A PROOF OF THE RIEMANN HYPOTHESIS

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Abstract. A proof of the Riemann hypothesis is obtained for zeta functions constructed in Fourier analysis on locally compact skew–fields. The skew–fields are the algebra of quaternions whose coordinates are real numbers and the algebra of quaternions whose coordinates are elements of a p–adic field for every prime p. Fourier analysis is also applied in locally compact algebras which are finite Cartesian products of locally compact skew–fields and in quotient spaces defined by a summation originating in the construction of Jacobian theta functions. The Riemann hypothesis is a consequence of the maximal accretive property of a Radon transformation relating Fourier analysis on a locally compact skew–field with Fourier analysis on a maximal commutative subfield. The maximal accretive property of the Radon transformation is preserved in Cartesian products but need not be preserved in quotient spaces. A proof of the Riemann hypothesis is obtained for zeta functions constructed in quotient spaces having the maximal accretive property. When the maximal accretive property fails in a quotient space, the domain of the Radon transformation is decomposed by a symmetry into two invariant subspaces in one of which the maximal accretive property is satisfied. The Riemann hypothesis is proved for the zeta function generated by the quotient space. A zeta function generated in Fourier analysis on skew–fields is constructed from the Euler zeta function by an analogue of the duplication formula for the gamma function. A proof of the Riemann hypothesis is obtained for the Euler zeta function.

1. Generalization of the Gamma Function

The gamma function is an analytic function of s in the complex plane with the exception of singularities at the nonpositive integers which satisfies the recurrence relation

$$s\Gamma(s) = \Gamma(s+1).$$

A generalization of the gamma function is obtained with the factor of s in the recurrence relation replaced by an arbitrary function of s which is analytic and has positive real part in the right half–plane. The hypergeometric function theory of the gamma function is generalized in a context which includes zeta functions.

An analytic weight function is defined as a function $W(z)$ of z which is analytic and without zeros in the upper half–plane.

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Hilbert spaces of functions analytic in the upper half–plane were introduced in Fourier analysis by Hardy. The weighted Hardy space $\mathcal{F}(W)$ is defined as the Hilbert space of functions $F(z)$ of z analytic in the upper half–plane such that the least upper bound

$$\|F\|^2_{\mathcal{F}(W)} = \sup \int_{-\infty}^{+\infty} |F(x+iy)/W(x+iy)|^2 dx$$

taken over all positive y is finite. The least upper bound is obtained in the limit as y decreases to zero. The classical Hardy space is obtained when $W(z)$ is identically one. Multiplication by $W(z)$ is an isometric transformation of the classical Hardy space onto the weighted Hardy space with analytic weight function $W(z)$.

An isometric transformation of the weighted Hardy space $\mathcal{F}(W)$ into itself is defined by taking a function $F(z)$ of z into the function

$$F(z)(z-w)/(z-w^-)$$

of z when w is in the upper half–plane. The range of the transformation is the set of elements of the space which vanish at w. A continuous linear functional on the weighted Hardy space $\mathcal{F}(W)$ is defined by taking a function $F(z)$ of z into its value $F(w)$ at w whenever w is in the upper half–plane. The function

$$W(z)W(w)^-/[2\pi i(w^- - z)]$$

of z belongs to the space when w is in the upper half–plane and acts as reproducing kernel function for function values at w.

A Hilbert space of functions analytic in the upper half–plane which has dimension greater than one is isometrically equal to a weighted Hardy space if an isometric transformation of the space onto the subspace of functions which vanish at w is defined by taking $F(z)$ into

$$F(z)(z-w)/(z-w^-)$$

when w is in the upper half–plane and if a continuous linear functional is defined on the space by taking $F(z)$ into $F(w)$ for w of the upper half–plane.

Examples of weighted Hardy spaces in are constructed from the Euler gamma function. An analytic weight function

$$W(z) = \Gamma(s)$$

is defined by

$$s = \tfrac{1}{2} - iz.$$

A maximal accretive transformation is defined in the weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space.

A linear relation with domain and range in a Hilbert space is said to be accretive if the sum

$$\langle a, b\rangle + \langle b, a\rangle \geq 0$$

of scalar products in the space is nonnegative whenever (a, b) belongs to the graph of the relation. A linear relation is said to be maximal accretive if it is not the proper restriction of an accretive linear relation with domain and range in the same Hilbert space. A maximal accretive transformation with domain and range in a Hilbert space is a transformation which is a maximal accretive relation with domain and range in the Hilbert space.

Theorem 1. *A maximal accretive transformation is defined in a weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space if, and only if, the function*

$$W(z - \tfrac{1}{2}i)/W(z + \tfrac{1}{2}i)$$

of z admits an extension which is analytic and has nonnegative real part in the upper half–plane.

Proof of Theorem 1. A Hilbert space $\mathcal{H}$ whose elements are functions analytic in the upper half–plane is constructed when a maximal accretive transformation is defined in the weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space. The space $\mathcal{H}$ is constructed from the graph of the adjoint of the transformation which takes $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space.

An element

$$F(z) = (F_+(z), F_-(z))$$

of the graph is a pair of analytic functions of z, which belong to the space $\mathcal{F}(W)$, such that the adjoint takes $F_+(z)$ into $F_-(z)$. The scalar product

$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{F}(W)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{F}(W)}$$

of elements $F(z)$ and $G(z)$ of the graph is defined as the sum of scalar products in the space $\mathcal{F}(W)$. Scalar self–products are nonnegative in the graph since the adjoint of a maximal accretive transformation is accretive.

An element $K(w, z)$ of the graph is defined by

$$K_+(w, z) = W(z)W(w - \tfrac{1}{2}i)^-/[2\pi i(w^- + \tfrac{1}{2}i - z)]$$

and

$$K_-(w, z) = W(z)W(w + \tfrac{1}{2}i)^-/[2\pi i(w^- - \tfrac{1}{2}i - z)]$$

when w is in the half–plane

$$1 < iw^- - iw.$$

The identity

$$F_+(w + \tfrac{1}{2}i) + F_-(w - \tfrac{1}{2}i) = \langle F(t), K(w, t)\rangle$$

holds for every element

$$F(z) = (F_+(z), F_-(z))$$

of the graph. An element of the graph which is orthogonal to itself is orthogonal to every element of the graph.

An isometric transformation of the graph onto a dense subspace of $\mathcal{H}$ is defined by taking

$$F(z) = (F_+(z), F_-(z))$$

into the function

$$F_+(z + \tfrac{1}{2}i) + F_-(z - \tfrac{1}{2}i)$$

of z in the half–plane

$$1 < iz^- - iz.$$

The reproducing kernel function for function values at w in the space $\mathcal{H}$ is the function

$$[W(z + \tfrac{1}{2}i)W(w - \tfrac{1}{2}i)^- + W(z - \tfrac{1}{2}i)W(w + \tfrac{1}{2}i)^-]/[2\pi i(w^- - z)]$$

of z in the half–plane when w is in the half–plane.

Division by $W(z + \frac{1}{2}i)$ is an isometric transformation of the space $\mathcal{H}$ onto a Hilbert space $\mathcal{L}$ whose elements are functions analytic in the half–plane and which contains the function

$$[\varphi(z) + \varphi(w)^-]/[2\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w when w is in the half–plane,

$$\varphi(z) = W(z - \tfrac{1}{2}i)/W(z + \tfrac{1}{2}i).$$

A Hilbert space with the same reproducing kernel functions is given an axiomatic characterization in the Poisson representation [1] of functions which are analytic and have positive real part in the upper half–plane. The argument applies to the present space $\mathcal{L}$ whose elements are functions analytic in the smaller half–plane.

The function

$$[F(z) - F(w)]/(z - w)$$

of z belongs to $\mathcal{L}$ whenever the function $F(z)$ of z belongs to $\mathcal{L}$ if w is in the smaller half–plane. The identity

$$0 = \langle F(t), [G(t) - G(\alpha)]/(t - \alpha)\rangle_{\mathcal{L}} - \langle [F(t) - F(\beta)]/(t - \beta), G(t)\rangle_{\mathcal{L}}$$
$$-(\beta - \alpha^-)\langle [F(t) - F(\beta)]/(t - \beta), [G(t) - G(\alpha)]/(t - \alpha)\rangle_{\mathcal{L}}$$

holds for all functions $F(z)$ and $G(z)$ which belong to $\mathcal{L}$ when α and β are in the smaller half–plane.

An isometric transformation of the space $\mathcal{L}$ into itself is defined by taking a function $F(z)$ of z into the function

$$F(z) + (w - w^-)[F(z) - F(w)]/(z - w)$$

of z when w is in the smaller half–plane.

The same conclusion holds when w is in the upper half–plane by the preservation of the isometric property under iterated compositions. The elements of $\mathcal{L}$ are functions which have analytic extensions to the upper half–plane. The computation of reproducing kernel functions applies when w is in the upper half–plane. The function $\varphi(z)$ of z has an analytic extension with nonnegative real part in the upper half–plane.

Since multiplication by $W(z + \frac{1}{2}i)$ is an isometric transformation of the space $\mathcal{L}$ onto the space $\mathcal{H}$, the elements of $\mathcal{H}$ have analytic extensions to the upper half–plane. The function

$$[W(z + \tfrac{1}{2}i)W(w - \tfrac{1}{2}i)^- + W(z - \tfrac{1}{2}i)W(w + \tfrac{1}{2}i)^-]/[2\pi i(w^- - z)]$$

of z belongs to the space when w is in the upper half–plane and acts as reproducing kernel function for function values at w.

The argument is reversed to construct a maximal accretive transformation in the weighted Hardy space $\mathcal{F}(W)$ when the function $\phi(z)$ of z admits an extension which is analytic and has positive real part in the upper half–plane. The Poisson representation constructs a Hilbert space $\mathcal{L}$ whose elements are functions analytic in the upper half–plane and which contains the function

$$[\phi(z) + \phi(w)^-]/[2\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w when w is in the upper half–plane. Multiplication by $W(z + \frac{1}{2}i)$ acts as an isometric transformation of the space $\mathcal{L}$ onto a Hilbert space $\mathcal{H}$ whose elements are functions analytic in the upper half–plane and which contains the function

$$[W(z + \tfrac{1}{2}i)W(w - \tfrac{1}{2}i)^- + W(z - \tfrac{1}{2}i)W(w + \tfrac{1}{2}i)^-]/[2\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w when w is in the upper half–plane.

A transformation is defined in the space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z + i)$ whenever the functions of z belong to the space. The graph of the adjoint is a space of pairs

$$F(z) = (F_+(z), F_-(z))$$

of elements of the space such that the adjoint takes the function $F_+(z)$ of z into the function $F_-(z)$ of z. The graph contains

$$K(w, z) = (K_+(w, z), K_-(w, z))$$

with

$$K_+(w, z) = W(z)W(w - \tfrac{1}{2}i)^-/[2\pi i(w^- + \tfrac{1}{2}i - z)]$$

and

$$K_-(w, z) = W(z)W(w + \tfrac{1}{2}i)^-/[2\pi i(w^- - \tfrac{1}{2}i - z)]$$

when w is in the half–plane
$$1 < iw^- - iw.$$
The elements $K(w,z)$ of the graph span the graph of a restriction of the adjoint. The transformation in the space $\mathcal{F}(W)$ is recovered as the adjoint of the restricted adjoint.

A scalar product is defined on the graph of the restricted adjoint so that an isometric transformation of the graph of the restricted adjoint into the space $\mathcal{H}$ is defined by taking
$$F(z) = (F_+(z), F_-(z))$$
into
$$F_+(z + \tfrac{1}{2}i) + F_-(z - \tfrac{1}{2}i).$$
The identity
$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{F}(W)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{F}(W)}$$
holds for all elements
$$F(z) = (F_+(z), F_-(z))$$
and
$$G(z) = (G_+(z), G_-(z))$$
of the graph of the restricted adjoint. The restricted adjoint is accretive since scalar self–products are nonnegative in its graph. The adjoint is accretive since the transformation in the space $\mathcal{F}(W)$ is the adjoint of its restricted adjoint.

The accretive property of the adjoint is expressed in the inequality
$$\|F_+(t) - \lambda^- F_-(t)\|_{\mathcal{F}(W)} \leq \|F_+(t) + \lambda F_-(t)\|_{\mathcal{F}(W)}$$
for elements
$$F(z) = (F_+(z), F_-(z))$$
of the graph when λ is in the right half–plane. The domain of the contractive transformation which takes the function
$$F_+(z) + \lambda F_-(z)$$
of z into the function
$$F_+(z) - \lambda^- F_-(z)$$
of z is a closed subspace of the space $\mathcal{F}(W)$. The maximal accretive property of the adjoint is the requirement that the contractive transformation be everywhere defined for some, and hence every, λ in the right half–plane.

Since $K(w,z)$ belongs to the graph when w is in the half–plane
$$1 < iw^- - iw,$$

an element $H(z)$ of the space $\mathcal{F}(W)$ which is orthogonal to the domain of the accretive transformation satisfies the identity

$$H(w - \tfrac{1}{2}i) + \lambda H(w + \tfrac{1}{2}i) = 0$$

when w is in the upper half–plane. The function $H(z)$ of z admits an analytic extension to the complex plane which satisfies the identity

$$H(z) + \lambda H(z + i) = 0.$$

A zero of $H(z)$ is repeated with period i. Since

$$H(z)/W(z)$$

is analytic and of bounded type in the upper half–plane, the function $H(z)$ of z vanishes everywhere if it vanishes somewhere.

The space of elements $H(z)$ of the space $\mathcal{F}(W)$ which are solutions of the equation

$$H(z) + \lambda H(z + i) = 0$$

for some λ in the right half–plane has dimension zero or one. The dimension is independent of λ.

If τ is positive, multiplication by

$$\exp(i\tau z)$$

is an isometric transformation of the space $\mathcal{F}(W)$ into itself which takes solutions of the equation for a given λ into solutions of the equation with λ replaced by

$$\lambda \exp(\tau).$$

A solution $H(z)$ of the equation for a given λ vanishes identically since the function

$$\exp(-i\tau z)H(z)$$

of z belongs to the space for every positive number τ and has the same norm as the function $H(z)$ of z.

The transformation which takes $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space $\mathcal{F}(W)$ is maximal accretive since it is the adjoint of its adjoint, which is maximal accretive.

This completes the proof of the theorem.

The theorem has no equivalent formulation. A maximal accretive transformation is defined in a weighted Hardy space $\mathcal{F}(W)$ for some real number h by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space if, and only if, the function

$$W(z + \tfrac{1}{2}ih)/W(z - \tfrac{1}{2}ih)$$

of z admits an extension which is analytic and has nonnegative real part in the upper half–plane.

Another theorem is obtained in the limit of small h. A maximal accretive transformation is defined in a weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $iF'(z)$ whenever the functions of z belong to the space if, and only if, the function

$$iW'(z)/W(z)$$

of z has nonnegative real part in the upper half–plane. The proof of the theorem is similar to the proof of Theorem 1. A maximal accretive transformation is defined in a weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $iF'(z)$ whenever the functions of z belong to the space if, and only if, the modulus of $W(x+iy)$ is a nondecreasing function of positive y for every real number x.

An Euler weight function is defined as an analytic weight function $W(z)$ such that a maximal accretive transformation is defined in the weighted Hardy space $\mathcal{F}(W)$ whenever h is in the interval $[-1,1]$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space.

If a function $\phi(z)$ of z is analytic and has positive real part in the upper half–plane, a logarithm of the functions is defined continuously in the half–plane with values in the strip of width π centered on the real line. The inequalities

$$-\pi \leq i \log \phi(z)^- - i \log \phi(z) \leq \pi$$

are satisfied. A function $\phi_h(z)$ of z which is analytic and has nonnegative real part in the upper half–plane is defined when h is in the interval $(-1,1)$ by the integral

$$\log \phi_h(z) = \sin(\pi h) \int_{-\infty}^{+\infty} \frac{\log \phi(z-t)dt}{\cos(2\pi it)+\cos(\pi h)}.$$

An application of the Cauchy formula in the upper half–plane shows that the function

$$\frac{\sin(\pi h)}{\cos(2\pi iz)+\cos(\pi h)} = \int_{-\infty}^{+\infty} \exp(2\pi itz) \ \frac{\exp(\pi ht - \exp(-\pi ht)}{\exp(\pi t)-\exp(-\pi t)} \ dt$$

of z is the Fourier transform of a function

$$\frac{\exp(\pi ht)-\exp(-\pi ht)}{\exp(\pi t)-\exp(-\pi t)}$$

of positive t which is square integrable with respect to Lebesgue measure and is bounded by h when h is in the interval $(0,1)$.

The identity

$$\phi_{-h}(z) = \phi_h(z)^{-1}$$

is satisfied. The function

$$\phi(z) = \lim \phi_h(z)$$

of z is recovered in the limit as h increases to one. The identity

$$\phi_{a+b}(z) = \phi_a(z - \tfrac{1}{2}ib)\phi_b(z + \tfrac{1}{2}ia)$$

when a, b, and $a + b$ belong to the interval $(-1, 1)$ is a consequence of the trigonometric identity

$$\frac{\sin(\pi a + \pi b)}{\cos(2\pi iz) + \cos(\pi a + \pi b)}$$
$$= \frac{\sin(\pi a)}{\cos(2\pi iz + \pi b) + \cos(\pi a)} + \frac{\sin(\pi b)}{\cos(2\pi iz - \pi a) + \cos(\pi b)}.$$

An Euler weight function $W(z)$ is defined within a constant factor by the limit

$$iW'(z)/W(z) = \lim \frac{\log \phi_h(z)}{h} = \pi \int_{-\infty}^{+\infty} \frac{\log \phi(z - t) dt}{1 + \cos(2\pi it)}.$$

as h decreases to zero. The identity

$$W(z + \tfrac{1}{2}ih) = W(z - \tfrac{1}{2}ih)\phi_h(z)$$

applies when h is in the interval $(-1, 1)$. The identity reads

$$W(z + \tfrac{1}{2}i) = W(z - \tfrac{1}{2}i)\phi(z)$$

in the limit as h increases to one.

An Euler weight function $W(z)$ is constructed which satisfies the identity

$$W(z + \tfrac{1}{2}i) = W(z - \tfrac{1}{2}i)\phi(z)$$

for a given nontrivial function $\phi(z)$ of z which is analytic and has nonnegative real part in the upper half–plane.

If a maximal accretive transformation is defined in a weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space, then the identity

$$W(z + \tfrac{1}{2}i) = W(z - \tfrac{1}{2}i)\phi(z)$$

holds for a function $\phi(z)$ of z which is analytic and has nonnegative real part in the upper half–plane. The analytic weight function $W(z)$ is the product of an Euler weight function and an entire function which is periodic of period i and has no zeros.

If $W(z)$ is an Euler weight function, the maximal accretive transformation defined for h in the interval $[0, 1]$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space $\mathcal{F}(W)$ is subnormal: The transformation is the restriction to an invariant subspace of a normal transformation in the larger Hilbert space $\mathcal{H}$ of (equivalence classes of) Baire functions $F(x)$ of real x for which the integral

$$\|F\|^2 = \int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt$$

converges. The passage to boundary value functions maps the space $\mathcal{F}(W)$ isometrically into the space $\mathcal{H}$.

For given h the function

$$\phi(z) = W(z + \tfrac{1}{2}ih)/W(z - \tfrac{1}{2}ih)$$

of z is analytic and has nonnegative real part in the upper half–plane. A transformation T is defined to take $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space $\mathcal{F}(W)$. A transformation S is defined to take $F(z)$ into

$$\varphi(z + \tfrac{1}{2}ih)F(z)$$

whenever the functions of z belong to the space $\mathcal{F}(W)$. The adjoint

$$T^* = S^{-1}TS^*$$

of T is computable from the adjoint S^* of S on a dense subset of the space $\mathcal{F}(W)$ containing the reproducing kernel functions for function values. The transformation $T^{-1}ST$ takes $F(z)$ into

$$\varphi(z - \tfrac{1}{2}ih)F(z)$$

on a dense set of elements $F(z)$ of the space $\mathcal{F}(W)$. The transformation $T^{-1}T^*$ is the restriction of a contractive transformation of the space $\mathcal{F}(W)$ into itself. The transformation takes $F(z)$ into $G(z)$ when the boundary value function $G(x)$ is the orthogonal projection of

$$F(x)\phi(x - \tfrac{1}{2}ih)^-/\phi(x - \tfrac{1}{2}ih)$$

in the image of the space $\mathcal{F}(W)$.

An isometric transformation of the space $\mathcal{H}$ onto itself is defined by taking a function $F(x)$ of real x into the function

$$F(x)\phi(x - \tfrac{1}{2}ih)^-/\varphi(x - \tfrac{1}{2}ih)$$

of real x. A dense set of elements of $\mathcal{H}$ are products

$$\exp(-iax)F(x)$$

for nonnegative numbers a with $F(x)$ the boundary value function of a function $F(z)$ of z which belongs to the space $\mathcal{F}(W)$. A normal transformation is defined in the space $\mathcal{H}$ as the closure of a transformation which is computed on such elements. The transformation takes the function

$$\exp(-iax)F(x)$$

of real x into the function

$$\exp(-iah)\exp(-iax)G(x)$$

of real x for every nonnegative number a when $F(z)$ and

$$G(z) = F(z + ih)$$

are functions of z in the upper half–plane which belong to the space $\mathcal{F}(W)$.

This completes the verification of subnormality for the maximal accretive transformation.

An entire function $E(z)$ of z is said to be of Hermite class if it has no zeros in the upper half–plane and if the modulus of $E(x + iy)$ is a nondecreasing function of positive y for every real number x. The Hermite class is also known as the Pólya class. Entire functions of Hermite class are limits of polynomials having no zeros in the upper half–plane [1]. Such polynomials appear in the Stieltjes representation of positive linear functions on polynomials.

A linear functional on polynomials with complex coefficients is said to be nonnegative if it has nonnegative values on polynomials whose values on the real axis are nonnegative. A positive linear functional on polynomials is a nonnegative linear functional on polynomials which does not vanish identically. A nonnegative linear functional on polynomials is represented as an integral with respect to a nonnegative measure μ on the Baire subsets of the real line. The linear functional takes a polynomial $F(z)$ into the integral

$$\int F(t)d\mu(t).$$

Stieltjes examines the action of a positive linear functional on polynomials of degree less than r for a positive integer r. A polynomial which has nonnegative values on the real axis is a product

$$F^*(z)F(z)$$

of a polynomial $F(z)$ and the conjugate polynomial

$$F^*(z) = F(z^-)^-.$$

If the positive linear functional does not annihilate

$$F^*(z)F(z)$$

for any nontrivial polynomial $F(z)$ of degree less than r, a Hilbert space exists whose elements are the polynomials of degree less than r and whose scalar product

$$\langle F(t), G(t)\rangle$$

is defined as the action of the positive linear functional on the polynomial

$$G^*(z)F(z).$$

Stieltjes shows that the Hilbert space of polynomials of degree less than r is contained isometrically in a weighted Hardy space $\mathcal{F}(W)$ whose analytic weight function $W(z)$ is a polynomial of degree r having no zeros in the upper half–plane.

An axiomatization of the Stieltjes spaces is stated in a general context [1]. Hilbert spaces are examined whose elements are entire functions and which have these properties:

(H1) Whenever an entire function $F(z)$ of z belongs to the space and has a nonreal zero w, the entire function

$$F(z)(z - w^-)/(z - w)$$

of z belongs to the space and has the same norm as $F(z)$.

(H2) A continuous linear functional on the space is defined by taking a function $F(z)$ of z into its value $F(w)$ at w for every nonreal number w.

(H3) The entire function

$$F^*(z) = F(z^-)^-$$

of z belongs to the space and has the same norm as $F(z)$ whenever the entire function $F(z)$ of z belongs to the space.

An example of a Hilbert space of entire functions which satisfies the axioms is obtained when an entire function $E(z)$ of z satisfies the inequality

$$|E(x - iy)| < |E(x + iy)|$$

for all real x when y is positive. A weighted Hardy space $\mathcal{F}(W)$ is defined with analytic weight function

$$W(z) = E(z).$$

A Hilbert space $\mathcal{H}(E)$ which is contained isometrically in the space $\mathcal{F}(W)$ is defined as the set of entire functions $F(z)$ of z such that the entire functions $F(z)$ and $F^*(z)$ of z belong to the space $\mathcal{F}(W)$. The entire function

$$[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

of z belongs to the space $\mathcal{H}(E)$ for every complex number w and acts as reproducing kernel function for function values at w.

A Hilbert space $\mathcal{H}$ of entire functions which satisfies the axioms (H1), (H2), and (H3) is isometrically equal to a space $\mathcal{H}(E)$ if it contains a nonzero element. The proof applies reproducing kernel functions which exist by the axiom (H2).

For every nonreal number w a unique entire function $K(w, z)$ of z exists which belongs to the space and acts as reproducing kernel function for function values at w. The function does not vanish identically since the axiom (H1) implies that some element of the space has a nonzero value at w when some element of the space does not vanish identically. The scalar self–product $K(w, w)$ of the function $K(w, z)$ of z is positive. The axiom (H3) implies the symmetry

$$K(w^-, z) = K(w, z^-)^-.$$

If λ is a nonreal number, the set of elements of the space which vanish at λ is a Hilbert space of entire functions which is contained isometrically in the given space. The function

$$K(w, z) - K(w, \lambda)K(\lambda, \lambda)^{-1}K(\lambda, z)$$

of z belongs to the subspace and acts as reproducing kernel function for function values at w. The identity

$$\begin{aligned} &[K(w, z) - K(w, \lambda)K(\lambda, \lambda)^{-1}K(\lambda, z)](z - \lambda^-)(w^- - \lambda) \\ &= [K(w, z) - K(w, \lambda^-)K(\lambda^-, \lambda^-)^{-1}K(\lambda^-, z)](z - \lambda)(w^- - \lambda^-) \end{aligned}$$

is a consequence of the axiom (H1).

