

Differential Equations and Dynamical Systems

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Contents

1	Introduction	4
1.1	Qualitative theory of differential equations and dynamical systems	6
1.2	Topics of this lecture notes	7
2	Topological classification of dynamical systems	11
2.1	Equivalences of dynamical systems	12
2.1.1	Discrete time dynamical systems	13
2.1.2	Continuous time dynamical systems	15
2.2	C^k classification of linear systems	17
2.3	C^0 classification of linear systems	19
2.3.1	Continuous time case in $n = 1$ dimension	19
2.3.2	Discrete time case in $n = 1$ dimension	19
2.3.3	Continuous time case in n dimension	21
2.3.4	Discrete time case in n dimension	28
2.4	Exercises	32
3	Local classification, normal forms and the Hartman-Grobman theorem	35
3.1	Hartman–Grobman theorem	37
3.1.1	STEP 1 of the proof of Theorem 3.5.	39
3.1.2	STEP 2 of the proof of Theorem 3.5.	40
3.1.3	STEP 3 of the proof of Theorem 3.5.	41
3.1.4	STEP 4 of the proof of Theorem 3.5.	42
3.2	Normal forms	46
3.3	Exercises	52
4	Stable, unstable and center manifold theorems	53
4.1	Stable and unstable manifold theorem	53
4.1.1	General approach	54
4.1.2	Global stable and unstable manifolds	59
4.2	Center manifold theorem	60
4.2.1	General approach	61

4.2.2	Approximation of the center manifold	62
4.3	Exercises	64
5	Global phase portrait, periodic orbits, index of a vector field	70
5.1	Investigating the global phase portrait by using the local ones	70
5.1.1	Global phase portraits of one dimensional systems	71
5.1.2	Global phase portraits of two dimensional systems	74
5.2	Periodic orbits	88
5.2.1	Existence of periodic orbits	88
5.2.2	Local phase portrait in a neighbourhood of a periodic orbit	92
5.3	Index theory for two dimensional systems	98
5.4	Behaviour of trajectories at infinity	103
5.5	Exercises	107
6	Introduction to bifurcation theory and structural stability	109
6.1	Normal forms of elementary bifurcations	110
6.2	Necessary conditions of bifurcations	122
6.3	Structural stability	125
6.3.1	Structural stability of one dimensional systems	127
6.3.2	Structural stability of higher dimensional systems	130
7	One co-dimensional bifurcations, the saddle-node and the Andronov-Hopf bifurcations	133
7.1	Saddle-node bifurcation	133
7.2	Andronov–Hopf bifurcation	136
7.2.1	Construction of the Lyapunov function	138
7.2.2	Andronov–Hopf bifurcation for linear parameter dependence	142
7.2.3	Andronov–Hopf bifurcation with general parameter dependence and with linear part in Jordan canonical form	144
7.2.4	Andronov–Hopf bifurcation for arbitrary parameter dependence	147
7.2.5	Case study for finding the Andronov–Hopf bifurcation	148
7.3	Computing bifurcation curves in two-parameter systems by using the parametric representation method	151
7.3.1	The parametric representation method	151
7.3.2	Bifurcation curves in two-parameter systems	156
8	Discrete dynamical systems, symbolic dynamics, chaotic behaviour	161
8.1	Discrete dynamical systems	161
8.1.1	Logistic map	163
8.1.2	Chaotic orbits	165
8.1.3	Symbolic dynamics	168

9	Reaction-diffusion equations	171
9.1	Stationary solutions of reaction-diffusion equations	172
9.2	Travelling wave solutions of reaction-diffusion equations	175
9.2.1	Existence of travelling waves	175
9.2.2	Stability of travelling waves	176
	Bibliography	187

Chapter 1

Introduction

Preface

These lecture notes are based on the series of lectures that were given by the author at the Eötvös Loránd University for master students in Mathematics and Applied Mathematics about the qualitative theory of differential equations and dynamical systems. The prerequisite for this was an introductory differential equation course. Hence, these lecture notes are mainly at masters level, however, we hope that the material will be useful for mathematics students of other universities and for non-mathematics students applying the theory of dynamical systems, as well.

The prerequisite for using this lecture notes is a basic course on differential equations including the methods of solving simple first order equations, the theory of linear systems and the methods of determining two-dimensional phase portraits. While the basic existence and uniqueness theory is part of the introductory differential equation course at the Eötvös Loránd University, not, these ideas are not presented in these lecture notes. However, the reader can follow the lecture notes without the detailed knowledge of existence and uniqueness theory. The most important definitions and theorems studied in the introductory differential equation course are briefly summarized in the Introduction below. For the interested reader we suggest to consult a textbook on differential equations, for example Perko's book [\[19\]](#).

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1.1 Qualitative theory of differential equations and dynamical systems

The theory of differential equations is a field of mathematics that is more than 300 years old, motivated greatly by challenges arising from different applications, and leading to the birth of other fields of mathematics. We do not aim to show a panoramic view of this enormous field, we only intend to reveal its relation to the theory of dynamical systems. According to the author's opinion the mathematical results related to the investigation of differential equation can be grouped as follows.

- Deriving methods for solving differential equations analytically and numerically.
- Prove the existence and uniqueness of solutions of differential equations.
- Characterize the properties of solutions without deriving explicit formulas for them.

There is obviously a significant demand, coming from applications, for results in the first direction. It is worth to note that in the last 50 years the emphasis is on the numerical approximation of solutions. This demand created and motivated the birth of a new discipline, namely the numerical methods of differential equations. The question of existence and uniqueness of initial value problems for ordinary differential equations was answered completely in the first half of the twentieth century (motivating the development of fixed point theorems in normed spaces). Hence today's research in the direction of existence and uniqueness is aimed at boundary value problems for non-linear ordinary differential equations (the exact number of positive solutions is an actively studied field) and initial-boundary value problems for partial differential equations, where the question is answered only under restrictive conditions on the type of the equation. The studies in the third direction go back to the end of the nineteenth century, when a considerable demand to investigate non-linear differential equations appeared, and it turned out that these kind of equations cannot be solved analytically in most of the cases. The start of the qualitative theory of differential equations was first motivated by Poincaré, who aimed to prove some qualitative properties of solutions without deriving explicit formulas analytically. The change in the attitude of studying differential equations can be easily interpreted by the simple example $\dot{x} = x$, $\dot{y} = -y$. This system can be obviously solved analytically, the traditional approach is to determine the solutions as $x(t) = e^t x_0$, $y(t) = e^{-t} y_0$. The new approach, called qualitative theory of differential equations, gives a different answer to this question. Instead of solving the differential equations it provides the phase portrait as shown in Figure 1.1. This phase portrait does not show the time dependence of the solutions, but several important properties of the solutions can be obtained from it. Moreover, the phase portrait can be determined also for non-linear systems, for which the analytic solution is not available. Thus a system of ordinary differential equations is considered as a dynamical system, the orbits of which are to be

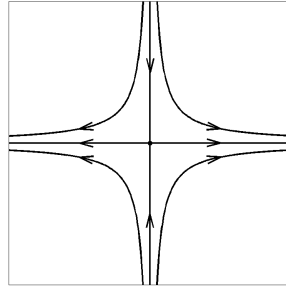


Figure 1.1: The phase portrait of the system $\dot{x} = x$, $\dot{y} = -y$, the so-called saddle point.

characterized, mainly from geometric or topological point of view. Based on this idea, the qualitative theory of differential equations and the theory of dynamical systems became closely related. This is shown also by the fact that in the title of modern monographs the expression "differential equation" is often accompanied by the expression "dynamical system". A significant contribution to the development of qualitative theory was the invention of chaotic systems and the opportunity of producing phase portraits by numerical approximation using computers. The use of the main tools of qualitative theory has become a routine. Not only the students in physics, chemistry and biology, but also students in economics can use the basic tools of dynamical system theory. Because of the wide interest in applying dynamical systems theory several monographs were published in the last three decades. These were written not only for mathematicians but also for researchers and students in different branches of science. Among several books we mention the classical monograph by Guckenheimer and Holmes [11], an introduction to chaos theory [1], the book by Hale and Koçak [12], by Hubbard and West [16], and by Perko [19]. Seydel's book [22] introduces the reader to bifurcation theory. The emphasis in the following books is more on proving the most important and relevant theorems. Hence for mathematics student the monographs by Chow and Hale [8], by Chicone [7], by Hirsch, Smale and Devaney [15], by Robinson [20] and by Wiggins [27] can be recommended.

1.2 Topics of this lecture notes

In this section we introduce the mathematical framework, in which we will work, and list briefly the topics that will be dealt with in detail. The object of our study is the system of autonomous ordinary differential equations (ODEs)

$$\dot{x}(t) = f(x(t))$$

where $x : \mathbb{R} \rightarrow \mathbb{R}^n$ is the unknown function and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a given continuously differentiable function that will be referred to as right hand side (r.h.s). Most of the ordinary differential equations can be written in this form and it is impossible to list all physical, chemical, biological, economical and engineering applications where this kind of system of ODEs appear.

The solution of the differential equation satisfying the initial condition $x(0) = p$ is denoted by $\varphi(t, p)$. It can be proved that φ is a continuously differentiable function satisfying the assumptions in the definition of a (continuous time) dynamical system below.

Definition 1.1.. *A continuously differentiable function $\varphi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called a (continuous time) dynamical system if it satisfies the following properties.*

- For all $p \in \mathbb{R}^n$ the equation $\varphi(0, p) = p$ holds,
- For all $p \in \mathbb{R}^n$ and $t, s \in \mathbb{R}$ the relation $\varphi(t, \varphi(s, p)) = \varphi(t + s, p)$ holds.

A dynamical system can be considered as a model of a deterministic process, \mathbb{R}^n is the state space, an element $p \in \mathbb{R}^n$ is a state of the system, and $\varphi(t, p)$ is the state, to which the system arrives after time t starting from the state p . As it was stated above, the solution of the above system of autonomous ordinary differential equations determines a dynamical system (after rescaling of time, if it is necessary). On the other hand, given a dynamical system one can define a system of autonomous ordinary differential equations, the solution of which is the given dynamical system. Hence autonomous ordinary differential equations and (continuous time) dynamical systems can be considered to be equivalent notions. We will use them concurrently in this lecture notes.

The above definition of the dynamical system can be extended in several directions. One important alternative is when time is considered to be discrete, that is \mathbb{Z} is used instead of \mathbb{R} . This way we get the notion of (discrete time) dynamical systems.

Definition 1.2.. *A continuous function $\varphi : \mathbb{Z} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called a discrete time dynamical system if it satisfies the following properties.*

- For all $p \in \mathbb{R}^n$ the equation $\varphi(0, p) = p$ holds,
- For all $p \in \mathbb{R}^n$ and $k, m \in \mathbb{Z}$ the relation $\varphi(k, \varphi(m, p)) = \varphi(k + m, p)$ holds.

As the notion of (continuous time) dynamical systems was derived from autonomous ordinary differential equations, the notion of discrete time dynamical systems can be derived from difference equations. A difference equation can be defined by a map. Namely, let $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous function and consider the difference equation (or recursion)

$$x_{k+1} = g(x_k)$$

with initial condition $x_0 = p$. The function $\varphi(k, p) = x_k$ satisfies the properties in the definition of the discrete time dynamical system. Thus a difference equation determines a discrete time dynamical system. On the other hand, if φ is a discrete time dynamical system, then introducing $g(p) = \varphi(1, p)$ one can easily check that $x_k = \varphi(k, p)$ is the solution of the recursion $x_{k+1} = g(x_k)$. This is expressed with the formula $\varphi(k, p) = g^k(p)$, where g^k denotes the composition $g \circ g \circ \dots \circ g$ k times. (If k is negative then the inverse function of g is applied.)

Continuous and discrete time dynamical systems often can be dealt with concurrently. Then the set of time points is denoted by \mathbb{T} as a common notation for \mathbb{R} and \mathbb{Z} . In the case of continuous time the notion "flow", while in the case of discrete time the notion "map" is used. The main goal in investigating dynamical systems is the geometric characterization of the orbits that are defined as follows. The orbit of a point p is the set

$$\{\varphi(t, p) : t \in \mathbb{T}\}$$

that is a curve in the continuous case and a sequence of points in the discrete case.

After defining the main concepts used in the lecture notes let us turn to the overview of the topics that will be covered.

In the next section we show how the differences and similarities of phase portraits can be exactly characterized from the geometrical and topological point of view. In order to do so we introduce the notion of topological equivalence, that is an equivalence relation in the set of dynamical systems. After that, we investigate the classes determined by the topological equivalence and try to find a simple representant from each class. The main question is how to decide whether two systems are equivalent if they are given by differential equations and the dynamical system φ is not known. The full classification can be given only for linear systems.

Nonlinear systems are classified in Section 3. However, in this case only the local classification in the neighbourhood of equilibrium points can be carried out. The main tool of local investigation is linearization, the possibility of which is enabled by the Hartman–Grobman theorem. If the linear part of the right hand side does not determine the phase portrait, then those nonlinear terms that play crucial role in determining the equivalence class the system belongs to can be obtained by using the normal form theory.

The stable, unstable and center manifold theorems help in the course of investigating the local phase portrait around steady states. These theorems are dealt with in Section 4. The manifolds can be considered as the generalizations of stable, unstable and center subspaces introduced for linear systems. These manifolds are invariant (similarly to the invariant subspaces), that is trajectories cannot leave them. The stable manifold contains those trajectories that converge to the steady state as $t \rightarrow +\infty$, while the unstable manifold contains those points, from which trajectories converge to the steady state as $t \rightarrow -\infty$. The center manifold enables us to reduce the dimension of the complicated part of the phase space. If the phase portrait can be determined in this lower dimensional manifold, then it can also be characterized in the whole phase space.

Tools for investigating the global phase portrait are dealt with in Section 5. First, an overview of methods for determining two dimensional phase portraits is presented. Then periodic solutions are studied in detail. Theorems about existence and non-existence of periodic solutions in two dimensional systems are proved first. Then the stability of periodic solutions is studied in arbitrary dimension. In the end of that section we return to two dimensional systems and two strong tools of global investigation are shown. Namely, the index of a vector field and compactification by projecting the system to the Poincaré sphere.

In the next sections we study dynamical systems depending on parameters, especially the dependence of the phase portraits on the value of the parameters. Methods are shown that can help to characterize those systems, in which a sudden qualitative change in the phase portrait appears at certain values of the parameters. These qualitative changes are called bifurcations. We deal with the two most important one co-dimensional bifurcations, the fold bifurcation and the Andronov–Hopf bifurcation, in detail.

An important chapter of dynamical systems theory is chaos. Our goal is to define what chaos means and investigate simple chaotic systems. The tools are developed mainly for discrete time dynamical systems, hence these are investigated separately in section 8. Methods for investigating fixed points, periodic orbits and their stability are shown. We introduce one of the several chaos definitions and prove that some one dimensional maps have chaotic orbits. As a tool we introduce symbolic dynamics and show how can it be applied to prove the appearance of chaos.

The last section presents an extension of dynamical systems theory to systems with infinite dimensional phase space. This can typically happen in the case of partial differential equations. In that section semilinear parabolic partial differential equations are studied that are often called reaction-diffusion equations. The existence and stability of stationary solutions (that correspond to equilibrium points) is studied first. Then, another type of orbit, the travelling wave is studied that is important from the application point of view. For these solutions again the question of existence and stability is dealt with in detail.

