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Theory of the Alternating-Gradient Synchrotron[1, 2]

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The equations of motion of the particles in a synchrotron in which the field gradient index

$$n = -(r/B)\,\partial B/\partial r$$

varies along the equilibrium orbit are examined on the basis of the linear approximation. It is shown that if n alternates rapidly between large positive and large negative values, the stability of both radial and vertical oscillations can be greatly increased compared to conventional accelerators in which n is azimuthally constant and must lie between 0 and 1. Thus aperture requirements are reduced. For practical designs, the improvement is limited by the effects of constructional errors; these lead to resonance excitation of oscillations and consequent instability if $2\nu_x$ or $2\nu_z$ or $\nu_x+\nu_z$ is integral, where ν_x and ν_z are the frequencies of horizontal and vertical betatron oscillations, measured in units of the frequency of revolution.

The mechanism of phase stability is essentially the same as in a conventional synchrotron, but the radial amplitude of synchrotron oscillations is reduced substantially. Furthermore, at a "transition energy" $E_1 \approx \nu_x Mc^2$ the stable and unstable equilibrium phases exchange roles, necessitating a jump in the phase of the radiofrequency accelerating voltage. Calculations indicate that the manner in which this jump is performed is not very critical.

1. INTRODUCTION

The particles in a circular magnetic accelerator, such as a synchrotron, cyclotron, or betatron, are confined to the vicinity of their equilibrium orbit by magnetic focusing forces. These forces are conventionally obtained by shaping the magnetic field in such a way that

$$0 < n < 1, \tag{1.1}$$

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[1] This paper is a revised version of a report written by us in 1953 and privately circulated at that time. Many if not most of the results obtained here have also been obtained independently by numerous other authors, especially members of the accelerator design groups at CERN, Geneva; Saclay, France; Harwell, England; and Cambridge, Massachusetts. No attempt has been made here to allocate credit for every single result. Comprehensive accounts of the theory of betatron oscillations, using somewhat different approaches from ours, may be found in references 9, 13, and 14.

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where

$$n = -(r/B)(\partial B/\partial r) \tag{1.2}$$

is the field gradient index. Increasing n strengthens the vertical focusing forces at the expense of the radial, while decreasing n has the opposite effect; the inequalities (1.1) impose limits on the strength of both focusing forces.

It has been shown [1, 2] that these limitations on the strength of the focusing forces can be overcome by letting the field gradient vary azimuthally, that is, by abandoning the axial symmetry that has characterized the fields of accelerators in the past (straight sections in some synchrotrons, of course, also represent some deviation from axial symmetry, but this has been more a perturbation than an essential feature). It was shown in reference [2] (hereafter referred to as CLS) that by letting n in (1.2) alternate between large positive and large negative values at suitable azimuthal intervals, one can obtain focusing forces an order of magnitude stronger than in an accelerator in which (1.1) is satisfied.

In the present paper we shall examine the characteristics of synchrotrons incorporating this "strong-focusing" or "alternating-gradient" scheme in more detail than was given in CLS. We are concerned with oscillations of two types: "betatron" oscillations, whose behavior is governed by the properties of the guide field and which are independent of the accelerating field, and "synchrotron" oscillations arising from the acceleration process. Considering these problems separately is justified [3] as long as the frequency of betatron oscillations is large compared to that of synchrotron oscillations, which is the case here as well as in most existing synchrotrons.

2. STABILITY OF BETATRON OSCILLATIONS

The characteristics of betatron oscillations are essentially the same whether the magnetic fields are stationary or slowly varying with time. We shall therefore assume in this and the following sections that we are dealing with stationary magnetic fields. The effects of adiabatic variation of parameters will be discussed in Section 3d.

We consider a magnetic field $\mathbf{B}(\mathbf{r})$ which has the property that there is a plane such that $\mathbf{B}$ at all points of the plane is perpendicular to the plane. This plane is called the *median plane* and is taken to be horizontal. (In Section 4c we shall abandon this condition of the existence of the median plane.) We further assume that there is a closed curve in this plane such that a particle of a certain magnetic rigidity p/e can move on this curve. We call this curve the *equilibrium orbit*.

In order to be usable as a guide field for accelerators, the magnetic field must be such that the motion of a particle near the equilibrium orbit is *stable* in the following sense: if a particle, whose momentum is appropriate to the given equilibrium orbit, is started with a small intial displacement and a small initial angle from the equilibrium orbit, it will remain near the equilibrium orbit for all time.

We characterize the position of a point P near the equilibrium orbit by the following set of curvilinear coordinates:

s = the distance along the equilibrium orbit measured from some fixed reference point to that point on the orbit closest to the point P,

x = the horizontal component of the displacement of P from the equilibrium orbit (taken to be positive in the outward direction),

z = the vertical component of the displacement.

The motion of a particle near the orbit may be expressed in terms of s as the independent coordinate (see Appendix). If all terms of second and higher orders in x, z, and their derivatives are neglected, the equations of motion may be written in the form

$$\frac{d^2x}{ds^2} = -\frac{1-n(s)}{\rho^2(s)}\,x, \tag{2.1}$$

$$\frac{d^2z}{ds^2} = -\frac{n(s)}{\rho^2(s)}\,z, \tag{2.2}$$

where

$$\rho(s) = pc/eB_z(s, 0, 0) \tag{2.3}$$

is the radius of curvature of the equilibrium orbit at s, and

$$n(s) = -\frac{\rho(s)}{B_z(s, 0, 0)}\left.\frac{\partial B_z(s, x, 0)}{\partial x}\right|_{x=0} = -\frac{\rho^2}{pc/e}\frac{\partial B_z}{\partial x} \tag{2.4}$$

is the field gradient at s.

Since the equilibrium orbit is closed, the quantities $n(s)$ and $\rho(s)$ are periodic functions of s, and (2.1) and (2.2) are examples of Hill's equation, i.e., linear equations with periodic coefficients and without first derivative terms. We write both equations in the form

$$\frac{d^2y}{ds^2} = -K(s)\,y, \tag{2.5}$$

where y represents either horizontal or vertical displacement, and where K satisfies the periodicity relation

$$K(s + C) = K(s). \tag{2.6}$$

Here C is the circumference of the equilibrium orbit.

In the alternating-gradient synchrotron the magnet ideally consists of N identical sections or "unit cells," so that K also satisfies the stronger periodicity relation

$$K(s+L)=K(s); \qquad L=C/N. \tag{2.7}$$

At this point it may be useful to review some of the properties of Hill's equation [4].

The solution of any linear second order differential equation of the form (2.5), whether or not K is periodic, is uniquely determined by the initial values of y and its derivative y',

$$\begin{aligned} y(s) &= ay(s_0)+by'(s_0), \\ y'(s) &= cy(s_0)+dt'(s_0), \end{aligned} \tag{2.8}$$

or, in matrix notation,

$$Y(s)=\begin{bmatrix} Y(s) \\ Y'(s) \end{bmatrix} = \mathrm{M}(s \mid s_0)\ Y(s_0)=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} y(s_0) \\ y'(s_0) \end{bmatrix}. \tag{2.9}$$

The usefulness of the matrix formulation (2.9) arises mainly from two features: In the first place, this formulation clearly separates the properties of the general solution of the problem from the features characterising any particular solution. That is, the matrix $\mathrm{M}(s \mid s_0)$ depends *only* on the function $K(s)$ between s_0 and s, and not on the particular solution. Secondly, the matrix for any interval made up of sub-intervals is just the product, calculated by the usual rules of matrix multiplication, of the matrices for the sub-intervals, that is,

$$\mathrm{M}(s_2 \mid s_0)=\mathrm{M}(s_2 \mid s_1)\ \mathrm{M}(s_1 \mid s_0), \tag{2.10}$$

as is easily verified.

The determinant of the matrix M is equal to unity, because the equation (2.5) does not contain any first derivative terms.

For the particular case of constant K the matrix takes the form

$$\mathrm{M}(s_0 \mid s)=\begin{bmatrix} \cos\phi & K^{-1/2}\sin\phi \\ -K^{1/2}\sin\phi & \cos\phi \end{bmatrix}, \tag{2.11}$$

where $\phi=K^{1/2}(s-s_0)$. If K is negative a more convenient way of writing this is

$$\mathrm{M}=\begin{bmatrix} \cosh\psi & (-K)^{-1/2}\sinh\psi \\ (-K)^{1/2}\sinh\psi & \cosh\psi \end{bmatrix}, \tag{2.12}$$

where $\psi=(-K)^{1/2}(s-s_0)$. For an interval of length l in which $K=0$,

$$\mathrm{M}=\begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix}. \tag{2.13}$$

For an interval in which K is piecewise constant the matrix is the product of the appropriate matrices of forms (2.11) to (2.13).

In the periodic systems we are considering here the matrices of particular interest are those which characterize the motion of the particle through a whole period. We write

$$\mathrm{M}(s) = \mathrm{M}(s + L \mid s); \tag{2.14}$$

this is the matrix for passage through one period, starting from s. Its elements are periodic functions of s with period L. The matrix for passage through one revolution is then

$$\mathrm{M}(s + NL \mid s) = [\mathrm{M}(s)]^N,$$

and that for passage through k revolutions is $[\mathrm{M}(s)]^{Nk}$.

In order for the motion to be stable as defined above, it is necessary and sufficient that all the elements of the matrix M^{Nk} remain bounded as k increases indefinitely. To obtain the condition for this, we consider the eigenvalues of the matrix $\mathrm{M}(s)$, that is, those numbers λ for which the characteristic matrix equation

$$\mathrm{M}\,Y = \lambda Y \tag{2.15}$$

possesses nonvanishing solutions. The eigenvalues are the solutions of the determinantal equation

$$|\mathrm{M} - \lambda \mathrm{I}| = 0, \tag{2.16}$$

or, more fully,

$$\lambda^2 - \lambda(a + d) + 1 = 0, \tag{2.17}$$

where we have made use of the fact that $\mathrm{Det}\ \mathrm{M} = ad - bc = 1$. If we write

$$\cos \mu = \tfrac{1}{2} Tr\ \mathrm{M} = \tfrac{1}{2}(a + d), \tag{2.18}$$

the two solutions of (2.17) are

$$\lambda = \cos \mu \pm i \sin \mu = e^{\pm i\mu}. \tag{2.19}$$

The quantity μ will be real if $|a + d| \leqslant 2$, and imaginary or complex if $|a + d| > 2$.

Let us now assume that $|a + d| \neq 2$. Then the matrix M may be written in a form which exhibits the eigenvalues and other properties explicitly. We define $\cos \mu$ by (2.18), and define α, β, and γ by

$$\begin{aligned} a - d &= 2\alpha \sin \mu, \\ b &= \beta \sin \mu, \\ c &= -\gamma \sin \mu; \end{aligned} \tag{2.20}$$

the condition Det M = 1 becomes

$$\beta\gamma - \alpha^2 = 1. \tag{2.21}$$

We resolve the ambiguity of the sign of $\sin\mu$ by requiring β to be positive if $|\cos\mu| < 1$ and by requiring $\sin\mu$ to be positive imaginary if $|\cos\mu| > 1$. The definition of μ is still ambiguous to the extent that any multiple of 2π may be added to μ without changing the matrix. This ambiguity will be resolved later.

The matrix M may now be written as

$$\mathrm{M} = \begin{bmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{bmatrix} = \mathrm{I}\cos\mu + \mathrm{J}\sin\mu \tag{2.22}$$

where I is the unit matrix, and

$$\mathrm{J} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix} \tag{2.23}$$

is a matrix with zero trace and unit determinant, satisfying

$$\mathrm{J}^2 = -\mathrm{I}. \tag{2.24}$$

It should be noted that the trace of M, and therefore μ, is independent of the reference point s. For, by virtue of (2.10), we have for any s_1 and s_2

$$\mathrm{M}(s_2 + L \mid s_1) = \mathrm{M}(s_2)\,\mathrm{M}(s_2 \mid s_1) = \mathrm{M}(s_2 \mid s_1)\,\mathrm{M}(s_1), \tag{2.25}$$

so that

$$\mathrm{M}(s_2) = \mathrm{M}(s_2 \mid s_1)\,\mathrm{M}(s_1)[\,\mathrm{M}(s_2 \mid s_1)\,]^{-1}. \tag{2.26}$$

Thus $\mathrm{M}(s_1)$ and $\mathrm{M}(s_2)$ are related by a similarity transformation, and therefore have the same trace and the same eigenvalues. On the other hand, the matrix $\mathrm{M}(s)$ as a whole does depend on the reference point s. Thus the elements α, β, γ of the matrix J are functions of s, periodic with period L.

Because of (2.25), the combination $\mathrm{I}\cos\mu + \mathrm{J}\sin\mu$ has properties similar to those of the complex exponential $e^{i\mu} = \cos\mu + i\sin\mu$; in particular, it is easily seen that, for any μ_1 and μ_2

$$(\mathrm{I}\cos\mu_1 + \mathrm{J}\sin\mu_1)(\mathrm{I}\cos\mu_2 + \mathrm{J}\sin\mu_2) = \mathrm{I}\cos(\mu_1 + \mu_2) + \mathrm{J}\sin(\mu_1 + \mu_2). \tag{2.27}$$

The kth power of the matrix M is thus

$$\mathrm{M}^k = (\mathrm{I}\cos\mu + \mathbf{J}\sin\mu)^k = \mathrm{I}\cos k\mu + \mathrm{J}\sin k\mu, \tag{2.28}$$

and the inverse is

$$\mathrm{M}^{-1} = \mathrm{I}\cos\mu - \mathrm{J}\sin\mu. \tag{2.29}$$

It follows from (2.28) that if μ is real the matrix elements of M^k do not increase indefinitely with increasing k but rather oscillate; on the other hand, if μ is not real, $\cos k\mu$ and $\sin k\mu$ increase exponentially, and therefore the matrix elements do the same. Therefore the motion is stable if μ is real, i.e., if $|a+d|<2$, and unstable if $|a+d|>2$.

In conventional circular accelerators, $K(s)$ is constant, $=n/R^2$ for vertical and $(1-n)/R^2$ for horizontal oscillations, and $L=2\pi R$. Thus

$$\mu_x = 2\pi(1-n)^{1/2}$$
$$\mu_z = 2\pi n^{1/2},$$

and the stability condition reduces to the well-known inequality (1.1). If N equal straight sections of length l are introduced, the matrix for a unit cell is (2.11) multiplied by (2.13), and

$$\cos\mu = \cos\phi - \frac{Nl\phi}{4\pi R}\sin\phi \tag{2.31}$$

as is well known [5].