An entire function $E(z)$ of z exists such that the identity

$$K(w, z) = [E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

holds for all complex z when w is not real. The entire function can be chosen with a zero at λ when λ is in the lower half–plane. The function is then unique within a constant factor of absolute value one. A space $\mathcal{H}(E)$ exists and is isometrically equal to the given space $\mathcal{H}$.

Examples [1] of Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3) are constructed from the analytic weight function

$$W(z) = a^{iz}\Gamma(\tfrac{1}{2} - iz)$$

for every positive number a. The space is contained isometrically in the weighted Hardy space $\mathcal{F}(W)$ and contains every entire function $F(z)$ such that the functions $F(z)$ and $F^*(z)$ of z belong to the space $\mathcal{F}(W)$. The space of entire functions is isometrically equal to a space $\mathcal{H}(E)$ whose defining function $E(z)$ is a confluent hypergeometric function [1]. Properties of the space motivate the definition of a class of Hilbert spaces of entire functions.

An Euler space of entire functions is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) such that a maximal accretive transformation is defined in the space for every h in the interval $[-1, 1]$ by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space.

Theorem 2. *A maximal accretive transformation is defined in a Hilbert space $\mathcal{H}(E)$ of entire functions for a real number h by taking $F(z)$ into $F(z + ih)$ whenever the functions* *of z belong to the space if, and only if, a Hilbert space $\mathcal{H}$ of entire functions exists which contains the function*

$$[E(z+\tfrac{1}{2}ih)E(w-\tfrac{1}{2}ih)^- - E^*(z+\tfrac{1}{2}ih)E(w^-+\tfrac{1}{2}ih)]/[2\pi i(w^- - z)]$$
$$+ [E(z-\tfrac{1}{2}ih)E(w+\tfrac{1}{2}ih)^- - E^*(z-\tfrac{1}{2}ih)E(w^- - \tfrac{1}{2}ih)]/[2\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w for every complex number w.

Proof of Theorem 2. The space $\mathcal{H}$ is constructed from the graph of the adjoint of the transformation which takes $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space. An element

$$F(z) = (F_+(z), F_-(z))$$

of the graph is a pair of entire functions of z, which belong to the space $\mathcal{H}(E)$, such that the adjoint takes $F_+(z)$ into $F_-(z)$. The scalar product

$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{H}(E)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{H}(E)}$$

of elements $F(z)$ and $G(z)$ of the graph is defined as a sum of scalar products in the space $\mathcal{H}(E)$. Scalar self–products are nonnegative since the adjoint of a maximal accretive transformation is accretive.

An element

$$K(w,z) = (K_+(w,z), K_-(w,z))$$

of the graph is defined for every complex number w by

$$K_+(w,z) = [E(z)E(w-\tfrac{1}{2}ih)^- - E^*(z)E(w^-+\tfrac{1}{2}ih)]/[2\pi i(w^-+\tfrac{1}{2}ih - z)]$$

and

$$K_-(w,z) = [E(z)E(w+\tfrac{1}{2}ih)^- - E^*(z)E(w^- - \tfrac{1}{2}ih)]/[2\pi i(w^- - \tfrac{1}{2}ih - z)].$$

The identity

$$F_+(w+\tfrac{1}{2}ih) + F_-(w-\tfrac{1}{2}ih) = \langle F(t), K(w,t)\rangle$$

holds for every element

$$F(z) = (F_+(z), F_-(z))$$

of the graph. An element of the graph which is orthogonal to itself is orthogonal to every element of the graph.

A partially isometric transformation of the graph onto a dense subspace of the space $\mathcal{H}$ is defined by taking

$$F(z) = (F_+(z), F_-(z))$$

into the entire function

$$F_+(z+\tfrac{1}{2}ih) + F_-(z-\tfrac{1}{2}ih)$$

of z. The reproducing kernel function for function values at w in the space $\mathcal{H}$ is the function

$$[E(z + \tfrac{1}{2}ih)E(w - \tfrac{1}{2}ih)^- - E^*(z + \tfrac{1}{2}ih)E(w^- + \tfrac{1}{2}ih)]/[2\pi i(w^- - z)]$$
$$+ [E(z - \tfrac{1}{2}ih)E(w + \tfrac{1}{2}ih)^- - E^*(z - \tfrac{1}{2}ih)E(w^- - \tfrac{1}{2}ih)]/[2\pi i(w^- - z)]$$

of z for every complex number w.

This completes the construction of a Hilbert space $\mathcal{H}$ of entire functions with the desired reproducing kernel functions when the maximal accretive transformation exists in the space $\mathcal{H}(E)$. The argument is reversed to construct the maximal accretive transformation in the space $\mathcal{H}(E)$ when the Hilbert space of entire functions with the desired reproducing kernel functions exists.

A transformation is defined in the space $\mathcal{H}(E)$ by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space. The graph of the adjoint is a space of pairs

$$F(z) = (F_+(z), F_-(z))$$

such that the adjoint takes the function $F_+(z)$ of z into the function $F_-(z)$ of z. The graph contains

$$K(w, z) = (K_+(w, z), K_-(w, z))$$

with

$$K_+(w, z) = [E(z)E(w - \tfrac{1}{2}ih)^- - E^*(z)E(w^- + \tfrac{1}{2}ih)]/[2\pi i(w^- + \tfrac{1}{2}ih - z)]$$

and

$$K_-(w, z) = [E(z)E(w + \tfrac{1}{2}ih)^- - E^*(z)E(w^- - \tfrac{1}{2}ih)]/[2\pi i(w^- - \tfrac{1}{2}ih - z)]$$

for every complex number w. The elements $K(w, z)$ of the graph span the graph of a restriction of the adjoint. The transformation in the space $\mathcal{H}(E)$ is recovered as the adjoint of its restricted adjoint.

A scalar product is defined on the graph of the restricted adjoint so that an isometric transformation of the graph of the restricted adjoint into the space $\mathcal{H}$ is defined by taking

$$F(z) = (F_+(z), F_-(z))$$

into

$$F_+(z + \tfrac{1}{2}ih) + F_-(z - \tfrac{1}{2}ih).$$

The identity

$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{H}(E)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{H}(E)}$$

holds for all elements

$$F(z) = (F_+(z), F_-(z))$$

of the graph of the restricted adjoint. The restricted adjoint is accretive since scalar self–products are nonnegative in its graph. The adjoint is accretive since the transformation in the space $\mathcal{H}(E)$ is the adjoint of its restricted adjoint.

The accretive property of the adjoint is expressed in the inequality

$$\|F_+(t) - \lambda^- F_-(t)\|_{\mathcal{H}(E)} \leq \|F_+(t) + \lambda F_-(t)\|_{\mathcal{H}(E)}$$

for elements

$$F(z) = (F_+(z), F_-(z))$$

of the graph when λ is in the right half–plane. The domain of the contractive transformation which takes the function

$$F_+(z) + \lambda F_-(z)$$

of z into the function

$$F_+(z) - \lambda^- F_-(z)$$

of z is a closed subspace of the space $\mathcal{H}(E)$. The maximal accretive property of the adjoint is the requirement that the contractive transformation be everywhere defined for some, and hence every, λ in the right half–plane.

Since $K(w, z)$ belongs to the graph for every complex number w, an entire function $H(z)$ of z which belongs to the space $\mathcal{H}(E)$ and is orthogonal to the domain is a solution of the equation

$$H(z) + \lambda H(z + i) = 0.$$

The function vanishes identically if it has a zero since zeros are repeated periodically with period i and since the function

$$H(z)/E(z)$$

of z is of bounded type in the upper half–plane. The space of solutions has dimension zero or one. The dimension is zero since it is independent of λ.

The transformation which takes $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space $\mathcal{H}(E)$ is maximal accretive since it is the adjoint of its adjoint, which is maximal accretive.

This completes the proof of the theorem.

The defining function $E(z)$ of an Euler space of entire functions is of Hermite class since the function

$$E(z - \tfrac{1}{2}ih)/E(z + \tfrac{1}{2}ih)$$

of z is of bounded type and of nonpositive mean type in the upper half–plane when h is in the interval $(0, 1)$. Since the function is bounded by one on the real axis, it is bounded by one in the upper half–plane. The modulus of $E(x + iy)$ is a nondecreasing function of positive y for every real x. An entire function $F(z)$ of z which belongs to the space $\mathcal{H}(E)$ is of Hermite class if it has no zeros in the upper half–plane and if the inequality

$$|F(x - iy)| \leq |F(x + iy)|$$

holds for all real x when y is positive.

In a given Stieltjes space $\mathcal{H}(E)$ multiplication by z is the transformation which takes $F(z)$ into $zF(z)$ whenever the functions of z belong to the space. Multiplication by z need not be a densely defined transformation in the space, but if it is not, the orthogonal complement of the domain of multiplication by z has dimension one. If $E(z) = \Lambda(z) - iB(z)$ for entire functions $A(z)$ and $B(z)$ of z which are real for real z, an entire function

$$S(z) = A(z)u + B(z)v$$

of z which belongs to the orthogonal complement of the domain of multiplication by z is a linear combination of $A(z)$ and $B(z)$ with complex coefficients u and v. This result is a consequence of the identity

$$[K(w,z)S(w) - K(w,w)S(z)]/(z-w) = [K(w^-,z)S(w^-) - K(w^-,w^-)S(z)]/(z-w^-)$$

which characters functions $S(z)$ of z which belong to the space and are linear combinations of u and v. The identity

$$v^-u = u^-v$$

is then satisfied.

When multiplication by z is not densely defined in a Stieltjes space with defining function

$$E(b,z) = A(b,z) - iB(b,z)$$

and when the domain of multiplication by z contains a nonzero element, the closure of the domain of multiplication by z is a Stieltjes space with defining function

$$E(a,z) = A(a,z) - iB(a,z)$$

which is contained isometrically in the given space. The defining function can be chosen so that the matrix equation

$$(A(b,z), B(b,z)) = (A(a,z), B(a,z)) \begin{pmatrix} 1 - \pi uv^-z & \pi uu^-z \\ -\pi vv_-z & 1 + \pi vu^-z \end{pmatrix}$$

holds for complex numbers u and v such that

$$v^-u = u^-v.$$

A Stieltjes space of dimension r whose elements are the polynomials of degree less than r has a polynomial

$$E(r,z) = A(r,z) - iB(z,z)$$

of degree r as defining function. A Stieltjes space of dimension n whose elements are the polynomials of degree n and which is contained isometrically in the given space exists for every positive integer n less than r. The defining function

$$E(n,z) = A(n,z) - iB(n,z)$$

of the space can be chosen so that the matrix equation

$$(A(n+1,z),B(n+1,z)) = (A(n,z),B(n,z))\begin{pmatrix} 1-\pi u_n v_n^- z & \pi u_n u_n^- z \\ -\pi v_n v_n^- z & 1+\pi v_n u_1^- z \end{pmatrix}$$

is satisfied. The initial defining function can be chosen so that the equation hold when n is zero with

$$(A(0,z),B(0,z)) = (1,0).$$

A Stieltjes space with defining function

$$E(t,z) = (n+1-t)E(n+1,z) + (t-n)E(n,z)$$

exists when $n \le t \le n+1$. The space is contained contractively in the Stieltjes space with defining function $E(n+1,z)$ and contains isometrically the Stieltjes space with defining function $E(n,z)$.

A nondecreasing metric function

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

of t in the interval $[0,r]$ is defined by

$$m(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$m(n+1) - m(n) = \begin{pmatrix} \pi u_n u_n^- & \pi v_n u_n^- \\ \pi v_n u_n^- & \pi v_n v_n^- \end{pmatrix}$$

for every nonnegative integer n less than r, and

$$m(t) = (n+1-t)m(n+1) + (t-n)m(n)$$

when $n < t < n+1$.

The differential equation

$$(A'(t,z),B'(t,z))I = z(A(t,z),B(t,z))m'(t)$$

is satisfied when t is in an interval $(n,n+1)$ with the prime indicating differentiation with respect to t and

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since $A(t,z)$ and $B(t,z)$ are continuous functions of t, the integral equation

$$(A(b,z),B(b,z))I - (A(a,z),B(a,z))I = z\int_a^b (A(t,z),B(t,z))dm(t)$$

is satisfied when a and b are in the interval $[0, r]$.

The integral equation for Stieltjes spaces of finite dimension admits a generalization to Stieltjes spaces of infinite dimension. The generalization applies a continuous function of positive t whose values are matrices

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

with real entries such that the matrix inequality

$$m(a) \leq m(b)$$

holds when a is less than b. It is assumed that $\alpha(t)$ is positive when t is positive, that

$$\lim \alpha(t) = 0$$

as t decreases to zero, and that the integral

$$\int_0^1 \alpha(t) d\gamma(t)$$

is finite.

The matrix

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is applied in the formulation of the integral equation. When a is positive, the integral equation

$$M(a, b, z)I - I = z \int_a^b M(a, t, z) dm(t)$$

admits a unique continuous solution

$$M(a, b, z) = \begin{pmatrix} A(a, b, z) & B(a, b, z) \\ C(a, b, z) & D(a, b, z) \end{pmatrix}$$

as a function of b greater than or equal to a for every complex number z. The entries of the matrix are entire functions of z which are self–conjugate and of Hermite class for every b. The matrix has determinant one. The identity

$$M(a, c, z) = M(a, b, z)M(b, c, z)$$

holds when $a \leq b \leq c$.

A bar is used to denote the conjugate transpose

$$M^- = \begin{pmatrix} A^- & C^- \\ B^- & D^- \end{pmatrix}$$

of a square matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with complex entries and also for the conjugate transpose

$$c^- = (c_+^-, c_-^-)$$

of a column vector

$$c = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

with complex entries. The space of column vectors with complex entries is a Hilbert space of dimension two with scalar product

$$\langle u, v \rangle = v^- u = v_+^- u_+ + v_-^- u_- .$$

When a and b are positive with a less than or equal to b, a unique Hilbert space $\mathcal{H}(M(a,b))$ exists whose elements are pairs

$$F(z) = \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

of entire functions of z such that a continuous transformation of the space into the Hilbert space of column vectors is defined by taking $F(z)$ into $F(w)$ for every complex number w and such that the adjoint takes a column vector c into the element

$$[M(a,b,z)IM(a,b,w)^- - I]c/[2\pi(z - w^-)]$$

of the space.

An entire function

$$E(c,z) = A(c,z) - iB(c,z)$$

of z which is of Hermite class exists for every positive number c such that the self–conjugate entire functions $A(c,z)$ and $B(c,z)$ satisfy the identity

$$(A(b,z), B(b,z)) = (A(a,z), B(a,z))M(a,b,z)$$

when a is less than or equal to b and such that the entire functions

$$E(c,z) \exp[\beta(c)z]$$

of z converge to one uniformly on compact subsets of the complex plane as c decreases to zero.

A space $\mathcal{H}(E(c))$ exists for every positive number c. The space $\mathcal{H}(E(a))$ is contained contractively in the space $\mathcal{H}(E(b))$ when a is less than or equal to b. The inclusion is isometric on the orthogonal complement in the space $\mathcal{H}(E(a))$ of the elements which are linear combinations

$$A(a, z)u + B(a, z)v$$

with complex coefficients u and v. These elements form a space of dimension zero or one since the identity

$$v^- u = u^- v$$

is satisfied.

A positive number b is said to be singular with respect to the function $m(t)$ of t if it belongs to an interval (a, c) such that equality holds in the inequality

$$[\beta(c) - \beta(a)]^2 \leq [\alpha(c) - \alpha(a)][\gamma(c) - \gamma(a)]$$

with $m(b)$ unequal to $m(a)$ and unequal to $m(c)$. A positive number is said to be regular with respect to $m(t)$ if it is not singular with respect to the function of t.

If a and c are positive numbers such that a is less than c and if an element b of the interval (a, c) is regular with respect to $m(t)$, then the space $\mathcal{H}(M(a, b))$ is contained isometrically in the space $\mathcal{H}(M(a, c))$ and multiplication by $M(a, b, z)$ is an isometric transformation of the space $\mathcal{H}(M(b, c))$ onto the orthogonal complement of the space $\mathcal{H}(M(a, b))$ in the space $\mathcal{H}(M(a, c))$.

If a and b are positive numbers such that a is less than b and if a is regular with respect to $m(t)$, then the space $\mathcal{H}(E(a))$ is contained isometrically in the space $\mathcal{H}(E(b))$ and an isometric transformation of the space $\mathcal{H}(M(a, b))$ onto the orthogonal complement of the space $\mathcal{H}(E(a))$ in the space $\mathcal{H}(E(b))$ is defined by taking

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

into

$$\sqrt{2}\ [A(a, z)F_+(z) + B(a, z)F_-(z)].$$

A function $\tau(t)$ of positive t with real values exists such that the function

$$m(t) + Iih(t)$$

of positive t with matrix values is nondecreasing for a function $h(t)$ of t with real values if, and only if, the functions

$$\tau(t) - h(t)$$

and

$$\tau(t) + h(t)$$

of positive t with real values are nondecreasing. The function $\tau(t)$ of t, which is continuous and nondecreasing, is called a greatest nondecreasing function such that

$$m(t) + Ii\tau(t)$$

is nondecreasing. The function is unique within an added constant.

If a and b are positive numbers such that a is less than b, multiplication by

$$\exp(ihz)$$

is a contractive transformation of the space $\mathcal{H}(E(a))$ into the space $\mathcal{H}(E(b))$ for a real number h, if, and only if, the inequalities

$$\tau(a) - \tau(b) \leq h \leq \tau(b) - \tau(a)$$

are satisfied. The transformation is isometric when a is regular with respect to $m(t)$.

An analytic weight function $W(z)$ may exist such that multiplication by

$$\exp(i\tau(c)z)$$

is an isometric transformation of the space $\mathcal{H}(E(c))$ into the weighted Hardy space $\mathcal{F}(W)$ for every positive number c which is regular with respect to $m(t)$. The analytic weight function is unique within a constant factor of absolute value one if the function

$$\alpha(t) + \gamma(t)$$

of positive t is unbounded in the limit of large t. The function

$$W(z) = \lim E(c,z) \exp(i\tau(c)z)$$

can be chosen as a limit as c increases to infinity uniformly on compact subsets of the upper half–plane.

If multiplication by

$$\exp(i\tau z)$$

is an isometric transformation of a space $\mathcal{H}(E)$ into the weighted Hardy space $\mathcal{F}(W)$ for some real number τ and if the space $\mathcal{H}(E)$ contains an entire function $F(z)$ whenever its product with a nonconstant polynomial belongs to the space, then the space $\mathcal{H}(E)$ is isometrically equal to the space $\mathcal{H}(E(c))$ for some positive number c which is regular with respect to $m(t)$.

A construction of Euler spaces of entire functions is made from Euler weight functions when a hypothesis is satisfied.

Theorem 3. *If for some real number τ a nontrivial entire function $F(z)$ of z exists such that the functions*

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

of z belong to the weighted Hardy space $\mathcal{F}(W)$ of an Euler weight function $W(z)$, then an Euler space of entire functions exists such that multiplication by $\exp(i\tau z)$ *is an isometric transformation of the space into the weighted Hardy space and such that the space contains every entire function $F(z)$ of z such that the functions*

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

of z belong to the weighted Hardy space.

Proof of Theorem 3. The set of entire functions $F(z)$ such that the functions

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

of z belong to the weighted Hardy space is a vector space with scalar product determined by the isometric property of multiplication as a transformation of the space into the weighted Hardy space. The space is shown to be a Hilbert space by showing that a Cauchy sequence of elements $F_n(z)$ of the space converge to an element $F(z)$ of the space.

Since the elements

$$\exp(i\tau z)F_n(z)$$

and

$$\exp(i\tau z)F_n^*(z)$$

of the weighted Hardy space form Cauchy sequences, a function $F(z)$ of z which is analytic separately in the upper half–plane and the lower half–plane exists such that the limit functions

$$\exp(i\tau z)F(z) = \lim \exp(i\tau z)F_n(z)$$

and

$$\exp(i\tau z)F^*(z) = \lim \exp(i\tau z)F_n^*(z)$$

of z belong to the weighted Hardy space. Since

$$|z - z^-|^{\frac{1}{2}} \exp(i\tau z)F(z)/W(z) = \lim |z - z^-|^{\frac{1}{2}} \exp(i\tau z)F_n(z)/W(z)$$

and

$$|z - z^-|^{\frac{1}{2}} \exp(i\tau z)F^*(z)/W(z) = \lim |z - z^-|^{\frac{1}{2}} \exp(i\tau z)F_n^*(z)/W(z)$$

uniformly in the upper half–plane and since the functions

$$\log |F_n(z)/W(z)|$$

and

$$\log |F_n^*(z)/W(z)|$$

of z are subharmonic in the half–plane

$$-1 < iz^- - iz,$$

the convergence of

$$F(z) = \lim F_n(z)$$

is uniform on compact subsets of the complex plane. The limit function $F(z)$ of z is analytic in the complex plane.

A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) is obtained such that multiplication by $\exp(i\tau z)$ is an isometric transformation of the space into the weighted Hardy space. Since the space contains a nonzero element by hypothesis, it is isometrically equal to a space $\mathcal{H}(E)$.

The space is shown to be an Euler space of entire functions by showing that a maximal accretive transformation is defined in the space for h in the interval $[-1, 1]$ by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space. The accretive property of the transformation is a consequence of the accretive property in the weighted Hardy space.

Maximality is proved by showing that every element of the space is a sum

$$F(z) + F(z + ih)$$

of functions $F(z)$ and $F(z + ih)$ of z which belong to the space.

Since a maximal accretive transformation exists in the weighted Hardy space, every element of the Hilbert space of entire functions is in the upper half–plane a sum

$$F(z) + F(z + ih)$$

of functions $F(z)$ and $F(z + ih)$ of z such that the functions

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F(z + ih)$$

of z belong to the weighted Hardy space. The function $F(z)$ of z admits an analytic continuation to the complex plane. The decomposition applies for all complex z.

The entire function

$$F^*(z) + F^*(z - ih)$$

of z belongs to the Hilbert space of entire functions since the space satisfies the axiom (H3). An entire function $G(z)$ of z exists such that

$$F^*(z) + F^*(z - ih) = G(z) + G(z + ih)$$

and such that the functions

$$\exp(i\tau z)G(z)$$

and

$$\exp(i\tau z)G(z+ih)$$

of z belong to the weighted Hardy space.

Vanishing of the entire function

$$F^*(z) - G(z+ih) = G(z) - F^*(z-ih)$$

of z implies the desired conclusion that the functions $F(z)$ and $F(z+ih)$ of z as well as the functions $G(z)$ and $G(z+ih)$ of z belong to the Hilbert space of entire functions. Vanishing is proved by showing boundedness of the function in the strip

$$-2h < iz^- - iz < 0$$

since the function is periodic of period $2ih$ with modulus which is periodic of period ih.

It can be assumed that the functions

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)G(z)$$

of z are elements of norm at most one in the weighted Hardy space. The inequalities

$$2\pi|F(z)|^2 \leq |\exp(-i\tau z)W(z)|^2/(iz^- - iz)$$

and

$$2\pi|G(z)|^2 \leq |\exp(-i\tau z)W(z)|^2/(iz^- - iz)$$

apply when z is in the upper half–plane. Since the inequalities

$$2\pi|F^*(z)|^2 \leq |\exp(i\tau z)W^*(z)|^2/(iz - iz^-)$$

and

$$2\pi|G(z+ih)|^2 \leq \exp(2\pi h)|\exp(-i\tau z)W(z+ih)|^2/(2h + iz^- - iz)$$

apply when z is in the strip, the inequality

$$\pi|F^*(z) - G(z+ih)|^2 \leq |\exp(i\tau z)W^*(z)|^2/(iz - iz^-)$$
$$+ \exp(2\pi h)|\exp(-i\tau z)W(z+ih)|^2/(2h + iz^- - iz)$$

applies when z is in the strip.

Boundedness of the entire function

$$F^*(z) - G(z+ih)$$

of z in the complex plane follows from the subharmonic property of the logarithm of its modulus. The entire function is a constant which vanishes because of the identity

$$F^*(z) - G(z+ih) = G(z) - F^*(z-ih).$$

This completes the proof of the theorem.

The hypotheses of the theorem are satisfied by an Euler weight function $W(z)$ which satisfies the identity

$$W(z+\tfrac{1}{2}i) = W(z-\tfrac{1}{2}i)\varphi(z)$$

for a function $\phi(z)$ of z which is analytic and has nonnegative real part in the upper half–plane if $\log \phi(z)$ has nonnegative real part in the upper half–plane. The modulus of $W(x+iy)$ is then a nondecreasing function of positive y for every real x.

Since the weight function can be multiplied by a constant, it can be assumed to have value one at the origin. The phase $\psi(x)$ is defined as the continuous function of real x with value zero at the origin such that

$$\exp(i\psi(x))W(x)$$

is positive for all real x. The phase function is a nondecreasing function of real x which is identically zero if it is constant in any interval.

