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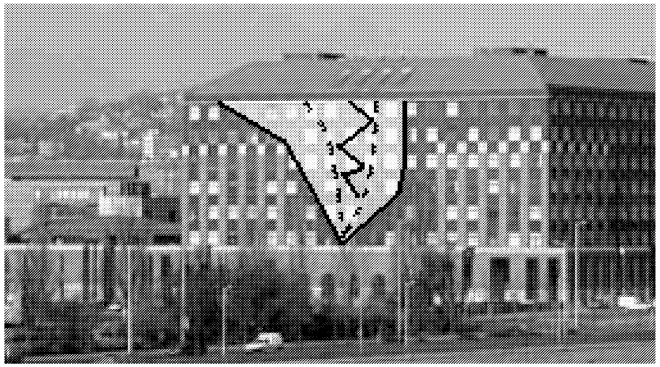

New criss-cross type algorithms for linear complementarity problems with sufficient matrices

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Abstract. We generalize new criss-cross type algorithms for linear complementarity problems (LCPs) given with sufficient matrices. Most LCP solvers require apriori information about the input matrix. The sufficiency of a matrix is hard to be checked (no polynomial time method is known). Our algorithm is similar to Zhang's linear programming, and Akkeleş-Balogh-Illés's criss-cross type algorithm for LCP-QP problems.

We modify our basic algorithm in such a way that can start with any matrix M, without having information about the property of the matrix (sufficiency, bisymmetricity, positive definitness, etc) in advance. Even in this case our algorithm terminates with one of the following cases in finite number of steps: it solves the LCP problem, solves its dual problem, or gives a certificate that the input matrix is not sufficient, so cycling can occur.

Although our algorithm is more general than that of Akkeleş and Illés's, the finiteness proof has benn simplified.

Furhermore, the finiteness proof of our algorithm gives a new, constructive proof to Fukuda and Terlaky's LCP duality theorem as well.

Keywords: linear complementarity problem, sufficient matrix, criss-cross algorithm, alternative and EP theorems.

Mathematics Subject Classification 2000: **49M35**, **90C20**.

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1 Introduction

The *linear complementarity problem* (LCP) is as follows: find vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, such that

$$-M\mathbf{u} + \mathbf{v} = \mathbf{q}, \quad \mathbf{u}\,\mathbf{v} = \mathbf{0}, \quad \mathbf{u},\ \mathbf{v} \geq \mathbf{0} \tag{1}$$

where $M \in \mathbb{R}^{n\times n}$, $\mathbf{q} \in \mathbb{R}^n$ and $\mathbf{u}\,\mathbf{v} = (u_1v_1, \dots, u_nv_n)$.

The linear complementarity problem is one of the most studied areas of mathematical programming. Large number of practical applications, the wide range of unsolved problems both theoretical and algorithmical makes it an attractive field of research.

The linear complementarity problem is $\mathbb{NP}$-complete, because the feasibility problem of linear equations with binary variables can be described as LCP problem too. The LCP problem remains $\mathbb{NP}$-complete, even if we restrict the matrix M to the class of negative semidifinite matrices [3], indeed it remains so on the class of P_0 matrices [12], too. The class of *sufficient matrices* is part of the class of P_0 matrices.

There are several different pivot algorithms to solve LCP problems. The criss-cross algorithm is one of those which was developed independently – for different optimization problems– by Chang, Terlaky and Wang. Since then, the criss-cross method become a class of algorithms, which differ in the rule of cross algorithm variants is based on the index selection rule and the orthogonality theorem.

Linear complementarity problems can be obtained from the Karush–Kuhn–Tucker optimality conditions for quadratic programming. For convex quadratic objective functions, the M matrix of the LCP problem is bisymmetric. Akkeleş, Balogh and Illés [1] developed their new criss-cross algorithm for such LCP problems. Their index selection rules are the *LIFO* (last-in-first-out) and the *most-often-selected-variable* rules. It is an interesting question, which is the widest class of matrices, to which the criss-cross algorithm with the above mentioned selection rules can be extended, reserving finiteness.

The class of sufficient matrices were introduced by Cottle, Pang and Venkateswaran [6]. Sufficient matrices can be interpreted as the generalization of P and PSD matrixes. Väliaho [17] has shown, that the class of sufficient matrices is the same as the class of P_* matrices. It was proved by den Hertog, Roos and Terlaky [9], that the sufficient matrices are exactly those, to which the criss-cross algorithm with minimal index selection rule can be applied to solve the LCP problem with any right hand side vector.

Our first aim is to generalize the criss-cross algorithm with the LIFO and most-often-selected-variable rules to the class of sufficient matrices. In addition, we simplify the proof of finiteness presented in [1]. The advantage of these selection rules, is that they offer reasonable freedom in variable selection (mostly at the beginning of the algorithm), providing us the possibility to avoid numerically instable pivots.

For the moment, there is no known efficient algorithm to decide whether a matrix is sufficient or not. (Väliaho [16] developed an inductive method to check sufficiency, but the algorithm is not polynomial). Most algorithms for LCP problems developed so far has the practically unattractive property, that they need the apriori information that the matrix is sufficient. Fukuda, Namiki and Tamura [7] gave the first algorithm, based on the LCP duality theorem of Fukuda and Terlaky [8] and modified to the form of EP theorems, which did not required apriori information on the sufficiency of the matrix. If the algorithm couldn't proceed, or began to cycle, it provides a polynomial size certificate, that the input matrix is not sufficient.

Our second aim is to extend the generalized algorithm such that for an arbitrary matrix M and right hand side $\mathbf{q}$, it either solves the LCP problem, or provides a polynomial size certificate that the matrix M is not sufficient. This property improves the value of the algorithm significantly, because it makes it applicable to a wide range of LCP problems, without requiring apriori information on the properties of the matrix. Indeed, the improved freedom of the pivot position selection compared to the minimal index rule gives the possibility to avoid numerically instable pivots, extending the practical applicability further.

The proof of this modified algorithm is a new, constructive proof for the LCP duality theorem in the stronger EP form.

Finally, let us fix the notations used in this paper:

$\mathbf{x}, x_i$	bold characters for vectors, normal for scalars
$\mathbf{v}\,\mathbf{u}$	the coordinate product (Hadamard) of $\mathbf{v}$ and $\mathbf{u}$ vectors
M	the matrix of the LCP problem, $M \in \mathbb{R}^{n\times n}$
B	basis, an $n \times n$ nonsingular submatrix of $[-M, I] \in \mathbb{R}^{n\times 2n}$
$\bar{M}$	short pivot tableau for a given basis
M_B	short pivot tableau for a given basis if the notation makes it necessary to emphasize that it belongs to basis B
$\bar{\mathbf{m}}_j$	the j^{th} column of matrix $\bar{M}$
$\bar{\imath}$	$= n+i$ if $i \in \{1,\dots,n\}$ and $= i-n$ if $i \in \{n+1,\dots,2n\}$
J_B	indices belonging to basis B
$J_N := \overline{J_B}$	indices outside basis B
$\oplus, \ominus, +, -$	nonnegative, nonpositive, positive, negative element
$< \dots >$	spanned vectorspace

2 Sufficient matrices

The research on the linear complementarity problems, and the efficiency of the solution methods are based on the properties of the matrix M. In the following, we survey some properties of the sufficient matrices.

We will use the following technical type definition.

Definition 2.1. For an $M \in \mathbb{R}^{n\times n}$ matrix, and the subset $\alpha = \{\alpha_1,\dots,\alpha_k\} \subseteq \{1,\dots,n\}$ of indices, the rectangular submatrix $M_{\alpha\alpha}$ is called a principal submatrix.

Now, we define the *sufficient matrices*, [6]

Definition 2.2. The $M \in \mathbb{R}^{n\times n}$ matrix is called column sufficient if there exists no $\mathbf{x} \in \mathbb{R}^n$ vector, such that

$$\begin{cases} x_i\,(M\mathbf{x})_i \leq 0 & \text{for all indices } i \in \{1,\dots,n\} \\ x_j\,(M\mathbf{x})_j < 0 & \text{for at least one index } j \in \{1,\dots,n\} \end{cases} \tag{2}$$

and we call it row sufficient if its transpose is column sufficient.

