OPTIMALITY OF COMBINED SOURCE-CHANNEL CODING SYSTEMS WITH MULTIPLE SOURCES AND PARALLEL CHANNELS

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ABSTRACT

Source-channel coding in multichannel systems may be a good candidate for making robust and efficient transmission systems. In this paper, the theoretical limit (OPTA) for such systems is found for Gaussian sources and AWGN channels. In general, the number of sources and channels is different and their individual bandwidths are arbitrary. Such systems can implement any bandwidth change, in practical systems the bandwidth change can be performed using only a small set of nonlinear mappings. The paper also describes the loss in performance due to a finite number of theoretical mappings between signal and channel spaces.

1. INTRODUCTION

In this paper, we study the use of combined source-channel coding when multiple sources are to be transmitted over parallel channels. The sources may have different bandwidths and variances, and the channels may likewise have individual bandwidths and noise levels. The problem at hand is then to find the optimal system performance given an available total amount of channel power.

Multiple sources may originate from different users, but another important case is when one signal is decomposed into subsources, e.g. to remove redundancy, as is the case in transform- and subband coding [6].

Multiple channels with a power constraint is typical when transmitting all channels over the same medium as in code division multiple access (CDMA) and orthogonal frequency division multiplex (OFDM).

One of the key features of combined source-channel coding is its possible robustness.

It is common to use the signal-to-noise ratio (SNR) as a way of measuring the quality of the received signal in a communication system. In dB, this is defined by

$$\mathrm{SNR} = 10\log_{10}\left(\frac{\sigma_X^2}{\sigma_D^2}\right), \tag{1}$$

where σ_X^2 and σ_D^2 are the signal and noise variances, respectively. Here we will try to optimize the total SNR for K parallel, independent, and identically distributed (i.i.d) Gaussian sources, with bandwidths W_i, which are being transmitted over L parallel, additive, white Gaussian noise (AWGN) channels, with bandwidths B_i. Depending on the ratio

$$\alpha = \frac{\sum_{i=1}^{L} B_i}{\sum_{i=1}^{K} W_i}, \tag{2}$$

bandwidth compression ($\alpha < 1$) or expansion ($\alpha > 1$) is obtained.

2. CHANNEL CAPACITY

For L parallel AWGN channels that are of the Nyquist type, and where channel i has bandwidth B_i, the capacity is given by

$$C(P) = \sum_{i=1}^{L} B_i \ln\left(1 + \frac{\sigma_{Y_i}^2}{\sigma_{N_i}^2}\right), \tag{3}$$

given in nats/second[1], when $\sigma_{Y_i}^2$ is the power for channel i, and $\sigma_{N_i}^2$ is the noise power for channel i. For an overall power constraint,

$$\sum_{i=1}^{L} \sigma_{Y_i}^2 \leq P, \tag{4}$$

the capacity is maximized when the total power is distributed according to the Water-filling method [1]. An example of this is given in Figure 1. The parameter φ indicates the "water level". The power allocated to each channel is $\sigma_{Y_i}^2 = (\varphi - \sigma_{N_i}^2)^+$, where the plus sign indicates that only channels receiving positive power are included.

[1] *Nats* occurs because we use the natural logarithm for the capacity calculation. By using the binary logarithm we would obtain bits. Nats are used to emphasize that no bits are involved in this theory.

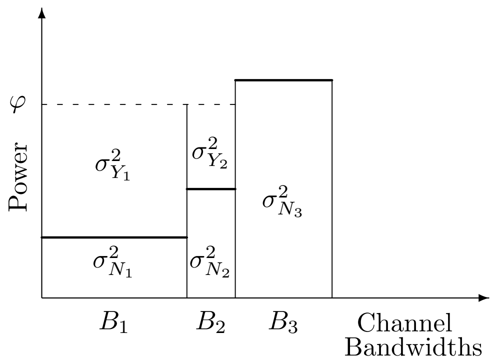

Figure 1: *Principle of water-filling.*

The easiest way of calculating Equation 3 is by selecting φ and then to find $C(P)$ and P for that value.

