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Massless One-Loop Scalar Three-Point Integral and Associated Clausen, Glaisher and L-Functions*

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Abstract

The massless one-loop three-point integral is obtained in terms of associated Clausen functions. The expression is manifestly symmetric in the three external variables. The main features of associated Clausen functions and their series expansions are presented. We also introduce the L-functions as analytical extensions of the associated Clausen functions.

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The one-loop three-point integral has been obtained by other authors previously [1]. The result is usually expressed in terms of dilogarithms, also known as Spence functions. However, the obtained formula lacks explicit symmetry under the permutation of the three external momenta, and conceals the structure of the real part of the integral.

Here, we obtain the massless one-loop three-point integral in terms of associated Clausen functions. Our expression manifests the symmetry under the permutation of the three external momenta and provides a transparent real part. (The real part of the integral is actually given by the imaginary part of the function $\mathrm{F}(p_1, p_2, p_3)$ defined below. See Eq. (1).) Since one-loop Feynman integrals are in increasing demand, and also since the various associated functions introduced here are not as well-documented as the polylogarithmic functions [2, 3], we have decided to communicate our results here to facilitate future reference.

We have employed only standard integration techniques in obtaining our formula; therefore, we shall present the result without derivation [4]. The massless one-loop three-point integral in question is (see also Fig. 1):

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k_1^2 k_2^2 k_3^2} \equiv -\frac{i}{(4\pi)^2}\, \mathrm{F}(p_1, p_2, p_3) \quad , \tag{1}$$

where p_1, p_2, p_3 are the external momenta of the three-point function. It is convenient to introduce the following variables:

$$\begin{aligned} \delta_i &= p_{i-1} \cdot p_{i+1} = (p_i^2 - p_{i-1}^2 - p_{i+1}^2)/2 \quad , \\ \mathcal{R} &= \delta_1\delta_2 + \delta_2\delta_3 + \delta_3\delta_1 = (2\ p_1^2\ p_2^2 + 2\ p_2^2\ p_3^2 + 2\ p_3^2\ p_1^2 - p_1^4 - p_2^4 - p_3^4)/4 \quad , \\ \rho &= \sqrt{|\mathcal{R}|} \quad . \end{aligned} \tag{2}$$

The subindices are understood to be modulo-3. That is, $p_4 \equiv p_1$ and $p_0 \equiv p_3$.

The exact form of the function $F(p_1, p_2, p_3)$ depends on the kinematic region of the three external momenta. In general, we can classify a kinematic region as trigonometric or hyperbolic, according to the signature of the variable $\mathcal{R}$.

1) Trigonometric case ($\mathcal{R} > 0$)

$$\begin{aligned} F(p_1, p_2, p_3) &= \frac{1}{\rho}\left[\mathrm{Cl}_2(2\phi_1) + \mathrm{Cl}_2(2\phi_2) + \mathrm{Cl}_2(2\phi_3)\right] \quad , \\ \phi_i &= \arctan\left(\frac{\rho}{\delta_i}\right) \quad , \end{aligned} \tag{3}$$

where $\mathrm{Cl}_2(x)$ is the Clausen function, which will be described later. The trigonometric case can happen only in the completely spacelike ($p_1^2, p_2^2, p_3^2 < 0$) and the completely timelike ($p_1^2, p_2^2, p_3^2 > 0$) regions. Geometrically, in the completely spacelike region the angles ϕ_1, ϕ_2 and ϕ_3 correspond to the three internal angles of a triangle with sides $\sqrt{-p_1^2}$, $\sqrt{-p_2^2}$ and $\sqrt{-p_3^2}$ (see Fig. 2), and ρ is twice the area of the triangle. Thus, in the completely spacelike region we have

$$\phi_1 + \phi_2 + \phi_3 = \pi \quad . \tag{4}$$

In the completely timelike region we have the same identity with the opposite sign:

$$\phi_1 + \phi_2 + \phi_3 = -\pi \quad . \tag{5}$$

Note that $F(p_1, p_2, p_3)$ contains no imaginary part in the trigonometric case, as one would expect in the completely spacelike and timelike regions.