Chapter 2

Topological classification of dynamical systems

The evolution of the theory of differential equations started by developing methods for solving differential equations. Using these methods the solution of different special differential equations can be given analytically, i.e. formulas can be derived. However, it turned out that the solution of systems of ordinary differential equations can be obtained analytically only in very special cases (e.g. for linear systems), and even in the case when formulas can be derived for the solutions, it is difficult to determine the properties of the solutions based on the formulas. For example, two dimensional linear systems can be characterized by plotting trajectories in the phase plane instead of deriving formulas for the solutions. It does not mean that the trajectories are plotted analytically, instead the main characteristics of the trajectories are shown, similarly to the case of plotting graphs of functions in calculus, when only the monotonicity, local maxima, minima and the convexity are taken into account, the exact value of the function does not play important role. Thus the construction of the phase portrait means that we plot the orbits of a system that is equivalent to the original system in a suitable sense, that somehow expresses that the phase portraits of the two systems "look like similar". In this section our aim is to define the equivalence of dynamical systems making the notion of "look like similar" rigorous.

We will define an equivalence relation for dynamical systems $\varphi : \mathbb{R} \times M \rightarrow M$. Then the goal is to determine the classes given by this equivalence relation, to find a representant from each class, the phase portrait of which can be easily determined, and finally, to derive a simple method that helps to decide if two systems are equivalent or not.

2.1 Equivalences of dynamical systems

Two dynamical systems will be called equivalent if their orbits can be mapped onto each other by a suitable mapping. First we define the classes of mappings that will be used. The equivalence relations for discrete and continuous time dynamical systems will be defined concurrently, hence we use the notation \mathbb{T} for \mathbb{R} or for \mathbb{Z} .

Definition 2.1.. *Let $M, N \subset \mathbb{R}^n$ be open sets. A function $h : M \rightarrow N$ is called a homeomorphism (sometimes a C^0 -diffeomorphism), if it is continuous, bijective and its inverse is also continuous. The function is called a C^k -diffeomorphism, if it is k times continuously differentiable, bijective and its inverse is also k times continuously differentiable.*

Definition 2.2.. *Let $M, N \subset \mathbb{R}^n$ be open connected sets. The dynamical systems $\varphi : \mathbb{T} \times M \rightarrow M$ and $\psi : \mathbb{T} \times N \rightarrow N$ are called C^k equivalent, (for $k = 0$ topologically equivalent), if there exists a C^k -diffeomorphism $h : M \rightarrow N$ (for $k = 0$ a homeomorphism), that maps orbits onto each other by preserving the direction of time. This is shown in Figure 2.1. In more detail, this means that there exists a continuous function $a : \mathbb{T} \times M \rightarrow \mathbb{T}$, for which $t \mapsto a(t, p)$ is a strictly increasing bijection and for all $t \in \mathbb{T}$ and $p \in M$ we have*

$$h(\varphi(t, p)) = \psi(a(t, p), h(p)).$$

One can get different notions of equivalence by choosing functions a and h having special properties. The above general equivalence will be referred to as equivalence of type 1. Now we define important special cases that will be called equivalences of type 2, 3 and 4.

Definition 2.3.. *The dynamical systems φ and ψ are called C^k flow equivalent (equivalence of type 2), if in the general definition above, the function a does not depend on p , that is there exists a strictly increasing bijection $b : \mathbb{T} \rightarrow \mathbb{T}$, for which $a(t, p) = b(t)$ for all $p \in M$. Thus the reparametrization of time is the same on each orbit.*

Definition 2.4.. *The dynamical systems φ and ψ are called C^k conjugate (equivalence of type 3), if in the general definition above $a(t, p) = t$ for all $t \in \mathbb{T}$ and $p \in M$. Thus there is no reparametrization of time along the orbits. In this case the condition of equivalence takes the form*

$$h(\varphi(t, p)) = \psi(t, h(p)).$$

Definition 2.5.. *The dynamical systems φ and ψ are called C^k orbitally equivalent (equivalence of type 4), if in the general definition above $M = N$ and $h = \text{id}$, that is the orbits are the same in the two systems and time is reparametrized.*

The definitions above obviously imply that the different types of equivalences are related in the following way.

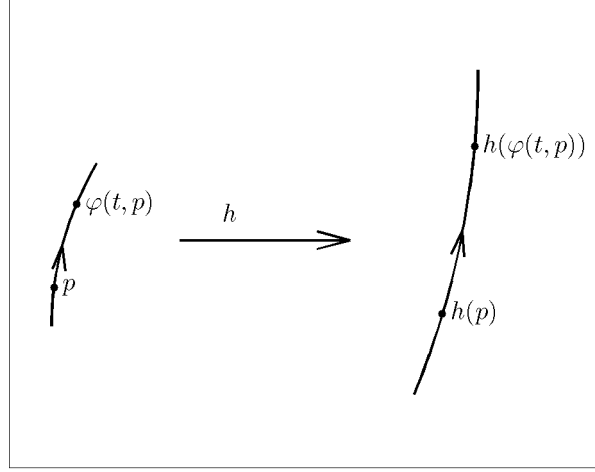


Figure 2.1: The orbits of topologically equivalent systems can be taken onto each other by a homeomorphism.

Proposition 2.1. 1. If the dynamical systems φ and ψ are C^k conjugate, then they are C^k flow equivalent.

2. If the dynamical systems φ and ψ are C^k flow equivalent, then they are C^k equivalent.

3. If the dynamical systems φ and ψ are orbitally equivalent, then they are C^k equivalent.

Summarizing, the equivalences follows from each other as follows

$$3 \Rightarrow 2 \Rightarrow 1, \quad 4 \Rightarrow 1.$$

2.1.1 Discrete time dynamical systems

First we show that for discrete time dynamical systems there is only one notion of equivalence, as it is formulated in the proposition below.

Let $\varphi : \mathbb{Z} \times M \rightarrow M$ and $\psi : \mathbb{Z} \times N \rightarrow N$ be discrete time dynamical systems. Let us define function f and g by $f(p) = \varphi(1, p)$ and $g(p) = \psi(1, p)$. Then the definition of a dynamical system simply implies that $\varphi(n, p) = f^n(p)$ and $\psi(n, p) = g^n(p)$, where f^n and g^n denote the composition of the functions with themselves n times, $f^n = f \circ f \circ \dots \circ f$ and similarly for g .

Proposition 2.2. *The statements below are equivalent.*

1. *The dynamical systems φ and ψ are C^k conjugate.*
2. *The dynamical systems φ and ψ are C^k flow equivalent.*
3. *The dynamical systems φ and ψ are C^k equivalent.*
4. *There exists a C^k -diffeomorphism $h : M \rightarrow N$, for which $h \circ f = g \circ h$.*

Proof. According to the previous proposition the first three statements follow from each other from top to the bottom. First we prove that the last statement implies the first. Then it will be shown that the third statement implies the last.

Using that $h \circ f = g \circ h$ one obtains

$$h \circ f^2 = h \circ f \circ f \underset{h \circ f = g \circ h}{=} g \circ h \circ f \underset{h \circ f = g \circ h}{=} g \circ g \circ h = g^2 \circ h.$$

Similarly, the condition $h \circ f^{n-1} = g^{n-1} \circ h$ implies $h(f^n(p)) = g^n(h(p))$, that is $h(\varphi(n, p)) = \psi(n, h(p))$ holds for all n and p that is exactly the C^k conjugacy of the dynamical systems φ and ψ .

Let us assume now that the dynamical systems φ and ψ are C^k equivalent. Let us observe first that if $r : \mathbb{Z} \rightarrow \mathbb{Z}$ is a strictly increasing bijection, then there exists $k \in \mathbb{Z}$, such that $r(n) = n + k$ for all $n \in \mathbb{Z}$. Namely, strict monotonicity implies $r(n+1) > r(n)$, while since r is a bijection there is no integer between $r(n+1)$ and $r(n)$, thus $r(n+1) = r(n) + 1$. Then introducing $k = r(0)$ we get by induction that $r(n) = n + k$. Thus for the function a in the definition of C^k equivalence holds that for all $p \in M$ there exists an integer $k_p \in \mathbb{Z}$, for which $a(n, p) = n + k_p$. Thus the C^k equivalence of φ and ψ means that for all $n \in \mathbb{Z}$ and for all $p \in M$

$$h(\varphi(n, p)) = \psi(n + k_p, h(p))$$

holds, that is

$$h(f^n(p)) = g^{n+k_p}(h(p)).$$

Applying this relation for $n = 0$ we get $h(p) = g^{k_p}(h(p))$. Then applying with $n = 1$ equation

$$h(f(p)) = g^{1+k_p}(h(p)) = g(g^{k_p}(h(p))) = g(h(p))$$

follows, that yields the desired statement. \square

Proposition 2.3. *The dynamical systems φ and ψ are orbitally equivalent if and only if they are equal.*

Proof. If the two dynamical systems are equal, then they are obviously orbitally equivalent. In the opposite way, if they are orbitally equivalent, then they are C^k equivalent, hence according to the previous proposition $h \circ f = g \circ h$. On the other hand, $h = id$ implying $f = g$, thus $\varphi(n, p) = \psi(n, p)$ for all $n \in \mathbb{Z}$, which means that the two dynamical systems are equal. \square

Definition 2.6.. *In the case of discrete time dynamical systems, i.e. when $\mathbb{T} = \mathbb{Z}$, the functions f and g and the corresponding dynamical systems are called C^k conjugate if there exists a C^k -diffeomorphism $h : M \rightarrow N$, for which $h \circ f = g \circ h$.*

Remark 2.1. *In this case the functions f and g can be transformed to each other by a coordinate transformation.*

Proposition 2.4. *In the case $k > 1$, if f and g are C^k conjugate and $p \in M$ is a fixed point of the map f (in this case $h(p)$ is obviously a fixed point of g), then the matrices $f'(p)$ and $g'(h(p))$ are similar.*

Proof. Differentiating the equation $h \circ f = g \circ h$ at the point p and using that $f(p) = p$ and $g(h(p)) = h(p)$ we get $h'(p)f'(p) = g'(h(p))h'(p)$. This can be multiplied by the inverse of the matrix $h'(p)$ (the existence of which follows from the fact that h is a C^k -diffeomorphism) yielding that the matrices $f'(p)$ and $g'(h(p))$ are similar. \square

Remark 2.2. *According to the above proposition C^k conjugacy (for $k \geq 1$) yields finer classification than we need. Namely, the maps $f(x) = 2x$ and $g(x) = 3x$ are not C^k conjugate (since the eigenvalues of their derivatives are different), while the phase portraits of the corresponding dynamical systems $x_{n+1} = 2x_n$ and $x_{n+1} = 3x_n$ are considered to be the same (all trajectories tend to infinity). We will see that they are C^0 conjugate, hence the above proposition is not true for $k = 0$.*

2.1.2 Continuous time dynamical systems

Let us turn now to the study of continuous time dynamical systems, i.e. let $\mathbb{T} = \mathbb{R}$ and let $\varphi : \mathbb{R} \times M \rightarrow M$ and $\psi : \mathbb{R} \times N \rightarrow N$ be continuous time dynamical systems. Then there are continuously differentiable functions $f : M \rightarrow \mathbb{R}^n$ and $g : N \rightarrow \mathbb{R}^n$, such that the solutions of $\dot{x} = f(x)$ are given by φ and those of $\dot{y} = g(y)$ are given by ψ .

Proposition 2.5. *1. Let $k \geq 1$. Then the dynamical systems φ and ψ are C^k conjugate if and only if there exists a C^k diffeomorphism $h : M \rightarrow N$ C^k , for which $h' \cdot f = g \circ h$ holds.*

2. Assume that the function $t \mapsto a(t, p)$ is differentiable. Then the dynamical systems φ and ψ are orbitally equivalent if and only if there exists a continuous function $v : M \rightarrow \mathbb{R}^+$, for which $g = f \cdot v$.

3. In Proposition 2.1 the converse of the implications are not true.

Proof. 1. First, let us assume that the dynamical systems φ and ψ are C^k conjugate. Then there exists a C^k diffeomorphism $h : M \rightarrow N$, for which $h(\varphi(t, p)) = \psi(t, h(p))$. Differentiating this equation with respect to t we get $h'(\varphi(t, p)) \cdot \dot{\varphi}(t, p) = \dot{\psi}(t, h(p))$. Since φ is the solution of $\dot{x} = f(x)$ and ψ is the solution of $\dot{y} = g(y)$, we have

$$h'(\varphi(t, p)) \cdot f(\varphi(t, p)) = g(\psi(t, h(p))).$$

Applying this for $t = 0$ yields

$$h'(\varphi(0, p)) \cdot f(\varphi(0, p)) = g(\psi(0, h(p))),$$

that is $h'(p) \cdot f(p) = g(h(p))$. Thus the first implication is proved. Let us assume now that there exists a C^k -diffeomorphism $h : M \rightarrow N$, for which $h' \cdot f = g \circ h$ holds. Let us define $\psi^*(t, q) := h(\varphi(t, h^{-1}(q)))$ and then prove that this function is the solution of the differential equation $\dot{y} = g(y)$. Then by the uniqueness of the solution $\psi^* = \psi$ is implied, which yields the desired statement since substituting $q = h(p)$ into the definition of ψ^* we get that φ and ψ are C^k conjugate. On one hand, $\psi^*(0, q) := h(\varphi(0, h^{-1}(q))) = q$ holds, on the other hand,

$$\begin{aligned} \dot{\psi}^*(t, q) &= h'(\varphi(t, h^{-1}(q))) \cdot \dot{\varphi}(t, h^{-1}(q)) \\ &= h'(h^{-1}(\psi^*(t, q))) \cdot f(h^{-1}(\psi^*(t, q))) = g(\psi^*(t, q)), \end{aligned}$$

which proves the statement.

2. First, let us assume that the dynamical systems φ and ψ are orbitally equivalent. Then $\varphi(t, p) = \psi(a(t, p), p)$, the derivative of which with respect to t is $\dot{\varphi}(t, p) = \dot{\psi}(a(t, p), p) \cdot \dot{a}(t, p)$. Since φ is the solution of $\dot{x} = f(x)$ and ψ is the solution of $\dot{y} = g(y)$, we have $f(\varphi(t, p)) = g(\psi(a(t, p), p))$. Applying this for $t = 0$ yields $f(p) = g(p) \cdot \dot{a}(0, p)$, which proves the statement by introducing the function $v(p) = \dot{a}(0, p)$. Assume now that there exists a function $v : M \rightarrow \mathbb{R}^+$, for which $g = f \cdot v$. Let $p \in \mathbb{R}^n$ and let $x(t) = \varphi(t, p)$. Introducing

$$b(t) = \int_0^t \frac{1}{v(x(s))} ds,$$

we have $\dot{b}(t) = 1/v(x(t)) > 0$, hence the function b is invertible, the inverse is denoted by $a = b^{-1}$. (This function depends also on p , hence it is reasonable to use the notation $a(t, p) = b^{-1}(t)$.) Let $y(t) = x(a(t, p))$, then

$$\dot{y}(t) = \dot{x}(a(t, p)) \dot{a}(t, p) = f(x(a(t, p))) \frac{1}{\dot{b}(a(t, p))} = f(y(t))v(y(t)) = g(y(t)).$$

Hence y is the solution of the differential equation $\dot{y}(t) = g(y(t))$ and satisfies the initial condition $y(0) = p$, therefore $y(t) = \psi(t, p)$. Thus using the definition $y(t) = x(a(t, p))$ we get the relation $\psi(t, p) = \varphi(a(t, p), p)$ that was to be proved.