When the phase function vanishes identically, the modulus of $W(x+iy)$ is a constant as a function of positive y for every real x. The weight function is then the restriction of a self–conjugate entire function of Pólya class. For every positive number τ an entire function

$$F(z) = W(z)\sin(\tau z)/z$$

is obtained such that the functions

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

of z belong to the space $\mathcal{F}(W)$. No nonzero entire function $F(z)$ exists such that the functions $F(z)$ and $F^*(z)$ belong to the space $\mathcal{F}(W)$.

When the phase function does not vanish identically, an entire function $E(z)$ of Hermite class which has no real zeros exists such that $E(x)$ is real for a real number x if, and only if, $\psi(x)$ is an integral multiple of π, and then

$$\exp(i\psi(x))E(x)$$

is positive. Such an entire function is unique within a factor of a self–conjugate entire function of Hermite class. The factor is chosen so that the function

$$E(z)/W(z)$$

of z has nonnegative real part in the upper half–plane. The entire functions $E(z)$ and $E^*(z)$ are linearly independent. A nontrivial entire function

$$F(z) = [E(z) - E^*(z)]/z$$

is obtained such that the functions $F(z)$ and $F^*(z)$ of z belong to the space $\mathcal{F}(W)$.

The same conclusions are obtained under a weaker hypothesis.

Theorem 4. *If an Euler weight function $W(z)$ satisfies the identity*

$$W(z + \tfrac{1}{2}i) = W(z - \tfrac{1}{2}i)\phi(z)$$

for a function $\phi(z)$ of z which is analytic and has nonnegative real part in the upper half–plane such that

$$\sigma(z) + \log\phi(z)$$

has nonnegative real part in the upper half–plane for a function $\sigma(z)$ of z which is analytic and has nonnegative real part in the upper half–plane such that the least upper bound

$$\sup \int_{-\infty}^{+\infty} \mathcal{R}\sigma(x+iy)dx$$

taken over all positive y is finite, then for every positive number τ a nontrivial entire function $F(z)$ exists such that the functions

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

of z belong to the weighted Hardy space $\mathcal{F}(W)$.

Proof of Theorem 4. It can be assumed that the symmetry condition

$$\sigma^*(z) = \sigma(-z)$$

is satisfied since otherwise $\sigma(z)$ can be replaced by $\sigma(z)+\sigma^*(-z)$. When h is in the interval $(0,1)$, the integral

$$\log\phi_h(z) = \sin(\pi h)\int_{-\infty}^{+\infty} \frac{\log\phi(z-t)dt}{\cos(2\pi it) + \cos(\pi h)}$$

defines a function $\phi_h(z)$ of z which is analytic and has nonnegative real part in the upper half–plane such that

$$W(z + \tfrac{1}{2}ih) = W(z - \tfrac{1}{2}ih)\phi_h(z).$$

The integral

$$\mathcal{R}\sigma_h(z) = \sin(\pi h) \int_{-\infty}^{+\infty} \frac{\mathcal{R}\sigma(z-t)dt}{\cos(2\pi it) + \cos(\pi h)}$$

and the symmetry condition

$$\sigma_h^*(z) = \sigma_h(-z)$$

define a function $\sigma_h(z)$ which is analytic and has nonnegative real part in the upper half–plane. The function

$$\sigma_h(z) + \log \phi_h(z)$$

of z has nonnegative real part in the upper half–plane since the function

$$\sigma(z) + \log \phi(z)$$

has nonnegative real part in the upper half–plane by hypothesis.

An analytic weight function $U(z)$ which admits an analytic extension without zeros to the half–plane $iz^- - iz > -1$ is defined within a constant factor by the identity

$$\log U(z + \tfrac{1}{2}ih) - \log U(z - \tfrac{1}{2}ih) = \sigma_h(z)$$

for h in the interval $(0,1)$ and by the symmetry

$$U^*(z) = U(-z).$$

The analytic weight function

$$V(z) = U(z)W(z)$$

has an analytic extension without zeros to the half–plane $iz^- - iz > -1$. The modulus of $U(x+iy)$ and the modulus of $V(x+iy)$ are nondecreasing functions of positive y for every real x.

Since

$$\frac{\partial}{\partial y} \log |U(x+iy)| = \pi \int_{-\infty}^{+\infty} \frac{\mathcal{R}\sigma(x+iy-t)dt}{1 + \cos(2\pi it)}$$

the integral

$$\int_{-\infty}^{+\infty} \frac{\partial}{\partial y} \log |U(x+iy)| dx = \int_{-\infty}^{+\infty} \mathcal{R}\sigma(x+iy)dx$$

is a bounded function of positive y. The phase $\psi(x)$ is the continuous, nondecreasing, odd function of real x such that

$$\exp(i\psi(x))U(x)$$

is positive for all real x. Since

$$\frac{\partial}{\partial y} \log |U(x+iy)| = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{d\psi(t)}{(t-x)^2 + y^2}$$

when y is positive, the inequality

$$\psi(b) - \psi(a) \leq \sup \int_{-\infty}^{+\infty} \mathcal{R}\sigma(x+iy)dx$$

holds when a is less than b with the least upper bound taken over all positive y.

The remaining arbitrary constant in $U(z)$ is chosen so that the integral representation

$$\log U(z) = \frac{1}{2\pi} \int_0^\infty \log(1 - z^2/t^2) d\psi(t)$$

holds when z is in the upper half–plane with the logarithm of $1 - z^2/t^2$ defined continuously in the upper half–plane with nonnegative values when z is on the upper half of the imaginary axis. The inequality

$$|U(z)| \leq |U(i|z|)$$

holds when z is in the upper half–plane since

$$|1 - z^2/t^2| \leq 1 + z^- z/t^2.$$

If a positive integer r is chosen so that the inequality

$$\int_{-\infty}^{+\infty} \mathcal{R}\sigma(x+iy)dx \leq 2\pi r$$

holds for all positive y, then the function

$$U(z)/(z+i)^r$$

is bounded and analytic in the upper half–plane.

Since the modulus of $V(x+iy)$ is a nondecreasing function of positive y for every real x, there exists for every positive number τ a nontrivial entire function $G(z)$ such that the functions

$$\exp(i\tau z)G(z)$$

and

$$\exp(i\tau z)G^*(z)$$

of z belong to the weighted Hardy space $\mathcal{F}(V)$. Since the entire function

$$G(z) = F(z)P(z)$$

is the product of an entire function $F(z)$ and a polynomial $P(z)$ of degree r, a nontrivial entire function $F(z)$ is obtained such that the functions

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

of z belong to the weighted Hardy space $\mathcal{F}(W)$.

This completes the proof of the theorem.

The Hilbert spaces of entire functions constructed from an Euler weight function are Euler spaces of entire functions.

Theorem 5. *A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) and which contains a nonzero element is an Euler space of entire functions if it contains an entire function whenever its product with a nonconstant polynomial belongs to the space and if multiplication by* $\exp(i\tau z)$ *is for some real number* τ *an isometric transformation of the space into the weighted Hardy space* $\mathcal{F}(W)$ *of an Euler weight function* $W(z)$.

Proof of Theorem 5. It can be assumed that τ vanishes since the function

$$\exp(-i\tau z)W(z)$$

is an Euler weight function whenever the function $W(z)$ of z is an Euler weight function.

The given Hilbert space of entire functions is isometrically equal to a space $\mathcal{H}(E)$ for an entire function $E(z)$ which has no real zeros since an entire function belongs to the space whenever its product with a nonconstant polynomial belongs to the space.

An accretive transformation is defined in the space $\mathcal{H}(E)$ when h is in the interval $[0, 1]$ by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space since the space is contained isometrically in the space $\mathcal{F}(W)$ and since an accretive transformation is defined in the space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space. It remains to prove the maximal accretive property of the transformation in the space $\mathcal{H}(E)$.

The ordering theorem for Hilbert spaces of entire functions applies to spaces which satisfy the axioms (H1), (H2), and (H3) and which are contained isometrically in a weighted Hardy space $\mathcal{F}(W)$ when a space contains an entire function whenever its product with a nonconstant polynomial belongs to the space. One space is properly contained in the other when the two spaces are not identical.

A Hilbert space $\mathcal{H}$ of entire functions which satisfies the axioms (H1) and (H2) and which contains a nonzero element need not satisfy the axiom (H3). Multiplication by $\exp(iaz)$ is for some real number a an isometric transformation of the space onto a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3).

A space $\mathcal{H}$ which satisfies the axioms (H1) and (H2) and which is contained isometrically in the space $\mathcal{F}(W)$ is defined as the closure in the space $\mathcal{F}(W)$ of the set of those elements of the space which functions $F(z + ih)$ of z for functions $F(z)$ of z belonging to the space $\mathcal{H}(E)$. An example of a function $F(z + ih)$ of z is obtained for every element of the space $\mathcal{H}(E)$ which is a function $F(z)$ of z such that the function $z^2F(z)$ of z belongs to the space $\mathcal{H}(E)$. The space $\mathcal{H}$ contains an entire function whenever its product with a nonconstant polynomial belongs to the space.

The function

$$E(z)/W(z)$$

of z is of bounded type in the upper half–plane and has the same mean type as the function

$$E(z + ih)/W(z + ih)$$

of z which is of bounded type in the upper half–plane. Since the function

$$W(z + ih)/W(z)$$

of z is of bounded type and has zero mean type in the upper half–plane, the function

$$E(z + ih)/E(z)$$

of z is of bounded type and of zero mean type in the upper half–plane.

If a function $F(z)$ of z is an element of the space $\mathcal{H}(E)$ such that the functions

$$F(z)/W(z)$$

and

$$F^*(z)/W(z)$$

of z have equal mean type in the upper half–plane, and such that the functions

$$G(z) = F(z + ih)$$

and

$$G^*(z) = F^*(z - ih)$$

of z belong to the space $\mathcal{F}(W)$, then the functions

$$G(z)/W(z)$$

and

$$G^*(z)/W(z)$$

of z have equal mean type in the upper half–plane. It follows that the space $\mathcal{H}$ satisfies the axiom (H3).

Equality of the spaces $\mathcal{H}$ and $\mathcal{H}(E)$ follows when the space $\mathcal{H}$ is contained in the space $\mathcal{H}(E)$ and when the space $\mathcal{H}(E)$ is contained in the space $\mathcal{H}$.

The function $F(z + ih)$ of z belongs to the space $\mathcal{H}(E)$ whenever the function $F(z)$ of z belongs to the space $\mathcal{H}$ and the function $F(z+ih)$ of z belongs to the space $\mathcal{F}(W)$ since the spaces $\mathcal{H}$ and $\mathcal{H}(E)$ satisfy the axiom (H3). If the space $\mathcal{H}(E)$ is contained in the space $\mathcal{H}$, then the space $\mathcal{H}$ is contained in the space $\mathcal{H}(E)$. If the space $\mathcal{H}$ is contained in the space $\mathcal{H}(E)$, then the space $\mathcal{H}(E)$ is contained in the space $\mathcal{H}$.

Since the transformation T which takes $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space $\mathcal{H}(E)$ is subnormal, the domain of the adjoint T^* of T contains the domain of T. A dense subspace of the graph of T^* is determined by elements of the domain of T^* which belong to the domain of T. The accretive property of T implies the accretive property of T^*. The maximal accretive property of T follows since T is the adjoint of T^*.

This completes the proof of the theorem.

Hypergeometric functions appear in the construction of the Stieltjes spaces for an Euler weight function when a rational function appears in the recurrence relation as the factor which is an analytic function with positive real part in the upper half–plane. When the Stieltjes spaces are known for a given Euler weight function, they can be obtained for related Eulerweight functions.

Assume that for some λ in the upper half–plane an isometric transformation of the weighted Hardy space defined by an Euler weight function $W_-(z)$ onto the set of functions which vanish at λ in the weighted Hardy space defined by an Euler weight function $W_+(z)$ takes a function $F(z)$ of z into the function $(1 - z/\lambda)F(z)$. Since multiplication by a constant of absolute value one does not change the weighted Hardy space of an Euler weight function, it can be assumed that

$$(1 - z/\lambda^-)W_-(z) = W_+(z).$$

Stieltjes spaces of entire functions which are contained contractively in the weighted Hardy space defined by $W_+(z)$ are defined by entire functions $E_+(t, z)$ for positive t. The functions have value one at the origin, depend continuously on t, and satisfy the equation

$$(A_+(b, z), B_+(b, z))I - (A_+(a, z), B_+(a, z))I = z \int_a^b (A_+(t, z), B_+(t, z))dm_+(t)$$

when a and b are positive for a continuous nondecreasing matrix function

$$m_+(t) = \begin{pmatrix} \alpha_+(t) & \beta_+(t) \\ \beta_r(t) & \gamma_+(t) \end{pmatrix}$$

of positive t. The Stieltjes space defined by $E_r(t, z)$ is contained isometrically in the weighted Hardy space defined by $W_+(z)$ when t is regular with respect to m_+. The union of the Stieltjes spaces is dense in the weighted Hardy space. The intersection of the Stieltjes spaces has dimension at most one.

The Stieltjes space defined by $E_r(t, z)$ is assumed to have dimension greater than one for every positive number t. A Stieltjes space exists for every positive number t such that multiplication by $1 - z/\lambda$ is an isometric transformation of the space onto the set of functions which vanish at λ in the Stieltjes space defined by $E_+(t, z)$. The space is contained contractively in the weighted Hardy space defined by $W_-(z)$ with an isometric inclusion when t is regular with respect to m_+. The defining function $E_-(t, z)$ of the space can be chosen so that

$$(1 - z/\lambda)[B_-(t, z)A_+(t, \lambda) - A_-(t, z)B_+(t, \lambda)] = B_+(t, z)A_+(t, \lambda) - A_+(t, z)B_+(t, \lambda).$$

The function has value one at the origin and depends continuously on t. The equation

$$(A_-(b, z), B_-(b, z))I - (A_-(a, z), B_-(a, z))I = z \int_a^b (A_-(t, z), B_-(t, z))dm_-(t)$$

holds when a and b are positive for a continuous nondecreasing matrix function

$$m_-(t) = \begin{pmatrix} \alpha-(t) & \beta_-(t) \\ \beta_-(t) & \gamma_-(t) \end{pmatrix}$$

of positive t. The union of the Stieltjes space is dense in the weighted Hardy space defined by $W_-(t,z)$. The intersection of the Stieltjes spaces has at most dimension one.

A matrix $P(t)$ with real entries and determinant one is defined for every positive t by the equation

$$|\lambda| \mathcal{R}[iB_+(t,\lambda)A_+(t,\lambda)^-]P(t)$$
$$= \begin{pmatrix} \mathcal{R}[i\lambda A_+(t,\lambda)B_+(t,\lambda)^-] & \\ sR[i\lambda B_+(t,\lambda)B_+(t,\lambda)^-] & \\ -\mathcal{R}[i\lambda A_+(t,\lambda)A_+(t,\lambda)^-] & -\mathcal{R}[i\lambda B_+(t,\lambda)A_+(t,\lambda))^-] \end{pmatrix}.$$

The matrix depends continuously on t. The identity

$$m_-(b) - m_-(a) = \int_a^b P(t)dm_+(t)P(t)^-$$

holds when a and b are positive.

2. Fourier Analysis on the Complex Skew–Plane

The Stieltjes spaces constructed from the gamma function apply to Fourier analysis for the complex skew–plane. The complex skew–plane is a vector space of dimension four over the real numbers which contains the complex plane as a vector subspace of dimension two. The multiplicative structure of the complex plane as a field is generalized as the multiplicative structure of the complex skew–plane as a skew–field. The conjugation of the complex plane is an automorphism which extends as an anti–automorphism of the complex skew–plane.

An element

$$\xi = t + ix + jy + kz$$

of the complex skew–plane has four real coordinates x, y, z, and t. The conjugate is

$$\xi^- = t - ix - jy - kz.$$

The multiplication table

$$ij = k, jk = i, ki = j$$
$$ji = -k, kj = -i, ki = -j$$

defines a conjugated algebra in which every nonzero element has an inverse. An automorphism of the skew–field is an inner automorphism which is defined by an element with conjugate as inverse. A plane is a maximal commutative subalgebra. Every plane is isomorphic to every other plane under an automorphism of the skew–field. The complex plane is the subalgebra of elements which commute with i.

The topology of the complex skew–plane is derived from the topology of the real line as is the topology of the complex plane. Addition and multiplication are continuous as transformations of the Cartesian product of the space with itself into the space. The topology of the real line is derived from Dedekind cuts. A real number t divides the real line into two open half–lines (t, ∞) and $(-\infty, t)$. The intersection of open half–lines is an open interval (a, b) when it is nonempty and not a half–line. A open subset of the line is a union of open intervals. The topology of the plane is the Cartesian product topology of Dedekind topologies of two coordinate lines. The topology of the complex skew–plane is the Cartesian product topology of the Dedekind topologies of four coordinate lines.

The canonical measure for the complex skew–plane is derived from the canonical measure for the real line as is the canonical measure for the complex plane. In all cases the canonical measure is defined on the Baire subsets of the space defined as the smallest class of sets containing the open sets and the closed sets and containing countable unions and countable intersections of sets of the class. A measure preserving transformation of the space onto itself is defined by taking ξ into $\xi + \eta$ for every element η of the space. This condition determines the canonical measure within a constant factor.

The canonical measure for the real line is Lebesgue measure, which assigns measure one to the interval $(0, 1)$. The canonical measure for the complex plane is the Cartesian product measure of the canonical measure of two coordinate lines, which assigns measure π to the unit disk $z^- z < 1$. The canonical measure for the complex skew–plane is the Cartesian product measure of the canonical measures of four coordinate lines, which assigns measure $\frac{1}{2}\pi^2$ to the unit disk $\xi^- \xi < 1$.

Multiplication by an element ξ of the space multiplies the canonical measure by $|\xi|$ in the case of the line, by $|\xi|^2$ in the case of the plane, and by $|\xi|^4$ in the case of the skew–plane with $|\xi|$ the nonnegative solution of the equation

$$|\xi|^2 = \xi^- \xi.$$

The modulus $|\xi|$ of ξ defines a metric on the space whose topology is identical with the Dedekind topology. The identity

$$|\xi\eta| = |\xi||\eta|$$

holds for all elements ξ and η of the space.

The ring of integers is a subset of the real line whose special properties include an Euclidean algorithm for greatest common divisors. If α is an integer and if β is a nonzero integer, an integer γ exists such that the inequality

$$(\alpha - \beta\gamma)^2 < \beta^2$$

is satisfied. An ideal of integers which contains a nonzero element contains a nonzero element β which minimizes the positive integer β^2. Every element α of the ideal is a product

$$\alpha = \beta\gamma$$

for an integer γ.

The ring of Gauss integers is a subset of the complex plane which admits an Euclidean algorithm. A Gauss integer

$$x + iy$$

is a complex number whose coordinates x and y are integers. If α is a Gauss integer and if β is a nonzero Gauss integer, a Gauss integer γ exists such that the inequality

$$(\alpha - \beta\gamma)^-(\alpha - \beta\gamma) < \beta^-\beta$$

is satisfied. An ideal of Gauss integers which contains a nonzero element contains a nonzero element β which minimizes the positive integer $\beta^-\beta$. Every element α of the ideal is a product

$$\alpha = \beta\gamma$$

for a Gauss integer γ.

The ring of Hurwitz integers is a subset of the complex skew–plane which admits an Euclidean algorithm. A Hurwitz integer is a quaternion

$$t + ix + jy + kz$$

whose coordinates are all integers or all halves of odd integers. If α is a Hurwitz integer and if β is a nonzero Hurwitz integer, a Hurwitz integer γ exists such that the inequality

$$(\alpha - \beta\gamma)^-(\alpha - \beta\gamma) < \beta^-\beta$$

is satisfied. A right ideal of Hurwitz integers which contains a nonzero element contains a nonzero element β which minimizes the positive integer $\beta^-\beta$. Every element α of the right ideal is a product

$$\alpha = \beta\gamma$$

for a Hurwitz integer γ.

The group of Hurwitz integers with conjugate as inverse contains twenty–four elements with a normal subgroup of eight elements whose quotient group is cyclic of three elements.

Integral elements of the complex skew–plane are defined as the integral elements of a skew–field which contains the Hurwitz integers and which has finite dimension as a vector space over the Gauss field whose elements are complex numbers with rational numbers as coordinates. The skew–field is given the discrete topology.

A discrete skew–field is defined as the set of elements of the complex skew–plane which are sums

$$a + kb$$

for elements a and b of a discrete field. A construction of the discrete field is made from the algebra of polynomials with coefficients in the Gauss field.

The polynomial algebra has an Euclidean algorithm. If $A(z)$ is a polynomial and if $B(z)$ is a nonzero polynomial, a polynomial $C(z)$ exists such that the degree of the polynomial

$$A(z) - B(z)C(z)$$

is less than the degree of $B(z)$.

An ideal of the polynomial algebra which contains a nonzero element contains a nonzero element $B(z)$ of least degree. The minimal polynomial is chosen with one as the coefficient to the highest power of z which has a nonzero coefficient. An element

$$A(z) = B(z)C(z)$$

of the ideal is the product of the minimal polynomial with a polynomial $C(z)$.

The discrete field is assumed to contain ξ^- whenever it contains ξ. If ξ is an element of the discrete field a homomorphism of the polynomial algebra into the discrete field is defined by taking $P(z)$ into $P(\xi)$. The kernel of the homomorphism is a maximal ideal whose quotient field is mapped isomorphically into the discrete field. The homomorphism commutes with conjugation if ξ is self–conjugate. The ideal is generated by a nonzero polynomial whose degree is no greater than the dimension of the discrete field as a vector space over the Gauss field. The minimal polynomial is chosen with one as the coefficient of the highest power of z which has a nonzero coefficient. The minimal polynomial has rational numbers as coefficients when ξ is self–conjugate. An element ξ of the discrete field is said to be integral if the coefficients of its minimal polynomial are Gauss integers.

The image of the polynomial algebra need not contain every element of the discrete field. But the elements of the discrete field which are obtained belong to a subfield which can be used to replace the Gauss field in the previous construction. An element of the discrete field is found which defines a homomorphism of the polynomial algebra onto a larger subfield of the discrete field. Iteration produces an element of the discrete field which generates the discrete field by a homomorphism of the polynomial algebra. The generating element can be chosen self–conjugate.

Sums and products of integral elements of the discrete field are integral. The conjugate of an integral element of the discrete field is integral.

An element

$$a + kb$$

of the discrete skew–field is said to be integral if $a + a^-$, $a - a^-$, $b + b^-$, $b - b^-$ are integral elements of the discrete field whose quotients by two are all integral or all nonintegral. Sums and products of integral elements of the discrete skew–field are integral. The conjugate of an element of the discrete skew–field is integral. An element of the complex skew–plane with rational numbers as coordinates is an integral element of the discrete skew–field if, and only if, it is a Hurwitz integer.

A simplification occurs when all roots of the minimal polynomial of a generating element of the discrete field belong to the field. The automorphisms of the discrete field then acts as a transitive group of permutations of the roots of the minimal polynomial. An automorphism of the discrete field commutes with conjugation and takes integral elements into integral elements. The automorphism admits a unique extension as an automorphism of the discrete skew–field which leaves k fixed. The extension commutes with conjugation and takes integral elements into integral elements. If a Hurwitz integer has conjugate as inverse, an inner automorphism of the discrete skew–field which commutes with conjugation and takes integral elements into integral elements is defined by taking ξ into $\omega^- \xi \omega$. Every automorphism of the discrete skew–field which takes integral elements into integral elements is the composition in either order of the inner automorphism defined by a Hurwitz integer with conjugate as inverse and an automorphism which takes the discrete field into itself and leaves k fixed.

The automorphisms of the discrete skew–field permit the construction on a scalar product for the discrete skew–field as a right vector space over the Gauss field. A linear functional on the skew–field is a transformation L of the skew–field into the Gauss field which takes $\xi\alpha + \eta\beta$ into $(L\xi)\alpha + (L\eta)\beta$ for all elements ξ and η of the skew–field and all elements α and β of the Gauss field.

The scalar product of elements ξ and η of the skew–field is the element $\langle \xi, \eta \rangle$ of the Gauss field defined by

$$\langle \xi, \eta \rangle \sum 1 = \sum \sigma(\eta)^- \sigma(\xi)$$

with summation over the automorphisms σ of the skew field which take integral elements into integral elements. The sum on the left is the number of automorphisms.

The properties of a scalar product are satisfied: A linear functional on the skew–field is defined by taking ξ into $\langle \xi, \eta \rangle$ for every element η of the skew–field symmetry

$$\langle \eta, \xi \rangle = \langle \xi, \eta \rangle^-$$

holds for all elements ξ and η of the skew–field. Positivity states that the scalar self–product

$$\langle \xi, \xi \rangle > 0$$

is positive for every nonzero element of the skew–field.

The definition of a scalar product is initially made for a discrete skew–field when all roots of the minimal polynomial of the discrete field belong to the field. A discrete skew–field which does not satisfy the hypothesis is however contained in a discrete skew–field which satisfies the hypothesis. The scalar product for the given skew–field as defined as the restriction of the scalar product for the extension field.