2) Hyperbolic case ($\mathcal{R} < 0$)

$$\mathrm{F}(p_1, p_2, p_3) = \frac{1}{\rho}\left[\widetilde{\mathrm{Clh}}_2(2\phi_1) + \widetilde{\mathrm{Clh}}_2(2\phi_2) + \widetilde{\mathrm{Clh}}_2(2\phi_3) + i\pi\phi_1\theta(p_1^2) + i\pi\phi_2\theta(p_2^2) + i\pi\phi_3\theta(p_3^2)\right] \quad , \tag{6}$$

where $\theta(x)$ is the step function

$$\theta(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} \quad , \tag{7}$$

$$\phi_i = \frac{1}{2}\ln\left|\frac{\delta_i + \rho}{\delta_i - \rho}\right| = \begin{cases} \mathrm{arctanh}(\rho/\delta_i), & \text{if } p_{i-1}^2 p_{i+1}^2 > 0 \\ \mathrm{arctanh}(\delta_i/\rho), & \text{if } p_{i-1}^2 p_{i+1}^2 < 0 \end{cases} \quad , \tag{8}$$

and

$$\widetilde{\mathrm{Clh}}_2(2\phi_i) = \begin{cases} \mathrm{Clh}_2(2\phi_i), & \text{if } p_{i-1}^2 p_{i+1}^2 > 0 \\ \not{\mathrm{C}}\mathrm{lh}_2(2\phi_i), & \text{if } p_{i-1}^2 p_{i+1}^2 < 0 \end{cases} \quad , \tag{9}$$

where $\mathrm{Clh}_2(x)$ is the hyperbolic Clausen function and $\not{\mathrm{C}}\mathrm{lh}_2(x)$ is the alternating hyperbolic Clausen function. The definitions and properties of these functions are discussed later. The hyperbolic case can happen in kinematic regions with any signature ($p_1^2, p_2^2, p_3^2 \gtrless 0$). For the hyperbolic case we have the following identity

$$\phi_1 + \phi_2 + \phi_3 = 0 \quad . \tag{10}$$

Thus, despite its appearance, Eq. (6) contains no imaginary part in the completely timelike region.

In summary, in the definite-signature regions (completely spacelike or timelike regions), we encounter both the trigonometric case and the hyperbolic case, whereas in the mixed-signature regions (some of the external momenta are space-like and some are timelike), we can have only the hyperbolic case. The numerical evaluation of the various associated Clausen functions can be performed with the help of the series expansions given below. We have checked our result numerically against direct Feynman parameter integrals in all kinematic regions.

Next, we give the definition and the main properties of the associated Clausen functions [5].

1) <u>(Trigonometric) Clausen function.</u>

- definition

$$\mathrm{Cl}_2(x) \equiv -\int_0^x \ln|2\sin(x/2)|dx = \sum_1 \frac{\sin nx}{n^2} \tag{11}$$

- periodicity

$$\mathrm{Cl}_2(x+2n\pi) = \mathrm{Cl}_2(x)\ , \qquad n = 0, \pm1, \pm2, \ldots \tag{12}$$

- parity

$$\mathrm{Cl}_2(-x) = -\mathrm{Cl}_2(x) \tag{13}$$

- zeros

$$x = n\pi\ , \qquad n = 0, \pm1, \pm2, \ldots \tag{14}$$

- maxima

$$\begin{aligned} x_{\max} &= \frac{\pi}{3} + 2n\pi\ , \quad n = 0, \pm1, \pm2, \ldots \\ \mathrm{Cl}_2(x_{\max}) &= 1.01494160\ldots \end{aligned} \tag{15}$$

- minima

$$x_{\min} = -\frac{\pi}{3} + 2n\pi \;, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$\mathrm{Cl}_2(x_{\min}) = -1.01494160\ldots \tag{16}$$

- duplication formula

$$\mathrm{Cl}_2(2x) = 2\ \mathrm{Cl}_2(x) - 2\ \mathrm{Cl}_2(\pi - x) \tag{17}$$

- special values

$$\mathrm{Cl}_2(\frac{\pi}{2}) = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \ldots = G = 0.91596559\ldots$$
$$\mathrm{Cl}_2(\frac{\pi}{3}) = \frac{3}{2}\ \mathrm{Cl}_2(\frac{2\pi}{3}) = 1.01494160\ldots \tag{18}$$

where G is Catalan's constant.