In alternating gradient synchrotrons [1, 2] the simplest magnet arrangement is that of CLS:

$$\begin{aligned} &\rho = \text{const} = R, \\ &n = n_1, \qquad 0 < s < \frac{\pi R}{N}, \\ &n = -n_2, \qquad \frac{\pi R}{N} < s < \frac{2\pi R}{N}. \end{aligned} \tag{2.32}$$

(The notation here is slightly different from that of CLS.)

In this case the matrix for one period is the product of (2.11) and (2.12). Computing its trace we find, for vertical oscillations,

$$\cos\mu_z = \cos\phi_z \cosh\psi_z - \frac{n_1-n_2}{2(n_1n_2)^{1/2}}\sin\phi_z \sinh\psi_z, \tag{2.33}$$

where

$$\phi_z = \pi n^{1/2}/N \qquad \text{and} \qquad \psi_z = \pi n^{1/2}/N,$$

and for horizontal oscillations,

$$\cos\mu_x = \cos\phi_x \cosh\psi_x - \frac{2-n_1+n_2}{[(n_2+1)(n_1-1)]^{1/2}}\sin\phi_x \sinh\psi_x, \tag{2.34}$$

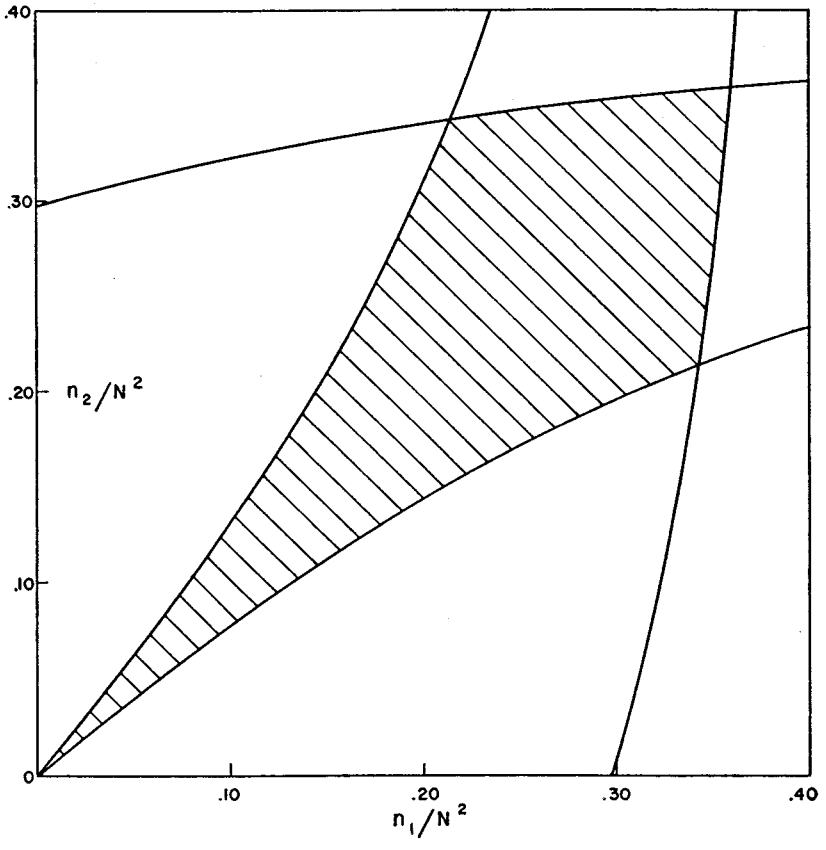

FIG. 1. Region of stability for radial and vertical oscillations.

where

$$\phi_x = \pi(n_2+1)^{1/2}/N \qquad \text{and} \qquad \psi_x = \pi(n_1-1)^{1/2}/N.$$

If $n_1 \gg 1$ and $n_2 \gg 1$, (2.34) is obtained from (2.33) by interchanging n_1 and n_2, and the stability criteria depend only on n_1/N^2 and n_2/N^2. Both modes are stable provided n_1/N^2 and n_2/N^2 lie within the "necktie" shaped region of Fig. 1.

3. AMPLITUDES OF BETATRON OSCILLATIONS

The motivation for proposing the alternating gradient configuration for the magnetic guide field in accelerators was the expectation that the effective focusing forces would be much stronger than in the corresponding conventional accelerators, leading to oscillations of smaller amplitudes around the equilibrium orbit, and

consequently smaller aperture requirements. In this section we shall first give this statement a precise meaning, and then investigate its validity quantitatively.

(a) *Phase-Amplitude Form of the Solution*

We may attempt to find solutions of Hill's equation (2.5) which have the form

$$y_1(s) = w(s)\, e^{i\psi(s)}, \tag{3.1}$$

where, for the moment, we impose no particular conditions on the functions w and ψ. It is easily verified by substitution that, if w and ψ satisfy

$$w'' + Kw - \frac{1}{w^3} = 0 \tag{3.2}$$

and

$$\psi' = \frac{1}{w^2}, \tag{3.3}$$

then y_1 as defined by (3.1) is indeed a solution, as is

$$y_2(s) = w(s)\, e^{-i\psi(s)}, \tag{3.4}$$

and that y_1 and y_2 are linearly independent. Therefore any solution of (2.5) is a linear combination of y_1 and y_2. We can therefore write the matrix $\mathrm{M}(s_2 \mid s_1)$ in terms of the solutions y_1 and y_2 or, what amounts to the same thing, in terms of the functions w and ψ. We obtain

$$\mathrm{M}(s_2 \mid s_1) = \begin{bmatrix} \dfrac{w_2}{w_1}\cos\psi - w_2 w_1'\sin\psi, & w_1 w_2 \sin\psi \\ -\dfrac{1 + w_1 w_1' w_2 w_2'}{w_1 w_2}\sin\psi - \left(\dfrac{w_1'}{w_2} - \dfrac{w_2'}{w_1}\right)\cos\psi, & \dfrac{w_1}{w_2}\cos\psi + w_1 w_2'\sin\psi \end{bmatrix} \tag{3.5}$$

where ψ stands for $\psi(s_2) - \psi(s_1)$, w_1 for $w(s_1)$, etc.

We now consider the case where $s_2 - s_1$ is just one period of $K(s)$, i.e., $s_2 - s_1 = L$. The matrix M is then identical with the matrix (2.22).[3] If we now require that $w(s)$ be a *periodic* function of s, then $w_1 = w_2$ and $w_1' = w_2'$, and the forms (3.5) and (2.22) are identical provided we make the identifications

$$\psi(s_2) - \psi(s_1) = \mu \tag{3.6}$$

$$w^2 = \beta, \tag{3.7}$$

$$ww' = -\alpha, \tag{3.8}$$

[3] We again exclude the case were $|\cos\mu| = \pm 1$.

from which follows automatically

$$\frac{1+(ww')^2}{w^2}=\frac{1+\alpha^2}{\beta}=\gamma. \tag{3.9}$$

This identification is legitimate if we can show that $\beta^{1/2}$—which is, of course, periodic—satisfies the differential equation (3.2) and that

$$\beta'=-2\alpha. \tag{3.10}$$

To prove this, consider the matrix for the transformation from $s+ds$ to $s+L+ds$. This matrix is, by (2.26),

$$\mathrm{M}(s+ds)=\mathrm{M}(s+ds\mid s)\,\mathrm{M}(s)[\,\mathrm{M}(s+ds\mid s)\,]^{-1}. \tag{3.11}$$

For infinitesimal ds,

$$\mathrm{M}(s+ds\mid s)=\begin{bmatrix}1 & ds\\ -K(s)\,ds & 1\end{bmatrix}. \tag{3.12}$$

Substituting (3.12) and (2.22) in (3.11) we find

$$\mathrm{M}(s+ds)=\mathrm{M}(s)+\begin{bmatrix}(K\beta-\gamma)\sin\mu & -2\alpha\sin\mu\\ -2K\alpha\sin\mu & -(K\beta-\gamma)\sin\mu\end{bmatrix}ds, \tag{3.13}$$

so that (3.10) is indeed valid, and furthermore

$$\alpha'=-\frac{1}{2}\beta''=K\beta-\gamma=K\beta-\frac{1+\alpha^2}{\beta} \tag{3.14}$$

and

$$\gamma'=2K\alpha. \tag{3.15}$$

With the aid of (3.10) and (3.14) it is easily verified that $\beta^{1/2}$ does indeed satisfy (3.2), and is therefore a periodic solution of that equation. Now (3.7) and (3.8) are justified, while (3.6) becomes the very important relation

$$\mu=\int_0^L\frac{ds}{\beta}. \tag{3.16}$$

(3.16) may be regarded as the definition of μ. It is consistent with the previous definition (2.18), but has the advantage of being unambiguous, while (2.18) only defines μ modulo 2π.

If we consider an accelerator of circumference $C = NL$ with N identical unit cells, the phase change per revolution is, of course, $N\mu$. A useful number is

$$\nu = \frac{N\mu}{2\pi} = \frac{1}{2\pi}\int_s^{s+C} \frac{ds}{\beta}; \tag{3.17}$$

this is the number of betatron oscillation wavelengths in one revolution. (In the European literature on accelerators this number is often denoted by Q.) A useful interpretation of ν is as the frequency of betatron oscillations measured in units of the frequency of revolution; we shall generally refer to ν simply as the frequency of oscillations.

The two particular solutions y_1 and y_2 may now be written as

$$y_{\substack{1\\2}} = \beta^{1/2}(s)\, e^{\pm i\nu\phi(s)}, \tag{3.18}$$

where

$$\phi(s) = \int \frac{ds}{\nu\beta} \tag{3.19}$$

is a function which increases by 2π every revolution and whose derivative is periodic. The general solution of (2.5) is

$$y(s) = a\beta^{1/2} \cos[\nu\phi(s) + \delta], \tag{3.20}$$

where a and δ are arbitrary constants. This is a pseudo-harmonic oscillation with varying amplitude $\beta^{1/2}(s)$ and varying instantaneous wavelength

$$\lambda = 2\pi\beta(s). \tag{3.21}$$

Incidentally, the relation (3.21) between the amplitude and the wavelength is formally just the same as in the WKB solution of the problem of the harmonic oscillator with varying wavelength; however, the relation between the wavelength and the parameters of the differential equation is not as simple as in the WKB problem.

In the treatment given here it has been tacitly assumed that $\beta(s)$ never vanishes, so that there are no singularities or ambiguities in the integral $\int ds/\beta$. This is the case when the motion is stable, i.e., when $|\cos\mu| < 1$. For then α, β, and γ are real and finite; it then follows from (2.21) that β (and γ) cannot vanish. In the unstable case ($|\cos\mu| > 1$, β imaginary) the solutions can still be written in the form (3.18), but appropriate conventions have to be specified for integrating around the zeros of $\beta(s)$. On the boundary between stable and unstable regions ($|\cos\mu| = 1$) the treatment given here breaks down altogether. We do not propose to treat the unstable and boundary cases in detail here.

(b) *Admittance*

From the form (3.20) of the solution of the equation of motion it follows that the quantity

$$W = \frac{1}{\beta}[\, y^2 + (\alpha y + \beta y')^2 \,] = \gamma y^2 + 2\alpha y y' + \beta y'^2 \tag{3.22}$$

is constant, independent of s. Therefore the largest displacement is attained where β has its maximum value.

In a given accelerator, the motion is restricted by the walls of the vacuum chamber, or other obstructions, to a certain region around the equilibrium orbit, let us say to $|y| < a$. Then all particles whose initial conditions are such that

$$W < W_0 = \frac{a^2}{\beta_{\max}}$$

will perform oscillations that remain within the vacuum chamber. Following Sigurgeirsson [6], we define the *admittance* of the system as the area of that region of (y, y') phase space for which any particles injected with initial values within the region will remain within the vacuum chamber. This area is evidently the area of the ellipse (3.22), with $W = W_0$, that is,

$$A = \text{admittance} = \pi a^2/\beta_{\max}. \tag{3.23}$$

It is therefore desirable to design an accelerator with as small a value of $\beta_{\max}$ as possible. The advantage of the alternating-gradient design is precisely that $\beta_{\max}$ can be made smaller than in a conventional accelerator of the same radius.

From (3.19) we see that, if $\beta(s)$ were constant, it would be equal to

$$\bar{\beta} = C/2\pi\nu = R/\nu, \tag{3.24}$$

where $R = C/2\pi$ is the mean radius of the accelerator. In the general case, the maximum value of β will exceed $\bar{\beta}$ by some factor, which we call the "form factor"

$$F = \beta_{\max}/\beta_{\text{Av}} = \nu\beta_{\max}/R. \tag{3.25}$$

The form factor F can generally be kept fairly small (say about 1.5), and therefore the admittance of an alternating gradient machine is mainly governed by the oscillation frequency ν.

In conventional accelerators $\nu = n^{1/2}$ for vertical and $(1-n)^{1/2}$ for horizontal oscillations; both these frequencies are less than 1. In alternating gradient accelerators we can make ν large, thus achieving a larger admittance for a given aperture or alternatively a smaller aperture for a given admittance.

It follows from (3.20) that the maximum y at any particular s is proportional to $\beta^{1/2}$. Therefore, if the vacuum chamber semiaperture a is constant all around the

machine, the particles will reach the walls only where $\beta = \beta_{\max}$; elsewhere they will only attain a maximum amplitude

$$a(\beta/\beta_{\max})^{1/2}.$$

This makes it permissible to insert structures within the aperture of the vacuum chamber without reducing the space available to the beam—provided these structures are placed where $\beta < \beta_{\max}$ and are close enough to the walls.

(c) *Approximate Calculations*

The general solution of Hill's equation

$$\frac{d^2y}{ds^2} + K(s)\, y = 0 \tag{2.5}$$

is characterized by the amplitude function

$$w(s) = \beta^{1/2}(s), \tag{3.26}$$

which is periodic in s with the same period C as $K(s)$. We wish to find a method for obtaining $\beta(s)$ and

$$\nu = \frac{1}{2\pi}\int_0^C \frac{ds}{\beta} \tag{3.19}$$

approximately in the case where $\beta(s)$ does not fluctuate very much about its mean value.

Equations (3.12) to (3.14) can be combined to yield a single third-order differential equation for β:

$$\beta''' + 4K\beta' + 2K'\beta = 0. \tag{3.27}$$

The amplitude function $\beta(s)$ is that particular solution of (3.27) which is periodic and is normalized so that

$$\beta\gamma - \alpha^2 = \tfrac{1}{2}\beta\beta'' - \tfrac{1}{4}\beta'^2 + K\beta^2 = 1. \tag{3.28}$$

[Equation (3.28) has three linearly independent solutions; the other two are, in general, the squares of the normal solutions of (2.5).]

