2) \\
\gamma_2 & := & (x_1, x_2, y_1, y_2) & \longmapsto & (x_1, x_2, y_1, y_2 + 1) \\
\gamma_3 & := & (x_1, x_2, y_1, y_2) & \longmapsto & (x_1 + 1, x_2, y_1, y_2) \\
\gamma_4 & := & (x_1, x_2, y_1, y_2) & \longmapsto & (x_1, x_2 + y_2, y_1 + 1, y_2).
\end{array}$$

(iii) KT *is a a flat 2-torus bundle over a 2-torus. Kodaira had shown that* KT *has a complex structure. However,* $\pi_1(KT) = \Gamma$, *hence* $H^1(\mathbb{R}^4/\Gamma;\mathbb{Z}) = \Gamma/[\Gamma,\Gamma]$ *has rank 3.*

(iv) *Let* $\rho : \mathbb{Z} \oplus \mathbb{Z} \to \mathrm{Diff}(T^2)$ *be a representation of the group* $\mathbb{Z} \oplus \mathbb{Z}$ *into the diffeomorphism group of a group* T^2 *defined by*

$$(1,0) \longrightarrow id \text{ and } (0,1) \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

where the matrix denotes the transformation of T^2 covered by the linear transformation of $\mathbb{R}^2$. This representation determines a T^2-bundle over T^2,

$$T^2 \longrightarrow KT \longrightarrow T^2, \text{ and } KT = \tilde{T}^2 \times_{\mathbb{Z}\oplus\mathbb{Z}} T^2,$$

where $\mathbb{Z} \oplus \mathbb{Z}$ acts on $\tilde{T}^2$ by covering transformations and on T^2 by ρ.

We can compute that the first Betti number is $\beta_1 = 3$. Since compact Kähler manifolds have even odd Betti numbers, KT is not Kähler.

Remark 2.41. *The Kodaira-Thurston manifold KT has an integral symplectic form ω, that is $[\omega] \in H^2(KT, \mathbb{Z})$. Hence it follows that (KT, ω) can be embedded symplectically into $\mathbb{CP}^5$, see [28, p. 251]. We write $\tilde{\mathbb{CP}}^5$ for the symplectic blow up of $\mathbb{CP}^5$ along the image of KT. $\tilde{\mathbb{CP}}^5$ is simply-connected, since $\mathbb{CP}^5$ is. Since $H^3(\tilde{\mathbb{CP}}^5)$ is isomorphic to $H^1(KT)$, we deduce that $\tilde{\mathbb{CP}}^5$ is symplectic but has no Kähler structure, as $b^3(\tilde{\mathbb{CP}}^5)$ is odd.*

2.4 J-holomorphic Curves

We have seen under what conditions a manifold carries an almost complex structure. As we saw in Section 1.1, that an almost complex manifold will only have many holomorphic functions if the almost complex structure J is integrable, i.e., the Nijenhuis tensor $N_J \equiv 0$. So, in general, almost complex manifolds will have few J-holomorphic functions. However, they will have plenty of J-holomorphic *curves*. Since we have seen that, in general, it is difficult to know whether an almost complex manifold admits integrable complex structures, it is very important to analyze structures that are independent of this feature.

Definition 2.42 (J-holomorphic Curves)**.** *A J-holomorphic curve on a manifold M is a map f from a compact Riemann surface Σ, with complex structure j, to M, such that*

$$df \circ j = J \circ df : T\Sigma \longrightarrow TM$$

(i.e., df is a complex linear bundle map).

Remark 2.43. *If M is endowed with a symplectic structure ω, and J and ω are compatible, then smoothly embedded J-holomorphic curves are also symplectically embedded.*

J-holomorphic curves are holomorphic with respect to the non-integrable almost complex structure J. The use of J-holomorphic curves allows us to define invariants that can distinguish one symplectic manifold from another. Gromov-Witten invariants are based on J-holomorphic curves and they are important in String Theory and Mirror Symmetry. J-holomorphic curves make it possible to study the global structure of symplectic manifolds.

Chapter 3

Quotient Constructions

In this chapter we analyze how to construct quotient spaces of manifolds with some given features under the action of a group.

We have seen that in many cases the properties of a given manifold are inherited by its submanifolds, e.g., all complex submanifolds of a Kähler manifold are also Kähler.

We have also seen that considering a manifold as the quotient of another manifold under a group action can provide useful information about its features, i.e., $S^1 \times S^{2n-1} = \left(\mathbb{C}^n \backslash \{0\}\right)/\mathbb{Z}$ or the characterization of $\mathbb{CP}^n$ as the quotient of $\mathbb{C}^{n+1}\backslash\{0\}$ under multiplication by complex scalars.

In general, we will be interested in analyzing under which conditions the quotient space is a manifold and which features it inherits from the original manifold after taking the quotient under the group action.

We are going to consider four different types of quotient constructions:

- Symplectic quotients, in which we construct a moment map.
- Kähler quotients, in which we construct a Kähler structure on the submanifold given by the quotient space under the group action.
- GIT quotients, in which we choose a linearization of the action of the group and consider the orbits under its action.
- Hyperkähler quotients which are rarer, but have important applications.

3.1 Symplectic Group Actions, Moment Maps, and Quotients

Symmetries are modelled using group actions. Thus we are interested in studying the action of a group G on a manifold M and in identifying the orbits of the action.

Definition 3.1 (Group action on M)**.** *A group action of a Lie group G on a manifold M is given by a map*

$$\begin{array}{rccc} \Psi: & G \times M & \longrightarrow & M \\ & (\gamma, x) & \longmapsto & \gamma \cdot x \end{array}$$

satisfying:

(i) *The map $(\gamma, x) \longmapsto \gamma \cdot x$ is smooth,*

(ii) $\gamma \cdot (\delta \cdot x) = (\gamma \cdot \delta) \cdot x$*, for all $\gamma, \delta \in G$,*

(iii) $1 \cdot x = x$.

*If the Lie group G acts by symplectomorphisms on the manifold M, then the action is called **symplectic**.*

Definition 3.2. *Let (M, ω) be a symplectic manifold, G a Lie group, $\mathfrak{g}$ the Lie algebra and $\Psi : G \to Sympl(M, \omega)$ a symplectic action. The action Ψ is **Hamiltonian** if the exists a map*

$$\mu : M \longrightarrow \mathfrak{g}^*$$

that satisfies:

(i) *For each $X \in \mathfrak{g}$, let*

- $\mu^X : M \longrightarrow \mathbb{R}$, $\mu^X(p) := \langle \mu(p), X \rangle$*, be the component of μ along X*
- $X^\#$ *be the vector field on M generated by the one-parameter subgroup* $\{\exp tX | t \in \mathbb{R}\} \subseteq G$.

Then $d\mu^X = \iota_{X^\#}\omega$, i.e., μ^X is a Hamiltonian function for the vector field $X^\#$.

(ii) *μ is equivariant with respect to the given action Ψ of G on M and the coadjoint action Ad^* of G on $\mathfrak{g}^*$*

$$\mu \circ \Psi_g = Ad_g^* \circ \mu.$$

*The vector (M, ω, G, μ) is then called a **Hamiltonian G-space** and μ is a **moment map**.*

We are interested in the conditions that guarantee that a symplectic action of a Lie group G is Hamiltonian, so that we can construct a moment map.