- expansion around $x = 0$

$$\begin{aligned}\mathrm{Cl}_2(x) &= -x\ln|x| + x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} B_{2k}}{2k(2k+1)!} x^{2k+1} \\ &= -x\ln|x| + x + \frac{x^3}{72} + \frac{x^5}{14400} + \frac{x^7}{1270080} \\ &\quad + \frac{x^9}{87091200} + \frac{x^{11}}{5269017600} + \ldots\end{aligned} \tag{19}$$

where B_n are Bernoulli numbers [6],

- expansion around $x = \pi$; define $\overline{x} = x - \pi$

$$\begin{aligned}\mathrm{Cl}_2(x) &= -(\ln 2)\overline{x} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(2^{2k}-1) B_{2k}}{2k(2k+1)!} \overline{x}^{2k+1} \\ &= -(\ln 2)\overline{x} + \frac{\overline{x}^3}{24} + \frac{\overline{x}^5}{960} + \frac{\overline{x}^7}{20160} + \frac{\overline{x}^9}{5806080} + \frac{31\overline{x}^{11}}{159667200} \\ &\quad + \frac{691\overline{x}^{13}}{49816166400} + \frac{5461\overline{x}^{15}}{5230697472000} + \ldots\end{aligned} \tag{20}$$

2) (Trigonometric) Alternating Clausen function. Although this function is not used in the scalar three-point integral, we have included it here for completeness.

- definition

$$\not{C}l_2(x) \equiv -\int_0^x \ln|2\cos(x/2)|dx = \sum_1 \frac{(-1)^n \sin nx}{n^2} \tag{21}$$

- relation to Clausen function.

$$\not{C}l_2(x) = \mathrm{Cl}_2(x+\pi) \tag{22}$$

Since $\not{C}l_2(x)$ is simply the half-period translation of $\mathrm{Cl}_2(x)$, all the properties of $\not{C}l_2(x)$ can be easily obtained from those of $\mathrm{Cl}_2(x)$; therefore we will not give them separately here.

3) Hyperbolic Clausen function.

- definition

$$\mathrm{Clh}_2(x) \equiv -\int_0^x \ln|2\sinh(x/2)|dx = \sum_1 \frac{\sinh nx}{n^2} \tag{23}$$

The series should be considered formal, since it is not convergent for real values of x.

- parity

$$\mathrm{Clh}_2(-x) = -\mathrm{Clh}_2(x) \tag{24}$$

- zeros

$$x = 0, \pm 2.49879679\ldots \tag{25}$$

- maximum and minimum

$$\begin{aligned} x_{\max} = -x_{\min} &= 2\ln\left(1/2 + \sqrt{5}/2\right) = 0.96242365\ldots \\ \mathrm{Cl}_2(x_{\max}) &= -\mathrm{Cl}_2(x_{\min}) = 0.98695978\ldots \end{aligned} \tag{26}$$

- expansion around $x = 0$

$$\begin{aligned} \mathrm{Clh}_2(x) &= -x\ln|x| + x - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k+1)!} x^{2k+1} \\ &= -x\ln|x| + x - \frac{x^3}{72} + \frac{x^5}{14400} - \frac{x^7}{1270080} \\ &\quad + \frac{x^9}{87091200} - \frac{x^{11}}{5269017600} + \ldots \end{aligned} \tag{27}$$

- large-x expansion. For $x > 0$

$$\mathrm{Clh}_2(x) = -\frac{x^2}{4} + \pi^2/6 - \sum_1 \frac{e^{-nx}}{n^2} \tag{28}$$

4) <u>Alternating Hyperbolic Clausen function.</u>

- definition

$$\not{C}\mathrm{lh}_2(x) \equiv -\int_0^x \ln|2\cosh(x/2)|dx = \sum_1 \frac{(-1)^n \sinh nx}{n^2} \tag{29}$$

The series should be considered formal, since it is not convergent for real values of x.

