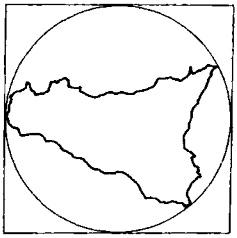
Atti del 2° Simposio Internazionale

PROBLEMI ATTUALI DELL'ANALISI E DELLA FISICA MATEMATICA

dedicato alla memoria del Prof. Gaetano Fichera



Taormina, 15–17 ottobre 1998

A cura di
Paolo Emilio Ricci
del
Dipartimento di Matematica
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INDICE

Introduzione

Questo volume raccoglie gli atti del Simposio che si è tenuto a Taormina, nei giorni 15–16–17 Ottobre, 1998, in Memoria del Prof. Gaetano Fichera, ed è stato stampato grazie ad un contributo del Consiglio Nazionale delle Ricerche.

L'iniziativa di celebrare la Memoria dell'illustre matematico scomparso con un Simposio a Lui dedicato ha ricevuto il patrocinio dell'Accademia Nazionale dei Lincei, del Ministero della Ricerca Scientifica e Tecnologica e dell'International Federation of Nonlinear Analysts (I.F.N.A.D.I.).

Il Simposio è stato parzialmente finanziato su fondi delle Università di Roma "La Sapienza" e di Catania, ed è stato sponsorizzato dai gruppi nazionali del CNR (G.N.A.F.A. e G.N.I.M.) e dal Banco di Sicilia.

Il Comitato Scientifico è stato costituito dai Proff. Robert P. Gilbert (Newark–DE, U.S.A.), Richard B. Hetnarski (Rochester–N.Y., U.S.A.), Enrico Magenes (Pavia), Vladimir G. Maz'ya (Linköping–Svezia), Olga Oleinik (Mosca–Russia).

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Hanno partecipato al Simposio più di sessanta studiosi italiani e stranieri. Sono state svolte, su invito, quindici relazioni.

Questo volume comprende la maggior parte delle relazioni svolte al Simposio dai conferenzieri invitati ed alcuni lavori di allievi o estimatori del Prof. Fichera che hanno aderito all'invito di contribuire con un loro articolo al volume.

Vogliamo rivolgere un caloroso ringraziamento non solo agli Enti e alle Istituzioni che hanno contribuito all'iniziativa, ma anche a tutti coloro che si sono prodigati per la buona riuscita del Simposio e, in particolare, al Prof. Salvatore Bonafede dell'Università di Catania ed al Dott. Pierpaolo Natalini dell'Università di Roma III, che hanno intensamente lavorato durante la fase organizzativa.

Roma, 2.2.2000

Paolo E. Ricci

Intervento di apertura del Simposio

Paolo Emilio Ricci

Signore e Signori, cortesi ospiti, cari Colleghi ed amici, gentili Signora Matelda e Dott. Massimo, siamo qui riuniti, a poco più di un anno dalla scomparsa del Prof. Gaetano Fichera, per celebrarne la memoria e ricordare in Lui non solo la figura di un grande Matematico, ma anche le doti di uno spirito libero e le insigni qualità di un Maestro che noi tutti abbiamo conosciuto e ammirato.

Questo Simposio avrebbe dovuto svolgersi lo scorso anno, in questa stessa sede, in occasione del suo 75° compleanno. Purtroppo la sua improvvisa scomparsa ci ha privato del piacere di dividere con Lui la gioia di un tale evento.

Quelli di Voi che erano presenti alla precedente edizione del Simposio, tenutasi in occasione del suo 70° compleanno, certamente ricordano la gioiosa partecipazione con cui Egli accolse i Colleghi e gli amici che erano venuti a presentargli gli auguri nella sua terra di Sicilia, che tanto amava.

Ebbene oggi, in Sua vece e a Suo nome, voglio ringraziare tutti gli illustri Colleghi che hanno accettato di onorarne la memoria intervenendo a questo Simposio.

Voglio ringraziare tutti i membri del Consiglio Scientifico e quelli del Comitato Organizzatore, ed in particolar modo il Prof. Francesco Nicolosi, che mi ha per primo incoraggiato a intraprendere questa iniziativa e aiutato, con fraterna amicizia, nell'impresa.

Devo anche ricordare con gratitudine l'Accademia Nazionale dei Lincei e il Ministero dell'Università e della Ricerca Scientifica e Tecnologica, che hanno accettato di patrocinare l'iniziativa; gli Enti promotori e finanziatori, e precisamente le Università degli Studi di Roma "La Sapienza" e l'Università degli Studi di Catania, che hanno fornito il personale e i mezzi necessari; i Gruppi del Consiglio Nazionale delle Ricerche di Analisi Funzionale e Applicazioni (GNAFA) e di Informatica Matematica (GNIM) che hanno assicurato il loro contributo; il Banco di Sicilia che, con grande sensibilità, ha offerto il suo sostegno.