The modulus is a function defined on the discrete skew–field which vanishes at the origin and which defines a homomorphism of the group of nonzero elements into the multiplicative group of positive real numbers. The modulus is determined by its values on self–conjugate elements of the skew–field since it satisfies the identity

$$\lambda(\xi)^2 = \lambda(\xi^- \xi).$$

A nonzero self–conjugate element ξ defines a linear transformation of the space of self–conjugate elements, treated as a vector space over the rational numbers, into itself by taking η into $\xi\eta$. The modulus $\lambda(\xi)$ is defined as the positive rational number which is the determinant of the transformation. When ξ is integral, $\lambda(\xi)$ is equal to the number of elements in the quotient ring of the ring of self–conjugate integral elements by the ideal generated by ξ.

A nontrivial ideal of the ring of self–conjugate integral elements of the discrete skew–field has a finite quotient ring and is uniquely determined by the number of elements of the quotient ring. The quotient ring modulo ρ of the ring of self–conjugate integral elements is defined as the unique quotient ring with ρ elements when such a quotient ring exists.

An ideal of the ring of integral elements of the complex skew–plane is generated by the ideal of self–conjugate integral elements. The quotient ring modulo ρ of the ring of integral elements of the complex skew–plane is defined as the quotient ring by the generated ideal. When ρ is odd, elements of the quotient ring modulo ρ of the ring of integral elements of the complex skew–plane are represented by quaternions

$$t + ix + jy + kz$$

whose coordinates are elements of the quotient ring modulo ρ of the ring of self–conjugate integral elements. The same representation applies when n is even except the coordinates are divided by two when they are all odd.

When r and s are relatively prime positive integers such that quotient rings modulo r and modulo s exist of the ring of integral elements of the complex skew–plane, a quotient ring modulo rs exists and is isomorphic to the Cartesian product of the quotient ring modulo r and the quotient ring modulo s.

When a quotient ring modulo ρ exists for a positive integer ρ which has only one prime divisor, there is a divisor n of ρ divisible by p such that a quotient ring modulo n exists for the ring of integral elements of the complex skew–plane. A quotient ring modulo n^k exists for every nonnegative integer k. A positive integer k exists such that

$$\rho = n^k.$$

When p is a prime and ρ is the least positive power of p for which a quotient ring modulo ρ exists, the ring of self–conjugate integral elements of the complex skew–plane modulo ρ is a field.

When ρ is odd, a skew–conjugate integral element

$$ia + jb + kc$$

of the complex skew–plane modulo ρ is represented by a quaternion with coordinates in the field of self–conjugate integral elements. The group of nonzero elements is cyclic of order $\rho - 1$. Half of the elements are squares of elements of the group and half are not. A choice of coordinates can be made so that the sum

$$a^2 + b^2 + c^2$$

is equal to any desired self–conjugate element. When the sum is not minus the square of a self–conjugate element, the set of integral elements of the complex skew–plane modulo ρ which commute with the skew–conjugate element is a field.

If κ is the choice of a nonzero skew–conjugate element which anti–commutes with

$$ia + jb + kc,$$

then every integral element

$$\alpha + \kappa\beta$$

of the complex skew–plane modulo ρ is a unique sum with α and β in the commuting field. Since the identity

$$\gamma\kappa = \kappa\gamma^-$$

holds for every element γ of the commuting field, the identity

$$(\alpha + \kappa\beta)^-(\alpha + \kappa\beta) = \alpha^-\alpha - \kappa^2\beta^-\beta$$

holds for all elements α and β of the commuting field. The equation

$$\alpha^-\alpha = \kappa^2\beta^-\beta$$

admits a solution in nonzero elements α and β of the commuting field since there exists no finite skew–field. An integral element ξ of the complex skew–plane modulo ρ is invertible if, and only if, $\xi^-\xi$ is nonzero. The number of integral elements of the complex skew–plane modulo ρ which are nonzero and noninvertible is

$$(\rho - 1)(\rho + 1)^2.$$

When ρ is the even prime, the set of integral elements of the complex skew–plane modulo ρ which commute with

$$\tfrac{1}{2} + \tfrac{1}{2}i + \tfrac{1}{2}j + \tfrac{1}{2}k$$

is a field of four elements. When ρ is a greater power of the even prime, every self–conjugate integral element of the complex skew–plane modulo ρ is the square of a self–conjugate integral element of the complex skew–plane modulo p. The set of self–conjugate integral elements of the complex skew–plane modulo ρ is the only subfield of the ring of integral elements of the complex skew–plane modulo ρ.

The group of nonzero elements of the discrete skew–field has normal subgroups whose quotient groups are finite. A normal subgroup is generated for every positive integer r by the nonzero integral elements ξ of the discrete skew–field such that for every prime divisor p of r the greatest power of p which is a divisor of $\lambda(\xi^-\xi)$ is a power of the greatest power of p which is a divisor of r. A fundamental domain for the equivalence relation defined by the subgroup is the set of nonzero integral elements ξ of the skew–field such that for every prime p the greatest power of p which is a divisor of $\lambda(\xi^-\xi)$ is less than the greatest power of p which is a divisor of r. The intersection of the normal subgroups is the group of elements of the discrete skew–field with conjugate as inverse.

The canonical measure for the complex plane is the Cartesian product measure of the Lebesgue measures for two coordinate lines. The canonical measure for the complex skew–plane is the Cartesian product measure of the Lebesgue measures of four coordinate lines.

The Fourier transformation for the real line is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure into itself which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(2\pi i\xi\eta) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \int \exp(-2\pi i\xi\eta) g(\eta) d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

The Fourier transformation for the complex plane is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure into itself which takes an integrable function into the continuous function

$$g(\xi) = \int \exp(\pi i(\xi^-\eta + \eta^-\xi)) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \int \exp(-\pi i(\xi^-\eta + \eta^-\xi)) g(\eta) d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

The Fourier transformation for the complex skew–plane is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure into itself which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(\pi i^-(\xi^-\eta + \eta^-\xi)) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \int \exp(-\pi i(\xi^-\eta + \eta^-\xi)) g(\eta) d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

The Fourier transformation for the complex skew–plane commutes with the isometric transformations of the Hilbert space onto itself which are defined by taking a function $f(\xi)$ of ξ into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ for every element ω of the complex skew–plane with conjugate as inverse. The Hilbert space decomposes into the orthogonal sum of invariant subspaces for the transformations taking a function $f(\xi)$ of ξ into the function $f(\omega\xi)$ of ξ for every element ω of the complex skew–plane with conjugate as inverse.

A homomorphism of the multiplicative group of nonzero elements of the complex skew–plane onto the multiplicative group of the positive half–line is defined by taking ξ into $\xi^-\xi$. The identity

$$\int |f(\xi^-\xi)|^2 d\xi = \pi^2 \int |f(\xi)|^2 \xi d\xi$$

holds with integration on the left with respect to the canonical measure for the complex skew–plane and with integration on the right with respect to Lebesgue measure for every Baire function $f(\xi)$ of ξ in the positive half–line.

The Hilbert space of homogeneous polynomials of degree ν is the set of functions

$$\xi = t + ix + jy + kz$$

of ξ in the complex skew–plane which are linear combinations of monomials

$$x^a y^b z^c t^d$$

whose exponents are nonnegative integers with sum

$$\nu = a + b + c + d.$$

The monomials are an orthogonal set with

$$\frac{a!b!c!d!}{\nu!}$$

as the scalar self–product of the monomial with exponents $a, b, c,$ and d. The function

$$2^{-\nu}(\eta^-\xi + \xi^-\eta)^\nu$$

of ξ in the complex skew–plane belongs to the space for every element η of the complex skew–plane and acts as reproducing kernel function for function values at η.

Isometric transformations of the Hilbert space of homogeneous polynomials of degree ν into itself are defined by taking a function $f(\xi)$ of ξ in the complex skew–plane into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ in the complex skew–plane for every element ω of the complex skew–plane with conjugate as inverse.

The Laplacian
$$\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$
takes homogeneous polynomials of degree ν into homogeneous polynomials of degree $\nu - 2$ when ν is greater than one and annihilates polynomials of smaller degree. The Laplacian commutes with the transformations which take a function $f(\xi)$ of ξ in the complex skew–plane into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ in the complex skew–plane for every element ω of the complex skew–plane with conjugate as inverse.

A homogeneous polynomial of degree ν is said to be harmonic if it is annihilated by the Laplacian. Homogeneous polynomials $f(\omega\xi)$ and $f(\xi\omega)$ of degree ν are harmonic for every element ω of the complex skew–plane with conjugate as inverse if the homogeneous polynomial $f(\xi)$ of degree ν is harmonic.

The Hilbert space of homogeneous harmonic polynomials of degree ν is the orthogonal complement in the space of homogeneous polynomials of degree ν of product of $\xi^-\xi$ with homogeneous polynomials of degree $\nu - 2$ when ν is greater than one. The space of homogeneous polynomials of degree ν has dimension
$$(\nu + 1)(\nu + 2)(\nu + 3)/6.$$
The space of homogeneous harmonic polynomials of degree ν has dimension
$$(\nu + 1)^2.$$

The function
$$\frac{(\eta^-\xi)^{\nu+1} - (\xi^-\eta)^{\nu+1}}{\eta^-\xi - \xi^-\eta}$$
of ξ in the complex skew–plane belongs to the space of homogeneous harmonic polynomials of degree ν for every element η of the complex skew–plane and acts as reproducing kernel function for function values at η.

The spaces of homogeneous harmonic polynomials of degree ν are contained in a Hilbert space of harmonic functions for every nonnegative integer ν. Homogeneous harmonic polynomials of unequal degree are orthogonal. The elements of the space are harmonic in the unit disk $\xi^-\xi < 1$. The function
$$\frac{1}{(1 - \eta^-\xi)(1 - \xi^-\eta)}$$
of ξ in the disk belongs to the space when η is an element of the disk and acts as reproducing kernel function for function values at η.

Isometric transformations of the space onto itself are defined for every element ω of the complex skew–plane with conjugate as inverse by taking a function $f(\xi)$ of ξ in the disk into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ in the disk.

The boundary of the disk, which is the set of elements of the complex skew–plane with conjugate as inverse, is a compact Hausdorff space in the subspace topology inherited from the complex skew–plane. The canonical measure for the space is the essentially unique nonnegative measure on its Baire subsets such that measure preserving transformations are defined on multiplication left or right by an element of the space. Uniqueness is obtained by stipulating that the full space has measure

$$\pi^2.$$

A homomorphism of the multiplicative group of nonzero elements of the complex skew–plane onto the positive half–line is defined by taking ξ into $\xi^- \xi$. The canonical measure for the complex skew–plane is mapped to the measure whose value on a Baire set E is the integral

$$\pi^2 \int t dt$$

over the set E. The canonical measure for the complex skew–plane is the Cartesian product measure of the canonical measure for the boundary of the disk and the image measure on Baire subsets of the positive half–line.

A continuous function $f(\omega)$ of ω on the boundary of the disk admits a continuous extension as a harmonic function

$$f(\xi) = \frac{1}{\pi^2} \int \frac{f(\omega) d\omega}{(1 - \omega^- \xi)(1 - \xi^- \omega)}$$

of ξ in the disk defined by integration with respect to the canonical measure for the boundary of the disk. The extended function is continuous in the closed disk.

A function

$$f(\xi) = \frac{1}{\pi^2} \int \frac{d\mu(\omega)}{(1 - \omega^- \xi)(1 - \xi^- \omega)}$$

of ξ in the disk which is harmonic and has nonnegative values is represented by a nonnegative measure μ on the Baire subsets of the boundary of the disk. The value

$$\mu(E) = \lim \int f(t\omega) d\omega$$

of the measure on a regular open set E is the limit as t increases to one of integrals over E with respect to the canonical measure for the boundary.

The complementary space to the complex plane in the complex skew–plane is the set of elements η of the complex skew–plane which satisfy the identity

$$\xi \eta = \eta \xi^-$$

for every element ξ of the complex plane. An element η of the complex skew–plane is skew–conjugate:

$$\eta^- = -\eta.$$

Multiplication on left or right by η is an injective transformation of the complex plane onto the complementary space for every nonzero element η of the complementary space. The transformation is continuous when the complementary space is given the topology inherited from the complex skew–plane. The transformation takes the canonical measure for the complex plane into $\eta^-\eta$ times the measure defined as the canonical measure for the complementary space.

An element of the complex skew–plane is the unique sum $\alpha + \beta$ of an element α of the complex plane and an element β of the complementary space. The topology of the complex skew–plane is the Cartesian product topology of the topology of the complex plane and the topology of the complementary space. The canonical measure for the complex skew–plane is the Cartesian product measure of the canonical measure for the complex plane and the canonical measure for the complementary space.

Radon transformations for the complex skew–plane are maximal accretive transformations in the Hilbert space of square integrable functions with respect to the canonical measure for the complex skew–plane.

A Radon transformation of harmonic ϕ is defined for a harmonic polynomial ϕ of degree ν which has norm one in the Hilbert space of harmonic functions of order ν. The domain and range of the transformation are contained in the Hilbert space of functions $f(\xi)$ of ξ in the complex skew–plane which are square integrable with respect to the canonical measure for the complex skew–plane and which satisfy the identity

$$\phi(\xi)f(\omega\xi) = \phi(\omega\xi)f(\xi)$$

for every element ω of the complex skew–plane with conjugate as inverse.

The Radon transformation of harmonic ϕ takes a function $f(\xi)$ of ξ in the complex skew–plane into a function $g(\xi)$ of ξ in the complex skew–plane when the identity

$$g(\omega\xi)/\phi(\omega\xi) = \int f(\omega\xi + \omega\eta)/\phi(\omega\xi + \omega\eta)d\eta$$

holds when ξ is in the complex plane for every element ω of the complex skew–plane with conjugate as inverse with integration with respect to the canonical measure for the complementary space to the complex plane in the complex skew–plane.

The integral is interpreted as a limit of integrals over disks $\eta^-\eta < n$ in the complementary space. The limit is taken in the metric topology of the Hilbert space of square integrable functions with respect to the canonical measure for the complex skew–plane.

Spectral analysis of the Radon transformation of harmonic ϕ is given by the Laplace transformation of harmonic ϕ. The domain of the Laplace transformation of harmonic ϕ is contained in the Hilbert space of functions $f(\xi)$ of ξ in the complex skew–plane which are square integrable with respect to the canonical measure and which satisfy the identity

$$\phi(\xi)f(\omega\xi) = \phi(\omega\xi)f(\xi)$$

for every element ω of the complex skew–plane with conjugate as inverse.

The function

$$\phi(\xi)\exp(\pi i\xi^-\xi)$$

of ξ in the complex skew–plane is an eigenfunction of the Radon transformation of harmonic ϕ for the eigenvalue

$$i/z$$

when z is in the upper half–plane.

The canonical measure for the upper half–plane is defined as the restriction to Baire subsets of the upper half–plane of the canonical measure for the complex plane.

A function

$$f(\xi) = \phi(\xi)h(\xi^-\xi)$$

of ξ in the complex skew–plane which belongs to the domain of the Laplace transformation of harmonic ϕ is parametrized by a function $h(z)$ of z in the upper half–plane admitting an extension to the complex plane satisfying the identity

$$h(\omega z) = h(z)$$

for every element ω of the complex plane with conjugate as inverse. The identity

$$\int |f(\xi)|^2 d\xi = \pi \int |\xi|^\nu |h(\xi)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the complex skew–plane and integration on the right with respect to the canonical measure for the upper half–plane.

An element of the range of the Laplace transformation of harmonic ϕ for the complex skew–plane is an analytic function

$$h^\wedge(z) = \pi \int_0^\infty t^\nu h(t)\exp(\pi itz)tdt$$

of z in the upper half–plane defined by a function $h(z)$ of z in the upper half–plane which admits an extension to the complex plane satisfying the identity

$$h(\omega z) = h(z)$$

for every element ω of the complex plane with conjugate as inverse such that the integral

$$\int |\xi|^\nu |h(\xi)|^2 d\xi$$

with respect to the canonical measure for the upper half–plane converges. The identity

$$\int_0^\infty \int_{-\infty}^{+\infty} |h^\wedge(x+iy)|^2 y^\nu dxdy = (2\pi)^{-\nu}\Gamma(1+\nu)\int_0^\infty t^\nu |h(t)|^2 tdt$$

is satisfied.

The range of the Laplace transformation of harmonic ϕ contains every function $h^\wedge(z)$ analytic in the upper half–plane for which the integral on the left converges. The range is a Hilbert space whose scalar self–product is defined by the integral.

A continuous linear functional is defined on the space by taking a function $h^\wedge(z)$ of z into its value $h^\wedge(w)$ at an element w of the upper half–plane. The reproducing kernel function

$$\frac{1+\nu}{4\pi}\ [\tfrac{1}{2}\ i(w^- - z)]^{-2-\nu}$$

for function values at w is a function of z in the upper half–plane obtained from the integral

$$\frac{1+\nu}{4\pi}\ (2\pi)^{-\nu}\Gamma(1+\nu)[\tfrac{1}{2}i(w^- - z)]^{-2-\nu} = \pi \textstyle\int_0^\infty t^\nu \exp(\pi i t(z - w^-))tdt.$$

The identity

$$h^\wedge(w) = \frac{1+\nu}{4\pi} \int_0^\infty \int_{-\infty}^{+\infty} \frac{h^\wedge(x+iy)y^\nu dxdy}{[\frac{1}{2}\ i(x - iy - w)]^{2+\nu}}$$

holds when w is in the upper half–plane.

The adjoint of the transformation which takes $h^\wedge(z)$ into $(z/i)h^\wedge(z)$ whenever the functions of z belong to the range of the Laplace transformation of harmonic ϕ takes the function

$$[\tfrac{1}{2}\ i(w^- - z)]^{-2-\nu}$$

of z into the function

$$(w/i)^-[\tfrac{1}{2}\ i(w^- - z)]^{-2-\nu}$$

of z for every element w of the upper half–plane.

The transformation which takes $h^\wedge(z)$ into $(z/i)h^\wedge(z)$ whenever the functions of z belong to the range of the Laplace transformation of harmonic ϕ is maximal accretive. The maximal accretive property is a consequence of the computation of reproducing kernel functions. A Hilbert space exists whose elements are functions analytic in the upper half–plane and which contains the function

$$(iw^- - iz)[\tfrac{1}{2}\ i(w^- - z)]^{-2-\nu}$$

as reproducing kernel function for function values at w when w is in the upper half–plane.

The adjoint of the Radon transformation of harmonic ϕ takes a function $f(\xi)$ of ξ in the complex skew–plane into a function $g(\xi)$ of ξ in the complex skew–plane when the identity

$$\int \phi(\xi)^- g(\xi)\exp(\pi i z\xi^-\xi)d\xi = (i/z)\int \phi(\xi)^- f(\xi)\exp(\pi i z\xi^-\xi)d\xi$$

holds when z is in the upper half–plane with integration with respect to the canonical measure for the complex skew–plane. The transformation is maximal accretive.

The Fourier transform for the complex skew–plane of the function

$$\phi(\xi)\exp(\pi iz\xi^-\xi)$$

of ξ in the complex skew–plane is the function

$$i^\nu(i/z)^{2+\nu}\phi(\xi)\exp(-\pi iz^{-1}\xi^-\xi)$$

of ξ in the complex skew–plane when z is in the upper half–plane. Since the Fourier transformation commutes with the transformations which take a function $f(\xi)$ of ξ into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ for every element ω of the complex skew–plane with conjugate as inverse, it is sufficient to make the verification when

$$\phi(t+ix+jy+kz) = (t+ix)^\nu.$$

The verification reduces to showing that the Fourier transform for the complex plane of the function

$$\xi^\nu\exp(\pi iz\xi^-\xi)$$

of ξ in the complex plane is the function

$$i^\nu(i/z)^{1+\nu}\xi^\nu\exp(-\pi iz^{-1}\xi^-\xi)$$

of ξ in the complex plane. It is sufficient by analytic continuation to make the verification when z lies on the imaginary axis. It remains by a change of variable to show that the Fourier transform of the function

$$\xi^\nu\exp(-\pi\xi^-\xi)$$

of ξ in the complex plane is the function

$$i^\nu\xi^\nu\exp(-\pi\xi^-\xi)$$

of ξ in the complex plane.

The desired identity follows since

$$i^\nu\xi^\nu\exp(-\pi\xi^-\xi) = \sum_{k=0}^\infty \xi^\nu\int\frac{(\pi i\xi^-\xi)^k(\pi i\eta^-\eta)^{\nu+k}}{k!(\nu+k)!}\exp(-\pi\eta^-\eta)d\eta$$

where

$$\exp(\pi i(\xi^-\eta+\eta^-\xi)) = \sum_{n=0}^\infty\frac{(\pi i\xi^-\eta+\pi i\eta^-\xi)^n}{n!}$$

and

$$(\pi i\xi^-\eta+\pi i\eta^-\xi)^n = \sum_{k=0}^n\frac{(\pi i\eta^-\xi)^{n-2k}(\pi i\xi^-\xi)^k(\pi i\eta^-\eta)^k}{k!(n-k)!}$$

where

$$\int (\pi i \eta^- \eta)^{\nu+k} \exp(-\pi \eta^- \eta) d\eta = i^{\nu+k}(\nu+k)!$$

and

$$i^\nu \exp(-\pi \xi^- \xi) = \sum_{k=0}^\infty \ i^{\nu+k} \ \frac{(\pi i \xi^- \xi)^k}{k!}$$

Integrations are with respect to the canonical measure for the complex plane. Interchanges of summation and integration are justified by absolute convergence.

If a function $f(\xi)$ of ξ in the complex skew–plane is square integrable with respect to the canonical measure and satisfies the identity

$$\phi(\xi) f(\omega \xi) = \phi(\omega \xi) f(\xi)$$

for every element ω of the complex skew–plane with conjugate as inverse, then its Fourier transform is a function $g(\xi)$ of ξ in the complex skew–plane which is square integrable with respect to the canonical measure and which satisfies the identity

$$\phi(\xi) g(\omega \xi) = \phi(\omega \xi) g(\xi)$$

for every element ω of the complex skew–plane with conjugate as inverse. The Laplace transforms of harmonic ϕ are functions $F(z)$ and $G(z)$ of z in the upper half–plane which satisfy the identity

$$G(z) = i^\nu (i/z)^{2+\nu} F(-1/z).$$

A construction of Euler weight functions is obtained on applying the Mellin transformation. The Mellin transformation reformulates the Fourier transformation for the real line on the multiplicative group of the positive half–line. Analytic weight functions constructed from the gamma function appear when the Mellin transformation is adapted to the domain of the Laplace transformation of harmonic ϕ.

The Mellin transform of harmonic ϕ of the function $f(\xi)$ of ξ in the complex skew–plane is an analytic function $F(z)$ of z in the upper half–plane which is defined when $f(\xi)$ vanishes in the disk $\xi^- \xi < a$ for some positive number a. The function is defined by the integral

$$\pi F(z) = \int_0^\infty g(it) t^{\frac{1}{2}\nu - iz} dt.$$

Since

$$g(iy) = \pi \int_0^\infty t^{\frac{1}{2}\nu} h(t) \exp(-\pi t y) t dt$$

when y is positive, the identity

$$F(z)/W(z) = \int_0^\infty h(t) t^{\frac{1}{2}\nu + iz} dt$$

holds with Euler weight function

$$W(z) = \pi^{-\frac{1}{2}\nu-1+iz}\Gamma(\tfrac{1}{2}\nu + 1 - iz).$$

The identity

$$\int_{-\infty}^{+\infty} |F(x+iy)/W(x+iy)|^2 dx = 2\pi \int_0^\infty |h(t)|^2 t^{\nu-2y} t dt$$

holds when y is positive.

The analytic function

$$a^{-iz}F(z)$$

of z in the upper half–plane belongs to the weighted Hardy space $\mathcal{F}(W)$ since the function $f(\xi)$ of ξ in the complex skew–plane vanishes when $\xi^-\xi < a$. An analytic function $F(z)$ of z in the upper half–plane such that the function

$$a^{-iz}F(z)$$

of z belongs to the space $\mathcal{F}(W)$ is the Mellin transform of a function $f(\xi)$ of ξ in the complex skew–plane which belongs to the domain of the Laplace transformation of harmonic ϕ and vanishes in the disk $\xi^-\xi < a$.

The adjoint of the Radon transformation for the complex skew–plane takes elements of the domain of the Laplace transformation of harmonic ϕ which vanish in the disk $\xi^-\xi < a$ into elements of the space which vanish in the disk. The transformation acts as a maximal accretive transformation on the subspace. The adjoint is a maximal accretive transformation which is unitarily equivalent to multiplication by

$$i/z$$

in the Hilbert space which is the image of the subspace under the Laplace transformation of harmonic ϕ. When $\xi^-\xi < 1$ is the unit disk, the transformation is unitarily equivalent to the transformation which takes $F(z)$ into $F(z - i)$ whenever the functions of z belong to the space $\mathcal{F}(W)$.

3. Harmonic Analysis on an r–adic Skew–Plane

An r–adic skew–plane is the ring of quotients of the completion of the ring of integral elements in the complex skew–plane in the r–adic topology: If ρ is a positive integer whose prime divisors are divisors of r such that a ring of integral elements modulo ρ exist, the discrete topology is the unique topology with respect to which the ring is a Hausdorff space since the ring is finite. The Hausdorff space is compact. Addition and multiplication are continuous as transformations of the Cartesian product of the ring with itself into the ring. Conjugation is continuous as a transformation of the ring into itself.