- parity

$$\not{C}\mathrm{lh}_2(-x) = -\not{C}\mathrm{lh}_2(x) \tag{30}$$

- zero

$$x = 0 \tag{31}$$

- expansion around $x = 0$

$$\begin{aligned}\not{C}\mathrm{lh}_2(x) &= -(\ln 2)x - \sum_{k=1}^{\infty} \frac{(2^{2k}-1)B_{2k}}{2k(2k+1)!} x^{2k+1} \\ &= -(\ln 2)x - \frac{x^3}{24} + \frac{x^5}{960} - \frac{x^7}{20160} + \frac{x^9}{5806080} - \frac{31x^{11}}{159667200} \\ &\quad + \frac{691x^{13}}{49816166400} - \frac{5461x^{15}}{5230697472000} + \ldots\end{aligned} \tag{32}$$

- large-x expansion. For $x > 0$

$$\not{C}\mathrm{lh}_2(x) = -\frac{x^2}{4} - \pi^2/12 - \sum_{1} \frac{(-1)^n e^{-nx}}{n^2} \tag{33}$$

In Fig. 3 we plot the functions $\mathrm{Cl}_2(x)$, $\mathrm{Clh}_2(x)$ and $\not{C}\mathrm{lh}_2(x)$ in the interval $-6 \leq x \leq 6$. Notice the approximately sinusoidal nature of $\mathrm{Cl}_2(x)$. The derivative of $\mathrm{Cl}_2(x)$ at zero is infinite.

Another set of functions closely related to the associated Clausen functions are the associated Glaisher functions [2]. We include their basic feature here for completeness. All these functions have even parity, and their defining series are given by

$$\begin{aligned}
\mathrm{Gl}_2(x) &= \sum_1 \frac{\cos nx}{n^2} \ , \qquad \not{\mathrm{G}}\mathrm{l}_2(x) = \sum_1 \frac{(-1)^n \cos nx}{n^2} \ , \\
\mathrm{Glh}_2(x) &= \sum_1 \frac{\cosh nx}{n^2} \ , \qquad \not{\mathrm{G}}\mathrm{lh}_2(x) = \sum_1 \frac{(-1)^n \cosh nx}{n^2} \ ,
\end{aligned} \tag{34}$$

where the two hyperbolic series are only formal. The trigonometric Glaisher functions are periodic with period 2π, and in the interval $[0, \pi]$ they are given by

$$\begin{aligned}
\mathrm{Gl}_2(x) &= \frac{1}{4}(\pi - x)^2 - \frac{\pi^2}{12} \ , \\
\not{\mathrm{G}}\mathrm{l}_2(x) &= \frac{x^2}{4} - \frac{\pi^2}{12} \ .
\end{aligned} \tag{35}$$

The hyperbolic Glaisher functions are explicitly given by

$$\begin{aligned}
\mathrm{Glh}_2(x) &= -\frac{x^2}{4} + \frac{\pi^2}{6} \ , \\
\not{\mathrm{G}}\mathrm{lh}_2(x) &= -\frac{x^2}{4} - \frac{\pi^2}{12} \ .
\end{aligned} \tag{36}$$

The massless three-point integral can also be expressed in terms of a complex analytical function, thus avoiding the division into subcases [7]. For all kinematic regions, the function $\mathrm{F}(p_1, p_2, p_3)$ has the following expression

$$\mathrm{F}(p_1, p_2, p_3) = \frac{1}{\rho}\left[\mathrm{Lsin}_2(2\phi_1) + \mathrm{Lsin}_2(2\phi_2) + \mathrm{Lsin}_2(2\phi_3)\right] \ , \tag{37}$$

with

$$\phi_i = \arctan\left(\frac{\rho}{\delta_i}\right) \equiv \frac{i}{2}\ln\left(\frac{\delta_i - i\rho - i\epsilon}{\delta_i + i\rho - i\epsilon}\right) \ . \tag{38}$$

The variables δ_i and ρ are defined as before, and ϵ is infinitesimally small and positive. The conventions for the imaginary part of logarithms and negative square roots can be taken to be $\mathrm{Im}\ln(-|x|) = i\pi$ and $\sqrt{-|x|} = i\sqrt{|x|}$.