We define two conditions for the existence of a moment map:

(a) Let $x \in \mathfrak{g}$ and let $\xi(x)$ be a vector field. $\xi(x) \cdot \omega$ needs to be exact for all $x \in \mathfrak{g}$. Given $\xi(x) \cdot \omega$ is closed, so we need $[\xi(x) \cdot \omega] = 0$ in $H^1(M, \mathbb{R})$.

(b) Suppose (a) holds, then there exists $\mu : M \to \mathbb{R}$ with $\mathrm{d}_\mu x = \iota_{X^\#}\omega$ for all $x \in \mathfrak{g}$ where we need to choose μ equivariant.

Sufficient conditions for (a) to hold:

- $H^1(M; \mathbb{R}) = 0$; rov
- ω is exact.

If (a) holds, a sufficient conditions for (b) to hold is that G is compact.

Proposition 3.3 (Existence of moment maps)**.** *Let (M, ω) be a symplectic manifold. If $H^1(M; \mathbb{R}) = 0$ and G is compact, then a moment map exists.*

Corollary 3.4. *If ω is exact and invariant and G is compact a moment map always exists.*

However, the obstruction to the existence of a moment map lies in the Lie algebra cohomology. A moment map μ always exists if G is semisimple, even when $H^1(M) \neq 0$.

Theorem 3.5. (Audin [3, III.2.1]) *Let M be a compact symplectic manifold endowed with an action of S^1. Assume the action is Hamiltonian. Then it has fixed points.*

Theorem 3.6. *A symplectic S^1-action on a closed 4-manifold is Hamiltonian iff it has fixed points.*

Theorem 3.7 (Frankel's Theorem, McDuff [27])**.** *A circle action which preserves the complex structure and the Kähler form on a compact Kähler manifold M is Hamiltonian if and only if it has fixed points.*

This is however not the case for symplectic manifolds in which we find examples of actions with fixed points that are not Hamiltonian since they are not exact. So we are interested in the conditions that must be set on a symplectic action to make it Hamiltonian.

Proposition 3.8. *A symplectic torus action on a compact symplectic $2n$-dimensional manifold of Lefschetz type is Hamiltonian if and only if it has fixed points.*

Remark 3.9. *In the presence of the Lefschetz type condition, see Theorem 1.35, a symplectic circle action $S^1 \times M \to M$ on a closed symplectic manifold M has fixed points if and only if the orbit map $\iota : S^1 \to M$ is trivial on homology.*

3.1.1 Marsden-Weinstein-Meyer Reduction

We often do not want to distinguish between points which lie in the same orbit of a group action and take the quotient M/G creating a quotient space. That is, if we consider the manifold modulo the orbits of the group action, so that we consider the quotient of the manifold under the action of the group G, we obtain a new manifold. Under certain conditions the quotient space inherits properties from the original manifold.

Symmetries in a mechanical system correspond to conserved quantities. We will be interested in studying the space generated by the moment map μ of the action of G on M and in analyzing the orbit space $\mu^{-1}(0)/G$. The quotient is constructed using the moment map of the action and identifying the orbits.

Theorem 3.10 (Marsden-Weinstein-Meyer). *Let (M, ω, G, μ) be a Hamiltonian G-space for a compact Lie group G. Let $\iota : \mu^{-1}(0) \hookrightarrow M$ be the inclusion map. Assume that G acts freely on $\mu^{-1}(0)$. Then*

- *the orbit space $M_{red} = \mu^{-1}(0)/G$ is a manifold.*
- *$\pi : \mu^{-1}(0) \to M_{red}$ is a principal G-bundle.*
- *there is a unique symplectic form ω_{red} on M_{red} satisfying $\iota^*\omega = \pi^*\omega_{red}$.*

The orbit space $\mu^{-1}(0)/G$ together with the symplectic form ω_{red} is a new symplectic manifold, which we write as the Marsden-Weinstein quotient $M//G$. This new symplectic manifold has dimension $\dim(M//G) = \dim M - 2\dim G$.

Remark 3.11. *If a moment map exists, then a symplectic quotient exists, but may be singular.*

Remark 3.12. *If the action of the compact Lie group G acting on M is free, then M/G is a manifold and $\pi : M \to M/G$ is a principal bundle. Note that singularities may occur when the action of G on M is not free.*

Example 3.13. *We define the action of S^1 on the symplectic manifold $(\mathbb{C}^{n+1}, \omega)$ as follows,*

$$\begin{array}{ccc} S^1 \times \mathbb{C}^{n+1} & \longrightarrow & \mathbb{C}^{n+1} \\ (e^{it}, (z_0, \dots, z_n)) & \longmapsto & (e^{it} z_0, \dots, e^{it} z_n) \end{array}$$

where the moment map is given by

$$\mu(z) = -\frac{1}{2}|z|^2 + \frac{1}{2}.$$

Thus we find

$$\mu^{-1}(0) = S^{2n+1},$$

so that

$$\mu^{-1}(0)/S^1 = S^{2n+1}/S^1 \cong \mathbb{CP}^n,$$

and we write

$$(\mathbb{C}^{n+1}, \omega)//S^1 = (\mathbb{CP}^n, \omega).$$

However, depending on how we define the group G to act on the manifold M, we obtain different quotient spaces.

Example 3.14 (Quotient space with singularities)**.**

- *Let us consider the action of S^1 on $\mathbb{C}^2$ as follows,*

  $$\begin{array}{ccc} S^1 \times \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ \big(e^{i\theta}, (z_1, z_2)\big) & \longmapsto & (e^{ik\theta} z_1, e^{i\theta} z_2) \end{array}$$

  where $k > 0$ is an integer.

  In this case the moment map is given by

  $$\mu(z) = -\frac{1}{2}\left(k|z_1|^2 + |z_2|^2\right)$$

  $\xi < 0$ is a regular value, $\mu^{-1}(\xi)$ is an ellipsoid. The stabilizer of $(z_1, z_2) \in \mu^{-1}(\xi)$ is $\{1\}$ if $z_2 \neq 0$ and $\mathbb{Z}_k$ if $z_2 = 0$. It has one cone singularity of type k with angle $\frac{2\pi}{k}$. $\mu^{-1}(\xi)/S^1$ is a teardrop orbifold. This is the weighted projective line $\mathbb{CP}^1_{k,1}$.

- *Let us consider the action of S^1 on $\mathbb{C}^2$ as follows,*

  $$\begin{array}{ccc} S^1 \times \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ \big(e^{i\theta}, (z_1, z_2)\big) & \longmapsto & (e^{ik\theta} z_1, e^{i\ell\theta} z_2), \end{array}$$

  for $k, \ell > 0$ coprime.