Un particolare pensiero va alla Memoria del Prof. Calogero Vinti che, quale membro del GNAFA, sostenne subito, con fervente partecipazione, questa iniziativa, ma non poté, purtroppo, vederla realizzata. Un'altra dolorosa mancanza è quella del Prof. Sigmund Prössdorf, che avrebbe dovuto onorarci con la sua presenza e che è invece prematuramente scomparso.

Non tutti i Colleghi invitati hanno potuto intervenire, talvolta per motivi di salute, come nel caso dei Proff. Luigi Caprioli, Olga A. Oleinik, Ian N. Sneddon, Ronald S. Rivlin, ovvero a causa di precedenti, improrogabili impegni di lavoro, o di motivi familiari di varia natura, come i Proff. Piero Bassanini, Lucilla Bassotti Rizza, Francesco Calogero, Italo Capuzzo Dolcetta, Piervittorio Ceccherini, Luciano Daboni, Catterina Dagnino, W. Norbert Everitt, Giorgio Ferrarese, Walter K. Hayman, Peter Lax, André Lichnerowicz, Pierre Louis Lions, Silvio Maracchia, Enrico Magenes, Claus Müller, Angelo Pescarini, Gino Roghi, Francesco Succi e Rosanna Succi Cruciani, Luigi Tanzi Cattabianchi, Wolfgang Tutschke.

Ringrazio questi Colleghi per aver voluto esprimere, con cortesi parole, il loro rincrescimento per non aver potuto essere presenti.

In particolare il Prof. Enrico Magenes, trattenuto a Pavia da un impreвisto impegno di lavoro, mi ha più volte pregato di portare a voi tutti il suo più caloroso saluto e i migliori auguri per la riuscita del Simposio.

Il presidente dell'Accademia dei Lincei Prof. Edoardo Vesentini, essendo impossibilitato a intervenire, ha pregato il Prof. Giuseppe Grioli, membro dell'Accademia, di rappresentarlo.

[È seguito un breve intervento del Prof. Grioli, che ha ricordato la sua lunga, fraterna amicizia con il Prof. Fichera.]

A nome dei numerosi Colleghi stranieri con i quali il Prof. Fichera ha intrattenuto rapporti di amicizia e di operosa collaborazione dò ora la parola al Prof. Vladimir Maz'ya, che ha accettato di far rivivere alcuni suoi ricordi che lo legano allo scomparso.

[È seguito l'intervento del Prof. Maz'ya, riportato alle pagg. 1-4 di questo Volume.]

È ora la volta di ricordare il Prof. Fichera quale insigne Maestro. Numerosi allievi del Professore hanno raggiunto la cattedra universitaria e molti nostri Colleghi hanno iniziato la loro carriera accademica avendolo quale relatore di una tesi di laurea.

A nome di tutti noi allievi, prego ora la Prof. Lucilla Bassotti Rizza di prendere la parola.

[È seguito un breve intervento della Prof. Bassotti che ha ricordato la figura del Prof. Fichera, a nome di tutti i suoi allievi.]

Questa cerimonia inaugurale volge ora al termine. È stata certo più triste di quella che si svolse in questo stesso luogo nel 1992, ma ugualmente densa di contenuti, di ricordi e di affetti.

Ancora grazie a tutti Voi per la cortese e fattiva collaborazione. Grazie soprattutto a Lei, Signora Matelda!

In memory of Gaetano Fichera

Vladimir Maz'ya

Gaetano Fichera passed away on the first of June 1996 at the age of 74. The world of science lost an illustrious researcher, inspiring teacher, and outstanding personality. He left behind his beloved companion Matelda Fichera, who took care of him and helped him in his everyday work during 44 years of their life together.

Gentile Signora Matelda,
Caro Signor Massimo Fichera,

I am happy and honoured to greet you at this picturesque place which Gaetano was so fond of. His untimely death became a terrible blow for me and my wife Tatyana. I still hear his voice and see his smile, but he is not with us any more. Gaetano called me his friend which was exactly what I felt for him. I think he would not mind that I start with a few personal reminiscences.

Fichera's name entered my life at the very beginning of my mathematical career. In 1963 I came across the Russian translation of his memoir on elliptic-parabolic equations. I still keep an old copybook with my notes on that paper, where Fichera applied abstract functional analytic ideas and new a priori estimates to a very general class of equations. As many of Fichera's other pioneering works, that one gave rise afterwards to a rich domain in the theory of partial differential equations. I remember quite well the inspiration I felt while reading it. Since then I followed almost everything he wrote.