The r–adic topology of the ring of integral elements of the complex skew–plane is the least topology with respect to which the projection onto the quotient ring modulo ρ is continuous for every positive integer ρ whose prime divisors are divisors of r such that a quotient ring modulo ρ exists. The ring of integral elements of the complex skew–plane is a Hausdorff space in the r–adic topology. Addition and multiplication are continuous as transformations of the Cartesian product of the ring with itself into the ring. Conjugation is continuous as a transformation of the ring into itself.

The ring of integral elements of the r–adic skew–plane is the compact Hausdorff space which is the completion of the ring of integral elements of the complex skew–plane in its r–adic topology. Addition and multiplication extend continuously as transformations of the Cartesian product of the ring with itself into the ring. Conjugation extends continuously as a transformation of the ring into itself.

The r–adic skew–plane is the ring of quotients of the ring of its integral elements with denominators positive integers whose prime divisors are divisors of r. The r–adic topology of the r–adic skew–plane is the least topology for which multiplication by ρ is a continuous open mapping for every positive integer ρ whose prime divisors are divisors of r and for which the topology of the ring of integral elements of the r–adic skew–plane is the subspace topology inherited from the r–adic skew–plane.

Addition is continuous as a transformation of the Cartesian product of the r–adic skew–plane with itself into the r–adic skew–plane. Multiplication by an element of the r–adic skew–plane is continuous as a transformation of the r–adic skew–plane into itself. Conjugation is continuous as a transformation of the r–adic skew–plane into itself.

The r–adic line is the ring of self–conjugate elements of the r–adic skew–plane. The r–adic skew–plane is the algebra of quaternions

$$t + ix + jy + kz$$

with coordinates in the r–adic line. The r–adic line is a closed subspace of the r–adic skew–plane which is a Hausdorff space in the subspace topology. The ring of integral elements of the r–adic line is a compact open subset of the r–adic line.

The r–adic modulus of an invertible integral element ξ of the r–adic line is the positive number $\lambda_r(\xi)$ whose inverse $\lambda_r(\xi)^{-1}$ is the number of elements in the quotient ring of the ring of integral elements of the r–adic line by the ideal generated by ξ. The identity

$$\lambda_r(\xi\eta) = \lambda_r(\xi)\lambda_r(\eta)$$

holds for all invertible elements ξ and η of the r–adic line. The r–adic modulus of an invertible element ξ of the r–adic line is defined so that the identity holds for all invertible integral elements ξ and η of the r–adic line. A noninvertible element of the r–adic line is given infinite r–adic modulus.

The r–adic modulus of an invertible element ξ of the r–adic skew–plane is defined as the positive solution $\lambda_r(\xi)$ of the equation

$$\lambda_r(\xi)^2 = \lambda_r(\xi^- \xi).$$

The identity

$$\lambda_r(\xi\eta) = \lambda_r(\xi)\lambda_r(\eta)$$

holds for all invertible elements ξ and η of the r–adic skew–plane. The r–adic modulus of a noninvertible element of the r–adic skew–plane is infinite.

A dense additive subgroup of a p–adic line is represented by real numbers. The function $\exp(2\pi i\xi)$ of ξ in the subgroup is a homomorphism of the subgroup into the multiplicative group of complex numbers of absolute value one which is continuous for the p–adic topology since its kernel is closed. The function admits a unique continuous extension as a function of ξ in the p–adic line which is a homomorphism of its additive group into the multiplicative group of complex numbers of absolute value one.

The function $\exp(2\pi i\xi)$ of ξ in the r–adic line is defined as the unique continuous homomorphism of its additive group into the multiplicative group of complex numbers of absolute value one which agrees with the definition given when ξ belongs to a component p–adic line.

The canonical measure for the r–adic line is the unique nonnegative measure on its Baire subsets such that a measure preserving transformation is defined by taking ξ into $\xi + \eta$ for every element η of the line and such that the value of the measure on the set of integral elements is one. Multiplication by an invertible element ξ of the r–adic line multiplies the canonical measure for the line by a factor of $\lambda_r(\xi)$.

The canonical measure for the r–adic skew–plane is the unique nonnegative measure on its Baire subsets such that a measure preserving transformation is defined by taking ξ into $\xi + \eta$ for every element η of the skew–plane and such that the value of the measure on the set of integral elements is one when r is odd and two when r is even. Multiplication by a nonzero element ξ of the r–adic skew–plane multiplies the canonical measure by a factor of $\lambda_r(\xi)^4$. Conjugation is a measure preserving transformation. The canonical measure for a r–adic skew–plane is the Cartesian product measure of the canonical measures of four coordinate r–adic lines. The canonical measure for the r–adic skew–plane is the Cartesian product measure of the canonical measures for component p–adic skew–planes.

The Fourier transformation for the r–adic skew–plane is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the r–adic skew–plane into itself which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(\pi i(\xi^-\eta + \eta^-\xi))f(\eta)d\eta$$

of ξ defined by integration with respect to the canonical measure for the r–adic skew–plane. Fourier inversion states that the integral

$$f(\xi) = \int \exp(-\pi i(\xi^-\eta + \eta^-\xi))g(\eta)d\eta$$

with respect to the canonical measure for the r–adic skew–plane represents the function $f(\xi)$ of ξ when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

An r–adic plane is a maximal commutative subring of the r–adic skew–plane which is isomorphic to the Cartesian product of p–adic planes taken over the prime divisors p of r. A p–adic plane is determined by the choice of an integral element ι_p of the complex skew–plane such that $\lambda(\iota_p^- \iota_p)$ is the least positive integer n which is divisible by p and has no other prime divisors such that a ring of self–conjugate integral elements of the complex skew–plane modulo n exists. The elements of the p–adic plane are the elements of the p–adic skew–plane which commute with ι_p. An element

$$\xi = \alpha + \iota_p \beta$$

of the p–adic plane is a sum with α and β elements of the p–adic line. The element ξ of the p–adic plane is integral if, and only if, the elements α and β of the p–adic line are integral.

The ring of integral elements of the r–adic plane is a compact Hausdorff space in the subspace topology inherited from the r–adic skew–plane. Addition and multiplication are continuous as transformations of the Cartesian product of the ring with itself into the ring. In the subspace topology inherited from the r–adic skew–plane the r–adic plane is a Hausdorff space which contains the ring of integral elements as an open and closed subset containing the origin. A continuous transformation of the r–adic plane into itself is defined by taking ξ into $\xi + \eta$ for every element η of the r–adic plane.

The canonical measure for the r–adic plane is the unique nonnegative measure on its Baire subsets such that a measure preserving transformation of the space into itself is defined by taking ξ into $\xi + \eta$ for every element η of the space and such that the ring of integral elements has measure one. Multiplication by an element ξ of the r–adic plane multiplies the canonical measure by a factor of $\lambda_r(\xi)^2$.

The conjugation of the r–adic skew–plane acts as a continuous isomorphism of the r–adic plane onto itself. The set of self–conjugate elements of the r–adic plane is the r–adic line. If ι is the element of the r–adic plane whose p–adic component is ι_p for every prime divisor p of r, then an element

$$\xi = \alpha + \iota \beta$$

of the r–adic plane has coordinates α and β in the r–adic line. The topology of the r–adic plane is the Cartesian product topology of the r–adic topologies of two r–adic lines. The canonical measure for the r–adic plane is a constant multiple of the Cartesian product measure of the canonical measures of two r–adic lines.

The complementary space to the r–adic plane in the r–adic skew–plane is the set of elements η of the r–adic skew–plane which satisfy the identity

$$\xi \eta = \eta \xi^-$$

for every element ξ of the r–adic plane. An element η of the complementary space to the r–adic plane in the r–adic skew–plane is skew–conjugate:

$$\eta^- = -\eta.$$

Multiplication on left or right by an invertible element of the complementary space is an injective transformation of the r–adic plane onto the complementary space. The transformation is continuous when the complementary space is given the topology inherited from the r–adic skew–plane. The canonical measure for the complementary space is defined as the image of the canonical measure for the r–adic plane under multiplication by a unit of the complementary space.

An element of the r–adic skew–plane is the unique sum $\alpha + \beta$ of an element α of the r–adic plane and an element β of the complementary space. The topology of the r–adic skew–plane is the Cartesian product topology of the topology of the r–adic plane and the topology of the complementary space. The canonical measure for the r–adic skew–plane is the Cartesian product measure of the canonical measure for the r–adic plane and the canonical measure for the complementary space.

The Fourier transformation for the r–adic plane is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the r–adic plane into itself which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(\pi i(\xi^- \eta + \eta^- \xi)) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \int \exp(-\pi i(\xi^- \eta + \eta^- \xi)) g(\eta) d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous. The Fourier transformation for the r–adic skew–plane commutes with the transformation which takes a function $f(\xi)$ of ξ into the functions $f(\omega\xi)$ of ξ for every element ω of the r–adic skew–plane with conjugate as inverse.

The group of invertible elements of the r–adic skew–plane contains closed normal subgroups with finite quotient groups. A normal subgroup is defined by every positive integer ν whose prime divisors are divisors of r. The normal subgroup is generated by the invertible integral elements ξ of the r–adic skew–plane such that for every prime divisor p of ν the greatest power of p which is a divisor of $\lambda_r(\xi^- \xi)^{-1}$ is a power of the greatest power of p which is a divisor of ν.

A fundamental domain for the equivalence relation defined by the subgroup is the set of invertible integral elements ξ of the r–adic skew–plane such that for every prime divisor p of r the greatest power of p which is a divisor of $\lambda_r(\xi^- \xi)^{-1}$ is less than or equal to the greatest power of p which is a divisor of ν with equality only when p is not a divisor of ν. The intersection of the normal subgroups is the group of units of the r–adic skew–plane.

The quotient group defined by ν is a compact Hausdorff space in the discrete topology. The canonical measure for the quotient group is counting measure divided by the number of elements in the group. An invertible integral element ω of the r–adic skew–plane defines a measure preserving transformation ξ into $\omega\xi$ of the quotient group into itself. The conjugation of the r–adic skew–plane defines a measure preserving conjugation of the quotient group into itself. An automorphism of the discrete skew–field which takes Hurwitz integers into Hurwitz integers defines a measure preserving automorphism of the quotient group.

If ν' is a divisor of ν such that for every prime divisor p of μ' the greatest power of p which is a divisor of ν is a power of the greatest power of p which is a divisor of ν', a normal subgroup of the quotient group defined by ν is defined as the set of elements ξ such that for every prime p the greatest power of p which is a divisor of $\lambda_r(\xi^-\xi)^{-1}$ is a power of the greatest power of p which is a divisor of ν'. The quotient group of the quotient group defined by ν by the normal subgroup is isomorphic to the quotient group defined by ν'.

The projection of the quotient group defined by ν onto the quotient group defined by ν' takes the canonical measure into the canonical measure and commutes with conjugation.

A function defined on the quotient group defined by ν' is treated as a function defined on the quotient group defined by ν which has equal values at elements which project into the same element of the quotient group defined by ν'. The Hilbert space of functions which are square integrable with respect to the canonical measure for the quotient group defined by ν' is contained isometrically in the Hilbert space of functions which are square integrable with respect to the canonical measure for the quotient group defined by ν.

A function defined on the quotient group defined by ν is said to be harmonic of order ν if it is orthogonal to functions defined on the quotient group defined by ν' for every proper divisor ν' of ν such that for every prime divisor p of ν' the greatest power of p which is a divisor of ν is a power of the greatest power of p which is a divisor of ν'.

For every invertible integral element ω of the r–adic skew–plane an isometric transformation of the Hilbert space of harmonic functions of order ν into itself is defined by taking a function $f(\xi)$ of ξ in the quotient group defined by ν into the function $f(\omega\xi)$ of ξ in the quotient group defined by ν.

An isometric transformation of the Hilbert space of harmonic functions of order ν into itself is defined by taking a function $f(\xi)$ of ξ in the quotient group defined by ν into the function $f(\xi^-)$ of ξ in the quotient group defined by ν.

A function defined on the quotient group defined by ν is treated as a function defined on the r–adic skew–plane which vanishes at noninvertible elements and which has equal values at invertible elements which project into the same element of the quotient group defined by ν.

The Radon transformation of harmonic ϕ for the r–adic skew–plane is a transformation with domain and range in the Hilbert space of functions $f(\xi)$ of ξ in the r–adic skew–plane which are square integrable with respect to the canonical measure for the r–adic skew–plane, which vanish at noninvertible elements, and which satisfy the identity

$$\phi(\xi)f(\omega\xi) = \phi(\omega\xi)f(\xi)$$

for every element ω of the r–adic skew–plane with conjugate as inverse.

The Radon transformation of harmonic ϕ takes a function $f(\xi)$ of ξ into the function $g(\xi)$ of ξ when the identity

$$g(\omega\xi)/\phi(\omega\xi) = \int f(\omega\xi + \omega\eta)/\phi(\omega\xi + \omega\eta)d\eta$$

holds for every element ω of the r–adic skew–plane with conjugate as inverse when ξ is in the r–adic plane with integration with respect to the canonical measure for the complementary space to the r–adic plane in the r–adic skew–plane.

The integral is interpreted as a limit of the integrals over the set of elements ξ of the complementary space such that

$$\lambda_r(\xi^- \xi) \leq n$$

for a positive integer n whose prime divisors are divisors of r. Convergence is in the metric topology of the Hilbert space of square integrable functions with respect to the canonical measure for the r–adic skew–plane as n becomes eventually divisible by every positive integer whose prime divisors are divisors of r.

The Laplace transformation of harmonic ϕ for the r–adic skew–plane gives a spectral analysis of the adjoint of the Radon transformation of harmonic ϕ for the r–adic skew–plane. The domain of the Laplace transformation of harmonic ϕ is the Hilbert space of functions $f(\xi)$ of ξ in the r–adic skew–plane which are square integrable with respect to the canonical measure for the r–adic skew–plane, which vanish at noninvertible elements, and which satisfy the identity

$$\phi(\xi)f(\omega\xi) = \phi(\omega\xi)f(\xi)$$

for every element ω of the r–adic skew–plane with conjugate as inverse. An r–adic half–plane is applied in the parametrization of these functions.

An r–adic half–plane is defined as a maximal commutative subalgebra of the r–adic skew–plane which is isomorphic to the Cartesian product of p–adic half–planes taken over the prime divisors p of r. A p–adic half–plane is defined as a maximal commutative subalgebra of the p–adic skew–plane whose ring of integral elements is a field modulo n for the least positive integer n divisible by p with no other prime divisor such that a ring of self–conjugate integral elements of the complex skew–plane modulo n exists. When p is an odd prime, the p–adic half–plane is chosen so that every element of the p–adic skew–plane is a sum

$$\alpha + \iota_p \beta$$

with α and β in the p–adic half–plane and so that the identity

$$\gamma \iota_p = \iota_p^- \gamma$$

holds for every skew–conjugate element of the half–plane.

An r–adic half–plane is a locally compact Hausdorff space in the subspace topology inherited from the r–adic skew–plane. The canonical measure for the r–adic half–plane is the unique nonnegative measure on its Baire subsets such that a measure preserving transformation is defined by taking ξ into $\xi + \eta$ for every element η of the r–adic half–plane and such that the set of integral elements has measure one.

A Laplace transformation of harmonic ϕ is defined when a harmonic function ϕ of order ν for the r–adic skew–plane has norm one in the Hilbert space of harmonic functions is of order ν. The domain of the Laplace transformation of harmonic ϕ is the set of functions $f(\xi)$ of ξ in the r–adic skew–plane which are square integrable with respect to the canonical measure for the r–adic skew–plane, which vanish at noninvertible elements of the r–adic skew–plane, and which satisfy the identity

$$\phi(\xi) f(\omega\xi) = \phi(\omega\xi) f(\xi)$$

for every element ω of the r–adic skew–plane with conjugate as inverse.

A function

$$f(\xi) = \phi(\xi) h(\xi^- \xi)$$

of ξ in the r–adic skew–plane which belongs to the domain of the Laplace transformation of harmonic ϕ is parametrized by a function $h(\xi)$ of ξ in the r–adic half–plane which is square integrable with respect to the canonical measure for the r–adic half–plane.

The function $h(\xi)$ of ξ is initially defined on the r–adic line. The transformation of the r–adic skew–plane into the r–adic line which takes ξ into $\xi^- \xi$ takes the canonical measure for the r–adic skew–plane into a nonnegative measure μ on the Baire subsets of the r–adic line. The identity

$$\int |f(\xi)|^2 d\xi = \int |h(\xi)|^2 d\mu$$

holds with integration on the left with respect to the canonical measure for the r–adic skew–plane and with integration on the right with respect to the measure μ.

The extension of functions defined on the r–adic line to functions defined on the r–adic half–plane is made by a decomposition with respect to characters for the r–adic half–plane.

parities defined by prime divisors p of r. A function $h(\xi)$ of ξ in the r–adic line is said to have parity if

$$h(\omega\xi) = h(\xi)$$

or if

$$h(\omega\xi) = -h(\xi)$$

whenever ω is an element of the r–adic line with itself as inverse.

The Hilbert space of functions which are square integrable with respect to μ is the orthogonal sum of subspaces defined parity. The extension to the r–adic half–plane of a function defined on the r–adic line is made by characters for the r–adic half–plane.

If ρ is a positive integer whose prime divisors are divisors of r, a character modulo ρ for the r–adic half–plane is a function $\chi(\xi)$ of ξ in the r–adic half–plane which vanishes at nonintegral elements, which satisfies the identity

$$\chi(\xi\eta) = \chi(\xi)\chi(\eta)$$

for all integral elements ξ and η, which has equal values at integral elements which are congruent modulo ρ, and which has a nonzero value at an integral element when, and only when, the element is invertible modulo ρ.

A character modulo ρ is said to be primitive modulo ρ if no character modulo ρ' exists for a proper divisor ρ' of ρ which agrees with the given character on integral elements which are invertible modulo ρ.

Primitive characters modulo ρ whose nonzero values are fourth roots of unity are applied for extension. The values are square roots of unity on elements of the r–adic line with self inverse. Characters are equivalent for extension purposes if they agree on elements of the r–adic line with self–inverse. The choice of a character is made in every equivalence class so that a product of chosen characters is a chosen character. The choice reduces to the choice when ρ has only one prime divisor, in which case the choice is between two characters which are conjugates of each other.

The group of elements of the r–adic half–plane with conjugate as inverse is compact in the subspace topology inherited from the r–adic half–plane. A unique nonnegative measure is defined on the Baire subsets of the group such that a measure preserving transformation is defined by taking ξ into $\omega\xi$ for every element ω of the group and such that the value of the measure on the group is one.

The group of invertible elements of the r–adic half–plane is locally compact in the subspace topology inherited from the half–plane. A unique nonnegative measure μ' is defined on the Baire subsets of the group such that a measure preserving transformation is defined by taking ξ into $\omega\xi$ for every element ω of the group with conjugate as inverse and such that for every Baire subset C of the r–adic line the set C' of elements of the r–adic half–plane which are products of an element of C and an element with conjugate as inverse the measure of C with respect to μ is equal to the measure of C' with respect to μ'.

If ω is an invertible element of the r–adic half–plane, the transformation which takes ξ into $\omega\xi$ multiplies the measure μ' by a factor of $\lambda_r(\omega)^2$. The measure μ' is a constant multiple of the canonical measure for the r–adic half–plane.

The set of integral elements of the r–adic half–plane has measure one for the canonical measure. When r is odd, the set of integral elements of the r–adic half–plane has measure one for the measure μ'. The measure μ' is equal to the canonical measure when r is odd. When r is even, the set of integral elements of the r–adic half–plane has measure two for the measure μ'. The measure μ' is is equal to twice the canonical measure when r is even.

When a function

$$f(\xi) = \phi(\xi)h(\xi^-\xi)$$

of ξ in the r–adic skew–plane is parametrized by a function $h(\xi)$ of ξ in the r–adic half–plane, the identity

$$\int |f(\xi)|^2 d\xi = \int |h(\xi)|^2 d\xi$$

holds when r is odd and the identity

$$\int |f(\xi)|^2 d\xi = 2 \int |h(\xi)|^2 d\xi$$

holds when r is even with integration on the left with respect to the canonical measure for the r–adic skew–plane and with integration on the right with respect to the canonical measure for the r–adic half–plane.

The Fourier transformation for the r–adic half–plane is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure into itself which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(\pi i(\xi^- \eta + \eta^- \xi)) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \int \exp(-\pi i(\xi^- \eta + \eta^- \xi)) g(\eta) d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

The Laplace transform of harmonic ϕ for the r–adic skew–plane of a function

$$f(\xi) = \phi(\xi) h(\xi^- \xi)$$

of ξ in the r–adic skew–plane is defined as the function $g(\xi)$ of ξ in the r–adic half–plane such that the function $\frac{1}{2} g(2\xi)$ of ξ is the Fourier transform of the function $h(\xi)$ of ξ in the r–adic half–plane. The function $g(\xi)$ of ξ is square integrable with respect to the canonical measure for the r–adic half–plane. The identity

$$\int |f(\xi)|^2 d\xi = 2 \int |g(\xi)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the r–adic skew–plane and with integration on the right with respect to the canonical measure for the r–adic half–plane.

The adjoint of the Radon transformation of harmonic ϕ takes a function

$$f_1(\xi) = \phi(\xi) h_1(\xi^- \xi)$$

of ξ in the r–adic skew–plane into a function

$$f_2(\xi) = \phi(\xi) h_2(\xi^- \xi)$$

of ξ in the r–adic skew–plane if, and only if, the Laplace transforms satisfy the identity

$$g_2(\xi) = \lambda_r(\xi)^{-1} g_1(\xi)$$

for ξ in the r–adic line. The proof applies Fourier analysis on an r–adic plane.

If n is an even positive integer whose prime divisors are divisors of r and if γ is an invertible element of the r–adic line, the Fourier transform of the function of ξ in the r–adic plane which is equal to

$$\exp(-\pi i \gamma \xi^- \xi)$$

when $n\gamma\xi^-\xi$ is integral and which vanishes otherwise is the function of ξ in the r–adic plane which is equal to

$$\lambda_r(\gamma)^{-1} \exp(\pi i \gamma^{-1} \xi^- \xi)$$

when $n\gamma^{-1}\xi^-\xi$ is integral and which vanishes otherwise.

The desired identity

$$\exp(\pi i \xi^- \xi) = \int \exp(\pi i(\xi^- \eta + \eta^- \xi)) \exp(-\pi i \eta^- \eta) d\eta$$

can be written

$$1 = \int \exp(-\pi i(\xi - \eta)^-(\xi - \eta)) d\eta$$

and reduces by change of variable to the case ξ equal to zero.

The isometric property of the Fourier transformation implies that the identity holds with the left side replaced by a constant of absolute value one and that the constant is independent of n. The constant is computed by an elementary argument when n is equal to two.

If n is an even positive integer whose prime divisors are divisors of r and if γ is an invertible element of the r–adic line, the function of ξ in the r–adic skew–plane which is equal to

$$\exp(-\pi i \gamma \xi^- \xi)$$

when $n\gamma\xi^-\xi$ is integral and which vanishes otherwise is an eigenfunction of the Radon transformation for the r–adic skew–plane for the eigenvalue

$$\lambda_r(\gamma)^{-1}.$$

The proof applies the identity

$$\exp(-\pi i \gamma(\xi + \eta)^-(\xi + \eta)) = \exp(-\pi i \gamma \xi^- \xi) \exp(-\pi i \gamma \eta^- \eta)$$

which holds when ξ is an element of the r–adic plane and η is an element of the complementary space to the r–adic plane in the r–adic skew–plane. The element

$$n\gamma(\xi + \eta)^-(\xi + \eta)$$

of the r–adic line is integral if, and only if the elements

$$n\gamma\xi^-\xi$$

and

$$n\gamma\eta^-\eta$$

of the r–adic line are integral.

The integral

$$\int \exp(-\pi i\gamma\eta^-\eta)d\eta$$

with respect to the canonical measure for the complementary space to the r–adic plane in the r–adic skew–plane over the set of elements η such that $n\gamma\eta^-\eta$ is integral is equal to the integral with respect to the canonical measure for the r–adic plane over the set of elements η such that $n\gamma\eta^-\eta$ is integral. The integral is equal to

$$\lambda_r(\gamma)^{-1}$$

by the computation of Fourier transforms for the r–adic plane.

The domain of the Laplace transformation of harmonic ϕ is the orthogonal sum of invariant subspaces for the adjoint of the Radon transformation of harmonic ϕ which are characterized as eigenfunctions of the transformation for given eigenvalues. The eigenvalues are the rational numbers which are equal to

$$\lambda_r(\gamma)^{-1}$$

for an invertible element γ of the r–adic line. The eigenfunctions for the eigenvalue are the functions $f(\xi)$ of ξ in the r–adic skew–plane whose Laplace transform vanishes at elements ξ of the r–adic half–plane such that $\gamma^{-1}\xi$ is not a unit.

The Radon transformation of harmonic ϕ for the ring of integral elements of the r–adic skew–plane is a restriction of the Radon transformation of harmonic ϕ for the r–adic skew–plane. The domain and range of the transformation are contained in the Hilbert space of functions $f(\xi)$ of ξ in the r–adic skew–plane which are square integrable with respect to the canonical measure for the r–adic skew–plane, which satisfy the identity

$$\phi(\xi)f(\omega\xi) = \phi(\omega\xi)f(\xi)$$

for every element ω of the r–adic skew–plane with conjugate as inverse, and which vanish at nonintegral elements of the r–adic skew–plane.