We now write

$$K(s) = \tfrac{1}{2}\epsilon g(s) \tag{3.29}$$

and

$$\beta = a[\,1 + \epsilon f_1(s) + \epsilon^2 f_2(s) + \cdots\,], \tag{3.30}$$

that is, we regard the focusing function $K(s)$ as "small" in some sense, and hope to obtain the deviation from constancy of the amplitude function $\beta(s)$ as a power series in the smallness parameter ϵ.

Substituting in (3.27) we obtain the recursion relation

$$f_n''' = -2f_{n-1}'g - f_{n-1}g', \tag{3.31}$$

where we must impose the condition that f_n is periodic and has zero mean value.

It is not entirely obvious that this periodicity condition can be met. It is necessary for this that the right hand side of (3.31) be a periodic function with zero mean. We can prove by induction that this is the case. For $n=1$, we have, from (3.31),

$$f_1''' = -g'. \tag{3.32}$$

Since g' is the derivative of a periodic function, it has zero mean, and therefore (3.32) can be solved for f_1. We may now write (3.32) in the form

$$f_n''' = -2(f_{n-1}g)' - (f_{n-1}f_1)''' + 3(f_{n-1}'f_1')' + f_{n-1}'''f_1 \tag{3.33}$$

so that the mean value is

$$\langle f_n''' \rangle = +\langle f_{n-1}'''f_1 \rangle. \tag{3.34}$$

We now note that, for any r and s,

$$\langle f_r'''f_s \rangle = \langle f_{r-1}'''f_{s+1} \rangle. \tag{3.35}$$

This relation may be verified by using the recursion relation (3.31) for f_r''' and integrating by parts. Applying (3.35) to (3.34) $n-2$ times we find

$$\langle f_{n-1}'''f_1 \rangle = \langle f_1'''f_{n-1} \rangle. \tag{3.36}$$

On the other hand, a straightforward triple partial integration yields

$$\langle f_{n-1}'''f_1 \rangle = -\langle f_1'''f_{n-1} \rangle. \tag{3.37}$$

Therefore $\langle f_{n-1}'''f_1 \rangle = 0$, and (3.31) has a periodic solution.

We now choose the normalizing constant so that (3.28) is satisfied. Thus, from (3.19),

$$\begin{aligned} \nu^2 &= \frac{1}{8\pi^2}\left(ff'' - \frac{1}{2}f^2 + \epsilon g f^2\right)\left[\int_0^C \frac{ds}{f}\right]^2 \\ &= \frac{1}{16\pi^2}\langle 2\epsilon g f^2 - 3f^2 \rangle \left[\int_0^C \frac{ds}{f}\right]^2. \end{aligned} \tag{3.38}$$

Expanding (3.38) in a power series in ϵ, and making use of the relations satisfied by the functions f_n, we obtain

$$\nu^2 = \frac{C^2}{16\pi^2}\left[2\epsilon\bar{g} + \epsilon^2\langle f_1'^2\rangle + \epsilon^3(\langle f_1'^2 f_1\rangle + \bar{g}\langle f_1^2\rangle) + \cdots\right]. \tag{3.39}$$

Thus we have obtained the amplitude function and the frequency of oscillation in terms of integrals derived from the focusing function $g(s)$. The procedure given here is particularly useful when $\epsilon\,|f_1|$ is small, so that the amplitude function $\beta(s)$ is nearly constant.

Up to the second order in ϵ our result is identical with that obtained from the "smooth approximation" given by Symon [7]. It has the advantage that it is easy to see how to obtain higher approximations.

The first term in (3.39) is the focusing or defocusing effect of the mean restoring force $\bar{g}$; the second term shows that fluctuations about the mean value will always produce an additional focusing term. In alternating-gradient accelerators the first term is generally small or zero, and the focusing is mainly obtained from the second term.

As an example, consider the CLS configuration (2.32), with $\epsilon g = 2n/R^2$.

In this case we obtain from the first two terms of (3.39)

$$\nu^2 = \frac{n_1 - n_2}{2} + \frac{\pi^2}{48}\frac{(n_1+n_2)^2}{N^2}. \tag{3.40}$$

This is precisely what is obtained by expanding the left hand side of (2.33) in a power series in μ_z and the right hand side in a power series in n_1 and n_2, ignoring terms of higher than second order in n_1 and n_2 or fourth order in μ_z, and noting that $\nu = N\mu/2\pi$.

(d) *Adiabatic Damping*

As the particle is accelerated, its mass and velocity increase. Furthermore, the shape of the magnetic field may change (for example, because of saturation effects) and, therefore, its focusing properties will change. It is, therefore, desirable to investigate what happens to the amplitude of oscillations under these circumstances.

The effect of the increase in mass and velocity is the same as in conventional circular accelerations, namely, a damping of the amplitude proportional to $p^{1/2}$, where p is the momentum of the particle [8]. The interesting question is: What happens when the focusing properties of the field change slowly, that is, when the matrix M changes slowly from one unit cell to the next?

Suppose we have

$$Y_{k+1} = \mathrm{M}_k\, Y_k, \tag{3.41}$$

$$Y_{k+2} = \mathrm{M}_{k+1}\, Y_{k+1} = (\mathrm{M}_k + \epsilon\mathrm{m})\; Y_{k+1}. \tag{3.42}$$

We want to find an invariant, i.e., a quadratic form U whose coefficient matrix V can be calculated from the matrixes M and m, such that

$$U_{k+1} = U_k, \tag{3.43}$$

except for terms of second and higher order in ϵ. Here V_{k+1} must be obtained from the elements of M_{k+1} by the same prescription by which V_k is obtained from M_k.

Let us define, for any 2×2 matrix M, its "conjugate"

$$\bar{\mathrm{M}} = \begin{bmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{bmatrix}. \tag{3.44}$$

Then $\mathrm{M} + \bar{\mathrm{M}} = \mathrm{Tr\ M}$ and $\mathrm{M}\bar{\mathrm{M}} = \mathrm{Det\ M}$; if M is unimodular, $\bar{\mathrm{M}}$ equals M^{-1}.

The invariant W [Eq. (3.22)] may be written

$$2W = \frac{1}{\sin \mu} [X, \mathrm{V}_0 Y], \tag{3.45}$$

where V_0 is the matrix

$$\mathrm{V}_0 = \mathrm{S}(\mathrm{M} - \bar{\mathrm{M}}) \tag{3.46}$$

and

$$\mathrm{S} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{3.47}$$

We, therefore, expect the adiabatic invariant to be of the form

$$U_k = [Y_k, \mathrm{V}_k Y_k], \tag{3.48}$$

with

$$\begin{aligned} \mathrm{V}_k &= a_k[\mathrm{S}(\mathrm{M}_k - \bar{\mathrm{M}}_k) + \epsilon \mathrm{S}(\mathrm{b} - \bar{b})], \\ \mathrm{V}_{k+1} &= (a_k + \alpha\epsilon)\, \mathrm{S}[\mathrm{M}_k - \bar{\mathrm{M}}_k + \epsilon(\mathrm{m} - \mathrm{m}) + \epsilon(\mathrm{b} - \bar{\mathrm{b}})]. \end{aligned} \tag{3.49}$$

The invariance condition $\mathrm{U}_{k+1} = \mathrm{U}_k$ leads, if terms of order ϵ^2 are neglected, to

$$\alpha(\mathrm{M} - \bar{\mathrm{M}}) + a\bar{\mathrm{M}}(\mathrm{m} - \bar{\mathrm{m}})\, \mathrm{M} + \bar{\mathrm{M}}(\mathrm{b} - \bar{\mathrm{b}})\, \mathrm{M} - (\mathrm{b} - \bar{\mathrm{b}}) = 0, \tag{3.50}$$

where we have left out the subscript k.

The solution of (3.50) is

$$\frac{\alpha}{a} = \frac{\mathrm{mM} + \bar{\mathrm{M}}\bar{\mathrm{m}}}{2 - \mathrm{M}^2 - \bar{\mathrm{M}}^2} = -\frac{\Delta(\sin \mu)}{\sin \mu}, \tag{3.51}$$

and

$$\mathrm{b} = \frac{\bar{\mathrm{M}}(\mathrm{Mm} - \mathrm{mM})}{4 \sin^2 \mu}, \tag{3.52}$$

where we have made use of the fact that both M and $\mathrm{M} + \epsilon\mathrm{m}$ are unimodular and, therefore, $\mathrm{m}\bar{\mathrm{M}} + \mathrm{M}\bar{\mathrm{m}} = 0$.

It follows from (3.51) that the adiabatic invariant is

$$\mathrm{V} = \frac{1}{\sin \mu} \mathrm{S}[\,\mathrm{M} - \bar{\mathrm{M}} + \epsilon(\mathrm{b} - \bar{\mathrm{b}})\,]; \tag{3.53}$$

this equals $2W$ as defined by (3.18) except for the small correction

$$\epsilon \mathrm{S}(\mathrm{b} - \bar{\mathrm{b}})/\sin \mu.$$

Thus, to lowest order in ϵ,

$$U = \gamma y^2 + 2\alpha y y' + \beta y'^2 + \frac{\epsilon}{2 \sin \mu} [\, Y, \mathrm{S}(\mathrm{b} - \bar{\mathrm{b}})\ Y] \tag{3.54}$$

is an adiabatic invariant as the focusing fields change slowly, and the amplitude $Y_{\max}$ varies approximately as ${\beta_{\max}}^{1/2}$. Combining this result with the amplitude variation proportional to $p^{-1/2}$ arising from acceleration, we have the result: As the energy and the focusing field change, the amplitude of oscillation varies as

$$(\beta_{\max}/p)^{1/2}.$$

4. EFFECTS OF MAGNET IMPERFECTIONS

In an actual magnet the fields will differ somewhat from the ideal design. Therefore a particle which originally starts out on the ideal equilibrium orbit will, in general, not stay exactly on that orbit but will deviate from it. The magnet is still usable for an accelerator provided the following requirements are met at all times during the acceleration cycle:

(1) There exists a closed orbit, the "displaced equilibrium orbit," which the particle can follow, and which is located well within the aperture of the machine.

(2) Oscillations about this displaced equilibrium orbit are stable.

(a) *Displacement of Equilibrium Orbits*

Let y be the displacement—horizontal or vertical—from the ideal equilibrium orbit. Then the equation of motion of the particle is of the form

$$\frac{d^2y}{ds^2} + K(s)\, y = F(s), \tag{4.1}$$

where we have neglected nonlinear terms, and where $F(s)$ is a measure of the deviation of the field on the ideal orbit from its ideal value:

$$F(s) = \frac{\Delta B}{B\rho}. \tag{4.2}$$

Here $B\rho$ is the magnetic rigidity of the particle, and $\Delta B = B_r$ for vertical oscillations and $\Delta B = B_z - B_0$ for radial oscillations, both measured on the ideal orbit.

The inhomogeneous equation (4.1) may be solved in terms of the solutions of the homogeneous equation (2.5), which are

$$\beta^{1/2} \cos\left(\int \frac{ds}{\beta} + \delta\right) = \beta^{1/2} \cos(\nu\phi + \delta). \tag{4.3}$$

We assume that the homogeneous solution is known, i.e., that we know the function $\beta(s)$.

We now introduce the new variables

$$\eta = \beta^{-1/2} y, \tag{4.4}$$

$$\phi = \int \frac{ds}{\nu\beta}. \tag{4.5}$$

Using the relations (3.12) to (3.14), the differential equation transforms to

$$\frac{d^2\eta}{d\phi^2} + \nu^2\eta = \nu^2\beta^{3/2}F(s). \tag{4.6}$$

The forcing term on the right hand side can be regarded as a function of the new independent variable ϕ, periodic with period 2π in ϕ corresponding to the period C in s. We have thus reduced the problem of the forced oscillations of Hill's equation to the forced oscillations of a harmonic oscillator.

The periodic solution of (4.6) is

$$\eta(\phi) = \frac{\nu}{2 \sin \pi\nu} \int_{\phi}^{\phi + 2\pi} f(\psi) \cos \nu(\pi + \phi - \psi)\, d\psi, \tag{4.7}$$

where $f(\psi) = \beta^{3/2}F(s)$. Thus the displacement of the closed orbit becomes infinite (in the linear approximation) when ν is integral, i.e., when the perturbing force (which is necessarily periodic with the period of the circumference) is in resonance with the free betatron oscillations. Since small field deviations are unavoidable, the magnet must be designed so that ν is not integral for either mode.

Corresponding to the invariant W of Section 3 we have the quantity

$$V(\phi) = \eta^2 + (\eta/\nu)^2 = \gamma y^2 + 2\alpha y y' + \beta y'^2. \tag{4.8}$$

From (4.7) we see that

$$V(\phi) = \frac{\nu^2}{4 \sin^2 \pi\nu} \int_{\phi}^{\phi+2\pi} \int_{\phi}^{\phi+2\pi} f(\psi)\, f(\chi) \cos \nu(\psi - \chi)\, d\psi\, d\chi. \tag{4.9}$$

Another useful formulation of the problem is in terms of Fourier components. Let

$$f(\phi) = \beta^{3/2} F(s) = \sum_k f_k e^{ik\phi}, \tag{4.10}$$

with

$$f_k = \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, e^{-ik\phi}\, d\phi = \frac{1}{2\pi\nu} \int_0^C \beta^{1/2} F(s)\, e^{-ik\phi}\, ds. \tag{4.11}$$

Then the periodic solution of (4.6) is

$$\eta = \sum \frac{\nu^2 f_k}{\nu^2 - k^2}\, e^{ik\phi}. \tag{4.12}$$

This formulation clearly exhibits the resonance properties of the solution. We see that the orbit is most sensitive to those Fourier components of the perturbation whose order is close to the free oscillation frequency ν, and that the Fourier components must be taken with respect to the phase variable ϕ rather than the geometrical variable s.

