*LYAPUNOV FUNCTIONS FOR THE PROBLEM OF LUR'E IN AUTOMATIC CONTROL**

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1. About 1950, Lur'e[1] initiated the study of a class of (closed-loop) control systems whose governing equations are

$$dx/dt = Fx - g\varphi(\sigma), \qquad d\xi/dt = -\varphi(\sigma), \qquad \sigma = h'x + \rho\xi. \tag{L}$$

In (L), σ, ξ, ρ are real scalars, x, g, h are real n-vectors, and F is a real $n \times n$ matrix. The prime denotes the transpose. F is stable (all its eigenvalues have negative real parts). $\varphi(\sigma)$ is a real-valued, continuous function which belongs to the class A_κ: $\varphi(0) = 0, 0 < \sigma\varphi(\sigma) < \sigma^2\kappa$.

We ask: Is the equilibrium state $x = 0$ of (L) g.a.s. (globally asymptotically stable) for any $\varphi \in A_\kappa$?

2. This problem is related to the well-known 1946 conjecture of Aizerman: *If* (L) *is g.a.s. for every linear* $\varphi \in A_\kappa$, *then it is also g.a.s. for any* $\varphi \in A_\kappa$. In this crude form, however, Aizerman's conjecture was found to be false, and Lur'e was led to consider a more special situation:[1, 2]

PROBLEM OF LUR'E. *Find conditions on* ρ, g, h, *and* F *which are necessary and sufficient for the existence of a Lyapunov function* V *of a special type* (*namely* V = *a quadratic form in* (x, σ) *plus the integral of* $\varphi(\sigma)$) *which assures g.a.s. of* (L) *for any* $\varphi \in A_\infty$.

This is essentially an algebraic problem.

3. Even if $\varphi(\sigma) = \epsilon\sigma$, with $\epsilon > 0$ and arbitrarily small, (L) can be g.a.s. only if $\rho > 0$. This follows easily by examining the characteristic equation of (L) when $\varphi(\sigma) = \epsilon\sigma$. Henceforth, it will be always assumed that $\rho > 0$.

4. The best information available to date concerning the Problem of Lur'e is the highly important 1961

THEOREM OF POPOV.[3] *Assume that F is stable and that $\rho > 0$. Then* (L) *is g.a.s. if the condition*

$$\text{Re}(2\alpha\rho + i\omega\beta)[h'(i\omega I - F)^{-1}g + \rho/i\omega] \geqq 0 \quad \textit{for all real } \omega \tag{P}$$

holds for $2\alpha\rho = 1$ *and some* $\beta \geqq 0$.

Popov has also studied, but did not resolve, the question of existence of a Lyapunov function which assures g.a.s. whenever (P) holds. We shall settle this question completely and at the same time solve the Problem of Lur'e.

5. In the same paper, Popov proved also: *Consider the most general function* $V(x, \sigma)$ *which is a quadratic form in* (x, σ) *plus a multiple of the integral of* $\varphi(\sigma)$:

$$V(x, \sigma) = x'Px + \alpha(\sigma - h'x)^2 + \beta \int_0^\sigma \varphi(\sigma)d\sigma + \sigma w'x \quad (\alpha, \beta \text{ real}). \tag{1}$$

If for any $\varphi \in A_\epsilon$ ($\epsilon > 0$) *the function* $V \geqq 0$ *and* $\dot{V}$ (*its derivative along solutions of* (L)) *is* $\leqq 0$, *then* $w = 0$.

Assuming $w = 0$, V will be nonnegative for any $\varphi \in A_\infty$ if and only if $\alpha \geqq 0$,

$\beta \geqq 0$, and $P = P' \geqq 0$ (nonnegative definite). From (L) and (1) (with $w = 0$), we get

$$\dot{V}(x, \sigma) = x'(PF + F'P)x - 2\varphi(\sigma)x'(Pg - \alpha\rho h - (^1/_2)\beta F'h) - \beta(\rho + h'g)\varphi^2(\sigma) - 2\alpha\rho\sigma\varphi(\sigma). \quad (2)$$

$\dot{V} \leqq 0$ for any $\varphi \in A_\infty$ implies $\gamma = \beta(\rho + h'g) \geqq 0$. If

$$\text{(a) } Q = -PF - F'P, \qquad \text{(b) } \sqrt{\gamma}q = r = Pg - \alpha\rho h - (^1/_2)\beta F'h, \quad (3)$$

defines Q, q, and r, we can write $\dot{V}$ as

$$\dot{V}(x, \sigma) = -[x'(Q - qq')x + (\sqrt{\gamma}\varphi(\sigma) + q'x)^2 + 2\alpha\rho\sigma\varphi(\sigma)]. \quad (4)$$

If $\gamma > 0$, $\dot{V} \leqq 0$ for any $\varphi \in A_\infty$ if and only if $Q - qq' \geqq 0$. If $\gamma = 0$, $\dot{V} \leqq 0$ for any $\varphi \in A_\infty$ if and only if $r = 0$ and $Q \geqq 0$. (In this case, q is not defined by (3b) but may be picked always so that $Q \geqq qq'$.)