The M matrix is called sufficient, if it is both column and row sufficient at the same time.

It can be shown, that the column sufficient matrices are exactly those, for which the solution set of linear complementarity problems is convex [6].

The class of sufficient matrices were introduced by Cottle et al. [6]. They have shown, that these are generalizations of P and positive semidefinite matrices. They have also shown, that sufficient matrices are specially structured P_0 matrices.

Later, den Hertog et al. [9] proved that sufficient matrices are exactly those, for which the criss-cross algorithm with minimal index rule can solve linear complementarity problems for any right hand side vector $\mathbf{q}$, in finite number of iterations.

To show some important properties of the sufficient matrices, we will need the definition of *strictly sign reversing*, and *strictly sign preserving* vectors, [7]:

Definition 2.3. We call a vector $\mathbf{x} \in \mathbb{R}^{2n}$ strictly sign reversing, if

$$\begin{array}{ll} x_i x_{\bar{i}} \leq 0 & \text{for all indices } i = 1, \dots, n \\ x_i x_{\bar{i}} < 0 & \text{for at least one index } i \in \{1, \dots, n\} \end{array}$$

We call a vector $\mathbf{x} \in \mathbb{R}^{2n}$ strictly sign preserving, if

$$\begin{array}{ll} x_i x_{\bar{i}} \geq 0 & \text{for all indices } i = 1, \dots, n \\ x_i x_{\bar{i}} > 0 & \text{for at least one index } i \in \{1, \dots, n\} \end{array}$$

Let us introduce subspaces

$$V := \{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^{2n} \mid [-M, I](\mathbf{u}, \mathbf{v}) = \mathbf{0}\}$$

and

$$V^{\perp} := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2n} \mid [I, M^T](\mathbf{x}, \mathbf{y}) = \mathbf{0}\}$$

Obviously, V and $V^{\perp}$ are orthogonal complementary subspaces of $\mathbb{R}^{2n}$.

Lemma 2.1. *The matrix $M \in \mathbb{R}^{n \times n}$ is sufficient if and only if there exists no strictly sign reversing vector in V, and there exists no strictly sign preserving vector in $V^{\perp}$. [7]*

Let us introduce the *short pivot tableaus*.

Definition 2.4. Let S be a given set of vectors, its components indexed by J. Furthermore, let $J_B \subseteq J$ such that the vectors indexed by it define a basis of S. Let $J_N = J \setminus J_B$ be the indices of the vectors in S outside the basis B. Then the short pivot tableau belonging to J_B is the $M \in \mathbb{R}^{|J_B| \times |J_N|}$ matrix, where m_{ij} is the coefficient of the vector belonging to $j \in J_N$ for $i \in J_B$.

Next lemma shows the sign structure of sufficient matrices. This structure is the only significant property we use in the rest of the paper.

Lemma 2.2. *(Cottle, Pang and Venkateswaran [6]) Let M be a sufficient matrix, B a basis, $\bar{M} = [\bar{m}_{ij} : i \in J_B, j \in J_N]$ the corresponding short pivot tableau. Then*

(a) *$\bar{m}_{i\bar{i}} \geq 0$ for all $i \in J_B$; furthermore*
(b) *for all $i \in J_B$, if $\bar{m}_{i\bar{i}} = 0$ then $\bar{m}_{i\bar{j}} = \bar{m}_{j\bar{i}} = 0$ or $\bar{m}_{i\bar{j}} \cdot \bar{m}_{j\bar{i}} < 0$ for all $j \in J_B$, $j \neq i$.*

We have to mention that the proof of the lemma above is constructive, so if the given structure of the matrix is violated, from the tableau $\bar{M}$ we can easily obtain certificate that M is not sufficient. The coding size of this proof is bounded by a polynomial of the coding size of M.

By the *permutation* of $M \in \mathbb{R}^{n \times n}$ we mean the matrix $P^T M P$, where P is a permutation matrix.

Lemma 2.3. *[9] Let $M \in \mathbb{R}^{n \times n}$ be a row (column) sufficient matrix. Then*

1. *any permutation of matrix M is row (column) sufficient,*
2. *the product DMD is row (column) sufficient, where $D \in \mathbb{R}_+^{n \times n}$ is a diagonal matrix,*
3. *every principal submatrix of M is row (column) sufficient.*

It is easy to prove that if M is sufficient, then after any number of arbitrary pivots, the matrix $\bar{M}$ is also sufficient, so the class of sufficient matrices is closed to the pivot operation, namely their properties are preserved during the criss-cross type algorithms, too.

For a matrix $M \in R^{n \times n}$, and $\alpha \subseteq \{1, \dots, n\}$, if $M_{\alpha\alpha}$ is nonsingular, then we will denote the block pivot operation belonging to α by η_α.

Lemma 2.4. *Let $M_{\alpha\alpha}$ be a nonsingular submatrix of the row (column) sufficient matrix M. Then the matrix $M' = \eta_\alpha(M)$ is row (column) sufficient. [9]*

So the class of sufficient matrices is closed also to the operation of block pivot.

3 The alternative theorem of the linear complementarity problem

The decision problem, whether an arbitrary linear complementarity problem has a solution or not, is in $\mathbb{NP}$, and not always in *co*-$\mathbb{NP}$. For some matrix classes it belongs to *co*-$\mathbb{NP}$. [3] The class of sufficient matrices is such a class. Let us formulate (1) again:

$$\left.\begin{array}{r}(\mathbf{u},\mathbf{v}) \in V(M,\mathbf{q}) := \{(\mathbf{u},\mathbf{v}) : -M\mathbf{u}+\mathbf{v}=\mathbf{q}\} \\ \mathbf{u}\,\mathbf{v}=\mathbf{0}, \quad \mathbf{u},\,\mathbf{v}\geq \mathbf{0}\end{array}\right\} \qquad (P-LCP)$$

In the theory of optimization, a naturally arising question is that if the problem $(P-LCP)$ has no solution, then is there another problem, defined by the same input data, which has.

Fukuda and Terlaky [8] answered this question in a very general form, for complementarity problems on oriented matroids.

Following the approach of Fukuda and Terlaky, we define the problem $(D-LCP)$

$$\left.\begin{array}{r}(\mathbf{x},\mathbf{y}) \in V(M,\mathbf{q})^{\perp} := \{(\mathbf{x},\mathbf{y}) : \mathbf{x}+M^T\mathbf{y}=\mathbf{0}, \;\; \mathbf{q}^T\mathbf{y}=-1\} \\ \mathbf{x}\,\mathbf{y}=\mathbf{0}, \;\; \mathbf{x},\,\mathbf{y}\geq \mathbf{0}\end{array}\right\} \qquad (D-LCP)$$

To characterize the feasibility of the linear complementarity problems, we can formulate the theorem of Fukuda and Terlaky as follows:[1]

Theorem 3.1. *For a sufficient matrix* $M \in \mathbb{R}^{n\times n}$*, and a vector* $\mathbf{q} \in \mathbb{R}^n$*, exactly one of the following statements hold:*
(1) the (P-LCP) problem has a feasible complementary solution $(\mathbf{u},\mathbf{v})$*,*
(2) the (D-LCP) problem has a feasible complementary solution $(\mathbf{x},\mathbf{y})$*.*

Proof. Suppose that both of them can be solved, and let $(\mathbf{u},\mathbf{v})$ be the solution of $(P-LCP)$ and $(\mathbf{x},\mathbf{y})$ be the solution of $(D-LCP)$. Then from the condition

$$-M\mathbf{u}+\mathbf{v}=\mathbf{q}$$

after taking left matrix product with $\mathbf{y}^T$ we get

$$-\mathbf{y}^T M\mathbf{u}+\mathbf{y}^T\mathbf{v}=\mathbf{y}^T\mathbf{q}=-1.$$

Using the first condition of the $(D-LCP)$ problem and the nonnegativity of our variables, we have the relation

$$0 \leq \mathbf{x}^T\mathbf{u}+\mathbf{y}^T\mathbf{v}=-1$$

which is a contradiction.

