GROUP DYNAMICS OF THE HYDROGEN ATOM†

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I. Introduction

Present ideas about the interactions between elementary particles are based on the concept of virtual processes which can be pictured by means of diagrams. These diagrams occur either in a perturbation expansion of the scattering amplitude as derived from field theory or in the unitarity condition imposed on an analytic S-matrix theory. In either case, the hope is that starting with a few—in some sense fundamental—particles and taking into account all their virtual processes, one will eventually be able to calculate the complete scattering matrix and to recover all observed resonance spectra from its singularity structure. To solve this problem completely, a summation of infinitely many diagrams would, in general, be necessary. While for weak enough interactions we know about hierarchies of strengths of diagrams and a finite subset of them may be enough to

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approximate the observations as well as desired, our knowledge about such hierarchies for strong interactions is quite rudimentary.

The number of observed resonances has been increasing continuously over the past years. Their masses and spins become larger hand in hand, and the spectrum of their probably quantum number starts exhibiting high regularities. Since this spectrum is generated by a large set of strong virtual processes, it is quite suggestive to ask whether or not one can use information on the spectrum directly to take into account the effect of all the corresponding diagrams and thus to obtain stringet restrictions upon the structure of the scattering amplitudes, or even to determine it completely. During the past year there has been a considerable success in setting up such a theory for the most fundamental possible interaction—the three-particle vertex. The guidelines of how such a theory should be formulated have been derived almost exclusively from the study of the dynamics of the best known quantum mechanical system—the non-relativistic hydrogen atom. The algebraic structure of its interaction with an external photon has become the model case of a three-particle vertex in which some external particle can excite another one over a large highly regular range of available quantum states. It is therefore quite worthwhile to study this structure in detail.

The goal of our study can be formulated in the following way: We want to be able to express all information contained in the Schrödinger theory of the hydrogen atom in a completely algebraic language in which no typically quantum mechanical variables, like internal coordinates, occur any longer but rather energy-momentum variables of interaction vertices which permit a relativistic generalization. The observation that there is a non-compact group O(4,1), whose maximally degenerate representation has a spectrum being in one-to-one correspondence with the hydrogen spectrum, will be the basis for our discussion (Section II). As we shall show, there exists an extension of this group, O(4,2), which, moreover, allows for a very simple description of all transition form factors of the process

$$H^{**} \to H^{*} + \gamma$$

in terms of simple group operations. (Sections III and IV.) In Section VI we shall give relativistic infinite component wave equations containing only external coordinates of the H atom but carrying complete information on the internal structure through its current operator. An important relation between the conservation of this current and the H atom mass spectrum will be discussed. In Section VII the free states of the H atom and their connection with the bound states will be given. Finally, the algebraic structure of the interaction will be described in such a way that it can readily be generalized to particle physics (Section VIII).

II. Group Theoretical Description of the Quantum Numbers of the Hydrogen Atom States

It has been known for quite some time that there exists a representation of the group O(4,1) containing the quantum numbers of the H atom in a natural way.[1] Recall that O(4,1) is the group of rotations in five dimensions with the metric

$$g = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}, \tag{II.1}$$

and is generated by the Lie algebra of antisymmetric O(4,1) tensors $L_{\alpha\beta}$ ($\alpha,\beta = 1,2,\ldots,5$) with the commutation rules

$$[L_{\alpha\beta}, L_{\alpha\gamma}] = ig_{\alpha\alpha}L_{\beta\gamma}. \tag{II.2}$$

The interesting representation can be given in terms of creation and annihilation operators of spin $\frac{1}{2}$:

$$a_r, a_r^\dagger;\ b_r, b_r^\dagger \qquad (r = 1,2)$$

satisfying

$$[a_r, a_s^\dagger] = \delta_{rs}\ ; \quad [b_r, b_s^\dagger] = \delta_{rs}\ , \tag{II.3}$$

and Pauli matrices

$$\sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix},\ \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},\ C = i\sigma_2, \tag{II.4}$$

as

$$\begin{aligned} L_{ij} &= \frac{1}{2}\left(a^\dagger\sigma_k a + b^\dagger\sigma_k b\right) \\ L_{i4} &= -\frac{1}{2}\left(a^\dagger\sigma_i a - b^\dagger\sigma_i b\right) \\ L_{i5} &= -\frac{1}{2}\left(a^\dagger\sigma_i C b^\dagger - aC\sigma_i b\right) \\ L_{45} &= \frac{1}{2i}\left(a^\dagger C b^\dagger - aCb\right). \end{aligned} \tag{II.5}$$

The representation space is then spanned by the basis

$$|n_1 n_2 m\rangle = \left[n_1!(n_2+|m|)!n_2!(n_1+|m|)!\right]^{-1/2} \times \begin{cases} a_1^{\dagger\, n_2+m} a_2^{\dagger\, n_1} b_1^{\dagger\, n_1+m} b_2^{\dagger\, n_2} |0\rangle & m \geqq 0 \\ a_1^{\dagger\, n_2} a_2^{\dagger\, n_1-m} b_1^{\dagger\, n_1} b_2^{\dagger\, n_2-m} |0\rangle & m \leqq 0 \end{cases} \quad \text{for} \tag{II.6}$$

The representation is obviously irreducible and also unitary, since all $L_{\alpha\beta}$ are Hermitian. The Casimir operators of O(4,1) have the values:

$$\begin{aligned} C_2 &= L_{\alpha\beta}L^{\alpha\beta} = -4 \\ C_4 &= L_{\alpha\beta}L^{\beta\gamma}L_{\gamma\delta}L^{\delta\alpha} = 0. \end{aligned} \tag{II.7}$$

The operators L_k, L_{i4} generate an O(4) subgroup which keeps the total number of $a^\dagger$ and $b^\dagger$, or also the operator

$$N = \tfrac{1}{2}(a^\dagger a + b^\dagger b + 2), \tag{II.8}$$

invariant. On the states $|n_1 n_2 m\rangle$ we find

$$N|n_1 n_2 m\rangle = n|n_1 n_2 m\rangle = (n_1+n_2+|m|+1)|n_1 n_2 m\rangle. \tag{II.9}$$

The other diagonal operators are

$$L_3|n_1 n_2 m\rangle = m|n_1 n_2 m\rangle \tag{II.10}$$

$$M_3|n_1 n_2 m\rangle = (n_1-n_2)|n_1 n_2 m\rangle. \tag{II.11}$$

If we do the following identification of generators

$$\begin{aligned} L &\equiv \text{orbital angular momentum} \\ N &\equiv \frac{1}{\sqrt{-2H}} \end{aligned} \tag{II.12}$$

where H is the Hamiltonian of the H-atom, we see that we thus have generated with O(4,1) the complete bound state Hilbert space of the hydrogen atom in the parabolic basis which is used for the theory of the Stark effect. In position space, the wave functions are given by[2])

$$u_{n_1 n_2 m}(\xi,\eta,\phi) = e^{im\phi} N_{n_1 n_2 m} e^{-i(\xi+\eta)/2} \left(\frac{\xi\eta}{n^2}\right)^{|m|/2} L^{|m|}_{n_1+|m|}(\xi/n) L^{|m|}_{n_2+|m|}(\eta/n) \qquad \text{(II. 13)}$$

where ξ,η are the parabolic coordinates:

$$\begin{aligned} \xi &= r + z \\ \eta &= r - z, \end{aligned} \qquad \text{(II. 14)}$$

ϕ is the azimuthal angle, and $N_{n_1 n_2 m}$ is the normalization constant:

$$N_{n_1 n_2 m} = \frac{(-)^{n_2+(|m|-m)/2}}{\sqrt{\pi} n^2} \left[\frac{n_1! n_2!}{(n_1+|m|)!^3 (n_2+|m|)!^3}\right]^{1/2} \qquad \text{(II. 15)}$$

In this representation, identifying the states $|n_1 n_2 m\rangle$ with $u_{n_1 n_2 m}(\xi,\eta,\phi)$, the angular momentum L and the Hamiltonian H become:

$$\underline{L} = \underline{r} \times \underline{p}, \qquad H = \frac{\underline{p}^2}{2} - \frac{1}{r}, \qquad \text{(II. 16)}$$

while the position representation of $R_i \equiv L_{i4}$ is the so-called Rumge Lentz vector:

$$\underline{R} = \tfrac{1}{2}(\underline{p}\times\underline{L} - \underline{L}\times\underline{p}) - \hat{r}. \qquad \text{(II. 17)}$$

One may then ask: What form does the usual wave functions $\Psi_{n\ell m}(\underline{x})$ (on which L^2 and L_3 are diagonal) take on our representation space? For this, one has just to observe that

$$L = J + K, \quad R = -J + K \qquad \text{(II. 18)}$$

defines L,R in terms of the commuting $O(3)\times O(3)$ generators of a- and b-spin:

$$\underline{J} = \tfrac{1}{2} a^\dagger \underline{\sigma} a, \quad \underline{K} = \tfrac{1}{2} b^\dagger \underline{\sigma} b. \qquad \text{(II. 19)}$$

On $|n_1 n_2 m\rangle$, $O(3)\times O(3)$ is diagonal since all a's, b's commute among each other and therefore couple totally symmetrically to $j=k=n_1+n_2+m = n-1/2$. One can also easily read off Eq. (II.6), by counting up- and down-states, that

$$j_3 = \tfrac{1}{2}(n_2-n_1+m), \quad k_3 = \tfrac{1}{2}(n_1-n_2+m). \tag{II.20}$$

If one now uses the fact that J and K commute, the basis on which L^2 is diagonal is just given by means of Wigner's 3-j symbols

$$|n\ell m\rangle = (-)^m(2\ell+1)^{\frac{1}{2}}\begin{pmatrix} (n-1)/2 & (n-1)/2 & \ell \\ \frac{1}{2}(n_2-n_1+m), & \frac{1}{2}(n_1-n_2 m), & -m\end{pmatrix}|n_1 n_2 m\rangle. \tag{II.21}$$

In x-space one has to identify $|n\ell m\rangle$ with

$$\Psi_{n\ell m}(\underline{x}) = N_{n\ell}\, e^{-r/n}\left(\frac{r}{n}\right)^{\ell} F\left(-n_r, 2\ell+2, 2\,\frac{r}{n}\right)^2 \tag{II.22}$$

where

$$N_{n\ell} = \frac{2^{\ell+1}}{(2\ell+1)!\,n^2}\left[\frac{(n+\ell)!}{n_r!}\right]^{\frac{1}{2}} \tag{II.23}$$

and n_r is the radial quantum number $n_r = n-\ell-1$.

Observe now that the group O(4,1) can be extended unitarily to O(4,2) on the same Hilbert space by introducing the additional (Hermitian) operators

$$\begin{aligned} L_{i6} &= \frac{1}{2i}\left(a^{\dagger}\sigma_i C b^{\dagger} + aC\sigma_i b\right) \\ L_{46} &= \frac{1}{2}\left(a^{\dagger} C b^{\dagger} + aCb\right) \\ L_{56} &= N \end{aligned} \tag{II.24}$$

which close with the generators (II.5) under the commutation rules (II.2) with α, β, γ, etc. running now from 1 to 6 and the metric

$$g = \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ &&& 1 && \\ &&&& -1 & \\ &&&&& -1 \end{pmatrix} \tag{II.25}$$

The group O(4,2) will be sufficient to describe in a simple manner all the properties of the H-atom with respect to electromagnetic transitions. To show this, we shall make extensive use of the O(2,1) × O(2,1) subgroup of O(4,2) containing only the generators:

$$L_{35}, L_{34}, L_{45}; L_{56}, L_{36}, L_{46}. \tag{II.26}$$

Define the operators

$$\begin{aligned}
N_1^+ &= -a_2^\dagger b_1^\dagger \\
N_1^- &= -a_2 b_1 \\
N_1^3 &= \frac{1}{2}\left(a_2^\dagger a_2 + b_1^\dagger b_1 + 1\right) = \frac{1}{2}\left(N_{a_2} + N_{b_1} + 1\right) \\
N_2^+ &= a_1^\dagger b_2^\dagger \\
N_2^- &= a_1 b_2 \\
N_2^3 &= \frac{1}{2}\left(a_1^\dagger a_1 + b_2^\dagger b_2 + 1\right) \equiv \frac{1}{2}\left(N_{a_1} + N_{b_2} + 1\right)
\end{aligned} \tag{II.27}$$

and, as usual, their cartesian combinations

$$\begin{aligned}
N_i^1 &= \frac{1}{2}\left(N_i^+ + N_i^-\right) \\
N_i^2 &= \frac{1}{2i}\left(N_i^+ - N_i^-\right).
\end{aligned} \tag{II.28}$$

Then we see that also these operators commute according to O(2,1) × O(2,1) rules:

$$\left[N_{\substack{1\\2}}^i N_{\substack{1\\2}}^j\right] = i g_{kk} N_{\substack{1\\2}}^k \qquad (i, j, k = \text{cyclic, running from 1 to 3})$$

$$\left[N_1^i, N_2^j\right] = 0, \qquad g = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}. \tag{II.29}$$

In terms of N_i^j, the operators (II.26) are

$$L_{34} = N_1^3 - N_2^3$$
$$L_{35} = N_1^1 - N_2^1$$
$$L_{45} = N_1^2 + N_2^2$$
$$L_{36} = -N_1^2 + N_2^2$$
$$L_{46} = N_1^1 + N_2^1$$
$$L_{56} = N_1^3 + N_2^3. \qquad \text{(II. 30)}$$

On the basis $|n_1 n_2 m\rangle$ the matrix elements of $N_{\frac{1}{2}}$ have a very simple representation, as can be seen by direct application of (II.27) onto (II.6):

$$N_1^{\pm}|n_1 n_2 m\rangle = -\left[\left(n_1 + \{{}^1_0\}\right)\left(n_1 + m + \{{}^1_0\}\right)\right]^{\frac{1}{2}} |n_1 \pm 1, n_2 m\rangle$$

$$N_2^{\pm}|n_1 n_2 m\rangle = \left[\left(n_2 + \{{}^1_0\}\right)\left(n_2 + m + \{{}^1_0\}\right)\right]^{\frac{1}{2}} |n_1 n_2 \pm 1, m\rangle$$

$$N_r^3 |n_1 n_2 m\rangle = (2n_r + m)|n_1 n_2 m\rangle \qquad (r = 1, 2). \qquad \text{(II. 31)}$$

The whole O(4,2) can be given in x-space representation if we determine N_i^j as functions of x. Using (II.31) and

$$L^m_{n_1+1+m}(\xi) = \frac{n_1+m+1}{n_1+1}\left(\xi \frac{\partial}{\partial \xi} + n_1 + m + 1 - \xi\right) L^m_{n_1+m}(\xi)$$

$$L^m_{n_1-1+m}(\xi) = \frac{1}{(n_1+m)^2}\left(\xi \frac{\partial}{\partial \xi} - n_1\right) L^m_{n_1+m}(\xi) \qquad \text{(II. 32)}$$

one easily finds before the states $u_{n_1 n_2 m}(\xi, \eta, \phi)$

$$N_1^+ = -D_{n/n+1}\left(\xi \frac{\partial}{\partial \xi} - \xi + \frac{\xi}{2n} + \frac{L_3}{2} + n_1 + 1\right)\left(\frac{n}{n+1}\right)^2 \Big\}$$

$$N_1^- = -D_{n/n+1}\left(\xi\frac{\partial}{\partial\xi} + \frac{\xi}{2n} - \frac{L_3}{2} - n_1\right)\left(\frac{n}{n+1}\right)^2 \Bigg\} \qquad \text{(II. 33)}$$

where $D_{n/n+1}$ is the dilatation operator defined by

$$D_a u(x) = u(ax). \qquad \text{(II. 34)}$$

III. Algebraic Substitution of Schrödinger Theory

In the last section we gave a representation of O(4,2) whose states could be brought in one-to-one correspondence with the states of the hydrogen atom. Given such a representation we want to analyze now the problem of how much additional information is needed to substitute Schrödinger theory.

Schrödinger theory provides us with two independent informations:

1) It gives the energy as a function of the internal quantum numbers.
2) It gives the wave function of the internal constituents of the system.