Channel characterization

For a single channel, the most common quality measure, is the channel signal-to-noise ratio (CSNR), which in dB is defined by

$$\text{CSNR} = 10\log_{10}\left(\frac{\sigma_Y^2}{\sigma_N^2}\right). \tag{5}$$

For multiple channels under a common power constraint, there is no straight forward extension of this measure. From the discussion above we have learned that the number of channels actually being used depends on the total signal power. The channel noise is therefore not uniquely defined. We can avoid the whole problem by calculating the channel capacity in terms of total channel power, P, only. However, when applying dB scales, we need some reference. Here we choose the total noise power as a reference

$$\sigma_N^2 = \sum_{i=1}^{L}\sigma_{N_i}^2, \tag{6}$$

irrespective of whether some channels are discarded for signal transmission due to their high noise level. In this case we can still use Equation 5.

3. THE RATE DISTORTION FUNCTION

For K parallel, zero mean, i.i.d Gaussian sources, with variances $\sigma_{X_i}^2$ and bandwidths W_i, and where source i is sampled using $2W_i$ samples per second, the rate in nats/second is given by

$$R(D) = \sum_{i=1}^{K} W_i \ln\left(\frac{\sigma_{X_i}^2}{\sigma_{D_i}^2}\right), \tag{7}$$

where

$$\sigma_{D_i}^2 = \begin{cases} \lambda, & \text{for } \lambda < \sigma_{X_i}^2, \\ \sigma_{X_i}^2, & \text{for } \lambda \geq \sigma_{X_i}^2. \end{cases} \tag{8}$$

λ is chosen so that $\sum_{i=1}^{K}\sigma_{D_i}^2 = D$. $\sigma_{D_i}^2$ is the distortion for source i. D is the total distortion summed over all the sources. This process is referred to as reverse water-filling [1]. An example of this is given in Figure 2.

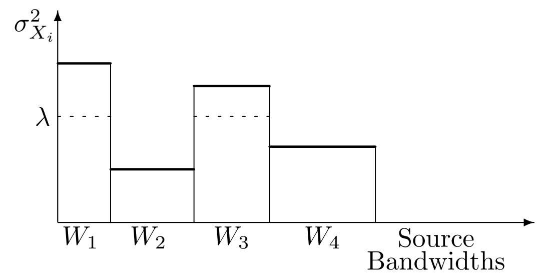

Figure 2: *Principle of reverse water-filling.*

Here λ is the "water level". If the signal power is above this level, the optimal noise allocation is given by λ. But if the signal power in a component is lower, no resources are allocated to that component and the noise will accordingly be equal to the signal power.

For computing the rate distortion function it is convenient, similar to the channel capacity calculations, to first choose λ, and then calculate both R and D from there. By systematically varying λ the desired range of values on the rate distortion function can be calculated.

3.1. OPTA

By combining the channel capacity $C(P)$ with the source rate $R(D)$ (Equations 3 and 7), we get what is called the OPTA curve (OPTA = optimal performance theoretically attainable). By doing this, the capacity of the transmission medium is set equal to the minimum rate possible for producing a certain distortion.

As an example, assume that three sources with variances $\sigma_{X_1}^2 = 250$, $\sigma_{X_2}^2 = \sigma_{X_3}^2 = 1$, and equal bandwidth $W_i = 1$ Hz ($i = 1, 2, 3$), are to be transmitted over two channels, each with the same bandwidths as the individual sources $B_i = 1$ Hz ($i = 1, 2$), and noise levels $\sigma_{N_1}^2 = 1$ and $\sigma_{N_2}^2 = 2$. This experiment obviously involves a bandwidth compression by 3:2. Using Equations 7 and 3, the rate distortion function and channel capacity are drawn in Figure 3, a and b, respectively.

OPTA can be derived from these plots by first selecting a desired signal-to-noise ratio in the received signal. The corresponding rate can be found from Figure 3 a. This rate is transferred to Figure 3 b, and the