We prove that one of problems $(P-LCP)$ and $(D-LCP)$ is feasible. To do this, we use the criss-cross algorithm with the new selection rule, where

[1]Fukuda and Terlaky [8] call their result as a *duality theorem*, but we think the *alternative theorem* represents its meaning more precisely.

finiteness of the algorithm will complete the proof of the alternative theorem. □

This also means that if the matrix M is sufficient and rational, then if the problem $(P-LCP)$ has no feasible complementary solution then it is well characterized, and a polynomial size certificate can be given, namely the solution of the problem $(D-LCP)$.

3.1 The criss-cross type algorithm

First, we generalize the algorithm of Akkeleş, Balogh and Illés [1] to the class of sufficient matrices. Let us denote with $\mathcal{I} := \{u_1, u_2, ..., u_N, v_1, v_2, ..., v_N\} \cup \{q\}$ the set of variables, while $I := \{1, 2, ..., N, \bar{1}, \bar{2}, ..., \bar{N}\} \cup \{q\}$ is the set of the corresponding indices, where $N = n+m$ and $|\mathcal{I}| = |I| = 2N+1$. To simplify the notation, for any $\alpha \in I$ let $\bar{\alpha}$ be its complementary variable's index, and let $\bar{\bar{\alpha}} = \alpha$ for all $\alpha \in I \setminus \{q\}$, so the complementary pair of $\bar{\alpha}$ is α.

The initial complementary solution of the linear complementarity problem (1) is $\mathbf{u} = \mathbf{0}$, $\mathbf{v} = \mathbf{q}$. Let the matrix of the short pivot tableau be denoted by $\bar{M}$.

Our aim is to generate a feasible complementary solution from our initial solution, using pivot operations. Lemmas 2.3. and 2.4. ensure that the sufficiency of the matrix is preserved during the algorithm. Moreover, we only make such pivot operations that preserve complementarity, too.

Criss-cross type algorithm

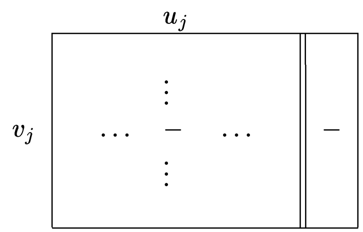
Input:
The problem (1), where M is sufficient, $\bar{M} := -M$, $\bar{q} := q$, $r := 1$, and the counter vector $\mathbf{s} = (0, \dots, 0)$ of the siye $2n$.
Begin
  $J := \{\alpha \in I : \bar{q}_\alpha < 0\}$
  While $(J \neq \emptyset)$ **do**
    $J_{\max} := \{\beta \in J : s_{r-1}(\beta) \geq s_{r-1}(\alpha),$ for all $\alpha \in J\}$
    let $k \in J_{\max}$ arbitrary index
    If $(\bar{m}_{kk} < 0)$ **then**
      diagonal pivot on $\bar{m}_{kk}$
      update vector $\mathbf{s}$
      $r := r + 1$
    Else
      $K := \{\alpha \in I : \bar{m}_{k\alpha} < 0\}$
      If $(K = \emptyset)$ **then**
        Stop: The LCP problem has no feasible solution
      Else
        $K_{\max} = \{\beta \in K : s_{r-1}(\beta) \geq s_{r-1}(\alpha),$ for all $\alpha \in K\}$
        let $l \in K_{\max}$ arbitrary
        exchange pivot on $\bar{m}_{kl}$ and m_{lk}
        update $\mathbf{s}$
        $r := r + 2$
      Endif
    Endif
  Endwhile
  Stop: A feasible complementary solution has been computed.
End

We explain the pivot operations (diagonal and exchange pivot) and the role of the counter vector $\mathbf{s}$ when we analyze the algorithm.

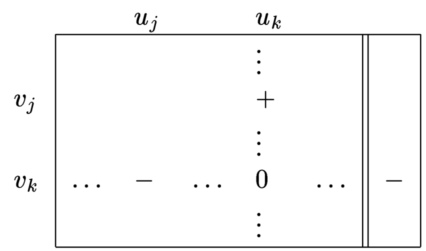
Let us assume that v_j is a basic variable and the value of variable v_j is infeasible. If $\bar{m}_{jj} < 0$ then we perform *diagonal pivot*, where variable u_j enters the basis, while variable v_j leaves it.

If $\bar{m}_{jj} = 0$, then we have to pivot on such an index k, for which $\bar{m}_{jk} < 0$. The solution obtained after a pivot like this is not complementary

any more, so to restore the complementarity of the solution we have to pivot on position (k, j) as well. Lemma 2.2. ensures that in such a case $\bar{m}_{kj} > 0$. These two pivots together are called *exchange pivot*.



Diagonal pivot



Exchange pivot

During an exchange pivot, variables u_j and u_k enter the basis, while v_j and v_k leave it. (see the figure of exchange pivot) We say that u_k and v_k are chosen actively, while u_j and v_j are chosen passively.

The pivot rule LIFO is handled by the help of the vector $\mathbf{s}_r : I \to \mathbb{N}_0^{2N}$ in paper [1]:

$$s_r(\alpha) = \begin{cases} r, & \text{if the variable indexed by } \alpha \in I \text{ moves in iteration } r \\ s_{r-1}(\alpha), & \text{otherwise.} \end{cases}$$

Let $\mathbf{s}_0(\alpha) = 0$ for all $\alpha \in I$. Obviously, $\mathbf{s}_r \geq \mathbf{s}_{r-1}$ and $\mathbf{s}_r \neq \mathbf{s}_{r-1}$.

The algorithm starts from the trivial complementary solution, and because it only makes diagonal and exchange pivots, it preserves complementarity. The sufficiency of matrix M ensures that (lemma 2.2) if we have to do exchange pivot, then the sign of the chosen pivot elements will be as desired. The algorithm terminates only if there is no solution, or if it has found the solution. So we have to prove that it is *finite*. Because the number of possible bases is finite, we have to show that the criss-cross type algorithm with LIFO pivot rule does not *cycle*.

3.2 Orthogonality theorem

We generalize Akkeleş, Balogh and Illés's [1] proof to the class of sufficient matrices, while simplifying it at the same time, thus making it possible to be modified in the sense of EP theorems too. Most of our proofs are based on the well known orthogonality theorem.

Let us define the vectors $\mathbf{t}^{(i)}$, $j \in J_B$ and $\mathbf{t}_j$, $j \in J_N \cup \{q\}$ as follows:

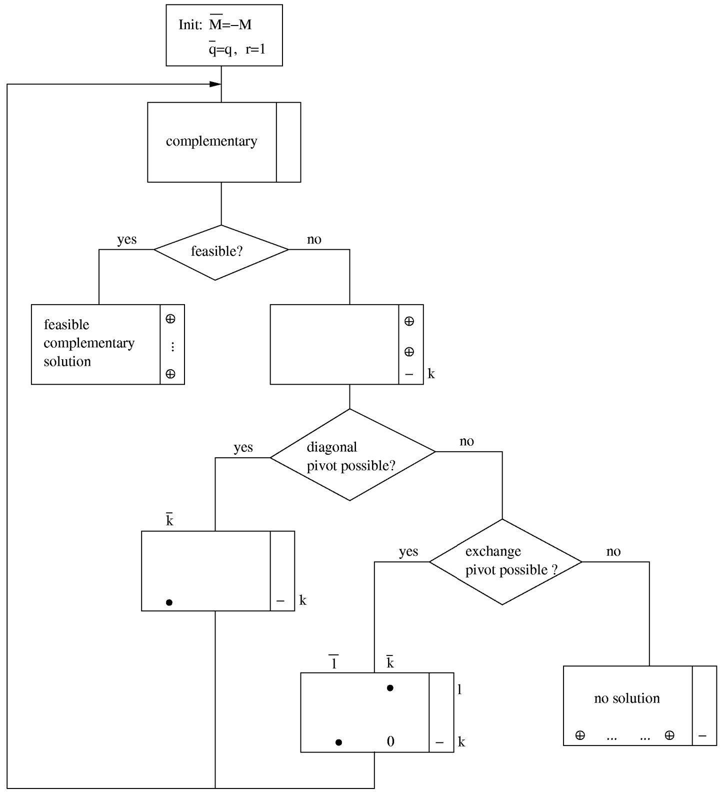


Figure 1: Flow chart of the algorithm

$$\left(\mathbf{t}^{(i)}\right)_k = \begin{cases} \bar{m}_{ik}, & \text{if } k \in J_N \cup \{q\} \\ 1, & \text{if } k = i \\ 0, & \text{otherwise} \end{cases}$$

and

$$(\mathbf{t}_j)_k = \begin{cases} \bar{m}_{kj}, & \text{if } k \in J_B \\ -1, & \text{if } k = j \\ 0, & \text{otherwise} \end{cases}$$