The ring of integral elements of the r–adic skew–plane is a compact Hausdorff space in the subspace topology inherited from the r–adic skew–plane. The canonical measure for the ring is the restriction to Baire subsets of the ring of the canonical measure for the r–adic skew–plane.

The spectral theory of the adjoint of the Radon transformation for the ring applies the quotient ring modulo r of the ring of integral elements of the r–adic skew–plane. A function defined on the quotient ring is treated as a function defined on the ring which has equal values at elements which are congruent modulo r. The canonical measure for the ring of integral elements of the r–adic skew–plane modulo r is counting measure divided by the number of elements in the ring. The projection of the ring of integral elements of the r–adic skew–plane onto the quotient ring modulo r is a continuous open mapping which is a homomorphism of conjugated ring structure and which maps the canonical measure into the canonical measure.

The ring of integral elements of the r–adic plane is contained in the ring of integral elements of the r–adic skew–plane. The ring is a compact Hausdorff space in the subspace topology inherited from the r–adic plane, which is identical with the subspace topology inherited from the ring of integral elements of the r–adic skew–plane. The canonical measure for the ring is the restriction to Baire subsets of the ring of the canonical measure for the r–adic plane.

The ring of integral elements of the r–adic plane modulo r is the quotient ring of the ring of integral elements of the r–adic plane modulo the ideal generated by r. The ring of integral elements of the r–adic plane modulo r is isomorphic to the image of the ring of integral elements of the r–adic plane in the ring of integral elements of the r–adic skew–plane modulo r. The ring of integral elements of the r–adic plane modulo r is treated as a subring of the ring of integral elements of the r–adic skew–plane modulo r. The canonical measure for the quotient ring is counting measure divided by the number of elements in the ring. The projection of the ring of integral elements of the r–adic plane onto the quotient ring is a continuous open mapping which is a homomorphism of conjugated ring structure and which maps the canonical measure into the canonical measure.

The complementary space to the ring of integral elements of the r–adic plane in the ring of integral elements of the r–adic skew–plane is defined as the set of integral elements of the complementary space to the r–adic plane in the r–adic skew–plane. The set of integral elements of the complementary space is an additive group which is a compact Hausdorff space in the subspace topology inherited from the complementary space to the r–adic plane in the r–adic skew–plane. The canonical measure for the set of integral elements of the complementary space is the restriction to its Baire subsets of the canonical measure for the complementary space.

The complementary space to the ring of integral elements of the r–adic plane modulo r in the ring of integral elements of the r–adic skew–plane modulo r is defined as the image in the ring of integral elements of the r–adic skew–plane modulo r of the set of integral elements of the complementary space to the r–adic plane in the r–adic skew–plane.

The complementary space modulo r has a finite number of elements and is given the discrete topology. The set is an additive group whose canonical measure is counting measure divided by r^2. The projection of the set of integral elements of the complementary space to the r–adic plane in the r–adic skew–plane onto the complementary space modulo r is a continuous open mapping which is a homomorphism of additive structure and which maps the canonical measure into the canonical measure.

The Radon transformation for the ring of integral elements of the r–adic skew–plane takes a function $f(\xi)$ of ξ in the ring into a function $g(\xi)$ of ξ in the ring when the identity

$$g(\omega\xi)/\phi(\omega\xi) = \int f(\omega\xi + \omega\eta)/\phi(\omega\xi + \omega\eta)d\eta$$

holds for every element ω of the r–adic skew–plane with conjugate as inverse when ξ is an integral element of the r–adic plane with integration with respect to the canonical measure for the complementary space to the ring of integral elements of the r–adic plane in the ring of integral elements of the r–adic skew–plane.

The Radon transformation for the ring of integral elements of the r–adic skew–plane modulo r is a restriction of the Radon transformation for the ring of integral elements of the r–adic skew–plane. The domain and range of the transformation are contained in the Hilbert space of functions $f(\xi)$ of ξ in the ring of integral elements of the r–adic skew–plane modulo r which are square integrable with respect to the canonical measure for the ring of integral elements of the r–adic skew–plane modulo r and which satisfy the identity

$$\phi(\xi)f(\omega\xi) = \phi(\omega\xi)f(\xi)$$

for every element ω of the r–adic skew–plane with conjugate as inverse.

The Radon transformation for the ring of integral elements of the r–adic skew–plane modulo r takes a function $f(\xi)$ of ξ in the ring into a function $g(\xi)$ of ξ in the ring when the identity

$$g(\omega\xi)/\phi(\omega\xi) = \int f(\omega\xi + \omega\eta)/\phi(\omega\xi + \omega\eta)d\eta$$

holds for every element ω of the r–adic skew–plane with conjugate as inverse when ξ is an element of the ring of integral elements of the r–adic plane modulo r with integration with respect to the canonical measure for the complementary space to the ring of integral elements of the r–adic plane modulo r in the ring of integral elements of the r–adic skew–plane modulo r.

A function

$$f(\xi) = \phi(\xi)h(\xi^-\xi)$$

of ξ in the ring of integral elements of the r–adic skew–plane which is square integrable with respect to the canonical measure for the ring, which vanishes at noninvertible elements of the r–adic skew–plane, and which satisfies the identity

$$\phi(\xi)f(\omega\xi) = \phi(\omega\xi)f(\xi)$$

for every element of the r–adic skew–plane with conjugate as inverse is parametrized by a function $h(\xi)$ of half–integral elements ξ of the r–adic half–plane.

The set of integral elements of the r–adic half–plane is subring which is a compact Hausdorff space in the subspace topology inherited from the r–adic half–plane. The canonical measure for the subring is the restriction to Baire subsets of the subring of the canonical measure for the r–adic half–plane.

The function $h(\xi)$ of integral elements ξ of the r–adic half–plane is square integrable with respect to the canonical measure for the ring of integral elements of the r–adic half–plane. When r is even, the identity

$$\int |f(\xi)|^2 d\xi = 2 \int |h(\xi)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the ring of integral elements of the r–adic skew–plane and with integration on the right with respect to the canonical measure for the ring of integral elements of the r–adic half–plane. The factor of two is deleted when r is odd.

The Fourier transformation for the ring of integral elements of the r–adic half–plane applies a quotient space of the r–adic half–plane. Elements ξ and η of the r–adic half–plane are said to be congruent modulo 1 if they differ by an integral element $\eta - \xi$ of the r–adic half–plane.

The r–adic half–plane modulo 1 is the quotient space of the r–adic half–plane defined by the equivalence relation. The quotient space is an additive group which has the discrete topology and whose canonical measure is counting measure. The projection of the r–adic half–plane onto the r–adic half–plane modulo 1 is a continuous open mapping which is a homomorphism of additive structure and which takes the canonical measure into the canonical measure. A function defined on the r–adic half–plane modulo 1 is treated as a function defined on the r–adic half–plane which has equal values at elements which are congruent modulo 1.

An example

$$\exp(\pi i(\xi^- \eta + \eta^- \xi))$$

of a function of ξ in the r–adic half–plane modulo 1 is obtained when η is an integral element of the r–adic half–plane.

The Fourier transformation for the ring of integral elements of the r–adic half–plane is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the ring onto the Hilbert space of square integrable functions with respect to the canonical measure for the r–adic half–plane modulo 1 which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(\pi i(\xi^- \eta + \eta^- \xi)) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure for the group. Fourier inversion

$$f(\xi) = \int \exp(-\pi i(\xi^- \eta + \eta^- \xi)) g(\eta) d\eta$$

applies with integration with respect to the canonical measure for the r–adic half–plane modulo 1 when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

If for some signature χ for the r–adic half–plane the function $f(\xi)$ of ξ in the integral elements of the r–adic half–plane satisfies the identity

$$f(\omega\xi) = \chi(\omega)f(\xi)$$

for every element ω of the r–adic half–plane with conjugate as inverse, then the function $g(\xi)$ of ξ in the r–adic half–plane modulo 1 satisfies the identity

$$g(\omega\xi) = \chi(\omega)^{-}g(\xi)$$

for every element ω of the r–adic half–plane with conjugate as inverse.

The Laplace transformation of harmonic ϕ for the ring of integral elements of the r–adic skew–plane is defined to take a function

$$f(\xi) = \phi(\xi)h(\xi^{-}\xi)$$

of ξ in the ring into the function $g(\xi)$ of ξ in the r–adic half–plane modulo 2 such that the function $\frac{1}{2}g(2\xi)$ in the r–adic half–plane modulo 1 is the Fourier transform of the function $h(\xi)$ of ξ in the ring of integral elements of the r–adic half–plane. When r is even, the identity

$$\int |f(\xi)|^2 d\xi = 2\int |g(\xi)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the ring and with integration on the right with respect to the canonical measure for the r–adic half–plane modulo 2.

The r–adic modulus

$$\lambda_r(\xi) = \min \lambda_r(\eta)$$

of an element ξ of the r–adic half–plane modulo 2 is defined as the minimum r–adic modulus of elements η of the r–adic half–plane which represent ξ.

The adjoint of the Radon transformation of harmonic ϕ for the ring of integral elements of the r–adic skew–plane takes a function

$$f_1(\xi) = \phi(\xi)h_1(\xi^{-}\xi)$$

of ξ in the ring into a function

$$f_2(\xi) = \phi(\xi)h_2(\xi^{-}\xi)$$

of ξ in the ring if, and only if, the Laplace transforms for the ring satisfy the identity

$$g_2(\xi) = \lambda_r(\xi)^{-1}g_1(\xi)$$

for ξ in the r–adic half–plane modulo 2.

The adjoint of the Radon transformation for the ring of integral elements of the r–adic skew–plane is a nonnegative transformation in the Hilbert space of square integrable functions with respect to the canonical measure for the ring. The transformation is extended by its adjoint.

A function

$$f(\xi) = \phi(\xi)h(\xi^-\xi)$$

of ξ in the ring of integral elements of the r–adic skew–plane modulo r which is square integrable with respect to the canonical measure for the ring modulo r and which satisfies the identity

$$\phi(\xi)f(\omega\xi) = \phi(\omega\xi)f(\xi)$$

for every element ω of the r–adic skew–plane with conjugate as inverse is parametrized by a function $h(\xi)$ of integral elements ξ of the r–adic half–plane which has equal values at elements which are congruent modulo r.

Integral elements ξ and η of the r–adic half–plane are said to be congruent modulo r if they differ by an element $\eta - \xi$ which is the product of r and an integral element of the r–adic half–plane. The quotient space for the equivalence relation is the ring of integral elements of the r–adic half–plane modulo r.

The quotient space contains a finite number of elements and is given the discrete topology. The canonical measure is counting measure divided by the number of elements in the quotient space.

The projection of the ring of integral elements of the r–adic half–plane onto the ring modulo r is a continuous open mapping which is a homomorphism of ring structure and which takes the canonical measure into the canonical measure. A function defined on the ring modulo r is treated as a function defined on the ring which has equal values at elements which are congruent modulo r.

When r is finite, an example of a function of ξ in the ring modulo r is

$$\exp(\pi i(\xi^-\eta + \eta^-\xi))$$

when η is an element of the r–adic half–plane such that $r\eta$ is an integral element of the r–adic half–plane: Multiplication by r annihilates the element of the r–adic half–plane modulo 1 represented by η.

The r–adic half–plane modulo r^{-1} is defined as the quotient group of the r–adic half–plane modulo the subgroup of elements whose product by r is integral. A homomorphism of the r–adic half–plane modulo 1 onto r–adic half–plane modulo r^{-1} is defined by taking the image of an element of the r–adic half–plane in the r–adic half–plane modulo 1 into its image in the r–adic half–plane modulo r. The kernel of the homomorphism is the set of elements of the r–adic half–plane modulo 1 whose product by r vanishes. The group of r–annihilated elements of the r–adic half–plane modulo 1 is isomorphic to the quotient group of the group of elements of the r–adic half–plane whose product by r is integral by the subgroup of integral elements of the r–adic half–plane. The group of r–annihilated elements of the r–adic half–plane modulo 1 contains a finite number of elements and is given the discrete topology. The canonical measure for the group is counting measure.

The Fourier transformation for the ring of integral elements of the r–adic half–plane modulo r is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the ring onto the Hilbert space of square integrable functions with respect to the canonical measure for the r–annihilated subgroup of the r–adic half–plane modulo 1 which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(\pi i(\xi^- \eta + \eta^- \xi)) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure for the ring. Fourier inversion

$$f(\xi) = \int \exp(-\pi i(\xi^- \eta + \eta^- \xi)) g(\eta) d\eta$$

holds with integration with respect to the canonical measure for the r–annihilated subgroup when the function $g(\xi)$ of ξ is integrable and the function $f(\xi)$ of ξ is continuous.

The Laplace transformation of harmonic ϕ for the ring of integral elements of the r–adic skew–plane modulo r takes a function

$$f(\xi) = \phi(\xi) h(\xi^- \xi)$$

of ξ in the ring into the function $g(\xi)$ of ξ in the $2r$–annihilated subgroup of the r–adic half–plane modulo 2 such that the function $\frac{1}{2}\, g(2\xi)$ of ξ in the r–annihilated subgroup of the r–adic half–plane modulo 1 which is the Fourier transform of the function $h(\xi)$ in the ring of integral elements of the r–adic half–plane modulo r.

The identity

$$\int |f(\xi)|^2 d\xi = 2 \int |g(\xi)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the ring of integral elements of the r–adic skew–plane modulo r and with integration on the right with respect to the canonical measure for the $2r$–annihilated subgroup of the r–adic half–plane modulo 2.

The adjoint of the Radon transformation of harmonic ϕ for the ring of integral elements of the r–adic skew–plane modulo r takes a function

$$f_1(\xi) = \phi(\xi) h_1(\xi^- \xi)$$

of ξ in the ring into a function

$$f_2(\xi) = \phi(\xi) h_2(\xi^- \xi)$$

of ξ in the ring if, and only if, the Laplace transforms for the ring satisfy the identity

$$g_2(\xi) = \lambda_r(\xi)^{-1} g_1(\xi)$$

for ξ in the $2r$–annihilated subgroup of the r–adic half–plane modulo 2.

The adjoint of the Radon transformation for the ring of integral elements of the r–adic skew–plane modulo r is a nonnegative transformation which is extended by its adjoint.

4. Harmonic Analysis on an r–Adelic Skew–Plane

An r–adelic skew–plane is defined for every positive integer r after its definition when r is one.

The 1–adelic skew–plane is the completion of the discrete skew–field in the metric topology of its scalar product. The discrete skew–field is a vector space over the Gauss field with multiplication by an element of the Gauss field on the right of an element of the discrete skew–field. The 1–adelic skew–plane is a vector space over the field of complex numbers since the Gauss field is dense in the field of complex numbers and since multiplication is continuous when the Gauss field has the subspace topology inherited from the complex plane and the discrete skew–field has the metric topology defined by its scalar product.

Conjugation is a conjugate linear transformation of the discrete skew–field into itself which extends continuously as a conjugation of the 1–adelic skew–plane.

The isometric transformation ξ into $k\xi$ of the discrete skew–field into itself has a unique continuous extension as an isometric transformation ξ into $k\xi$ of the 1–adelic skew–plane into itself.

An element

$$\gamma = \alpha + k\beta$$

of the complex skew–plane is a linear combination of elements α and β of the complex plane which defines the product

$$\gamma\xi = \alpha\xi + \beta k\xi$$

as a linear combination of elements of the 1–adelic skew–plane.

The product

$$\eta = \gamma\xi$$

of an element ξ of the 1–adelic skew–plane and an element γ of the complex skew–plane is defined by the conjugate product

$$\eta^- = \xi^-\gamma^-\gamma^-.$$

The 1–adelic plane is a Hilbert space which is contained isometrically in the 1–adelic skew–plane and whose elements are the elements of the 1–adelic skew–plane on which multiplication on left and right by elements of the complex plane agree. An orthogonal basis for the discrete field as a vector space over the Gauss field is an orthogonal basis for the 1–adelic plane as a vector space over the field of complex numbers.

If a set of elements γ of the discrete field is an orthogonal basis for the discrete field, every element

$$\xi = \sum \gamma\xi_\gamma$$

of the 1–adelic skew–plane is a unique sum over the elements γ of the orthogonal basis with coefficients ξ_γ in the complex skew–plane. The identity

$$\langle \xi, \xi \rangle = \sum \langle \gamma, \gamma \rangle \xi_\gamma^- \xi_\gamma$$

holds with summation over the elements γ of the orthogonal basis. The element ξ belongs to the 1–adelic plane if, and only if, the elements ξ_γ belong to the complex plane.

The canonical measure for the 1–adelic skew–plane is a nonnegative measure on its Baire subsets such that a measure preserving transformation is defined by taking ξ into $\xi + \eta$ for every element η of the 1–adelic skew–plane. The measure is mapped into a constant multiple of the Cartesian product measure of canonical measures for complex skew–planes by taking ξ into the element of the Cartesian product which has component ξ_γ at γ for every element of the orthogonal basis. The canonical measure for the 1–adelic skew–plane is normalized so that the constant is the product

$$\prod \langle \gamma, \gamma \rangle^2$$

taken over the elements γ of the orthogonal basis.

The canonical measure for the 1–adelic plane is a nonnegative measure on its Baire subsets such that a measure preserving transformation is defined by taking ξ into $\xi + \eta$ for every element η of the 1–adelic plane. The measure is mapped into a constant multiple of the Cartesian product measure of canonical measures for complex planes by taking ξ into the element of the Cartesian product which has ξ_γ as component for every element γ of the orthogonal basis. The canonical measure for the 1–adelic plane is normalized so that the constant is the product

$$\prod \langle \gamma, \gamma \rangle$$

taken over the elements γ of the orthogonal basis.

The complementary space to the 1–adelic plane in the 1–adelic skew–plane is its orthogonal complement. Multiplication by k is an isometric transformation of the 1–adelic plane onto its complementary space. The canonical measure for the complementary space is a nonnegative measure on its Baire subsets such that a measure preserving transformation is defined by taking ξ into $\xi + \eta$ for every element η of the space. The measure is the image of the canonical measure for the 1–adelic plane under the transformation which takes ξ into $k\xi$. The canonical measure for the 1–adelic skew–plane is the Cartesian product measure for the 1–adelic plane and the canonical measure for its complementary space.

Conjugation is a measure preserving transformation of the 1–adelic skew–plane into itself. Every isometric transformation of the 1–adelic skew–plane into itself is measure preserving. An example of an isometric transformation of the 1–adelic skew–plane into itself is the continuous extension of an automorphism of the discrete skew–field which takes Hurwitz integers into Hurwitz integers. If the transformation takes ξ into η, it takes $\omega\xi$ into $\omega\eta$ and $\xi\omega$ into $\eta\omega$ for every element ω of the complex skew–plane. Similar conclusions apply to the 1–adelic plane.

The Fourier transformation for the 1–adelic skew–plane is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the 1–adelic skew–plane into itself which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(\pi i \langle \xi, \eta \rangle + \pi i \langle \eta, \xi \rangle) f(\eta)$$

of ξ defined by integration with respect to the canonical measure. Fourier inversion states that

$$f(\xi) = \int \exp(-\pi i\langle \xi, \eta\rangle - \pi i\langle \eta, \xi\rangle) g(\eta) d\eta$$

with integration with respect to the canonical measure when the function $f(\xi)$ of ξ is continuous and the function $g(\xi)$ of ξ is integrable.

The Fourier transformation for the 1–adelic plane is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the 1–adelic plane into itself which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(\pi i\langle \xi, \eta\rangle + \pi i\langle \eta, \xi\rangle) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure. Fourier inversion states that

$$f(\xi) = \int \exp(-\pi i\langle \xi, \eta\rangle - \pi i\langle \eta, \xi\rangle) g(\eta) d\eta$$

with integration with respect to the canonical measure when the function $f(\xi)$ of ξ is continuous and the function $g(\xi)$ of ξ is integrable.

Harmonic functions of order ν for the 1–adelic skew–plane are defined when an orthogonal basis is chosen for the discrete field. The degree of the harmonic polynomial is a family of nonnegative integers ν_γ parametrized by elements γ of the orthogonal basis. A harmonic polynomial of degree ν is a continuous function $\phi(\xi)$ of $\xi = \sum \gamma\xi_\gamma$ in the 1–adelic skew–plane which is a harmonic polynomial of degree ν_γ in the variable ξ_γ for every γ when other variables are held fixed.

A harmonic polynomial of degree ν is said to be a monomial if it is a product of monomials of degree ν_γ in the variables ξ_γ. The Hilbert space of harmonic polynomials of degree ν is a Hilbert space which has an orthogonal basis consisting of monomials of degree ν and in which the scalar self–product of a monomial is the product of the scalar self–products of monomials in the variables ξ_γ.

Isometric transformations of the Hilbert space of harmonic polynomials of degree ν into itself are defined by taking a function $f(\xi)$ of ξ in the 1–adelic skew–plane into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ in the 1–adelic skew–plane for every element ω of the complex skew–plane with conjugate as inverse.

A Radon transformation of harmonic ϕ for the 1–adelic skew–plane is defined when a harmonic polynomial of degree ν for the 1–adelic skew–plane does not vanish identically. The domain and range of the Radon transformation of harmonic ϕ are contained in the Hilbert space of functions $f(\xi)$ of ξ in the 1–adelic skew–plane which are square integrable with respect to the canonical measure and which satisfy the identity

$$\phi(\xi) f(\omega\xi) = \phi(\omega\xi) f(\xi)$$

for every element ω of the complex skew–plane with conjugate as inverse.

The Radon transformation of harmonic ϕ takes a function $f(\xi)$ of ξ in the 1–adelic skew–plane into a function $g(\xi)$ of ξ in the 1–adelic skew–plane when the identity

$$g(\omega\xi)/\phi(\omega\xi) = \int f(\omega\xi + \omega\eta)/\phi(\omega\xi + \omega\eta)d\eta$$

holds for ξ in the 1–adelic plane for every element ω of the complex skew–plane with integration with respect to the canonical measure for the complementary space to the 1–adelic plane in the 1–adelic skew–plane.

The integral is interpreted as a limit of integrals over disks $\langle \eta, \eta \rangle < n$ in the complementary space. The limit is taken in the metric topology of the Hilbert space of square integrable functions with respect to the canonical measure for the 1–adelic skew–plane.

The function

$$\phi(\xi) \exp(\pi i z \langle \xi, \xi \rangle)$$

of ξ in the 1–adelic skew–plane is an eigenfunction of the Radon transformation of harmonic ϕ for the eigenvalue

$$i/z$$

when z is in the upper half–plane.

The harmonic function is assumed to have norm one in the Hilbert space of harmonic polynomials of degree ν. The domain of the Laplace transformation of harmonic ϕ is the set of functions

$$f(\xi) = \phi(\xi)h(\langle \xi, \xi \rangle)$$

of ξ in the 1–adelic skew–plane which are square integrable with respect to the canonical measure and which are parametrized by a function $h(z)$ of z in the upper half–plane admitting an extension to the complex plane satisfying the identity

$$h(\omega z) = h(z)$$

for every element ω of the complex plane with conjugate as inverse.

The inequality

$$\pi_\infty \int |\xi|^{\nu_\infty} |h(\xi)|^2 d\xi \leq \int |f(\xi)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the complex skew–plane and with integration on the right with respect to the canonical measure for the 1–adelic skew–plane with

$$2 + \nu_\infty = \sum (2 + \nu_\gamma)$$

defined as a sum and

$$\frac{\pi_\infty}{2 + \nu_\infty} = \prod \frac{\pi}{2 + \nu_\gamma}$$

defined as a product over the elements γ of the orthogonal basis. Every function $h(z)$ of z in the upper half–plane for which the integral on the left converges parametrizes a function $f(\xi)$ of ξ in the 1–adelic skew–plane for which equality holds.

An element of the range of the Laplace transformation of harmonic ϕ for the 1–adelic skew–plane is an analytic function

$$h^\wedge(z) = \pi \int_0^\infty t^{\nu_\infty} h(t) \exp(\pi i t z) t dt$$

of z in the upper half–plane defined by a function $h(z)$ of z in the upper half–plane which admits an extension to the complex plane satisfying the identity

$$h(\omega z) = h(z)$$

for every element ω of the complex plane with conjugate as inverse such that the integral

$$\int |\xi|^{\nu_\infty} |h(\xi)|^2 d\xi$$

with respect to the canonical measure for the upper half–plane converges. The identity

$$\int_0^\infty \int_{-\infty}^{+\infty} |h^\wedge(x+iy)|^2 y^{\nu_\infty} dx dy = (2\pi)^{-\nu_\infty} \Gamma(1+\nu_\infty) \int_0^\infty t^{\nu_\infty} |h(t)|^2 t dt$$

is satisfied.

The orthogonal complement of the kernel of the Laplace transformation of harmonic ϕ is an invariant subspace for the adjoint of the Radon transformation of harmonic ϕ on which the adjoint is a maximal accretive transformation. The adjoint of the Radon transformation takes a function $f(\xi)$ of ξ which is orthogonal to the kernel into a function $g(\xi)$ of ξ which is orthogonal to the kernel when the identity

$$\int \phi(\xi)^- g(\xi) \exp(\pi i z \langle \xi, \xi \rangle) d\xi = (i/z) \int \phi(\xi)^- f(\xi) \exp(\pi i z \langle \xi, \xi \rangle) d\xi$$

holds when z is in the upper half–plane with integration with respect to the canonical measure for the 1–adelic skew–plane.