3. In order to prove this statement we show counterexamples.

- (i) Let us introduce the matrices $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$. Then the phase portraits of the differential equations $\dot{x} = Ax$ and $\dot{y} = By$ are the same, since both of them are centers, however, the periods of the solutions in these two systems are different. Hence in order to map the orbits of the two systems onto each other time reparametrisation is needed. This means that the two systems are C^k flow-equivalent, however, they are not C^k conjugate.
- (ii) Assume that both φ and ψ has a pair of periodic orbits and the ratio of the periods is different in the two systems. Then they are not C^k flow-equivalent, but they can be C^k equivalent.
- (iii) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$. Then both $\dot{x} = Ax$ and $\dot{y} = By$ determines a saddle point, that is they are C^0 equivalent, however their orbits are not identical, hence they are not orbitally equivalent. \square

2.2 C^k classification of linear systems

In this section we classify continuous time linear systems of the form $\dot{x} = Ax$ and discrete time linear systems of the form $x_{n+1} = Ax_n$ according to equivalence relations introduced in the previous section. Let us introduce the spaces

$$L(\mathbb{R}^n) = \{A : \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ linear mapping}\}$$

and

$$GL(\mathbb{R}^n) = \{A \in L(\mathbb{R}^n) : \det A \neq 0\}$$

for the continuous and for the discrete time cases. If $A \in L(\mathbb{R}^n)$, then the matrix A is considered to be the right hand side of the linear differential equation $\dot{x} = Ax$, while $A \in GL(\mathbb{R}^n)$ is considered to be a linear map determining the discrete time system $x_{n+1} = Ax_n$. Thus the space $L(\mathbb{R}^n)$ represents continuous time and $GL(\mathbb{R}^n)$ represents discrete time linear systems. In the linear case the dynamical system can be explicitly given in terms of the matrix. If $A \in L(\mathbb{R}^n)$, then the dynamical system determined by A (that is the solution of the differential equation $\dot{x} = Ax$) is $\varphi(t, p) = e^{At}p$. If $A \in GL(\mathbb{R}^n)$, then the dynamical system generated by A (that is the solution of the recursion $x_{n+1} = Ax_n$) is $\psi(n, p) = A^n p$. In the following the equivalence of the matrices will be meant as the equivalence of the corresponding dynamical systems. Moreover, we will use the notion below.

Definition 2.7.. *The matrices A and B are called linearly equivalent, if there exist $\alpha > 0$ and an invertible matrix P , for which $A = \alpha PBP^{-1}$ holds.*

Proposition 2.6. *Let $\mathbb{T} = \mathbb{R}$ and $k \geq 1$.*

1. *The matrices $A, B \in L(\mathbb{R}^n)$ are C^k conjugate, if and only if they are similar.*
2. *The matrices $A, B \in L(\mathbb{R}^n)$ are C^k equivalent, if and only if they are linearly equivalent.*

Proof. 1. Assume that the matrices A and B are C^k conjugate, that is there exists a C^k -diffeomorphism $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$, such that $h(\varphi(t, p)) = \psi(t, h(p))$, i.e. $h(e^{At}p) = e^{Bt}h(p)$. Differentiating this equation with respect to p we get $h'(e^{At}p) \cdot e^{At} = e^{Bt}h'(p)$, then substituting $p = 0$ yields $h'(0)e^{At} = e^{Bt} \cdot h'(0)$. Differentiating now with respect to t leads to $h'(0)e^{At} \cdot A = e^{Bt} \cdot B \cdot h'(0)$, from which by substituting $t = 0$ we get $h'(0)A = B \cdot h'(0)$. The matrix $h'(0)$ is invertible, because h is a diffeomorphism, hence multiplying the equation by this inverse we arrive to $A = h'(0)^{-1}Bh'(0)$, i.e. the matrices A and B are similar. Assume now, that the matrices A and B are similar, that is there exists an invertible matrix P , for which $A = P^{-1}BP$. Then the linear function $h(p) = Pp$ is a C^k -diffeomorphism taking the orbits onto each other by preserving time, namely $Pe^{At}p = Pe^{P^{-1}BPt}p = e^{Bt}Pp$.

2. Assume that the matrices A and B are C^k equivalent, that is there exists a C^k -diffeomorphism $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and a differentiable function $a : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$, such that $h(\varphi(t, p)) = \psi(a(t, p), h(p))$, that is $h(e^{At}p) = e^{Ba(t, p)}h(p)$. Differentiating this equation with respect to p and substituting $p = 0$ yields $h'(0)e^{At} = e^{Ba(t, 0)} \cdot h'(0)$. Differentiating now with respect to t we get $h'(0)e^{At} \cdot A = e^{Ba(t, 0)} \cdot B\dot{a}(t, 0) \cdot h'(0)$, from which by substituting $t = 0$ we obtain $h'(0)A = B\dot{a}(0, 0) \cdot h'(0)$. The matrix $h'(0)$ is invertible, because h is a diffeomorphism, hence multiplying the equation by this inverse and introducing $\alpha = \dot{a}(0, 0)$ we arrive to $A = \alpha h'(0)^{-1}Bh'(0)$, i.e. the matrices A and B are linearly equivalent. Assume now, that the matrices A and B are linearly equivalent, that is there exists an invertible matrix P and $\alpha > 0$, for which $A = \alpha P^{-1}BP$. Then the linear function $h(p) = Pp$ is a C^k -diffeomorphism taking the orbits onto each other with time reparametrisation $a(t, p) = \alpha t$, namely $Pe^{At}p = Pe^{\alpha P^{-1}BPt}p = e^{B\alpha t}Pp$. \square

Remark 2.3. *According to the above proposition the classification given by C^k conjugacy and equivalence is too fine when $k \geq 1$. Namely, the matrices $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$ are neither C^k conjugate nor C^k equivalent, however, they both determine a stable node, hence we do not want them to be in different classes since the behaviour of the trajectories is the same in the two systems. We will see that this cannot happen to matrices that are C^0 conjugate, that is the above proposition does not hold for $k = 0$.*

Proposition 2.7. *Let $\mathbb{T} = \mathbb{Z}$ and $k \geq 1$. The matrices $A, B \in GL(\mathbb{R}^n)$ are C^k conjugate, if and only if they are similar.*

Proof. Assume that the matrices A and B are C^k conjugate, that is there exists a C^k -diffeomorphism $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$, for which $h(Ap) = Bh(p)$. Differentiating this equation with respect to p yields $h'(Ap)A = Bh'(p)$, then substituting $p = 0$ we get $h'(0)A = Bh'(0)$. The matrix $h'(0)$ is invertible, because h is a diffeomorphism, hence multiplying the equation by this inverse $A = h'(0)^{-1}Bh'(0)$, that is the matrices A and B are similar. Assume now, that the matrices A and B are similar, that is there exists an invertible matrix P , for which $A = P^{-1}BP$. Then the linear function $h(p) = Pp$ is a C^k -diffeomorphism taking the orbits onto each other, namely $PAp = BPp$. \square

2.3 C^0 classification of linear systems

In this section the following questions will be studied.

1. How can it be decided if two matrices $A, B \in L(\mathbb{R}^n)$ are C^0 equivalent or C^0 conjugate?
2. How can it be decided if two matrices $A, B \in GL(\mathbb{R}^n)$ are C^0 conjugate?

First, we answer these questions in one dimension.

2.3.1 Continuous time case in $n = 1$ dimension

Let us consider the differential equation $\dot{x} = ax$. If $a < 0$, then the origin is asymptotically stable, i.e. all solutions tend to the origin as $t \rightarrow \infty$. If $a > 0$, then the origin is unstable, i.e. all solutions tend to infinity as $t \rightarrow \infty$. If $a = 0$, then every point is a steady state. These phase portraits are shown in Figure 2.2 for positive, zero and negative values of a . Thus the linear equations $\dot{x} = ax$ and $\dot{y} = by$, in which $a, b \in \mathbb{R}$, are C^0 equivalent if and only if $\text{sgn } a = \text{sgn } b$. (The homeomorphism can be taken as the identity in this case.)

2.3.2 Discrete time case in $n = 1$ dimension

Let us consider the discrete time dynamical system given by the recursion $x_{n+1} = ax_n$ for different values of $a \in \mathbb{R} \setminus \{0\}$. We note that the set $GL(\mathbb{R})$ can be identified with the set $\mathbb{R} \setminus \{0\}$. Since this recursion defines a geometric sequence, the behaviour of the trajectories can easily be determined. C^0 equivalence divides $GL(\mathbb{R})$ into the following six classes.

1. If $a > 1$, then for a positive initial condition x_0 the sequence is strictly increasing, hence 0 is an unstable fixed point.
2. If $a = 1$, then every point is a fixed point.

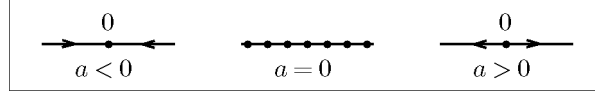


Figure 2.2: Three classes of continuous time linear equations in one dimension.

3. If $0 < a < 1$, then 0 is a stable fixed point, every solution converges to zero monotonically.
4. If $-1 < a < 0$, then 0 is a stable fixed point, every solution converges to zero as an alternating sequence. Hence this is not conjugate to the previous case since a homeomorphism takes a segment onto a segment.
5. If $a = -1$, then the solution is an alternating sequence.
6. If $a < -1$, then 0 is an unstable fixed point, however, the sequence is alternating, hence this case is not conjugate to the case $a > 1$.

For the rigorous justification of the above classification we determine the homeomorphism yielding the conjugacy.

To given numbers $a, b \in \mathbb{R} \setminus \{0\}$ we look for a homeomorphism $h : \mathbb{R} \rightarrow \mathbb{R}$, for which $h(ax) = bh(x)$ holds for all x . The homeomorphism h can be looked for in the form

$$h(x) = \begin{cases} x^\alpha & \text{if } x > 0 \\ -(-x)^\alpha & \text{if } x < 0 \end{cases}$$

If $a, b > 0$ and $x > 0$, then from equation $h(ax) = bh(x)$ we get $a^\alpha x^\alpha = bx^\alpha$, hence $a^\alpha = b$, i.e. $\alpha = \frac{\ln b}{\ln a}$. The function h is a homeomorphism, if $\alpha > 0$, that holds, if a and b lie on the same side of 1. Thus if $a, b > 1$, then the two equations are C^0 conjugate,

and similarly, if $a, b \in (0, 1)$, then the two equations are also C^0 conjugate. (It can be seen easily that the equation $h(ax) = bh(x)$ holds also for negative values of x .) One can prove similarly, that if $a, b < -1$ or if $a, b \in (-1, 0)$, then the two equations are C^0 conjugate. Thus using the above homeomorphism $h(x) = |x|^\alpha \text{sgn}(x)$ we can prove that C^0 conjugacy divides $GL(\mathbb{R})$ at most into six classes. It is easy to show that there are in fact six classes, that is taking two elements from different classes they are not C^0 conjugate, i.e. it cannot be given a homeomorphism, for which $h(ax) = bh(x)$ holds for all x .

2.3.3 Continuous time case in n dimension

Let us consider the system of linear differential equations $\dot{x} = Ax$, where A is an $n \times n$ matrix. The C^0 classification is based on the stable, unstable and center subspaces, the definitions and properties of which are presented first. Let us denote by $\lambda_1, \lambda_2, \dots, \lambda_n$ the eigenvalues of the matrix (counting with multiplicity). Let us denote by u_1, u_2, \dots, u_n the basis in \mathbb{R}^n that yields the real Jordan canonical form of the matrix. The general method for determining this basis would need sophisticated preparation, however, in the most important special cases the basis can easily be given as follows. If the eigenvalues are real and different, then the basis vectors are the corresponding eigenvectors. If there are complex conjugate pairs of eigenvalues, then the real and imaginary part of the corresponding complex eigenvector should be put in the basis. If there are eigenvalues with multiplicity higher than 1 and lower dimensional eigenspace, then generalised eigenvectors have to be put into the basis. For example, if λ is a double eigenvalue with a one dimensional eigenspace, then the generalised eigenvector v is determined by the equation $Av = \lambda v + u$, where u is the unique eigenvector. We note, that in this case v is a vector that is linearly independent from u and satisfying $(A - \lambda I)^2 v = 0$, namely $(A - \lambda I)^2 v = (A - \lambda I)u = 0$. Using this basis the stable, unstable and center subspaces can be defined as follows.

Definition 2.8.. Let $\{u_1, \dots, u_n\} \subset \mathbb{R}^n$ be the basis determining the real Jordan canonical form of the matrix A . Let λ_k be the eigenvalue corresponding to u_k . The subspaces

$$E_s(A) = \langle \{u_k : \text{Re}\lambda_k < 0\} \rangle, \quad E_u(A) = \langle \{u_k : \text{Re}\lambda_k > 0\} \rangle,$$

$$E_c(A) = \langle \{u_k : \text{Re}\lambda_k = 0\} \rangle$$

are called the stable, unstable and center subspaces of the linear system $\dot{x} = Ax$. ($\langle \cdot \rangle$ denotes the subspace spanned by the vectors given between the brackets.)

The most important properties of these subspaces can be summarised as follows.

Theorem 2.9.. The subspaces $E_s(A)$, $E_u(A)$, $E_c(A)$ have the following properties.

1. $E_s(A) \oplus E_u(A) \oplus E_c(A) = \mathbb{R}^n$
2. They are invariant under A (that is $A(E_i(A)) \subset E_i(A)$, $i = s, u, c$), and under e^{At} .
3. For all $p \in E_s(A)$ we have $e^{At}p \rightarrow 0$, if $t \rightarrow +\infty$, moreover, there exists $K, \alpha > 0$, for which $|e^{At}p| \leq Ke^{-\alpha t}|p|$, if $t \geq 0$.
4. For all $p \in E_u(A)$ we have $e^{At}p \rightarrow 0$, if $t \rightarrow -\infty$, moreover, there exists $L, \beta > 0$, for which $|e^{At}p| \leq Le^{\beta t}|p|$, if $t \leq 0$.

The invariant subspaces can be shown easily for the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ determining a saddle point. Then the eigenvalues of the matrix are 1 and -1 , the corresponding eigenvectors are $(1, 0)^T$ and $(0, 1)^T$. Hence the stable subspace is the vertical coordinate axis and the unstable subspace is the horizontal coordinate axis, as it is shown in Figure 2.3.

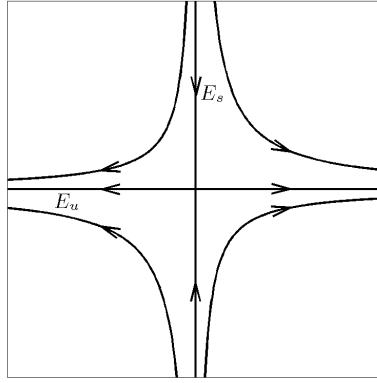


Figure 2.3: The stable and unstable subspaces for a saddle point.

The dimensions of the stable, unstable and center subspaces will play an important role in the C^0 classification of linear systems. First, we introduce notations for the dimensions of these invariant subspaces.

Definition 2.10.. Let $s(A) = \dim(E_s(A))$, $u(A) = \dim(E_u(A))$ and $c(A) = \dim(E_c(A))$ denote the dimensions of the stable, unstable and center subspaces of a matrix A , respectively.

The spectrum, i.e. the set of eigenvalues of the matrix A will be denoted by $\sigma(A)$. The following set of matrices is important from the classification point of view. The elements of

$$EL(\mathbb{R}^n) = \{A \in L(\mathbb{R}^n) : \operatorname{Re} \lambda \neq 0, \forall \lambda \in \sigma(A)\},$$

are called hyperbolic matrices in the continuous time case.

First, these hyperbolic systems will be classified according to C^0 -conjugacy. In order to carry out that we will need the Lemma below.