Once a representation of the group O(4,2) has been found on the Hilbert space of the states $|n\ell m\rangle$, the first information must be supplied by a relation:

1a) $$H = H(n\ell m). \qquad \text{(III. 1)}$$

In order to recover the wave functions from the O(4,2) representation states $|n\ell m\rangle$ we have to know either:

2a) The diagonal operators L_{56}, L^2, L_3 as a function of the position and momentum operators x and p:

$$L_{56} = L_{56}(x,p)$$
$$L^2 = L^2(x,p)$$
$$L_3 = L_3(x,p) \qquad \text{(III. 2a)}$$

(these relations were determined in the last section), or

2b) x, p as a function of the generators of O(4,2), i.e.,

$$x = x(L_{ab})$$
$$p = p(L_{ab}). \qquad \text{(III. 2b)}$$

Both specifications 2a) and 2b) involve the internal coordinates of the system and thus are not suitable in this form for a relativistic generalization to elementary particle physics. Observe, however, that the relation (III. 2b) can be used to describe the electromagnetic structure of the system. Let $\vec{x}_e, \vec{x}_p, \vec{X}, \vec{x}$ denote the electron, proton, center of mass and relative coordinate $(=\vec{x}_e - \vec{x}_p)$ of the H-atom, respectively, then the wave function of the atom moving with momentum $\vec{q}$ is given by

$$\phi_{\vec{q}}(\vec{x}_e, \vec{x}_p) = \frac{1}{\sqrt{V}} e^{i\vec{q}\vec{X}} \psi_n(\vec{x}) \tag{III. 3}$$

where $\psi_n(\vec{x})$ is the usual Schrödinger wave function for the relative motion of electron and proton. The electronic current for the transition of the atom from momentum $\vec{q}$ to rest is then given by the Fourier transforms:

$$\rho(\vec{q}) = \int \phi^{\prime *}_{\vec{o}}(\vec{x}_e, \vec{x}_p) \phi_{\vec{q}}(\vec{x}_e, \vec{x}_p) e^{-i\vec{q}\vec{x}_e} d\vec{x}_e \, d\vec{x}_p$$

$$I^i(\vec{q}) = \frac{1}{2m_e i} \int \left[\phi^{\prime *}_{\vec{o}}(x_e x_p) \frac{\overleftrightarrow{\partial}}{\partial x_e^{\,i}} \phi_{\vec{q}}(x_e x_p)\right] e^{-i\vec{q}\vec{x}_e} d\vec{x}_e \, d\vec{x}_p . \tag{III. 4}$$

Remember that this current describes the amplitude for the coupling of a hydrogen to an external photon with momentum $\vec{q}$ and polarization vector ϵ^μ via the scalar product:

$$A(q) = \frac{1}{c} \rho(q) \epsilon^o - I^i(q) \epsilon^i . \tag{III. 5}$$

Going from $\vec{x}_e$ and $\vec{x}_p$ to the center-of-mass and internal coordinates X and x, we can express ρ and I^i as

$$\rho(\vec{q}) = \int \psi^*_{n'}(\vec{x}) e^{-i\vec{q}(m_p/(m_p+m_e))\vec{x}} \psi_n(\vec{x}) d\vec{x} \tag{III. 6}$$

$$I^i(\vec{q}) = K^i(\vec{q}) + \frac{q_i}{2m_e} \rho(\vec{q}) \tag{III. 7}$$

with the auxiliary vector:

$$K^i(\vec{q}) = \frac{1}{m_e i} \int \psi^*_{n'}(x) \frac{\partial}{\partial x^i} e^{-i\vec{q}(m_p/(m_p+m_e))\vec{x}} \psi_n(\vec{x}) d\vec{x} . \tag{III. 8}$$

From these equations we see that if we eliminate $\vec{x}$ and $\partial/\partial\vec{x}$ using the equations (III. 2b), we can obtain a completely algebraic prescription for obtaining the electromagnetic current as a function of the momentum transfer q. But this current is now enough to replace all Schrödinger theory:

First, the wave functions of the atom can be obtained from $\rho(\vec{q})$ by a simple Fourier transform which gives all the products

$$\psi^*_{n'}(x)\psi_n(x)$$

and therefore all wave functions $\psi_n(x)$. Second, the functions $\rho(\vec{q})$, $I^i(\vec{q})$ form an electromagnetic current and thus have to be conserved (since the photon has spin one). Therefore, if the photon goes in the 3-direction, the ratio I^3/ρ has to fulfill

$$\frac{I^3(\vec{q})}{\rho(\vec{q})} = \frac{M' - M - q^2/2M}{q} \tag{III.9}$$

and from this relation we can calculate the energy as a function of the internal quantum numbers. Thus we see that an algebraic theory of the electromagnetic current for all momentum transfers can completely substitute the Schrödinger equation of a system. Such a theory will be formulated in the following section.

IV. Electromagnetic Current in O(4,2) Description

We shall proceed here according to the program designed in Section III. We first find the operators $\vec{x}$ and $\vec{p}$ as a function of the generators L_{ab} of O(4,2) and express then the electromagnetic current for all transitions $n \to n'$ in terms of group and Lie algebra operations.[3)]

To find $\vec{x}$ and $\vec{p}$, the most elegant approach makes use of the natural representation of O(4,1) on the space of normed homogeneous functions $f(z_a)$ $(a = 1, \ldots, 5)$ in the five-dimensional parameter space of O(4,2).[4)] The scalar product is defined as

$$(f, g) = 2 \int f^*(z)\, g(z)\, \delta(z^2)\, dz \tag{IV.1}$$

which projects out the irreducible part of the representation that remains on the light cone.

A finite group element G of O(4,1) transforms f(z) into

$$f'(z) \equiv U(G)\, f(z)\, U^{-1}(G) = f(zG). \tag{IV.2}$$

G can be written as

$$G_{cd} = \left(e^{-i\alpha_{ab}L_{ab}}\right)_{cd} \tag{IV.3}$$

with

$$\left(L_{ab}\right)_{cd} = i\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right), \qquad \text{(IV.4)}$$

and its x-representation is

$$L_{ab} = \frac{1}{i}\left(z_a\partial_b - z_b\partial_a\right) \qquad \text{(IV.5)}$$

$$U(G) = e^{-i\alpha_{ab}L_{ab}} \qquad \text{(IV.6)}$$

It is easy to verify that a function can be normed if and only if their degree of homogeneity N satisfies the conditions:

$$N = -\frac{3}{2} + i\nu, \quad \nu = \text{real} \left(= -\frac{p-1}{2} + i\nu \quad \text{for} \quad O(p,1)\right) \qquad \text{(IV.7)}$$

or

$$-3 < N < 0 \qquad \left(-(p-1) < N < 0 \text{ for } O(p,1)\right). \qquad \text{(IV.8)}$$

In such an irreducible representation, the parameter z_5 can be removed out of the operators and states defining new functions Φ by

$$f(z) = z_5^N \Phi(\xi_\mu) \qquad \text{(IV.9)}$$

where we have introduced homogeneous coordinates

$$\xi = \frac{z_\mu}{z_5}. \qquad \text{(IV.10)}$$

On the functions $\Phi(\xi_\mu)$ the finite transformations are then given by the factor representation

$$U(G)\,\Phi(\xi)\,U^{-1}(G) = \left(\xi_\mu G^\mu{}_5 + G^5{}_5\right)^N \Phi(\xi^*G) \qquad \text{(IV.11)}$$

where

$$(\xi^*G)_\mu \equiv U(G)\xi_\mu U^{-1}(G) = \frac{\xi_\nu G^\nu{}_\mu + G^5{}_\mu}{\xi_\nu G^\nu{}_5 + G^5{}_5}. \qquad \text{(IV.12)}$$

The infinitesimal generators are then

$$L_{\mu\nu} = \frac{1}{i}\left(\xi_\mu\partial_\nu + \xi_\nu\partial_\mu\right)$$

$$L_{\mu 5} = -\frac{1}{i}\left[\xi_\mu(N - (\xi\partial)) + \partial_\mu\right] \qquad \text{(IV.13)}$$

and the scalar product becomes

$$(\Phi', \Phi) \equiv (f', f) = \int dz_5 z_5^{2N+3} \Phi'^*(\xi)\, \Phi(\xi) d\Omega \tag{IV.14}$$

where

$$d\Omega = 2\delta(\xi^2 - 1) d\xi . \tag{IV.15}$$

z_5 can in general not be eliminated out of the scalar product. A complete orthogonal set of functions in the representation space is given by

$$f_{N,n,\alpha} = z_5^{N-n+1} y_{N,n-1,\alpha}(z_\mu) \tag{IV.16}$$

where the $y_{N,n-1,\alpha}$'s are homogeneous polynomials of degree n-1 in z_μ; i.e., they can be written in terms coordinate tensors

$$y_{N,n-1,\alpha}(z_\mu) = z_{\mu_1} \cdot \ldots \cdot z_{\mu_{n-1}} \tag{IV.17}$$

and in terms of four-dimensional spherical harmonics as

$$y_{N,n-1,\alpha}(z_\mu) = (z_5)^{n-1} y_{N,n-1,\alpha}(\xi_\mu). \tag{IV.18}$$

Observe that $y_{N,n-1,\alpha}$ solves the potential equation in four dimensions:

$$\partial^2 y_{N,n-1,\alpha}(z_\mu) = 0, \tag{IV.19}$$

which is therefore invariant under the operations of O(4,1) for every N.

The Casimir operators of O(4,1) are

$$C_2 = L_{ab}L^{ab} = 2\left[(3 + (z\partial))(z\partial)\right] = 2\left[(p - 1 + z\partial)(z\partial)\right] \quad \text{for } O(p,1) \tag{IV.20}$$

and

$$C_4 = L_{ab}L^{bc}L_{cd}L^{da} = 0 \tag{IV.21}$$

which give in front of functions of homogeneity N, from the Euler equation:

$$(z\partial) f(z) = Nf(z) \tag{IV.22}$$

$$C_2 = 2N(3+N) \tag{IV.23}$$

$$C_4 = 0. \tag{IV.24}$$

We see that $N=-2$ gives the same representation as the one used in (II.5) (see (II.7)). This is also clear for another reason. On the H-atom representation there exists a five-vector $(L_{\mu 6}, L_{56})$ transforming like z_a in the z-representation of O(4,1). But the operator z_a raises the degree of homogeneity from N to N+1. Hence, it can only then exist inside a single representation if N and N+1 are equivalent. From the Casimir operators C_2, C_4 we see that this is the case only for $N=-2(-p/2$ for $O(p,1))$.

The functions $f_{N,n,\alpha}$ can be chosen to be eigenstates of L^2, L_3. They also satisfy

$$R^2 f_{N,n,\alpha} = \frac{n^2-1}{2} f_{N,n,\alpha}. \tag{IV.25}$$

Hence, with the labels $\alpha = \ell, m; N=-2$ we have the correspondence:

$$f_{-2,n,\ell,m}(z_a) = \langle z | n\ell m\rangle. \tag{IV.26}$$

Therefore we can formally introduce a completeness relation:

$$\int d^5 z\ \delta(z^2)\, |z\rangle\langle z| = 1. \tag{IV.27}$$

Observe now that, since there exists for any O(p,1) an operator Γ_5 on the representation space with $N=-p/2$ which transforms like z_5 and is completely expressible as a function of ξ, we can define a new scalar product replacing (IV.14) by

$$(\Phi', \Phi) = \int \Phi'^*(\xi)\Gamma_5^{2N+3}\Phi(\xi)d\Omega_\xi \tag{IV.28}$$

and eliminate in this way the fifth variable z_5 completely from the representation. It is only in this case that one can represent O(p,1) unitarily on a space with p variables. Γ_5^{2N+3} is the metric which makes the non-Hermitian operators $L_{\mu 5}$ equivalent to Hermitian ones. For the particular case of the H-atom representation, the invariant scalar product then becomes

$$\begin{aligned}(\Phi', \Phi) &= \int \Phi'^*(\xi)\Gamma_5^{-1}\,\Phi(\xi)d\Omega_\xi \\ &= \int \Phi'^*(\xi)L_{56}^{-1}\,\Phi(\xi)d\Omega_\xi.\end{aligned} \tag{IV.29}$$

Analogously to what we did in Eqs. (IV.25) and (IV.26), we can now introduce the special homogeneous functions in ξ:

$$\Phi_{-2,n,\ell,m}(\xi)$$

and identify them with the H-atom states $|n\ell m\rangle$ through the formal definition:

$$\Phi_{n\ell m}(\xi) \equiv \Phi_{-2,n,\ell,m} = \langle \xi | n\ell m\rangle \qquad \text{(IV.30)}$$

with the completeness relation

$$\int d\Omega_\xi \, |\xi\rangle \, L_{56}^{-1} \langle \xi| = 1. \qquad \text{(IV.31)}$$

The explicit connection between hydrogen wave functions and $\Phi(\xi)$'s can now easily be given. Fock[5] observed that the n-dependent stereographic projection of the wave functions $\Psi_{n\ell m}(p)$ in momentum space onto the surface of a sphere in four-dimension with unit radius defined by

$$\vec{\xi} = \frac{2p_n\vec{p}}{p^2+p_n^2}, \quad \xi_4 = \frac{p^2-p_n^2}{p^2+p_n^2} \quad p_n = \frac{1}{n} \qquad \text{(IV.32)}$$

or its inverse:

$$\vec{p} = p_n \frac{\vec{\xi}}{1-\xi_4} \quad p^2 = p_n^2 \frac{1+\xi_4}{1-\xi_4} \qquad \text{(IV.33)}$$

and

$$\Phi_{n\ell m}(\vec{\xi}) = \frac{1}{4} \frac{(p_n^2+p^2)^2}{p_n^{3/2}} \Psi_{n\ell m}(\vec{q})$$

$$= p_n^{3/2}(1-\xi_4)^{-2}\Psi_{n\ell m}\left(p_n \frac{\vec{\xi}}{1-\xi_4}\right), \qquad \text{(IV.34)}$$

transforms the Schrödinger equation (in atomic units:

$$\mu = \frac{m_e m_p}{m_e+m_p} = 1, \quad e = \hbar = 1)$$

$$\left(\frac{p^2}{2} - E\right)\Psi_{n\ell m} = \frac{1}{2\pi^2}\int d\vec{q}\,|\vec{q}-\vec{p}|^{-2}\Psi_{n\ell m}(q) \qquad \text{(IV.35)}$$

into the integral equation for four-dimensional spherical harmonics:

$$\Phi_{n\ell m}(\xi) = \frac{1}{2p_n\pi^2}\int d\Omega_\eta (\eta-\xi)^{-2}\Phi_{n\ell m}(\eta) \qquad \text{(IV.36)}$$

which therefore are just our $\Phi_{n\ell m}(\xi)$ defined in Eq. (IV.30). The physical scalar product is not equal to the invariant one. For equal principal quantum numbers it is found to be

$$(\Psi'_n, \Psi_n)_{phys} = \int d\Omega_\xi\, \Phi_n'^*(\xi)\,\Phi_n(\xi) \qquad \text{(IV.37)}$$

since the (n-dependent) spherical angle is

$$d\Omega_\xi^n = 2\delta(\xi^2-1)d\xi = \frac{(2p_n)^3}{(p^2+p_n^2)^3}\,d^3p \qquad \text{(IV.38)}$$

giving, together with (IV.34),

$$(\Psi'_n, \Psi_n)_{phys} = \int d^3p\,\Psi_n'^*\Psi_n\,. \qquad \text{(IV.39)}$$

For $n'\neq n$, the scalar product (IV.37) obviously gives zero, since the functions Φ', Φ are spherical harmonics of different degree. In this case $d\Omega_\xi$ becomes a more involved function of the momentum and $p_n, p_{n'}$ than (IV.38).