The function $\text{Lsin}_2(z)$ is the analytical extension of the function $\text{Cl}_2(x)$ to the entire complex plane. For a number of reasons, we have introduced a new notation for this function and other analytically extended functions. First of all, the new notation emphasizes the form of the defining series of these functions. Secondly, Clausen and Glaisher functions are real functions whereas the L-functions are complex functions. This distinction is very clear in the case of the hyperbolic Glaisher function. For real x

$$\text{Glh}_2(x) = -\frac{x^2}{4} + \frac{\pi^2}{6} \ , \tag{39}$$

whereas

$$\text{Lcosh}_2(x) = -\frac{x^2}{4} + \frac{\pi^2}{6} - i\frac{\pi}{2}|x| \ . \tag{40}$$

Another argument in favor of a new notation is that, in the case of Clausen functions, their L-function partners are not the naïve analytical continuation of their defining integrals as given in Eqs. (11), (21), (23) and (29). It seems best to keep Clausen-Glaisher functions real, and name their analytical partners differently.

Keeping the definition of L-functions separate from Clausen-Glaisher functions also avoids the staggered definition used in Ref. [2], for example

$$\text{Cl}_{2m}(x) = \sum_1 \frac{\sin nx}{n^{2m}} \ , \tag{41}$$

but

$$\text{Cl}_{2m+1}(x) = \sum_1 \frac{\cos nx}{n^{2m+1}} \ . \tag{42}$$

Finally, as we will see shortly, all the L-functions are naturally defined in terms of the $\mathrm{Lexp}_2(x)$ function. It thus appears appropriate to use the new notation to reflect this relationship.

Next, we give the list of L-functions [8] and their basic properties.

$$
\begin{aligned}
\mathrm{Lexp}_m(z) &\equiv \mathrm{Li}_m(e^z) = \sum_1 \frac{e^{nz}}{n^m} \quad , \\
\mathrm{Lsin}_m(z) &\equiv \frac{1}{2i}\left[\mathrm{Lexp}_m(iz) - \mathrm{Lexp}_m(-iz)\right] = \sum_1 \frac{\sin nz}{n^m} \quad , \\
\mathrm{Lcos}_m(z) &\equiv \frac{1}{2}\left[\mathrm{Lexp}_m(iz) + \mathrm{Lexp}_m(-iz)\right] = \sum_1 \frac{\cos nz}{n^m} \quad , \\
\mathrm{Lsinh}_m(z) &\equiv \frac{1}{2}\left[\mathrm{Lexp}_m(z) - \mathrm{Lexp}_m(-z)\right] = \sum_1 \frac{\sinh nz}{n^m} \quad , \\
\mathrm{Lcosh}_m(z) &\equiv \frac{1}{2}\left[\mathrm{Lexp}_m(z) + \mathrm{Lexp}_m(-z)\right] = \sum_1 \frac{\cosh nz}{n^m} \quad .
\end{aligned}
\tag{43}
$$

The various series given above should be considered formal. All L-functions are periodic. The period of $\mathrm{Lexp}_m(z)$, $\mathrm{Lsinh}_m(z)$ and $\mathrm{Lcosh}_m(z)$ is $2\pi i$, whereas the period of $\mathrm{Lsin}_m(z)$ and $\mathrm{Lcos}_m(z)$ is 2π. The alternating L-functions ($\not{\mathrm{L}}$-functions) are defined as the half-period shifts of the L-functions,