Fichera's paper on existence theorems in elasticity from the *Handbuch der Physik* was translated into Russian and appeared as a book in 1974 with Mikhlin's introduction. His extensive work on singularities of the equilibrium electric potential of a cube was published in *Uspehi Matem. Nauk* in 1975. He was famous in the USSR, his research being congenial to the Soviet school of partial differential equations and structural mechanics.

I knew Fichera only through his works until September 1971 when Mikhlin introduced me to him at Muskhelishvili's 80th anniversary symposium in Tbilisi. It was just shaking hands, nothing more. Fichera was an honourable guest of that pompous conference, always surrounded by bigwigs of the Soviet academic hierarchy, and I was too shy to approach him with mathematical discussions. However, he remembered me and during the following years I received a few letters from him.

One of these, of January 20, 1976, with an invitation to publish a survey paper on elliptic equations in domains with nonsmooth boundaries in one of the Italian mathematical journals, requires a clarifying remark.

In order to control and reduce the connections of Soviet scientists with the West, the academic bureaucracy allowed publications abroad very selectively and rather seldom. Permission, first from one's own university administration, and secondly, from the Ministry of Higher Education in Moscow was necessary for sending an article to a foreign journal. A complicated procedure to obtain permission had been invented which included, in particular, an odd requirement to present an invitation from the editorial board of the journal. The purpose of Fichera's letter mentioned above was to provide me with such an invitation.

In 1976–1977, we corresponded about a joint article of ours for the *Applicable Analysis.* That was a brief scientific biography of Solomon G. Mikhlin written on the occasion of his seventieth birthday. Fichera respected Mikhlin as a person and admired his work. Mikhlin was elected a foreign Member of the *Accademia dei Lincei* in 1981, but he was not allowed to visit Italy, so the Ficheras brought the tiny golden lynx to his apartment in Leningrad. Tatyana and I were the only guests at that inauguration ceremony.

The following morning we took them to the Russian Museum, but instead of enjoying the art Gaetano and I went through the rooms and passages talking exclusively about asymptotic behaviour of solutions to the Lamé system near polyhedral vertices. Of course, this was all my fault.

The next day, when they had to leave Leningrad, Gaetano did something very unusual for us. To explain this, I need to stray a bit from the topic. By that time, when the publishing policy of *Nauka* became explicitly antisemitic, all my books were rejected without comment after a long delay. Among them was my and Tatyana's manuscript devoted to a new theory of multipliers in spaces of differentiable functions.

After learning that the text was turned down Gaetano suggested to promote it in the West and smuggled it through customs. The whole operation was illegal, since we had no permission, but he did not hesitate. The book was published by Pitman in 1985.

I remember Gaetano Fichera with love and admiration as a man of courage, intellectual richness, kind heart and moral grandeur.

And he was a brilliant mathematician, whose work extends over 55 years, not counting the years of World War II. He was a bearer of the great Italian tradition in mathematics, and not only as a pure mathematician of the highest level but also a philosopher of science. For him, mathematics and its applications were inseparable.

One person can hardly do justice to the whole of Fichera's heritage. He contributed to mathematical elasticity, partial and ordinary differential equations, calculus of variations, functional analysis, approximation theory, potential theory, measure theory and integration, complex function theory, exterior differential forms, and numerical analysis here mention briefly only a few of his works, which are closest to my own interests.

Fichera applied methods of functional analysis and potential theory to elasticity in order to prove existence theorems for the mixed boundary value problem, where displacements and tractions are prescribed on two disjoint parts of the boundary. These results, published in 1950, were generalised to nonhomogeneous anisotropic media in his classical Springer *Lecture Notes.*

In an article of 1961 he proved an estimate for the displacement vector which is known now as the Fichera maximum principle. At the same time he developed a method of boundary integral equations for solving boundary value problems posed for higher order elliptic equations in a plane domain.

In 1963 Fichera wrote the first work containing an existence and uniqueness theorem for the so-called unilateral problem stated by Signorini. Here, an elastic body is supported by a frictionless surface and the area of contact should be found along with the solution. This work became the starting point for numerous subsequent studies which resulted in the extensive theory of variational inequalities.

In his 1903 lectures on the propagation of waves Hadamard gave a condition guaranteeing wave propagation in any direction (the so-called Legendre–Hadamard condition). Two years later, Hadamard's friend, the physicist Duhem, noted a gap in Hadamard's arguments. Fichera discovered a simple proof based on the general theory of elliptic systems. It was typical for him to apply sophisticated mathematical techniques to difficult problems of continuum mechanics.

Before passing to Fichera's contributions to partial differential equations I must remind you that he entered the field in the 1940s, when functional analytic methods only began to penetrate into it. Fichera became an ardent advocate of these methods. A good example is his theory of elliptic–parabolic equations, which impressed me so much in my youth. As early as in 1947, Fichera studied singularities of solutions to the mixed boundary value problems for second order elliptic equations. In particular, he introduced certain orthogonality conditions for right–hand sides which are necessary and sufficient for the smoothness of weak solutions. Much later similar ideas appeared in the general theory of elliptic boundary value problems in domains with corners and conic vertices.