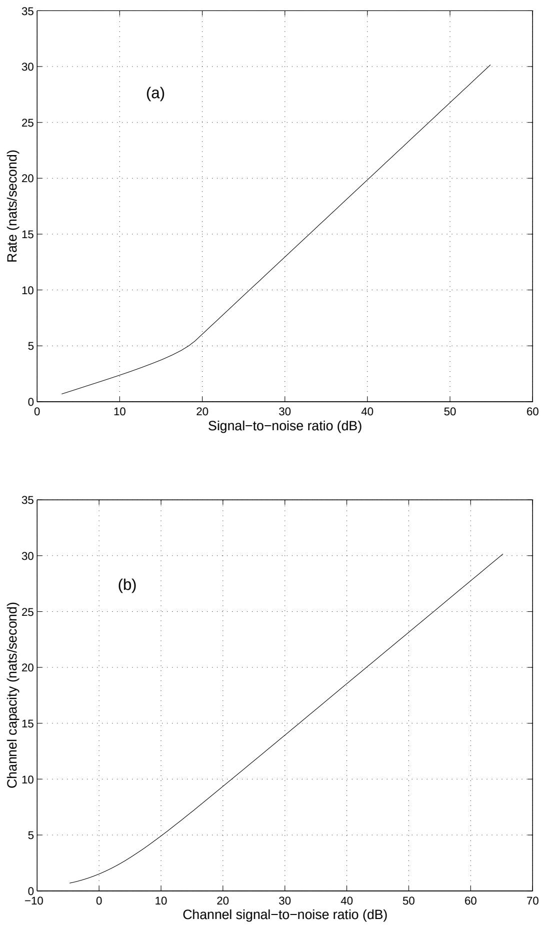

Figure 3: a) *Rate distortion function for a three-source signal.* b) *Channel capacity for a two-channel system.*

required channel signal-to-noise ratio is found. This is equivalent to the the channel power. If doing this for the given example, the dashed curve in Figure 4 is found. The other curves can be derived likewise by varying $\sigma^2_{X_1}$.

By studying the curves in Figure 3 and Figure 4, we see that there are points where the curves break. These points are where the different sources are included, as the CSNR increases. For high CSNR in Figure 4, we see that all the curves have equal slope. This means that the same number of sources and channels are used.

4. LIMITS FOR SYSTEMS BASED ON FINITE MAPPINGS

OPTA does not give any indication on how to practically achieve the obtained limit, nor how we can actually lower the signal bandwidth. In practice one must resort to compression and efficient channel rep-

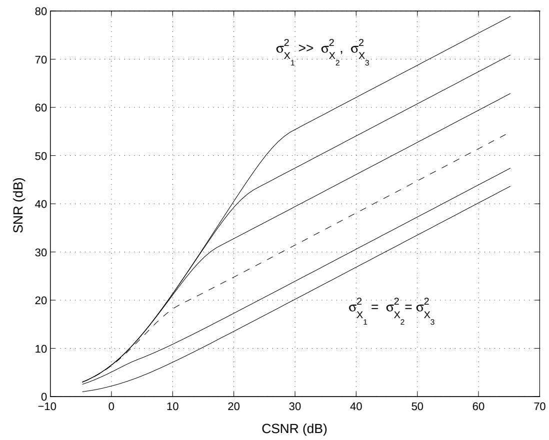

Figure 4: *OPTA for* 3 *sources and* 2 *channels, varying* $\sigma^2_{X_1}$. $\sigma^2_{N_i}$, $i = 1, 2$ *and* $\sigma^2_{X_i}$, $i = 2, 3$ *are fixed.*

resentations.

The most direct way of obtaining bandwidth reduction is through nonlinear mappings [5]. Only a limited number of dimension changes can be derived without extreme complexities. This means that in order to obtain a certain bandwidth change, we have to limit our choices. Assume, e.g. that we want to make a 3:2 reduction of bandwidth. In terms of signals and channels with equal bandwidths, this means that three sources must be transmitted over two channels. This can be obtained in a number of ways, but the simplest choices would be to either skip one of the source signals, or to combine two of them and transmit the last source directly.

A basic question is then, what is the loss in performance due to a finite number of available mappings? Let us consider the case described above with 3:2 dimensional change. Also assume that the used mappings perform according to OPTA for dimension change 2:1 and no dimension change (1:1). By combining 2:1 mapping and 1:1 mapping, we get a system of dimension change 3:2.

An example of this is given in Figure 5. Here the additional decision is made that source no.1 is being sent on channel no.1. Since the number of mappings is limited, it will no longer be optimal to distribute power according to the Water-filling method. The figure illustrates how one can find the optimal distribution of the available power in this case.