The quantum mechanical operators $\vec{x}$ and $\vec{p}$ have a complicated n-dependent form before the functions $\Phi_{n\ell m}(\xi)$. Observe, however, that we can introduce the alternative states

$$\bar{\Phi}_{n\ell m}(\xi) = \frac{1}{4}\,\frac{(p^2+a^2)^2}{a^2}\,\Psi_n(p) \qquad \text{(IV.40)}$$

together with the fixed stereographic projection which one obtains by substituting $p_n\equiv a$ in Eqs. (IV.32), (IV.33). The physical scalar product becomes then

$$(\Psi', \Psi)_{phys} = \frac{1}{a}\int d\Omega_\xi\, \bar{\Phi}'^*(\xi)(1-\xi_4)\,\bar{\Phi}(\xi)$$

$$= \int d\Omega_\xi\, \bar{\Phi}'^*(\xi)\,\frac{1}{L_{56}}\,\frac{1}{a}\,(L_{56}-L_{46})\,\bar{\Phi}(\xi) \qquad \text{(IV.41)}$$

or, in terms of the invariant scalar product (IV.29),

$$(\Psi', \Psi)_{\text{phys}} = \left(\overline{\Phi}', \frac{1}{a}(L_{56}-L_{46})\overline{\Phi}\right). \tag{IV.42}$$

Thus the physical scalar product is obtained by using the charge operator $\rho = 1/a(L_{56}-L_{46})$ as a metric in the invariant scalar product. For the operators x_i and p_i we find

$$x_i = i\frac{\partial}{\partial p_i} = i\frac{\partial \xi_\mu}{\partial p_i}\frac{\partial}{\partial \xi_\mu} = \frac{1}{a}(L_{i5} - L_{i4}) \tag{IV.43}$$

and from (IV.33)

$$p_i = a\frac{\xi}{1-\xi_4} = a(L_{56} - L_{46})^{-1}L_{i6} \tag{IV.44}$$

whose physical matrix elements are from (IV.41) expressible in terms of the invariant scalar product as $(\Psi', p_i\Psi)_{\text{phys}} = (\overline{\Phi}', L_{i6}\overline{\Phi})$. The connection of the states $\overline{\Phi}_{n\ell m}(\xi)$ with $\Phi(\xi)$ can be given by the "tilting" operation

$$T_n n\,\overline{\Phi}_{n\ell m}(\xi) = \Phi_{n\ell m} \tag{IV.45}$$

with

$$T_n = e^{i\theta_n L_{45}}, \qquad \theta_n = \ln na. \tag{IV.46}$$

Applying this to $\overline{\Phi}(\xi)$ we indeed find

$$\begin{aligned} e^{i\theta_n L_{45}}\overline{\Phi}_n(\xi) &= (\operatorname{ch}\ln na - \operatorname{sh}\ln na\ \xi_4)^{-2}\,\overline{\Phi}_n(\xi^* T_n) \\ &= \frac{4}{n^2a^2}\left[(1-\xi_4)\left(1+\frac{p_n^2}{a^2}\,p^2\right)\right]^{-2}\overline{\Phi}_n(\xi^* T_n) \\ &= \frac{p_n^{3/2}}{\sqrt{n}}(1-\xi_4)^{-2}\,\Psi_n\left(p_n\frac{\xi_i}{1-\xi_4}\right) \end{aligned} \tag{IV.47}$$

But the functions on the right side are according to (IV.30) just the spherical harmonics in ξ up to a factor $1/\sqrt{n}$. Hence,

$$T_n n\overline{\Phi}_n(\xi) = \sqrt{n}\,\Phi_n(\xi). \tag{IV.48}$$

We see from this equation that $T_n n\overline{\Phi}_n(\xi)$ has the norm one in the invariant scalar product (IV.29), $\overline{\Phi}$ itself has the invariant norm n^{-2}. Observe that the tilter dilates the p in the wave function by p_n/a.[3)]

We have thus established a one-to-one correspondence between our states $|n\ell m\rangle$ and functions in the four-dimensional ξ space. This correspondence can be summarized by

$$\sqrt{n}\,\Phi_{n\ell m}(\xi) = \langle \xi | n\ell m\rangle, \tag{IV.49}$$

$$\bar{\Phi}_{n\ell m}(\xi) \equiv \langle \xi | \bar{n}\ell m\rangle \tag{IV.50}$$

with

$$|\bar{n}\ell m\rangle = \frac{1}{n} e^{-i\theta_n L_{45}} |n\ell m\rangle; \qquad \theta_n = \log an \tag{IV.51}$$

and

$$\int \overline{d\Omega}_{\xi} |\xi\rangle L_{56}^{-1} \langle \xi| = 1. \tag{IV.52}$$

Observe that the tilted states $|\bar{n}\ell m\rangle$ are renormalized by a factor $1/n$ in order that they have unit norm in the physical scalar product.

The physical quantities $\vec{x}$ and $\vec{p}$ (in atomic units) are represented on the physical states $|\bar{n}\ell m\rangle$ by

$$x_i = \frac{1}{a}(L_{i5} - L_{i4}) \tag{IV.53}$$

$$p_i = a(L_{56} - L_{46})^{-1} L_{i6} \tag{IV.54}$$

if the physical scalar product is defined via the charge density operator

$$\rho = \frac{1}{a}(L_{56} - L_{46}). \tag{IV.55}$$

Thus, if T is some observable, then the matrix element $(\psi_{n'}, T\psi_n)$ becomes

$$\langle \bar{n}' | \rho T | \bar{n}\rangle. \tag{IV.56}$$

Note that since the diagonal elements of ρ give the charge of the different states, the renormalization factor $1/n$ in (IV.51) guarantees constant charge for all states $|\bar{n}\ell m\rangle$.

With this information we now immediately find the group theoretical expressions for ρ and I^i by inserting (IV.53)-(IV.55) into (III.6)-(III.8). The result is (in atomic units)[6]

$$\rho(\vec{q}) = \langle \bar{n}| \frac{1}{a}(L_{56} - L_{46}) e^{-i(q_i/m_e)(1/a)(L_{i5} - L_{i4})} |\bar{n}\rangle \tag{IV.57}$$

$$I^i(\vec{q}) = K^i(\vec{q}) + \frac{\vec{q}}{2m_e}\rho(\vec{q}) \tag{IV.58}$$

$$K^i(\vec{q}) = \frac{1}{m_e}\langle \bar{n}'|L_{i6}e^{-i(q_i/m_e)(1/a)(L_{i5}-L_{i4})}|\bar{n}\rangle. \tag{IV.59}$$

Observe that

$$\exp\left[-i\frac{q_i}{m_e}\frac{1}{a}(L_{i5}-L_{i4})\right]$$

represents a Galilean subgroup on the O(4,2) Hilbert space. Since $q_i/(m_e+m_p)$ is the velocity of the H-atom, the operator

$$M_i \equiv -\frac{m_p}{a}(L_{i5}-L_{i4}) \tag{IV.60}$$

has to be identified with the Galilean generators. A physical state $|\bar{n}\rangle$ that is multiplied with the element

$$\exp\left[i\frac{q_i}{m_e}\frac{1}{a}(L_{i5}-L_{i4})\right]$$

of the Galilean group will be called "boosted" to momentum q and denoted by $|\bar{n},q\rangle$. Since the current (ρ, I^i) is a Galilean four-vector; i.e.,

$$[M_i, I^i] = i\rho$$

$$[M_i\rho] = 0, \tag{IV.61}$$

we can write the current operator in a form that exhibits this property in a better way than Eqs. (IV.57)-(IV.59). For this we go into a general frame of reference where the initial atom moves with momentum p, the final one with $p' = p + q$. Then the operators become

$$\rho = -\frac{m_p}{m_e}\frac{1}{a}(L_{56}-L_{46}) + \frac{m_p+m_e}{m_e}\frac{1}{a}(L_{56}-L_{46}). \tag{IV.62}$$

$$I^i = \frac{1}{m_e}L_{i6} + \frac{(p'+p)^i}{2m_e}\frac{1}{a}(L_{56}-L_{46}). \tag{IV.63}$$

The first and second parts of ρ and I^i now form separately a Galilean vector (IV.61). While the first part is completely algebraic, the second part involves the momentum vectors p' and p. Such a current is generally called a convective current. The first part of I_i

describes the internal motion of the system. The convective part gives the current due to the motion of the electron cloud as a whole.

Let us summarize the result of this section: The algebraic structure of the electromagnetic current can be characterized in the following way.

1) There exists a certain Galilean subgroup of O(4,2).

2) There is a vector operator with respect to this subgroup which is the sum of an algebraic and a convective part.

3) The physical states are given by the tilted and renormalized basis states of the representation. A tilter is a non-compact rotational invariant group operation. The renormalization factor is necessary to guarantee constancy of change within a multiplet.

4) The current at momentum transfer q is obtained as the matrix elements of the vector operator between a physical state at rest and one that is boosted by the Galilean transformation to momentum q.

This is the form of algebraic rules which has meanwhile successfully been applied to particle physics[7)] to find electromagnetic[8)] and pionic form[9)] factors of baryons. In order to make the theory relativistic, one has to use a Lorentz subgroup instead of a Galilean subgroup, and corresponding vector or pseudoscalar operators to represent the external photon or pion.

We want to point out at this place that in the composition of the vector operator, the convective part has proved to be very important for particle physics applications. While a theory without such a term can explain the shapes of the electromagnetic form factors rather well, magnetic moments turn out to have the wrong sign and the mass spectrum decreases with n. With a convective part, the fit to the nucleon form factors and the baryon mass spectrum becomes, however, excellent.

V. Evaluation of the Current

In this section we shall evaluate the matrix elements of the currents, using the formula (IV.57) and (IV.59), for arbitrary bound-bound transitions $n \to n'$. In atomic physics, these currents have been calculated for the transitions $1 \to n$ by Mohr and Massey[10)] by means of the integral expression in x space (III.6). It was not until recently that this integral was done for the general case.[11)] While integrations produce the current in a rather complicated and cumbersome form, our group theoretical formulation will lead to a highly symmetric result expressed in terms of products of hypergeometric functions.[7),12)]

The evaluation of the expressions (IV.57) and (IV.59) is simplified if we insert the explicit form of the physical states (IV.51) and move both tilting operators together. In this way we obtain the alternative relations:

$$\rho(\vec{q}) = \frac{1}{n} \langle n' |(L_{56} - L_{46}) e^{-i\theta_{n'n} L_{45}} e^{-i(q_i/m_e)n(L_{i5}-L_{i4})} | n \rangle . \tag{V.1}$$

$$K^i(\vec{q}) = \frac{1}{m_e n n'} \langle n' | L_{i6} e^{-i\theta_{n'n} L_{45}} e^{-i(q_i/m_e)n(L_{i5}-L_{i4})} | n \rangle , \tag{V.2}$$

where $\theta_{n'n} = \log n/n'$.

Inserting a complete set of intermediate states behind the Lie algebra operators and denoting the finite group transformation by G, i.e.,

$$G(\vec{q}) \equiv e^{-i\theta_{n'n} L_{45}} e^{-i(q_i/m_e)n(L_{i5}-L_{i4})} , \tag{V.3}$$

both expressions have the form

$$\langle n' | \mathcal{O} | n'' \rangle \langle n'' | G(\vec{q}) | n \rangle . \tag{V.4}$$

The matrix elements of $\mathcal{O}$ are easily evaluated in the parabolic basis using the creation and annihilation operator representation (II.5), (II.24), and for $m \geqq 0$ one finds ($|n\rangle$ stands for $|n_1 n_2 m\rangle$):

$$\begin{aligned}
\langle n | (L_{56}-L_{46}) | n \rangle &= n \\
\langle n | (L_{56}-L_{46}) | n_1+1, n_2, m \rangle &= \tfrac{1}{2}\left[(n_1+1)(n_1+m+1)\right]^{\frac{1}{2}} \\
\langle n | (L_{56}-L_{46}) | n_1-1, n_2, m \rangle &= \tfrac{1}{2}\left[n_1(n_1+m)\right]^{\frac{1}{2}} \\
\langle n | (L_{56}-L_{46}) | n_1, n_2+1, m \rangle &= -\tfrac{1}{2}\left[(n_2+1)(n_2+m+1)\right]^{\frac{1}{2}} \\
\langle n | (L_{56}-L_{46}) | n_1, n_2-1, m \rangle &= -\tfrac{1}{2}\left[n_2(n_2+m)\right]^{\frac{1}{2}} .
\end{aligned} \tag{V.5}$$

Similarly one finds for L_{i6} ($m \geqq 0$), defining as usual

$$L_6^{\pm} = L_{16} \pm iL_{26} , \tag{V.6}$$

$$\left.\begin{aligned}
\langle n | L_6^{+} | n_1 n_2 m-1 \rangle &= i\left[(n_1+m)(n_2+m)\right]^{\frac{1}{2}} \\
\langle n | L_6^{+} | n_1+1, n_2+1, m-1 \rangle &= i\left[(n_1+1)(n_2+1)\right]^{\frac{1}{2}}
\end{aligned}\right\}$$

$$\left.\begin{aligned}\langle n|L_6^-|n_1n_2m+1\rangle &= -i\Big[(n_1+m+1)(n_2+m+1)\Big]^{\frac{1}{2}}\\ \langle n|L_6^-|n_1-1,\, n_2-1,\, m+1\rangle &= -i\Big[n_1n_2\Big]^{\frac{1}{2}}\end{aligned}\right\} \qquad (V.7)$$

$$\begin{aligned}\langle n|L_{36}|n_1n_2-1,m\rangle &= \frac{1}{2i}\Big[(n_2+m)n_2\Big]^{\frac{1}{2}}\\ \langle n|L_{36}|n_1-1,n_2m\rangle &= \frac{1}{2i}\Big[(n_1+m)n_1\Big]^{\frac{1}{2}}\\ \langle n|L_{36}|n_1n_2+1,m\rangle &= \frac{1}{2i}\Big[(n_2+m+1)(n_2+1)\Big]^{\frac{1}{2}}\\ \langle n|L_{36}|n_1+1,\, n_2m\rangle &= -\frac{1}{2i}\Big[(n_1+m+1)(n_1+1)\Big]^{\frac{1}{2}}\end{aligned} \qquad (V.8)$$

What remains is to determine the matrix elements of $G(\vec{q})$. Without loss of generality we can assume that $\vec{q}$ points in the z direction; the other matrix elements may be obtained by a simple rotation. Then $G(\vec{q})$ becomes:

$$G(q) = e^{-i\theta_{n'n}L_{45}}e^{-i(q/m_e)n(L_{35}-L_{34})}. \qquad (V.9)$$

It has the advantage of leaving the magnetic quantum number m invariant. Representations of finite group transformation in O(4,2) are not known in general. In the particular case of our maximally degenerate representation we can, however, evaluate the matrix elements of G(q) using O(2,1) subgroups of O(4,2) discussed in (II.29), etc. Defining the operators K_i as

$$K_1 = L_{45},\; K_2 = -L_{35},\; K_3 = L_{34},$$

we see that they close to an O(2,1) subalgebra of the O(2,1) × O(2,1) algebra of (II.30), satisfying

$$[K_1,K_2] = -iK_3,\; [K_2,K_3] = iK_1,\; [K_3,K_1] = iK_2. \qquad (V.10)$$

K_3 is diagonal on the $|n_1n_2m\rangle$ basis with the eigenvalue n_1-n_2. The other two operators can be written more explicitly in terms of the generators N_i^j as

$$K_1 = N_1^2 + N_2^2$$

$$K_2 = -\left(N_1^1 - N_2^1\right). \tag{V.11}$$

Consider now the representation of elements of the O(2,1) groups formed by N_i^j. An irreducible representation of the N_1^i algebra, for instance, which contains a state $|n_1 n_2 m\rangle$ contains all the states

$$|0, n_2, m\rangle, \ldots, |\infty, n_1, n_2\rangle \tag{V.12}$$

with the eigenvalues of N_1^3 being $n_1 + m+1/2$. Then, in the notation of Bargmann[13] (see the Appendix), the matrix element of

$$e^{-iN_1^2\beta}$$

is just a function v_{mn}^k with the Casimir operator $k=(m+1)/2$.

$$\langle n_1' n_2' m' | e^{-iN_1^2\beta} | n_1 n_2 m\rangle = v^{(m+1)/2}_{n_1+((m+1)/2),\, n_1+((m+1)/2)}\left(\mathrm{sh}\frac{\beta}{2}\right)\delta_{n_2' n_2}\delta_{m'm}. \tag{V.13}$$