In 1974, Fichera approached the classical problem of the equilibrium distribution of an electric charge on the surface of a cube. That was another of his important contributions to the nonsmooth elliptic theory.

Fichera's enlightening article of 1977 on the Saint-Venant principle crucially influenced the development of this interesting topic. I mention also his ingenious method of two-sided approximation of the eigenvalues of positive operators in Hilbert space and his later studies of materials with memory, which occupied him during many years.

Here I finish my list of mathematical achievements by Gaetano Fichera, as incomplete as it is. During his life he did much more and at the end of it he was not ready to interrupt his labour.

Among his remarkable gifts was that of a historian of mathematics. Now he himself entered the annals of its history.

Cauchy theory and the continua of Cosserat: new points of view

GIUSEPPE GRIOLI

Introduction

In a continuous model of a tri–dimensional body, that according to modern Physics has a corpuscolar constitution, the basic problem is how to characterize the deformation from which the stress depends. The simplest and most frequent way is to associate to each point of the continuum model a displacement $\underline{u}$. One obtains Cauchy model in which the stress is characterized by a symmetrical matrix, t_{rs}.

A less simple but interesting model is Cosserat Continuum in which the geometry of deformation is characterized by two fields: the displacement $\underline{u}$ and the rotation R. It is assumed that the forces that a portion of the body exerts on the other through a surface element is represented by the stress t_{rs} (in general non–symmetric) and by a couple stress, ψ_{rs}. Of course, Cosserat's model better approaches the corpuscolar conception of matter than the Cauchy's.

From a phenomenological point of view, Cosserat Continuum correspond to a material constituted by a many small rigid particles. One may refine Cosserat's theory assuming that the elementary particles have a homogeneous deformation, and so on.

It is evident that a better approximation in the valuation of the deformation results in a better characterization of the internal forces and thermodynamical functions, for example, the free energy (Helmotz function).

In the usual theory of Cauchy and Cosserat continuous bodies, and also of more complex microstructures, the size of the elementary particles of corpuscolar matter is absent. In my opinion, this lessen those theories. I think that a continuum model is more realistic if, in some way, size is taken into account. In the following I will show how to reach this end, considering only Cosserat elastic bodies but it is evident that many thought are valid for more complex microstructures and also for non elastic bodies. To be precise, I want to show the kind of dependence of the free energy on the parameter, h, which represents the greatest size of the elementary particles. The consequences are very remarkable. In particular, the analytical problems of the determination of the fields $\underline{u}$ and R split themselves into two distinct problems, also in the case of Dynamics. In general, being h very small, it is possible to neglect a certain power of h and, under appropriate conditions of regularity, to express the stress, displacements, rotations and so on, as polynomials in h with coefficients characterized by analytical sistems which are simpler than that of Cosserat's traditional theory. Of course, the first degree polynomials give interesting results. The Cauchy and Cosserat theories are unified and one may have a non–symmetric stress also in problems of Cauchy's type.

In Continuum Mechanics a subtle point is the problem of the boundary conditions. Also in more simple theory of Cauchy the significant cases in which one knows mathematical expressions for the forces that the external world exercises on the body through its boundary are very few. All the more reason that happens for the external couples in Cosserat theory. This question will be considered in the following.

Since my main object is to test the new procedure, I shall confine myself to talking about the dependence of the free energy on the parameter h and the consequences on the constitutive equations and, therefore, on the Thermdynamics, in general. For this aim it will be enough to consider equilibrium problems.

1 On the deformation of a Cosserat Continuum

Let C be a reference configuration for a tri–dimensional body, C' the deformed one and c and c' the positions of an elementary particle of the body in C and C' respectively.

I denote by: G and G' the center of mass of the particles in the positions c and c' P and P' two corresponding points of c and c', $\underline{u} = GG'$ the displacement GG', R the matrix which characterizes the rotation of c, X_i, x_i, $(i = 1, 2, 3)$, the coordinates of G and G' with respect to a rectangular cartesian coordinate system.

As is well known, it is possible to characterize the rotation R by a vector $\underset{\sim}{Q}$. The components of $\underline{u}$ and R are

$$u_i = x_i - X_i \ , \quad R_{rs} = \frac{1}{1+Q^2}\left[(1-Q^2)\delta_{rs} + 2e_{rls}Q_l + 2Q_rQ_s\right] \ , \qquad (1.1)$$

where Q_i are the components of $\underset{\sim}{Q}$ and e_{rls} and δ_{rs} denote respectively the Ricci's tensor and the Krönecker delta.