  In this case the moment map is given by

  $$\mu(z) = -\frac{1}{2}\left(k|z_1|^2 + \ell|z_2|^2\right).$$

– *Stabilizer of* $(z_1, 0)$ *is* $\mathbb{Z}_k$ *for* $z_1 \neq 0$

– *Stabilizer of* $(0, z_2)$ *is* $\mathbb{Z}_\ell$ *for* $z_2 \neq 0$

– *Stabilizer of* (z_1, z_2) *is* $\{1\}$ *for* $z_1, z_2 \neq 0$

$\mu^{-1}(\xi)/S^1$ *has two cone singularities of type* k *and* ℓ*. This is* $\mathbb{CP}^1_{k,\ell}$*.*

- *However, if we model the action of* S^1 *on* $\mathbb{C}^2$ *as* $e^{i\theta}(z_1, z_2) \mapsto (e^{i\theta}z_1, e^{i\theta}z_2)$*, then the quotient* $\mu^{-1}(\xi)/S^1 = S^3/S^1 = \mathbb{CP}^1$*.*

3.1.2 The Atiyah-Guillemin-Sternberg Convexity Theorem

The Atiyah-Guillemin-Stenberg convexity theorem asserts that when G is a torus $\mathbb{T}^n$ and M is a symplectic compact manifold, the moment map image $\mu(M)$ is a convex polytope. The proof uses Morse theory, and relies on the fact that moment maps for symplectic forms are non-degenerate.

Theorem 3.15 (Atiyah-Guillemin-Sternberg)**.** *Let* (M, ω) *be a compact connected symplectic manifold, and let* $\mathbb{T}^n$ *be an* n*-torus. Suppose that* $\Psi : \mathbb{T}^n \to Sympl(M, \omega)$ *is a Hamiltonian action with moment map* $\mu : M \to \mathbb{R}^n$*. Then*

- *the level sets of* μ *are connected.*
- *the image of* μ *is convex.*
- *the image of* μ *is the convex hull of the images of the fixed points of the action*

The image $\mu(M)$ *of the moment map is called the* ***moment polytope****.*

Example 3.16. *Let us consider the action of the torus* $\mathbb{T}^2$ *on* $\mathbb{CP}^2$ *as defined by*

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0 : z_1 : z_2] = [z_0 : e^{i\theta_1}z_1 : e^{i\theta_2}z_2],$$

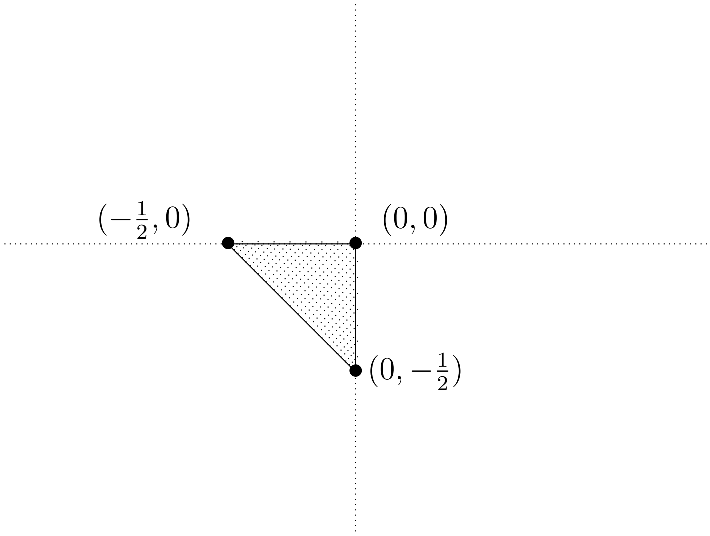
which has moment map $\mu : \mathbb{CP}^2 \to \mathbb{R}^2$ *of the form*

$$\mu[z_0 : z_1 : z_2] = -\frac{1}{2}\left(\frac{|z_1|^2}{\sum_{i=0}^2 |z_i|^2}, \frac{|z_2|^2}{\sum_{i=0}^2 |z_i|^2}\right),$$

and their moment map values are

$$\begin{array}{ccc} [1:0:0] & \longmapsto & (0,0) \\ [0:1:0] & \longmapsto & (-\frac{1}{2}, 0) \\ [0:0:1] & \longmapsto & (0, -\frac{1}{2}). \end{array}$$

The corresponding moment polytope is



Definition 3.17 (Effective actions)**.** *The G-action is called **effective** if each element $g \neq 1$ in G moves at least one $x \in M$.*

Example 3.18. *The S^1-action on S^3 defined by $t(z_1, z_2) = (t^{m_1} z_1, t^{m_2} z_2)$ is effective if and only if m_1 and m_2 are relatively prime.*

Proof. If $k > 1$ divides m_1 and m_2, then we can write $t(z_1, z_2) = (t^{m_1} z_1, t^{m_2} z_2) = \left(\left(t^k\right)^{m_1/k} z_1, \left(t^k\right)^{m_2/k} z_2 \right)$ which has fixed points for $t = e^{\frac{2\pi i}{k}}$, i.e., t does not move any point of S^3. □

Corollary 3.19. *Under the conditions of the convexity theorem, if the $\mathbb{T}^n$-action is effective, then there must be at least $n+1$ fixed points.*

Proposition 3.20. *Let $(M, \omega, \mathbb{T}^n, \mu)$ be a Hamiltonian $\mathbb{T}^n$-space. If the action of $\mathbb{T}^n$ is effective, then $\dim M \geq 2n$.*

Definition 3.21 (Toric symplectic manifold)**.** *A toric symplectic manifold is a compact connected symplectic manifold (M^n, ω) equipped with an effective Hamiltonian action of a torus $\mathbb{T}^n$ of dimension equal to half the dimension of the manifold, and with a choice of a corresponding moment map $\mu : M \to \mathbb{R}^n$.*

3.1.3 Delzant Theorem

We do not have a classification of symplectic manifolds, but we do have a classification of toric symplectic manifolds in terms of combinatorial data.

Theorem 3.22 (Delzant). *Let the group $G = \mathbb{T}^n$ act on the compact symplectic manifold (M, ω). If $\dim G = \frac{1}{2}\dim M$, the moment polytope $\mu(M^{2n})$ determines the Hamiltonian manifold (M, ω, μ) up to isomorphism.*

Corollary 3.23. *Hamiltonian $\mathbb{T}^n$-spaces where $\dim M = 2n$ are classified by the image of the moment map. In this case, the polyhedron determines the manifold.*

Definition 3.24 (Delzant Polytope). *A Delzant polytope Δ in $\mathbb{R}^n$ is a convex polytope satisfying*

- *it is simple: there are n edges meeting at each vertex.*
- *it is rational: the edges meeting at the vertex p are of the form $p+tu_i, 0 \leq t < \infty$ for $u_i \in \mathbb{Z}^n$, $i = 1, \ldots, n$.*
- *it is smooth: $u_1, \ldots, u_n$ can be chosen to be a basis for $\mathbb{Z}^n$.*

Proposition 3.25. *Toric manifolds are classified by Delzant polytopes. There is a one-one correspondence*

$$\begin{array}{ccc} \text{toric manifolds} & \longleftrightarrow & \text{Delzant polytopes} \\ (M^{2n}, \omega, \mathbb{T}^n, \mu) & \longleftrightarrow & \mu(M) \end{array}$$

3.1.4 Constructions of Toric Symplectic Manifolds

In this section we give some explicit examples of the construction of different Delzant polytopes arising from different toric actions. We also consider how the blow up process of a manifold at a point changes the corresponding polytope.