Lemma 2.11.. 1. If $s(A) = n$, then the matrices A and $-I$ are C^0 conjugate.

2. If $u(A) = n$, then the matrices A and I are C^0 conjugate.

Proof. We prove only the first statement. The second one follows from the first one if it is applied to the matrix $-A$. The proof is divided into four steps.

a. The solution of the differential equation $\dot{x} = Ax$ starting from the point p is $x(t) = e^{At}p$, the solution of the differential equation $\dot{y} = -y$ starting from the same point is $y(t) = e^{-t}p$. According to the theorem about quadratic Lyapunov functions there exists a positive definite symmetric matrix $B \in \mathbb{R}^{n \times n}$, such that for the corresponding quadratic form $Q_B(p) = \langle Bp, p \rangle$ it holds that $L_A Q_B$ is negative definite. We recall that $(L_A Q_B)(p) = \langle Q'_B(p), Ap \rangle$. The level set of the quadratic form Q_B belonging to the value 1 is denoted by $S := \{p \in \mathbb{R}^n : Q_B(p) = 1\}$.

b. Any non-trivial trajectory of the differential equation $\dot{x} = Ax$ intersects the set S exactly once, that is for any point $p \in \mathbb{R}^n \setminus \{0\}$ there exists a unique number $\tau(p) \in \mathbb{R}$, such that $e^{A\tau(p)}p \in S$. Namely, the function $V^*(t) = Q_B(e^{At}p)$ is strictly decreasing for any $p \in \mathbb{R}^n \setminus \{0\}$ and $\lim_{+\infty} V^* = 0$, $\lim_{-\infty} V^* = +\infty$. The function $\tau : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ is continuous (by the continuous dependence of the solution on the initial condition), moreover $\tau(e^{At}p) = \tau(p) - t$.

c. Now, the homeomorphism taking the orbits of the two systems onto each other can be given as follows

$$h(p) := e^{(A+I)\tau(p)}p, \quad \text{if } p \neq 0, \quad \text{and } h(0) = 0.$$

This definition can be explained as follows. The mapping takes the point first to the set S along the orbit of $\dot{x} = Ax$. The time for taking p to S is denoted by $\tau(p)$. Then it takes this point back along the orbit of $\dot{y} = -y$ with the same time, see Figure 2.4.

d. In this last step, it is shown that h is a homeomorphism and maps orbits to orbits. The latter means that $h(e^{At}p) = e^{-t}h(p)$. This is obvious for $p = 0$, otherwise, i.e. for $p \neq 0$ we have

$$h(e^{At}p) = e^{(A+I)\tau(e^{At}p)}e^{At}p = e^{(A+I)(\tau(p)-t)}e^{At}p = e^{(A+I)\tau(p)}e^{-t}p = e^{-t}h(p).$$

Thus it remains to prove that h is a homeomorphism. Since $L_{-I}Q_B = Q_{-2B}$ is negative definite, the orbits of $\dot{y} = -y$ intersect the set S exactly once, hence h is bijective (its inverse can be given in a similar form). Because of the continuity of the function τ the functions h and h^{-1} are continuous at every point except 0. Thus the only thing that remained to be proved is the continuity of h at zero. In order to that we show that

$$\lim_{p \rightarrow 0} e^{\tau(p)}e^{A\tau(p)}p = 0.$$

Since $e^{A\tau(p)}p \in S$ and S is bounded, it is enough to prove that $\lim_{p \rightarrow 0} \tau(p) = -\infty$, that is for any positive number T there exists $\delta > 0$, such that it takes at least time T to get from the set S to the ball $B_\delta(0)$ along a trajectory of $\dot{x} = Ax$. In order to that we prove that there exists $\gamma < 0$, such that for all points $p \in S$ we have $e^{\gamma t} \leq Q_B(e^{At}p)$, that is the convergence of the solutions to zero can be estimated also from below. (Then obviously $|e^{At}p|$ can also be estimated from below.) Let C be the negative definite matrix, for which $L_A Q_B = Q_C$. The negative definiteness of C and the positive definiteness of B imply that there exist $\alpha < 0$ and $\beta > 0$, such that $Q_C(p) \geq \alpha|p|^2$ and $Q_B(p) \geq \beta|p|^2$ for all $p \in \mathbb{R}^n$. Let $V^*(t) := Q_B(e^{At}p)$ (for an arbitrary point $p \in S$), then $\dot{V}^*(t) = Q_C(e^{At}p)$, hence $\dot{V}^*(t)Q_B(e^{At}p) = V^*(t)Q_C(e^{At}p)$ implying $\beta\dot{V}^*(t) \geq \alpha V^*(t)$. Let $\gamma := \frac{\alpha}{\beta}$, then Gronwall's lemma implies that $V^*(t) \geq e^{\gamma t}$, that we wanted to prove. \square

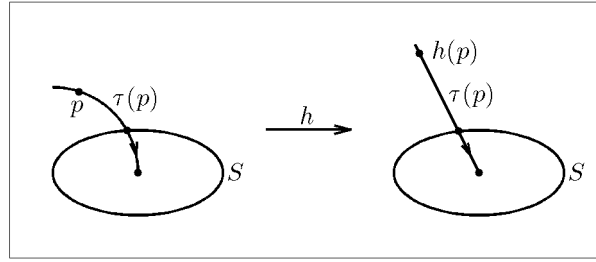


Figure 2.4: The homeomorphism h taking the orbits of $\dot{x} = Ax$ to the orbits of $\dot{y} = -y$.

Using this lemma it is easy to prove the theorem below about the classification of hyperbolic linear systems.

Theorem 2.12.. *The hyperbolic matrices $A, B \in EL(\mathbb{R}^n)$ are C^0 conjugate, and at the same time C^0 equivalent, if and only if $s(A) = s(B)$. (In this case, obviously, $u(A) = u(B)$ holds as well, since the center subspaces are zero dimensional.)*

The C^0 classification is based on the strong theorem below, the proof of which is beyond the framework of this lecture notes.

Theorem 2.13. (Kuiper). *Let $A, B \in L(\mathbb{R}^n)$ be matrices with $c(A) = c(B) = n$. These are C^0 equivalent, if and only if they are linearly equivalent.*

The full classification below follows easily from the two theorems above.

Theorem 2.14.. *The matrices $A, B \in L(\mathbb{R}^n)$ are C^0 equivalent, if and only if $s(A) = s(B)$, $u(A) = u(B)$ and their restriction to their center subspaces are linearly equivalent (i.e. $A|_{E_c}$ and $B|_{E_c}$ are linearly equivalent).*

Example 2.1. *The space of two-dimensional linear systems, that is the space $L(\mathbb{R}^2)$ is divided into 8 classes according to C^0 equivalence. We list the classes according to the dimension of the center subspaces of the corresponding matrices.*

1. *If $c(A) = 0$, then the dimension of the stable subspace can be 0, 1 or 2, hence there are three classes. The simplest representants of these classes are*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

corresponding to the unstable node (or focus), saddle and stable node (or focus), respectively. (We recall that the node and focus are C^0 conjugate.) The phase portraits belonging to these cases are shown in Figures 2.5, 2.6 and 2.7.

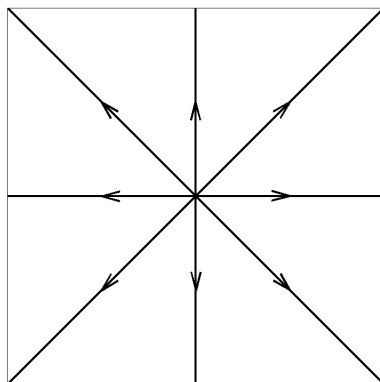


Figure 2.5: Unstable node.

2. *If $c(A) = 1$, then the dimension of the stable subspace can be 0 or 1, hence there are two classes. The simplest representants of these classes are*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The phase portraits belonging to these cases are shown in Figures 2.8 and 2.9.

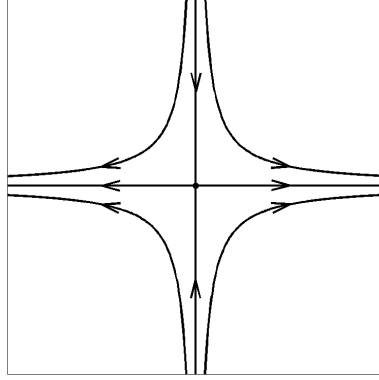


Figure 2.6: Saddle point.

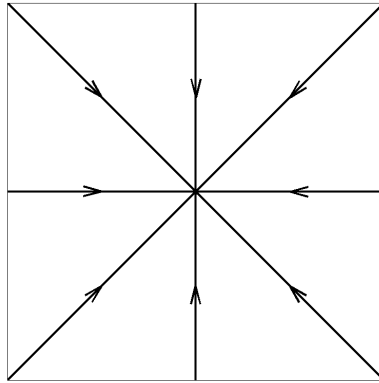


Figure 2.7: Stable node.

3. If $c(A) = 2$, then the classes are determined by linear equivalence. If zero is a double eigenvalue, then we get two classes, and all matrices having pure imaginary eigenvalues are linearly equivalent to each other, hence they form a single class. Hence there are 3 classes altogether, simple representants of which are

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The last one is the center. The phase portraits belonging to these cases are shown in Figures 2.10, 2.11 and 2.12.

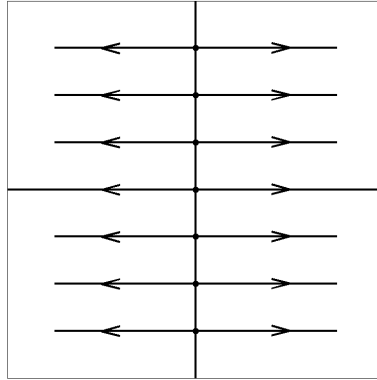


Figure 2.8: Infinitely many unstable equilibria.

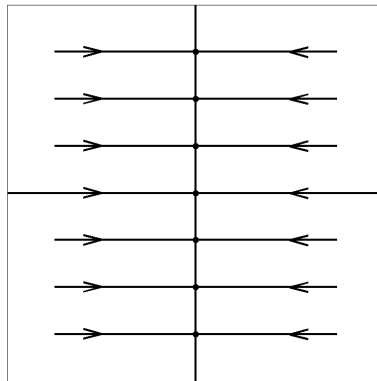


Figure 2.9: Infinitely many stable equilibria.

It can be shown similarly that the space $L(\mathbb{R}^3)$ of 3-dimensional linear systems is divided into 17 classes according to C^0 equivalence.

The space $L(\mathbb{R}^4)$ of 4-dimensional linear systems is divided into infinitely many classes according to C^0 equivalence, that is there are infinitely many different 4-dimensional linear phase portraits.

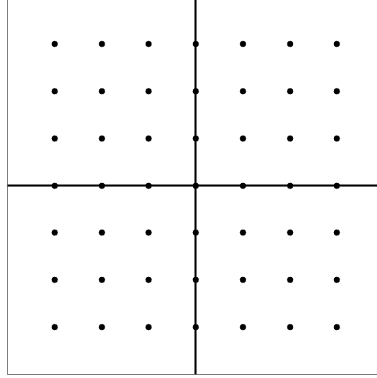


Figure 2.10: Every point is an equilibrium.

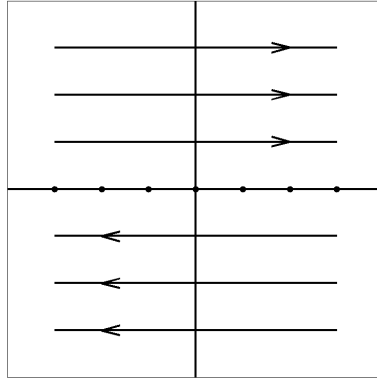


Figure 2.11: Degenerate equilibria lying along a line.

2.3.4 Discrete time case in n dimension

Consider the system defined by the linear recursion $x_{k+1} = Ax_k$, where A is an $n \times n$ matrix. The C^0 classification uses again the stable, unstable and center subspaces, that will be defined first, for discrete time systems. Let us denote the eigenvalues of the matrix with multiplicity by $\lambda_1, \lambda_2, \dots, \lambda_n$. Let u_1, u_2, \dots, u_n denote that basis in \mathbb{R}^n in which the matrix takes it Jordan canonical form. Using this basis the stable, unstable and center subspaces can be defined as follows.

Definition 2.15.. Let $\{u_1, \dots, u_n\} \subset \mathbb{R}^n$ be the above basis and let λ_k be the eigenvalue

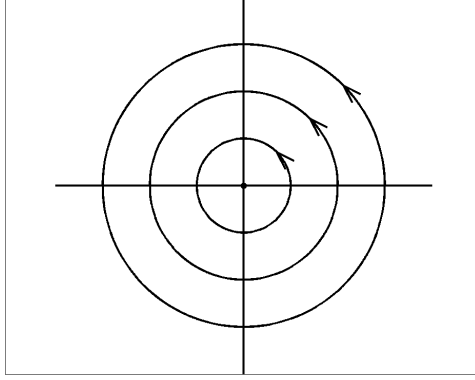


Figure 2.12: Center.

corresponding to u_k (note that u_k may not be an eigenvector). The subspaces

$$E_s(A) = \langle \{u_k : |\lambda_k| < 1\} \rangle, \quad E_u(A) = \langle \{u_k : |\lambda_k| > 1\} \rangle,$$

$$E_c(A) = \langle \{u_k : |\lambda_k| = 1\} \rangle$$

are called the stable, unstable and center subspaces belonging to the matrix $A \in GL(\mathbb{R}^n)$. (The notation $\langle \cdot \rangle$ denotes the subspace spanned by the vectors between the brackets.)

The most important properties of these subspaces are summarised in the following theorem.

Theorem 2.16.. *The subspaces $E_s(A)$, $E_u(A)$, $E_c(A)$ have the following properties.*

1. $E_s(A) \oplus E_u(A) \oplus E_c(A) = \mathbb{R}^n$
2. They are invariant under A (that is $A(E_i(A)) \subset E_i(A)$, $i = s, u, c$).
3. For any $p \in E_s(A)$ we have $A^n p \rightarrow 0$, if $n \rightarrow +\infty$.
4. For any $p \in E_u(A)$ we have $A^{-n} p \rightarrow 0$, if $n \rightarrow +\infty$.

The dimensions of the stable, unstable and center subspaces will play an important role in the C^0 classification of linear systems. First, we introduce notations for the dimensions of these invariant subspaces.

Definition 2.17.. *Let $s(A) = \dim(E_s(A))$, $u(A) = \dim(E_u(A))$ and $c(A) = \dim(E_c(A))$ denote the dimensions of the stable, unstable and center subspaces of a matrix A , respectively.*

The following set of matrices is important from the classification point of view. The elements of

$$HL(\mathbb{R}) = \{A \in GL(\mathbb{R}^n) : |\lambda| \neq 1 \ \forall \lambda \in \sigma(A)\},$$

are called hyperbolic matrices in the discrete time case.

In the discrete time case only the hyperbolic systems will be classified according to C^0 -conjugacy. In order to carry out that we will need the Lemma below.

Lemma 2.18.. *Let the hyperbolic matrices $A, B \in HL(\mathbb{R}^n)$ be C^0 conjugate, that is there exists a homeomorphism $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$, for which $h(Ax) = Bh(x)$ for all $x \in \mathbb{R}^n$. Then the following statements hold.*

1. $h(0) = 0$,
2. $h(E_s(A)) = E_s(B)$, that is h takes the stable subspace to stable subspace; $h(E_u(A)) = E_u(B)$, that is h takes the unstable subspace to unstable subspace
3. $s(A) = s(B)$, $u(A) = u(B)$.