Definition 3.1. We call the above defined $\mathbf{t}^{(i)}$ vector a fundamental circuit, and the $\mathbf{t}_j$ vector as a fundamental cocircuit. [7]

We can now state the orthogonality theorem, [11]:

Theorem 3.2. *For a matrix $M \in \mathbb{R}^{n\times n}$ and its arbitrary bases B' and B'', the vectors $\mathbf{t}'^{(i)}$ and $\mathbf{t}''_j$ belonging to $M_{B'}$ and $M_{B''}$ respectively are orthogonal.*

3.3 Almost terminal tableaus

Let us assume that there exists an example, that the algorithm is not finite. The number of bases is finite, namely $\binom{2n}{n}$, so the algorithm makes infinite number of iterations only if cycling occurs. Let us consider a cycling example with minimal size. In such example, because of the minimality, every variable moves during the cycle.

Assume, that every variable has already moved at least once. Then $|J_{\max}| = |K_{\max}| = 1$ in every further iteration, because during any pivot, we assign to the moving variables such $\mathbf{s}$ counter values, that have not yet assigned, and for variable pairs with the same assigned value, exactly one of them is in basis, (the other should be nonbasic variable).

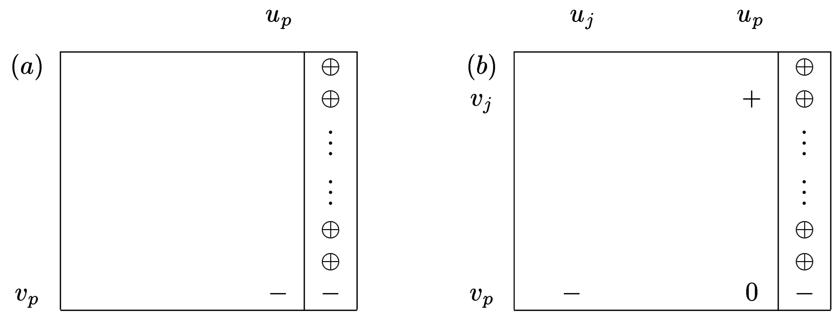
Consider the r^{th} iteration. Consider the least valued basic variable v_p according to the order provided by $\mathbf{s}$, then

$$p = \text{argmin}\,\{i \in I_B \,:\, s_r(i) \le s_r(j),\ \forall\, j \in I_B\}.$$

The $\mathbf{s}$ value of variable v_p cannot change while it remains in basis, so because of our pivot rule, if we make a pivot in its row in iteration $r' > r$, then (i) p is not feasible and (ii) its $\mathbf{s}$ value is maximal among the infeasible variables. Condition (ii) is satisfied if and only if v_p is the only infeasible variable in iteration r'. The short pivot tableaus for this case can be as follows:

1. The algorithm chooses u_p to enter the basis.

   The diagonal element $\bar{m}_{pp} < 0$, so a diagonal pivot is possible [tableau (a)]: u_p enters the basis, while v_p leaves it.



After the diagonal pivot, the values in **s** change according to the rule:

$$s_{r'}(\alpha) = \begin{cases} r', & \text{if } \alpha \in \{p, \bar{p}\}, \\ s_{r'-1}(\alpha), & \text{otherwise} \end{cases}$$

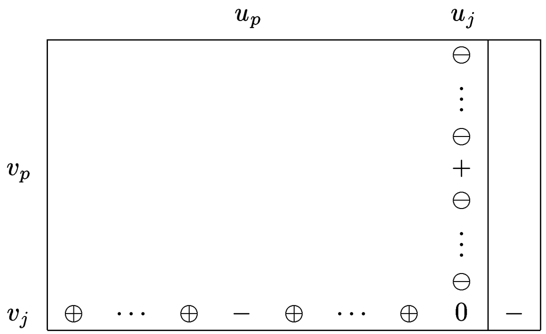
2. The algorithm chooses variable u_p to enter the basis, but $\bar{m}_{pp} = 0$ so an exchange pivot is necessary, [tableau (b)]. Variables u_p and u_j enter the basis, while variables v_p and v_j leave it.

The column of **q** is the same as in tableau (a). In this case, it is unimportant whether u_j or v_j is in the basis. We consider the case, when v_j is in the basis. For the other case, we only need to switch the role of j and $\bar{j}$ in the definition of $s_{r'+1}(\alpha)$:

$$\mathbf{s}_{r'+1}(\alpha) = \begin{cases} r', & \text{if } \alpha \in \{\bar{p}, j\}, \\ r'+1, & \text{if } \alpha \in \{p, \bar{j}\}, \\ s_{r'-1}(\alpha), & \text{otherwise} \end{cases}$$

3. The algorithm chooses a variable u_j to enter the basis, but $\bar{m}_{jj} = 0$ so an exchange pivot is needed, and the algorithm chooses the variable u_p as well, (tableau (c)).

(c)



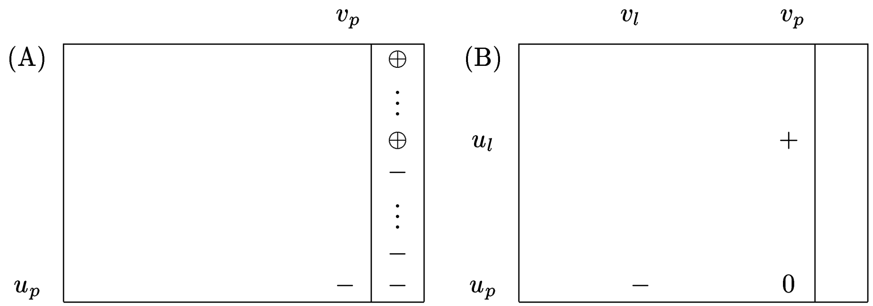
According to our pivot rule, in the row of v_j only in the columns of u_p and **q** can be negative elements, and because $\bar{m}_{jj} = 0$, using lemma 2.2 we can also fill in the sign structure of u_j's column. In this case, it is once again unimportant whether u_j or v_j is in the basis. We consider the case, when v_j is in the basis, for the other case, we again switch the role of j and $\bar{j}$ in the definition of $s_{r'+1}(\alpha)$:

$$s_{r'+1}(\alpha) = \begin{cases} r', & \text{if } \alpha \in \{p, \bar{j}\}, \\ r'+1, & \text{if } \alpha \in \{\bar{p}, j\}, \\ s_{r'-1}(\alpha), & \text{otherwise} \end{cases}$$

For later use we state that in cases 1. and 2. we used only the pivot rule when filled in the sign structure, while we used the sufficiency of the matrix in the third case for the column of u_j.