$$
\begin{aligned}
\not{\mathrm{L}}\mathrm{exp}_m(z) &\equiv \mathrm{Lexp}_m(z + i\pi) = \sum_1 \frac{(-1)^n e^{nz}}{n^m} \quad , \\
\not{\mathrm{L}}\mathrm{sin}_m(z) &\equiv \mathrm{Lsin}_m(z + \pi) = \sum_1 \frac{(-1)^n \sin nz}{n^m} \quad , \\
\not{\mathrm{L}}\mathrm{cos}_m(z) &\equiv \mathrm{Lcos}_m(z + \pi) = \sum_1 \frac{(-1)^n \cos nz}{n^m} \quad , \\
\not{\mathrm{L}}\mathrm{sinh}_m(z) &\equiv \mathrm{Lsinh}_m(z + i\pi) = \sum_1 \frac{(-1)^n \sinh nz}{n^m} \quad , \\
\not{\mathrm{L}}\mathrm{cosh}_m(z) &\equiv \mathrm{Lcosh}_m(z + i\pi) = \sum_1 \frac{(-1)^n \cosh nz}{n^m} \quad .
\end{aligned}
\tag{44}
$$

The $\mathrm{Lexp}_m(z)$ function satisfies the following recursion relation

$$\mathrm{Lexp}_m(z) = \int\limits_{-\infty}^{z} \mathrm{Lexp}_{m-1}(z)dz \quad . \tag{45}$$

The first three $\mathrm{Lexp}_2(z)$ functions are given by

$$\begin{aligned} \mathrm{Lexp}_0(z) &= \frac{e^z}{1-e^z} \quad , \\ \mathrm{Lexp}_1(z) &= -\ln(1-e^z) \quad , \\ \mathrm{Lexp}_2(z) &= -\int\limits_{-\infty}^{z} \ln(1-e^z)dz \quad . \end{aligned} \tag{46}$$

Similarly, we have

$$\not{\mathrm{L}}\mathrm{exp}_m(z) = \int\limits_{-\infty}^{z} \not{\mathrm{L}}\mathrm{exp}_{m-1}(z)dz \quad , \tag{47}$$

and

$$\begin{aligned} \not{\mathrm{L}}\mathrm{exp}_0(z) &= -\frac{e^z}{1+e^z} \quad , \\ \not{\mathrm{L}}\mathrm{exp}_1(z) &= -\ln(1+e^z) \quad , \\ \not{\mathrm{L}}\mathrm{exp}_2(z) &= -\int\limits_{-\infty}^{z} \ln(1+e^z)dz \quad . \end{aligned} \tag{48}$$

The explicit form of other L-functions can be similarly obtained. We will not reproduce them here.

In the following, we will concentrate on the case $m = 2$. The function $\mathrm{Lexp}_2(z)$ has branch cuts on the positive semiaxes where $\mathrm{Im}\ z = 2n\pi i$, $n = 0, \pm 1, \pm 2, \ldots$, and the function $\not{\mathrm{L}}\mathrm{exp}_2(z)$ has branch cuts on the positive semiaxes where $\mathrm{Im}\ z =$

$(2n+1)\pi i, n = 0, \pm 1, \pm 2, \ldots$. On the real axis, we choose the imaginary part of $\mathrm{Lexp}_2(x)$ to be

$$\mathrm{Im}\ \mathrm{Lexp}_2(x) = -i\pi x\ \theta(x)\ . \tag{49}$$

Around the origin, the two functions have the following series expansion

$$\begin{aligned}
\mathrm{Lexp}_2(z) &= -z\ln(-z) + z - \frac{z^2}{4} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k+1)!} z^{2k+1}\ , \\
\not{L}\mathrm{exp}_2(z) &= -\frac{\pi}{12} - (\ln 2) z - \frac{z^2}{4} - \sum_{k=1}^{\infty} \frac{(2^{2k}-1)B_{2k}}{2k(2k+1)!} z^{2k+1}\ ,
\end{aligned} \tag{50}$$

where B_n are Bernoulli numbers [6] defined through the generating function

$$\frac{t}{e^t - 1} = \sum_0 B_n \frac{t^n}{n!}\ . \tag{51}$$

We have $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_4 = -1/30$, etc. The series expansions for other L-functions follow easily from those in Eqs. (50).