Similarly, one obtains for $e^{-iN_2^2\beta}$

$$\langle n_1' n_2' m' | e^{-iN_2^2\beta} | n_1 n_2 m\rangle = v^{(m+1)/2}_{n_1'+((m+1)/2),\, n_1+((m+1)/2)}\left(-\mathrm{sh}\frac{\beta}{2}\right)\delta_{n_1' n_1}\delta_{m'm}$$

and therefore for $e^{-iK_1\beta}$

$$\langle n_1' n_2' m' | e^{-iK_1\beta} | n_1 n_2 m\rangle = v^{(m+1)/2}_{n_1'+((m+1)/2),\, n_1+((m+1)/2)}\left(+\mathrm{sh}\frac{\beta}{2}\right)$$

$$\cdot v^{(m+1)/2}_{n_2'+((m+1)/2),\, n_2+((m+1)/2)}\left(-\mathrm{sh}\frac{\beta}{2}\right)\delta_{m'm}. \tag{V.14}$$

Knowing this, we can find the matrix elements of the operator G(q) immediately by parametrizing it in Euler form. Inserting K_i from (V.11), we have to find Euler angles α, β, γ such that

$$G = e^{-i\theta K_1} e^{i(qn/m_e)(K_2+K_3)} = e^{-i\alpha K_3} e^{-i\beta K_1} e^{-i\gamma K_3}. \tag{V.15}$$

One does this most easily in the 2×2 quaternion representation of $O(2,1)$, substituting

$$K_1 = \frac{i\sigma_1}{2}, \quad K_2 = \frac{i\sigma_2}{2}, \quad K_3 = \frac{\sigma_3}{2}, \tag{V.16}$$

from which one finds for the left side of Eq. (V.15) the expression

$$e^{-i\theta K_1} e^{i(nq/m_e)(K_2+K_3)}$$

$$= \mathrm{ch}\frac{\theta}{2} + \left[\sigma_1 + \frac{nq}{2m_e}(1 - \mathrm{cth}(\theta/2))(\sigma_2 - i\sigma_3)\right] \mathrm{sh}\,\frac{\theta}{2} \tag{V.17}$$

while the "Euler quaternion" is

$$e^{-i\alpha K_3} e^{-i\beta K_1} e^{-i\gamma K_3} = \cos\frac{\alpha+\gamma}{2}\,\mathrm{ch}\,\frac{\beta}{2} + \sigma_1 \cos\frac{\alpha-\gamma}{2}\,\mathrm{sh}\,\frac{\beta}{2}$$

$$+ \sigma_2 \sin\frac{\alpha-\gamma}{2}\,\mathrm{sh}\frac{\beta}{2} - i\sigma_3 \sin\frac{\alpha+\gamma}{2}\mathrm{ch}\frac{\beta}{2}. \tag{V.18}$$

Comparison of (V.17) with (V.18) gives the four equations:

$$\mathrm{ch}\,\frac{\theta}{2} = \cos\frac{\alpha+\gamma}{2}\,\mathrm{ch}\,\frac{\beta}{2} \tag{V.19a}$$

$$\mathrm{sh}\,\frac{\theta}{2} = \cos\frac{\alpha-\gamma}{2}\,\mathrm{sh}\,\frac{\beta}{2} \tag{V.19b}$$

$$\frac{nq}{2m_e}\left(1 - \mathrm{cth}\,\frac{\theta}{2}\right)\mathrm{sh}\,\frac{\theta}{2} = \sin\frac{\alpha-\gamma}{2}\,\mathrm{sh}\,\frac{\beta}{2} \tag{V.19c}$$

$$\frac{nq}{2m_e}\left(1 - \mathrm{cth}\,\frac{\theta}{2}\right)\mathrm{sh}\,\frac{\theta}{2} = \sin\frac{\alpha+\gamma}{2}\,\mathrm{ch}\,\frac{\beta}{2}\,. \tag{V.19d}$$

Squaring b) and c) and adding them we obtain

$$\mathrm{sh}^2\,\frac{\beta}{2} = \mathrm{sh}^2\,\frac{\theta}{2}\left(1 + \frac{n^2q^2}{4m_e^2}\left(1 - \mathrm{cth}\,\frac{\theta}{2}\right)^2\right). \tag{V.20}$$

If we insert $\theta = \log n/n$, we find

$$\mathrm{sh}\,\frac{\beta}{2} = \frac{1}{2\sqrt{n'n}}\left[(n'-n)^2 + \frac{q^2}{m_e^2}\,n'^2 n^2\right]^{\frac{1}{2}} \tag{V.21}$$

$$\mathrm{ch}\,\frac{\beta}{2} = \frac{1}{2\sqrt{n'n}}\left[(n'+n)^2 + \frac{q^2}{m_e^2}\,n'^2 n^2\right]^{\frac{1}{2}}. \tag{V.22}$$

For α we then obtain

$$\sin\begin{Bmatrix}\alpha\\ \gamma\end{Bmatrix} = \begin{Bmatrix}-n\\ n'\end{Bmatrix}\frac{q/m_e}{\mathrm{sh}\,\beta} \tag{V.23}$$

$$\cos\begin{Bmatrix}\alpha\\ \gamma\end{Bmatrix} = -\frac{1}{2n'n}\left[(n'^2-n^2) \pm n'^2 n^2 \frac{q^2}{m_e^2}\right]\Big/\mathrm{sh}\,\beta. \tag{V.24}$$

The phase of β has been fixed such that in the limit $q \to 0$

$$\alpha \to \begin{Bmatrix}-\pi\\ -\pi/2\\ 0\end{Bmatrix}, \qquad \gamma \to \begin{Bmatrix}+\pi\\ +\pi/2\\ 0\end{Bmatrix} \tag{V.25}$$

for $n' \gtreqless n$, respectively, as we can see from Eqs. (V.23) and (V.24).

With these angles, the matrix elements of the finite transformation G becomes in the $|n_1 n_2 m\rangle$ basis:

$$G^m_{n_1'n_2'n_1n_2} = \langle n_1'n_2'm|G(q^2)|n_1n_2m\rangle = e^{-i(n_1'-n_2')\alpha}e^{-i(n_1-n_2)\gamma}$$

$$\cdot\, v^{(m+1)/2}_{n_2'+((m+1)/2),\,n_1+((m+1)/2)}\left(+\mathrm{sh}\frac{\beta}{2}\right)$$

$$\cdot\, v^{(m+1)/2}_{n_2'+((m+1)/2,\,n_2+((m+1)/2)}\left(-\mathrm{sh}\,\frac{\beta}{2}\right)\delta_{m'm}. \tag{V.26}$$

Collecting the different terms in Eqs. (V.4)-(V.8) and changing to the $|n\ell m\rangle$ basis according to (II.21), we finally obtain for $\rho(q)$:

$$\underset{n'\ell'm,n\ell m}{\rho(q)} = \frac{1}{n}\left[(2\ell'+1)(2\ell+1)\right]^{\frac{1}{2}}\sum_{k',k}\begin{pmatrix}(n'-1)/2 & (n'-1)/2 & \ell'\\ m/2-k' & m/2+k' & -m\end{pmatrix}$$

$$\times\begin{pmatrix}(n-1)/2 & (n-1)/2 & \ell\\ m/2-k & m/2+k & -m\end{pmatrix}$$

$$\times\Big\{n'h^{o,(\ell'-\ell)}_{k',k}(\alpha,\gamma)v^{(m+1)/2}_{n'/2+k',\,n/2+k}(\beta) +$$

$$+\frac{1}{2}\left[(n'+1+2k')^2-m^2\right]^{\frac{1}{2}} h_{k',k}^{+,(\ell'-\ell)}(\alpha,\gamma)\, v^{(m+1)/2}_{n'/2+k'+1,\ n/2+k}(\beta)$$

$$+\frac{1}{2}\left[(n'-1+2k')^2-m^2\right]^{\frac{1}{2}} h_{k',k}^{-,(\ell'-\ell)}(\alpha,\gamma)\, v^{(m+1)/2}_{n'/2+k'-1,\ n/2+k}(\beta)\Big\}$$

$$\times\, v^{(m+1)/2}_{n'/2-k',\ n/2-k}(-\beta) \qquad \text{(V.27)}$$

where we have introduced the function

$$h_{k',k}^{\{\overset{+}{\underset{-}{0}}\},(i)}(\alpha,\gamma) = \begin{bmatrix}\cos\\ -i\sin\end{bmatrix}\left[\left(2k'+\begin{Bmatrix}+1\\0\\-1\end{Bmatrix}\right)\alpha+2k\gamma\right] \quad \text{for} \quad (-)^i = \begin{bmatrix}+1\\-1\end{bmatrix}. \qquad \text{(V.28)}$$

In a similar way we obtain

$$I^{\pm}_{n'\ell'm\pm1,n\ell m}(q) = K^{\pm}(q) = \mp\frac{i}{n'n}\frac{1}{m_e}\left[(2\ell'+1)(2\ell+')\right]^{\frac{1}{2}}$$

$$\times\sum_{k'k}\begin{pmatrix}(n'-1)/2 & (n'-1)/2 & \ell'\\ (m\pm1)/2-k' & (m\pm1)/2+k' & -m\mp1\end{pmatrix}$$

$$\times\begin{pmatrix}(n-1)/2 & (n-1)/2 & \ell\\ m/2-k & m/2+k & -m\end{pmatrix}$$

$$\times\left\{\left[\left(\frac{n'+m}{2}\right)^2-k'^2\right]^{\frac{1}{2}} h_{k',k}^{o,(\ell'-\ell+1)}(\alpha,\gamma)\right.$$

$$\times\, v^{(m+1)/2}_{(n'\mp1)/2+k',\ n/2+k}(\beta)\, v^{(m+1)/2}_{(n'\mp1)/2-k',\ n/2-k}(-\beta)$$

$$+\left[\left(\frac{n'-m}{2}\right)^2-k'^2\right]^{\frac{1}{2}} h_{k',k}^{o,(\ell'-\ell+1)}(\alpha,\gamma)$$

$$\left.\times\, v^{(m+1)/2}_{(n'\pm1)/2+k',\ n/2+k}(\beta)\, v^{(m+1)/2}_{(n'\pm1)/2-k',\ n/2-k}(-\beta)\right\} \qquad \text{(V.29)}$$

and

$$K^3(q) = \frac{i}{n'n}\frac{1}{m_e}\left[(2\ell'+1)(2\ell+')\right]^{\frac{1}{2}} \sum_{k'k} \begin{pmatrix} (n'-1)/2 & (n'-1)/2 & \ell \\ m/2-k' & m/2+k' & -m \end{pmatrix}$$

$$\times \begin{pmatrix} (n-1)/2 & (n-1)/2 & \ell \\ m/2-k & m/2+k & -m \end{pmatrix}$$

$$\times \left\{ \frac{1}{2}\left[\left(n'+1+2k'\right)^2\right]^{\frac{1}{2}} h^{+,(\ell'-\ell+1)}_{k',k}(\alpha,\gamma)\, v_{n'/2+k'+1,\ n/2+k}{}^{(\beta)}\right.$$

$$- \frac{1}{2}\left[\left(n'-1+2k'\right)^2 - m^2\right]^{\frac{1}{2}} h^{-,(\ell'-\ell+1)}_{k',k}(\alpha,\gamma)$$

$$\left. \times\, v_{n'/2+k'-1,\ n/2+k}{}^{(\beta)}\right\} v_{n'/2-k',\ n/2-k}{}^{(-\beta)}. \qquad \text{(V.30)}$$

We have plotted $\rho_{n'\ell'm,n\ell m}(q)$ for some of the excitations at the end.

Every factor $v \cdot v$ contains a term $\mathrm{ch}^{-(n'+n)}\beta/2$ which has a singularity when

$$\mathrm{ch}(\beta/2) = 0. \qquad \text{(V.31)}$$

From (V.22) we see that this happens at

$$q^2 = \frac{-(n'+n)^2}{n^2 n'^2} m_e^2 \qquad \text{(V.32)}$$

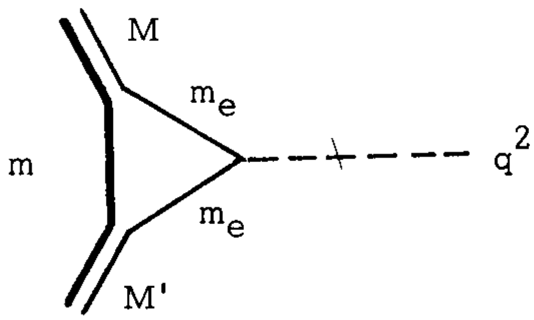
which can be written in terms of the binding energies of the states $|n\ell m\rangle$

$$B_n \equiv -\frac{1}{2n^2} \qquad \text{(V.33)}$$

as

$$q^2 = -2\left(\sqrt{B_n} + \sqrt{B_{n'}}\right)^2 m_e^2. \qquad \text{(V.34)}$$

Observe that this position of the singularity coincides up to order $B/M \cong 10^{-8}$ exactly with the anomalous threshold of the diagram



which can be calculated from the Cutkosky rules to lie at[14)]

$$\cos\theta_1 = \cos(\theta_2 + \theta_3) \tag{V.35}$$

with

$$\cos\theta_1 = \frac{2m_e^2 - q^2}{2m_e^2}$$

$$\cos\theta_2 = \frac{m_e^2 + m^2 - M^2}{2m_e m}$$

$$\cos\theta_3 = \frac{m_e^2 + m^2 - M'^2}{2m_e m} \; .$$

Solving this we obtain

$$t \equiv q^2 = 2m_e^2 - \frac{1}{2m_e^2 m^2}\Big\{\left(m_e^2+m^2-M^2\right)\left(m_e^2+m^2-M'^2\right)$$
$$- \Big[\left(2M^2\left(m_e^2+m^2\right)-\left(m_e^2-m^2\right)^2 - M^4\right)$$
$$\times \left(2M'^2\left(m_e^2+m^2\right) - \left(m_e^2-m^2\right) - M'^4\right)\Big]^{\frac{1}{2}}\Big\} \; . \tag{V.36}$$

For $M = M'$ this reduces to

$$t = 4m_e^2 - \frac{1}{m^2}\left[m_e^2 + m^2 - M^2\right]^2 . \tag{V.37}$$

If, as in the case of the H-atom, M differs from $m + m_e$ only by a very small binding energy B,

$$M = m + m_e - B \tag{V.38}$$

and find up to order $(B/(m+m_e))^2$

$$t = 2\frac{m_e}{m}(m_e + m)(\sqrt{B} + \sqrt{B'})^2 \tag{V.39}$$

which reduces for $M = M'$ to

$$t = 8\frac{m_e}{m}(m_e + m)B = 8\frac{m_e^2}{\mu}B. \tag{V.40}$$

We see that (V.32) coincides exactly with this since the units are chosen as atomic units ($\mu = 1$).

Observe that knowing the masses of the constituents of a quantum mechanical system the position of the anomalous threshold singularity of a form factor gives the complete information on its mass spectrum. For particle physics, however, our ignorance about constituents unfortunately prevents us from making use of this information, and one has to go back to the relation (III.9) following from current conservation to determine mass spectra.