The Fourier transform for the 1–adelic skew–plane of the function

$$\phi(\xi) \exp(\pi i z \langle \xi, \xi \rangle)$$

of ξ in the 1–adelic skew–plane is the function

$$i^{\nu_\infty} (i/z)^{2+\nu_\infty} \phi(\xi) \exp(-\pi i z^{-1} \langle \xi, \xi \rangle)$$

of ξ in the l–adelic skew–plane when z is in the upper half–plane.

The r–adelic skew–plane is the Cartesian product of the 1–adelic skew–plane and the r–adic skew–plane. An element ξ of the r–adelic skew–plane has a component ξ_+ in the 1–adelic skew–plane and a component ξ_- in the r–adic skew–plane. The r–adelic skew–plane is a conjugated ring with coordinate addition and multiplication.

The sum $\xi + \eta$ of elements ξ and η of the r–adelic skew–plane is the element of the r–adelic skew–plane whose component in the 1–adelic skew–plane is the sum

$$\xi_+ + \eta_+$$

of components in the 1–adelic skew–plane and whose component in the r–adic skew–plane is the sum

$$\xi_- + \eta_-$$

of components in the r–adic skew–plane.

The product $\xi\eta$ of elements ξ and η of the r–adelic skew–plane is the element of the r–adelic skew–plane whose component in the 1–adelic skew–plane is the product

$$\xi_+\eta_+$$

of components in the 1–adelic skew–plane and whose component in the r–adic skew–plane is the product

$$\xi_-\eta_-$$

of components in the r–adic skew–plane.

The conjugate of an element ξ of the r–adelic skew–plane is the element ξ^- of the r–adelic skew–plane whose component in the 1–adelic skew–plane is the conjugate

$$\xi_+^-$$

of the component in the 1–adelic skew–plane and whose component in the r–adic skew–plane is the conjugate

$$\xi_-^-$$

of the component in the r–adic skew–plane.

The r–adelic skew–plane is a locally compact Hausdorff space in the Cartesian product topology of the topology of the 1–adelic skew–plane and the topology of the r–adic skew–plane. Addition is continuous as a transformation of the Cartesian product of the r–adelic skew–plane with itself into the r–adelic skew–plane. Multiplication by an element of the r–adelic skew–plane is a continuous transformation of the r–adelic skew–plane into itself. Conjugation is a continuous transformation of the r–adelic skew–plane into itself.

The canonical measure for the r–adelic skew–plane is the Cartesian product measure of the canonical measure for the 1–adelic skew–plane and the canonical measure for the r–adic skew–plane. The measure is defined on Baire subsets of the r–adelic skew–plane. A measure preserving transformation of the r–adelic skew–plane into itself is defined by taking ξ into $\xi + \eta$ for every element η of the r–adelic skew–plane. Measure preserving transformations of the r–adelic skew–plane into itself are defined by taking ξ into $\omega\xi$ and into $\xi\omega$ for every element ω of the r–adelic skew–plane with has conjugate as inverse.

The Fourier transformation for the r–adelic skew–plane is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the r–adelic skew–plane into itself which takes an integrable function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane into the continuous function

$$g(\xi_+, \xi_-) = \int \exp(\pi i \langle \xi_+, \eta_+ \rangle + \pi i \langle \eta_+, \xi_+ \rangle) \exp(\pi i(\xi_-^- \eta_- + \eta_-^- \xi_-)) f(\eta_+, \eta_-) d\eta$$

of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi_+, \xi_-) = \int \exp(-\pi i \langle \xi_+, \eta_+ \rangle - \pi i \langle \eta_+, \xi_+ \rangle) \exp(-\pi i(\xi_-^- \eta_- + \eta_-^- \xi_-)) g(\eta_+, \eta_-) d\eta$$

applies with integration with respect to the canonical measure when the function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ is integrable and the function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ is continuous.

The r–adelic half–plane is the set of elements (z, ξ) of the r–adelic skew–plane whose component z in the complex skew–plane belongs to the upper half–plane and whose component ξ in the r–adic skew–plane belongs to the r–adic half–plane. The topology of the r–adelic half–plane is the Cartesian product topology of the topology of the upper half–plane and the topology of the r–adic half–plane. The canonical measure for the r–adelic half–plane is the Cartesian product measure of the canonical measure for the upper half–plane and the canonical measure for the r–adic half–plane.

Harmonic functions of order $\nu = (\nu_+, \nu_-)$ for the r–adelic skew–plane are defined when harmonic functions of order ν_+ are defined for the 1–adelic skew–plane and harmonic functions of order ν_- are defined for the r–adic skew–plane.

A harmonic function of order ν for the r–adelic skew–plane is a continuous function $\phi(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane such that for every element ξ_- of the r–adic skew–plane the function of ξ_+ in the 1–adelic skew–plane is a harmonic function of order ν_+ and for every element ξ_+ of the 1–adelic skew–plane the function of ξ_- in the r–adic skew–plane is a harmonic function of order ν_-.

The set of harmonic functions of order ν for the r–adelic skew–plane is a Hilbert space whose scalar product is defined from the scalar product of the Hilbert space of harmonic functions of order ν_+ for the 1–adelic skew–plane and the scalar product for the Hilbert space of harmonic functions of order ν_- for the r–adic skew–plane.

The Hilbert space of harmonic functions of order ν_+ for the 1–adelic skew–plane admits an orthogonal basis whose elements are monomials. The Hilbert space of harmonic functions of order ν for the r–adelic skew–plane is the orthogonal sum of subspaces determined by the monomials. A subspace contains the product of the monomial with a harmonic function of order ν_- for the r–adic skew–plane. The scalar product of two elements of the subspace is the scalar self–product of the monomials multiplied by the scalar product of the harmonic functions for the r–adic skew–plane.

Multiplication on left or right by an element of the r–adelic skew–plane with conjugate as inverse on the argument of a harmonic function of order ν is an isometric transformation of the Hilbert space of harmonic functions of order ν into itself. The dimension of the

Hilbert space of harmonic functions of order ν for the 1–adelic skew–plane is the product of the dimension of the Hilbert space of harmonic functions of order ν_+ for the 1–adelic skew–plane and the Hilbert space of harmonic functions of order ν_- for the r–adic skew–plane.

Hecke operators are self–adjoint transformations of the Hilbert space of harmonic functions of order ν for the r–adelic skew–plane into itself. An isometric transformation of the Hilbert space into itself is defined by every nonzero element ω of the discrete skew–field by taking a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane into the function $f(\xi_+\omega, \xi_-\omega)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane.

An integral element of the discrete skew–field with integral inverse has conjugate as inverse. Define μ as the number of elements of the discrete skew–field with conjugate as inverse.

A Hecke operator $\Delta(n)$ is defined for every positive integer n whose prime divisors are divisors of r such that a ring of self–conjugate integral elements of the complex skew–plane modulo n exists. The transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane into the function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane defined by the sum

$$\mu g(\xi_+, \xi_-) = \sum f(\xi_+\omega, \xi_-\omega)$$

over the integral elements ω of the discrete skew–field such that

$$n = \lambda(\omega^-\omega).$$

The identity

$$\Delta(m)\Delta(n) = \sum k\Delta(mn/k^2)$$

holds for all positive integers m and n whose prime divisors are divisors of r such that $\Delta(m)$ and $\Delta(n)$ are defined with summation over the common odd divisors k of m and n.

The Hecke operator $\Delta(1)$ is the orthogonal projection of the Hilbert space of harmonic functions of order ν onto the subspace of functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane which satisfy the identity

$$f(\xi_+, \xi_-) = f(\xi_+\omega, \xi_-\omega)$$

for every integral element ω of the discrete skew–plane with conjugate as inverse.

The kernel of $\Delta(1)$ is contained in the kernel of $\Delta(n)$, and the range of $\Delta(n)$ is contained in the range of $\Delta(1)$, for every positive integer n whose prime divisors are divisors of r such that $\Delta(n)$ is defined.

Hecke operators act as self–adjoint transformations in the range of $\Delta(1)$. The range of $\Delta(1)$ is the orthogonal sum of invariant subspaces whose elements are characterized as eigenfunctions of $\Delta(n)$ for a real eigenvalue $\tau(n)$ for every positive integer n whose prime divisors are divisors of r such that $\Delta(n)$ is defined.

Eigenvalues defining an invariant subspace satisfy the identity

$$\tau(m)\tau(n) = \sum k\tau(mn/k^2)$$

for all positive integers m and n whose prime divisors are divisors of r such that $\Delta(m)$ and $\Delta(n)$ are defined with summation over the common odd divisors k of m and n. The eigenvalue $\tau(n)$ is one when n is one.

A Radon transformation of harmonic ϕ is defined for the r–adelic skew–plane when a nontrivial harmonic function ϕ of order ν defines an eigenfunction for Hecke operators. The transformation is defined by integration with respect to the canonical measure for the complementary space to the r–adelic plane in the r–adelic skew–plane.

The r–adelic plane is the set of elements $\xi = (\xi_+, \xi_-)$ of the r–adelic skew–plane whose component ξ_+ in the 1–adelic skew–plane belongs to the 1–adelic plane and whose component ξ_- in the r–adic skew–plane belongs to the r–adic plane. The r–adelic plane is isomorphic to the Cartesian product of the complex 1–adelic plane and the r–adic plane. The subspace topology of the r–adelic plane inherited from the r–adelic skew–plane is identical with the Cartesian product topology of the topology of the 1–adelic plane and the topology of the r–adic plane. The canonical measure for the r–adelic plane is the Cartesian product measure of the canonical measure for the 1–adelic plane and the canonical measure for the r–adic plane.

The complementary space to the r–adelic plane in the r–adelic skew–plane is the set of elements $\xi = (\xi_+, \xi_-)$ of the r–adelic skew–plane whose component ξ_+ in the 1–adelic skew–plane belongs to the complementary space to the 1–adelic plane in the 1–adelic skew–plane and whose component ξ_- in the r–adic skew–plane belongs to the complementary space to the r–adic plane in the r–adic skew–plane. The complementary space to the r–adelic plane in the r–adelic skew–plane is isomorphic to the Cartesian product of the complementary space of the 1–adelic plane in the 1–adelic skew–plane and the complementary space to the r–adic plane in the r–adic skew–plane.

The topology which the r–adelic plane inherits from the r–adelic skew–plane is identical with the Cartesian product topology of the topology of the 1–adelic plane and the topology of the r–adic plane. The canonical measure for the r–adelic plane is the Cartesian product measure of the canonical measure for the 1–adelic plane and the canonical measure for the r–adic plane.

The topology which the complementary space to the r–adelic plane in the r–adelic skew–plane inherits from the r–adelic skew–plane is identical with the Cartesian product topology of the topology of the complementary space of the 1–adelic plane in the 1–adelic skew–plane and the topology of the complementary space to the r–adic plane in the r–adic skew–plane.

The canonical measure for the complementary space to the r–adelic plane in the r–adelic skew–plane is the Cartesian product measure of the canonical measure for the complementary space to the 1–adelic plane in the 1–adelic skew–plane and the canonical measure for the complementary space to the r–adic plane in the r–adic skew–plane.

The Radon transformation of harmonic ϕ has domain and range in the Hilbert space of functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane which satisfy the identity

$$\phi(\xi_+, \xi_-) f(\omega_+ \xi_+, \omega_- \xi_-) = \phi(\omega_+ \xi_+, \omega_- \xi_-) f(\xi_+, \xi_-)$$

for every element $\omega = (\omega_+, \omega_-)$ of the r–adelic skew–plane with conjugate as inverse.

The transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane into the function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane when the identity

$$g(\omega_+ \xi_+, \omega_- \xi_-) / \phi(\omega_+ \xi_+, \omega_- \xi_-)$$

$$= \int f(\omega_+ \xi_+ + \omega_+ \eta_+, \omega_- \xi_- + \omega_- \eta_-) / \phi(\omega_+ \xi_+ + \omega_+ \eta_+, \omega_- \xi_- + \omega_- \eta_-) d\eta$$

holds with integration with respect to the canonical measure for the complementary space to the r–adelic plane in the r–adelic skew–plane for every element $\xi = (\xi_+, \xi_-)$ of the r–adelic plane and every element $\omega = (\omega_+, \omega_-)$ of the r–adelic skew–plane with conjugate as inverse.

The integral is taken in the metric topology of the Hilbert space of square integrable functions with respect to the canonical measure for the r–adelic skew–plane of integrals over compact subsets of the complementary space to the r–adelic plane in the r–adelic skew–plane.

The Laplace transformation of harmonic ϕ for the r–adelic skew–plane is a spectral analysis of the adjoint of the Radon transformation of harmonic ϕ for the r–adelic skew–plane in an invariant subspace. A Laplace transformation of harmonic ϕ is defined for a harmonic function ϕ of order ν which has norm one in the Hilbert space of harmonic functions of order ν.

The domain of the Laplace transformation of harmonic ϕ is contained in the Hilbert space of functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane which are square integrable with respect to the canonical measure for the r–adelic skew–plane, which satisfy the identity

$$\phi(\xi_+, \xi_-) f(\omega_+ \xi_+, \omega_- \xi_-) = \phi(\omega_+ \xi_+, \omega_- \xi_-) f(\xi_+, \xi_-)$$

for every element $\omega = (\omega_+, \omega_-)$ of the r–adelic skew–plane with conjugate as inverse, and which vanish at noninvertible elements of the r–adelic skew–plane.

A function

$$f(\xi_+, \xi_-) = \phi(\xi_+, \xi_-) h(\langle \xi_+, \xi_+ \rangle, \xi_-^- \xi_-)$$

of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane which belongs to the domain of the Laplace transformation of harmonic ϕ is parametrized by a function $h(z, \xi)$ of (z, ξ) in the r–adelic half–plane.

For every element ξ of the r–adic half–plane the function of z in the upper half–plane admits an extension to the complex plane which satisfies the identity

$$h(\omega z, \xi) = h(z, \xi)$$

for every element ω of the complex plane with conjugate as inverse. For every element z of the upper half–plane the function of ξ in the r–adic half–plane satisfies the identity

$$h(z, \omega\xi) = h(z, \xi)$$

for every element ω of the r–adic half–plane with conjugate as inverse which is the square of an element with conjugate as inverse. If for some primitive character χ modulo ρ for the r–adic half–plane whose values are fourth roots of unity, the identity

$$h(z, \omega\xi) = \chi(\omega)h(z, \xi)$$

holds for every element ξ of the r–adic line and every element ω of the r–adic line with itself as inverse, then the identity holds for every element ω of the r–adic half–plane with conjugate as inverse and every element ξ of the r–adic half–plane.

The identity

$$\int |f(\xi_+, \xi_-)|^2 d\xi = \pi_\infty \int \ |\xi_+|^{\nu_\infty} |h(\xi_+, \xi_-)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the r–adelic skew–plane and with integration on the right with respect to the canonical measure for the r–adelic half–plane.

The Laplace transformation of harmonic ϕ of the function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane is the function

$$g(z, \xi) = 2 \int \exp(\pi i |\eta_+| z) \exp(\tfrac{1}{2}\pi i(\xi^- \eta_- + \eta_-^- \xi)) |\eta_+|^{\nu_\infty} h(\eta_+, \eta_-) d\eta$$

of (z, ξ) in the r–adelic half–plane defined by integration with respect to the canonical measure for the r–adelic half–plane when the integral is absolutely convergent. The integral is otherwise defined to maintain the identity

$$\int |\tfrac{1}{2}\eta_+ - \tfrac{1}{2}\eta_+^-|^{\nu_\infty} |g(\eta_+, \eta_-)|^2 d\xi = (2\pi)^{-\nu_\infty} \Gamma(1 + \nu_\infty) \textstyle\int |\eta_+|^{\nu_\infty} |h(\eta_+, \eta_-)|^2 d\eta$$

with integration with respect to the canonical measure for the r–adelic half–plane.

A computation of the adjoint of the Radon transformation of harmonic ϕ for the r–adelic skew–plane is made from the Laplace transformation of harmonic ϕ for the r–adelic skew–plane. The adjoint of the Radon transformation takes a function $f_1(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane into the function $f_2(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the r–adelic skew–plane if, and only if, the Laplace transforms of harmonic ϕ are functions $g_1(z, \xi)$ and $g_2(z, \xi)$ of (z, ξ) in the r–adelic half–plane which satisfy the identity

$$g_2(z, \xi) = (i/z)\lambda_r(\xi)^{-1} g_1(z, \xi).$$

The adjoint of the Radon transformation of harmonic ϕ for the r–adelic skew–plane is a maximal accretive transformation in the orthogonal complement of the kernel of the

Laplace transformation. The adjoint of the Radon transformation of harmonic ϕ is examined in subspaces for the same property.

An element $\xi = (\xi_+, \xi_-)$ of the r–adelic skew–plane is treated as integral if its r–adic component ξ_- is integral. The set of integral elements of the r–adelic skew–plane is a ring which is a locally compact Hausdorff space in the topology inherited from the r–adelic skew–plane. The canonical measure for the ring is the restriction to Baire subsets of the ring of the canonical measure for the r–adelic skew–plane.

The ring of integral elements of the r–adelic skew–plane is isomorphic to the Cartesian product of the 1–adelic skew–plane and the ring of integral elements of the r–adic skew–plane. The topology of the ring of integral elements of the r–adelic skew–plane is the Cartesian product topology of the topology of the 1–adelic skew–plane and the topology of the ring of integral elements of the r–adic skew–plane. The canonical measure for the ring of integral elements of the r–adelic skew–plane is the Cartesian product measure of the canonical measure for the 1–adelic skew–plane and the canonical measure for the ring of integral elements of the r–adic skew–plane.

The set of integral elements of the r–adelic plane is a ring which is a locally compact Hausdorff space in the topology inherited from the r–adelic plane. The canonical measure for the ring is the restriction to Baire subsets of the ring of the canonical measure for the r–adelic plane.

The ring of integral elements of the r–adelic plane is isomorphic to the Cartesian product of the 1–adelic plane and the ring of integral elements of the r–adic plane. The topology for the ring of integral elements of the r–adelic plane is the Cartesian product topology of the topology of the 1–adelic plane and the topology of the ring of integral elements of the r–adic plane. The canonical measure for the ring of integral elements of the r–adelic plane is the Cartesian product measure of the canonical measure for the 1–adelic plane and the canonical measure for the ring of integral elements of the r–adic plane.

The complementary space to the ring of integral elements of the r–adelic plane in the ring of integral elements of the r–adelic skew–plane is the set of integral elements of the complementary space to the r–adelic plane in the r–adelic skew–plane. The complementary space to the ring of integral elements of the r–adelic plane in the ring of integral elements of the r–adelic skew–plane is a locally compact Hausdorff space in the subspace topology inherited from the complementary space to the r–adelic plane in the r–adelic skew–plane. The canonical measure for the complementary space to the ring of integral elements of the r–adelic plane in the ring of integral elements of the r–adelic skew–plane is the restriction to its Baire subsets of the canonical measure for the complementary space to the r–adelic plane in the r–adelic skew–plane.

The complementary space to the ring of integral elements of the r–adelic plane in the ring of integral elements of the r–adelic skew–plane is isomorphic to the Cartesian product of the complementary space to the 1–adelic plane in the 1–adelic skew–plane and the complementary space to the ring of integral elements of the r–adic plane in the ring of integral elements of the r–adic skew–plane. The topology of the complementary space to the ring of integral elements of the r–adelic plane in the ring of integral elements of the r–adelic skew–plane is the Cartesian product topology of the topology of the complementary space to the 1–adelic plane in the 1–adelic skew–plane and the topology of the complementary space to the ring of integral elements of the r–adic plane in the ring of integral elements of the r–adic skew–plane. The canonical measure for the complementary space to the ring of integral elements of the r–adelic plane in the ring of integral elements of the r–adelic skew–plane is the Cartesian product measure of the canonical measure for the complementary space to the 1–adelic plane in the 1–adelic skew–plane and the canonical measure for the complementary space to the ring of integral elements of the r–adic plane in the ring of integral elements of the r–adic skew–plane.

The Radon transformation of harmonic ϕ for the ring of integral elements of the r–adelic skew–plane is a transformation whose domain and range are contained in a Hilbert space which is the domain of the Laplace transformation of harmonic ϕ for the ring of integral elements of the r–adelic skew–plane. The elements of the domain of the Laplace transformation are functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the r–adelic skew–plane which are square integrable with respect to the canonical measure for the ring and which satisfy the identity

$$\phi(\xi_+, \xi_-)f(\omega_+\xi_+, \omega_-\xi_-) = \phi(\omega_+\xi_+, \omega_-\xi_-)f(\xi_+, \xi_-)$$

for every element $\omega = (\omega_+, \omega_-)$ of the r–adelic skew–plane with conjugate as inverse.

The Radon transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the ring into a function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the ring when the identity

$$g(\omega_+\xi_+, \omega_-\xi_-)/\phi(\omega_+\xi_+, \omega_-\xi_-)$$
$$= \int f(\omega_+\xi_+ + \omega_+\eta_+, \omega_-\xi_- + \omega_-\eta_-)/\phi(\omega_+\xi_+ + \omega_+\eta_+, \omega_-\xi_- + \omega_-\eta_-)d\eta$$

for every element $\omega = (\omega_+, \omega_-)$ of the r–adelic skew–plane with conjugate as inverse for every element $\xi = (\xi_+, \xi_-)$ of the ring of integral elements of the r–adelic plane with integration with respect to the canonical measure for the complementary space to the ring of integral elements of the r–adelic plane in the ring of integral elements of the r–adelic skew–plane. The integral is interpreted as a limit in the metric topology of the domain of the Laplace transformation of integrals over compact subsets of the complementary space.

An element $\xi = (\xi_+, \xi_-)$ of the r–adelic half–plane is treated as integral if its r–adic component ξ_- is an integral element of the r–adic half–plane. The set of integral elements of the r–adelic half–plane is an additive subgroup which is a locally compact Hausdorff space in the subspace topology inherited from the r–adelic half–plane. The canonical measure for the subgroup is the restriction to Baire subsets of the subgroup of the canonical measure for the r–adelic half–plane.

The group of integral elements of the r–adelic half–plane is isomorphic to the Cartesian product of the upper half–plane and the group of integral elements of the r–adic half–plane. The topology of the group of integral elements of the r–adelic half–plane is the Cartesian product topology of the topology of the upper half–plane and the topology of the group of integral elements of the r–adic half–plane. The canonical measure for the group of integral elements of the r–adelic half–plane is the Cartesian product measure of the canonical measure for the upper half–plane and the canonical measure for the group of integral elements of the r–adic half–plane.

A function
$$f(\xi_+, \xi_-) = \phi(\xi_+, \xi_-) h(\langle \xi_+, \xi_+ \rangle, \xi_-^- \xi_-)$$
of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the r–adelic skew–plane which belongs to the domain of the Laplace transformation of harmonic ϕ for the ring is parametrized by a function $h(z, \xi)$ of (z, ξ) in the group of integral elements of the r–adelic half–plane whose product with $|z|^{\frac{1}{2}\nu_\infty}$ is square integrable with respect to the canonical measure for the group. For every integral element ξ of the r–adic half–plane the function $h(z, \xi)$ of z in the upper half–plane admits an extension to the complex plane which satisfies the identity

$$h(\omega z, \xi) = h(z, \xi)$$

for every element ω of the complex plane with conjugate as inverse. If for a primitive character χ modulo ρ for the r–adic half–plane with fourth roots of unity as nonzero values, the identity
$$h(z, \omega \xi) = \chi(\omega) h(z, \xi)$$
holds for every element ω of the r–adic line with itself as inverse, then the identity holds for every element ω of the r–adic half–plane with conjugate as inverse.

The identity
$$\int |f(\xi_+, \xi_-)|^2 d\xi = \pi_\infty \int |\xi_+|^{\nu_\infty} |h(\xi_+, \xi_-)|^2 d\xi$$
holds with integration on the left with respect to the canonical measure for the ring of integral elements of the r–adelic skew–plane and with integration on the right with respect to the canonical measure for the group of integral elements of the r–adelic half–plane.

Elements $\xi = (\xi_+, \xi_-)$ and $\eta = (\eta_+, \eta_-)$ of the r–adelic half–plane are treated as congruent modulo 1 if the components
$$\xi_+ = \eta_+$$
in the upper half–plane are equal and the components ξ_- and η_- in the r–adic half–plane are congruent modulo 1. The quotient space modulo the equivalence relation is the r–adelic half–plane modulo 1.

The r–adelic half–plane modulo 1 is isomorphic to the Cartesian product of the upper half–plane and the r–adic half–plane modulo 1. The r–adelic half–plane modulo 1 is an additive group whose topology is the Cartesian product topology of the upper half–plane and the topology of the r–adic half–plane modulo 1. The canonical measure for the r–adelic half–plane modulo 1 is the Cartesian product measure of the canonical measure for the upper half–plane and the canonical measure for the r–adic half–plane modulo 1.