6. Our solution of the Lur'e Problem will utilize and extend results of Popov,[3] Yakubovich,[4] and LaSalle.[5] In addition, the following observation is of crucial technical importance.

By the writer's canonical structure theorem,[6] F, g, h defining a linear subsystem of (L) may be replaced by F_{BB}, g_B, and h_B (notations of ref. 6), without loss of generality as far as the g.a.s. of (L) is concerned. In fact, $h'(i\omega I - F)^{-1}g$ in (P) is equal to $h_B'(i\omega I - F_{BB})^{-1}g_B$.

Hence it may and *it will be assumed without loss of generality that the pair* (F, g) *is completely controllable and* (F, h') *is completely observable.*

All that is needed from controllability theory[7] in the subsequent discussion is the lemma:

The following statements are equivalent: (i) (F, g) *is completely controllable;* (ii) $\det [g, Fg, \ldots, F^{n-1}g] \neq 0$; (iii) $x'[\exp Ft]g \equiv 0$ *for all* t *implies* $x = 0$; (iv) g *does not belong to any proper* F*-invariant subspace of* R^n.

By definition, (F, h') *is completely observable if and only if* (F', h) *is completely controllable.*

7. THEOREM (Solution of the Problem of Lur'e). *Consider* (L), *where* $\rho > 0$, F *is stable,* (F, g) *is completely controllable, and* (F, h') *is completely observable. We seek a suitable Lyapunov function* V *from the class defined by* (1).

(A) $V > 0$ *and* $\dot{V} \leqq 0$ *for any* $\varphi \in A_\infty$ *(hence* V *is a Lyapunov function which assures Lyapunov stability of* $x = 0$ *of* (L) *for any* $\varphi \in A_\infty$*) if and only if* $w = 0$ *and there exist real constants* α, β *such that* $\alpha \geqq 0$, $\beta \geqq 0$, $\alpha + \beta > 0$, *and* (P) *holds.*

(B) *Suppose* V *satisfies the preceding conditions. Then* V *is a Lyapunov function which assures g.a.s. of* (L) *if and only if either* (i) $\alpha \neq 0$ *or* (ii) $\alpha = 0$ *and the equality sign in* (P) *occurs only at those values of* ω *where* $\text{Re}\{h'(i\omega I - F)^{-1}g\} \geqq 0$.

(C) *There is an "effective" procedure for computing* V.

The constants α, β whose existence is required are precisely those used in (1) to define V.

8. The principal tool in the proof of the theorem is the following result, itself of great interest in linear system theory:

MAIN LEMMA. *Given a real number* γ, *two real n-vectors* g, k, *and a real* $n \times n$ *matrix* F. *Let* $\gamma \geqq 0$, F *stable, and* (F, g) *completely controllable. Then* (i) *a real n-vector* q *satisfying*

$$\text{(a)}\quad F'P + PF = -qq', \qquad \text{(b)}\quad Pg - k = \sqrt{\gamma}q \tag{5}$$

exists if and only if

$$(^1/_2)\gamma + \operatorname{Re}\{k'(i\omega I - F)^{-1}g\} \geqq 0 \quad \textit{for all real } \omega. \tag{6}$$

Moreover, (ii) $X_1 = \{x : x'Px = 0\}$ *is the linear space of unobservable states*[6] *relative to* (F, k'); (iii) q *can be "effectively" computed;* (iv) (5) *implies* (6) *even if* qq' *is replaced by* $qq' + R$, *where* $R = R' \geqq 0$.

Observe that (5a) and the stability of F imply that P is symmetric, nonnegative definite.

9. *Proof of the Main Lemma:* Necessity: Add and subtract $i\omega I$ from (5a). Multiply (5a) by $(i\omega I - F)^{-1}$ on the right and by $(-i\omega I - F')^{-1}$ on the left. Using (5b) yields

$$2\operatorname{Re}\{k'(i\omega I - F)^{-1}g\} = |q'(i\omega I - F)^{-1}g|^2 - 2\sqrt{\gamma}\operatorname{Re}\{q'(i\omega I - F)^{-1}g\}, \tag{7}$$

which implies (6). Adding $R = R' \geqq 0$ to qq' in (5a) does not diminish the right-hand side of (7). Hence (iii).