The real and imaginary parts of $\mathrm{Lexp}_2(z)$ can be obtained by Kummer's formula (see Ref. [2])

$$\begin{aligned}
\mathrm{Lexp}_2(x+iy) = &-\frac{1}{2}\int_0^x \ln\left(1 - 2e^x \cos y + e^{2x}\right) dx \\
&+ i\left\{xy' + \frac{1}{2}\mathrm{Cl}_2(2y) + \frac{1}{2}\mathrm{Cl}_2(2y') + \frac{1}{2}\mathrm{Cl}_2(2y'')\right\}\ ,
\end{aligned} \tag{52}$$

where

$$y' = \arctan\left(\frac{e^x \sin y}{1 - e^x \cos y}\right)\ , \quad y'' = \pi - y' - y''\ . \tag{53}$$

The separation of other L-functions into real and imaginary parts can be obtained by using the previous formula.

We have given here only some basic features of the L-functions. However, since they are defined from the polylogarithms, many other properties of polylogarithms are translated directly to L-functions. We refer the reader to Refs. [2-3] for other potential properties of L-functions.

In summary, we have provided an analytically and numerically desirable expression for the massless three-point scalar integral in terms of associated Clausen functions and discussed the main features of these functions and their analytically extended partners, the L-functions. The simplicity shown in Eq. (37) hints at the potential usefulness of these functions in other Feynman diagram calculations.

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[3] K. S. Kölbig, J. A. Mignaco, E. Remiddi, B.I.T. 10(1970)38, K. S. Kölbig, SIAM J. Math. Anal. 17(1986)1232.

[4] The intermediate Feynman parameter integral can be expressed in term of the Clausen function by the formula (A.3.2.3) on page 305 of Ref. [2].

[5] The trigonometric Clausen and Glaisher functions are fully described in Ref. [2]. However, the other associated functions seem to lack handy references. The notation of these functions here is tentative.

[6] See, for instance, I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products, Academic Press, London, 1980, p. xxix.

[7] However, the evaluation of the massless three-point integral is more practical with the real Clausen functions given previously.

[8] These functions can also be named the associated polylogarithmic exponential functions. The L-notation is inspired from the definition of $\mathrm{Lexp}_m(z)$ in terms of the polylogarithm $\mathrm{Li}_m(z)$. In fact, in this notation, the polylogarithm can be interpreted as the "L-identity" function.

[9] Z. Bern, L. J. Dixon and D. A. Kosower, preprint Pitt/92-04.

FIGURE CAPTIONS

1) One-loop Feynman diagram associated to the massless three-point function.

2) Geometrical interpretation of the angles ϕ_1, ϕ_2, ϕ_3 and the variable ρ in the completely spacelike region.

3) Plot of the Clausen function, the hyperbolic Clausen function and the alternating hyperbolic Clausen function.