For completeness we finally give the results in a form more suitable for practical applications. Consider an inelastic collision of an electron with a hydrogen atom of the form

$$\vec{k}_n + \left[(n\ell m), \vec{o}\right] \to \vec{k}_{n'} + \left[(n'\ell'm'), \vec{q}\right], \tag{V.41}$$

where $\vec{k}_n$, $\vec{k}_{n'}$ denote the initial and final momentum of the electron and the atom is assumed to be at rest before the collision. The momentum transfer is

$$\vec{q} = \vec{k}_n - \vec{k}_{n'}. \tag{V.42}$$

The cross section for this process then is in the Born approximation:

$$\frac{d\sigma}{d\Omega} = \frac{4}{q^4}\frac{k_{n'}}{k_n}\left|\rho_{n'\ell'm', n\ell m}(q)\right|^2. \tag{V.43}$$

In this formula the azimuthal quantum numbers m' and m are measured with respect to $\vec{q}$. If we instead use the z-axis for quantization and let k_n go in z-direction, ρ has to be replaced by

$$\rho_{n'\ell' m_z', n\ell m_z}(\vec{q}) = e^{i(m_z'-m_z)\varphi}\sum_m d^{\ell'}_{m_z' m}(\theta) d^{\ell}_{m_z m}(\theta)\rho_{n'\ell' m, n\ell m}(q)$$

$$= (-)^{m_z'-m}\sum_{L,m}\langle \ell' m_z' \ell, -m_z | L, m_z'-m_z\rangle \qquad (V.44)$$

$$\langle \ell' m\, \ell, -m | L0\rangle\ (4\pi/2L+1)^{\frac{1}{2}} Y_m^L(\theta,\phi)\rho_{n'\ell' m, n\ell m}(q).$$

With this quantization the cross section depends only on ϑ and we can introduce

$$K = |\vec{q}| = \sqrt{k_n^2 - k_n'^2 - 2k_n k_n' \cos\vartheta} \qquad (V.45)$$

to rewrite it in the form

$$\frac{d\sigma}{dK} = \frac{8\pi}{k_n^2}\frac{1}{K^3}|\rho|^2. \qquad (V.46)$$

Thus the total cross section is

$$\sigma = \int_{K_{min}}^{K_{max}} dK \frac{d\sigma}{dK} = \frac{8\pi}{k_n^2}\int \frac{dK}{K^3}|\rho|^3$$

with

$$K_{min} = \frac{\Delta E}{\bar{k}}$$

$$K_{max} = 2\bar{k} \qquad (V.47)$$

where

$$\Delta E = E_{n'} - E_n \qquad (V.48)$$

is the energy transfer (recoil has been neglected) and

$$\bar{k} = \tfrac{1}{2}(k_n + k_{n'}) \qquad (V.49)$$

is the average electron momentum. The experimentalists usually measure σ as a function of k_n.

VI. Infinite Component Wave Equations

The dynamical information contained in the algebraic formulation of the electromagnetic current can be expressed in an esthetical and compact way by the use of an infinite component wave equation.[15] Since the state $|n\ell m\rangle$ satisfies

$$L_{56}|n\ell m\rangle = n|n\ell m\rangle, \tag{VI.1}$$

we find for the physical, tilted states $|\bar{n}\ell m\rangle$ the equation

$$\frac{n}{2a}\left[\left(\frac{1}{n^2}+a^2\right)L_{56}-\left(\frac{1}{n^2}-a^2\right)L_{46}\right]|\bar{n}\ell m\rangle = n|\bar{n}\ell m\rangle, \tag{VI.2}$$

or, using the energy $E_n = -1/2n^2$,

$$\left[\left(E_n-\frac{a^2}{2}\right)L_{56}-\left(E_n+\frac{a^2}{2}\right)L_{46}+a\right]|\bar{n}\ell m\rangle = 0. \tag{VI.3}$$

The Galilean boosted states

$$|\bar{n}\ell m, p\rangle = e^{-i(p_i/m_e)(1/a)(L_{i5}-L_{i4})}|\bar{n}\ell m\rangle \tag{VI.4}$$

can therefore be obtained as a solution of[4]

$$\left[\left(E_n-\frac{p_i^2}{2m_e^2}\right)\left(L_{56}-L_{46}\right)-\frac{a^2}{2}\left(L_{56}+L_{46}\right)-a\frac{p_i}{m_e}L_{i6}+a\right]|\bar{n}\ell m,p\rangle = 0. \tag{VI.5}$$

If one now introduces the electromagnetic coupling into this equation by the minimal substitution

$$\vec{p} \to \vec{p}-\vec{A}, \tag{VI.6}$$

one finds the additional term linear in $\vec{A}$:

$$-A_i\left[aL_{i6}+\frac{p_i}{m_e}(L_{56}-L_{46})\right] \tag{VI.7}$$

which is exactly what the spatial current I^i of Equation (IV.63) would give between states moving both with momentum $p' = p$.

Thus, Eq. (VI.5), together with the prescription of minimal electromagnetic coupling, completely describes the internal structure of the hydrogen atom. It is therefore equivalent to the Schrödinger equation with an external electromagnetic field.

Notice, however, that the renormalization factor of the tilted states, $1/n$ (see IV.51), cannot be obtained from this equation. This factor has to be determined separately by imposing the requirement of constant charge in a multiplet.

There also exists a relativistic current and a corresponding wave equation for the hydrogen atom. This current is characterized by the property to give, in the Galilean limit ($c \to \infty$), the correct non-relativistic matrix elements (Eqs. (V.28)-(V.30)).[6] The current is composed of an algebraic four-vector

$$\Gamma^{\mu} = (-L_{56}, L_{i6}) \qquad \text{(VI. 8)}$$

and a convective part

$$\frac{P^{\mu}}{2m_p} L_{46}, \qquad P^{\mu} = p'^{\mu} + p^{\mu}$$

in the form (in natural units $\mu = c = \hbar = 1$)

$$J^{\mu} = \frac{\alpha}{m_e}\left(\Gamma^{\mu} - \frac{P^{\mu}}{2m_p} L_{46}\right). \qquad \text{(VI. 9)}$$

The physical states have to be taken as

$$|\bar{n}\ell m\rangle = \frac{1}{n} e^{-i\ell g(n2(m_p/\alpha))L_{45}} |n\ell m\rangle. \qquad \text{(VI. 10)}$$

The Lorentz transformation under which J^{μ} behaves like a four-vector is

$$e^{-i\zeta^k L_{k5}},$$

where $\zeta = \mathrm{sh}^{-1}(p/M)$. Therefore the matrix elements of the current at arbitrary momentum transfer q are given by

$$I^{\mu} = \langle \bar{n}' | J^{\mu} | \bar{n}, q\rangle = \langle \bar{n}' | J^{\mu} e^{-i\zeta^k L_{k5}} | n\rangle \qquad \text{(VI. 11)}$$

which can be written in terms of the states $|n\rangle$, in atomic units, as

$$I^{o}(\zeta) = \frac{1}{\alpha}\frac{1}{n} \langle n' | (L_{56} - L_{46}) e^{-i\ell g(n/n')L_{45}} e^{-i(\zeta^k/\alpha) m_p n (L_{k5} - L_{k4})} | n\rangle + \mathcal{O}(\alpha)$$

$$\text{(VI. 12)}$$

$$I^i(\zeta) = \frac{1}{nn'}\,\frac{1}{m_e}\,\langle n'|L_{i6}\,e^{-i\ell_{\mathcal{G}}(n/n')L_{45}}\,e^{-i(\zeta^k/\alpha)m_p n(L_{k5}-L_{k4})}\,|n\rangle$$

$$+ (q^i/2m_e)\,\alpha I^o(\zeta) + \mathcal{O}^i(\alpha^2). \qquad \text{(VI. 13)}$$

But if we here go to the non-relativistic limit, $c = 1/\alpha \to \infty$, we obtain

$$\frac{\zeta}{\alpha}\,m_p \longrightarrow \frac{q}{m_e}\cdot\mu \qquad \text{(VI. 14)}$$

and, inserting

$$I^o = c\rho = \frac{1}{\alpha}\,\rho, \qquad \text{(VI. 15)}$$

we indeed recover in this limit the expressions (V. 1) and (V. 2).

This current can now be used to construct a relativistic wave equation which possesses the states (VI. 10) as solutions and produces, after minimal coupling with the electromagnetic field via the substitution

$$p^\mu \to p^\mu - A^\mu, \qquad \text{(VI. 16)}$$

the correct current (VI. 11). (The normalization factor $1/n$ again has to be determined by the constancy of charge requirement.) This equation is

$$\left(J^\mu p_\mu - \alpha\,\frac{m_p^2 - m_e^2}{2m_p m_e}\,L_{46} + \alpha^2\right)|\bar{n}\ell m, p\rangle \qquad \text{(VI. 17)}$$

Observe that according to this equation the current J^μ is conserved. In general, if J^μ is some current operator and if the boosted physical states satisfy an equation

$$\left(J^\mu p_\mu + \text{const. operator}\right)|\bar{n}, p\rangle = 0. \qquad \text{(VI. 18)}$$

Then the matrix elements

$$I^\mu = \langle\bar{n}, p'|\,J^\mu\,|\bar{n}, p\rangle \qquad \text{(VI. 19)}$$

fulfill automatically the current conservation law:

$$(p'-p)I^\mu = 0. \qquad \text{(VI. 20)}$$

In our discussion of the algebraic substitution of Schrödinger theory in Section III we concluded that given a method to construct the complete current of a system, its mass spectrum could be read off an equation like (III.9). From what we have learned in this section we can therefore make the alternative statement: Given a current operator and tilted physical states, this current is only conserved if the masses (and tilting angles) are adjusted in such a way that the boosted physical states satisfy an infinite component wave equation of the form (VI.18).[16]

Observe, however, that the wave equation (VI.17) has many more solutions than the ones corresponding to the H-atom. In fact, the set of discrete solutions of (VI.17) is

$$M_n^2 = \left(m_p^2 + m_e^2\right) \pm 2m_p m_e \sqrt{1 - \alpha^2/n^2} \tag{VI.21}$$

whose upper sign gives the hydrogen spectrum

$$M_n = \left[M - \frac{\mu\alpha^2}{2n^2} - \mathcal{O}(\alpha^4)\right], \tag{VI.22}$$

while the lower sign gives the unphysical masses:

$$M_n = \left(m_p - m_e + \frac{m_p m_e}{m_p - m_e}\frac{\alpha^2}{2n^2}\right) + \mathcal{O}(\alpha^4) \tag{VI.23}$$

which distinguishes from (VI.22) by m_e changing sign.

There is another infinity of solutions of both equations for the H-atom, relativistic as well as non-relativistic. They are found if one asks for energies $E > 0$ or masses $M > m_p + m_e$; i.e., for the continuous states of the H-atom. Then the equations (VI.5) and (VI.17) cannot be solved any more by tilting representation states on which L_{56} is diagonal, but we have to find eigenstates of L_{46}. These solutions will be discussed in the next section.

VII. The Continuous Spectrum

The infinite component wave equation of the H-atom (VI.3) is completely equivalent to the Schrödinger equation. Until now, we have restricted ourselves to the discussion of the bound state solutions of this equation, and the calculation of form factors was done only for bound-bound transitions. In this section we want to show how the continuous states can be included in the group dynamical treatment. It will turn out that all the results obtained generalize to apply also for the continuous states provided that instead of

Bargmann's v-functions, used in Section V for the representation of O(2,1), we employ a certain analytic continuation thereof.

Consider the wave equation (VI.3) for the H-atom at rest. If the energy E is positive, call the corresponding solution $|\bar{\alpha}\ell m\rangle$ and their energy E_α. Then we can perform a transformation

$$|\bar{\alpha}\rangle = \frac{1}{i}\sqrt{2E_\alpha}\; e^{-i\ell g(a/\sqrt{2E_\alpha})L_{45}}|\alpha\rangle \tag{VII.1}$$

and bring (VI.3) to the form

$$\left\{\left[\left(E_\alpha + \frac{a^2}{2}\right)^2 - \left(E_\alpha - \frac{a^2}{2}\right)^2\right]^{\frac{1}{2}} L_{46} - a\right\}|\alpha\rangle = 0 \tag{VII.2}$$

In order to solve this equation we need representation states on which L_{46} is diagonal. Since we know that the spectrum of energies (and thus that of L_{46}) is continuous, these states are not contained in the Hilbert space but they have δ-function normalization. The problem of finding eigenstates of the non-compact operator L_{46} in a unitary representation space with diagonal L_{56} has not yet been solved in general. In our maximally degenerate representation space, however, the problem can be reduced to that of changing the basis in O(2,1) from eigenstates of K_3 to those of the non-compact generator K_2 (see Appendix B).

Assume that this basis change has been performed. Let $|\nu)$ be the eigenstates of L_{46} with eigenvalue ν. The energy of this state is then, according to (VII.2),

$$E_\nu = \frac{1}{2\nu^2} \tag{VII.3}$$

and the physical states (VII.1) become

$$|\bar{\nu}) = \frac{i}{\nu}\, e^{-i\ell g\, \nu a\, L_{45}}|\nu). \tag{VII.4}$$

Since L_{46} commutes with L^2 and L_3, a complete set of labels is given by $|\nu\ell m)$ (the ℓm-basis). A parabolic basis is spanned by states with diagonal operators L_{46}, L_{35} and L_3 which we shall write as $|\nu,\lambda,m)$. Electromagnetic form factors are now given by matrix elements like the following:

$$\rho \equiv \frac{1}{\nu'\nu} (\nu'| e^{i\ell g \nu a L_{45}} \frac{1}{a} (L_{56} - L_{46}) e^{i(q_i/m_e)(1/a)(L_{i5}-L_{i4})} \times e^{-i\ell g \nu a L_{45}} |\nu), \qquad \text{(VII.5)}$$

et cetera, just as in Section IV.[17]

The evaluation of such matrix elements can in principle be done by inserting intermediate states $|n\ell m\rangle\langle n\ell m|$ on the right and left sides of the operator and finding the transformation matrix element $\langle n\ell m|\nu)$ between the different basis states.

These matrix elements can easily be calculated. Because of angular momentum conservation, the basis change from the states $|n\ m\rangle$ to eigenstates of L_{46}, L^2, L_3 is achieved by a relation

$$|\nu\ell m) = \sum_n \langle n\ell m|\nu\ell m)\, |n\ell m\rangle. \qquad \text{(VII.6)}$$

To determine the coefficients of this expansion we can use another O(2,1) subgroup similar to that of Eq. (V.10) consisting of the operators[18]

$$K_1 = L_{45},\quad K_2 = L_{46},\quad K_3 = L_{56} \qquad \text{(VII.7)}$$

which generate an irreducible representation of the discrete class $D^+_{\ell+1}$ on the ladder of states

$$|n = \ell+1, \ell, m\rangle,\quad |\ell+2, \ell, m\rangle,\quad |\ell+3, \ell, m\rangle, \ldots . \qquad \text{(VII.8)}$$

Changing from L_{56} to L_{46} diagonalization at fixed ℓm is then equivalent to changing from K_3 to K_2 diagonalization within every $D^+_{\ell+1}$ ladder. Thus the coefficients $\langle n\ell m|\nu\ell m\rangle$ are nothing but the matrix of basis change of the discrete class D^+_k, $\langle km|k\nu)$, which have been given in Appendix B, if one inserts the relevant values of Casimir operator $k=\ell+1$ and of the K_3 eigenvalues $m=n$, i.e.,

$$\langle n\ell m|\nu\ell m) = \langle \ell+1, n|\ell+1, \nu). \qquad \text{(VII.9)}$$

The corresponding relation for the parabolic basis $|\nu,\lambda,m)$,

$$|\nu\lambda m) = \sum_{n_1,n_2} \langle n_1 n_2 m|\nu\lambda m)\, |n_1 n_2 m\rangle, \qquad \text{(VII.10)}$$

is evaluated by means of the $O(2,1) \otimes O(2,1)$ representation

$D^{+}_{(m+1)/2} \otimes D^{+}_{(m+1)/2}$ of Eq. (II.27) generated by N_1^i, N_2^i on the square array of states

$$|n_1 = \frac{m+1}{2}, n_2 = \frac{m+1}{2}, m\rangle, \quad |\frac{m+1}{2}, \frac{m+3}{2}, m\rangle, \ldots$$
$$|n_1 = \frac{m+3}{2}, n_2 = \frac{m+1}{2}, m\rangle, \quad |\frac{m+3}{2}, \frac{m+3}{2}, m\rangle, \ldots . \quad \text{(VII.11)}$$

On every such array the change of diagonalization from N_1^3, N_2^3 with eigenvalues n_1, n_2 to N_1^1, N_2^1 with eigenvalues ν_1, ν_2 is accomplished by a matrix

$$\langle k = \frac{m+1}{2}, n_1 | \frac{m+1}{2}, \nu_1) \langle k = \frac{m+1}{2}, n_2 | \frac{m+1}{2}, \nu_2) \quad \text{(VII.12)}$$

From (II.30) we have

$$L_{35} = N_1^1 - N_2^1, \quad L_{46} = N_1^1 + N_2^1. \quad \text{(VII.13)}$$

Therefore in the new states, $\nu_1 - \nu_2$ has to be identified with the eigenvalue λ of L_{35} defined before Eq. (VII.5), while the eigenvalue ν of L_{46} is $\nu_1 + \nu_2$. Thus we obtain for the matrix $\langle n_1 n_2 m | \nu\lambda m)$ the expression

$$\langle n_1 n_2 m | \nu\lambda m) = 2 \langle \frac{m+1}{2}, n_1 | \frac{m+1}{2}, \frac{\nu+\lambda}{2}) \langle \frac{m+1}{2}, n_2 | \frac{m+1}{2}, \frac{\nu-\lambda}{2}) \quad \text{(VII.14)}$$

where the factor 2 has been introduced in order to change the normalization from the ν_1, ν_2 scale to the ν, λ scale.