Sufficiency, We exhibit a constructive procedure for finding q, hence V. Let a_k be the coefficient of s^k in the polynomial det $(sI - F) = \psi(s)$. Let $e_n = g$, $e_{n-1} = Fg + a_{n-1}g, \ldots, e_1 = F^{n-1}g + a_{n-1}F^{n-2}g + \ldots + a_0 g$. Because (F, g) is completely controllable, these vectors are linearly independent, hence form a basis for R^n. Relative to this basis, F, g, and h have the form

$$F = \begin{pmatrix} 0 & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & 0 & 1 \\ -a_0 & \ldots & & -a_{n-2} & -a_{n-1} \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{pmatrix}, \quad h = \begin{pmatrix} b_0 \\ \cdot \\ \cdot \\ b_{n-2} \\ b_{n-1} \end{pmatrix}.$$

Using the theory of the Laplace transformation, etc.,[8] it follows that

$$h'(sI - F)^{-1}g = (b_0 + \ldots + b_{n-1}s^{n-1})/\psi(s). \tag{8}$$

This formula identifies the components of any vector q (relative to the basis $e_1, \ldots, e_n$) with the numerator coefficients of the rational function $q'(sI - F)^{-1}g$.

Setting $s = i\omega$ and assuming (6), we can write

$$\gamma + 2\operatorname{Re}\{k'(i\omega I - F)^{-1}g\} = |\theta(i\omega)|^2/|\psi(i\omega)|^2 \geqq 0, \tag{9}$$

where θ is a polynomial in $i\omega$ of degree n with real coefficients.

θ is determined as follows. The numerator of the left-hand side of (9) is the polynomial $\Gamma(-\omega^2) = [\gamma - 2k'F(\omega^2 I + F^2)^{-1}g]\cdot[\det(\omega^2 I + F^2)]$. Since Γ has real coefficients and is nonnegative, its zeros λ_k are complex conjugate and of even multiplicity if real, negative. The zeros of $\Lambda(i\omega) = \Gamma(-\omega^2)$ are $\pm\sqrt{\lambda_k}$ and occur in complex conjugate pairs. The reflection of a pair of complex conjugate zeros of Λ about the imaginary axis is also a pair of zeros of Λ. Therefore $\theta(i\omega)$ exists and may be taken, e.g., as the product of all factors of Λ with left-half-plane zeros. θ so defined has complex conjugate zeros and therefore it is a polynomial with real coefficients. The above choice of θ is not unique, but convenient.

Since the leading coefficient of θ is $\sqrt{\gamma}$, $\nu = -\theta + \sqrt{\gamma}\psi$ is a polynomial of (formal) degree $n - 1$. If the coefficients of ν, arranged in the order of ascending powers, are identified with the vector q, then $\nu(i\omega)/\psi(i\omega) = q'(i\omega I - F)^{-1}g$ by (8). By retracing the steps of the necessity proof, it is easily verified that q so defined satisfies (5).

Let $X_1 = \{x : q'[\exp Ft]x \equiv 0\}$. By (5a), $x_1 \in X_1$ if and only if $x_1'Px_1 = 0$. Then (5b) implies $k'[\exp Ft]x_1 \equiv 0$. Hence, $X_1 \subset X_2 = \{x : k'[\exp Ft]x \equiv 0\}$. But it can be shown[8] that dim X_1 = {degree of the largest common divisor of ν, ψ} = {degree of largest common divisor of the numerator and denominator of $k'(i\omega I - F)^{-1}g$} = dim X_2. Hence, $X_1 = X_2$, which implies (ii) and completes the proof of the main lemma.

A weaker version of this lemma was proved by Yakubovich.[4]

10. *Proof of Part A of the Theorem:* Define $k = \alpha\rho h + (1/2)\beta F'h$.

Sufficiency. (a) *If* $\alpha \geqq 0$, $\beta \geqq 0$, *then condition* (P) *implies the following:* $\gamma \geqq 0$ *and there is a* q *satisfying* (3b). Indeed, if $\beta = 0$, then obviously $\gamma = 0$. If $\beta > 0$, then the left-hand side of (P) tends asymptotically to $\rho + h'g$ as $|\omega| \geqq \infty$ so that $\rho + h'g$ and hence γ must be nonnegative. By the definition of k, (P) is equivalent to (6). Since $\gamma \geqq 0$, the main lemma shows that q exists and satisfies (5*b*), which is the same as (3b).