Let G_1, G'_1 be the analogous points of G, G' for a particle that in C and C' is in the positions c_1, c'_1 very close to c and c'. Denoting by the comma the derivation with respect to X_i, one has

$$\begin{cases} u_r(G') = u_r + u_{r,l}\,\mathrm{d}X_l, \qquad R_{rs}(G') = R_{rs} + R_{rs,l}\,\mathrm{d}X_l\ , \\ Q_r(G') = Q_r + Q_{r,l}\,\mathrm{d}X_l\ . \end{cases} \tag{1.2}$$

Putting

$$\underline{\xi} = GG'\ , \quad \underline{\eta} = GP\ , \quad \underline{\eta}^{(1)} = G_1P_1 \tag{1.3}$$

and denoting by $\underline{a}$ ($a_{rs} = x_{r,s}$) the displacement gradient, one has

$$PP_1 = \underline{\xi} + \underline{\eta}_1 - \underline{\eta}\ ; \quad P'P_1' = a\underline{\xi} + R(\underline{\eta}_1, -\underline{\eta}) + R_{,i}\,\underline{\eta}_1\,\mathrm{d}X_i \tag{1.4}$$

Introducing the strain ε and the matrices

$$\nu = a^T R\ , \quad \nu^i = R^T R_i\ , \qquad (i = 1, 2, 3)\ , \tag{1.5}$$

one has

$$2\varepsilon = \nu\nu^T - 1 \tag{1.6}$$

and, after some computations,

$$\begin{cases} |PP_1|^2 = \xi^2 + 2\underline{\xi}\cdot(\underline{\eta}_1, -\underline{\eta}) + (\underline{\eta}_1 - \underline{\eta})^2\ , \\ |P'P_1'|^2 = [(1+2\varepsilon)\underline{\xi} + 2\nu(\underline{\eta}_1 - \underline{\eta})]\cdot\underline{\xi} + 2[\underline{\eta}_1 - \underline{\eta} + \nu^T\underline{\xi}]\cdot \\ \qquad\qquad \cdot\nu^i\underline{\eta}_1\,\mathrm{d}X_i + (\nu^i)^T\nu^l\underline{\eta}_1 - \underline{\eta}_1\,\mathrm{d}X_i\,\mathrm{d}X_l + (\underline{\eta}_1 - \underline{\eta})^2 \end{cases} \tag{1.7}$$

The necessary and sufficient conditions such that the displacement from C to C' be a rigid displacement are that

$$\nu \equiv 1\ , \quad \varepsilon \equiv 0\ , \quad \nu^i \equiv 0\ . \tag{1.8}$$

Denoting by δ the linear dilatation coefficient, one has

$$(1+\delta)^2 = \frac{|P'P_1'|^2}{|PP_1|^2}\ , \tag{1.9}$$

where $\delta = 0$ if and only if the displacement CC' is rigid.

I denote by h the greatest dimension of the elementary particles of the body. Putting

$$\underline{\eta} = h\underline{\eta}'\ , \underline{\eta}_1 = h\underline{\eta}_1'\ , \tag{1.10}$$

after some calculations, one finds

$$(1+\delta)^2 = \frac{\alpha(\varepsilon) + \beta(v, v^i)h + \gamma(\nu, \nu^i)h^2}{\overline{\alpha} + \overline{\beta}h + \overline{\gamma}h^2}\ , \tag{1.11}$$

where

$$\begin{cases} \alpha(\varepsilon) = (1+2\varepsilon)\underline{\xi}\cdot\underline{\xi} \ , \\ \beta(v,v^i) = 2\nu^T\underline{\xi}\cdot[\underline{\eta}'_1 - \underline{\eta}' + \nu^i\underline{\eta}'_1\,\mathrm{d}X_i] \ , \\ \gamma(\nu,\nu^i) = (\underline{\eta}'_1 - \underline{\eta}')^2 + \nu^i\underline{\eta}'_1\,\mathrm{d}X_i\cdot[2(\underline{\eta}'_1 - \underline{\eta}') + \nu^l\underline{\eta}'_1\,\mathrm{d}X_l] \ , \end{cases} \tag{1.12}$$

$$\overline{\alpha} = \xi^2 \ , \quad \overline{\beta} = 2\underline{\xi}\cdot(\underline{\eta}'_1 - \underline{\eta}') \quad \overline{\gamma} = (\underline{\eta}'_1 - \underline{\eta}')^2 \ . \tag{1.13}$$