Proof. 1. Substituting $x = 0$ into equation $h(Ax) = Bh(x)$ leads to $h(0) = Bh(0)$. This implies that $h(0) = 0$, because the matrix B is hyperbolic, that is 1 is not an eigenvalue.

2. If $x \in E_s(A)$, then $A^n x \rightarrow 0$ as $n \rightarrow \infty$, hence $h(A^n x) = B^n h(x)$ implies that $B^n h(x)$ tends also to zero. Therefore $h(x)$ is in the stable subspace of B . Thus we showed that $h(E_s(A)) \subset E_s(B)$. Using similar arguments for the function h^{-1} we get that $h^{-1}(E_s(A)) \subset E_s(B)$ yielding $E_s(B) \subset h(E_s(A))$. Since the two sets contain each other, they are equal $h(E_s(A)) = E_s(B)$.

3. Since there is a homeomorphism taking the subspace $E_s(A)$ to the subspace $E_s(B)$, their dimensions are equal, i.e. $s(A) = s(B)$, implying also $u(A) = u(B)$ since the dimensions of the center subspaces are zero. \square

In the case of continuous time linear systems we found that $s(A) = s(B)$ is not only a necessary, but also a sufficient condition for the C^0 conjugacy of two hyperbolic linear systems. Now we will investigate in the case of a one-dimensional and a two-dimensional example if this condition is sufficient or not for discrete time linear systems.

Example 2.2. *Consider the one-dimensional linear equations given by the numbers (one-by-one matrices) $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. Both have one dimensional stable subspace, that is $s(A) = s(B) = 1$, since the orbits of both systems are formed by geometric sequences converging to zero. However, as it was shown in Section 2.3.2, these two equations are not C^0 conjugate. It was proved there that C^0 conjugacy divides the space $GL(\mathbb{R})$ into six classes.*

This example shows that $s(A) = s(B)$ is not sufficient for the C^0 conjugacy of the two equations. Despite of this seemingly negative result, it is worth to investigate the following two dimensional example, in order to get intuition for the classification of hyperbolic linear systems.

Example 2.3. Consider the two-by-two matrices $A = \frac{1}{2}I$ and $B = -\frac{1}{2}I$, where I is the unit matrix. The stable subspace is two-dimensional for both systems, that is $s(A) = s(B) = 2$, since their orbits are given by sequences converging to zero. We will show that these matrices are C^0 conjugate. We are looking for a homeomorphism $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfying $h(\frac{1}{2}x) = -\frac{1}{2}h(x)$ for all $x \in \mathbb{R}^2$. It will be given in such a way that circles centered at the origin remain invariant, they will be rotated by different angles depending on the radius of the circle. Let us start from the unit circle with radius one and define h as the identity map on this circle, i.e. $h(x) = x$ is $|x| = 1$. Then equation $h(\frac{1}{2}x) = -\frac{1}{2}h(x)$ defines h along the circle of radius $1/2$, namely h rotates this circle by π , i.e. $h(x) = -x$ is $|x| = 1/2$. In the annulus between the two circles the homeomorphism can be defined arbitrarily. Then equation $h(\frac{1}{2}x) = -\frac{1}{2}h(x)$ defines h again in the annulus between the circles of radius $1/2$ and $1/4$. Once the function is known in this annulus the equation defines again its values in the annulus between the circles of radius $1/4$ and $1/8$. In a similar way, the values of h in the annulus between the circles of radius 1 and 2 are defined by the equation $h(\frac{1}{2}x) = -\frac{1}{2}h(x)$ based on the values in the annulus determined by the circles of radius $1/2$ and 1 . It can be easily seen that the angle of rotation on the circle of radius 2^k has to be $-k\pi$. Thus let the angle of rotation on the circle of radius r be $-\pi \log_2(r)$. This ensures that the angle of rotation is a continuous function of the radius and for $r = 2^k$ it is $-k\pi$. This way the function h can be given explicitly on the whole plane as follows

$$h(x) = R(-\pi \log_2(|x|))x, \quad \text{where} \quad R(\alpha) = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}$$

This function is obviously bijective, since the circles centered at the origin are invariant and the map is bijective on these circles. Its continuity is also obvious except at the origin. The continuity at the origin can be proved by using the above formula, we omit here the details of the proof.

We note that in 3-dimension the matrices $A = \frac{1}{2}I$ and $B = -\frac{1}{2}I$, where I is the unit matrix of size 3×3 , are not C^0 conjugate. Thus we found that $s(A) = s(B)$ is not a sufficient condition of C^0 conjugacy. The sufficient condition is formulated in the following lemma that we do not prove here.

Lemma 2.19.. Assume that $s(A) = s(B) = n$ (or $u(A) = u(B) = n$). Then A and B are C^0 conjugate, if and only if $\text{sgn det } A = \text{sgn det } B$.

This lemma enables us to formulate the following necessary and sufficient condition for the C^0 conjugacy of matrices.

Theorem 2.20.. *The hyperbolic linear maps $A, B \in HL(\mathbb{R}^n)$ are C^0 conjugate, if and only if*

- $s(A) = s(B)$
- $\operatorname{sgn} \det A|_{E_s(A)} = \operatorname{sgn} \det B|_{E_s(B)}$
- $\operatorname{sgn} \det A|_{E_u(A)} = \operatorname{sgn} \det B|_{E_u(B)}$

Example 2.4. *According to this theorem the matrices $A = \frac{1}{2}I$ and $B = -\frac{1}{2}I$, where I is the $n \times n$ unit matrix, are C^0 conjugate, if and only if n is even. Namely, in this case the determinant of the matrix $-I$ is also positive, while for odd values of n it is negative. Thus, as we have already shown, the matrices*

$$A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \text{ and } B = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

are C^0 conjugate.

Example 2.5. *Using the above theorem, it is easy to prove that the space $HL(\mathbb{R}^2)$ of hyperbolic linear maps is divided into 8 classes according to C^0 conjugacy. The proof of this is left to the Reader.*

2.4 Exercises

1. Which one of the following matrices in $L(\mathbb{R}^2)$ is C^1 conjugate to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$?

$$A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad B = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad C = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Answer: none of them.

2. Which one of the following matrices in $L(\mathbb{R}^2)$ is C^0 conjugate to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$?

$$A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad B = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad C = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Answer: C.

3. Which one of the following matrices in $L(\mathbb{R}^2)$ is C^0 equivalent to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$?

$$A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad B = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad C = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Answer: C.

4. Which one of the following matrices in $L(\mathbb{R}^3)$ is C^0 equivalent to the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$?

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 1 & 1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

Answer: B.

5. Which one of the following matrices in $GL(\mathbb{R}^2)$ is C^1 conjugate to the matrix $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$?

$$A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix}$$

Answer: none of them.

6. Which one of the following matrices in $GL(\mathbb{R}^2)$ is C^0 conjugate to the matrix $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$?

$$A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix}$$

Answer: C.

7. Which one of the following matrices in $L(\mathbb{R}^2)$ is hyperbolic, that is which one is in the set $EL(\mathbb{R}^2)$?

$$A = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$$

Answer: B.

8. Which one of the following matrices in $GL(\mathbb{R}^2)$ is hyperbolic, that is which one is in the set $HL(\mathbb{R}^2)$?

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \frac{1}{2} \\ -2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 2 \\ 6 & 5 \end{pmatrix}$$

Answer: A.

9. Which one of the following systems is orbitally equivalent to the linear differential equation belonging to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$?

$$\dot{x} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} x, \quad \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_1^3 + y_1 y_2^2 \\ -y_1^2 y_2 - y_2^3 \end{pmatrix}$$

Answer: the first one.

Chapter 3

Local classification, normal forms and the Hartman-Grobman theorem

Consider the n -dimensional system of autonomous differential equations

$$\dot{x}(t) = f(x(t)). \quad (3.1)$$

There is no general method for solving this system, hence the most important way of getting information about the solutions is to determine the phase portrait. The constant solutions $x(t) \equiv p$ can be obtained by solving the system of algebraic equations $f(p) = 0$. The solution p of this system is called an equilibrium or steady state of the dynamical system. The behaviour of trajectories in a neighbourhood of a steady state can be investigated by linearisation that can be explained simply as follows. Introducing the function $y(t) = x(t) - p$ the differential equation takes the form

$$\dot{y}(t) = \dot{x}(t) = f(x(t)) = f(p) + f'(p)y(t) + r(y(t)) = f'(p)y(t) + r(y(t)),$$

where r denotes the remainder term. For small y , i.e. when x is close to p , the remainder can be neglected with respect to the linear term (assuming that it is not too small). Hence it can be expected that the local phase portrait in a neighbourhood of the equilibrium p is determined by the linear equation

$$\dot{y}(t) = f'(p)y(t) \quad (3.2)$$

that is called the linearised equation at the point p , since it is given by the Jacobian of f . In order to make this argumentation rigorous we have to define two things. First, what does it mean that the linear term is not too small, second, what does the local phase portrait mean.

Concerning these questions, the stability of steady states is investigated in an introductory differential equation course. First, we briefly summarise these notions and the corresponding results.

Let us denote by $t \mapsto \varphi(t, p)$ the solution of (3.1) that satisfies the initial condition $x(0) = p$, and denote by $I(p)$ the interval in which this solution is defined.

Definition 3.1.. An equilibrium point $p \in \mathbb{R}^n$ of system (3.1) is called *stable*, if for any $\varepsilon > 0$ there exists $\delta > 0$, such that

$$q \in \mathbb{R}^n, \quad |q - p| < \delta, \quad t \geq 0 \text{ imply } |\varphi(t, q) - p| < \varepsilon.$$

The equilibrium is called *asymptotically stable*, if it is stable and $|\varphi(t, q) - p| \rightarrow 0$ as $t \rightarrow +\infty$ for the above q , see Figure 3.1. The equilibrium is called *unstable*, if it is not stable.

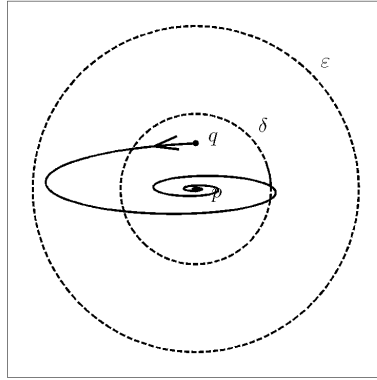


Figure 3.1: An asymptotically stable equilibrium.

The linearisation determines stability in the following cases.

Theorem 3.2..

1. If the real parts of the eigenvalues of the matrix $f'(p)$ are negative, then p is an asymptotically stable equilibrium of system (3.1).
2. If the matrix $f'(p)$ has at least one eigenvalue with positive real part, then p is an unstable equilibrium of system (3.1).

This theorem can be applied when the stable subspace is n -dimensional, or the unstable subspace is at least one dimensional. More general statements can be formulated for given dimensions of the invariant subspaces, these are the stable, unstable and center manifold theorems that will be dealt with in the next section. In this section we investigate the problem of finding the simplest system the local phase portraits of which is the same as that of the given system at the given point. In order to formulate this rigorously, we introduce the notion of local equivalence.

Definition 3.3.. Let (M, φ) , (N, ψ) be dynamical systems and $p \in M$, $q \in N$ be given points. The dynamical system φ at the point p and the dynamical system ψ at the point q are C^k equivalent (conjugate), if there is a neighbourhood U of point p , a neighbourhood V of the point q and a C^k -diffeomorphism $h : U \rightarrow V$ taking the orbits into each other (by preserving time) and satisfying $h(p) = q$.

The main idea of local investigation is that the terms of the power series expansion of f at the point p , i.e. $f(x) = f(p) + f'(p) \cdot (x - p) + \dots$ determines the local phase portrait at the point p . In this section the theorems related to this question are dealt with. These can be summarised briefly as follows.

- Flow-box theorem: The nonzero zeroth order term determines the local phase portrait.
- Hartman–Grobman theorem: A hyperbolic linear term determines the local phase portrait.
- Theory of normal forms: The resonant higher order terms determine the local phase portrait.

The flow-box theorem is formulated here, the two other theorems are dealt with in separate subsections. The flow-box theorem describes the behaviour at a non-equilibrium point, while the other two theorems deal with steady states. According to that theorem the local phase portrait at a non-equilibrium point is C^k conjugate to that of a system the orbits of which are parallel straight lines.

Theorem 3.4. (Flow-box theorem). *If $f(p) \neq 0$, then the system $\dot{x} = f(x)$ at the point p and the system $\dot{y} = f(p)$ at the origin are locally C^k conjugate (assuming that $f \in C^k$). That is, when $f(p) \neq 0$, i.e. p is not an equilibrium point, then the zeroth order term of the expansion determines the local phase portrait.*

3.1 Hartman–Grobman theorem

Let $D \subset \mathbb{R}^n$ be a connected open set, $f : D \rightarrow \mathbb{R}^n$ be a C^1 (continuously differentiable) function, $p^* \in D$ be an equilibrium point, i.e. $f(p^*) = 0$. The solution of the system

$$\dot{x}(t) = f(x(t)) \tag{3.3}$$

satisfying the initial condition $x(0) = p$ is denoted by $x(t) = \varphi(t, p)$. The notation $\varphi_t(p) = \varphi(t, p)$ will be used; then $\varphi_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$. The Hartman–Grobman theorem states that the linear part of the system at a hyperbolic equilibrium determines the local phase portrait up to topological equivalence. Let $A = f'(p^*)$ be the Jacobian matrix at p^* . Then the linearised system is

$$\dot{y}(t) = Ay(t). \tag{3.4}$$

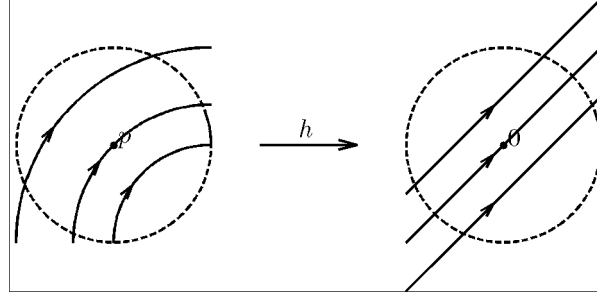


Figure 3.2: Flow-box theorem.

Theorem 3.5. (Hartman–Grobman). *Let D , f , p^* be given as above and assume that the matrix A is hyperbolic, i.e. the real parts of its eigenvalues are non-zero. Then system (3.3) at the point p^* and system (3.4) at the origin are locally topologically conjugate. That is there exist a neighbourhood $U \subset \mathbb{R}^n$ of p^* , a neighbourhood $V \subset \mathbb{R}^n$ of the origin and a homeomorphism $h : U \rightarrow V$, for which*

$$h(\varphi(t, p)) = e^{At}h(p) \quad (3.5)$$

for all $p \in U$ and for all $t \in \mathbb{R}$, for which $\varphi(t, p) \in U$ holds. That is, in a short form $h \circ \varphi_t = e^{At} \circ h$.

The theorem will be proved in the following four steps.

1. It is shown that $p^* = 0$ can be assumed without loss of generality.
2. The function f is extended to the whole space \mathbb{R}^n in such a way that it is equal to its own linear part outside a suitably chosen ball. It is shown that the extended system and the linear part are (globally) topologically conjugate. This statement is referred to as the global version of the Hartman–Grobman theorem.
3. The global version is reduced to the discrete time version of the Hartman–Grobman theorem.

4. Proof of the discrete time version of the Hartman–Grobman theorem, that is also referred to as Hartman–Grobman theorem for maps.

