A NOTE ON THE HAWKINS-SIMON CONDITIONS*)

M. KATO, G. MATSUMOTO, and T. SAKAI

The object of the present note is to provide an alternative proof of the theorem that the Hawkins-Simon conditions are necessary and sufficient for the static Leontief system to have positive solutions. Originally, the theorem is proved by Hawkins and Simon [2], and subsequently by Morishima [5], and Nikaido [6], [7] in a more general setting. Most of the books referring to the theorem, however, confine their remarks to the two-sector case for the convenience of a diagrammatic exposition.[1] The following proof will be performed in the wake of the original version by Hawkins and Simon.

Let us consider the system of nonhomogeneous linear equations:

$$\sum_{j=1}^{m} a_{ij}x_j = k_i \qquad (i=1,\cdots,m) \tag{1}$$

with $a_{ij} \leqq 0$ for all $i \neq j$. Then, the theorem to be proved is as follows:

THEOREM: A necessary and sufficient condition that the x_i satisfying (1) be all positive for any positive k_i is that all principal minors of the square matrix $A=(a_{ij})$ are positive, *i.e.*,

$$\begin{vmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & & \vdots \\ a_{s1} & \cdots & a_{ss} \end{vmatrix} > 0 \qquad (s=1,\cdots,m) \tag{2}$$

Proof. We first prove that the Hawkins-Simon conditions (2) is necessary for the system (1) to have solutions $x_i>0$ for any $k_i>0$, and then that (2) is sufficient.

(*Necessity*) If the system (1) has positive solutions, the first equation of the system (1) can be rewritten as

$$a_{11} = \frac{1}{x_1}(k_1 - \sum_{j=2}^{m} a_{1j}x_j) > 0\,. \tag{3}$$

This implies that (2) holds when $s=1$.

By using the first equation of the system (1), x_1 can be eliminated from the remaining equations. Thus, we obtain the subsystem of equations:

$$\sum_{j=2}^{m}\left(a_{ij} - \frac{a_{i1}}{a_{11}}a_{1j}\right)x_j = k_i - \frac{a_{i1}}{a_{11}}k_1 \qquad (i=2,\cdots,m) \tag{4}$$

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1) See, for example, Dorfman, Samuelson, and Sollow ([1], Chap. 9), Kuenne ([3], Chap. 6), and Morishima ([4], Chap. 2).

For simplicity, the system (4) is denoted by

$$(5) \qquad \sum_{j=2}^{m} a_{ij}^{(1)} x_j = k_i^{(1)} \qquad (\mathrm{i}=2, \cdots, m)$$

In view of the conditions that $a_{ij} \leqq 0$ for all $i \neq j$, and $a_{11} > 0$, we conclude that

$$(6) \qquad a_{ij}^{(1)} \leqq 0 \quad \text{for all } i=j, \quad \text{and} \quad k_i^{(1)} > 0 .$$

If the system (1) has positive solutions, the first equation of the subsystem (5) can be rewritten as

$$(7) \qquad a_{22}^{(1)} = \frac{1}{x_2}\left(k_2^{(1)} - \sum_{j=3}^{m} a_{2j}^{(1)} x_j\right) > 0 .$$

It is easily seen that this implies that the condition (2) holds for $s=2$, because

$$(8) \qquad a_{22}^{(1)} = \frac{1}{a_{11}} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} .$$

Similarly, by using the first equation of the subsystem (5), x_2 (the first variable of the subsystem) can be eliminated from the remaining $(m-2)$ equations. Thus, we obtain the second subsystem of $(m-2)$ equations whose off-diagonal coefficients and nonhomogeneous terms have the same sign as those of the system (1) or (5). In general, by using the first equation of the s-th subsystem of $(m-s)$ equations, x_{s+1} (the first variable of the s-th subsystem) can be eliminated out of the remaining $(m-s-1)$ equations. Thus, we shall obtain the $(s+1)$-th subsystem of $(m-s-1)$ equations whose off-diagonal coefficients and nonhomogeneous terms have the same sign as those of earlier subsystems.