The use of the basis transformations (VII.6) and (VII.14) for the evaluation of the currents is quite cumbersome. It is therefore desirable to pursue a more direct approach which treats free and bound states from the beginning on the same footing. Indeed, it is possible to find the matrix elements of the current in a form which is defined for all complex values of ν' and ν and has the property to give the free-free transitions for real ν', ν and to reduce to bound-free or to the bound-bound currents of Section V by analytic continuation of the corresponding quantum number ν to integer values of $-i\nu = n$.

Consider a non-unitary representation which is defined by the introduction of new states:

$$|-i\nu\rangle \equiv e^{+(\pi/2)L_{45}} |\nu). \quad \text{(VII.15)}$$

This transformation is not contained in the group but it can uniquely be specified by analytically continuing the group operation $e^{i\varphi L_{45}}$ to $\varphi = -i(\pi/2)$. Note that the transformation is non-unitary and therefore $|-i\nu\rangle$ will have scalar products different from $|\nu)$. Indeed, as shown in Appendix B,

$$\langle i\nu' | -i\nu\rangle = 2e^{-\pi\nu} \text{ sh } \pi\nu \ \delta(\nu'-\nu). \tag{VII.16}$$

Applying L_{56} to $|-i\nu\rangle$ we find

$$L_{56}|-i\nu\rangle = -i\nu|-i\nu\rangle \tag{VII.17}$$

which explains the notation used for the state. Since L^2 and L_3 commute with L_{45}, the transformation (VII.4) leaves the quantum numbers ℓm invariant and states like $|-i\nu,\ell m\rangle$ form a good basis of the Hilbert space. A parabolic set of quantum numbers, on the other hand, can now again be given by L_3, L_{34} just as in the case of bound states. Defining the parabolic states

$$|-i\nu, -i\lambda, m\rangle \equiv e^{(\pi/2)L_{45}}|\nu,\lambda,m), \tag{VII.18}$$

we find

$$\begin{aligned} L_3|-i\nu,-i\lambda,m\rangle &= m|-i\nu,-i\lambda,m\rangle \\ L_{34}|-i\nu,-i\lambda,m\rangle &= -i\lambda\,|-i\nu,-i\lambda,m\rangle. \end{aligned} \tag{VII.19}$$

The normalization of these states can be found to be (see Appendix B)

$$\langle i\nu',i\lambda',m'|-i\nu,-i\lambda,m\rangle$$

$$= 8e^{-\pi\nu}\begin{cases} \text{sh } \pi\frac{\nu+\lambda}{2} \text{ sh } \pi\frac{\nu-\lambda}{2} & \text{for } m = \text{odd} \\ \text{ch } \pi\frac{\nu+\lambda}{2} \text{ ch } \pi\frac{\nu-\lambda}{2} & \text{for } m = \text{even} \end{cases}. \tag{VII.20}$$

Observe that in terms of the non-orthogonal basis $|-i\nu\rangle$ the physical states (VII.1) can be written as

$$|\overline{-i\nu}\rangle = \frac{i}{\nu}\, e^{-i\,\ell g(-i\nu)a\,L_{45}}|-i\nu\rangle \tag{VII.21}$$

if it is understood that the first sheet of the logarithm is used with

$$|\arg(\ell g\ z)| < \pi. \tag{VII.22}$$

This is a formula completely analogous to the bound state case (IV.52) if one replaces n by $-i\nu$. The normalization of these states in the physical scalar product is the same as (VII.16) and (VII.20).

Since the infinite component wave equation is analytic in E_α it is clear that the physical states form a vector valued analytic function of ν which can uniquely be extended into the whole complex ν-plane. The bound states then obviously appear as the values of this function at positive integer values of $-i\nu$. If we agree on the convention that the norms of the bound state limits of this function are unity, the free states will in general have a norm

$$\langle +i\nu'\ell' m' | -i\nu \ell m\rangle = N_\nu^\ell \delta(\nu'-\nu)\,\delta_{\ell'\ell}\delta_{m'm} \tag{VII.23}$$

in the ℓm basis, and

$$\langle +i\nu' i\lambda' m' | -i\nu, -i\lambda, m\rangle = N_{\nu\lambda}^m \delta(\nu'-\nu)\delta(\lambda'-\lambda)\,\delta_{m'm} \tag{VII.24}$$

in the parabolic basis. It is interesting that the normalization constants fixed according to this convention turn out to be exactly the same as those given in (VII.16) and (VII.20) (see Appendix B).

In terms of the states $|-i\nu\rangle$ normalized in this way, we can now define new currents by the matrix elements:

$$\rho_{-i\nu',-i\nu}(q) = \frac{1}{\nu'\nu}\langle i\nu' | e^{i\ell g\, i\nu' a L_{45}} \frac{1}{a}(L_{56}-L_{46})$$

$$e^{-i(q_i/m_e)(1/a)(L_{i5}-L_{i4})} e^{-i\ell g(-i\nu)a L_{45}} | -i\nu\rangle \tag{VII.25}$$

and similar expressions for the spatial components I^i. These currents now have the remarkable property to describe transitions to free states normalized according to (VII.16) and (VII.20) if ν is a real number, while transitions to normalized bound states $|n\rangle$ can be obtained by continuing $-i\nu$ to integer values $-i\nu = n$.

In order to evaluate these equations for the currents, we can proceed in a way completely analogous to Section V. We first rewrite ρ and K^i in the form:

$$\rho_{-i\nu',-i\nu}(q) = \frac{i}{\nu}\langle i\nu' | \frac{1}{a}(L_{56}-L_{46})G | -i\nu\rangle$$

$$K^i_{-i\nu',-i\nu}(q) = \frac{1}{m_e\nu'\nu}\langle i\nu' | L_{i6}\, G | -i\nu\rangle \tag{VII.26}$$

where G is the same as in (V.3) except that n'n are continued to

$i\nu', -i\nu$, respectively. Then, inserting intermediate states through the completeness relation

$$\sum_m \int \frac{d\nu d\lambda}{N^m_{\nu\lambda}} \; |-i\nu, -i\lambda, m \times i\nu, i\lambda, m| = 1, \qquad \text{(VII.27)}$$

we can reduce the problem to that of calculating the matrix elements of the generators L_{56}, L_{46}, L_{i6} and of G. We have given the matrix elements of the generators in Appendix C. The operator G can again be decomposed in Euler angle form with, in general, complex angles. Equations (V.21)-(V.24) carry over without change if one continues n' and n to imaginary values along a path lying completely on the complex n plane cut from $-\infty$ to 0. Then the only problem left is to find an analytic continuation of Bargmann's v-functions such that

$$\langle +i\nu', +i\lambda', m| e^{-i(N_1^2+N_2^2)\beta} |-i\nu, -i\lambda, m\rangle$$

$$\equiv v^{(m+1)/2}_{-i(\nu'+\lambda')/2, -i(\nu+\lambda)/2}(\text{sh}\,\beta/2)\, v^{(m+1)/2}_{-i(\nu'-\lambda')/2, -i(\nu-\lambda)/2}(-\text{sh}\,\beta/2) \qquad \text{(VII.28)}$$

This problem is solved in Appendix B.

Let us consider this situation in position space. Since the Schrödinger equation is analytic in the energy, also the wave functions are analytic.

The bound state wave functions in the ℓm basis

$$R_{n\ell}(r) = \frac{2^{\ell+1}}{(2\ell+1)!} \sqrt{\frac{(n+\ell)!}{(n-\ell-1)!}} \, \frac{1}{n^2} \, e^{-r/n} \left(\frac{r}{n}\right)^{\ell}$$

$$\times F\left(-(n-\ell-1), 2\ell+2, 2r/n\right) \qquad \text{(VII.29)}$$

are the continuation of the free wave functions of energy $1/2\nu^2$

$$R_{-i\nu,\ell}(r) = -e^{i\ell(\pi/2)} \frac{2^{\ell+1}}{(2\ell+1)!} \sqrt{\frac{(-i\nu+\ell)!}{(-i\nu-\ell-1)!}} \, \frac{1}{\nu^2} \, e^{-ir/\nu} \left(\frac{r}{\nu}\right)^{\ell}$$

$$F\left(i\nu+\ell+1, 2\ell+2, 2i\, r/\nu\right) \qquad \text{(VII.30)}$$

which are normalized to

$$\int R_{+i\nu',\ell}(r)R_{-i\nu,\ell}(r)r^2 dr = N_\nu^\ell \delta(\nu'-\nu) \tag{VII. 31}$$

with the normalization factor

$$N_\nu^\ell = 2e^{-\pi\nu}|\sin\pi\nu| = |1 - e^{-2\pi\nu}| \tag{VII. 32}$$

which agrees with Eq. (VII. 16). In the parabolic case we find that the free wave functions

$$u_{i\nu,i\lambda,m} = -\frac{e^{\pm im\varphi}}{\sqrt{\pi}}\frac{1}{\nu^2}\sqrt{\frac{n_1!n_2!}{(n_1+m)!^3(n_2+m)!^3}}\, e^{-i(\xi+\eta)/2}\left(\frac{\xi\eta}{\nu^2}\right)^{m/2}$$

$$L^m_{n_1+m}(i\xi/\nu)L^m_{n_2+m}(i\eta/\nu) \tag{VII. 33}$$

with

$$n_1 = -\frac{m+1}{2} - \frac{i}{2}(\nu-\lambda)$$

$$n_2 = -\frac{m+1}{2} - \frac{i}{2}(\nu+\lambda)$$

continue properly to the bound wave functions $u_{n_1n_2m}$ for integer $n_1, n_2 \geqq 0$ and $-i\nu = n$, and that they have the normalization

$$\frac{1}{4}\int d\xi d\eta d\varphi(\xi+\eta)u_{i\nu' i\lambda' m'}(\xi\eta\varphi)u_{i\nu,i\lambda,m}(\xi\eta\varphi) = 8|1-(-)^{m+1}e^{-\pi(N+\lambda)}|$$
$$\times |1-(-)^{m+1}e^{-\pi(N-\lambda)}|$$
$$\times \delta(\nu'-\nu)\delta(\lambda'-\lambda), \tag{VII. 34}$$

which amounts to the normalization factor defined in (VII. 21) being

$$N^m_{\nu\lambda} = 8|1-(-)^{m+1}e^{-\pi(N+\lambda)}|\cdot|1-(-)^{m+1}e^{-\pi(N-\lambda)}| \tag{VII. 35}$$

just as in (VII. 20).

Finally we like to mention that states which behave asymptotically like outgoing or incoming plane waves in the direction $\hat{k}$ are characterized by

$$\hat{k}\cdot L \mid -i\nu, \hat{k}\,\pm\rangle = 0$$

$$\hat{k}_i \cdot L_{i4} \mid -i\nu, \hat{k}\,\pm\rangle = (\pm i\nu - 1) \mid -i\nu, \hat{k}\,\pm\rangle .^{19)} \quad \text{(VII.36)}$$

They are given in position space by

$$u_{k\pm} = e^{(i/2)(\pm kr + \vec{k}\cdot\vec{r})} \sqrt{\frac{\nu}{1-e^{-2\pi\nu}}}\ \frac{1}{(2\pi)^2 i} \int d\xi\, (\xi + \tfrac{1}{2})^{\pm i\nu} (\xi - \tfrac{1}{2})^{\mp i\nu - 1}\, e^{-i(\pm kr - \vec{k}\vec{r})\xi} \quad \text{(VII.37)}$$

with the normalization

$$\int u^*_{k'} u_k d^3x = \delta^3(k'-k)$$

and have no simple continuations to bound states like the other two cases. For $\vec{k}$ pointing in z direction, these states can, however, be reached from the parabolic states by continuing λ to $\pm\nu + i$.

We see that group theory and wave functions go nicely hand in hand also in the case of continuous states.

VIII. Summary

We have seen that the internal structure of the composite quantum mechanical system of the H-atom can be described completely in terms of simple group operations in the representation space of the non-compact group O(4,2). There are indications that the algebraic structure of this description may well be model independent and carry over to the physics of elementary particles. Let us, therefore, summarize the general features of this structure. For applications to particle physics one clearly has to substitute mentally the Lorentz group every time the Galilean group occurs in the following statements in order to construct a relativistically invariant theory.

Given a three-particle vertex in which an external interaction (like photon, pion, et cetera) can cause momentum transfer dependent transitions between a highly symmetric array of physical states, then the structure of the amplitude for these processes can be described in the following group theoretical way:

1) There is a (in general non-compact) group G which contains all possible states of the system at rest in a single unitary irreducible representation. The group G has to contain a Galilean sub-group, the "boosting group," rotational invariant non-compact operators, "tilting operators," and tensor operators possessing the transformation properties of the external interactions.

2) The physical states are tilted, renormalized states in the irreducible representation space.

3) The amplitude for any external interaction with momentum transfer q is given by the matrix elements of the corresponding tensor operator between physical states transformed to their true velocities before and after the collision via the boosting group.

4) The normalization factors in the physical states are fixed by the requirement of constancy of charge in a multiplet.

5) The tilting angles and masses of the particles are determined from the requirement of conservation of the electromagnetic current (which is equivalent to the physical states solving a certain infinite component wave equation).

Three-particle vertices of baryons and mesons built according to this structure have shown excellent agreement with experiment and have given many predictions (in fact, there are always infinitely many predictions for every vertex) using the group O(4,2). It will be interesting to see how many of these predictions will survive a closer experimental test.

Appendix A: Global Representations of O(2,1)

Let $K_1K_2K_3$ generate O(2,1) according to the commutation rules (II.29), and let $|m\rangle$ be states on which the operator K_3 is diagonal with the eigenvalue m, then one can classify all unitary representations by the value q of the Casimir operator

$$Q = -K_1^2 - K_2^2 + K_3^2 \tag{A.1}$$

or alternatively by using k such that

$$q = k(1-k) \tag{A.2}$$

as

1) Discrete classes D_k^+ or D_k^-, existing for $k = 1/2, 1, 3/2, \ldots$ et cetera, with the spectrum of m being m or $-m = k, k+1, k+2, \ldots$, respectively.

2) Continuous classes C_q^0 or $C_q^{1/2}$ existing for $1/4 \leq q < \infty$ or $0 < q < \infty$ with the spectrum

$$m = 0, \pm 1, \pm 2, \ldots$$

or

$$m = \pm 1/2, \pm 3/2, \pm 5/2, \ldots,$$

respectively.

If $K_1 = i\sigma_1/2$, $K_2 = i\sigma_2/2$, $K_3 = \sigma_3/2$ generate a (non-unitary) 2×2 representation of O(2,1), consisting of the elements

$$a = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \qquad \alpha\bar{\alpha} - \beta\bar{\beta} = 1, \tag{A.3}$$

then the representation of D_k^+ and the continuous classes is given by

$$v_{mn}(a) = \theta_{mn} \bar{\alpha}^{-m-n} \beta^{m-n} F\left(k-n, 1-n-k, 1+m-n, -\bar{\beta}\beta\right) \tag{A.4}$$

with

$$\theta_{mn} = \frac{1}{\Gamma(1+m-n)} \left[\frac{\Gamma(m+1-k)\Gamma(m+k)}{\Gamma(n+1-k)\,\Gamma(n+k)}\right]^{\frac{1}{2}}, \tag{A.5}$$

which is well defined for $m \geqq n$. For $m<n$ the limit $m-n \to$ negative integer has to be used. If one does a careful limiting procedure $k \to 1/2, 3/2, \ldots$, then (A.4) holds also for D_k^-. Explicitly, one finds from unitarity

$$v_{mn}(a) = v^*_{nm}(a^{-1}) \tag{A.6}$$

$$a^{-1} = \begin{pmatrix} \bar{\alpha} & -\beta \\ -\bar{\beta} & \alpha \end{pmatrix}$$

Hence, for $m \leqq n$ one can take v_{nm} from (A.4) with the substitution $\beta \to -\bar{\beta}$.