In practice we usually do not know the perturbing function $F(s)$ in detail, but know some of its statistical characteristics. In that case we can make statistical assertions about the equilibrium orbit. Consider, for example, a machine made of M magnets, with a field error ΔB_r at the position of the ideal equilibrium orbit in the rth magnet (assumed constant throughout that magnet). Let us define

$$\Delta y_r = \frac{F_r}{K_r} = \frac{1}{B\rho} \frac{\Delta B_r}{K_r}, \tag{4.13}$$

where we als assume that $K(s) = K_r$ is constant in the magnet; furthermore we assume that all magnets have the same absolute value of K. The significance of Δy_r is that it is that displacement from the ideal position of the magnet which will just cause a field error ΔB_r, regardless of whether the error is actually caused by a magnet displacement or by an error in the intrinsic characteristics of the magnet. Assume further that the errors in different magnets are uncorrelated, and that the mean square error is

$$\langle \Delta y^2 \rangle_{\mathrm{Av}} = \delta^2. \tag{4.14}$$

We now consider the expectation value of the amplitude of the displaced equilibrium orbit in an ensemble of machines having errors as described. The quantity V defined by (4.9) is a convenient measure of the square of the amplitude. Its expectation value is

$$\langle V(\phi)\rangle_{\mathrm{Av}} = \frac{\nu^2}{4\sin^2\pi\nu}\int_{\phi}^{\phi+2\pi}\int_{\phi}^{\phi+2\pi}\langle f(\psi)\,f(\chi)\rangle\cos\nu(\psi-\chi)\,d\psi\,d\chi. \quad (4.15)$$

Since, by hypothesis, errors in different magnets are uncorrelated, $\langle f(\psi)\,f(\chi)\rangle = 0$ unless ψ and χ lie within the same magnet. Let us assume that the length of the individual magnets is small compared to the wave length of betatron oscillations, i.e., small compared to β. Then the factor $\cos\nu(\psi-\chi)$ can be replaced by unity when ψ and χ are in the same magnet, f can be replaced by its value at the center of the magnet, namely $\beta^{3/2}K\Delta y$, and the interval of ϕ corresponding to the magnet is $L/\nu\beta$, where L is the lenth of the magnet. Thus the contribution of the rth magnet to the double integral in (4.15) is

$$\frac{K^2L^2\beta_r(\Delta y_r)^2}{\nu^2}. \quad (4.16)$$

The fluctuations in Δy from magnet to magnet are assumed uncorrelated; hence we may averages β_r and $\Delta y_r{}^2$ separately in averaging (4.16). The mean value of β_r is very nearly R/ν [see Eq. (3.24)]. The length of a magnet is $2\pi\rho/M$, where ρ is the radius of curvature in the magnets (ρ is less than R if there are field-free sections between the magnets). Furthermore, $K = \pm n/\rho^2$. Therefore

$$\langle V\rangle_{\mathrm{Av}} = \frac{\pi^2}{\sin^2\pi\nu}\frac{n^2R}{M\nu\rho^2}\delta^2. \quad (4.17)$$

The amplitude of oscillations is given by $Y = (\beta V)^{1/2}$. Again replacing β by R/ν, we find for the mean square amplitude

$$\langle Y^2\rangle_{\mathrm{Av}} = \frac{\pi^2}{\sin^2\pi\nu}\frac{R^2}{\rho^2}\frac{n^2}{\nu^2M}\delta^2. \quad (4.18)$$

For the design of an accelerator it is desirable to have an estimate, not just of the mean square amplitude, but of the largest amplitude that can reasonably be expected. It has been shown by Lüders [9] that the higher moments of the distribution of Y^2 satisfy

$$\langle Y^{2k}\rangle_{\mathrm{Av}} = k!\,[\langle Y^2\rangle]^k \quad (4.19)$$

for $k \ll M$. It follows that Y^2 has an exponential distribution (corresponding to a Rayleigh distribution in the amplitude Y). Thus the probability that Y^2 exceeds

four times the mean value (4.18) is about $e^{-4} = 0.02$. It is thus safe to assume that, with 98% probability, the displacement of the closed orbit will be less than

$$P = 2 \frac{\pi}{|\sin \pi\nu|} \frac{R}{\rho} \frac{|n|}{\nu M^{1/2}} F^{1/2} \tag{4.20}$$

times the root mean square equivalent displacement of the individual magnet, where F is the form factor defined by (3.25).

For the 30-Bev machine now under construction at Brookhaven, we have $R = 421$ ft, $\rho = 280$ ft, $|n| = 360$, $M = 240$, $= 8.75$, $F^{1/2} = 1.25$, so that the multiplication factor P equals 36. Thus if the errors in placement of the individual magnets are random and uncorrelated with rms displacement of, say, 0.02 inch, the resulting equilibrium orbit is unlikely to deviate by more than 0.72 inch.

If we consider machines with different values of n and M but similar configurations of the unit period, we note that the phase shift μ per period depends only on n/N^2, and that $\nu = N\mu/2\pi$ [Eq. (3.17)]. Let us assume that the parameters are varied so as to keep n/N^2 constant, and also so as to keep M/N (the number of magnets per unit cell) and ρ/R (the fraction of the circumference occupied by magnets) constant. Also adjust n in each case so that $\sin \pi\nu$ has the same value. Then n varies as N^2, ν varies as N, M varies as N, and F is constant; thus the factor P increases proportional to $N^{1/2}$ or $n^{1/4}$. This is in contrast with the amplitude factor for free oscillations which, as we saw in Section 3, decreases with increasing n.

This leads to a fundamental limitation on the strength of focusing that is practicable. The parameters of a machine have to be chosen so as to strike a compromise between the decreased aperture requirements for free betatron oscillations and the increased orbit deviations arising from errors in magnet placement. The very large values of n and ν proposed in our original paper [2] appeared feasible only because at the time of writing that paper we were not sufficiently aware of the importance of the effects of magnet errors.

(b) *Errors in Field Gradients*

The distribution of field gradients along the equilibrium orbit may also deviate from the ideal—due to variations in length of the individual magnet sectors, in magnet gap dimensions, in iron properties, etc. As a result the periodicity condition

$$K(s + C/N) = K(s) \tag{2.7}$$

is not exactly satisfied. However, the weaker periodicity relation

$$K(s + C) = K(s) \tag{2.6}$$

remains valid. The stability and amplitude considerations of sections 2 and 3 still apply, but the unit cell that must be considered is the whole revolution rather than the Nth part of it, as in the ideal machine.

The matrix for the transformation about one revolution is still of the form (2.23). It may be written as the product of the individual—no longer quite identical—unit cell matrices

$$\mathrm{M} = \prod_{i=1}^{N} \mathrm{M}_i, \tag{4.21}$$

where

$$\mathrm{M}_i = \mathrm{I} \cos \mu_i + \mathrm{J}_i \sin \mu_i \tag{4.22}$$

is the matrix for the ith unit cell.

The stability condition is again

$$|\mathrm{Tr}\ \mathrm{M}| \leqslant 2. \tag{4.23}$$

In a perfect machine, $\mathrm{Tr}\ \mathrm{M} = 2\cos(N\mu)$, where μ is the phase shift for one unit cell. If the imperfections are small, the matrix for the actual machine will differ only slightly from that for the perfect case; therefore the perturbations can cause (4.23) to be violated only if $\cos(N\mu)$ is near $+1$ or -1, that is, if $\nu = N\mu/2\pi$ is near an integral or half-integral value. Integral values of ν, as we have seen, already lead to difficulties because of large deviations of the equilibrium orbit; thus the main practical effect of gradient errors is to introduce instability in the vicinity of half-integral values of ν.

Another effect of the gradient errors will be to alter the amplitude function $\beta(s)$, and therefore the form factor F defined by (3.25) and the admittance of the system, even when stability is preserved.

To investigate these effects quantitatively, we write the equation of oscillation about the actual equilibrium orbit in the form

$$\frac{d^2y}{ds^2} + [K_0(s) + k(s)]\, y = 0, \tag{4.24}$$

where $K_0(s)$ is the focusing function for a perfect machine [satisfying the strong periodicity relation (2.7)], and $k(s)$ [satisfying only the weak periodicity relation (2.6)] is the perturbation. Suppose the solution for the perfect case ($k = 0$) is known and that the matrix for one complete revolution in the perfect case is

$$\mathrm{M}_0(s) = \mathrm{I} \cos \mu_0 + \mathrm{J}(s) \sin \mu_0; \qquad \mathrm{J}(s) = \begin{bmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{bmatrix}. \tag{4.25}$$

Consider now a short interval of length ds_1 near s_1. Its contribution to the matrix M_0 is, in the limit ${K_0}^{1/2}\, ds_1 \ll 1$,

$$\mathrm{m}_0 = \begin{bmatrix} 1 & ds_1 \\ -K_0(s_1)\, ds_1 & 1 \end{bmatrix}, \tag{4.26}$$

while its contribution to the actual matrix is

$$\mathrm{m} = \begin{bmatrix} 1 & ds_1 \\ -[K_0(s_1) + k(s_1)]\, ds_1 & 1 \end{bmatrix}. \tag{4.27}$$

If k were different from zero only in the interval ds_1, the actual matrix would be obtained from $\mathrm{M}_0(s)$ by replacing the contribution (4.26) by (4.27), that is, by multiplying $\mathrm{M}_0(s_1)$ on the left by

$$\mathrm{m}\mathrm{m}_0{}^{-1} = \begin{bmatrix} 1 & 0 \\ -k(s_1)\, ds_1 & 1 \end{bmatrix}. \tag{4.26}$$

Carrying out the multiplication, we find

$$\mathrm{Tr}\ \mathrm{M} = 2\cos\mu = 2\cos\mu_0 - (\beta \sin\mu_0)\, k(s_1)\, ds_1. \tag{4.29}$$

Thus the error k in the interval ds_1 contributes $-\beta(s_1)\, k(s_1) \sin\mu_0 \cdot ds_1$ to the trace of the matrix. Adding the contributions from errors $k(s)$ over the whole circumference we obtain

$$\Delta(\cos\mu) = -\frac{\sin\mu_0}{2} \int_0^C \beta(s)\, k(s)\, ds, \tag{4.30}$$

and the frequency shift is

$$\Delta\nu = \frac{\Delta\mu}{2\pi} = -\frac{\Delta(\cos\mu)}{2\pi \sin\mu_0} = \frac{1}{4\pi} \int_0^C \beta(s)\, k(s)\, ds. \tag{4.31}$$

The expressions (4.30) and (4.31) are only first approximations, since terms of second and higher orders in k are neglected. When $\sin\mu_0$ is near zero the second order approximation must be considered. If there is an error $k(s_1)\, ds_1$ near s_1 and $k(s_2)\, ds_2$ near s_2, the matrix at s_1 is

$$\mathrm{M}(s_1) = \begin{bmatrix} 1 & 0 \\ -k(s_1)\, ds_1 & 1 \end{bmatrix} \mathrm{B} \begin{bmatrix} 1 & 0 \\ -k(s_2)\, ds_2 & 1 \end{bmatrix} \mathrm{A}, \tag{4.32}$$

where A is the matrix of the unperturbed system from s_1 to s_2, and B the matrix from s_2 to $s_1 + C$. We may write

$$\begin{aligned} \mathrm{M}(s_1) = \mathrm{M}_0(s_1) &- (\mathrm{RBA})\, k(s_1)\, ds_1 \\ &- (\mathrm{BRA})\, k(s_2)\, ds_2 + (\mathrm{RBRA})\, k(s_1)\, k(s_2)\, ds_1\, ds_2, \end{aligned} \tag{4.33}$$

where

$$R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{4.34}$$

The trace of $M(s_1)$ is

$$2\cos\mu = 2\cos\mu_0 - (k_1\beta_1\, ds_1 + k_2\beta_2\, ds_2)\sin\mu_0 + B_{12}A_{12}k_1k_2\, ds_1\, ds_2, \tag{4.35}$$

where we have written k_1 for $k(s_1)$, etc.

From (3.5) we obtain

$$\begin{aligned} A_{12} &= (\beta_1\beta_2)^{1/2}\sin\nu(\phi_2-\phi_1), \\ B_{12} &= (\beta_1\beta_2)^{1/2}\sin\nu(2\pi-\phi_2+\phi_1) = (\beta_1\beta_2)^{1/2}\sin[\mu_0-\nu(\phi_2-\phi_1)]. \end{aligned} \tag{4.36}$$

Substituting in (4.35) and integrating, we find

$$\begin{aligned} \cos\mu - \cos\mu_0 = &-\frac{\sin\mu_0}{2}\int_0^C k(s)\,\beta(s)\,ds \\ &+\frac{1}{2}\int_0^C ds_1 \int_{s_1}^C ds_2\, k_1k_2\beta_1\beta_2 \sin\nu(\phi_2-\phi_1)\sin[\mu_0-\nu(\phi_2-\phi_1)]. \end{aligned} \tag{4.37}$$

We are now in a position to find the width of a *stopband*, i.e., the width of that range of $\nu = \mu_0/2\pi$ over which $|\cos\mu| > 1$. Consider the case where ν is nearly integer,

$$\nu = p + \epsilon,$$

with p an integer and ϵ small. Then, to second order in ϵ,

$$\sin\mu_0 = 2\pi\epsilon; \qquad \cos\mu_0 = 1 - 2\pi^2\epsilon^2.$$

We neglect terms of higher than the second order in ϵ and k combined. Therefore ν in the arguments of the sines in the double integral in (5.37) may be replaced by p, μ_0 by $2\pi p$, and the integral over the triangular region

$$\int_0^C ds_1 \int_{s_1}^C ds_2$$

equals half the integral over the square region

$$\int_0^C ds_1 \int_0^C ds_2.$$

Writing the sine functions in terms of complex exponentials we obtain, after some manipulation,

$$\cos\mu - 1 = -2\pi^2\epsilon^2 - \pi\epsilon J_0 + \tfrac{1}{8}(|J_{2p}|^2 - {J_0}^2), \tag{4.38}$$

where, for any n,

$$J_n = \int_0^C \beta(s)\, k(s)\, e^{-in\phi(s)}\, ds. \tag{4.39}$$

Solving for $\cos\mu - 1 = 0$, we find

$$\epsilon = -\frac{J_0 \pm |J_{2p}|}{4\pi} \tag{4.40}$$

so that the width of the stopband is

$$\delta\nu = \frac{|J_{2p}|}{2\pi} = \frac{1}{2\pi}\left| \int_0^C \beta(s)\, k(s)\, e^{2ip\phi(s)}\, ds \right|. \tag{4.41}$$

The derivation for half-integral stopbands ($\nu = p + \frac{1}{2}$) is exactly analogous with the result

$$\delta\nu = \frac{|J_{2p+1}|}{2\pi}. \tag{4.42}$$

We may express these integrals in terms of the phase angle ϕ as the independent variable, with

$$ds = \nu\beta\, d\phi.$$

Thus

$$J_n = \int_0^{2\pi} \nu\beta^2 k e^{-in\phi}\, d\phi \tag{4.43}$$

is $2\pi\nu$ times the nth exponential Fourier component of $\beta^2 k$, and the results of this section are

shift of frequency [Eq. (4.31)]

$$\Delta\nu = \tfrac{1}{2}\nu(\beta^2 k)_0; \tag{4.44}$$

width of stopband

$$\delta\nu = \nu|(\beta^2 k)_{2\nu}|. \tag{4.45}$$

Beat Factors. The amplitude function $\beta(s)$ will also be modified by gradient errors, and its maximum value, which determines the admittance of the system and also the expected peak deviations of the closed orbit due to field errors, will in general be larger in an imperfect machine than in a perfect one. Because the amplitudes of both free and forced oscillations, and hence aperture requirements, are proportional to $\beta^{1/2}$, it is desirable that the actual maximum value of β exceed the ideal maximum by as small a factor as possible.