Now we consider the moment when u_p leaves the basis for the first time after iteration r. Let this be iteration $r'' > r'$, and the corresponding basis be denoted by B''. The pivot tableau for this iteration can have three different structures:

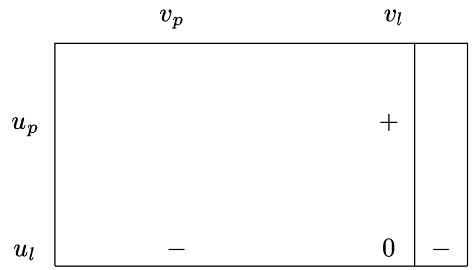
A. According to the pivot rule, we choose variable u_p to leave the basis, $\bar{m}_{pp} < 0$, so a diagonal pivot takes place.



B. The pivot rule chooses variable u_p to leave the basis, but $\bar{m}_{pp} = 0$ so an exchange pivot is needed: v_l (or u_l) enters the basis, while u_l(or v_l) leaves it.

C. The algorithm chooses variable u_l(or v_l), but $\bar{m}_{ll} = 0$ so an exchange pivot takes place, and v_p enters the basis, while u_l leaves it.

(C)



In the following, we show that if the matrix M is sufficient, then none of the tableaus $(a) - (c)$ can be followed by one of the tableaus $(A) - (C)$. For the sake of our later modified algorithm, we track where we use the sufficiency of the matrix too.

3.4 Auxiliary lemmas

We consider those cases first, which do not take into consideration the sufficiency of the matrix of the linear complementarity problem.

First, we show that tableau (c) cannot be followed by tableaus (A) or (B).

Lemma 3.3. *Let us denote the tableau of case* (c) *by* $M_{B'}$*, and the tableau of* A *(or* B*) by* $M_{B''}$*. Consider the vectors* $\mathbf{t}'^{(\bar{j})}$ *and* $\mathbf{t}''_q$*, read from the row of the basic variable* v_j *in tableau* $M_{B'}$*, and from the column of* $\mathbf{q}$ *in* $M_{B''}$*. Then*

$$(\mathbf{t}'^{(\bar{j})})^T\,\mathbf{t}''_q > 0$$

Proof. Let $J'' := \{\,\alpha \in J_{B''} \,:\, \bar{q}''_i < 0\,\}$. Because of the properties of vector $\mathbf{s}$, $J'' \subset J_{B'}$ and so $\mathbf{t}'_{\bar{j}i} = 0$ for all indices $i \in J'' \setminus \{\bar{j}, p\}$, thus

$$\sum_{i \in J'' \setminus \{\bar{j},p\}} t_{\bar{j}i} t''_{iq} = 0, \tag{3}$$

Taking into consideration that $s_{r'}(\bar{j}) = s_{r'}(p)$ and $s_{r''-1}(j) > s_{r''-1}(p)$, we know that t''_{jq} and $t''_{\bar{j}q}$ are nonnegative. It can be read from tableau (c) that $t'_{\bar{j}j} = 0$, $t'_{\bar{j}\bar{j}} = 1$, $t'_{\bar{j}\bar{p}} = 0$, $t'_{\bar{j}p} < 0$ and $t'_{\bar{j}q} < 0$ so

$$t'_{\bar{j}\bar{j}}\, t''_{\bar{j}q} + t'_{\bar{j}j} t''_{jq} + t'_{\bar{j}\bar{p}} t''_{\bar{p}q} + t'_{\bar{j}p}\, t''_{pq} + t'_{\bar{j}q}\, t''_{qq} = t'_{\bar{j}p} t''_{pq} - t'_{\bar{j}q} > 0, \tag{4}$$

because $t''_{qq} = -1$ by definition, and $t''_{pq} < 0$ according to the pivot rule of the algorithm. (tableaus (A) and (B)).

If $l \notin J'' \cup \{j, \bar{j}, p, \bar{p}, q\}$ then again from the tableaus we know that $t'_{\bar{j}l} \geq 0$ and by the definition of J'' it stands that $t''_{lq} \geq 0$, so

$$\sum_{l \notin J \cup \{j,\bar{j},p,\bar{p},q\}} t'_{\bar{j}l}\, t''_{lq} \geq 0. \tag{5}$$

Summing inequalities (3)-(5) we get our claim. □

From tableau (c) we consider the row of variable v_j, while in the structure of the tableaus (A) and (B) we do not make use of the sufficiency of the matrix, so the lemma uses only the pivot rule. Tableaus (c) and (A) (or (B)) are exclusive because of the orthogonality theorem and the lemma above, regardless of the sufficiency of the matrix.

We now prove that tableaus (a) and (b) cannot be followed by tableau (C).

Lemma 3.4. *Let us denote the tableau* (a) *(or* (b)*) by* $M_{B'}$*, and the tableau* (C) *by* $M_{B''}$*. Consider the vectors* $\mathbf{t}'_q$ *and* $\mathbf{t}''^{(l)}$*, belonging to the column of* $\mathbf{q}$ *in tableau* $M_{B'}$*, and to the row of* u_l *in tableau* $M_{B''}$*. Then*

$$(\mathbf{t}''^{(l)})^T\,\mathbf{t}'_q > 0$$

Proof. Like in the previous lemma, $J''_l := \{i \in I_{N''} : t''_{li} < 0\} \subset I_{N'}$, so $t'_{iq} = 0$ holds for every $i \in J''_l$, thus

$$\sum_{i \in J''_l} t''_{li}\, t'_{iq} = 0. \tag{6}$$

Furthermore, for an index $j \notin J_2 := J''_l \cup \{\bar{l}, l, \bar{p}, p, q\}$, $t''_{lj} \geq 0$ and $t'_{jq} \geq 0$, therefore

$$\sum_{j \notin J_2} t''_{lj} t'_{jq} \geq 0. \tag{7}$$

From tableaus $M_{B'}$ and $M_{B''}$ it can be read that $t'_{qq} = -1$, $t''_{ll} = 1$, $t''_{l\bar{l}} = t''_{lp} = t'_{pq} = 0$ and $t''_{l\bar{p}} < 0$, $t''_{lq} < 0$, $t'_{\bar{p}q} < 0$, $t'_{\bar{l}q} \geq 0$ and $t'_{lq} \geq 0$ so

$$t''_{l\bar{l}} t'_{\bar{l}q} + t''_{ll}\, t'_{lq} + t''_{l\bar{p}}\, t'_{\bar{p}q} + t''_{lp} t'_{pq} + t''_{lq}\, t'_{qq} = t'_{lq} + t''_{l\bar{p}}\, t'_{\bar{p}q} - t''_{lq} \geq t''_{l\bar{p}} t'_{\bar{p}q} - t''_{lq} > 0. \tag{8}$$

Summing relations (6) – (8) we get our claim. □

We can now consider tableaus where the sufficiency of the matrix plays an important role.

In the following, we show that tableaus (a) (or (b)) cannot be followed by tableaus (A) or (B).

Lemma 3.5. *Let the complementary solutions* $(\mathbf{u}', \mathbf{v}')$ *and* $(\mathbf{u}'', \mathbf{v}'')$*, belonging to tableaus* (a) *(or* (b)*) and* (A) *(or* (B)*), be given. Then*

$$(\mathbf{u}' - \mathbf{u}'')\; M\; (\mathbf{u}' - \mathbf{u}'') \leq \mathbf{0}$$

Proof. We prove all the four cases simultaneously.