If the system (1) has positive solutions, the first equation of the $(s-1)$-th subsystem can be rewritten as

$$(9) \qquad a_{ss}^{(s-1)} = \frac{1}{x_s}\left(k_s^{(s-1)} - \sum_{j=s+1}^{m} a_{sj}^{(s-1)} x_j\right) > 0 , \qquad (s=2, \cdots, m)$$

The above procedure of elimination is equivalent to such operations that a multiple of the elements of one row of the coefficient matrix (a_{ij}) is added to the corresponding elements of other rows. According to the well-known property of determinants, this does not alter the values of the principal minors of (a_{ij}). Thus, we have

$$(10) \qquad \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ a_{s1} & a_{s2} & \cdots & a_{ss} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ 0 & a_{22}^{(1)} & \cdots & a_{2s}^{(1)} \\ & & \ddots & \\ 0 & 0 & \cdots & a_{ss}^{(s-1)} \end{vmatrix}$$

$$= a_{11} a_{22}^{(1)} a_{33}^{(2)} \cdots a_{s-1.s-1}^{(s-2)} a_{ss}^{(s-1)} \qquad (s=2, \cdots, m)$$

This completes the proof of necessity.

(*Sufficiency*) In view of (10), we have

$$a_{11}>0\,,\quad a_{22}^{(1)}>0\,,\cdots,a_{m-1,m-1}^{(m-2)}>0\,,\quad \text{and}\quad a_{mm}^{(m-1)}>0\,, \tag{11}$$

if all principal minors of the matrix (a_{ij}) are positive.

To begin with, let us consider the last (*i.e.*, the $(m-1)$-th) subsystem:

$$a_{mm}^{(m-1)}x_m=k_m^{(m-1)}\,, \tag{12}$$

where $k_m^{(m-1)}>0$.

By virtue of (11), we can divide both sides of the equation (12) by $a_{mm}^{(m-1)}$ and obtain

$$x_m>0\,. \tag{13}$$

In order to work out the proof by mathematical induction, suppose that we have already obtained solutions:

$$x_m>0\,,\quad x_{m-1}>0,\cdots,x_{s+1}>0\,. \tag{14}$$

Now, let us consider the first equation of the $(s-1)$-th subsystem:

$$a_{ss}^{(s-1)}x_s=k_s^{(s-1)}-\sum_{j=s+1}^{m}a_{sj}^{(s-1)}x_j\,, \tag{15}$$

where the nonhomogeneous term $k_s^{(s-1)}$ is positive and the off-diagonal coefficients $a_{sj}^{(s-1)}$ are nonpositive. By (14), the right-hand-side of the equation is positive. By virtue of (11), we can divide both sides of the equation by $a_{ss}^{(s-1)}$ and obtain

$$x_s>0\,. \tag{16}$$

Thus the proof is complete, Q. E. D.

References

[1] Dorfman, R., P. A. Samuelson, and R. M. Solow, *Linear Programming and Economic Analysis* (McGraw-Hill, 1958).

[2] Hawkins, D., and H. A. Simon, "Note: Some Conditions of Macroeconomic Stability", *Econometrica* XVII, 1949, pp. 245–248.

[3] Kuenne, R. E., *The Theory of General Economic Equilibrium* (Princeton University Press, 1963).

[4] Morishima, M., *Sangyo Renkan Ron Nyumon*, (Introduction to Input-Output Analysis) (Sobunsha, 1956).

[5] ——————, *Equilibrium, Stability and Growth* (Oxford University Press, 1964).

[6] Nikaido, H., *Introduction to Sets and Mappings in Modern Economics.* (North-Holland, 1970).

[7] ——————, *Convex Structures and Economic Theory* (Academic Press, 1968).