Example 3.26. *The circle S^1 acts on $\mathbb{CP}^1$ by*

$$e^{i\theta} \cdot [z_0 : z_1] = [z_0 : e^{i\theta} z_1],$$

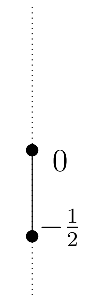
and has moment map

$$\begin{array}{rccc} \mu : & \mathbb{CP}^1 & \longrightarrow & \mathbb{R} \\ & [z_0 : z_1] & \longmapsto & -\frac{1}{2}\left(\frac{|z_1|^2}{|z_0|^2+|z_1|^2}\right), \end{array}$$

the fixed points are

$$\begin{array}{rcr} [1:0] & \longmapsto & 0 \\ [0:1] & \longmapsto & -\frac{1}{2} \end{array}$$

so that the corresponding moment polytope is



Example 3.27. *In this example we model two different actions of $G = \mathbb{T}^3$ on the manifold $\mathbb{CP}^3$ such that the first one is a toric action with the image of the moment map given by a Delzant polytope, while the second action is not effective and will not provide a toric action. We will see that in the second model, the image of the moment map is not Delzant.*

- *Let the torus $\mathbb{T}^3$ act on $\mathbb{CP}^3$ by*

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2 : e^{i\theta_3} z_3],$$

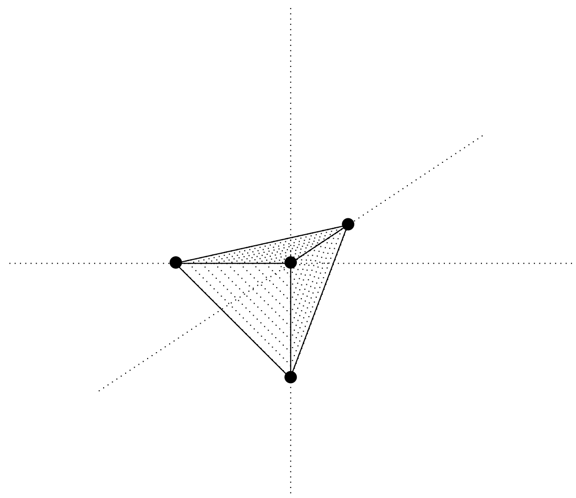
with moment map $\mu : \mathbb{CP}^3 \to \mathbb{R}^3$ of the form

$$\mu\left([z_0 : z_1 : z_2 : z_3]\right) = -\frac{1}{2}\left(\frac{|z_1|^2}{\sum_{i=0}^3 |z_i|^2}, \frac{|z_2|^2}{\sum_{i=0}^3 |z_i|^2}, \frac{|z_3|^2}{\sum_{i=0}^3 |z_i|^2}\right),$$

so that the fixed points are

$$\begin{array}{lcl} [1:0:0:0] & \longmapsto & (0,0,0) \\ [0:1:0:0] & \longmapsto & (-\frac{1}{2},0,0) \\ [0:0:1:0] & \longmapsto & (0,-\frac{1}{2},0) \\ [0:0:0:1] & \longmapsto & (0,0,-\frac{1}{2}). \end{array}$$

The corresponding moment polytope in $\mathbb{R}^3$ is



- *However, if we define the action of the torus $\mathbb{T}^3$ on $\mathbb{CP}^3$ by*

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : e^{2i\theta_1} z_1 : e^{4i\theta_2} z_2 : e^{6i\theta_3} z_3],$$

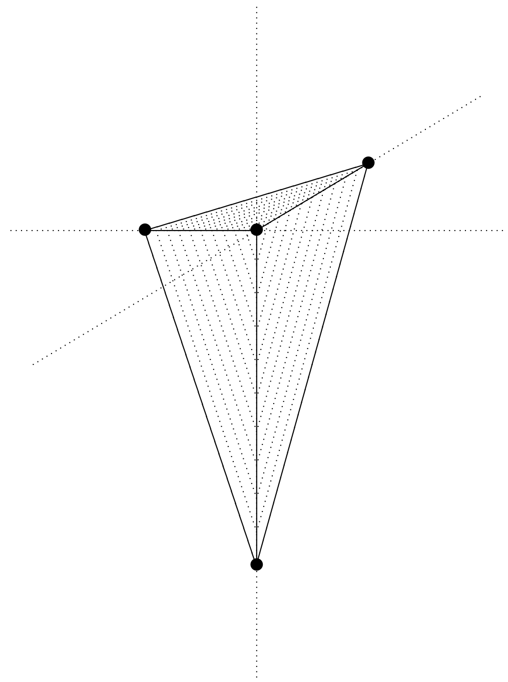
which has moment map

$$\mu\left([z_0 : z_1 : z_2 : z_3]\right) = -\left(\frac{|z_1|^2}{\sum_{i=0}^{3}|z_i|^2}, \frac{2|z_2|^2}{\sum_{i=0}^{3}|z_i|^2}, \frac{3|z_3|^2}{\sum_{i=0}^{3}|z_i|^2}\right),$$

so that the fixed points are

$$\begin{array}{ccc} [1:0:0:0] & \longmapsto & (0,0,0) \\ [0:1:0:0] & \longmapsto & (-1,0,0) \\ [0:0:1:0] & \longmapsto & (0,-2,0) \\ [0:0:0:1] & \longmapsto & (0,0,-3) \end{array}$$

and the corresponding moment polytope in $\mathbb{R}^3$ is



We can see that the action of the torus is not locally free so it does not provide a Delzant polytope. Indeed, this polytope is related to the one in the first part of the example by a linear map given by

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix},$$

whose inverse is not an integer matrix. In order to obtain a Delzant polytope, the corresponding matrix must be an invertible integer matrix with inverse an integer matrix.

- *Let us consider the map given by the matrix*

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The corresponding torus action is given by

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : e^{i\theta_1 + i\theta_2 + i\theta_3} z_1 : e^{i\theta_2 + i\theta_3} z_2 : e^{i\theta_3} z_3],$$

which has moment map

$$\mu\left([z_0 : z_1 : z_2 : z_3]\right) = -\frac{1}{2}\left(\frac{|z_1|^2}{\sum_{i=0}^3 |z_i|^2}, \frac{|z_1|^2 + |z_2|^2}{\sum_{i=0}^3 |z_i|^2}, \frac{|z_1|^2 + |z_2|^2 + |z_3|^2}{\sum_{i=0}^3 |z_i|^2}\right),$$

so that the fixed points are

$$\begin{array}{rcl} [1:0:0:0] & \longmapsto & (0,0,0) \\ [0:1:0:0] & \longmapsto & (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \\ [0:0:1:0] & \longmapsto & (0, -\frac{1}{2}, -\frac{1}{2}) \\ [0:0:0:1] & \longmapsto & (0, 0, -\frac{1}{2}) \end{array}$$

which defines a Delzant polytope.