Figure 6 shows an example of the loss in performance due to the limited number of mappings. Still assuming three source signals and two channels, two systems with fixed mappings are being compared to OPTA. The first system uses two different mappings, 2:1 and 1:1. This means that we are combining two source signals, and are transmitting the last source signal directly. The second system only uses one mapping, 1:1 for two of the signals. This means that

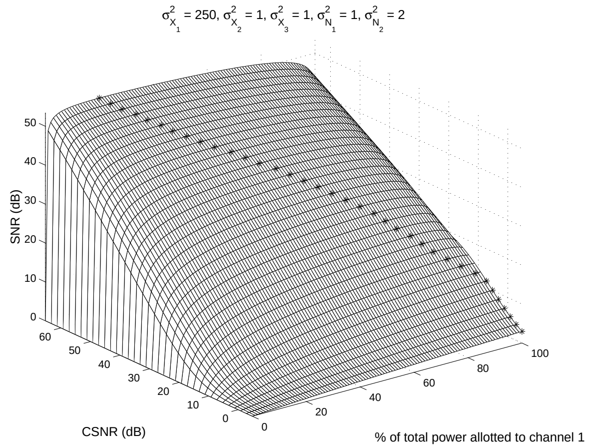

Figure 5: *Example of power distribution with* 3 *sources and* 2 *channels, using 2:1 and 1:1 mappings. The points marked with* $*$*, indicate the highest SNR, the optimal allotment.*

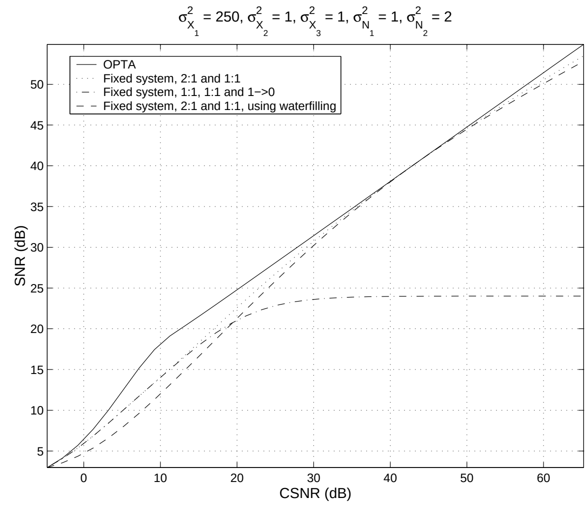

Figure 6: *Comparison between fixed systems and OPTA. Using two different fixed systems for dimension change 3:2.*

this system skips one of the source signals. Increasing CSNR above a certain level gives no gain as the skipped source signal will appear as distortion. The fourth curve gives an example of the performance for the first system with power distribution according to the Water-filling method.

The above theory is also an upper bound for any practical system with the same types of mappings. The practical mappings will perform worse than the result obtained for the individual OPTAs.

5. CONCLUSIONS

The theoretical limit (OPTA) for systems with AWGN channels, and Gaussian i.i.d sources has been found. Furthermore, limits for systems with a given structure where a certain class of theoretical mappings are available have been derived. This theory will be useful for systems using a set of nonlinear mappings to convert source symbols to channel symbols. The theory will both give upper bounds for practical system performance as well as guidelines on how the power should be distributed to the different mappings.

Although the presented theory seems to be limited to multiple sources and channels, it is also applicable for finding channel capacities for channels with varying signal-to-noise ratios as a function of frequency, and for finding rate-distortion functions for signals with non-flat power spectral densities. In such cases one can split a Gaussian source into several small subsources and estimate the rate accurately provided the sub-sources have small enough bandwidths. This is because the spectra are going to be almost flat within the sub-sources. Adding up the rates for the subsources under a common distortion constraint, it is possible to obtain an estimate for the rate for the overall source. This can be done in an similar way for an AWGN channel with uneven noise power density, or equivalently nonconstant CSNR as a function of frequency.

6. REFERENCES

[1] Cover, T.M. and Thomas, J.A , *Elements of information Theory*, John Wiley & Sons, Inc., New York, NY, USA, 1991.

[2] Coward, Helge, *Joint Source-Channel Coding: Development of Methods and Utilization in Image Communications*, Ph.D thesis, Norwegian University of Science and Technology, 2001.

[3] Fuldseth, Arild, *Robust subband video compression for noisy channels with multilevel signaling*, Ph.D thesis, Norwegian University of Science and Technology, 1997.

[4] Lervik, J. M., *Subband Image Communication over Digital Transparent and Analog Waveform Channels*, Ph.D thesis, Norwegian University of Science and Technology, 1996.

[5] Ramstad Tor A. *Shannon mappings for robust communication*, Telektronikk, no.1, pp 114-128, 2002.

[6] Tor Audun Ramstad, Sven Ole Aase, and John Håkon Husøy. *Subband Compression of Images - Principles and Examples.* ELSEVIER Science Publishers BV, North Holland, 1995.