From (1.11), (1.12), (1.13) it follows

$$\begin{cases} \delta^0 = \lim_{h\to 0}\delta = \left(\frac{\alpha}{\overline{\alpha}}\right)^{\frac{1}{2}} - 1 = \frac{\sqrt{(1+2\varepsilon)\underline{\xi}\cdot\underline{\xi}}}{|\underline{\xi}|} - 1 > -1 \ , \\ \delta^{(1)} = \lim_{h\to 0}\left(\frac{\partial\delta}{\partial h}\right) = \frac{1}{2(1+\delta^0)\xi^2}\Big\{2\nu^T\underline{\xi}\cdot\left[\underline{\eta}'_1 - \underline{\eta}' + \nu^i\underline{\eta}_1\,\mathrm{d}X_i\right] - \\ \qquad -(1+2\varepsilon)\underline{\xi}\cdot\underline{\xi}\frac{2\underline{\xi}\cdot(\underline{\eta}'_1 - \underline{\eta}')}{\xi^2}\Big\} \ . \end{cases} \tag{1.14}$$

It is easy to show that [assuming $|\xi| > 0$] when h tend to zero there exists the partial derivatives of δ with respect to h of any order. Further, for $h = 0$, the deformation of the body depends exclusively on the strain ε even when the Continuum is sensible to the rotation R. This circumstance has important consequences for the constitutive relations.

2 Some previous statements

In the classic theory of Cosserat Continua it is assumed that the inner forces that a part of the body exercises on the other through a surface element $d\sigma'$ are represented by the Cauchy stress $td\sigma'$ and the couple stress $pd\sigma'$, $(t = t_{rs}, p = p_{rs})$. Reasoning from the field equations one deduces the density of the work of the inner forces, $dl^{(i)}$, when the body goes from the present configuration C' to a $C' + \mathrm{d}C'$ which is very close. Denoting by du_r the variation of u_r and by $\underline{w}$ the rotation of a particle, one has

$$dl^{(i)} = t_{rs}(du_r)_{/s} + e_{rps}t_{ps}w_r + p_{rs}w_{r/s} \ , \tag{2.15}$$

where the bar denotes the derivative with respect to x_i.

I want to observe that an expression like (2.15) may be obtained without introducing the concept of couple stress. That is, in our case, assuming that the stress depends on translation and rotations and, therefore, assuming that the density of the work of the inner forces is a linear polynomial in the variables $(du)_{r/s}$, w_r, $w_{r/s}$:

$$dl^{(i)} = t_{rs}(du_r)_{/s} + q_r w_r + z_{rs}w_{r/s} \ , \tag{2.16}$$

where q_r and z_{rs} are parameters.

The physical principle that the work of the inner forces is equal to zero for any rigid displacement of the body requires as necessary and sufficient condition

$$q_r = e_{rps} t_{ps} \ . \tag{2.17}$$

In fact, for a rigid displacement one has

$$w_r = \frac{1}{2} e_{rlm} (du_m)_{/l} \ , \quad w_{r/s} = 0 \ , \quad (du_r)_{/s} + (du_s)_{/r} = 0 \ . \tag{2.18}$$

From (2.16), (2.18) it follows (2.17) and the expression (2.16) becomes (2.15).

As is well known, according to the axiom of objectivity, it is convenient to put

$$t_{rs} = \frac{1}{D} x_{r,l} x_{s,m} T_{lm} \ , \quad p_{rs} = \frac{1}{D} x_{r,l} x_{s,m} P_{lm} \ , \tag{2.19}$$

where $D > 0$ is the determinant jacobian $\|x_{r,s}\| > 0$ and T_{rs}, P_{rs} are two matrices depending on the deformation. Denoting by $\mathrm{d}Q_i$ the variation of Q_i corresponding to the displacement of the body from C' to $C' + \mathrm{d}C'$, one has

$$w_r = \frac{1}{2} e_{rls} \beta_{ls}^{(i)} \, \mathrm{d}Q_i \ , \quad \beta_{rs}^{(i)} = R_{rt} \frac{\partial R_{st}}{\partial Q_i} \ . \tag{2.20}$$

Therefore, the lagrangean expression $\overline{dl}^{(i)}$ of the density of work of the inner forces, according to (2.15), (2.20), is

$$\begin{aligned} \overline{dl}^{(i)} &= x_{r,l} T_{lm} du_{r,m} + \Big[x_{l,t} x_{s,m} T_{tm} \beta_{sl}^{(p)} + \\ &+ \frac{1}{2} e_{trs} x_{r,l} P_{li} \beta_{ts,i}^{(p)} \Big] \, \mathrm{d}Q_p + \frac{1}{2} e_{trs} x_{r,l} P_{li} \beta_{ts}^{(p)} \, \mathrm{d}Q_{p,i} \ . \end{aligned} \tag{2.21}$$

The lagrangean equations for the equilibrium of a Cosserat Continuum, deducible applying the general variational Mechanics principles, according to (2.21) are

$$(x_{r,l} T_{ls})_{,s} = F_r \ (\text{in } C); \quad x_{r,l} T_{ls} N_s = fr \ (\text{on } \sigma) \ , \tag{2.22}$$

$$(x_{r,l} P_{ls})_{,s} + e_{rps} x_{p,l} x_{s,m} T_{ml} = M_r \ (\text{in } C) \ ; x_{r,l} P_{ls} N_s = m_r \ (\text{on } \sigma) \ , \tag{2.23}$$

where F_r and M_r characterize the external body forces and couples and f_r and m_r the analogous surface ones. N_r denotes the components of the inner unit vector perpendicular to the boundary σ of C.