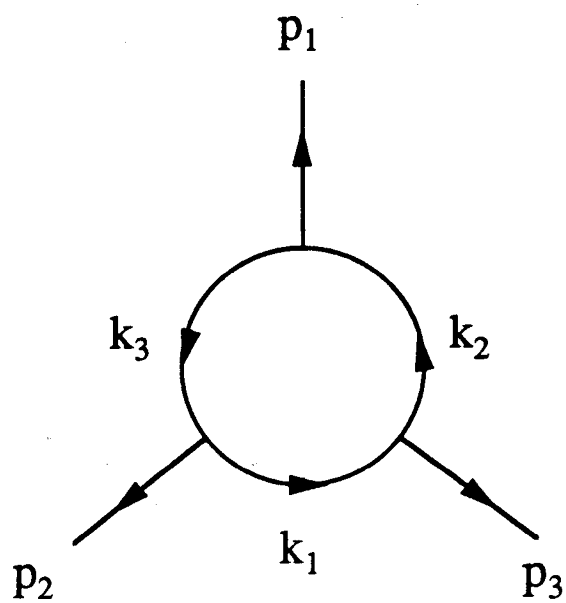
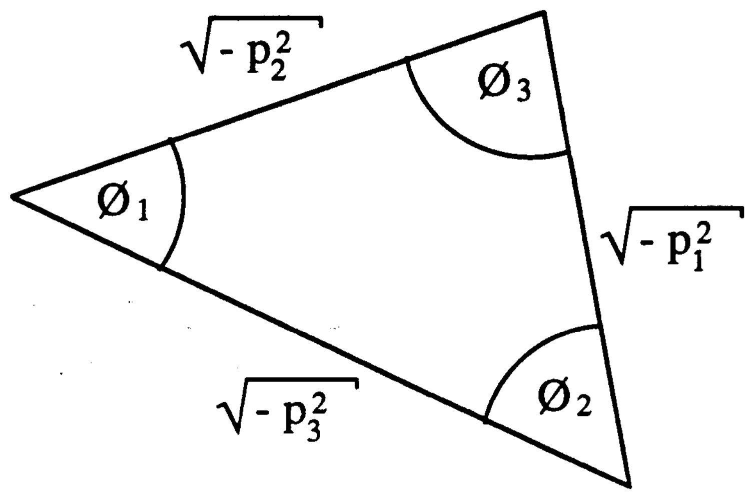
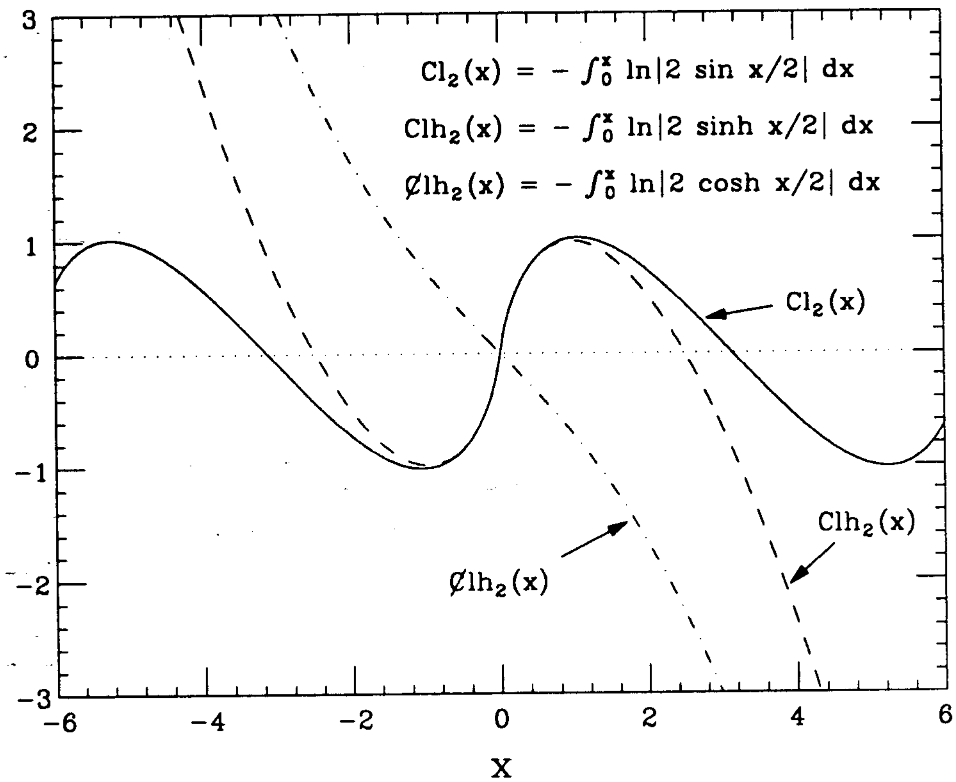


Fig. 1



$\rho = 2 \times$ Area of Triangle

Fig. 2



$Cl_2(x) = -\int_0^x \ln|2 \sin x/2|\, dx$
$Clh_2(x) = -\int_0^x \ln|2 \sinh x/2|\, dx$
$\not{C}lh_2(x) = -\int_0^x \ln|2 \cosh x/2|\, dx$
$Cl_2(x)$
$Clh_2(x)$
$\not{C}lh_2(x)$
3
2
1
0
−1
−2
−3
−6
−4
−2
0
2
4
6
x

Fig. 3