Example 3.28. *The action of* $\mathbb{T}^2$ *on* $\mathbb{CP}^1 \times \mathbb{CP}^1$ *as*

$$(e^{i\theta}, e^{i\eta}) \cdot ([z_0 : z_1], [w_0 : w_1]) = \left(\left[z_0 : e^{i\theta} z_1\right], \left[w_0 : e^{i\eta} w_1\right]\right)$$

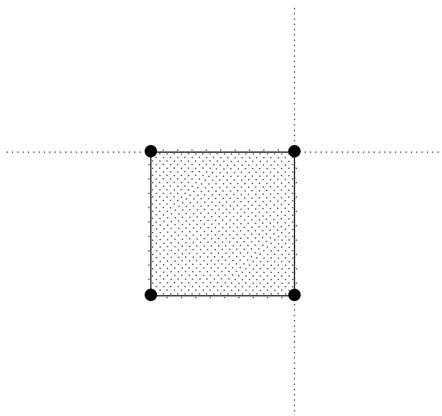
with moment map

$$\mu\left([z_0 : z_1], [w_0 : w_1]\right) = -\frac{1}{2}\left(\frac{|z_1|}{|z_0|^2 + |z_1|^2}, \frac{|w_1|}{|w_0|^2 + |w_1|^2}\right)$$

with fixed points

$$\begin{array}{rcl} ([1:0],[1:0]) & \longmapsto & (0,0) \\ ([1:0],[0:1]) & \longmapsto & (0, -\frac{1}{2}) \\ ([0:1],[1:0]) & \longmapsto & (-\frac{1}{2}, 0) \\ ([0:1],[0:1]) & \longmapsto & (-\frac{1}{2}, -\frac{1}{2}) \end{array}$$

The corresponding moment polytope is



Blow up Constructions

We are going to consider the Delzant polytopes of manifolds blown up at one point.

Example 3.29. *In this example we consider Hirzebruch surfaces. Let us consider the subset of* $\mathbb{CP}^1 \times \mathbb{CP}^2$

$$W_k = \{([z_0 : z_1], [w_0 : w_1 : w_2]) \in \mathbb{CP}^1 \times \mathbb{CP}^2 \mid z_0^k w_1 = z_1^k w_0\}$$

where $k \in \mathbb{N}$.

W_k *is a complex submanifold of* $\mathbb{CP}^1 \times \mathbb{CP}^2$*. The restriction of the projection* $\mathbb{CP}^1 \times \mathbb{CP}^2 \to \mathbb{CP}^1$ *to* W_k *is a bundle over* $\mathbb{CP}^1$ *with fiber* $\mathbb{CP}^1$*. Since* $\mathbb{CP}^1$ *and* $\mathbb{CP}^2$ *carry a symplectic structure, the manifold* $\mathbb{CP}^1 \times \mathbb{CP}^2$ *will be also symplectic and* W_k *is a symplectic submanifold.*

We now consider the action of $\mathbb{T}^2$ *on* $\mathbb{CP}^1 \times \mathbb{CP}^2$ *as*

$$(e^{i\theta}, e^{i\eta}) \cdot ([z_0 : z_1], [w_0 : w_1 : w_2]) = \left(\left[e^{i\theta} z_0 : z_1\right], \left[e^{ki\theta} w_0 : w_1 : e^{i\eta} w_2\right]\right)$$

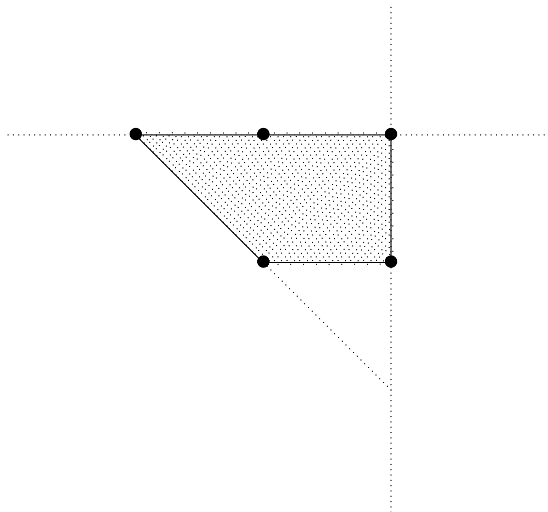
with moment map

$$\mu([z_0 : z_1], [w_0 : w_1 : w_2]) = \left(-\frac{1}{2}\left(\frac{|z_0|^2}{|z_0|^2+|z_1|^2} + \frac{k|w_0|^2}{|w_0|^2+|w_1|^2+|w_2|^2}\right), -\frac{1}{2}\frac{|w_2|^2}{|w_0|^2+|w_1|^2+|w_2|^2}\right)$$

so that the fixed points are

$$\begin{array}{lcl}
([1:0],[1:0:0]) & \longmapsto & (-\frac{1+k}{2}, 0) \\
([1:0],[0:1:0]) & \longmapsto & (-\frac{1}{2}, 0) \\
([1:0],[0:0:1]) & \longmapsto & (-\frac{1}{2}, -\frac{1}{2}) \\
([0:1],[1:0:0]) & \longmapsto & (-\frac{k}{2}, 0) \\
([0:1],[0:1:0]) & \longmapsto & (0, 0) \\
([0:1],[0:0:1]) & \longmapsto & (0, -\frac{1}{2}).
\end{array}$$

The image of the momentum mapping $\mu : W_k : \mathbb{R}^2$ *is shown below*



Note that when $k = 1$*, the manifold* W_1 *is the the projective plane blown up at a point.*

Symplectic Blow-up

Definition 3.30. *Let us consider the manifold $\mathbb{R}^{2n}$, that is, $(x_1, y_1, \ldots, x_n, y_n)$ and $0 \in \mathbb{R}^{2n}$. Let $B_\epsilon(0)$ be an open ball of radius ϵ about the origin 0. Then we define the symplectic blow up by*

$$\frac{\mathbb{R}^{2n}\backslash B_\epsilon(0)}{\sim} = \tilde{M}$$

where $\sim$ collapses the sphere $S_\epsilon(0)$ down to $\mathbb{CP}^{n-1}$.

The $U(1)$-action is defined by

$$(x_j, y_j) \longmapsto (\cos\theta x_j + \sin\theta y_j, \cos\theta y_j - \sin\theta x_j).$$

We define a map

$$\pi : S_\epsilon(0) \longrightarrow \mathbb{CP}^{n-1} = \frac{S_\epsilon(0)}{U(1)}$$

Then we find that a natural way to make $\tilde{M}$ into a manifold and $\tilde{\omega}$ into a symplectic form on $\tilde{M}$ such that $\tilde{\omega}|_{\tilde{M}\backslash\mathbb{CP}^{n-1}} \equiv \omega_{\mathbb{R}^{2n}}|_{\mathbb{R}^{2n}\backslash\bar{B}_\epsilon(0)}$.

Let (M, ω) be a symplectic manifold and $p \in M$ a point. By Darboux Theorem 2.3, we have $\exists p \in \mathcal{U} \subset M$ such that $(\mathcal{U}, \omega|_{\mathcal{U}}) \equiv (\mathcal{V}, \omega_0|_{\mathcal{V}})$ where $0 \in \mathcal{V} \subset \mathbb{R}^{2n}$.