The following notations will be used:

$$\begin{aligned} B_r &= \{p \in \mathbb{R}^n : |p| < r\} \\ C^0(\mathbb{R}^n, \mathbb{R}^n) &= \{g : \mathbb{R}^n \rightarrow \mathbb{R}^n : g \text{ is continuous} \} \\ C^1(\mathbb{R}^n, \mathbb{R}^n) &= \{g : \mathbb{R}^n \rightarrow \mathbb{R}^n : g \text{ is continuously differentiable} \} \\ C_b^0(\mathbb{R}^n, \mathbb{R}^n) &= \{g : \mathbb{R}^n \rightarrow \mathbb{R}^n : g \text{ is bounded and continuous} \} \end{aligned}$$

for $a \in C^0(\mathbb{R}^n, \mathbb{R}^n)$ and for $b \in C^1(\mathbb{R}^n, \mathbb{R}^n)$

$$\|a\|_0 = \sup_{\mathbb{R}^n} |a| \quad \|b\|_1 = \|b\|_0 + \|b'\|_0$$

Before proving the theorem we formulate the above mentioned versions of the Hartman–Grobman theorem.

Theorem 3.6. (Hartman–Grobman global version). *Let $A \in L(\mathbb{R}^n)$ be a hyperbolic linear map, $a \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $a(0) = 0$, $a'(0) = 0$ and let $f(p) = Ap + a(p)$. The solution of system (3.3) is denoted by $\varphi(\cdot, p)$. The equilibrium point is the origin. Then there exists a number $\nu > 0$, such that for a function a with compact support and satisfying $\|a\|_1 < \nu$, there exists a homeomorphism $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfying*

$$h(\varphi(t, p)) = e^{At}h(p) \tag{3.6}$$

for all $p \in \mathbb{R}^n$ and for all $t \in \mathbb{R}$.

Theorem 3.7. (Hartman–Grobman for maps). *Let $L \in GL(\mathbb{R}^n)$ be a hyperbolic linear map, that is the absolute value of its eigenvalues are not equal to 1 and to 0. Then there exists a number $\mu > 0$, such that for a function $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $\|F\|_1 < \mu$, there exists $g \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$, for which $H = id + g$ is a homeomorphism and*

$$H \circ (L + F) = L \circ H. \tag{3.7}$$

Now we turn to the proof of the Hartman–Grobman theorem (Theorem 3.5.).

3.1.1 STEP 1 of the proof of Theorem 3.5.

Proposition 3.1. *Assume that the Hartman–Grobman theorem is proved when $p^* = 0$. Then the theorem holds for any p^* .*

Proof. Let $l : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the translation $l(p) = p - p^*$. Let $y : \mathbb{R} \rightarrow \mathbb{R}^n$, $y(t) = x(t) - p^*$. Then $\dot{y}(t) = \dot{x}(t) = f(x(t)) = f(y(t) + p^*)$, that is y is a solution of the differential equation

$$\dot{y}(t) = g(y(t)), \quad (3.8)$$

$g = f \circ l^{-1}$. The equilibrium of this equation is the origin. Let $\psi(\cdot, q)$ denote the solution of this differential equation satisfying the initial condition $y(0) = q$. It is easy to see that $\psi(t, q) = \varphi(t, q + p^*) - p^*$, that is

$$l \circ \varphi_t = \psi_t \circ l. \quad (3.9)$$

Since by the hypothesis the Hartman–Grobman theorem is true for equation (3.8), there is a homeomorphism h_1 , for which

$$h_1 \circ \psi_t = e^{At} \circ h_1, \quad (3.10)$$

where $A = g'(0) = f'(p^*)$. Composing equation (3.10) with l from the right $h_1 \circ \psi_t \circ l = e^{At} \circ h_1 \circ l$. Applying (3.9) we get $h_1 \circ l \circ \varphi_t = e^{At} \circ h_1 \circ l$. Introducing $h = h_1 \circ l$ we get the desired statement, because, being the composition of two homeomorphisms, h itself is a homeomorphism. \square

3.1.2 STEP 2 of the proof of Theorem 3.5.

Using the extension lemma below we prove that the global version of the Hartman–Grobman theorem (Theorem 3.6.) implies the local version (Theorem 3.5.). The technical proof of the lemma will not be shown here.

Lemma 3.8.. *Let $f \in C^1(B_R, \mathbb{R}^n)$ and let $A = f'(0)$. For any number $\nu > 0$ there exist $r > 0$ and $a \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, for which*

1. $|p| < r$ implies $a(p) = f(p) - Ap$,
2. $|p| > 2r$ implies $a(p) = 0$,
3. $\|a\|_1 < \nu$.

Proof of Theorem 3.5.. According to Proposition 3.1 it is enough to prove the theorem in the case $p^* = 0$. Let $\nu > 0$ be the number given by Theorem 3.6. (it depends only on the matrix A). The extension lemma yields a number $r > 0$ and a function $a \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ to this value of ν . Let $\bar{f} = A + a$ and denote by $\bar{\varphi}(\cdot, p)$ the solution of the differential equation $\dot{x}(t) = \bar{f}(x(t))$. Then the functions f and \bar{f} coincide in the ball B_r , hence for all $p \in B_r$ and for all t satisfying $\varphi(t, p) \in B_r$ the equality $\bar{\varphi}(t, p) = \varphi(t, p)$ holds. The function a satisfies the assumptions of Theorem 3.6., hence according to that theorem there exists a homeomorphism $\bar{h} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, for which $\bar{h} \circ \bar{\varphi}_t = e^{At} \circ \bar{h}$. Then introducing $U = B_r$, $h = \bar{h}|_U$ and $V = h(B_r)$ we get the desired statement. \square

3.1.3 STEP 3 of the proof of Theorem 3.5.

In this section we prove that the global version of the Hartman–Grobman theorem (Theorem 3.6.) follows from the Hartman–Grobman theorem for maps.

Proof of Theorem 3.6.. Applying the variation of constants formula to the differential equation $\dot{x}(t) = Ax(t) + a(x(t))$ with initial condition $x(0) = p$ one obtains

$$\varphi(t, p) = e^{At}p + \int_0^t e^{A(t-s)}a(\varphi(s, p))ds .$$

Substituting $t = 1$

$$\varphi(1, p) = e^A p + \int_0^1 e^{A(1-s)}a(\varphi(s, p))ds . \quad (3.11)$$

Let

$$L = e^A \quad \text{and} \quad F(p) = \int_0^1 e^{A(1-s)}a(\varphi(s, p))ds .$$

Let us choose the number $\mu > 0$ to the matrix L according to Theorem 3.7.. Then it is easy to show that there exists a number $\nu > 0$, for which $\|a\|_1 < \nu$ implies $\|F\|_1 < \mu$.

Since the eigenvalues of the matrix A have non-zero real part, the absolute value of the eigenvalues of L are not equal to 1. Thus the assumptions of Theorem 3.7. are fulfilled. Therefore there exists a unique function $g \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$, for which

$$(\text{id} + g) \circ (L + F) = L \circ (\text{id} + g) . \quad (3.12)$$

Now we show that by choosing $h = \text{id} + g$ the statement to be proved, i.e. $h \circ \varphi_t = e^{At} \circ h$ holds. In order to prove that it is enough to show that the function

$$\alpha(p) = e^{-At}h(\varphi(t, p))$$

coincides with the function h . This will be verified if we prove that the function $\alpha - \text{id}$ satisfies (3.12) and $\alpha - \text{id} \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$ holds as well, because the function g is unique in the given function space. These two statement will be proved below.

According to equation (3.12)

$$h(\varphi(1, p)) = e^A h(p)$$

holds for all $p \in \mathbb{R}^n$. Hence

$$\begin{aligned} [\alpha \circ (L + F)](p) &= \alpha(\varphi(1, p)) = e^{-At}h(\varphi(t, \varphi(1, p))) = e^{-At}h(\varphi(t + 1, p)) = \\ &= e^{-At}h(\varphi(1, \varphi(t, p))) = e^{-At}e^A h(\varphi(t, p)) = e^A e^{-At}h(\varphi(t, p)) = (L \circ \alpha)(p), \end{aligned}$$

which proves that the function $\alpha - \text{id}$ satisfies (3.12). On the other hand,

$$(\alpha - \text{id})(p) = e^{-At}h(\varphi(t, p)) - p = e^{-At}(h(\varphi(t, p)) - \varphi(t, p)) + e^{-At}\varphi(t, p) - p .$$

The first term of the right hand side is bounded (in the variable p), since $h - \text{id}$ is bounded in \mathbb{R}^n . The second term is also bounded because for large values of $|p|$ we have $a(p) = 0$, hence the equation is linear there, implying $\varphi(t, p) = e^{At}p$. This proves that $\alpha - \text{id} \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$ that verifies the statement. \square

3.1.4 STEP 4 of the proof of Theorem 3.5.

Proof of Theorem 3.7. The proof is divided into five steps.

1. Let E_s and E_u be the stable and unstable subspaces belonging to the linear mapping L . It is known that these are invariant, i.e. $L(E_s) \subset E_s$ and $L(E_u) \subset E_u$, and they span the whole space, that is $E_s \oplus E_u = \mathbb{R}^n$. Let

$$L_s = L|_{E_s}, \quad L_u = L|_{E_u} .$$

It can be shown that by the suitable choice of the norm (or, in other words, by the suitable choice of the basis in the subspaces) one can achieve $\|L_s\| < 1$ and $\|L_u^{-1}\| < 1$. Let

$$r = \max\{\|L_s\|, \|L_u^{-1}\|\} < 1 .$$

2. In this step we prove that there exists a positive number $\mu > 0$, for which in the case $\|F\|_1 < \mu$ the function $L + F$ is invertible. In order to prove that we apply the following global inverse function theorem.

Global inverse function theorem Let $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ be a function, for which $\phi'(p)^{-1}$ exists for all $p \in \mathbb{R}^n$ and there exists $K > 0$, such that $\|\phi'(p)^{-1}\| \leq K$. Then ϕ is a homeomorphism.

Let $\phi = L + F$, then $\phi' = L + F'$. Hence there exists $\mu > 0$, such that the conditions of the global inverse function theorem hold for ϕ , if $\|F\|_1 < \mu$. (We do not check these conditions in detail.) Thus the function $L + F$ is a homeomorphism.

3. Now we transform equation (3.7) to a form, for which the Contraction Mapping Principle can be applied to determine the function g .

It is easy to see that (3.7) is equivalent to equation

$$F + g \circ (L + F) = L \circ g . \tag{3.13}$$

Composing this equation by the function $(L + F)^{-1}$ from right we get

$$g = -F \circ (L + F)^{-1} + L \circ g \circ (L + F)^{-1} , \tag{3.14}$$

and then composing by the function L^{-1} from left we arrive to

$$L^{-1} \circ F + L^{-1} \circ g \circ (L + F) = g . \tag{3.15}$$

Since $E_s \oplus E_u = \mathbb{R}^n$, for both F and g it can be introduced the functions $F_s, g_s : \mathbb{R}^n \rightarrow E_s$ and $F_u, g_u : \mathbb{R}^n \rightarrow E_u$, in such a way that

$$g = g_s + g_u \quad \text{and} \quad F = F_s + F_u$$

hold. It is obvious that $g \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$ implies $g_s, g_u \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$ as well. Define the operator T for a function $g \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$ as follows.

$$T(g) = L \circ g_s \circ (L + F)^{-1} - F_s \circ (L + F)^{-1} + L^{-1} \circ g_u \circ (L + F) + L^{-1} \circ F_u . \quad (3.16)$$

We show that if g is a fixed point of T , then $H = \text{id} + g$ is a solution of equation (3.7). Namely, for an arbitrary $p \in \mathbb{R}^n$

$$(L \circ g_s \circ (L + F)^{-1} - F_s \circ (L + F)^{-1})(p) \in E_s \quad \text{and} \quad (L^{-1} \circ g_u \circ (L + F) + L^{-1} \circ F_u)(p) \in E_u$$

hold. Hence according to $g = g_s + g_u$ the equality $T(g) = g$ can hold only if the following two equations hold

$$L \circ g_s \circ (L + F)^{-1} - F_s \circ (L + F)^{-1} = g_s \quad \text{and} \quad L^{-1} \circ g_u \circ (L + F) + L^{-1} \circ F_u = g_u .$$

These equations yield

$$L \circ g_s = g_s \circ (L + F) + F_s \quad \text{and} \quad L \circ g_u = g_u \circ (L + F) + F_u .$$

Adding these equations and using the linearity of L one obtains (3.13), which is equivalent to equation (3.7).

4. In this step we prove that the operator T maps the space $C_b^0(\mathbb{R}^n, \mathbb{R}^n)$ into itself, and choosing a suitable norm on the space the mapping T is a contraction. Hence the existence and uniqueness of the function g follows from the Contraction Mapping Principle.

The operator T is obviously defined on the whole space $C_b^0(\mathbb{R}^n, \mathbb{R}^n)$, and for every $g \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$ the function $T(g)$ is a continuous function on \mathbb{R}^n . We show that $T(g)$ is a bounded function verifying that T maps the space $C_b^0(\mathbb{R}^n, \mathbb{R}^n)$ into itself. The boundedness follows from the fact that each term in the right hand side of (3.16) is a bounded function. For example, in the case of the last term

$$|(L^{-1} \circ F_u)(p)| = |L^{-1} F_u(p)| \leq \|L^{-1}\| \|F\|_0$$

holds for all $p \in \mathbb{R}^n$. The proof is similar for the other terms.

Now we prove that T is a contraction. The norm is defined as follows. For $g \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$ let

$$\|g\|_* = \|g_s\|_0 + \|g_u\|_0 .$$

It can be easily proved that this defines a norm and the space $C_b^0(\mathbb{R}^n, \mathbb{R}^n)$ endowed with this norm is complete. In order to prove that T is contractive we use the relations

$$\begin{aligned}(T(g))_s &= L \circ g_s \circ (L + F)^{-1} - F_s \circ (L + F)^{-1} , \\ (T(g))_u &= L^{-1} \circ g_u \circ (L + F) + L^{-1} \circ F_u ,\end{aligned}$$

and the fact that for $g, \bar{g} \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$ we have that $(g + \bar{g})_s = g_s + \bar{g}_s$, and $(g + \bar{g})_u = g_u + \bar{g}_u$. For arbitrary $g, \bar{g} \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$

$$\begin{aligned}\|T(g) - T(\bar{g})\|_* &= \|(T(g) - T(\bar{g}))_s\|_0 + \|(T(g) - T(\bar{g}))_u\|_0 = \\ &\|L \circ g_s \circ (L + F)^{-1} - L \circ \bar{g}_s \circ (L + F)^{-1}\|_0 + \|L^{-1} \circ g_u \circ (L + F) - L^{-1} \circ \bar{g}_u \circ (L + F)\|_0 = \\ &\sup_{p \in \mathbb{R}^n} |L_s g_s((L + F)^{-1}(p)) - L_s \bar{g}_s((L + F)^{-1}(p))| + \\ &\sup_{p \in \mathbb{R}^n} |L_u^{-1} g_u((L + F)(p)) - L_u^{-1} \bar{g}_u((L + F)(p))| \leq \\ &\|L_s\| \|g_s - \bar{g}_s\|_0 + \|L_u^{-1}\| \|g_u - \bar{g}_u\|_0 \leq r(\|(g - \bar{g})_s\|_0 + \|(g - \bar{g})_u\|_0) \leq r \|g - \bar{g}\|_* .\end{aligned}$$

Thus $r < 1$ ensures that T is a contraction, hence it has a unique fixed point $g \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$.

