characterization of maximal ideals of the algebra of continuous functions on a compact set*

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Let X be a compact topological space and let $C(X)$ be the algebra of continuous real-valued functions on this space. In this entry, we shall examine the maximal ideals of this algebra.

Theorem 1. *Let X be a compact topological space and I be an ideal of $C(X)$. Then either $I = C(x)$ or there exists a point $p \in X$ such that $f(p) = 0$ for all $f \in I$.*

Proof. Assume that, for every point $p \in X$, there exists a continuous function $f \in I$ such that $f(p) \neq 0$. Then, by continuity, there must exist an open set U containing p so that $f(q) \neq 0$ for all $q \in U$. Thus, we may assign to each point $p \in X$ a continuous function $f \in I$ and an open set U of X such that $f(q) \neq 0$ for all $q \in U$. Since this collection of open sets covers X, which is compact, there must exists a finite subcover which also covers X. Call this subcover $U_1, \ldots, U_n$ and the corresponding functions $f_1, \ldots f_n$. Consider the function g defined as $g(x) = (f_1(x))^2 + \cdots + (f_n(x))^2$. Since I is an ideal, $g \in I$. For every point $p \in X$, there exists an integer i between 1 and n such that $f_i(p) \neq 0$. This implies that $g(p) \neq 0$. Since g is a continuous function on a compact set, it must attain a minimum. By construction of g, the value of g at its minimum cannot be negative; by what we just showed, it cannot equal zero either. Hence being bounded from below by a positive number, g has a continuous inverse. But, if an ideal contains an invertible element, it must be the whole algebra. Hence, we conclude that either there exists a point $p \in x$ such that $f(p) = 0$ for all $f \in I$ or $I = C(x)$. □

Theorem 2. *Let X be a compact Hausdorff topological space. Then an ideal is maximal if and only if it is the ideal of all points which go zero at a given point.*

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Proof. By the previous theorem, every non-trivial ideal must be a subset of an ideal of functions which vanish at a given point. Hence, it only remains to prove that ideals of functions vanishing at a point is maxiamal.

Let p be a point of X. Assume that the ideal of functions vanishing at p is properly contained in ideal I. Then there must exist a function $f \in I$ such that $f(p) \neq 0$ (otherwise, the inclusion would not be proper). Since f is continuous, there will exist an open neighborhood U of p such that $f(x) \neq 0$ when $x \in U$. By Urysohn's theorem, there exists a continuous function $h \colon X \to \mathbb{R}$ such that $f(p) = 0$ and $f(x) = 0$ for all $x \in X \setminus U$. Since I was assumed to contain all functions vanishing at p, we must have $f \in I$. Hence, the function g defined by $g(x) = (f(x))^2 + (h(x))^2$ must also lie in I. By construction, $g(g) > 0$ when $x \in U$ and when $g(x) \in X \setminus U$. Because X is compact, g must attain a minimum somewhere, hence is bounded from below by a positive number. Thus g has a continuous inverse, so $I = C(X)$, hence the ideal of functions vanishing at p is maximal. $\square$