If we choose $\epsilon > 0$ small enough such that $\overline{B_\epsilon(0)} \subset \mathcal{V}$ then we have

$$\tilde{M} = \frac{M\backslash B_\epsilon(0)}{\sim}$$

This manifold $\tilde{M}$ has reduced volume given by

$$vol(\tilde{M}, \tilde{\omega}) = vol(M) - vol(B_\epsilon(0)).$$

Example 3.31. *Let us consider the blowup of $\mathbb{CP}^2$ at $[1:0:0]$. We can write it as*

$$\tilde{\mathbb{CP}}^2 = \{([z_0 : z_1], [w_0 : w_1 : w_2]) \in \mathbb{CP}^1 \times \mathbb{CP}^2 \mid z_0 w_2 = z_1 w_1\}.$$

$\tilde{\mathbb{CP}}^2$ is a complex submanifold of $\mathbb{CP}^1 \times \mathbb{CP}^2$ and carries a Kähler form, so $\tilde{\mathbb{CP}}^2$ is a symplectic manifold.

We now define a compatible action of $\mathbb{T}^2$ on $\tilde{\mathbb{CP}}^2$ as

$$(e^{i\theta}, e^{i\eta}) \cdot ([z_0 : z_1], [w_0 : w_1 : w_2]) = \left(\left[e^{i\theta} z_0 : e^{i\eta} z_1\right], \left[w_0 : e^{i\theta} w_1 : e^{i\eta} w_2\right]\right)$$

with symplectic form

$$\alpha\omega_{\mathbb{CP}^1} + \omega_{\mathbb{CP}^2}$$

where $\alpha > 0$ is a parameter.

The moment map of the action is

$$\mu\left([z_0 : z_1], [w_0 : w_1 : w_2]\right) = \left(-\frac{1}{2}\left(\frac{\alpha|z_0|^2}{\sum_{i=0}^{1}|z_i|^2} + \frac{|w_1|^2}{\sum_{i=0}^{2}|w_i|^2}\right), -\frac{1}{2}\left(\frac{\alpha|z_1|^2}{\sum_{i=0}^{1}|z_i|^2} + \frac{|w_2|^2}{\sum_{i=0}^{2}|w_i|^2}\right)\right). \tag{3.1}$$

We now set $s = \frac{|w_1|^2}{|w_0|^2+|w_1|^2+|w_2|^2}$ *and* $t = \frac{|w_2|^2}{|w_0|^2+|w_1|^2+|w_2|^2}$ *for* $s, t \geq 0$ *and* $s + t \leq 1$ *so that we can write* (z_0, z_1) *in terms of* w_0, w_1 *and* w_2 *as*

$$\frac{|z_0|^2}{|z_0|^2 + |z_1|^2} = \frac{s}{s+t}, \quad \frac{|z_1|^2}{|z_0|^2 + |z_1|^2} = \frac{t}{s+t}.$$

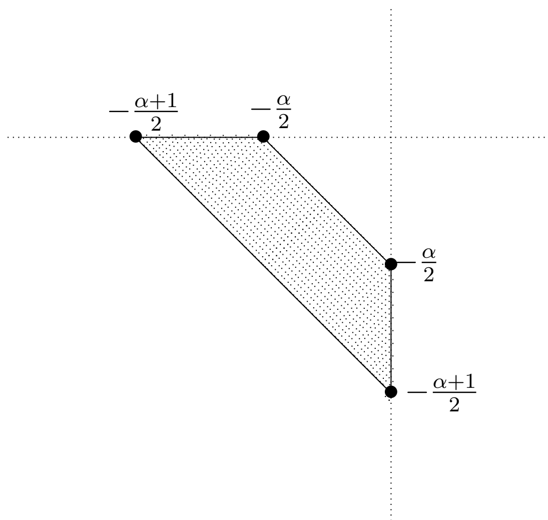
Then we can rewrite Equation (3.1) as

$$(s,t) \longmapsto \left(-\frac{1}{2}\left(\frac{\alpha s}{s+t} + s\right), -\frac{1}{2}\left(\frac{\alpha t}{s+t} + t\right)\right),$$

so that the fixed points are

$$\begin{array}{lcl}
(1,0) & \longmapsto & (-\frac{\alpha+1}{2}, 0) \\
(0,1) & \longmapsto & (0, -\frac{\alpha+1}{2}) \\
(\epsilon, 0) & \longmapsto & \left(-\frac{1}{2}\left(\frac{\alpha\epsilon}{\epsilon} + \epsilon\right), 0\right) = (-\frac{\alpha}{2}, 0) \text{ for } \epsilon \to 0 \\
(0, \epsilon) & \longmapsto & \left(0, -\frac{1}{2}\left(\frac{\alpha\epsilon}{\epsilon} + \epsilon\right)\right) = (0, -\frac{\alpha}{2}) \text{ for } \epsilon \to 0.
\end{array}$$

The image of the moment map is shown below



The volume of (M, ω) *decreases when it is blown up to* $(\tilde{M}, \tilde{\omega})$ *Audin [3].*

This is the blow-up of $\mathbb{CP}^2$ *at one point. In general, the blowing-up process produces a truncated polytope. Note that in Example 3.16 we modelled the torus action of* $\mathbb{T}^2$ *on* $\mathbb{CP}^2$ *and we found that the moment map image is a triangle. After blowing up* $\mathbb{CP}^2$ *at one point, we find that the corresponding image for the moment map is a truncated triangle, that is, a trapezoid.*

Depending on the values of α*, the area of the triangle that we cut off will vary. This corresponds to the size of the exceptional divisor.*

3.2 Kähler Quotients

The Kähler quotient is closely related to the symplectic quotient. Since all Kähler manifolds are symplectic we can use the Marsden-Weinstein-Meyer Reduction explained in Subsection 3.1.1 so that the quotient space will also be a symplectic manifold. However, Kähler manifolds have the special feature that its almost complex structure needs to be integrable, i.e., Kähler manifolds have holomorphic functions. So we will be interested in constructing quotient spaces that inherit these further properties and are also Kähler manifolds. We begin with an example of symplectic reduction applied to a Kähler manifold.

Example 3.32. *Let $M = \mathbb{C}^n$ with its standard Hermitian structure and the action of the circle $G = S^1$ by scalar multiplication. The moment map $\mu : \mathbb{C}^n \to \mathbb{R}$ is*

$$\mu(z) = -\frac{1}{2}|z|^2,$$

and $\mu^{-1}(0) = S^{2n-1}$. The symplectic quotient is

$$S^{2n-1}/S^1 = \mathbb{CP}^{n-1} = \left(\mathbb{C}^n \backslash \{0\}\right)/\mathbb{C}^*,$$

so that $M^s = \mathbb{C}^n \backslash \{0\}$ and $\mathbb{CP}^{n-1}$ inherits a natural Kähler metric, the Fubini Study metric.

We see that the quotient space also carries a Kähler structure. If we consider the action of **any** compact Lie group G on a Kähler manifold M, and we assume that G preserves both the metric, the complex structure, and the symplectic structure, then under mild conditions there is a moment map

$$\mu : M \longrightarrow \mathfrak{g}^*$$

where $\mathfrak{g}^*$ is the dual of the Lie algebra of G. The components of μ are Hamiltonian functions corresponding to the flows defined by one-parameter subgroups of G. μ is G-equivariant. The manifold

$$M_\xi = \mu^{-1}(\xi)/G$$

where ξ is a regular value for μ inherits a natural symplectic structure. M_ξ also inherits a Riemannian metric. Together with the symplectic form ω this defines an almost complex structure J which makes M_ξ a Kähler manifold [1].