5. It remained to show that the function $H = \text{id} + g$ is a homeomorphism. This will be proved by constructing the inverse of H as a solution of an equation similar to (3.7), and by proving that this equation has a unique solution. Repeating the steps 3. and 4. one can show that there exists a unique function $g^* \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$, such that

$$(L + F) \circ (\text{id} + g^*) = (\text{id} + g^*) \circ L . \quad (3.17)$$

On the other hand, substituting $F \equiv 0$ into equation (3.7) we get

$$L \circ (\text{id} + g) = (\text{id} + g) \circ L . \quad (3.18)$$

The only solution of this equation is $g \equiv 0$.

We show that if $g \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$ is the unique solution of equation (3.7), then the inverse of $H = \text{id} + g$ is the function $H^* = \text{id} + g^*$. Equations (3.7) and (3.17) yield

$$L \circ H \circ H^* = H \circ (L + F) \circ H^* = H \circ H^* \circ L .$$

Let $\bar{g} = H \circ H^* - \text{id}$. Then \bar{g} is a solution of (3.18). If we prove that $\bar{g} \in C_b^0(\mathbb{R}^n, \mathbb{R}^n)$, then by the uniqueness $\bar{g} \equiv 0$, that is $\text{id} = H \circ H^*$. This can be shown as follows.

$$\bar{g} = H \circ H^* - \text{id} = (\text{id} + g) \circ (\text{id} + g^*) - \text{id} = g^* + g \circ (\text{id} + g^*) .$$

The function in the right hand side is obviously continuous and bounded, which proves the statement. It can be proved similarly that $\text{id} = H^* \circ H$, hence H is a continuous bijection implying that it is a homeomorphism. \square

The examples below illustrate the necessity of the assumption on hyperbolicity. Namely, if the linear part is not hyperbolic, then the linearised system and the non-linear system may have different local phase portraits.

Example 3.1. *Consider the two dimensional system*

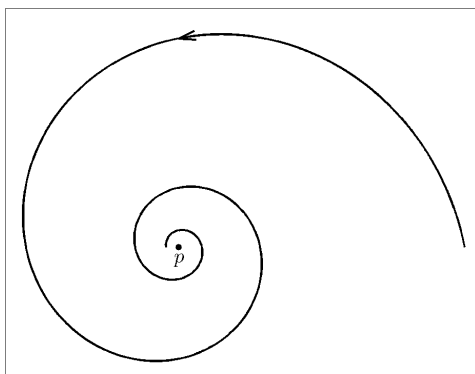
$$\dot{x} = -y - xy^2 - x^3, \quad (3.19)$$

$$\dot{y} = x - y^3 - x^2y. \quad (3.20)$$

The origin is an equilibrium of this system. The linearized system at the origin is given by the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which determines a center. Hence the linear part is not hyperbolic. The phase portrait of the non-linear system can be obtained by polar coordinate transformation. Introducing the functions r and ϕ by the transformation formulas $x(t) = r(t) \cos(\phi(t))$ and $y(t) = r(t) \sin(\phi(t))$, one obtains after differentiation

$$\dot{x} = \dot{r} \cos(\phi) - r\dot{\phi} \sin(\phi), \quad \dot{y} = \dot{r} \sin(\phi) + r\dot{\phi} \cos(\phi).$$

Multiplying the first equation by $\cos(\phi)$ and the second equation by $\sin(\phi)$, then adding the equations and using the differential equations for x and for y we arrive to $\dot{r} = -r^3$. Carrying out a similar calculation, but now the first equation is multiplied by $\sin(\phi)$ and the second one by $\cos(\phi)$ we get $\dot{\phi} = 1$. Hence the function r is strictly decreasing and tends to zero, while ϕ is a strictly increasing function tending to infinity. These facts obviously imply that the origin is a stable focus, as it can be seen in Figure 3.1.4. Thus the non-linear system is not locally topologically equivalent to its linear part at the origin.



The phase portrait of system (3.19)-(3.20).

Example 3.2. Consider the two dimensional system

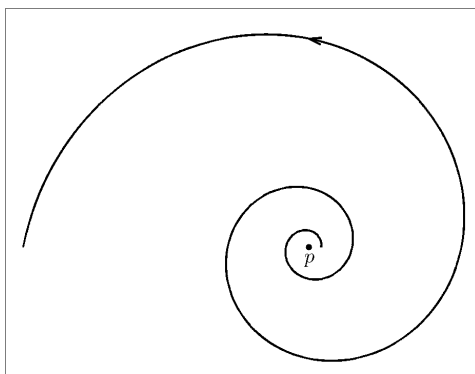
$$\dot{x} = -y + xy^2 + x^3, \quad (3.21)$$

$$\dot{y} = x + y^3 + x^2y. \quad (3.22)$$

The origin is an equilibrium of this system. The linearized system at the origin is given by the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which determines a center. Hence the linear part is not hyperbolic. The phase portrait of the non-linear system can be obtained again by polar coordinate transformation. Introducing the functions r and ϕ by the transformation formulas $x(t) = r(t) \cos(\phi(t))$ and $y(t) = r(t) \sin(\phi(t))$, one obtains after differentiation

$$\dot{x} = \dot{r} \cos(\phi) - r\dot{\phi} \sin(\phi), \quad \dot{y} = \dot{r} \sin(\phi) + r\dot{\phi} \cos(\phi).$$

Multiplying the first equation by $\cos(\phi)$ and the second equation by $\sin(\phi)$, then adding the equations and using the differential equations for x and for y we arrive to $\dot{r} = r^3$. Carrying out a similar calculation, but now the first equation is multiplied by $\sin(\phi)$ and the second one by $\cos(\phi)$ we get $\dot{\phi} = 1$. Hence the function r is strictly increasing and tends to infinity, and similarly ϕ is a strictly increasing function tending to infinity. These facts obviously imply that the origin is an unstable focus, as it can be seen in Figure 3.1.4. Thus the non-linear system is not locally topologically equivalent to its linear part at the origin.



The phase portrait of system (3.21)-(3.22).

3.2 Normal forms

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a k times continuously differentiable function that will be denoted shortly by $f \in C^k$. Sometimes it is assumed that f is infinitely many times differentiable,

this will be denoted by $f \in C^\infty$. If in addition, f is analytic, that is it is equal to the sum of its Taylor series, then the notation $f \in C^\omega$ is used.

The main idea of deriving normal forms can be briefly summarised as follows. In order to simplify the differential equation $\dot{x}(t) = f(x(t))$ let us introduce the function $y(t)$ by the transformation $x = h(y)$, where h is a diffeomorphism. The differential equation can be easily derived for the function y in the following way. Differentiating the relation $x(t) = h(y(t))$ we get $\dot{x} = h'(y) \cdot \dot{y}$, on the other hand, $\dot{x} = f(x) = f(h(y))$ implies $h'(y) \cdot \dot{y} = f(h(y))$, hence for y the differential equation $\dot{y}(t) = g(y(t))$ holds, where f and g are related by

$$h' \cdot g = f \circ h. \quad (3.23)$$

According to Proposition 2.5 this means that the dynamical systems corresponding to the differential equations $\dot{x}(t) = f(x(t))$ and $\dot{y}(t) = g(y(t))$ are C^k conjugate, once $h \in C^k$. The aim is to choose the diffeomorphism h in such a way that the Taylor expansion of g contains as few terms as possible (in this case the phase portrait of $\dot{y}(t) = g(y(t))$ is easier to determine than that of system $\dot{x}(t) = f(x(t))$). In other words, by the transformation h we try to find those terms in the expansion of f that play role in determining the phase portrait.

The local phase portrait will be investigated in a neighbourhood of an equilibrium point p . As a preliminary step the equilibrium is shifted to the origin and the linear part is transformed to Jordan canonical form. That is the function $y = x - p$ is introduced first. For this function the differential equation takes the form $\dot{y} = g(y)$, where $g(y) = f(y+p)$. In order to get the Jordan canonical form let us introduce the invertible matrix $P \in \mathbb{R}^{n \times n}$, and let y be defined by the linear transformation $x = Py$. Then $P\dot{y} = f(Py)$, that is $\dot{y} = g(y)$, where $g(y) = P^{-1} \cdot f(Py)$. Then the Jacobian of system $\dot{y} = g(y)$ at the origin $g'(0) = P^{-1} \cdot f'(0) \cdot P$ can be considered as the Jordan canonical form of the Jacobian $f'(0)$ with a suitably chosen matrix P . Hence we can assume without loss of generality that the equilibrium of system $\dot{x}(t) = f(x(t))$ is the origin (i.e. $f(0) = 0$) and the Jacobian $f'(0)$ is in Jordan canonical form.

Now, the aim is to transform the system to the simplest possible form. We illustrate first the method in the one dimensional case. Let $f(x) = Ax + a_r \cdot x^r + o(x^r)$, where $r \geq 2$ and $o(x^r)$ denotes a function containing only the terms that are higher order than r , that is a function, for which $o(x^r)/x^r$ converges to zero as $x \rightarrow 0$. Let us look for the function h in the form $h(x) = x + h_r \cdot x^r$. Let us check, based on equation (3.23), whether the function g can be chosen in the form $g(x) = Ax + o(x^r)$. Since $h'(x) = 1 + r \cdot h_r \cdot x^{r-1}$, therefore assuming $g(x) = Ax + o(x^r)$ the left hand side of (3.23) takes the form

$$h'(x)g(x) = Ax + Arh_r x^r + o(x^r),$$

on the other hand, the right hand side of (3.23) is

$$f(h(x)) = Ax + Ah_r x^r + a_r x^r + o(x^r).$$

If the coefficients of the terms of order r are equal, then $Arh_r = Ah_r + a_r$, from which the unknown coefficient h_r of function h can be determined as

$$h_r = \frac{a_r}{A(r-1)},$$

if $A \neq 0$ holds. Thus in the one dimensional case, if the linear part is non-zero, then all the higher order terms can be transformed out, which means that the system is locally C^∞ conjugate to its linear part. In fact, using the above procedure the nonlinear terms can be transformed out step by step after each other as follows. First, the second degree polynomial $H_2(x) = x + h_2 \cdot x^2$ is used, with $h_2 = a_2/A$. Then the function G_2 takes the form $G_2(x) = Ax + o(x^2)$ and satisfies (3.23), that is $H'_2 \cdot G_2 = f \circ H_2$. Then for the function $G_2(x) = Ax + b_3x^3 + o(x^3)$ we determine a function $H_3(x) = x + h_3 \cdot x^3$. In this function $h_3 = b_3/2A$, hence $G_3(x) = Ax + o(x^3)$ and satisfies (3.23), that is $H'_3 \cdot G_3 = G_2 \circ H_3$. Now let $h = H_2 \circ H_3$, then $h' = (H'_2 \circ H_3) \cdot H'_3$, therefore

$$\begin{aligned} f \circ h &= f \circ H_2 \circ H_3 = (H'_2 \cdot G_2) \circ H_3 = (H'_2 \circ H_3) \cdot (G_2 \circ H_3) = \\ &= (H'_2 \circ H_3) \cdot H'_3 \cdot G_3 = h' \cdot G_3. \end{aligned}$$

Thus we have $f \circ h = h' \cdot G_3$, which means that the transformation $h = H_3 \circ H_2$ leads to an equation that does not contain second and third degree terms. Following this procedure, assume that the function G_{k-1} containing only terms of degree $k-1$ and higher, is already determined. Then by a suitable coefficient h_k of the function H_k we can achieve that the function G_k determined by the equation

$$H'_k \cdot G_k = G_{k-1} \circ H_k$$

contains no terms of degree smaller than k . Hence the mapping h given by the infinite composition $h = H_2 \circ H_3 \circ \dots$ transforms the equation to its linear part, i.e. equation (3.23) holds with the linear function $g(x) = Ax$. Here a difficult question arises, namely the convergence of the infinite composition. At the end of this section a theorem will be formulated about the convergence. Now, we extend the procedure to the case of n -dimensional systems.

Thus let us consider the system $\dot{x}(t) = f(x(t))$, the equilibrium of which is the origin, and assume that the linearised system, i.e. the Jacobian $A = f'(0)$ is a diagonal matrix. That is

$$f(x) = Ax + a(x) + o(x^r),$$

where

$$A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

is diagonal and the nonlinear part $a(x)$ contains only terms of degree r and higher, that is $a(x)$ can be expressed as the linear combination of terms in the form

$$e_i \cdot x_1^{m_1} \cdot x_2^{m_2} \cdot \dots \cdot x_n^{m_n}, \quad m_1 + m_2 + \dots + m_n = r,$$

where $e_i \in \mathbb{R}^n$ is the i -th unit vector (its i -th coordinate is 1, its other coordinates are 0). For example, in the case $n = 2$, $r = 2$ the function $a(x)$ is in the space

$$V_2 = \text{span} \left\{ \begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2^2 \end{pmatrix} \right\}.$$

Thus system $\dot{x}(t) = f(x(t))$ can be written in the form

$$\begin{aligned} \dot{x}_1 &= \lambda_1 x_1 + a_{20} x_1^2 + a_{11} x_1 x_2 + a_{02} x_2^2 + o(x^3), \\ \dot{x}_2 &= \lambda_2 x_2 + b_{20} x_1^2 + b_{11} x_1 x_2 + b_{02} x_2^2 + o(x^3). \end{aligned}$$

For an arbitrary n and r the function $a(x)$ is an element of the similarly defined space

$$V_r = \text{span}\{e_i \cdot x_1^{m_1} \cdot x_2^{m_2} \cdot \dots \cdot x_n^{m_n} : m_1 + m_2 + \dots + m_n = r, i = 1, 2, \dots, n.\}$$

The homeomorphism h is searched in the form $h(x) = x + H(x)$, where $H(x) \in V_r$ is also a linear combination of terms of order r . The aim is to choose a function H , for which equation (3.23) holds with a function g that can be written in the form $g(x) = Ax + o(x^r)$. The left hand side of equation (3.23) is

$$(f \circ h)(x) = Ax + AH(x) + a(x + H(x)) + o(x^r) = Ax + AH(x) + a(x) + o(x^r)$$

because $a(x + H(x)) = a(x) + o(x^r)$, by using the power series expansion of a . On the other hand

$$h'(x) \cdot g(x) = (I + H'(x)) \cdot (Ax + o(x^r)) = Ax + H'(x) \cdot Ax + o(x^r).$$

Since the terms of degree r are equal on the left and right hand side, we get the following equation for H

$$H'(x)Ax - AH(x) = a(x).$$

This equation is referred to as the homological equation. Let us introduce the linear mapping $L_A : V_r \rightarrow V_r$ that associates to a function H the left hand side of the homological equation, i.e.

$$(L_A H)(x) = H'(x)Ax - AH(x).$$

The homological equation has a unique solution to each function $a \in V_r$, if and only if L_A is bijective, that is the following proposition holds.

Proposition 3.2. *If 0 is not an eigenvalue of the mapping L_A , then the terms of degree r can be transformed out.*

Let us investigate the effect of the mapping L_A on the basis elements of V_r . Introducing the notation $x^m = x_1^{m_1} \cdot x_2^{m_2} \cdot \dots \cdot x_n^{m_n}$ one can prove the following statement.

Lemma 3.9..

$$L_A(e_i x^m) = e_i x^m \left(\sum_{k=1}^n m_k \lambda_k - \lambda_i \right),$$

where the numbers m_k are the coordinates of m , and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues in the diagonal of the matrix A .