$$\begin{aligned}
(\mathbf{u}' - \mathbf{u}'')\; M\; (\mathbf{u}' - \mathbf{u}'') &= (\mathbf{u}' - \mathbf{u}'')\; (\mathbf{q} + M\,\mathbf{u}' - \mathbf{q} - M\,\mathbf{u}'') \\
&= (\mathbf{u}' - \mathbf{u}'')\; (\mathbf{v}' - \mathbf{v}'') \\
&= \mathbf{u}'\; \mathbf{v}' - \mathbf{u}'\; \mathbf{v}'' - \mathbf{u}''\; \mathbf{v}' + \mathbf{u}''\; \mathbf{v}'' = -\mathbf{u}'\; \mathbf{v}'' - \mathbf{u}''\; \mathbf{v}',
\end{aligned}$$

where the last equation holds because of the complementarity of the given solutions. Let $J'' := \{\,\alpha \in J_{B''} \,:\, \bar{q}''_i < 0\,\}$. According to the pivot rule, $s_{r''}(p) > s_{r''}(\alpha)$ for all $\alpha \in J'' \setminus \{p\}$, so these indices have not moved since basis B', so $\alpha \in J_{B'}$ and thus for all $i \in J'' \setminus \{p\}$, the value of u'_i (or v''_i) and u''_i (or v'_i) is zero:

$$u'_i\; v''_i + u''_i\; v'_i = 0 \tag{9}$$

From tableau (a) (or (b)), and tableau (A) (or (B)) it can be read that $u'_p = 0, v'_p < 0$ and $u''_p < 0$, $\mathbf{v}''_p = 0$ so,

$$u'_p\; v''_p + u''_p\; v'_p > 0, \tag{10}$$

Furthermore, for any $j \notin J''$ $u'_j, v'_j, u''_j, v''_j \geq 0$ holds, thus

$$u'_j\ v''_j + u''_j\ v'_j \geq 0. \tag{11}$$

Summarizing, the vector $(\mathbf{u}' - \mathbf{u}'')$ is such that $(\mathbf{u}' - \mathbf{u}'')\ M\ (\mathbf{u}' - \mathbf{u}'') \leq \mathbf{0}$. □

Note that the proof is constructive, because from the bases B' and B'', the vector $\mathbf{u}' - \mathbf{u}''$ proving the lack of sufficiency of our matrix can easily be obtained.

In the last lemma, we will investigate the case when tableau (c) would be followed by tableau (C).

Lemma 3.6. *Let us denote the tableau (c) by $M_{B'}$, and the tableau (C) by $M_{B''}$. Consider the vectors $\mathbf{t}'_j$ and $\mathbf{t}''^{(l)}$ belonging the column of u_j in tableau $M_{B'}$, and to the row of u_l in tableau $M_{B''}$. Then*

$$(\mathbf{t}''^{(l)})^T\ \mathbf{t}'_j < 0$$

Proof. Let $J''_l = \{i \in I_{N''}\ :\ t''_{li} < 0\} \setminus \{j\}$. Because our tableaus are complementary, and according to our pivot rule we chose the variable from the set of possible variables which moved last, thus the variables of indices J''_l have not moved since $M_{B'}$, so $\bar{J}''_l \subset I_{B'}$ and $J''_l \subset I_{N'}$, thus $t'_{ij} = 0$ if $i \in J''_l$. Summing these, we get

$$\sum_{i \in J''_l \cup \{q\}} t''_{li}\, t'_{ij} = 0. \tag{12}$$

Furthermore, if $i \notin J_1 := J''_l \cup \{q, p, \bar{p}, j, \bar{j}, l, \bar{l}\}$ then $t'_{ij} \leq 0$ according to tableau (c). By the definition of J''_l, $t''_{li} \geq 0$, so

$$\sum_{i \notin J_1} t''_{li}\, t'_{ij} \leq 0, \tag{13}$$

From tableaus $M_{B'}$ and $M_{B''}$, taking into consideration the definition of vector $\mathbf{t}$, we get that

$$t''_{l\bar{l}} = t'_{\bar{j}j} = t'_{qj} = 0,\ t''_{ll} = 1,\ t'_{jj} = -1\ \text{ and } t'_{lj} \leq 0,\ t''_{lp} < 0,\ t'_{pj} > 0$$

so

$$t''_{lq}\, t'_{qj} + t''_{lp}\, t'_{pj} + t''_{l\bar{p}}\, t'_{\bar{p}j} + t''_{lj}\, t'_{jj} + t''_{l\bar{j}}\, t'_{\bar{j}j} + t''_{ll} t'_{lj} + t''_{l\bar{l}}\, t'_{\bar{l}j} < -t''_{lj} \tag{14}$$

We have to show that $t''_{lj} \geq 0$. In the tableau of basis B' in case (c), we do an exchange pivot. In this case, $s_{r'}(\bar{j}) = s_{r'}(p) = r'$ and $s_{r'+1}(j) =$

$s_{r'+1}(\bar{p}) = r' + 1$. In the tableau of $M_{B''}$, $J''_{max} = \{\bar{p}\}$. Because the tableau is complementary, two cases are possible:

1. If $j \in I_{N''}$ then variable u_j has moved between iterations $(r' + 1)$ and r'', so $s_{r''}(j) > s_{r''}(\bar{p})$ and because of our pivot rule, this is possible only if $t''_{lj} \geq 0$.

2. If $j \in I_{B''}$, then $t''_{lj} = 0$.

Summing inequalities (12) – (14) we get our claim. □

3.5 Finiteness of the criss-cross type algorithm

In this section, we prove the finiteness of the criss-cross algorithm, making the proof of theorem 3.1 complete.

Theorem 3.7. *The criss-cross type algorithm is finite for the linear complementarity problem with sufficient matrices.*

Proof. Assume contrary, that the algorithm is not finite. Because a linear complementarity problem has finitely many different bases, and the algorithm in each iteration defines the next basis uniquely, the algorithm can have infinite number of iterations only if it is cycling.

Consider the minimal cycling example defined at the begining of the section. In this example, every variable moves during a cycle. Taking into consideration our lemmas, after variable u_p enters the basis in the case of basis B', it cannot leave it again:

If it enters in case (a) or (b), and leaves the basis in case (A) or (B), then lemma 3.5. contradicts to the sufficiency of matrix M.

If it enters in case (c), and leaves the basis in case (A) or (B), then lemma 3.3. contradicts to the orthogonality theorem.

If it entrers in case (c), and leaves the basis in case (C), then lemma 3.6. contradicts to the ortogonality theorem.

If it enters in case (b) or (c), and leaves the basis in case (C), then lemma 3.3. contradicts to the orthogonality theorem.

All possible cases lead to a contradiction, therefore the algorithm is finite.□

This tableau shows the cases in which we took into consideration the sufficiency of the matrix M, during the proof of finiteness of the criss-cross type algorithm.

	(a)	(b)	(c)
(A)	*	*	
(B)	*	*	
(C)			*

4 EP–theorems

We cannot expect to be able to solve every linear complementarity problems with an arbitrary matrix using the criss-cross type method. If our generalized criss-cross type algorithm would not be able to solve a linear complementarity problem then it would give a certificate that the matrix of the problem is not sufficient.

The modification of our criss-cross type algorithm is based on the theory developed by Cameron and Edmonds [2]. They introduced the so called *EP–theorems*.

The general form of an EP (Existentially Polynomial time) theorem is as follows:

$$[\forall \mathbf{x} : F_1(\mathbf{x}) \text{ or } F_2(\mathbf{x}) \text{ or } \ldots \text{ or } F_k(\mathbf{x})]$$

holds, where $F_i(\mathbf{x})$ is a statement of the form

$$F_i(\mathbf{x}) = [\exists \mathbf{y}_i \text{ for which } \|\mathbf{y}_i\| \leq \|\mathbf{x}\|^{n_i} \text{ and } f_i(\mathbf{x}, \mathbf{y}_i)].$$

We generalize the criss-cross type algorithm for general linear complementarity problem. Here $n_i \in \mathbb{Z}^+$, $\|\mathbf{z}\|$ donotes the coding size of $\mathbf{z}$, while $f(\mathbf{x}, \mathbf{y})$ is a predicate for which there is a polynomial-time algorithm.

Before formalizing the duality theorem of the linear complementarity problem in EP form, we need to state some definitions and theorems.

We call the *holder* of vector $\mathbf{x}$ the set $\{i \mid \mathbf{x}_i \neq 0\}$. From a given set of vectors, a vector with minimal holder is called a *circuit*.

We will use the *conform* decomposition of a vector, [7].