Proposition 3.33. *$M//G := \mu^{-1}(\xi)/G$ is a new Kähler manifold of dimension $\dim M - 2\dim G$.*

We can also construct the Kähler quotient using a different method.

Let (M, J, g) be a Kähler manifold and let G be a real compact Lie group acting on (M, J, g). We want to consider the complexification $G^{\mathbb{C}}$ of the Lie group G and the complexification $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ of the Lie algebra $\mathfrak{g}$. The G-action naturally extends to a holomorphic $G_{\mathbb{C}}$-action, with the additional generating vector fields $J\xi_M$, $\xi \in \mathfrak{g}$.

However, $M/G^{\mathbb{C}}$ may not be Hausdorff, so that we may need to restrict to the "stable"points of M, i.e. $M^{\mathrm{st}} \subset M$ and consider the quotient $M^{\mathrm{st}}/G^{\mathbb{C}}$ to obtain a completely holomorphic description of a complex manifold in $M//G$ as a Kähler quotient. We will stabibly in Section 3.3.

If everything works, we obtain a quotient $M//G$ that we can identify with $M/G^{\mathbb{C}}$ as a complex manifold.

Example 3.34. *Let us consider the manifold $\mathbb{C}^{a+b}$ with a $\mathbb{C}^*$-action defined by*

$$(z_1, \ldots, z_a, z_{a+1}, \ldots, z_{a+b}) \longmapsto (uz_1, \ldots, uz_a, u^{-1}z_{a+1}, \ldots, u^{-1}z_{a+b}).$$

The Kähler quotient is $\mathbb{C}^{a+b}//U(1)$. We construct a moment map

$$\mu = -\frac{1}{2}\left(|z_1|^2 + \ldots + |z_a|^2 - |z_{a+1}|^2 - \ldots - |z_{a+b}|^2\right)$$

and we find

$$\begin{array}{rcl} M_\xi & = & \mu^{-1}(\xi)/U(1) \\ & = & \{(z_1, \ldots, z_{a+b}) : |z_1|^2 + \ldots + |z_a|^2 - |z_{a+1}|^2 - \ldots - |z_{a+b}|^2 = -2\xi\}/U(1). \end{array}$$

- *For $\xi > 0$, we find that M_ξ is a $\mathbb{C}^a$ vector bundle over $\mathbb{CP}^{b-1}$.*
- *For $\xi < 0$, we find that M_ξ is a $\mathbb{C}^b$ vector bundle over $\mathbb{CP}^{a-1}$.*
- *At $\xi = 0$ we have a singularity, a complex cone on $\mathbb{CP}^{a-1} \times \mathbb{CP}^{b-1}$.*

Example 3.35 (Kähler quotient)**.** *Let G be the circle S^1 acting on $\mathbb{CP}^2$ acting via the representation*

$$\begin{array}{rccc} \rho : & S^1 & \longrightarrow & U(3) \\ & t & \longmapsto & \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix}. \end{array}$$

Then the moment map is defined as

$$\mu(\xi) = -\frac{1}{2}\left(\frac{|x_1|^2 + |x_2|^2}{|x_0|^2 + |x_1|^2 + |x_2|^2}\right).$$

Thus $[x_0 : x_1 : x_2] \in \mu^{-1}(\xi)$ *if and only if*

$$-(1+2\xi)\left(|x_1|^2 + |x_2|^2\right) = 2\xi|x_0|^2.$$

Therefore,

- $\mu^{-1}(\xi)$ *is empty if* $\xi < -\frac{1}{2}$ *or* $\xi > 0$.
- $-\frac{1}{2}, 0$ *are not regular values of* μ.
- *If* $-\frac{1}{2} < \xi < 0$ *then* $\mu^{-1}(c)$ *can be identified by taking* $x_0 = 1$ *with sphere of radius* $\left|\frac{2\xi}{1+2\xi}\right|$ *in* $\mathbb{C}^2 \subset \mathbb{CP}^2$ *so that the Kähler quotient is* $\mu^{-1}(c)/S^1 \cong \mathbb{CP}^1$.

3.3 GIT Quotients

GIT provides another view of the construction of quotients of a manifold under the action of a group. In this case, the manifold we consider is a complex projective variety M as in Subsection 1.3.1, and the group G is a complex reductive group.

Fundamental references for GIT are [24, 30].

Theorem 3.36. *A complex Lie group G is* ***reductive*** *iff it is the complexification of any maximal compact subgroup K.*

Example 3.37. *The following are some examples when we consider* $\mathbb{C}$,

$$\begin{array}{ccc} SO(n) & \longrightarrow & SO(n,\mathbb{C}) \\ SU(n) & \longrightarrow & SL(n,\mathbb{C}) \\ U(n) & \longrightarrow & GL(n,\mathbb{C}). \end{array}$$

If M is a projective algebraic variety (with Kähler class coming from a projective embedding) then M is the projective variety whose projective coordinate ring is essentially the G-invariant part of the projective coordinate ring of M [1].

To apply GIT to construct quotients, we consider the action of the group G on the projective coordinate ring $A(M)$ of the complex variety M. We require a linearisation of the action of G, i.e, an ample line bundle $\mathcal{L}$ on M and a lift of the action of G to $\mathcal{L}$. So for some projective embedding $M \subset \mathbb{CP}^n$ determined by $\mathcal{L}$ for $k \gg 0$, (see Subsection 1.3.2), the action of G on M extends to an action on $\mathbb{CP}^n$ given by a representation

$$\rho : G \longrightarrow GL(n+1,\mathbb{C})$$

to obtain an induced action of G on $\mathbb{C}[x_0,\ldots,x_n]$ and on $A(M)$, and taking for $\mathcal{L}^k$ the standard line bundle $\mathcal{O}(1)$ on $\mathbb{CP}^n$.

We consider the subring $A(M)^G$ of $A(M)$ consisting of the elements of $A(M)$ which are invariant by G. $A(M)^G$ is a graded complex algebra, and because G is reductive, we find that $A(M)^G$ is finitely generated, and hence we can associate a complex projective variety to it [30].

Definition 3.38. *Let M be a projective complex algebraic variety and G a reductive complex Lie group acting on M. Then we write $M//_{\mathrm{GIT}}\, G$ for the projective variety associated to the ring of invariants $A(M)^G$.*

Under some conditions, a Kähler quotient can be identified with the quotient variety associated by GIT to the complexified group action. And in the same way that we have to make a choice for the moment map to obtain the Kähler or symplectic quotient as seen in the examples of Subsection 3.1.4, we have to choose a linearization of the action of G to construct the GIT quotient $M//_{\mathrm{GIT}}\, G$, see [24].

Then we have the GIT quotient $M//_{\mathrm{GIT}}\, G$ if we work in algebraic geometry, and the Kähler quotient $\mu^{-1}(0)/G$ if we work in Kähler geometry. The resulting complex manifolds are biholomorphic away from singularities, and the two quotient constructions provide the same space, up to homeomorphism (and diffeomorphism away from singularities) [25].

We now discuss the fact that the inclusion of $A^G(M)$ in $A(M)$ defines a rational map $\phi : M \to M//G$ that will not in general be well defined everywhere on M, since there may be points of $M \subset \mathbb{CP}^n$ where every G-invariant polynomial vanishes. The points in M for which there exist some non-vanishing $f \in A^G(M)$ are called *semistable points*. The set of semistable points in M is written as M^{ss}.