Appendix B. Spinorial Method for the Representations of O(2,1) in Discrete and Continuous Basis [20),21)]

Let a_1^+, a_2^+ be two complex variables and $a_r \equiv \partial/\partial a_1^+$ $(r=1,2)$. Then the representation with Casimir operator $q = k(1-k)$ is generated by

$$K_1 = -\frac{i}{2} a^+ \sigma^1 a, \quad K_2 = -\frac{i}{2} a^+ \sigma^2 a, \quad K_3 = \frac{1}{2} a^+ \sigma^3 a \tag{B.1}$$

on the Hilbert space of functions

$$|k,m\rangle \equiv A_m^k a_1^{+\,m-k} a_2^{+\,-m-k} \tag{B.2}$$

where A_m^k is some normalization factor. Because of the possibility of an Euler decomposition of every group element we can restrict our discussion to the subgroup $h = e^{iK_1\varphi}$ which is represented by the 2×2 *matrix*

$$h = \begin{pmatrix} \text{ch } \varphi/2 & \text{sh } \varphi/2 \\ \text{sh } \varphi/2 & \text{ch } \varphi/2 \end{pmatrix} \tag{B.3}$$

on the space of

$$\begin{pmatrix} a_1^+ \\ a_2^+ \end{pmatrix},$$

i.e.,

$$e^{iK_1\varphi}\begin{pmatrix} a_1^+ \\ a_2^+ \end{pmatrix} e^{-iK_1\varphi} = h\begin{pmatrix} a_1^+ \\ a_2^+ \end{pmatrix}. \tag{B.4}$$

But then we find that

$$e^{iK_1\varphi}|k,m\rangle = N_m^k(\text{ch } \varphi a_1^+ + \text{sh } \varphi a_2^+)^{m-k}(\text{sh } \varphi a_1^+ + \text{ch } \varphi a_2^+)^{-m-k} \tag{B.5}$$

which can be expanded in states $|k,m\rangle$ by a series

$$e^{iK_1\varphi}|k,m\rangle = \sum_{m'} v_{m'm}^k(\varphi)\,|k,m'\rangle \tag{B.6}$$

where

$$v_{m'm}^k = \frac{A_m^k}{A_{m'}^k}\,\text{sh}^{m-m'}\varphi/2\ \text{ch}^{-2k+m'-m}\varphi/2$$

$$\times \sum_{p=o}^{\infty} \text{th}^{2p}\varphi/2\begin{pmatrix} -k+m \\ m-m'+p \end{pmatrix}\begin{pmatrix} -k-m \\ p \end{pmatrix} \tag{B.7}$$

and the sum can be collected to give a hypergeometric function. For $m \geqq m'$ one finds

$$v_{m'm}^k = \frac{A_m^k}{A_{m'}^k}\,\frac{1}{(m-m')}\,\frac{(-k+m)!}{(-k+m')!}\,\text{sh}^{m-m'}\varphi/2\ \text{ch}^{-2k+m'-m}\varphi/2$$

$$\times F\left(k-m',\ k+m,\ 1+m-m',\ \text{th}^2\varphi/2\right). \tag{B.8}$$

If one imposes the unitarity condition upon v:

$$v^k_{m'm}(\varphi) = v^k_{mm'}(-\varphi),$$

one finds that A^k_m has to be

$$A^k_m = (-)^{k-m}\left[\frac{(k+m-1)!}{(-k+m)!}\right]^{\frac{1}{2}} \tag{B.9}$$

and k in the range given in Appendix A. But then (B.8) exactly coincides with (A.4) if one remembers the relation

$$F\left(a,b,c,z\right) = \left(1-z\right)^{-a} F\left(a, c-b, c, \frac{z}{z-1}\right). \tag{B.10}$$

Representations which diagonalize K_2 can be found in a completely analogous fashion. Here one simply considers the states

$$|k\nu) \equiv A^k_\nu e^{(\pi/2)K_1}\left(a_1^+\right)^{i\nu-k}\left(a_2^+\right)^{-i\nu-k}$$

$$\equiv A^k_\nu \left(a_1 + ia_2^+\right)^{i\nu-k}\left(a_2 + ia_1^+\right)^{-i\nu-k} \tag{B.11}$$

which obviously diagonalize K_2.

The operation $e^{(\pi/2)K_1}$ amounts to a canonical, non-unitary transformation to the new variables[21])

$$c_1^+ = a_1^+ + ia_2^+, \qquad c_2^+ = a_2^+ + ia_1^+ \tag{B.12}$$

in terms of which K_i have the form

$$K_1 = -\frac{i}{2}c^+\sigma^1 c, \quad K_2 = \frac{i}{2}c^+\sigma^3 c, \quad K_3 = \frac{i}{2}c^+\sigma^2 c. \tag{B.13}$$

The matrix elements of $e^{iK_1\varphi}$ can now be obtained by forming a state like (B.5) and expanding in the form

$$e^{iK_1\varphi}|k,\nu) \equiv N_\nu^k\left(\operatorname{ch}\varphi c_1^+ + \operatorname{sh}\varphi c_2^+\right)^{i\nu-k}\left(\operatorname{ch}\varphi c_2^+ + \operatorname{sh}\varphi c_1^+\right)^{-i\nu-k}$$

$$= \int_{-\infty}^{\infty} d\nu' \, v_{\nu'\nu}(\varphi)\,|k,\nu') \tag{B.14}$$

The last step is done by means of the generalization of the binomial expansion

$$(1+\xi)^z = \frac{1}{2\pi\Gamma(-z)}\int_{-\infty}^{\infty}\Gamma(-z+i\nu)\Gamma(-i\nu)\xi^{i\nu}d\nu$$

$$|\arg \xi| < \pi \tag{B.15}$$

One finds[21)]

$$v^k_{\nu'\nu}(\varphi) = \frac{A^k_\nu}{A^k_{\nu'}}\frac{1}{2\pi}\operatorname{ch}^{-2k}\varphi/2$$

$$\times\Big\{\operatorname{th}^{i(\nu'-\nu)}\varphi/2\,\frac{\Gamma(k+i\nu')}{\Gamma(k+i\nu)}\,\Gamma(i(\nu'-\nu))F\left(k-i\nu,\, k+i\nu',\, 1+i(\nu'-\nu), \operatorname{th}^2\varphi/2\right)$$

$$+\operatorname{th}^{-i(\nu'-\nu)}\varphi/2\,\frac{\Gamma(k-i\nu')}{\Gamma(k-i\nu)}\,\Gamma(i(\nu-\nu'))F\left(k+i\nu,\, k-i\nu',\, 1-i(\nu'-\nu),\, \operatorname{th}^2\varphi/2\right)\Big\}.$$

$$\tag{B.16}$$

The normalization A^k_ν is fixed from unitarity to be

$$A_\nu = \frac{2^{-k/2}}{\sqrt{2\pi}}\left[\Gamma(k+i\nu)\,\Gamma(k-i\nu)\right]^{1/2} \tag{B.17}$$

The non-unitary basis $|k,-i\nu\rangle = e^{(\pi/2)K_1}|k,\nu)$ on which K_3 has the eigenvalue $-i\nu$ can then be shown to have the scalar products

$$\langle k,i\nu'|k,-i\nu\rangle = 2e^{\pi\nu}\begin{Bmatrix}\operatorname{sh}\nu\pi\\ \operatorname{ch}\nu\pi\end{Bmatrix}\times\delta(\nu'-\nu) \quad \text{for} \begin{cases}k = \text{integer}\\ k = \text{half integer}.\end{cases} \tag{B.18}$$

From this it follows immediately, by going to the corresponding O(2,1) or O(2,1) × O(2,1) subgroups, that the O(4,2) states defined in (VII.15) have the normalization (VII.16) in the ℓm-basis (where

$k = \ell + 1$) and (VII.20) in the parabolic basis (where $k = (m+1)/2$. Note that an extra factor two has to be introduced apart from the product of two expressions of the form (B.18) in order to change the normalization from the $(\nu+\lambda)/2$, $(\nu-\lambda)/2$ to the ν, λ scales.)

The matrix elements for the basis change $\langle k, m | k, \nu)$ can be obtained by expanding (B.11) in terms of states $|k, m\rangle$. One finds:

$$\langle k, m | k, \nu) = 2^k \frac{A_\nu^k}{A_m^k} e^{-(\pi/2)(\nu + im)} \frac{\Gamma(-k+i\nu+1)}{\Gamma(-k+m+1)} \times \frac{1}{\Gamma(i\nu - m+1)} F(k-m, k+i\nu, i\nu-m+1, -1), \tag{B.19}$$

which can also be obtained by continuing the matrix element

$$\langle k, +i\nu' | k, \nu) = (k, i\nu' | e^{(\pi/2)K_1} | k, \nu)$$

to $i\nu' = m$.

Appendix C. Representation of Some $O(2,1) \times O(2,1)$ Generators on the Continuous Basis.

Given the states of the parabolic basis $|\nu, \lambda, m)$, then by definition

$$L_{46} | \nu\lambda m) = \nu | \nu\lambda m)$$

$$L_{35} | \nu\lambda m) = \lambda | \nu\lambda m)$$

$$L_{12} | \nu\lambda m) = m | \nu\lambda m). \tag{C.1}$$

Observe that these states diagonalize the generators N_r^1 $(r = 1, 2)$ of the $O(2,1)$ subgroups (II.30)—namely,

$$N_r^{\,1} | \nu\lambda m) \equiv \nu_r | \nu\lambda m) = \frac{\nu - (-)^r \lambda \, | \nu\lambda m)}{2}. \tag{C.2}$$

Let us define the Hermitian "raising and lowering" operators

$$N_r^{\pm} = N_r^{\,2} \mp N_r^{\,3} \tag{C.3}$$

which satisfy

$$\left[N_r^{\,1}, N_r^{\,\pm} \right] = \pm i N_r^{\,\pm}. \tag{C.4}$$

$N_r^{\pm}$ raise and lower ν_r by i, respectively. From this, one finds

$$(\nu', \lambda', m' | N_r^+ O | \nu\lambda m) = + \left[\nu_r(\nu_r - i) + \frac{m^2-1}{4} \right] e^{-i(\partial/\partial \nu_r)}$$

$$(\nu', \lambda', m' | O | \nu\lambda m)$$

$$= + e^{-i(\partial/\partial \nu_r)} \left[\nu_r(\nu_r + i) + \frac{m^2-1}{4} \right] (\nu'\lambda' m' | O | \nu\lambda m)$$

$$(\nu'\lambda' m' | N_r^- O | \nu\lambda m) = -e^{i(\partial/\partial \nu_r)} (\nu'\lambda' m' | O | \nu\lambda m).$$

From this, all matrix elements of the $O(2,1) \times O(2,1)$ operators (II.30) are known.

Acknowledgement

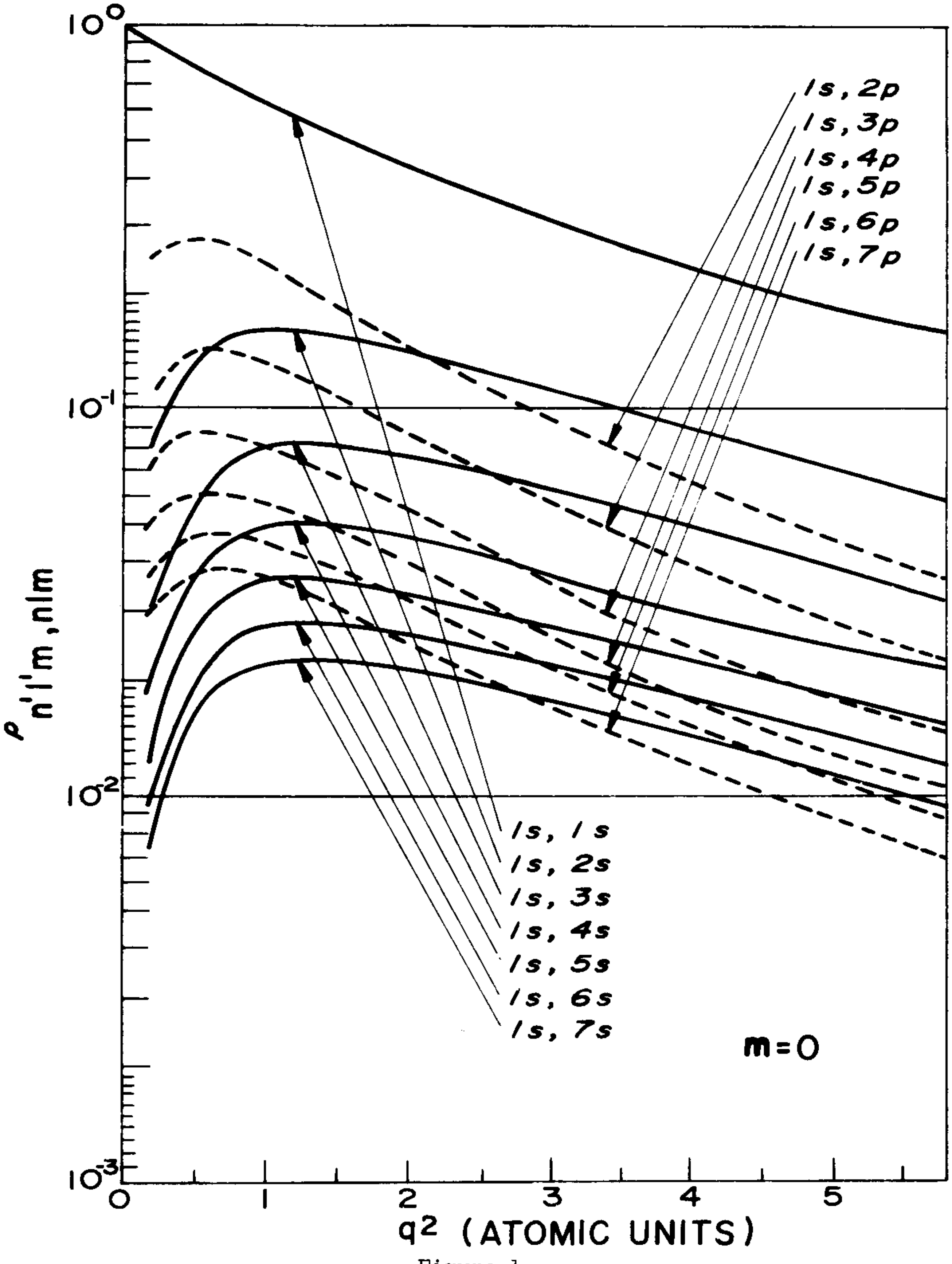
The author wishes to thank Professor A. O. Barut for many stimulating discussions.