(b) If $Q = qq'$ then P, Q satisfy (3a) because P, qq' satisfy (5a). Thus we have constructed a V of the form (1), and $V \geqq 0$ and $\dot{V} \leqq 0$ for any $\varphi \in A_\infty$.

(c) *If* $\alpha \geqq 0$, *and* (P) *holds, then* V *is positive definite if* $\alpha + \beta > 0$. Indeed, if either $\alpha = 0$ or $\beta = 0$, the pair (F, k') is completely observable because so is (F, h'). By (ii) of the main lemma $P > 0$. If both $\alpha, \beta > 0$, then again by (ii) of the main lemma $x'Px = 0$ only if $k'[\exp Ft]x \equiv 0$. But there is no $x \neq 0$ for which this condition can hold jointly with $h'x = 0$, because that would contradict complete observability of (F, h'). Hence, $P + \alpha hh' > 0$. Thus V is positive definite.

Necessity. Suppose $V > 0$ and $\dot{V} \leqq 0$. Then $\alpha \geqq 0$, $\beta \geqq 0$, and $\alpha + \beta > 0$ are certainly necessary; moreover, there must exist P, Q, and q satisfying (3) and we must have also $\gamma \geqq 0$, $Q = qq' + R$ ($R = R' \geqq 0$.) Since (3) corresponds to (5), it follows by (i) of the main lemma that (6) is satisfied. (6) is equivalent to (P), so that (P) is necessary.

11. *Proof of Part B of the Theorem.* Let V be the Lyapunov function constructed in §10. We recall Theorem VIII of ref. 9 (p. 66): *If* $V > 0$ *and* $\dot{V} \leqq 0$, *then every solution bounded for* $t > 0$ *tends to some invariant set contained in* $\dot{V} = 0$, Thus to establish g.a.s. of (L) we have to show that (a) every solution of (L) is bounded, and (b) the only invariant set of (L) in $\dot{V} = 0$ is $\{0\}$.

(a) This can be proved by exactly the same technique as was used by LaSalle[5] in similar context.

(b) We seek a solution $(x(t), \sigma(t))$ of (L), not identically zero, whose values lie in its own positive limit set as well as in $\dot{V} = 0$. Since V may be multiplied by a positive constant, there are two cases to be considered:

(i) Let $2\alpha\rho = 1$. By (4) $\dot{V} = 0$ only if $\sigma(t) \equiv 0$. so that $h'x(t) = -\rho\xi_0 =$ const. Moreover, $x(t) = [\exp Ft]x_0$ since $\varphi \equiv 0$. But $x_0 \neq 0$ would contradict complete observability of (F, h').

(ii) Now let $\alpha = 0$. By (4) $\dot{V} = 0$ implies

$$\sqrt{\gamma}\varphi(\sigma(t)) \equiv -q'x(t). \tag{10}$$

If $q'x(t) \equiv 0$, we have again the previous case. Otherwise $\gamma > 0$. Then $x(t)$ is the solution of the linear differential equation $dx/dt = (F + \gamma^{-1/2}gq')x$. By (a) above $x(t)$ is bounded. Hence $x(t)$ can lie in its own positive limit set only if it is almost periodic. Therefore at least one pair of eigenvalues of $F + \gamma^{-1/2}gq'$ must be $\pm i\omega_k \neq 0$, which implies that (6) holds with the equality sign at $\omega = \omega_k$. But then (10) and the requirement $\text{Re}\{h'(i\omega_k I - F)^{-1}g\} \geqq 0$, $k = 1, 2, \ldots$ are incompatible. Hence $\{0\}$ is the only invariant set in $\dot{V} = 0$.

On the other hand, the modified condition (P) in (*B-ii*) of the theorem is necessary for g.a.s. since it is the Nyquist stability criterion for linear functions in A_∞.

12. Even if we drop the assumption of complete controllability and observability of the subsystem (F, g, h), the theorem remains valid with respect to the completely controllable and completely observable state variables (x_B, σ). Since F is stable, $F_{AA}, \ldots, F_{DD}$ (see ref. 6) must be also stable. Thus, our theorem actually implies g.a.s. of the entire system (L), i.e., of the variables (x, σ). In particular, it implies Popov's theorem.

The question then arises whether the Lyapunov function (1), constructed on (x_B, σ), can be extended to (x, σ). If (P) holds as a strict inequality (so that $\dot{V} < 0$), this is quite easy to show and was explicitly pointed out by Morozan[10] using Yakubovich's version of our main lemma. But if $\dot{V} \leqq 0$, it seems unlikely that an explicit Lyapunov function can be constructed in general which specializes to (1) on (x_B, σ).

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