Proposition 3.39. *Two semistable points p_1, p_2 are called **s-equivalent** iff the closures $\overline{O_G(p_1)}$ and $\overline{O_G(p_2)}$ of the G-orbits of p_1 and p_2 meet in M^{ss} and we have $\phi(p_1) = \phi(p_2)$. Topologically, $M//G$ is the quotient of M^{ss} by the s-equivalence relation.*

Definition 3.40. *We define a point $p \in M$ to be a **stable point** if it has a neighbourhood in M^{ss} such that every G-orbit meeting this neighbourhood is closed in M^{ss}, and is of maximal dimension equal to the dimension of G.*

If $M^{st} = M^{ss}$ and we have no strictly semistable points, then $M//G = M^{st}/G$.

Example 3.41. *Let us consider the quotient of $\mathbb{C}^2$ under the action of the group $\mathbb{C}^*$. We are going to construct the quotient using GIT and Marsden-Weinstein-Meyer reduction.*

(i) *Let us consider the GIT quotient given by* $\mathbb{C}^2//_{\mathrm{GIT}}\,\mathbb{C}^*$ *where* $u \in \mathbb{C}^*$ *acts on* $(z_1, z_2) \in \mathbb{C}^2$ *by*

$$u : (z_1, z_2) \longmapsto (uz_1, u^{-1}z_2).$$

Then the orbits of the action are given by:

(a) $\{(z_1, z_2) : z_1 z_2 = c \text{ where } c \in \mathbb{C}\backslash\{0\}\}$.

(b) $\{(z_1, 0) : z_1 \neq 0\}$.

(c) $\{(0, z_2) : z_2 \neq 0\}$.

(d) $\{(0, 0)\}$.

As a topological space $\mathbb{C}^2/\mathbb{C}^*$ *is non-Hausdorff:* $\mathbb{C}$ *with a triple point at* 0*, so we need to restrict to some (semi) stable open subset before taking the quotient. There is a* $\mathbb{C}^*$*-equivariant line bundle* $\mathcal{L}$ *on* $\mathbb{C}^2$ *such that each linearization is of the form* $\mathcal{L}^k$ *for* $k \in \mathbb{Z}$*. There are three interesting cases:*

- $k > 0$,
- $k < 0$,
- $k = 0$.

Depending on the choice of linearisation, the behaviour of the orbits described in (a) *to* (d) *is as follows:*

- *When* $k > 0$*, the orbits described in* (a), (b) *are stable and the ones described in* (c), (d) *are unstable.*
- *When* $k < 0$*, the orbits described in* (a), (c) *are stable while* (b), (d) *are unstable.*
- *When* $k = 0$*, the orbit described in* (a) *is stable and* (b), (c), (d) *are semistable.*

The quotient is $\mathbb{C}$ *in each case.*

(ii) *If we now construct the symplectic quotient of* $\mathbb{C}^2$ *under the action of* S^1 *as in Example 3.13 taking* $n = 1$*, we obtain the following*

$$\mathbb{C}^2//\mathbb{C}^* = \mu^{-1}(0)/S^1 = S^3/S^1 \cong \mathbb{CP}^1,$$

where $\mu(z) = -\frac{1}{2}\left(|z_1|^2 + |z_2|^2\right) + \frac{1}{2}$ *is the corresponding moment map, and* $\mathbb{CP}^1$ *is considered as a symplectic manifold. With the appropriate linearisation,* $\mathbb{C}^2\backslash 0$ *is stable and* 0 *is unstable.*

Thus, more generally, we have the following quotient construction

$$\mathbb{C}^n//_{\mathrm{GIT}}\, \mathbb{C}^* = (\mathbb{C}^n\backslash 0)/\mathbb{C}^* = \mathbb{CP}^{n-1}.$$

In the following example, we find different choices for the linearisation of the action of G.

Example 3.42.

- *Let us consider the action of* $\mathbb{C}^*$ *on* $\mathbb{CP}^1$ *which we define as follows*

$$\begin{array}{lcl} \mathbb{C}^* \times \mathbb{CP}^1 & \longrightarrow & \mathbb{CP}^1 \\ t[z_1 : z_2] & \longmapsto & [z_1 : tz_2] \end{array}$$

  so the orbits are $\{[1:t], t \neq 0\}, \{[1:0]\}, \{[0:1]\}$ *and we have no good categorical quotient.*

- *Let us consider the action of* $\mathbb{C}^*$ *on* $\mathbb{C}^2\backslash\{0\}$ *which we define as follows*

$$\begin{array}{lcl} \mathbb{C}^* \times \mathbb{C}^2\backslash\{0\} & \longrightarrow & \mathbb{CP}^1 \\ t(z_1, z_2) & \longmapsto & [tz_1 : tz_2] \end{array}$$

  which we can identify with the map $\rho : \mathbb{C}^2\backslash\{0\} \to \mathbb{CP}^1$ *and defines a good categorical quotient.*

3.4 Hyperkähler Quotients

Let M be a hyperkähler manifold and let G be a compact Lie group of automorphisms of M. Using the 3 symplectic structures ω_I, ω_J, ω_K of M as defined in Subsection 1.3.3, we get 3 moment maps μ_I, μ_J, μ_K which we can combine into a single quaternionic moment map

$$\mu : M \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3, \tag{3.2}$$

which is G-equivariant. Let $\zeta \in \mathfrak{g}^* \otimes \mathbb{R}^3$ be fixed by G and assume this is a regular value of μ. Then the manifold

$$M_\zeta = \mu^{-1}(\zeta)/G$$

has 3 induced symplectic structures which, together with the induced metric, define a hyperkähler structure.

The following map $\mu_{\mathbb{C}}$ is the moment map with respect to the holomorphic symplectic form $\omega_{\mathbb{C}}$ of the action of the complex group $G^{\mathbb{C}}$:

$$\mu_{\mathbb{C}} = \mu_J + i\mu_K : M \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \mathbb{C}.$$

It is holomorphic with respect to the complex structure I on M.

Thus $\mu^{-1}(\zeta)$, where μ is the quaternionic moment map given in Equation 3.2, can be rewritten in the form

$$\mu_1^{-1}(a) \cap \mu_{\mathbb{C}}^{-1}(b)$$

for some $a \in \mathfrak{g}$ and $b \in \mathfrak{g} \otimes \mathbb{C}$.

We have that away from its singularities

$$\mu_J^{-1}(0) \cap \mu_K^{-1}(0) = (\mu_J + i\mu_K)^{-1}(0)$$

is a complex submanifold of M and $\left(\mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0)\right)/G$ is the hyperkähler quotient, so the hyperkähler quotient is the Kähler quotient of the Kähler manifold $\mu_{\mathbb{C}}^{-1}(b)$, see Hitchin [22].

Proposition 3.43. *Let M be a hyperkähler manifold and G a compact Lie group acting freely on $\mu^{-1}(\xi)$. Then $\mu^{-1}(\xi)/G$ is hyperkähler manifold, with dimension*

$$\dim \mu^{-1}(\zeta)/G = \dim M - 4 \dim G.$$

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