* * * *

Figures 1-5. The form factors of the charge distribution

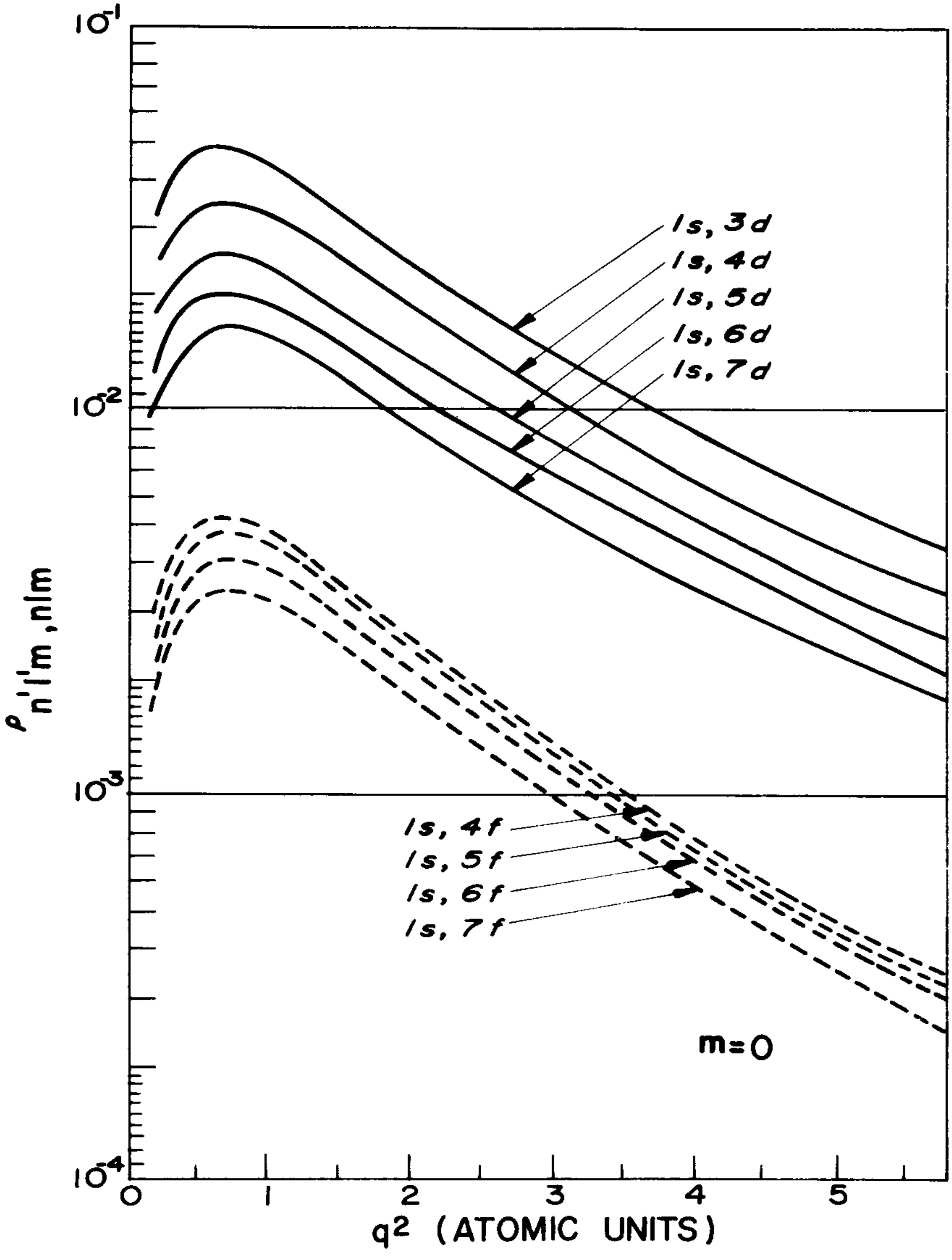
$$\rho_{n'\ell'm, n\ell m}(q) = \int \psi^*_{n'\ell'm} e^{iqz} \psi_{n\ell m}$$

are plotted in atomic units $e = \hbar = (m_e m_p)/(m_e + m_p) = 1$. [A very fast Fortran subroutine has been written which calculates the functions $\rho_{n'\ell'm, n\ell m}(q)$ according to formula (V.27) and the spatial currents $I^i_{n'\ell'm', n\ell m}(q)$ according to (V.29) and (V.30).]



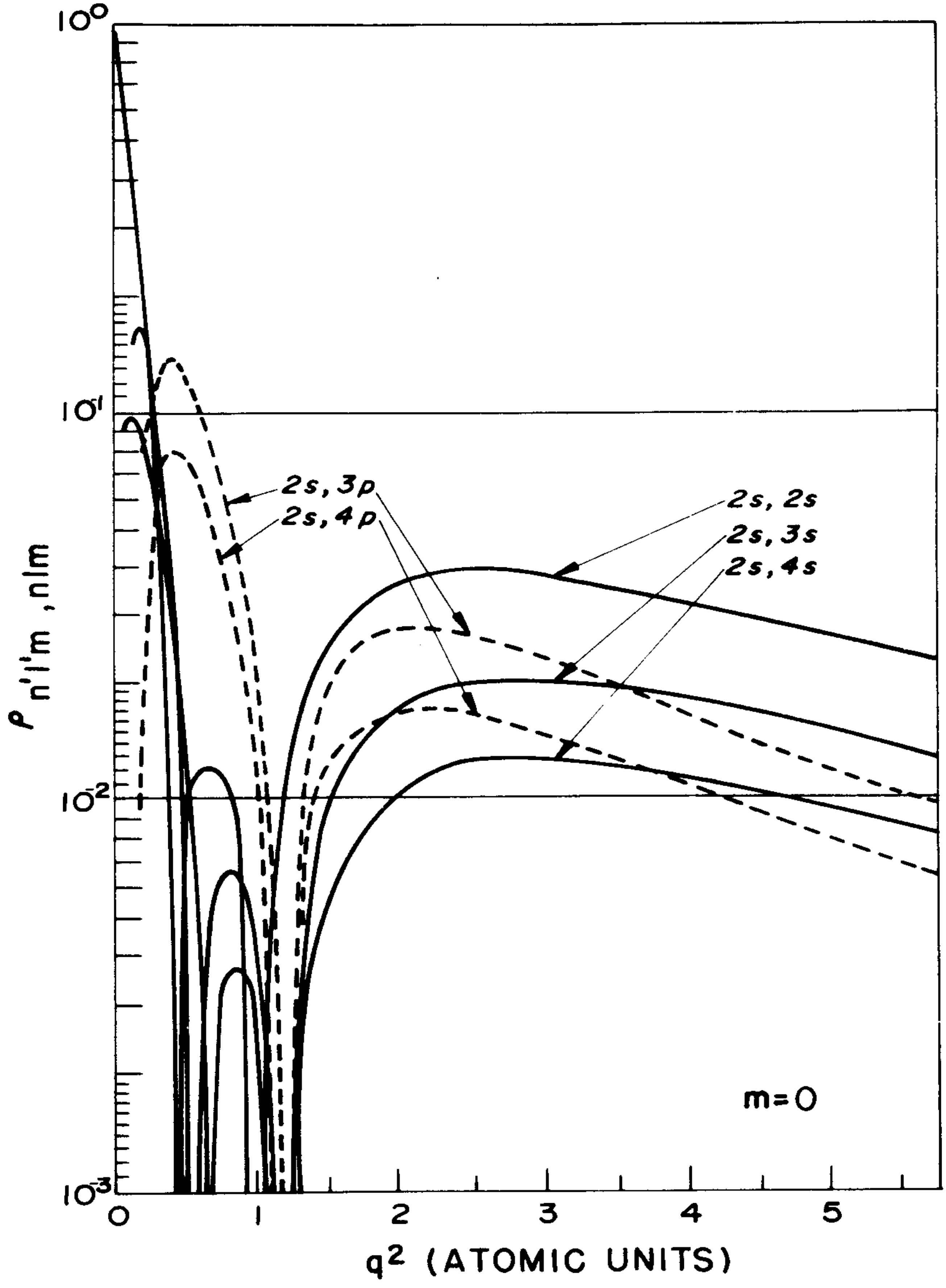
10^0
10^{-1}
10^{-2}
10^{-3}
$\rho_{n'l'm,nlm}$
1s, 2p
1s, 3p
1s, 4p
1s, 5p
1s, 6p
1s, 7p
1s, 1s
1s, 2s
1s, 3s
1s, 4s
1s, 5s
1s, 6s
1s, 7s
m=0
0
1
2
3
4
5
q^2 (ATOMIC UNITS)

Figure 1



1s, 3d
1s, 4d
1s, 5d
1s, 6d
1s, 7d
1s, 4f
1s, 5f
1s, 6f
1s, 7f
m=0
$\rho_{n'l'm,nlm}$
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
1
2
3
4
5
q^2 (ATOMIC UNITS)

Figure 2



10^0
10^{-1}
10^{-2}
10^{-3}
$\rho_{n'l'm,nlm}$
2s, 3p
2s, 4p
2s, 2s
2s, 3s
2s, 4s
m=0
0
1
2
3
4
5
q^2 (ATOMIC UNITS)

Figure 3

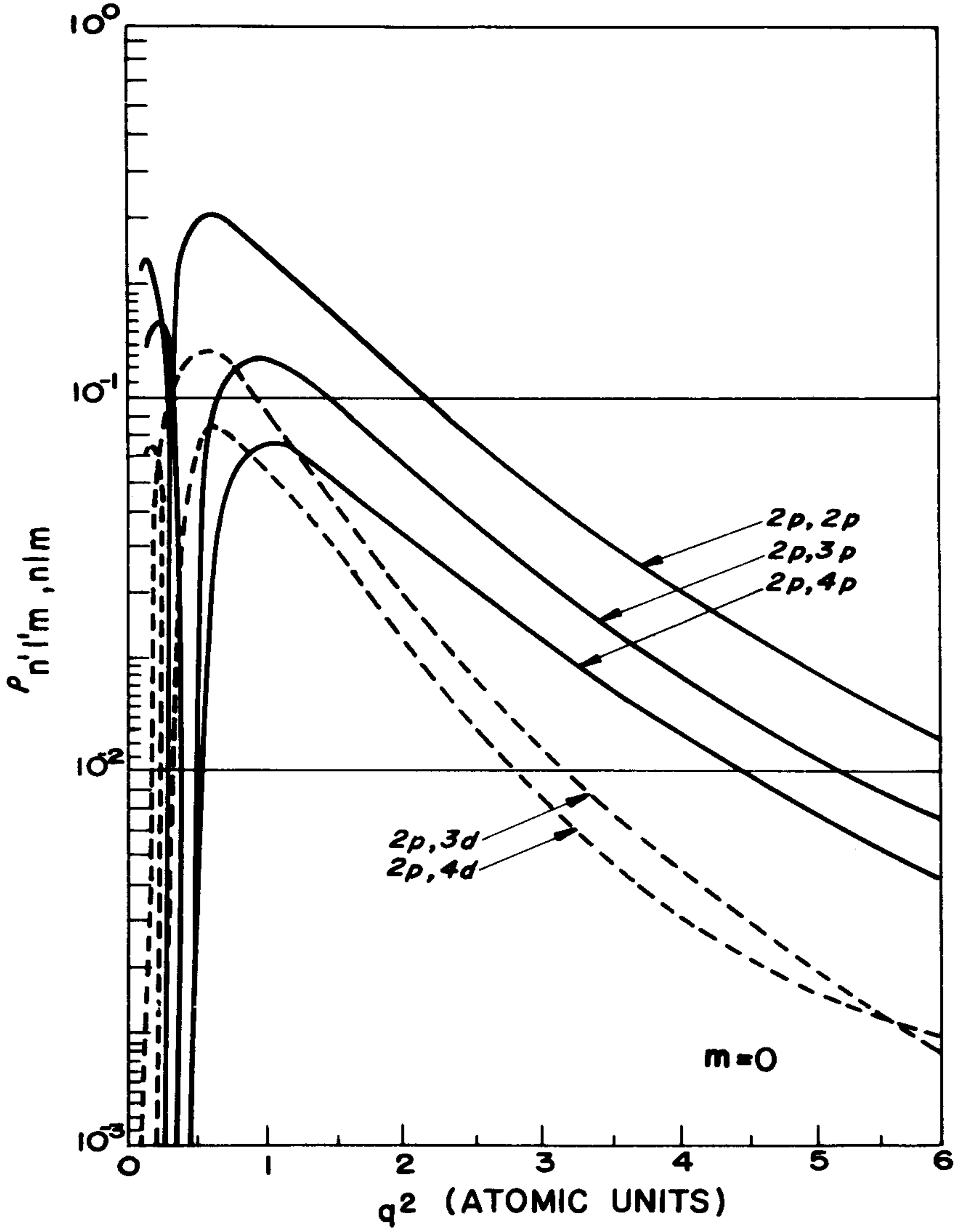
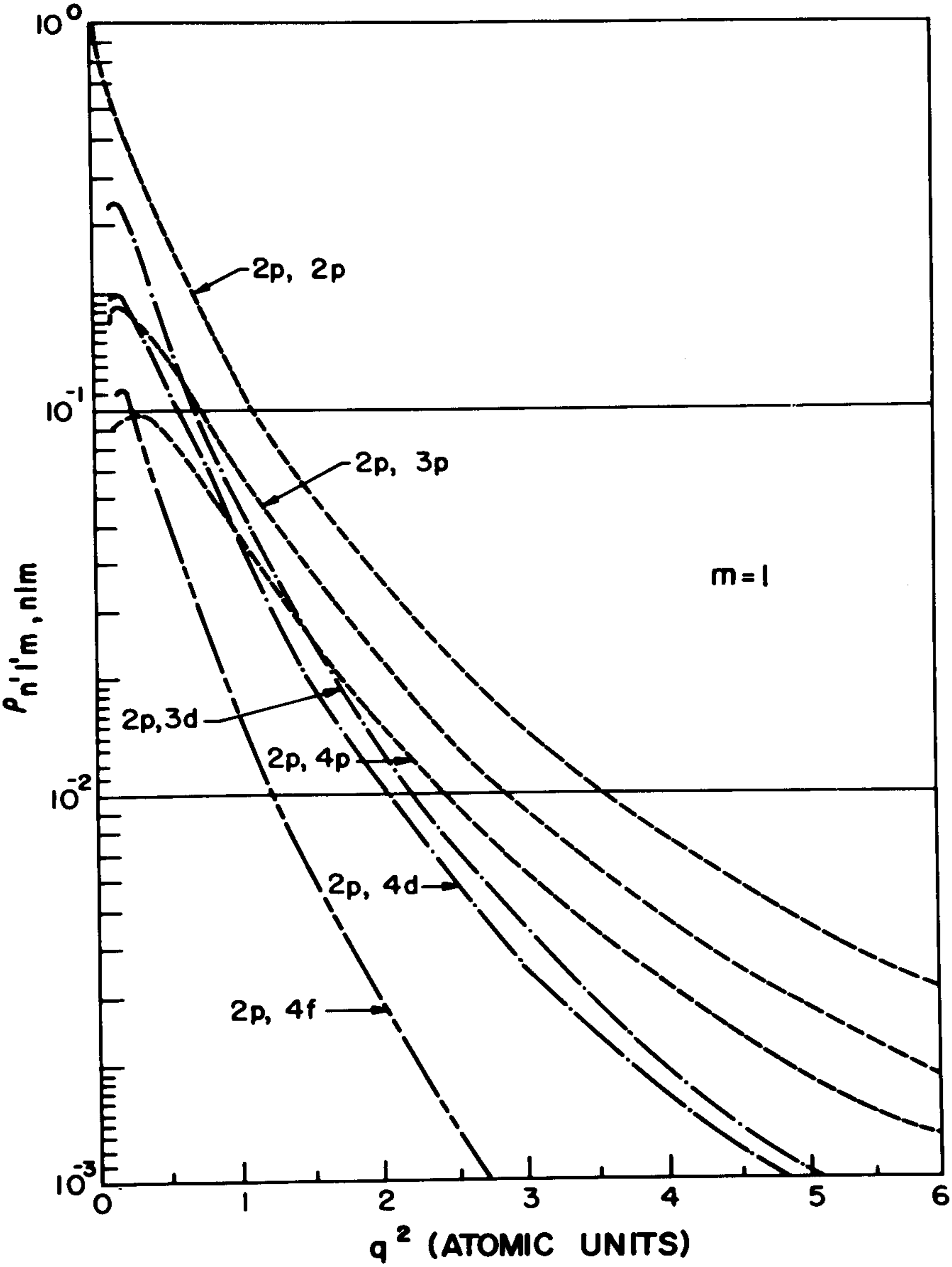

10^0
10^{-1}
10^{-2}
10^{-3}
$\rho_{n'l'm,nlm}$
2p, 2p
2p, 3p
2p, 4p
2p, 3d
2p, 4d
m = 0
0
1
2
3
4
5
6
q^2 (ATOMIC UNITS)

Figure 4



10^0
10^{-1}
10^{-2}
10^{-3}
$\rho_{n'l'm,nlm}$
2p, 2p
2p, 3p
m=1
2p,3d
2p, 4p
2p, 4d
2p, 4f
0
1
2
3
4
5
6
q² (ATOMIC UNITS)

Figure 5

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