We define the "beat factor"

$$G = [\,\beta(\text{actual})/\beta(\text{ideal})\,]_{\max} \tag{4.46}$$

for a machine in which the actual function $\beta(s)$ differs from the ideal one because of gradient errors.

To obtain the change in β at some particular azimuth s_1 we again consider first the contribution from the error $k(s_2)$ in an interval ds_2 at s_2. This is obtained from the matrix (4.33) with $k_1 = 0$:

$$\begin{aligned} \mathrm{M}(s_1) &= \mathrm{M}_0(s_1) = \mathrm{BRA}k(s_2)\,ds_2 \\ &= \mathrm{M}_0(s_1) - \begin{bmatrix} B_{12}A_{11} & B_{12}A_{12} \\ B_{22}A_{11} & B_{22}A_{12} \end{bmatrix} k(s_2)\,ds_2. \end{aligned} \tag{4.47}$$

The (12) element of $\mathrm{M}(s_1)$, which is what determines β, is, by (4.36),

$$\begin{aligned} M_{12}(s_1) &= M_{12}{}^0 - B_{12}A_{12}k(s_2)\,ds_2 \\ &= \beta_1 \sin\mu_0 - \beta_1\beta_2 \sin \nu(\phi_2-\phi_1)\sin[\,\mu_0 - \nu(\phi_2-\phi_1)\,]\,k(s_2)\,ds_2. \end{aligned} \tag{4.48}$$

Integrating over s_2, we obtain

$$\Delta M_{12}(s_1) = -\beta_1 \int_{s_1}^{s_1+C} ds_2\, k_2\beta_2 \sin \nu(\phi_2-\phi_1)\sin[\,\mu_0 - \nu(\phi_2-\phi_1)\,].$$

But

$$\Delta M_{12} = \Delta(\beta\sin\mu) = \Delta\beta\sin\mu_0 + \beta_1\cos\mu_0\cdot\Delta\mu.$$

Using (4.31) we solve for $\Delta\beta$, obtaining

$$\Delta\beta = \frac{\beta_1}{2\sin\mu_0}\int_{s_1}^{s_1+C} k(s_2)\,\beta(s_2)\cos 2\nu(\pi+\phi_2-\phi_1)\,ds_2 \tag{4.49}$$

$$= \frac{\nu\beta_1}{2\sin\mu_0}\int_{\phi_1}^{\phi_1+2\pi} k\beta^2\cos 2\nu(\pi+\phi-\phi_1)\,d\phi, \tag{4.50}$$

where we have changed to the phase $\phi = \int ds/\nu\beta$ as the independent variable. From (4.50) it is easily verified that the fractional change in β satisfies the differential equation

$$\frac{d^2}{d\phi^2}\frac{\Delta\beta}{\beta} + 4\nu^2\frac{\Delta\beta}{\beta} = -2\nu^2\beta^2 k(s), \tag{4.51}$$

which is very similar in form to the equation (4.6) satisfied by the displaced equilibrium orbit, but with 2ν, rather than ν as the frequency of free oscillations. We may again solve (4.51) by Fourier analysis: we have

$$\beta^2 k(s) = \frac{1}{2\pi\nu} \sum_{p=-\infty}^{\infty} J_p e^{ip\phi}, \tag{4.52}$$

where J_p is just the integral given by (4.30). Then the periodic solution of (4.51) is

$$\frac{\Delta\beta}{\beta} = -\frac{\nu}{4\pi} \sum_p \frac{J_p e^{ip\phi}}{\nu^2 - (p/2)^2}, \tag{4.53}$$

which is, of course, equivalent to (4.50). The form (4.53) shows that the amplitude function is most sensitive to those harmonic components of the error in $K(s)$, or rather in $\beta^2 K(s)$, whose order is nearest to 2ν, and that it becomes infinite when 2ν approaches an integer. If we consider only the leading term in (4.53), i.e., that value of p which is closest to 2ν, we have approximately

$$\frac{\Delta\beta}{\beta} = -\frac{|J_p| \cos(p\phi + \delta)}{4\pi(\nu - p/2)}, \tag{4.54}$$

where δ is a phase angle. Since the width of the stopband at $\nu = p/2$ is just $|J_p|/2\pi$ [Eq. (4.41)], we see that the beat factor produced by the errors is related to the width of the nearest stopband produced by them,

$$G = 1 + (\Delta\beta/\beta)_{\max} = 1 + \frac{(\delta\nu)_p}{2(\nu - p/2)}, \tag{4.55}$$

where $(\delta\nu)_p$ is the width of the stopband at $p/2$, and $\nu - p/2$ is the distance from the stopband.

Effect of Random Errors. As an example, we again consider the form the foregoing effects take when the errors in $n(s)$ or $K(s)$ are randomly distributed in M magnets, with no correlation between the errors in different magnets. We have the following root mean square ensemble averages for all orders p small compared to M,

$$\begin{aligned} \frac{1}{2\pi} \langle |J_p| \rangle_{\text{rms}} &= \frac{1}{2\pi} \left\langle \left| \oint k\beta e^{-ip\phi}\, ds \right| \right\rangle_{\text{rms}} \\ &= M^{1/2} \frac{R\rho}{\nu M} k_{\text{rms}} = \frac{R\,|n|}{\rho\nu M^{1/2}} \left\langle \frac{\Delta n}{n} \right\rangle_{\text{rms}}, \end{aligned} \tag{4.56}$$

where we have, in the averaging process, replaced β by R/ν and noted that the integration is over M intervals of average lenth $2\pi\rho/M$. For the machine now under

construction at Brookhaven ($R = 421$ ft, $\rho = 280$ ft, $n = 360$, $M = 240$, $\nu = 8.75$) we obtain

$$\frac{1}{2\pi}\langle J_p \rangle_{\text{rms}} = 4.0 \langle \Delta n \rangle_{\text{rms}}.$$

The stopband width is just this, while the shift in ν is $J_0/4\pi$. The beat factor, for $\nu = 8.75$, aries aqually from the stopbands at $\frac{17}{2}$ and $\frac{18}{2}$, and is

$$G = 1 + 4.0 \left\langle \frac{\Delta n}{n} \right\rangle_{\text{rms}}.$$

Thus if the variation in n from magnet to magnet were 1 percent (rms), we would, on the average, expect ν_x and ν_z to differ from the design values by 0.02 unit, have stopbands of width 0.04 unit, and have a beat factor of 1.04. Any particular machine might, of course, be worse than this, though it would be unlikely to be worse by more than factor of 2.

(c) *Coupling between Horizontal and Vertical Oscillations*

Up to this point we have assumed that horizontal and vertical deviations from the equilibrium orbit would be treated separately. This is the case when the magnetic field possesses a "median plane," i.e., a plane on which the field is everywhere perpendicular to the plane. However, the field of an actual magnet will deviate slightly from this condition; consequently the equilibrium orbit need not be a plane curve, and the two types of oscillations may be coupled.

In this case it is convenient to make use of the Hamiltonian formulation of the equations of motion. It is shown in the Appendix that the equations of motion, with the arc length s along the equilibrium orbit as the independent variable, can be derived from a Hamiltonian function

$$G(x, z, p_x, p_z, s), \tag{4.57}$$

which is periodic in s. Here x and z are the components of the displacement from the equilibrium orbit parallel and perpendicular to the oscillating plane, and p_x and p_z are canonically conjugate momenta. The equations of motion are

$$\begin{aligned} x' &= \frac{\partial G}{\partial p_x}, \qquad p_x' = -\frac{\partial G}{\partial x}, \\ z' &= \frac{\partial G}{\partial p_z}, \qquad p_z' = -\frac{\partial G}{\partial z}, \end{aligned} \tag{4.58}$$

where a prime, as usual, denotes differentiation by s.

To obtain linear equations of motion we expand G as a power series in x, p_x, z, p_z and neglect all terms higher than the second order. Since $x = z = 0$ is the equilibrium orbit, there are no first-order terms. Thus G is a homogeneous quadratic

function of x, p_x, z, p_z. Because of Maxwell's equations it may be written in the form

$$G = \tfrac{1}{2}[K_1 x^2 + K_2 z^2 + 2Mxz + (p_x - Qz)^2 + (p_z + Qx)^2]. \tag{4.59}$$

In the particular case where the equilibrium orbit still lies in a plane, these coefficients have the following physical significance

$$K_1 = \frac{(1-n)}{\rho^2}, \qquad K_2 = \frac{n}{\rho^2}$$

as in the uncoupled case; M/K_2 is the slope of the surface on which the field has zero radial component, and Q is proportional to the longitudinal or solenoidal component of the magnetic field on the equilibrium orbit. If the equilibrium orbit is not plane, these interpretations must be modified slightly, but the Hamiltonian still has the form (4.59).

When G is a quadratic form (regardless of whether it is of the particular form (4.59), or not), the equations of motion (4.58) are linear and therefore define a linear canonical transformation of (x, p_x, z, p_z) phase space at $s = s_1$ into phase space at $s = s_2$. Such a transformation can, of course, be represented by a (4×4) matrix M,

$$X(s_2) = \mathrm{M}(s_2 \mid s_1)\, X(s_1), \tag{4.60}$$

where $X(s)$ stands for the vector

$$\begin{bmatrix} x(s) \\ p_x(s) \\ z(s) \\ p_z(s) \end{bmatrix}.$$

The Hamiltonian quadratic form G can also be represented by a—symmetrical—4×4 matrix: in vector notation

$$G = \tfrac{1}{2}[X, \mathrm{G}X]. \tag{4.61}$$

The equations of motion (4.58) then become, in matrix notation,

$$X' = \mathrm{SG}\, X, \tag{4.62}$$

where

$$\mathrm{S} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{4.63}$$

It follows that, for any two solutions X_1 and X_2

$$\frac{d}{ds}[X_2, \mathrm{S}X_1] = [X_2', \mathrm{S}X_1] + [X_2, \mathrm{S}X_1'] = 0, \tag{4.64}$$

that is, the bilinear form

$$[X_2, \mathrm{S}X_1] = x_1 p_{x2} - x_2 p_{x1} + z_1 p_{z2} - z_2 p_{z1}$$

is invariant. Therefore, this form has the same value at $s = s_1$ and at $s = s_2$. But, by (4.60),

$$\begin{aligned}[X_2(s_2), \mathrm{S}X_1(s_2)] &= [\mathrm{M}X_2(_1), \mathrm{SM}X_1(s_1)] \\ &= [X_2(s_1)\, \tilde{\mathrm{M}}\mathrm{SM}X_1(s_1)] = [X_2(s_1), \mathrm{S}X_1(s_1)],\end{aligned} \tag{4.65}$$

where $\tilde{\mathrm{M}}$ is the matrix obtained from M by transposing rows and columns. Since this relation is satisfied for any two solutions X_1 and X_2 the matrix M must satisfy

$$\tilde{\mathrm{M}}\mathrm{SM} = \mathrm{S}. \tag{4.66}$$

This relation is due to Poincaré [10], who proved it for the matrix of partial derivatives $\partial x_i(s_2)/\partial x_k(s_1)$ for any canonical system, linear or not. Evidently, the theorem holds for systems of any number k of dimensions, provided S is the $(2k \times 2k)$ matrix obtained by writing

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

along the diagonal k times, and zero elsewhere.

Matrices M satisfying (4.66) are said to be *sympletic*. From the symplectic property of the transformation matrix we deduce that for each eigenvalue λ of the matrix M, its reciprocal must also be an eigenvalue. For if we take for X_1 and X_2 the eigensolutions X_i and X_k of

$$\mathrm{M}X = \lambda X,$$

corresponding to the eigenvalues λ_i and λ_k, we have, from (4.65)

$$(\lambda_i \lambda_k - 1)[X_i(s_1), \mathrm{S}X_k(s_1)] = 0. \tag{4.67}$$

Given the eigenvector X_i, (4.67) must hold for all eigenvectors X_k. But the bilinear form $[X_i, \mathrm{S}X_k]$ cannot vanish for all eigenvectors X_k, since any set of initial values is a linear combination of the four eigenvectors. Therefore at least one of the eigenvalues is the reciprocal of λ_i. Therefore, the eigenvalues may be arranged in reciprocal pairs.

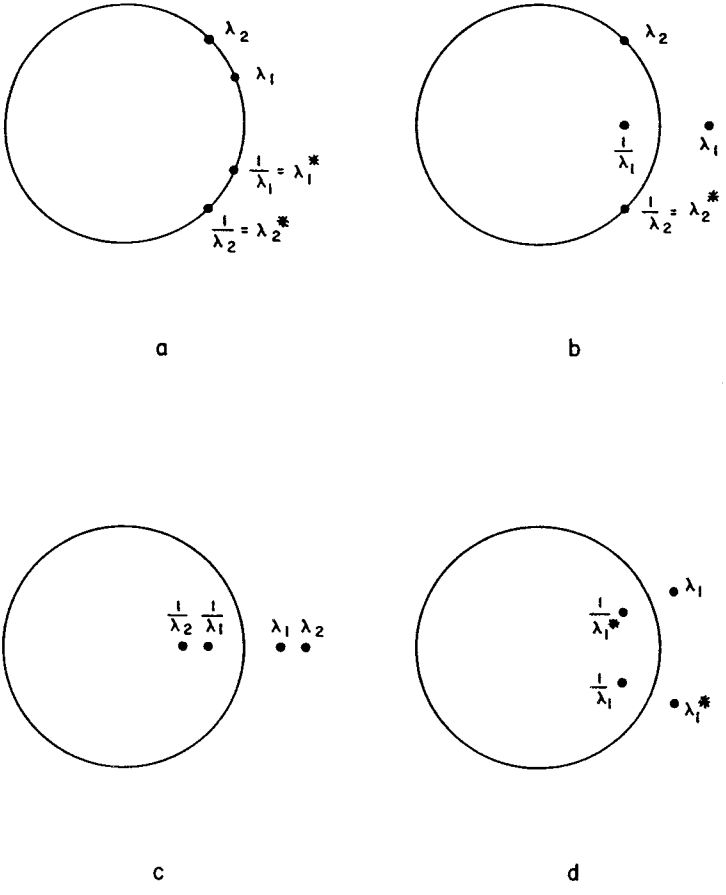

FIG. 2. Location of eigenvalues for two-dimensional linear oscillations. (a) Both modes stable. (b) One mode stable, one mode unstable. (c) Both modes unstable in absence of coupling. (d) Instability induced by coupling.