3 Constitutive equations. Their dependence on the parameter h

The axiom of objectivity or, more simply, the fact that the deformation is characterized by the matrices ν_{rs}, ν^i_{rs}, requires that the free energy, Z, depends on the deformation through these matrices. After all, keeping in mind the expression (2.21) of $\overline{dl}^{(i)}$, it is possible to show that the constitutive equations are

$$\begin{cases} x_{r,l}t_{ls} = -R_{rm}\frac{\partial Z}{\partial \nu_{sm}} , \\ x_{r,l}P_{ls} = -e_{mtl}R_{rt}\frac{\partial Z}{\partial \nu^s_{ml}} . \end{cases} \tag{3.24}$$

Of course, the function Z depends also on the temperature (or on the entropy) but for my purpose it is not necessary to point out this fact. On the contrary, it is important to emphasize the dependence of Z on the parameter h, due to the dependence of deformation on it [see (1.11)]. Therefore, I assume that is

$$Z = W(\nu, \nu^i, h) . \tag{3.25}$$

Previously it has been remarked that the coefficient of dilatation and its derivatives with respect to h admit limit for $h = 0$. Therefore, an analogous property for the function Z seems reasonable, at least under suitable conditions of anlytical regularity satisfied by many bodies.

It is not my aim to develop the general consequences of the assumption (3.25), but to associate with (3.25) a special hypothesis that, in my opinion, is physically acceptable and leads to some interesting developments of the theory. That is, keeping in mind that when $h = 0$ the coefficient of dilatation depends only on the strain [see (1.11), (1.14)] in the following I assume the basic hypothesis

$$\lim_{h\to 0} Z(\nu, \nu^i, h) = \overline{W}(\varepsilon) , \tag{3.26}$$

in any position C'. One may assume the existence of a Taylor's series for Z or, at least, a polynomial expression but, supposing that the value of the parameter h is very small, in order to see the consequences of the theory, in short, in the following I simply assume

$$Z(\nu, \nu^i, h) = \overline{W}(\varepsilon) + h\overline{\overline{W}}(\nu, \nu^i) . \tag{3.27}$$

From (1.6) the relations

$$\varepsilon = \frac{1}{2}(\nu\nu^T - \underline{1}) , \quad \frac{\partial \varepsilon_{rs}}{\partial \nu_{pq}} = \frac{1}{2}(\delta_{rp}\nu_{sq} + \delta_{sp}\nu_{rq}) , \tag{3.28}$$

follow. Therefore the constitutive equations become

$$\begin{cases} x_{r,l}T_{ls} = -R_{rm}\left[\frac{\partial \overline{W}}{\partial \varepsilon_{sl}}\nu_{lm} + h\frac{\partial \overline{\overline{W}}}{\partial \nu_{sm}}\right] , \\ x_{r,l}P_{ls} = -he_{mtl}R_{rt}\frac{\partial \overline{\overline{W}}}{\partial \nu^{s}_{ml}} . \end{cases} \tag{3.29}$$

Let A_{rs} be the co-factor of $x_{r,s}$ in the determinant $\|x_{r,s}\|$. From (3.29) follows

$$T_{rs} = -\frac{\partial \overline{W}}{\partial \varepsilon_{rs}} - \frac{h}{D}R_{pm}A_{pr}\frac{\partial \overline{\overline{W}}}{\partial \nu_{sm}} . \tag{3.30}$$

4 The successive field and boundary equations. Consequences

According to (3.29), (3.30), the field and boundary equations (2.22), (2.23) depend explicitely on h. Therefore, in every equilibrium or evolutive problem the solutions u_r, T_{rs}, Q_r, R_{rs}, ν_{rs}, ν^{i}_{rs} depend on h. Under suitable regularity conditions, according to what has been said in the previous section, I assume that u_r, $Q_r, \ldots$ are expressible by polynomials in h (or by Taylor's series). Putting for a function $g(u_r, \ldots, h)$,

$$g^{(i)} = \left(\frac{d^i g}{dh^i}\right)_{h=0} , \tag{4.31}$$

one has

$$u_r = u^0_r + hu^{(1)}_r + \ldots , \quad Q_r = Q^0_r + hQ^{(1)}_r + \ldots \tag{4.32}$$

and also for $T_{rs}, \ldots$.