Definition 4.1. Let $V \subseteq \mathbb{R}^n$ be an arbitrary linear subspace, $\mathbf{x}, \mathbf{x}^1, \ldots, \mathbf{x}^k$ vectors from subspace V. We say that vector $\mathbf{x}$ is conformally decomposed into vectors $\mathbf{x}^1, \ldots, \mathbf{x}^k$, if

$$\begin{aligned}
&\mathbf{x} = \mathbf{x}^1 + \ldots \mathbf{x}^k \quad \text{and} \\
&x_i = 0 \Longrightarrow x_i^1 = \ldots = x_i^k = 0, \\
&x_i > 0 \Longrightarrow x_i^1, \ldots, x_i^k \geq 0, \\
&x_i < 0 \Longrightarrow x_i^1, \ldots, x_i^k \leq 0 \quad \text{for all indices } i = 1, \ldots, n.
\end{aligned}$$

For a linear subspace, the following holds, [14].

Lemma 4.1. *Let V be a linear subspace in $\mathbb{R}^n$. Then any $\mathbf{x} \in V$ can conformly be decomposed to circuits $\mathbf{c}^1, \ldots, \mathbf{c}^k$ of V.*

With the lemma above, it can be shown that if M is not sufficient, then the certificate of non sufficiency can be given using a circuit, or the sum of two circuits, [7].

Lemma 4.2. *If M is not column (row) sufficient, then there exists a strictly sign reversing (preserving) circuit in subspace V ($V^\perp$), or there exists a strictly sign reversing (preserving) vector $\mathbf{x}$ is subspace V ($\mathbf{y}$ in $V^\perp$), that can be decomposed into the sum of two complementary circuits.*

Using these results, the following theorem can be proven.

Theorem 4.3. *Let the matrix $M \in \mathbb{R}^{n\times n}$ be not sufficient. Then there exists a certificate that M is not sufficient, whose encoding size is polynomially bounded by the encoding size of M.*

We want to state the duality theorem of linear complementarity problems in EP form. To do this, first we have to formalize it in such a way that the sufficiency of the matrix is no longer a condition of the theorem.

Theorem 4.4. *For any rational matrix $M \in \mathbb{Q}^{n\times n}$ and vector $\mathbf{q} \in \mathbb{Q}^n$ at least one of the following holds:*
(1) the problem (P-LCP) has a complementary, feasible solution $(\mathbf{u}, \mathbf{v})$.
(2) the problem (D-LCP) has a complementary, feasible solution $(\mathbf{x}, \mathbf{y})$.
(3) the matrix M is not sufficient.

he theorem is not in EP form yet. When the (1) or (2) holds, then the solution itself is polynomial sized. In case of (3), we have to show that there is a polynomial size certificate, that matrix M is not sufficient.

Now, we can state the LCP duality theorem in EP form, [7].

Theorem 4.5. *For any rational matrix $M \in \mathbb{Q}^{n\times n}$ and vector $\mathbf{q} \in \mathbb{Q}^n$ at least one of the following holds:*
(1) the problem (P-LCP) has a complementary, feasible solution $(\mathbf{u}, \mathbf{v})$, whose encoding size is polynomially bounded by the encoding size of M and $\mathbf{q}$.
(2) the problem (D-LCP) has a complementary, feasible solution $(\mathbf{x}, \mathbf{y})$, whose encoding size is polynomially bounded by the encoding size of M and $\mathbf{q}$.
(3) the matrix M is not sufficient, and there is a proof, whose encoding size is polynomially bounded by the encoding size of M.

Note that cases (1) and (2) are exclusive, while case (3) can hold with both case (1) or (2). Moreover, it is a naturally arising condition that the entries of the matrix should be rational numbers.

To prove the theorem, we have to modify our algorithm and prove its finiteness.

We modify our algorithm, so that it either solves problem $(P-LCP)$ or its dual, or proves the lack of sufficiency of the input matrix[2], giving a polynomial size certificate.

[2] There is no known efficient, polynomial algorithm to check the sufficiency of a matrix.

Lemma 2.2 ensures that if our matrix is sufficient, then the pivot operations can always be done, and if not, it provides the required proof that matrix M is not sufficient.

We now have to handle the case of cycling.

When **avoiding cycling**, note that in the proof of finiteness of the original algorithm, the minimality of the cycling example is not necessary. Let us consider an arbitrary cycling example. Let the index set of the variables involved in the cycling be R, and consider an iteration, when cycling has already began, every variable in R has already been moved, and the algorithm chooses a variable to enter the basis, with the smallest $\mathbf{s}$ value is the smallest among the variables in R outside the basis. Let the basis of this moment be B', and let the smallest $\mathbf{s}$ valued entering variable in R be u_p. Let B'' be the basis when u_p leaves next.

Then the structure of tableaus $M_{B'}$ and $M_{B''}$ restricted to indices R and vector $\mathbf{q}$, is exactly like in case $(a)-(c)$ and $(A)-(C)$. Between these two tableaus, a variable whose index is not in R has not moved. Thus in the product of the fundamental circuits in lemmas 3.3, 3.4 and 3.6, for the indices not in R and not q, from the corresponding variables exactly one is in basis, so the product for these indices is always zero. For the same reason, in the product of $-\mathbf{u}' \cdot \mathbf{v}'' - \mathbf{u}'' \cdot \mathbf{v}'$ in lemma 3.5, the entries for the indices not in R are zeros. So the proofs are valid for an arbitrary cycling example.

When **handling the lack of sufficiency**, remember that we only used sufficiency in lemmas 3.5 and 3.6. This last one used the sign property of the sufficient matrices, based on lemma 2.2. So if our algorithm checks that the required sign property is fulfilled during every exchange pivot (cases (c) and (C) refers to such pivots), then because of the orthogonality theorem, tableau (c) cannot be followed by tableau (C). If the required sign structure is violated, then the proof that matrix M is not sufficient is provided by the same lemma.

The modified criss-cross type algorithm

Input
  Let (1) be given. $\bar{M} = -M$, $\bar{\mathbf{q}} = \mathbf{q}$, $r = 1$, Initaliaze Q
Begin
  While $((J := \{\alpha \in I : \bar{\mathbf{q}}_\alpha < 0\}) \neq \emptyset)$ **do**
    $J_{\max} := \{\beta \in J : \mathbf{s}_{r-1}(\beta) \geq \mathbf{s}_{r-1}(\alpha)$, for all $\alpha \in J\}$
    Let $k \in J_{\max}$ be arbitrary
    Check $-\mathbf{u}' \cdot \mathbf{v}'' - \mathbf{u}'' \cdot \mathbf{v}'$ with the help of $Q(k)$
    If $(-\mathbf{u}' \cdot \mathbf{v}'' - \mathbf{u}'' \cdot \mathbf{v}' \lneqq \mathbf{0})$ **then**
      Stop: M is not sufficient, proof: $\mathbf{u}' - \mathbf{u}''$
    Endif
    If $(\bar{m}_{kk} < 0)$ **then**
      diagonal pivot on $\bar{m}_{kk}$
      $Q(k) = [J_B, \bar{\mathbf{m}}_q]$, $r := r + 1$
    ElseIf $(\bar{m}_{kk} > 0)$
      Stop: M is not sufficient, proof: like in lemma 2.2
    Else /* $\bar{m}_{kk} = 0$ */
      $K := \{\alpha \in I : \bar{m}_{k\alpha} < 0\}$
      If $(K = \emptyset)$ **then**
        Stop: DLCP solution //see remar
      Else
        $K_{\max} = \{\beta \in K : \mathbf{s}_{r-1}(\beta) \geq \mathbf{s}_{r-1}(\alpha)$, for all $\alpha \in K\}$
        Let $l \in K_{\max}$ be arbitrary
        If ($\mathbf{m}_k$ and $\mathbf{m}^k$ or $\mathbf{m}_l$ and $\mathbf{m}^l$ sign structure is violated)
          then Stop: M is not sufficient,
          proof: like in lemma 2.2
        Endif
        Exchange pivot on $\bar{m}_{kl}$ and $\bar{m}_{lk}$, modify $\mathbf{s}$
        $Q(k) = [\emptyset\,[J_B, \bar{\mathbf{m}}_q]]$, $Q(l) = [\emptyset, \mathbf{0}]$, $r := r + 2$
      Endif
    Endif
  EndWhile
  Stop: we have a complementary feasible solution
End

There remains the cases of tableau (a)-(b), and (A)-(B). Lemma 3.5 is based on the vectors

$$-\mathbf{u}^{'} \cdot \mathbf{v}^{''} - \mathbf{u}^{''} \cdot \mathbf{v}^{'} \tag{15}$$

referring to such tableaus $M_{B'}$ and $M_{B''}$, where the same variable moves during both pivot, and in both cases this variable was chosen actively (so not the second variable of an exchange pivot). Note that we do not need the whole tableau here, the only information we use is the column of $\mathbf{q}$ (the actual complementary solution) and the set of indices in basis. If vector (15) is strictly sign reversing, then as in the note after lemma 3.5, the evidence that matrix M is not sufficient is the vector $\mathbf{u}^{'} - \mathbf{u}^{''}$.