It follows that the product of the eigenvalues, and therefore the determinant of the matrix M, is equal to unity (Liouville's theorem). However, the symplectic condition is—for problems involving two or more degrees of freedom—a much stronger restriction than just the conservation of volume in phase space. Liouville's theorem imposes just *one* constraint on the $(2k)^4$ elements of M (where k is the dimensionality of the problem), while the symplectic condition imposes $k(2k-1)$ constraints [11]. Thus Liouville's theorem is equivalent to the symplectic condition only for one-dimensional systems.

Since M is real, the complex conjugate of an eigenvalue is also an eigenvalue. We thus have the following possibilities, assuming the eigenvalues to be distinct (Fig. 2):

[4] Other proofs of this fact have been given by Lüders [12], Sturrock [13], and Seiden [14].

(a) All four eigenvalues lie on the unit circle, forming two complex conjugate and reciprocal pairs.

(b) One reciprocal pair is real, the others are complex and on the unit circle.

(c) Two real reciprocal pairs.

(d) One eigenvalue, say λ_1, complex and not on the unit circle; the other eigenvalues must then be $\lambda_2 = 1/\lambda$, $\lambda_3 = \lambda_1^*$, $\lambda_4 = 1/\lambda_1^*$.

The motion is unstable if any eigenvalue is greater than 1 in absolute value. Thus only situation (a) is stable. Cases (a), (b), and (c) correspond to the uncoupled case with both modes stable, one mode stable and one unstable, and both modes unstable, respectively. Case (d) does not arise without coupling, and represents a type of instability that is generated only by the coupling.

We now ask: If the uncoupled motion is stable [case (a)], under what circurstances can the coupling lead to instability?

We assume that the coupling is weak. Then the matrix with coupling differs only slightly from the unperturbed matrix, and the same is true of the eigevalues. We exclude the case where the uncoupled system is near a resonance of the type we already know to be harmful (ν integral or half-integral, eigenvalues $= \pm 1$). Then a small change in the eigenvalues cannot lead to situation (b) or (c), and it can lead to (d) only if the eigenvalues for the uncoupled modes are nearly equal, i.e., if, approximately,

$$\cos \mu_x = \cos \mu_z. \tag{4.68}$$

Equation (4.68) means that either

$$\nu_x + \nu_z = \text{integer} \tag{4.69}$$

or

$$\nu_x - \nu_z = \text{integer}. \tag{4.70}$$

We shall now show that instability of type (d) cannot arise in case (4.70), but will in general arise in case (4.69)[2].

We look for quadratic forms in the variables x, p_x, z, p_z which are invariant under the transformation (4.60), analogous to W [Eq. (3.21)] in the one-dimensional case. Such a quadratic form is given by a symmetrical matrix U which must satisfy

$$(X, \mathrm{U}X) = (\mathrm{M}X, \mathrm{UM}X) = (X, \tilde{\mathrm{M}}\mathrm{UM}X)$$

or

$$\tilde{\mathrm{M}}\mathrm{UM} = \mathrm{U} \tag{4.71}$$

Using (4.66) we find that the matrix SU must commute with M. Therefore, the symmetric solutions of (4.71) are

$$\mathrm{U}_k = \mathrm{S}\mathrm{M}^k - \tilde{\mathrm{M}}^k\mathrm{S}, \tag{4.72}$$

where k is any integer; in general there are no other solutions except linear combinations of the matrices U_k.

Now consider the uncoupled case, where

$$\mathrm{M} = \begin{pmatrix} & & 0 & 0 \\ & \mathrm{A} & & \\ & & 0 & 0 \\ 0 & 0 & & \\ & & & \mathrm{D} \\ 0 & 0 & & \end{pmatrix}; \tag{4.73}$$

A and D are the matrices of form (2.23) for the x and z motion, respectively. In this case it is easily seen that

$$[X, \mathrm{U}_k X] = 2W_x \sin k\mu_x + 2W_z \sin k\mu_z, \tag{4.74}$$

where W_x and W_z are the invariants (3.18).

The presence of coupling will add small terms to (4.74).

When $\cos\mu_x \neq \cos\mu_z$, the invariance of (4.74) for all k implies that W_x and W_y must be separately invariant. The only effect of coupling is to produce a slight change in the forms W_x and W_z. On the other hand, when $\cos\mu_x = \cos\mu_z$, then either $\sin k\mu_x = \sin 5\mu_z$ for all k [case (4.70)] or $\sin k\mu_x = -\sin k\mu_z$ [case (4.69)]. Then the invariance (4.74) merely means that either

$$W_x + W_y$$

or

$$W_x - W_y$$

is invariant. In the former case, which corresponds to the *difference* $\nu_x - \nu_y$ being integral, the invariant is positive definite, since W_x and W_y are separately positive definite. The addition of small terms arising from the coupling does not alter the positive definite character. Therefore, the motion is bounded, and we have stability.

On the other hand, if $\nu_x + \nu_y$ is integral, the invariant is the difference between two positive definite quantities, and is therefore not definite. In that case, the coupling may (and in general will) induce instability.

To demonstrate that instability does indeed occur, and to estimate how strong it is, we must compute the eigenvalues of the matrix M. We write M in the form

$$\mathrm{M} = \begin{bmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{bmatrix}, \tag{4.75}$$

where A, B, C, D are 2×2 matrices. For any 2×2 matrix A we define its "symplectic conjugate," as in (3.44),

$$\bar{\mathrm{A}} = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tilde{\mathrm{A}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}, \tag{4.76}$$

and for a 4×4 matrix,

$$\bar{\mathrm{M}} = -\mathrm{S}\tilde{\mathrm{M}}\mathrm{S} = \begin{bmatrix} \bar{\mathrm{A}} & \bar{\mathrm{C}} \\ \bar{\mathrm{B}} & \bar{\mathrm{D}} \end{bmatrix}. \tag{4.77}$$

The symplectic condition (4.66) then leads to

$$\mathrm{Det\ A} + \mathrm{Det\ B} = \mathrm{Det\ A} + \mathrm{Det\ C} = \mathrm{Det\ D} + \mathrm{Det\ C} = 1 \tag{4.78}$$

and

$$\mathrm{A}\bar{\mathrm{C}} + \mathrm{B}\bar{\mathrm{D}} = 0. \tag{4.79}$$

Since the eigenvalues of M come in reciprocal pairs, it suffices to find the two quantities

$$2 \cos \mu_1 = \Lambda_1 = \lambda_1 + \frac{1}{\lambda_1} \tag{4.80}$$

and

$$2 \cos \mu_2 = \Lambda_2 = \lambda_2 + \frac{1}{\lambda_2}. \tag{4.81}$$

These are, of course, the eigenvalues of the matrix

$$\mathrm{M} + \mathrm{M}^{-1} = \mathrm{M} + \bar{\mathrm{M}} = \begin{bmatrix} \mathrm{A} + \bar{\mathrm{A}} & \mathrm{B} + \bar{\mathrm{C}} \\ \mathrm{C} + \bar{\mathrm{B}} & \mathrm{D} + \bar{\mathrm{D}} \end{bmatrix}. \tag{4.82}$$

These eigenvalues satisfy the characteristic equation

$$\Lambda^2 - \Lambda(\mathrm{A} + \bar{\mathrm{A}} + \mathrm{D} + \bar{\mathrm{D}}) + (\mathrm{A} + \bar{\mathrm{A}})(\mathrm{D} + \bar{\mathrm{D}}) - (\mathrm{B} + \bar{\mathrm{C}})(\bar{\mathrm{B}} + \mathrm{C}) = 0, \tag{4.83}$$

the solutions of which are

$$2\Lambda = (\mathrm{TrA} + \mathrm{TrD}) \pm [(\mathrm{TrA} - \mathrm{TrD})^2 + 4\ \mathrm{Det}(\mathrm{B} + \bar{\mathrm{C}})]^{1/2}. \tag{4.84}$$

If the expression under the square root sign is negative, Λ will be complex, and we are dealing with an instability of type (d), induced by the coupling. If the coupling is weak, the matrices B and C will be small, and therefore Λ can be complex only if $\mathrm{TrA} - \mathrm{TrD}$ is nearly zero and at the same time $\mathrm{Det}(\mathrm{B} + \bar{\mathrm{C}})$ is negative.

It follows in general from the relation (4.79) that, when $\mathrm{TrA} = \mathrm{TrD}$ and the unperturbed motion is stable ($|\mathrm{TrA}| < 2$), then, regardless of the details of the coupling, $\mathrm{Det}(\mathrm{B} + \bar{\mathrm{C}})$ is positive in the case of a difference resonance ($\nu_x - \nu_z =$ integer) and negative in the case of a sum resonance ($\nu_x + \nu_z =$ integer). To see this, we write B and $\bar{\mathrm{C}}$ in the form

$$\mathrm{B} = \mathrm{ED},\ \bar{\mathrm{C}} = -\bar{\mathrm{A}}\mathrm{E} \tag{4.85}$$

where, from (4.79),

$$\mathrm{E} = \mathrm{B}\bar{\mathrm{D}}/(\mathrm{D}\bar{\mathrm{D}}). \tag{4.86}$$

Thus

$$\mathrm{Det}(\mathrm{B} + \bar{\mathrm{C}}) = \mathrm{Det}(\mathrm{ED} - \bar{\mathrm{A}}\mathrm{E}). \tag{4.87}$$

It may now be verified that, if A and D have the same determinant and the same trace, and if E is any matrix whatsoever,

$$\begin{aligned}
&\mathrm{Det}(\mathrm{D} - \bar{\mathrm{A}})\ \mathrm{Det}(\mathrm{ED} - \bar{\mathrm{A}}\mathrm{E}) \\
&\quad = \mathrm{Det}[(\mathrm{D} - \bar{\mathrm{A}})(\mathrm{ED} - \bar{\mathrm{A}}\mathrm{E})] \\
&\quad = \tfrac{1}{4}\{\mathrm{Tr}[(\mathrm{D} - \bar{\mathrm{A}})(\mathrm{ED} - \bar{\mathrm{A}}\mathrm{E})]\}^2 + \tfrac{1}{16}[\mathrm{Det}(\mathrm{A} - \bar{\mathrm{A}})][\mathrm{Tr}(\mathrm{ED} - \bar{\mathrm{A}}\mathrm{E})]^2.
\end{aligned} \tag{4.88}$$

Now, if the unperturbed problem is stable, $\mathrm{Det}(\mathrm{A} - \bar{\mathrm{A}})$ is positive, equal to $4 \sin^2 \mu$. Thus (4.88) is positive, and therefore $\mathrm{Det}(\mathrm{B} + \bar{\mathrm{C}})$ has the same sign as $\mathrm{Det}(\mathrm{D} - \bar{\mathrm{A}})$. But it is easily seen that, when $\mathrm{TrA} = \mathrm{TrD}$,

$$\mathrm{Det}(\mathrm{D} - \bar{\mathrm{A}}) = \frac{(\beta_x + \beta_z)^2 + (\alpha_z \beta_x + \alpha_x \beta_z)^2}{\beta_x \beta_z} \sin \mu_x \sin \mu_z, \tag{4.89}$$

which is positive at a difference resonance ($\sin \mu_x = \sin \mu_z$) and negative at a sum resonance ($\sin \mu_x = -\sin \mu_z$).

We have thus proved that coupling will induce instability at a sum resonance but not at a difference resonance. The proof is independent of the form of the coupling; it holds whether the coupling is caused by twists of the magnets about the beam axis or by longitudinal magnetic field. The proof is also independent of perturbation theory; it holds just as long as the coupling is not strong enough to alter the signs

of $\mathrm{Det}(\mathrm{A}-\bar{\mathrm{A}})$ and $\mathrm{Det}(\mathrm{D}-\bar{\mathrm{A}})$. In these respects our proof is more general than the proofs given in refs. [12] and [14].

We now turn to a quantitative estimate of the strength of the instability. If a short section of magnet of length ds at s is rotated through a small angle θ, the effect is that the overall matrix at s must be multiplied by

$$\mathrm{I}+[K_z(s)-K_x(s)]\,\theta\,ds\begin{bmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\1&0&0&0\end{bmatrix}, \tag{4.90}$$

where I is the unit matrix and K_z and K_x are the focusing functions at s. Thus the perturbed matrix for the whole revolution is

$$\mathrm{M}=\mathrm{M}_0+\epsilon\,ds\begin{bmatrix}0&0&A_{12}&0\\0&0&A_{22}&0\\D_{12}&0&0&0\\D_{22}&0&0&0\end{bmatrix}, \tag{4.91}$$

where $\epsilon=(K_z-K_x)\,\theta$. This gives

$$\begin{aligned}\Delta&=\mathrm{Det}(\mathrm{B}+\bar{\mathrm{C}})=A_{12}D_{12}(\epsilon\,ds)^2\\&=\beta_x\beta_z\sin\mu_x\sin\mu_z(\epsilon\,ds)^2.\end{aligned} \tag{4.92}$$

This is, of course, positive at a difference resonance ($\sin\mu_x=\sin\mu_z$) and negative at a sum resonance ($\sin\mu_x=-\sin\mu_z$). Thus we have an example of instability arising at a sum resonance but not at a difference resonance.

Now consider again a random distribution of tilt angles in M magnets comprising a machine.

The expectation value of $\mathrm{Det}(\mathrm{B}+\bar{\mathrm{C}})$ will be the sum of terms like (4.92) for all the magnets. We have, then,

$$\langle\Delta\rangle=\frac{-16\pi^2n^2}{\nu^2M}\frac{R^2}{\rho^2}\sin^2\mu\langle\theta^2\rangle. \tag{4.93}$$

Instability will arise if

$$(\cos\mu_x-\cos\mu_z)^2<-\Delta.$$

There is, therefore, a stopband of width

$$\delta\nu=\frac{1}{2\pi}\delta\mu=\frac{2}{2\pi}\left(\frac{-\Delta}{\sin^2\mu}\right)^{1/2}. \tag{4.94}$$

Its root mean square width is, by (4.93),

$$\langle \delta\nu \rangle_{\text{rms}} = \frac{4\,|n|}{\nu M^{1/2}} \frac{R}{\rho} \langle \theta \rangle_{\text{rms}}. \tag{4.95}$$

For the Brookhaven parameters we obtain

$$\langle \delta\nu \rangle_{\text{rms}} = 16.0 \langle \theta \rangle_{\text{rms}}.$$

Thus, if the rms tilt is 10^{-3} radian, we obtain an rms stopband width of 0.016 unit.