It is interesting to consider the consequences of the approximation (3.27), (3.29), assuming that in (4.32) the terms in h^2 are negligible. According to (3.29), one has

$$(x_{r,l}T_{ls})^0 = -\left(x_{r,l}\frac{\partial \overline{W}}{\partial \varepsilon_{sl}}\right)^0 = f^0_{rs}(u^0_{p,q}) , \quad (x_{r,l}P_{ls})^0 = 0 . \tag{4.33}$$

The corresponding field and boundary equations are

$$f^0_{rs,s} = F_r , \ (\text{in } C) \qquad f^0_{rs}N_s = f_r \ (\text{on } \sigma) \tag{4.34}$$

$$\begin{cases} 0 = M^0_r & (\text{in } C) \\ 0 = m^0_r & \end{cases} \qquad 0 = m_r \ (\text{on } \sigma) \tag{4.35}$$

Therefore, the stress is symmetric and the parameter p_{rs} are equal to zero. The analytical problem, expressed by (4.34), coincides with that of the classical theory of Cauchy. The parameter Q_I^0 are undetermined in this approximation but do not influence the stress.

It seems that the conditions (4.35) represent a discrepancy in the theory, because they place conditions on the external loads. Nevertheless, in my opinion, physical reasons require that M_r and m_r depend on a parameter that goes to zero when h tends to zero. In fact, if reducing the body forces acting on the particle which is in c', due to the external world, to a force applied in G' and a couple M_r, it is reasonable to mantain that M_r depends on the dimension of c' and goes to zero when c' tends to a point. That happens, for example, if the particle in c' is a magnetic dipole in a magnetic field. Further, the vector m_r characterizes surface couples exerted on the body by other bodies through the boundary, while $-m_r$ characterizes the couple that the body exerts on the external world through the boundary and goes to zero when h tends to zero. In short, for physical concreteness, it seems reasonable that the quantities M_r, m_r go to zero when h tends to zero. For example, those may be proportional to h and hM_r and hm_r must be substituted to M_r and m_r in all the equations. In the following I will assume that hypothesis.

In second approximation, according to (3.29), one has

$$\begin{aligned}(x_{r,l}T_{ls})^{(1)} &= -\left[\left(\frac{\partial \overline{W}}{\partial \varepsilon_{ls}}\right)^0 x_{r,l}^{(1)} + \left(\frac{\partial^2 \overline{W}}{\partial \varepsilon_{ls}\partial \varepsilon_{vq}}\right)^0 x_{r,l}^0 \varepsilon_{vq}^{(1)}\right] \\ &- R_{rm}^0 \left(\frac{\partial \overline{\overline{W}}}{\partial v_{sm}}\right)^0 = f_{rsvq}^{(1)}(u_{i,l}^0)u_{v,q}^{(1)} + S_{rs}^{(1)}(u_{p,q}^0, Q_i^0, Q_{t,m}^0)\ ,\end{aligned} \tag{4.36}$$

$$(x_{r,l}P_{ls})^{(1)} = -e_{mtl}R_{rt}^0 \left(\frac{\partial \overline{\overline{W}}}{\partial \nu_{ml}^s}\right)^0 = \Psi_{rs}^{(1)}(u_{p,q}^0, Q_i^0, Q_{i,l}^0)\ . \tag{4.37}$$

First of all I observe that because of the fact that the asymmetric part of $T_{rs}^{(1)}$ depens only on $\overline{\overline{W}}$ [see (3.30)], one has

$$\begin{aligned}e_{rvs}\left(u_{v,l}x_{s,m}T_{ml}\right)^{(1)} &= -e_{rps}u_{p,l}^0 u_{s,m}^0 \left(\frac{A_{tm}}{D}R_{tq}\frac{\partial \overline{\overline{W}}}{\partial \nu_{lq}}\right)^0 = \\ &= -e_{rpt}x_{pl}^0 R_{t,q}\left(\frac{\partial \overline{\overline{W}}}{\partial \nu_{lq}}\right)^0 = L_r^0(u_{i,m}^0, Q_i^0, Q_{i,l}^0)\ .\end{aligned} \tag{4.38}$$

According to (4.36), (4.37), (4.38), the field and boundary equations are

$$\begin{cases}\left[f_{rspq}^{(1)}(u_{i,l}^0)u_{p,q}^{(1)} + S_{rs}^{(1)}(u_{i,l}^{(0)}, Q_i^{(0)}, Q_{i,m}^0)\right]_{,s} = 0\ , & (\text{in } C)\ , \\ \left[f_{rspq}^{(1)}u_{p,q}^{(1)} + S_{rs}^{(1)}\right]N_s = 0 & (\text{on } \sigma)\ ,\end{cases} \tag{4.39}$$