Let us introduce a list $Q(p)$ $(p = 1, \dots, n)$. For every entry of this list, belongs two vectors of dimension n. At the beginning,

$$Q(p) := \begin{bmatrix} [1, \dots, n] \\ [0, \dots, 0] \end{bmatrix} \quad p = 1, \dots, n$$

When variable u_l or v_l leaves the basis during a diagonal pivot or such an exchange pivot where this variable is active (variable selected first) then we modify the value of $Q(l)$ such, that to the first vector we write the *indices* of variables in basis before the actual pivot operation, while to the second vector we write the *values* of variables in basis before the pivot operation:

$$Q(l) := \begin{bmatrix} [\text{ indices of variables in basis }] \\ [\text{ values of variables in basis}] \end{bmatrix}$$

If variable u_l or v_l enters basis passively (as the second variable of an exchange pivot) then we modify the value of $Q(l)$ as:

$$Q(l) := \begin{bmatrix} [1, \dots, n] \\ [0, \dots, 0] \end{bmatrix}$$

Before the algorithm performs a pivot operation, it checks if the actively selected variable which enters the basis was chosen actively last time when it was living the basis or not. If yes, with the help of the list Q, it checks vector (15) and only after this does it modify the list Q. Because the complementary pairs of variables move together during the pivot operations, it is not necessary to provide space for both of them in the list Q.

Note that because the definition of the initial values of Q, and the modification of Q during a passive exchange pivot, it suffices to check product (15) during any pivot. If tableaus (a) (or (b)), is followed by tableau (A) (or (B)), then the product will always be zero.

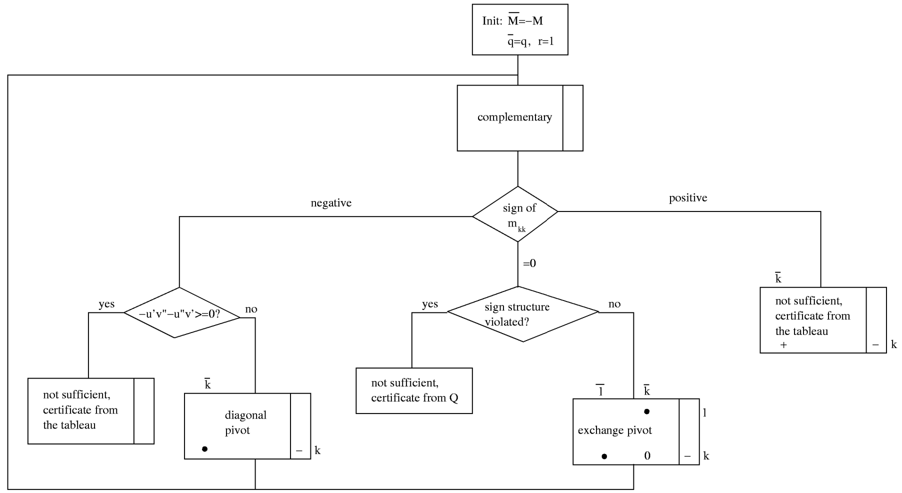


Figure 2: Flow chart of the modified algorithm

It would not be necessary every time to fill in the list Q. With a slight modification of the algorithm, we would be able to save storage space, as well In the worst case, the storage space required by the list Q would be the storage space required to store n^2 integer and n^2 real numbers.

By an operation $Q(j) = [\{I\}, \{\mathbf{h}\}]$ we mean that to the entry of j in the list Q, we write to the place of basis indices the list I, while to the place of $\mathbf{q}$ values of the vector $\mathbf{h}$.

We have to investigate the case, when $(P - LCP)$ has no solution. This occurs, when $K = \emptyset$. The structure of the pivot tableau is shown below. Consider the vector

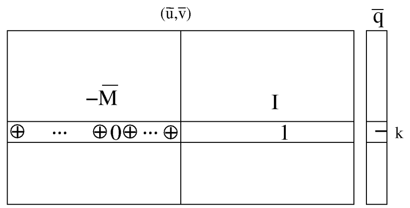
$$(\mathbf{x}', \mathbf{y}') = \mathbf{t}^{(k)} \mid_{J_N \cup J_B} .$$

Using the orthogonality theorem, we get that this vector is orthogonal to every row of $[-M^T \mid -I]$, in other words $M^T\mathbf{x}' + \mathbf{y}' = \mathbf{0}$. Applying the orthogonality theorem to the column of the right hand side vector $\mathbf{q}$ (in the starting basis), we have

$$(\mathbf{x}', \mathbf{y}')^T \mathbf{t}_q \mid_{J_N \cup J_B} = (\mathbf{x}', \mathbf{y}')^T (\mathbf{q}, \mathbf{0}) = \mathbf{x}'^T \mathbf{q} = q_k.$$

So the vector $(\mathbf{x}, \mathbf{y}) = (\mathbf{x}', \mathbf{y}')/(-q_k)$ is a solution to the problem $(D -$

LCP), because nonegativity and complementary follows from the structure of the pivot tableau.



$$(x,y)_i = \begin{cases} -\bar{m}_{ki}/\bar{q}_k & \text{if } i \in I_N \\ -1/\bar{q}_k & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

5 Summary

We generalized newcriss-cross type algorithms of Arif A. Akkeleş, L. Balogh and T. Illés [1] for linear complementarity problems with sufficient matrices. In meantime, we were able to simplify the finiteness proof of the algorithm, too. For better practical applicability, we modified the generalized algorithm so that the apriori information on the sufficiency of the matrix is not necessary. In case of lack of sufficiency, the algorithm cannot ensure finiteness, then it terminates and provides a polynomial size certificate that the matrix is not sufficient. We achieved our goals using the duality theorem of linear complementarity problems [8], and with its EP theorem form [7]. The algorithm because of Akkeleş et al.'s new pivot rules provides significant freedom in choosing the pivot position during the first part of the algorithm (when $\mid J_{max} \mid > 1$ and/or $\mid K_{max} \mid > 1$), so making it possible to avoid some numerically instable pivots.

The case of the most often selected variable rule can be handled similarly to the LIFO rule. We only need to define the vector counter **s** like

$$s_r(\alpha) = \begin{cases} s_{r-1}(\alpha) + 1, & \text{if } \alpha \in I \text{ moves in iteration } r \\ s_{r-1}(\alpha), & \text{otherwise} \end{cases}$$

Although we have significant freedom in choosing the pivot position at the beginning of the algorithm, the order of the variables will be fixed later (when $\mid J_{max} \mid = \mid K_{max} \mid = 1$).

With the help of the counter vector **s**, the minimal index rule can be defined as a special case, and the proofs remain the same. To do this, we need to fix the vector **s** as $s_r(i) = i$ for all $i \in I$. Our proof are simplier in this case.

The finiteness of the generalized criss-cross algorithm provides a new, constructive proof to the LCP duality theorem, while modified version provides a constructive proof to the LCP duality theorem in the EP form, as well.

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