The Laplace transform of harmonic ϕ of the function

$$f(\xi_+, \xi_-) = \phi(\xi_+, \xi_-) h(\langle \xi_+, \xi_+ \rangle, \xi_-^- \xi_-)$$

of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the r–adelic skew–plane is defined as the function

$$g(z, \xi) = 2 \int \exp(\pi i |\eta_+| z) \exp(\tfrac{1}{2}\pi i(\xi^- \eta_- + \eta_-^- \xi)) |\eta_+|^\nu h(\eta_+, \eta_-) d\eta$$

of (z, ξ) in the r–adelic half–plane modulo 1 defined by integration with respect to the canonical measure for the group of integral elements of the r–adelic half–plane. The identity

$$\int |\tfrac{1}{2}\eta_+ - \tfrac{1}{2}\eta_+^-|^{\nu_\infty} |g(\eta_+, \eta_-)|^2 d\eta = (2\pi)^{-\nu_\infty} \Gamma(1 + \nu_\infty) \int |\eta_+|^{\nu_\infty} |h(\eta_+, \eta_-)|^2 d\eta$$

holds with integration on the left with respect to the canonical measure for the r–adelic half–plane modulo 2 and with integration on the right with respect to the canonical measure for the group of integral elements of the r–adelic half–plane.

The adjoint of the Radon transformation of harmonic ϕ for the ring of integral elements of the r–adelic skew–plane takes a function $f_1(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the ring into a function $f_2(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the ring if, and only if, the Laplace transforms of harmonic ϕ satisfy the identity

$$g_2(z, \xi) = (i/z) \lambda_r(\xi)^{-1} g_1(z, \xi)$$

for every element ξ of the r–adic half–plane modulo 2.

The adjoint of the Radon transformation of harmonic ϕ for the ring of integral elements of the r–adelic skew–plane is an accretive transformation in the orthogonal complement of the kernel of the Laplace transformation of harmonic ϕ. The accretive property of the transformation is preserved in an invariant subspace.

Integral elements $\xi = (\xi_+, \xi_-)$ and $\eta = (\eta_+, \eta_-)$ of the r–adelic skew–plane are treated as congruent modulo r if their components

$$\xi_+ = \eta_+$$

in the 1–adelic skew–plane are equal and if their components ξ_- and η_- in the r–adic skew–plane are congruent modulo r. The quotient space is the ring of integral elements of the r–adelic skew–plane modulo r.

The ring is isomorphic to the Cartesian product of the 1–adelic skew–plane and the ring of integral elements of the r–adic skew–plane modulo r. The topology of the ring of integral elements of the r–adelic skew–plane modulo r is the Cartesian product topology of the topology of the 1–adelic skew–plane and the topology of the ring of integral elements of the r–adic skew–plane modulo r. The canonical measure for the ring of integral elements of the r–adelic skew–plane modulo r is the Cartesian product measure of the canonical measure for the 1–adelic plane and the canonical measure for the ring of integral elements of the r–adic skew–plane modulo r.

The projection of the ring of integral elements of the r–adelic skew–plane onto the ring of integral elements of the r–adelic skew–plane modulo r is a homomorphism of conjugated ring structure which is a continuous open mapping and which takes the canonical measure into the canonical measure.

A function defined on the ring of integral elements of the r–adelic skew–plane modulo r is treated as a function defined on the ring of integral elements of the r–adelic skew–plane which has equal values at elements which are congruent modulo r.

The ring of integral elements of the r–adelic plane modulo r is the quotient ring of the ring of integral elements of the r–adelic plane defined by congruence modulo r. The ring of integral elements of the r–adelic plane modulo r is isomorphic to the image of the ring of integral elements of the r–adelic plane in the ring of integral elements of the r–adelic skew–plane modulo r.

The ring of integral elements of the r–adelic plane modulo r is isomorphic to the Cartesian product of the 1–adelic plane and the ring of integral elements of the r–adic plane modulo r. The topology of the ring of integral elements of the r–adelic plane modulo r is the Cartesian product topology of the topology of the 1–adelic plane and the topology of the ring of integral elements of the r–adic plane modulo r. The canonical measure for the ring of integral elements of the r–adelic plane modulo r is the Cartesian product measure of the canonical measure for the 1–adelic plane and the canonical measure for the ring of integral elements of the r–adic plane modulo r.

The projection of the ring of integral elements of the r–adelic plane onto the ring of integral elements of the r–adelic plane modulo r is a homomorphism of conjugated ring structure which is a continuous open mapping and which takes the canonical measure into the canonical measure.

A function defined on the ring of integral elements of the r–adelic plane modulo r is treated as a function defined on the ring of integral elements of the r–adelic plane which has equal values at elements which are congruent modulo r.

The complementary space to the ring of integral elements of the r–adelic plane modulo r in the ring of integral elements of the r–adelic skew–plane modulo r is defined as the image in the ring of integral elements of the r–adelic skew–plane modulo r of the complementary space to the ring of integral elements of the r–adelic plane in the ring of integral elements of the r–adelic skew–plane. The complementary space to the ring of integral elements of the r–adelic plane modulo r in the ring of integral elements of the r–adelic skew–plane modulo r is isomorphic to the quotient space modulo r of the complementary space to the ring of integral elements of the r–adelic plane in the ring of integral elements of the r–adelic skew–plane.

The complementary space to the ring of integral elements of the r–adelic plane modulo r in the ring of integral elements of the r–adelic skew–plane modulo r is isomorphic to the Cartesian product of the complementary space to the 1–adelic plane in the 1–adelic skew–plane and the complementary space to the ring of integral elements of the r–adic plane modulo r in the ring of integral elements of the r–adic skew–plane modulo r.

The topology of the complementary space to the ring of integral elements of the r–adelic plane modulo r in the ring of integral elements of the r–adelic skew–plane modulo r is the Cartesian product topology of the topology of the complementary space to the 1–adelic plane in the 1–adelic skew–plane and the topology for the complementary space to the ring of integral elements of the r–adic plane modulo r in the ring of integral elements of the r–adic skew–plane modulo r.

The canonical measure for the complementary space to the ring of integral elements of the r–adelic plane modulo r in the ring of integral elements of the r–adelic skew–plane modulo r is the Cartesian product measure of the canonical measure for the complementary space to the 1–adelic plane in the 1–adelic skew–plane and the canonical measure for the complementary space to the ring of integral elements of the r–adic plane modulo r in the ring of integral elements of the r–adic skew–plane modulo r.

The projection of the complementary space to the ring of integral elements of the r–adelic plane in the ring of integral elements of the r–adelic skew–plane into the complementary space to the ring of integral elements of the r–adelic plane modulo r in the ring of integral elements of the r–adelic skew–plane modulo r is a homomorphism of additive structure which is a continuous open mapping and which maps the canonical measure into the canonical measure.

A function defined on the complementary space to the ring of integral elements of the r–adelic plane modulo r in the ring of integral elements of the r–adelic skew–plane modulo r is treated as a function defined in the complementary space to the ring of integral elements of the r–adelic plane in the ring of integral elements of the r–adelic skew–plane which has equal values at elements which are congruent modulo r.

The Radon transformation of harmonic ϕ for the ring of integral elements of the r–adelic skew–plane modulo r is a transformation with domain and range in the domain of the Laplace transformation of harmonic ϕ for the ring of integral elements of the r–adelic skew–plane modulo r.

The domain of the Laplace transformation of harmonic ϕ for the ring of integral elements of the r–adelic skew–plane modulo r is the Hilbert space of functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the r–adelic skew–plane modulo r which are square integrable with respect to the canonical measure for the ring and which satisfy the identity

$$\phi(\xi_+, \xi_-) f(\omega_+\xi_+, \omega_-\xi_-) = \phi(\omega_+\xi_+, \omega_-\xi_-) f(\xi_+, \xi_-)$$

for every element $\omega = (\omega_+, \omega_-)$ of the r–adelic skew–plane with conjugate as inverse.

The Radon transformation of harmonic ϕ takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the r–adelic skew–plane modulo r into a function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the r–adelic skew–plane modulo r when the identity

$$g(\omega_+\xi_+, \omega_-\xi_-)/\phi(\omega_+\xi_+, \omega_-\xi_-)$$
$$= \int f(\omega_+\xi_+ + \omega_+\eta_+, \omega_-\xi_- + \omega_-\eta_-)/\phi(\omega_+\xi_+ + \omega_+\eta_+, \omega_-\xi_- + \omega_-\eta_-) d\eta$$

holds for every element $\omega = (\omega_+, \omega_-)$ of the r–adelic skew–plane with conjugate as inverse when $\xi = (\xi_+, \xi_-)$ is an element of the ring of integral elements of the r–adelic plane modulo r and when integration is with respect to the canonical measure for the complementary space to the ring of integral elements of the r–adelic plane modulo r in the ring of integral elements of the r–adelic skew–plane modulo r.

The integral is interpreted as a limit of integrals over compact subsets of the complementary space to the ring of integral elements of the r–adelic plane modulo r in the ring of integral elements of the r–adelic skew–plane modulo r. The limit is taken in the metric topology of the Hilbert space of square integrable functions with respect to the canonical measure for the ring of integral elements of the r–adelic skew–plane modulo r.

Integral elements $\xi = (\xi_+, \xi_-)$ and $\eta = (\eta_+, \eta_-)$ of the r–adelic half–plane are treated as congruent modulo r if their components

$$\xi_+ = \eta_+$$

in the upper half–plane are equal and if their components ξ_- and η_- in the r–adic half–plane differ by an element $\eta_- - \xi_-$ of the r–adic half–plane which is the product of r and an integral element of the r–adic half–plane.

The quotient group of the group of integral elements of the r–adelic half–plane is the group of integral elements of the r–adelic half–plane modulo r. The group is isomorphic to the Cartesian product of the upper half–plane and the group of half–integral elements of the r–adic half–plane modulo r.

The topology of the group of integral elements of the r–adelic half–plane modulo r is the Cartesian product topology of the topology of the upper plane and the topology of the group of integral elements of the r–adic half–plane modulo r. The canonical measure for the group of integral elements of the r–adelic half–plane modulo r is the Cartesian product measure of the canonical measure for the upper half–plane and the canonical measure for the ring of integral elements of the r–adic half–plane modulo r.

The projection of the group of integral elements of the r–adelic half–plane onto the group of integral elements of the r–adelic half–plane modulo r is a homomorphism of additive structure which is a continuous open mapping and which maps the canonical measure into the canonical measure.

A function defined on the group of integral elements of the r–adelic half–plane modulo r is treated as a function defined on the group of integral elements of the r–adelic half–plane which has equal values at elements which are congruent modulo r.

A function

$$f(\xi_+, \xi_-) = \phi(\xi_+, \xi_-) h(\langle \xi_+, \xi_+ \rangle, \xi_-^- \xi_-)$$

of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the r–adelic skew–plane modulo r which belongs to the domain of the Laplace transformation of harmonic ϕ for the ring is parametrized by a function $h(z, \xi)$ of (z, ξ) in the group of integral elements of the r–adelic half–plane modulo r whose product with $|z|^{\frac{1}{2}\nu_\infty}$ is square integral with respect to the canonical measure for the group.

For every element ξ of the group of integral elements of the r–adic half–plane modulo r the function $h(z, \xi)$ of z in the upper half–plane admits an extension to the complex plane which satisfies the identity

$$h(\omega z, \xi) = h(z, \xi)$$

for every element ω of the complex plane with conjugate as inverse. If for a primitive character χ modulo ρ for the r–adic half–plane with fourth roots of unity as nonzero values the identity

$$h(z, \omega\xi) = \chi(\omega)h(z, \xi)$$

holds for every element ω of the r–adic line which is its own inverse, then the identity holds for every element ω of the r–adic half–plane with conjugate as inverse.

The identity

$$\int |f(\xi_+, \xi_-)|^2 d\xi = \pi_\infty \int |\xi_+|^{\nu_\infty} |h(\xi_+, \xi_-)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the ring of integral elements of the r–adelic skew–plane modulo r and with integration on the right with respect to the canonical measure for the group of integral elements of the r–adelic half–plane modulo r.

The Laplace transform of harmonic ϕ of the function

$$f(\xi_+, \xi_-) = \phi(\xi_+, \xi_-)h(\langle \xi_+, \xi_+ \rangle, \xi_-^- \xi_-)$$

of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the r–adelic skew–plane modulo r is defined as the function

$$g(z, \xi) = 2 \int \exp(\pi i |\eta_+| z) \exp(\tfrac{1}{2}\pi i(\xi^- \eta_- + \eta_-^- \xi)) |\eta_+|^{\nu_\infty} h(\eta_+, \eta_-) d\eta$$

of (z, ξ) in the r–adelic half–plane modulo 2 such that $r\xi$ vanishes defined by integration with respect to the canonical measure for the group of half–integral elements of the r–adelic half–plane modulo r.

The identity

$$\int |\tfrac{1}{2}\eta_+ - \tfrac{1}{2}\eta_+^-|^{\nu_\infty} |g(\eta_+, \eta_-)|^2 d\eta = (2\pi)^{-\nu_\infty} \Gamma(1 + \nu_\infty) \textstyle\int |\eta_+|^{\nu_\infty} |h(\eta_+, \eta_-)|^2 d\eta$$

holds with integration on the left with respect to the canonical measure for the r–adelic half–plane modulo 2 over the set of elements $\eta = (\eta_+, \eta_-)$ such that $r\eta_-$ vanishes and with integration on the right with respect to the canonical measure for the group of integral elements of the r–adelic half–plane modulo r.

The adjoint of the Radon transformation of harmonic ϕ for the ring of integral elements of the r–adelic skew–plane modulo r takes a function $f_1(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the ring into a function $f_2(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the ring if, and only if, the Laplace transforms of harmonic ϕ satisfy the identity

$$g_2(z, \xi) = (i/z)\lambda_r(\xi)^{-1} g_1(z, \xi)$$

for every element ξ of the r–adic half–plane modulo 2 such that $r\xi$ vanishes.

The adjoint of the Radon transformation of harmonic ϕ for the ring of integral elements of the r–adelic skew–plane modulo r is an accretive transformation in the orthogonal complement of the kernel of the Laplace transformation.

By hypothesis the harmonic ϕ is an eigenfunction of the Hecke operator $\Delta(n)$ for an eigenvalue $\tau(n)$ for every divisor n of r in the definition of the Jacobi space of harmonic ϕ for the r–adelic skew–plane. The space is the set of functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the r–adelic skew–plane modulo r which satisfy the identity

$$\phi(\xi_+, \xi_-) f(\omega_+ \xi_+, \omega_- \xi_-) = \phi(\omega_+ \xi_+, \omega_- \xi_-) f(\xi_+, \xi_-)$$

for every element $\omega = (\omega_+, \omega_-)$ of the r–adelic skew–plane with conjugate as inverse, which are square integrable with respect to the canonical measure for the ring over the set of elements of the ring whose r–adic component is a unit of the ring of integral elements of the r–adic skew–plane modulo r, and which satisfy the identity

$$f(\xi_+, \xi_-) = \sum f(\xi_+ \omega^{-1}, \xi_- \omega^{-1})$$

with summation over the integral elements ω of the complex skew–plane representing divisors

$$n = \omega^- \omega$$

of r such that $\xi_- \omega^{-1}$ is a unit of the ring of integral elements of the r–adic skew–plane modulo r.

The function

$$f(\xi_+, \xi_-) = \phi(\xi_+, \xi_-) h(\langle \xi_+, \xi_+ \rangle, \xi_-^- \xi_-)$$

of $\xi = (\xi_+, \xi_-)$ in the ring is parametrized by a function $h(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the group of integral elements of the r–adelic half–plane modulo r whose product with $|\xi_+|^{\frac{1}{2}\nu_\infty}$ is square integrable with respect to the canonical measure for the group over the set of elements of the group whose r–adic component is a unit of the ring of integral elements of the r–adic half–plane modulo r, and which satisfy the identity

$$h(\xi_+, \xi_-) = \tau(n) h(n^{-1} \xi_+, n^{-1} \xi_-)$$

when n is a divisor of r such that $n^{-1} \xi_-$ is a unit of the ring of integral elements of the r–adic half–plane modulo r.

The identity

$$\int |f(\xi_+, \xi_-)|^2 d\xi = 2\pi \int |h(\xi_+, \xi_-)|^2 d\xi$$

holds with integration on the left with respect to the canonical measure for the ring of integral elements of the r–adelic skew–plane modulo r over the set of elements of the ring whose r–adic component is a unit of the ring of integral elements of the r–adic skew–plane modulo r and with integration on the right with respect to the canonical measure for the group of integral elements of the r–adelic half–plane modulo r over the set of elements whose r–adic component is a unit of the ring of integral elements of the r–adic half–plane modulo r.

The domain and range of the Radon transformation for the Jacobi space of harmonic ϕ for the r–adelic skew–plane are contained in the Jacobi space for the r–adelic skew–plane. The transformation takes $f(\xi_+, \xi_-)$ into $g(\xi_+, \xi_-)$ when these functions of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the r–adelic skew–plane modulo r satisfy the identity

$$g(\omega\xi_+, \xi_-)/\phi(\omega\xi_+, \xi_-) = \int f(\omega\xi_+ + \omega\eta, \xi_-)/\phi(\omega\xi_+ + \omega\eta, \xi_-)d\eta$$

with integration with respect to the canonical measure for the complementary space to the complex plane in the complex skew–plane for every element $\xi = (\xi_+, \xi_-)$ of the r–adelic skew–plane whose component ξ_+ in the 1–adelic skew–plane belongs to the 1–adelic plane and for every element ω of the 1–adelic skew–plane with conjugate as inverse.

The Laplace transformation for the Jacobi space of harmonic ϕ for the r–adelic skew–plane is a spectral analysis of the adjoint of the Radon transformation for the Jacobi space of harmonic ϕ for the r–adelic skew–plane which is defined by the theta function of harmonic ϕ for the r–adelic skew–plane. The theta function

$$\theta(z, \xi) = \sum \tau(n) n^{\frac{1}{2}\nu_\infty} \exp(\pi i n z) \exp(\tfrac{1}{2}\pi i n(\xi + \xi^-))$$

is a function of (z, ξ) in the r–adelic half–plane defined by summation over the divisors n of r.

The Laplace transform of a function

$$f(\xi_+, \xi_-) = \phi(\xi_+, \xi_-) h(\langle \xi_+, \xi_+ \rangle, \xi_-^- \xi_-)$$

of $\xi = (\xi_+, \xi_-)$ in the ring of integral elements of the r–adelic skew–plane modulo r which belongs to the Jacobi space of harmonic ϕ for the r–adelic skew–plane is the function

$$g(z, \xi) = 2 \int \theta(|\eta_+| z, \eta_-^- \xi) |\eta_+|^{\nu_\infty} h(\eta_+, \eta_-) d\eta$$

of (z, ξ) in the r–adelic half–plane modulo 2 which is defined by integration with respect to the canonical measure for the group of integral elements of the r–adelic half–plane modulo r over the set of elements of the group whose r–adic component is a unit of the ring of integral elements of the r–adic half–plane modulo r.

The Jacobi space for the r–adelic half–plane is a Hilbert space whose elements are the Laplace transforms of elements of the Jacobi space of harmonic ϕ for the r–adelic skew–plane and which is the orthogonal sum of subspaces defined by primitive characters χ modulo p for the r–adic half–plane whose nonzero values are fourth roots of unity.

If χ is a primitive character modulo ρ for the r–adic half–plane, a function $g(z, \xi)$ of (z, ξ) in the r–adelic half–plane is said to be of character χ if the identity

$$g(z, \omega\xi) = \chi(\omega) g(z, \xi)$$

holds for every unit ω of the r–adic half–plane.

A function $\kappa(\chi, \xi)$ of ξ in the r–adic half–plane is defined which vanishes when the p–adic component of $p\xi$ is nonintegral for some prime divisor p of r which is not a divisor of p. The function is otherwise defined by products

$$\kappa(\chi, \xi) \prod (1 - p^2) = \rho^{-1} \chi(\rho\xi) \prod (1 - p^{-2})$$

on the left over the prime divisors p of r which are not divisors of ρ such that the p–adic component of ξ is nonintegral and on the right over all prime divisors p of r which are not divisors of ρ. The function vanishes when $r\xi$ is nonintegral and has equal values at elements of the r–adic half–plane which are congruent modulo 1.

The function is the Fourier transform of a function of ξ in the r–adic half–plane which vanishes when ξ is not a unit and which is otherwise the product of the conjugate character $\chi^-(\xi)$ and a constant of absolute value one. The identity

$$\sum |\kappa(\chi, \xi)|^2 = \prod (1 - p^{-2})$$

holds with summation over the equivalence classes of elements ξ of the r–adic half–plane modulo 1 and with the product taken over the prime divisors p of r.

A function

$$g(z, \xi) = \sum \tau(n) n^{\frac{1}{2}\nu_\infty} h^\wedge(nz) 2\kappa(\chi, \tfrac{1}{2}n\xi)$$

of (z, ξ) in the r–adelic half–plane modulo 2 which is of character χ and belongs to the Jacobi space of harmonic ϕ for the r–adelic half–plane is a sum over the divisors n of r with an analytic function

$$h^\wedge(z) = \pi \int_0^\infty \exp(\pi itz) t^{\nu_\infty} h(t) t dt$$

of z in the upper half–plane which is a Laplace transform of harmonic ϕ for the complex skew–plane.

The set of elements of character χ of the Jacobi space of harmonic ϕ for the r–adelic half–plane is a Hilbert space with the scalar self–product of a function $g(z, \xi)$ of (z, ξ) in the r–adelic half–plane modulo 2 defined as the integral

$$\int_0^\infty \int_{-\infty}^{+\infty} |h^\wedge(x + iy)|^2 y^{\nu_\infty} dx dy \prod (1 - p^{-2})$$

with the product taken over the prime divisors p of r.

The Jacobi space of harmonic ϕ for the r–adelic skew–plane is contained in the domain of the Laplace transformation of harmonic ϕ for the ring of integral elements of the r–adelic skew–plane modulo r. A function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the ring which belongs to the Jacobi space of harmonic ϕ for the r–adelic skew–plane satisfies the identity

$$\int |f(\xi_+, \xi_-)|^2 d\xi = \sum \tau(n)^2 \int |f(\xi_+, \xi_-)|^2 d\xi$$

with summation over the divisors n of r, with integration on the left with respect to the canonical measure for the ring of integral elements of the r–adelic skew–plane modulo r, and with integration on the right with respect to the same measure over the set of elements of the ring whose r–adic component is a unit of the ring of integral elements of the r–adic skew–plane modulo r.

The Laplace transformation for the Jacobi space of harmonic ϕ for the r–adelic skew–plane agrees with the Laplace transformation for the ring of integral elements of the r–adelic skew–plane modulo r. The Jacobi space of harmonic ϕ for the r–adelic half–plane is contained in the range of the Laplace transformation of harmonic ϕ for the ring of integral elements of the r–adelic skew–plane modulo r.

When a function

$$g(z,\xi) = \sum \tau(n) n^{\frac{1}{2}\nu_\infty} h^\wedge(nz) 2\kappa(\chi, \tfrac{1}{2}n\xi)$$

of (z,ξ) in the r–adelic half–plane modulo 2 belongs to the Jacobi space of harmonic ϕ for the r–adelic half–plane, the identity

$$\sum \int_0^\infty \int_{-\infty}^{+\infty} |g(x+iy,\xi)|^2 y^{\nu_\infty} dxdy = \sum \tau(n)^2 \int_0^\infty \int_{-\infty}^{+\infty} |h^\wedge(x+iy)|^2 y^{\nu_\infty} dxdy \prod (1-p^{-2})$$

holds with the product taken over the prime divisors p of r, with summation on the left over the equivalence classes of elements ξ of the r–adic half–plane modulo 2, and with summation on the right over the divisors n of r.

A function

$$g(z,\xi) = \sum \tau(n) n^{\frac{1}{2}\nu_\infty} h_n^\wedge(nz) 2\kappa(\chi, \tfrac{1}{2}n\xi)$$

of (z,ξ) in the r–adelic half–plane modulo 2 which is of character χ and belongs to the range of the Laplace transformation of harmonic ϕ for the ring of integral elements of the r–adelic skew–plane modulo r is a sum over the divisors n of r with for every divisor n of r an analytic function

$$h_n^\wedge(z) = \pi \int_0^\infty \exp(\pi i t z) t^{\nu_\infty} h_n(t) t dt$$

of z in the upper half–plane which is a Laplace transform of harmonic ϕ for the complex skew–plane.

The range of the Laplace transformation of harmonic ϕ for the ring of integral elements of the r–adelic skew–plane modulo r is a Hilbert space in which the scalar self–product of the function $g(z,\xi)$ of (z,ξ) in the r–adelic half–plane modulo 2 is the integral

$$\int_0^\infty \int_{-\infty}^{+\infty} \sum \tau(n)^2 |h_n^\wedge(x+iy)|^2 y^{\nu_\infty} dxdy \prod (1-p^{-2})$$

with summation over the divisors n of r and with the product taken over the prime divisors p of r.

The adjoint of the inclusion of the Jacobi space of harmonic ϕ for the r–adelic half–plane in the range of the Laplace transformation of harmonic ϕ for the ring of integral elements of the r–adelic skew–plane modulo r takes a function

$$\sum \tau(n) n^{\frac{1}{2}\nu_\infty} h_n^\wedge(nz) 2\kappa(\chi, \tfrac{1}{2}n\xi)$$

of (z, ξ) in the r–adelic half–plane modulo 2 into the function

$$\sum \tau(n) n^{\frac{1}{2}\nu_\infty} h^\wedge(nz) 2\kappa(\chi, \tfrac{1}{2}n\xi)$$

of (z, ξ) in the r–adelic half–plane modulo 2 defined by summation

$$h(z)(\sum \tau(n)^2)^{-\frac{1}{2}} = \sum \tau(n) h_n(z),$$

over the divisors n of r.

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