(d) *Nonlinear Effects*

The actual equations of motion will contain nonlinear terms as well. The detailed theory of the effects of nonlinearities is beyond the scope of this paper, but the following facts are worth mentioning.

Nonlinearities modify the behavior of the resonances we have just obtained from the linear theory. The shift of the equilibrium orbit, which is infinite according Eq. (4.7) when ν is integral, becomes finite, and the unstable oscillation arising when ν is half-integral or when $\nu_x + \nu_z$ is integral also becomes limited in amplitude. Both these effects arise because the nonlinearity causes the frequency ν to vary with amplitude; thus as the amplitude increases the frequencies change, and the resonant condition ceases to apply.

On the other hand, nonlinearities can also cause instability in situations which would be stable in the linear theory. Specifically, instability can arise if

$$a\nu_x + b\nu_z = \text{integer}$$

with a and b integral [15]. However, it has been shown by Moser [16] and Sturrock [17] that these instabilities will arise only if a and b are of the same sign, or if a or b is zero (analogous to the result in linear coupling resonance). Furthermore, if $a + b \geqslant 5$ the motion is generally stable even at resonance, because the detuning effects of the nonlinearity dominate the resonance effects. For $a + b = 3$, the system is generally unstable, while for $a + b = 4$, the system may be stable or unstable depending on the relative magnitudes of certain coefficients; detailed calculations show that it is more likely to be stable than unstable for practical machine designs. To summarize the effects of imperfections,

$$\nu_x \quad \text{or} \quad \nu_z = \text{integer}$$

imperfections will generally cause a large shift in the equilibrium orbit (infinite in the linear approximation). If

$$2\nu_x \quad \text{or} \quad 2\nu_z \quad \text{or} \quad \nu_x + \nu_z = \text{integer},$$

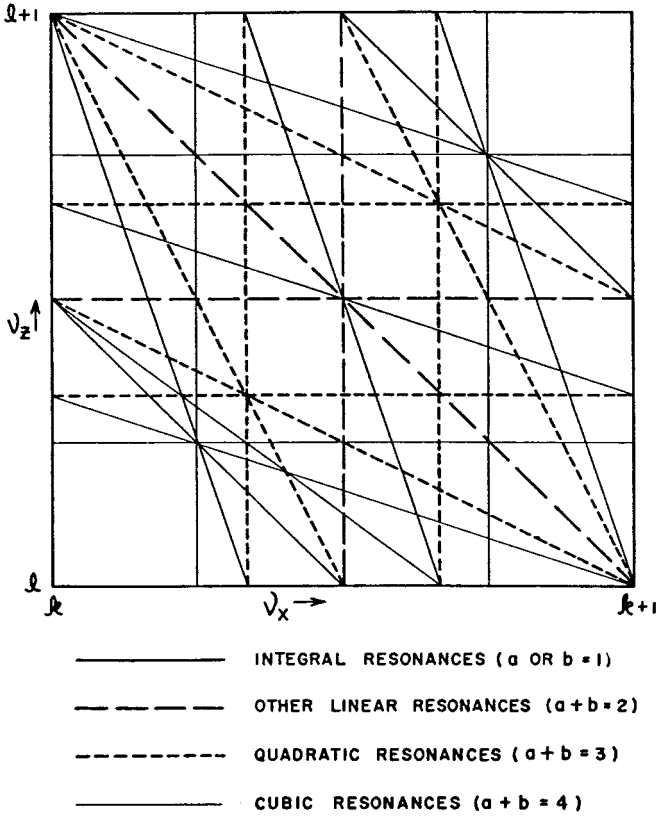

FIG. 3. Values of ν_x and ν_z at which constructional errors may induce instability.

oscillations about the equilibrium orbit are in general unstable. Nonlinearities produce further instabilities if

$$3\nu_x \quad \text{or} \quad 2\nu_x + \nu_z \quad \text{or} \quad \nu_x + 2\nu_z \quad \text{or} \quad 3\nu_z = \text{integer}$$

and may also do so if

$$4\nu_x \quad \text{or} \quad 3\nu_x + \nu_z \quad \text{or} \quad 2(\nu_x + \nu_z) \quad \text{or} \quad \nu_x + 3\nu_z \quad \text{or} \quad 4\nu_z = \text{integer}.$$

The values of ν_x and ν_z at which resonances may lead to instability are schematically shown in Fig. 3.

5. PHASE STABILITY

The synchrotron [18, 19] is made possible by the phase stability of the acceleration, i.e., by the fact that a particle which arrives at the accelerating gap at a phase of

the accelerating field different from the equilibrium phase experiences a phase acceleration or "phase restoring force" toward the equilibrium phase, provided the phase displacement and the momentum deviation of the particle are not too great.

Phase stability arises from the fact that the period of revolution of a particle with more (or less) than the synchronous momentum differs from that of the particle with the synchronous momentum. In the conventional synchrotron we have

$$\Delta t/t = (\Delta C/C) - (\Delta v/v) = [(1-n)^{-1} - (Mc^2/E)^2](\Delta p/p), \tag{5.1}$$

where Δt is the change in the period t of revolution, ΔC is the change in the orbit circumference C, and Δv is the change in the velocity v, associated with a change Δp in the momentum p.

In the alternating gradient synchrotron the relation between orbit circumference and momentum is altered, so that the term $(1-n)^{-1}$ in (5.1) must be replaced by a different quantity, depending on the field configuration. We shall see that this quantity, the "momentum compaction coefficients,"

$$\alpha = \frac{\Delta C/C}{\Delta p/p} \tag{5.2}$$

is small compared to unity in synchrotrons with strong alternating gradients. Then (5.1) is replaced by

$$\frac{\Delta t}{t} = \left(\alpha - \frac{{E_0}^2}{E^2}\right)\frac{\Delta p}{p} = \left(\frac{{E_0}^2}{{E_1}^2} - \frac{{E_0}^2}{E^2}\right)\frac{\Delta p}{p} = \eta\,\frac{\Delta p}{p}, \tag{5.3}$$

where

$$E_1 = Mc^2/\alpha^{1/2} \tag{5.4}$$

may be called the "transition energy," and E_0 is the rest energy.

The coefficient η of $\Delta p/p$ in (5.3) is negative for energies less than E_1 and changes sign as the particle is accelerated through E_1. As pointed out in CLS, this means that at this point the equilibrium phase angle shifts from the rising to the falling side of the voltage curve. To retain an accelerated beam beyond this energy it will be necessary to shift the phase of the applied radiofrequency voltage by the appropriate amount at the proper time.

The equation of phase oscillation may be obtained from the two equations [3, 5, 20]

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_s}\right) = \frac{eV}{2\pi}(\sin\phi - \sin\phi_0) \tag{5.5a}$$

$$\frac{d\phi}{dt} - \omega_1 = \eta h\omega_s\,\frac{\Delta p}{p} = \frac{\eta h\omega_s}{\beta^2}\,\frac{\Delta E}{E}, \tag{5.5b}$$

where the applied voltage is

$$V \sin \left[\int (h\omega_s + \omega_1)\, dt \right],$$

ω_s is the angular velocity of the particle, h the harmonic order (i.e., the applied frequency is designed to be h times the particle frequency), and ω_1 the frequency error, i.e., the difference between the actual applied frequency and its ideal value $h\omega_s$.

Combining the two Eqs. (5.5) we find

$$\frac{1}{h{\omega_0}^2} \frac{d}{dt} \left[\frac{E}{\eta} \left(\frac{d\phi}{dt} - \omega_1 \right) \right] = \frac{eV}{2\pi} (\sin \phi - \sin \phi_0) \tag{5.6}$$

where $\omega_0 = \omega_s/\beta$ is the circular frequency of a particle with velocity c. In the absence of frequency errors Eq. (5.6) leads to stable small oscillations about that phase at which $\sin \phi = \sin \phi_0$ and $\eta \cos \phi_0 < 0$, while the other phase at which $\sin \phi = \sin \phi_0$ is a position of unstable equilibrium.

Just as in the case of the conventional synchrotron, the phase oscillations are stable for the range of phases from $\pi - \phi_0$ to that phase ϕ_2 for which

$$\cos \phi_2 + \phi_2 \sin \phi_2 = -\cos \phi_0 + (\pi - \phi_0) \sin \phi_0. \tag{5.7}$$

The amplitude of the associated radial oscillations may be obtained by using (5.2) and (5.5b). The amplitude of oscillations reaching to the limits of phase stability is

$$\frac{\Delta r}{R} = \alpha \left\{ \frac{eV}{\pi h p \beta c \eta} [(\pi - 2\phi_0) \sin \phi_0 - 2 \cos \phi_0] \right\}^{1/2}. \tag{5.8}$$

This differs from the expression for the conventional synchrotron in that it contains the factor

$$\alpha / |\eta|^{1/2} \tag{5.9}$$

instead of

$$[(1-n)(1-(1-n)\, {E_0}^2/E^2)]^{-1/2}. \tag{5.10}$$

For nonrelativistic energies, $\eta \approx 1$, and (5.9) is very small compared to (5.10). Therefore the radial amplitude of synchrotron oscillations is greatly reduced in alternating gradient synchrotrons as compared to conventional synchrotrons. In fact, this reduction is so great that, for reasonable designs, the radial synchrotron oscillation amplitude is less than the betatron oscillation amplitude. This fact reduces the horizontal aperture requirement to the point where it is not much greater than the vertical requirement—in contrast to large conventional synchrotrons such as the Cosmotron at Brookhaven, which require horizontal apertures about five times their vertical apertures.

The small value of α does, however, introduce a complication. If the transition energy $E_1 = E_0/\alpha^{1/2}$ lies between the initial and final energies, η changes sign as the transition energy is passed, and the position of stable equilibrium jumps from ϕ_0 (lying between 0 and $\pi/2$) to $\pi - \phi_0 = \phi_1$.

Can the particles which were oscillating about ϕ_0 before the transition be made to oscillate stably about ϕ_1 afterwards?

Consider Eq. (5.6) before the transition and for small deviations in phase from ϕ_0. The equation becomes

$$\frac{d}{dt}\left[\frac{E}{-\eta}\frac{d\phi}{dt}\right] + \frac{h\omega_0{}^2 eV\cos\phi_0}{2\pi}(\phi - \phi_0) = 0. \tag{5.11}$$

As long as η varies slowly the solution is closely approximated by the adiabatic form

$$\phi - \phi_0 = A(-\eta/E)^{1/4}\cos\left(\int \Omega\, dt + \delta\right), \tag{5.12}$$

where

$$\Omega = (-\eta he V\cos\phi_0/2\pi E)^{1/2}\,\omega_0 \tag{5.13}$$

is the circular frequency of synchrotron oscillations, and δ and A are constants. Thus the amplitude of phase oscillations damps down to zero as $-\eta$ approaches zero. The concomitant radial oscillations are given, according to (5.5), by

$$\frac{\Delta r}{r} = -\frac{A\alpha}{\beta}\left(\frac{e^2V^2\cos^2\phi_0}{-\eta E^3h^2}\right)^{1/4}\sin\left(\int \Omega\, dt + \delta\right), \tag{5.14}$$

so that their amplitude appears to become infinite as $\eta \to 0$. Actually this form or the adiabatic approximation is, of course, not valid when η approaches zero. A better procedure is to approximate η/E by a linear function of t in that region; this leads to

$$\phi - \phi_0 = (-E_0/E)^{1/4}\, X^{1/2}[aJ_{2/3}(X) + bN_{2/3}(X)], \tag{5.15}$$

with $X = \int \Omega\, dt$; $J_{2/3}$ and $N_{2/3}$ are Bessel and Neumann functions of order $\frac{2}{3}$, and a and b are constants. For large values of X this is identical with (5.12); for $X \to 0$ (5.15) approaches

$$\phi - \phi_0 \to -\frac{2^{3/2}b}{3^{1/3}\Gamma(\frac{1}{3})}K, \tag{5.16}$$

while the radial amplitude approaches

$$\frac{\Delta r}{r} \to \frac{2^{3/2}(3^{1/2}a - b)\,\alpha}{3^{5/3}\Gamma(\frac{5}{3})\,K}\left(\frac{heVE_0\cos\phi_0}{2\pi E_1{}^2}\right)^{1/2}, \tag{5.17}$$

with

$$K=\left(\frac{2E_0{}^7eV\sin^2\phi_0}{h\pi E_1{}^8\cos\phi_0}\right)^{1/12}. \tag{5.18}$$

It is assumed here that the transition energy E_1 is relativistic.

The interesting quantities are the ratios of initial amplitudes to the amplitudes at transition. These are, for the phase oscillations,

$$\frac{(\phi-\phi_0)_1}{(\phi-\phi_0)_i}=\frac{2\pi^{1/2}}{3^{1/3}\Gamma(\frac{1}{3})}\left(\frac{E_i}{E_0\eta_i}\right)^{1/4}\cdot K \tag{5.19}$$

and for the radial oscillations

$$\frac{(\Delta r/R)_1}{(\Delta r/R)_i}=\frac{4\pi^{1/2}}{3^{5/2}\Gamma(\frac{5}{3})}\frac{\beta_\iota(E_0E_i{}^3\eta_i)^{1/4}}{E_1K}. \tag{5.20}$$

Numerically, for the parameters for the 30-Bev Brookhaven accelerator (injection at 50 Mev, $E_1\approx 9Mc^2$, $eV=190$ kev, $\sin\phi_0=\frac{1}{2}$, $h=12$), we find

$$\frac{(\phi-\phi_0)_1}{(\phi-\phi_0)_i}=0.103 \qquad \frac{(\Delta p/p)_1}{(\Delta p/p)_i}=0.42.$$

Thus at the transition energy the particles are very sharply bunched in phase; their momentum spread is fairly large but still relatively smaller than at injection, at least in this numerical example.

This sharp bunching makes it possible to reestablish phase stability beyond E_1 by shifting the phase of the applied rf at the time of transition. To see how this is possible we multiply (5.6) by η^2:

$$\dot\eta\frac{E}{h\omega_0{}^2}(\dot\phi-\omega_1)+\frac{\eta}{h\omega_0{}^2}\frac{d}{dt}[E(\dot\phi-\omega_1]\,])-\frac{\eta^2eV}{2\pi}(\sin\phi-\sin\phi_0)=0. \tag{5.21}$$

When η is small the only important term is the first. Then the phase of the particles can be shifted by simply shifting the phase of the rf by $\phi_1-\phi_0$, i.e., by introducing a perturbation ω_1 in the applied frequency for a time $(\phi_1-\phi_0)/\omega_1$.

The time during which this has to be done is not very critical, since the time constant for phase changes, i.e., the synchrotron oscillation period, is quite long. The second and third terms of (5.21) will be small compared to the first as long as

$$|\Omega\eta/\dot\eta|<1, \tag{5.22}$$

which is the case when

$$\frac{|E-E_1|}{E_1}<\left(\frac{E_1eV\sin^2\phi_0}{4\pi hE_0{}^2\cos\phi_0}\right)^{1/3}. \tag{5.23}$$

For our parameters this condition is $|E-E_1|<124$ Mev, which is satisfied during approximately 7 milliseconds.

This agrees with an expression found by Goldin and Koskarev [21].

The quantity α which determines the transition energy is found as follows: When the particle has momentum $p+\Delta p$, we have

$$\frac{d^2x}{ds^2}+\frac{(1-n)}{\rho^2}x=\frac{1}{\rho}\frac{\Delta p}{p}. \tag{5.24}$$

This is an inhomogeneous Hill's equation of the form studied in Section 4a. We solve it by the method of Fourier components used there: If

$$\frac{\beta^{3/2}}{\rho(s)}=\sum_k a_k e^{ik\phi(s)}, \tag{5.25}$$

then the displaced equilibrium orbit is

$$x=\frac{\Delta p}{p}\beta^{1/2}v^2\sum_k\frac{a_k e^{ik\phi}}{v^2-k^2} \tag{5.26}$$

and, of course,

$$a_k=\frac{1}{2\pi}\int_0^{2\pi}\frac{\beta^{3/2}}{\rho}e^{-ik\phi}\,d\phi. \tag{5.27}$$

The difference in length between this orbit and the original orbit $x=0$ is

$$\Delta C=\int_0^C\frac{x}{\rho}\,ds=v\int_0^{2\pi}\frac{\beta x}{\rho}\,d\phi, \tag{5.28}$$

where we use $\phi=\int ds/v\beta$ as the variable of integration. Using (5.25) and (5.26), we find

$$\Delta C=2\pi v^3\frac{\Delta p}{p}\sum_k\frac{|a_k{}^2|}{v^2-k^2} \tag{5.29}$$

and

$$\alpha=\frac{1}{2\pi R}\frac{\Delta C}{\Delta p/p}=\frac{v^3}{R}\sum_k\frac{|a_k{}^2|}{v^2-k^2}. \tag{5.30}$$

In most accelerator designs, the dominant term in (5.25) and, therefore, in (5.30), is the one with $k=0$. If the magnet is composed of sectors in all of which

the radius of curvature ρ is the same, separated by straight sections, we have approximately

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{\beta^{3/2}}{\rho}\, d\phi = \frac{1}{2\pi\nu}\int_0^{C} \frac{\beta^{1/2}}{\rho}\, ds \approx (R/\nu^3)^{1/2}, \tag{5.31}$$

where we have replaced β by its average value R/ν.

This leads to

$$\alpha = \frac{1}{\nu^2} \tag{5.32}$$

and therefore,

$$E_1 = \nu E_0.$$

If the number of periods of the orbit is N, the only terms with $k \neq 0$ in (5.30) will be those with $k = \pm N$, $\pm 2N$, etc. Since ν will generally be less than N, all the terms with $k > 0$ will then be negative. This fact has enabled Vladimirskii and Tarasov [22] to propose an accelerator design in which α is zero or negative, thus eliminating the transition energy. This is done by inserting K "compensating magnets" with reversed fields but the same gradients as would be called for in a design without compensating magnets. If ν is slightly less than K, the term $k = \pm K$ in (5.30) is large and negative, and can be made to cancel the leading term $k = 0$.

APPENDIX A: EXISTENCE OF EQUILIBRIUM ORBITS

Consider a magnetic field B in space. A particle of momentum p and electric charge e moving in this field will move along a path whose curvature vector is

$$\boldsymbol{\kappa} = \frac{e\mathbf{B} \times \boldsymbol{\alpha}}{pc}, \tag{A1}$$

where $\boldsymbol{\alpha}$ is the unit vector in the direction of motion. If a closed curve satisfies (A1) throughout its length it is a possible equilibrium orbit for a particle of momentum p; conversely any equilibrium orbit must satisfy (A1).

We shall now show that a curve which encloses the maximum flux as compared to neighboring curves of the same circumference satisfies (A1) for a suitable momentum, and is therefore a possible equilibrium orbit. Consider an arbitrary closed curve of circumference C. Let s be the are length measured along this curve, and let its location be given by

$$\mathbf{r} = \mathbf{r}_1(s) \tag{A2}$$

Any other curve near the given one may be described by the equation

$$\mathbf{r} = \mathbf{r}_2(s) = \mathbf{r}_1(s) + \mathbf{x}(s), \tag{A3}$$

where s is still the length of the first curve, and $\mathbf{x}(s)$ is a vector perpendicular to the first curve. The circumference of the curve (A3) is

$$C_2 = C + \int_0^C \mathbf{x} \cdot \boldsymbol{\kappa}\, ds, \tag{A4}$$

where $\boldsymbol{\kappa}(s)$ is the curvature vector of the first curve at s. If the second curve is restricted to be a curve of the same length as the first,

$$\int_0^C \mathbf{x} \cdot \boldsymbol{\kappa}\, ds = 0. \tag{A5}$$

But the flux enclosed between the two curves is

$$\Delta\Phi = \int_0^C \mathbf{B} \cdot \boldsymbol{\alpha} \times \mathbf{x}\, ds = \int_0^C \mathbf{x} \cdot \mathbf{B} \times \boldsymbol{\alpha}\, ds. \tag{A6}$$

If the original curve is an equilibrium orbit, (A1) holds, and therefore (A6) vanishes whenever (A5) does; hence the flux is stationary. Conversely, if the flux is stationary, (A6) must vanish whenever $\mathbf{x}(s)$ is chosen so that (A5) vanishes; therefore a relation of the form (A1) must hold, and the curve is a possible equilibrium orbit. But a curve of given length enclosing maximum flux for all curves of that length will surely exist if the field strength is bounded. Hence equilibrium orbits exist.

The maximum-flux orbits thus constructed will, in many cases, pass through the currents and iron which produce the magnetic field. The usable orbits will, in general, enclose an amount of flux that is stationary but not maximal. The existence of such orbits cannot necessarily be proved in the general case. However, in the special case of a field with mirror symmetry an equilibrium orbit may be found by constraining the orbit to lie in the plane of symmetry. The curve of given length enclosing maximum flux subject to this constraint will be an equilibrium orbit and will not intersect the iron or currents if these lie outside the plane of symmetry.

This result appears to be in contradiction to the result found in Section 4a, where we saw that when the oscillation frequency ν is integral small imperfections of the magnetic field led to infinite displacements of the equilibrium orbit. However, that result was based on the linear approximation to the equations of motion; in an actual magnetic field the equations of motion are never exactly linear, and the actual displacement of the equilibrium orbit is always finite (though comparatively large when ν is integral).

APPENDIX B: HAMILTONIAN FORMULATION OF THE EQUATIONS OF MOTION

We suppose that the motion of the particle takes place in the vicinity of a given curve in space—which may or may not be an equilibrium orbit as defined in Appendix A.

We introduce a special curvilinear system of coordinates in the vicinity of this reference curve. Let the reference curve be given by

$$\mathbf{r} = \mathbf{r}_0(s) \tag{B1}$$

where s is the are length measured along the curve from some fixed initial point. Then

$$\boldsymbol{\alpha}(s) = \frac{d\mathbf{r}_0(s)}{ds} \tag{B2}$$

is the unit vector tangent to the curve at s, and

$$\boldsymbol{\kappa}(s) = -\frac{d\boldsymbol{\alpha}(s)}{ds} = -\Omega(s)\,\boldsymbol{\beta}(s) \tag{B3}$$

is the curvature vector; $\Omega(s)$ is the numerical value of the curvature and $\boldsymbol{\beta}(s)$ is a unit vector. The plane containing the point $\mathbf{r}_0(s)$ and parallel to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is called the osculating plane; its unit normal is

$$\boldsymbol{\gamma}(s) = \boldsymbol{\alpha}(s) \times \boldsymbol{\beta}(s). \tag{B4}$$

Since $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ are unit vectors, it is easily seen that

$$\begin{aligned} \boldsymbol{\beta}' &= \Omega(s)\,\boldsymbol{\alpha}(s) + \omega(s)\,\boldsymbol{\gamma}(s), \\ \boldsymbol{\gamma}' &= -\omega(s)\,\boldsymbol{\beta}(s), \end{aligned} \tag{B5}$$

where a prime denotes differentiation with respect to s, and where $\omega(s)$ is a quantity known as the torsion of the curve—for a plane curve, $\omega(s)$ vanishes identically.

We now define the position of any point in space not too far from the reference curve by

$$\mathbf{r}(s, x, z) = \mathbf{r}_0(s) + x\boldsymbol{\beta}(s) + z\boldsymbol{\gamma}(s), \tag{B6}$$

where s, x, z form a system of orthogonal curvilinear coordinates. (B6) is unique for points whose distance from the reference curve is less than the radius of curvature $\rho = 1/\Omega$.

To express the equations of motion in Hamiltonian form in terms of these coordinates, we must find the momenta canonically conjugate to s, x, z. This is done by performing a contact transformation produced by the generating function (23)

$$F(p_X, p_Y, p_Z; s, x, z) = \mathbf{p} \cdot [\mathbf{r}_0(s) + x\boldsymbol{\beta}(s) + z\boldsymbol{\gamma}(s)], \tag{B7}$$

where p_X, p_Y, p_Z are the components of the linear momentum p in the Cartesian (XYZ) coordinate system. We obtain the canonical momenta

$$\begin{aligned} p_s &= \frac{\partial F}{\partial s} = \mathbf{p} \cdot [(1 + \Omega x)\,\boldsymbol{\alpha} + \omega(x\boldsymbol{\gamma} - z\boldsymbol{\beta})], \\ p_x &= \frac{\partial F}{\partial x} = \mathbf{p} \cdot \boldsymbol{\beta}, \\ p_z &= \frac{\partial F}{\partial z} = \mathbf{p} \cdot \boldsymbol{\gamma}. \end{aligned} \tag{B8}$$

The equations of motion are derived from the Hamiltonian

$$H = eV + c[m^2c^2 + (\mathbf{p} - e\mathbf{A})^2]^{1/2}, \tag{B9}$$

where V and $\mathbf{A}$ are the scalar and vector potentials of the electromagnetic field. In terms of the new variables this equals

$$\begin{aligned} H = eV + c\Big\{ & m^2c^2 + \frac{1}{(1+\Omega x)^2}\,[p_s - eA_s + \omega z(p_x - eA_x) \\ & - \omega x(p_z - eA_z)^2 + (p_x - eA_x)^2 + (p_z - eA_z)^2 \Big\}^{1/2}, \end{aligned} \tag{B10}$$

where A_s, A_x and A_z are obtained by inserting the vector $\mathbf{A}$ in (B8) instead of $\mathbf{p}$. Note that A_s and p_s are *not* the components of the vectors in the tangential direction.

The Hamiltonian equations of motion are

$$\begin{aligned} \dot{s} &= \frac{\partial H}{\partial p_s}, \qquad \dot{p}_s = -\frac{\partial H}{\partial s}, \\ \dot{x} &= \frac{\partial H}{\partial p_x}, \qquad \dot{p}_x = -\frac{\partial H}{\partial x}, \\ \dot{z} &= \frac{\partial H}{\partial p_z}, \qquad \dot{p}_z = -\frac{\partial H}{\partial z}. \end{aligned} \tag{B11}$$

Using the first of these equations, we can change to s as the independent variable instead of the time t. The equations transform to

$$\begin{aligned} x' &= -\frac{\partial p_s}{\partial p_x}, & p_x' &= \frac{\partial p_s}{\partial x}, \\ z' &= -\frac{\partial p_s}{\partial p_z}, & p_z' &= \frac{\partial p_s}{\partial z}, \\ t' &= \frac{\partial p_s}{\partial H}, & H' &= -\frac{\partial p_s}{\partial t}, \end{aligned} \tag{B12}$$

where primes denote differentiation with respect to s, and where p_s is expressed as a function of x, p_x, z, p_z, H, t, s, obtained by solving (B10).

Equations (B12) are again in Hamiltonian form, with the new Hamiltonian function

$$G = -p_s \tag{B13}$$

and the pairs of canonically conjugate variables

$$x,\ p_x; \qquad z,\ p_z; \qquad t,\ -H. \tag{B14}$$

In the particular case where the potentials V and $\mathbf{A}$ are independent of time, the new Hamiltonian G is also time-independent, and therefore t is an ignorable coordinate in the system (B12). In this case the transformation just performed reduces the number of degrees of freedom of the problem from three to two, but at the cost of having the new Hamiltonian dependent on the independent variable s. This procedure is important in classical dynamics [24, 25]. Because of the geometrical significance of the variable s, the dependence of the new Hamiltonian G on s is periodic.

The linearized equations of motion are obtained by expanding G as a power series in x, p_x, z, p_z and retaining only terms up to the second order. We consider a static magnetic field, so that $V = 0$ and $\mathbf{A}$ is independent of time. We may choose a gauge such that the power series expansions of the components of $\mathbf{A}$ are in the form

$$\begin{aligned} A_s &= ax + bz + cx^2 + dx\,z + ez^2 + \cdots, \\ A_x &= -fz + \cdots, \\ A_z &= fx + \cdots. \end{aligned} \tag{B15}$$

Terms of higher order than those given in (B15) do not appear in the expansion of G up to second order. The coefficients a, b, c, d, e, f, are periodic functions of s.

The only one of Maxwell's equations which imposes a restriction on the coefficients up to this order is

$$(\nabla \times \mathbf{B})_s = (\nabla \times \nabla \times \mathbf{A})_s = 0, \tag{B16}$$

which leads to

$$2(c+e) = \Omega a + 4\omega f. \tag{B17}$$

If the reference curve is taken to be an equilibrium orbit of a particle of momentum p, we have

$$b = 0, \qquad a = -\Omega p/e. \tag{B18}$$

In this case, by substituting (B15) in (B10) and (B12), we obtain a Hamiltonian G of the form (4.59), and the homogeneous equations of motion used in Sections 2 to 4.

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