Proof. Since the function H is given by

$$H(x) = e_i x^m = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_1^{m_1} x_2^{m_2} \dots x_n^{m_n} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

we have

$$AH(x) = e_i x^m = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_i x_1^{m_1} x_2^{m_2} \dots x_n^{m_n} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \lambda_i e_i x^m.$$

On the other hand

$$H'(x) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ m_1 \frac{x^m}{x_1} & m_2 \frac{x^m}{x_2} & \dots & m_n \frac{x^m}{x_n} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$Ax = \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \vdots \\ \lambda_n x_n \end{pmatrix},$$

implying

$$H'(x)Ax = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x^m \sum_{k=1}^n m_k \lambda_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} = e_i x^m \sum_{k=1}^n m_k \lambda_k.$$

The equations above yield

$$H'(x)Ax - AH(x) = e_i x^m \left(\sum_{k=1}^n m_k \lambda_k - \lambda_i \right),$$

that was to be proved. \square

According to this lemma the eigenvalues of the mapping L_A can be written in the form $\sum_{k=1}^n m_k \lambda_k - \lambda_i$, the corresponding eigenvectors are the functions of the form $e_i x^m$. Since these are exactly the functions that span the vector space V_r , we found so many eigenvalues as the dimension of the vector space V_r . Therefore all eigenvalues of L_A can be written in the form $\sum_{k=1}^n m_k \lambda_k - \lambda_i$. Hence if these are nonzero numbers, then the mapping L_A is bijective. Based on this formula let us introduce the following notion.

Definition 3.10.. *The set of eigenvalues of the matrix A are called resonant, if there exists $i \in \{1, 2, \dots, n\}$, and there are nonnegative integers m_1, m_2, \dots, m_n , for which $m_1 + m_2 + \dots + m_n \geq 2$ and $\lambda_i = \sum_{k=1}^n m_k \lambda_k$. If this holds, then the term $e_i x^m$ is called a resonant term.*

Thus if the eigenvalues of the matrix A are non-resonant, then the mapping L_A is bijective. This implies that there is a sequence of diffeomorphisms, the composition of which is a diffeomorphism that shows that the system and its linear part are C^∞ conjugate. (Consequently, they are C^k conjugate for any k .) The convergence of the infinite composition can be verified under suitable conditions. This is formulated in the next theorem.

Theorem 3.11. (Poincaré). *1. If the eigenvalues of the matrix A are non-resonant, then all non-linear terms can be formally transformed out from the system .*

2. If the eigenvalues of the matrix A are non-resonant and the convex hull of the eigenvalues in the complex plane does not contain the origin, then system $\dot{x} = f(x)$ and its linear part $\dot{y} = Ay$ are locally C^∞ conjugate at the origin.

Example 3.3. *If the system is $n = 2$ dimensional, then the resonant terms of degree 2 can be occur as follows.*

$$\begin{aligned} m = (0, 2), \quad 0 \cdot \lambda_1 + 2 \cdot \lambda_2 &= \lambda_1 \text{ or } \lambda_2 \\ m = (1, 1), \quad 1 \cdot \lambda_1 + 1 \cdot \lambda_2 &= \lambda_1 \text{ or } \lambda_2 \\ m = (2, 0), \quad 2 \cdot \lambda_1 + 0 \cdot \lambda_2 &= \lambda_1 \text{ or } \lambda_2 \end{aligned}$$

For example, in the case $\lambda_1 = 0$ or $\lambda_2 = 0$, we get resonance with $m = (1, 1)$, hence the term $x_1 x_2$ cannot be transformed out.

3.3 Exercises

1. If the system is $n = 2$ dimensional, and the eigenvalues of the matrix A are $\pm i$, then which of the terms below is resonant?

$$(A) \quad x_1 x_2, \quad (B) \quad x_1^2, \quad (C) \quad x_1 x_2^2$$

Answer: (C).

2. If the system is $n = 2$ dimensional, and the eigenvalues of the matrix A are ± 1 , then which of the terms below is resonant?

$$(A) \quad x_1 x_2, \quad (B) \quad x_1^2, \quad (C) \quad x_1 x_2^2$$

Answer: (C).

Chapter 4

Stable, unstable and center manifold theorems

For linear systems the stable, unstable and center subspaces were introduced in Definition 2.8.. These subspaces are invariant, that is trajectories do not leave them. In this section we show that in non-linear systems the role of invariant subspaces is taken over by invariant manifolds. The proof of the stable and unstable manifold theorem is less technical, this will be dealt with in the first section. The center manifold theorem and its applications are presented in a separate section.

4.1 Stable and unstable manifold theorem

As a motivation for the theorem let us investigate first the following two simple systems.

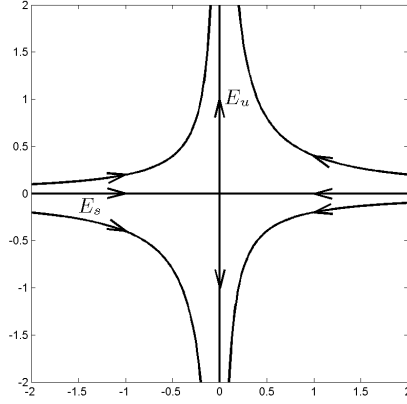
Example 4.1. *Let us consider the linear system*

$$\begin{aligned}\dot{x}_1 &= -x_1 \\ \dot{x}_2 &= x_2\end{aligned}$$

that has a saddle point. In this system the stable subspace E_s is the horizontal axis, while the unstable subspace E_u is the vertical axis as it is shown in Figure 4.1. Note that there is no other invariant one-dimensional subspace (i.e. there is no other line through the origin that is not left by the trajectories).

Stable and unstable subspaces in Example 4.1.

The example below shows how the invariant subspaces are changed by a non-linear perturbation of the above linear system.



Example 4.2. *Let us consider the non-linear system*

$$\begin{aligned}\dot{x}_1 &= -x_1, \\ \dot{x}_2 &= x_2 + x_1^2.\end{aligned}$$

The first equation is independent, its solution can be given as $x_1(t) = e^{-t}c_1$. Substituting this solution into the second equation we arrive to an inhomogeneous linear differential equation that can be solved as $x_2(t) = e^t \cdot c_2 + \frac{c_1^2}{3}(e^t - e^{-2t})$. These solutions satisfy the initial conditions $x_1(0) = c_1$, $x_2(0) = c_2$. Note that if $c_2 + \frac{c_1^2}{3} = 0$ holds for the initial conditions, then for any time t the same relation $x_2(t) + \frac{x_1^2(t)}{3} = 0$ holds. That is the set

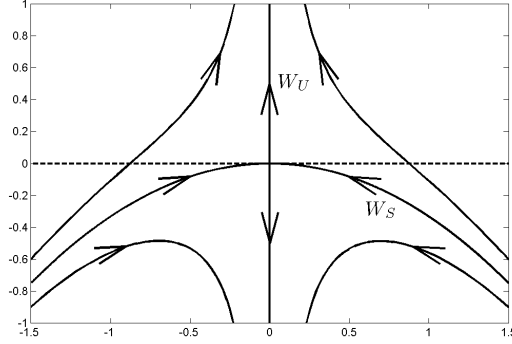
$$W_s = \{(c_1, c_2) \in \mathbb{R}^2 : c_2 + \frac{c_1^2}{3} = 0\}$$

is invariant, i.e. the trajectories do not leave it. The solutions starting from this set tend to the origin as time goes to infinity. This is why it is called stable manifold, this is the set that took over the role of the stable subspace. The invariant manifold is easier to determine, since the vertical axis is invariant and along this line trajectories tend to infinity. Thus the invariant manifold is the $W_u = \{(0, c_2) \in \mathbb{R}^2 : c_2 \in \mathbb{R}\}$ subspace. The invariant manifolds are shown in Figure 4.1.

Stable and unstable manifolds in Example 4.2.

4.1.1 General approach

Based on the above motivating examples let us turn now to the general case. In order to avoid the technical definition of differentiable manifolds, they will be substituted by



graphs of differentiable functions. We note that the manifold generalises the notion of curves and surfaces, hence instead of the abstract notion we can think of these simple geometrical objects. If the Reader wants to know more about manifolds, then we suggest to read Section 2.7 in Perko's book [19].

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuously differentiable function, for which $f(0) = 0$. Hence the origin is an equilibrium of system $\dot{x} = f(x)$. The solution starting from point p is denoted by $\varphi(t, p)$. Assume that the Jacobian $f'(0)$ has no eigenvalues with zero real part, i.e. the origin is a hyperbolic equilibrium. The dimension of the stable subspace E_s is denoted by k , and that of the unstable subspace E_u is $n - k$.

Theorem 4.1. (Stable and unstable manifold). *There is a neighbourhood U of the origin and there exist continuously differentiable functions $\Psi : E_s \cap U \rightarrow E_u$ and $\Phi : E_u \cap U \rightarrow E_s$, for which the k -dimensional local stable and $n - k$ -dimensional local unstable manifolds*

$$W_s^{loc} = \{(q, \Psi(q)) \in \mathbb{R}^n : q \in E_s \cap U\} \quad \text{and} \quad W_u^{loc} = \{(r, \Phi(r)) \in \mathbb{R}^n : r \in E_u \cap U\}$$

have the following properties. (We note that the vectors q and $\Phi(r)$ above have k coordinates, while the vectors r and $\Psi(q)$ have $n - k$ coordinates.)

1. W_s^{loc} is positively invariant, W_u^{loc} is negatively invariant.
2. The manifold W_s^{loc} is tangential to the subspace E_s at the origin, and the manifold W_u^{loc} is tangential to the subspace E_u at the origin.
3. If $p \in W_s^{loc}$, then $\lim_{t \rightarrow \infty} \varphi(t, p) = 0$.
4. If $p \in W_u^{loc}$, then $\lim_{t \rightarrow -\infty} \varphi(t, p) = 0$.

Proof. It is enough to prove the statement concerning the stable manifold, because the unstable manifold of the system $\dot{x} = f(x)$ is the same as the stable manifold of the system $\dot{x} = -f(x)$. This part of the proof will be divided into three steps.

STEP 1.

First, the linear part is transformed to Jordan canonical form. The Jordan canonical form of the matrix $f'(0)$ is:

$$J = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix},$$

where the real part of the eigenvalues of $B \in \mathbb{R}^{k \times k}$ are negative and the real part of the eigenvalues of $C \in \mathbb{R}^{n-k \times n-k}$ are positive. Let P be the matrix that transforms the Jacobian $f'(0)$ to Jordan canonical form, that is $Pf'(0)P^{-1} = J$. Introducing the new variable $\tilde{x} = Px$ one obtains

$$\begin{aligned} \dot{\tilde{x}} &= P\dot{x} = Pf(x) = Pf'(0)x + P(f(x) - f'(0)x) = \\ &= Pf'(0)P^{-1}\tilde{x} + P(f(P^{-1}\tilde{x}) - f'(0)P^{-1}\tilde{x}) \end{aligned}$$

that is

$$\dot{\tilde{x}} = J\tilde{x} + a(\tilde{x}), \quad (4.1)$$

where $a(0) = 0 \in \mathbb{R}^n$ and $a'(0) = 0 \in \mathbb{R}^{n \times n}$. Let us now consider the differential equation for the function \tilde{x} . The stable and unstable subspaces of this equation are

$$\tilde{E}_s = \{p \in \mathbb{R}^n : p_{k+1} = p_{k+2} = \dots p_n = 0\}, \quad \tilde{E}_u = \{p \in \mathbb{R}^n : p_1 = p_2 = \dots p_k = 0\}.$$

Thus the phase space \mathbb{R}^n can be split up as a direct sum of a k dimensional and an $n - k$ dimensional subspace. In order to define the stable manifold, the continuously differentiable functions $\psi_{k+1}, \psi_{k+2}, \dots, \psi_n : E_s \cap U \rightarrow \mathbb{R}$ of k variable will be given, in such a way that the set

$$\begin{aligned} \tilde{W}_s^{loc} &= \{(p_1, p_2, \dots, p_k, \psi_{k+1}(p_1, \dots, p_k), \dots, \psi_n(p_1, \dots, p_k)) : \\ &\quad (p_1, \dots, p_k, 0, \dots, 0) \in E_s \cap U\} \end{aligned}$$

will be positively invariant and the trajectories in it tend to the origin. Figure 4.1 shows the graph of ψ_3 in the case $n = 3$ and $k = 2$, that determines the two-dimensional stable manifold.

Since the new and the original variables are related by the transformation $\tilde{x} = Px$, the stable manifold \tilde{W}_s^{loc} corresponding to equation (4.1) yields the stable manifold of the original equation through the linear transformation $P^{-1}\tilde{W}_s^{loc} = W_s^{loc}$. Thus it is enough to prove the theorem for the equation (4.1), that is in the case when the linear part is in Jordan canonical form. In the next steps we assume that the linear part is in Jordan canonical form, hence we use the notation W_s^{loc} instead of \tilde{W}_s^{loc} .

STEP 2

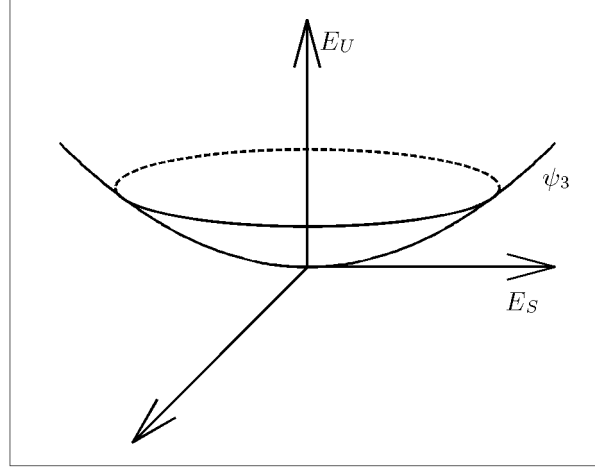


Figure 4.1: In the case $n = 3$ and $k = 2$ the stable manifold is given by the graph of the function ψ_3 .

Since system (4.1) is in the form of a direct sum, we introduce the following notations. For an arbitrary $v \in \mathbb{R}^n$ let $v = v_s + v_u$, where

$$v_s = \begin{pmatrix} v_1 \\ \vdots \\ v_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad v_u = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v_{k+1} \\ \vdots \\ v_n \end{pmatrix}.$$

Moreover, let

$$J_s = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \quad J_u = \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}.$$

If x is a solution of system (4.1) and $a = a_s + a_u$, then for the functions x_s and x_u

$$\dot{x}_s = J_s x_s + a_s(x) \tag{4.2}$$

$$\dot{x}_u = J_u x_u + a_u(x) \tag{4.3}$$

hold. Applying the variation of constants formula to equation (4.1) one obtains

$$x(t) = e^{Jt} p + \int_0^t e^{J(t-\tau)} a(x(\tau)) d\tau,$$

in which

$$e^{Jt} = \begin{pmatrix} e^{Bt} & 0 \\ 0 & e^{Ct} \end{pmatrix}.$$

The variation of constants formula can also be applied to equations (4.2) and (4.3).

$$x_s(t) = e^{J_s t} p_s + \int_0^t e^{J_s(t-\tau)} a(x(\tau)) d\tau \quad (4.4)$$

$$x_u(t) = e^{J_u t} p_u + \int_0^t e^{J_u(t-\tau)} a(x(\tau)) d\tau, \quad (4.5)$$

where

$$e^{J_s t} = \begin{pmatrix} e^{Bt} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad e^{J_u t} = \begin{pmatrix} 0 & 0 \\ 0 & e^{Ct} \end{pmatrix},$$

therefore $e^{J_s t} a_u = 0$, that is $e^{J_s t} a_s = e^{J_s t} a$, and similarly, because of $e^{J_u t} a_s = 0$ we get $e^{J_u t} a_u = e^{J_u t} a$.

In a point $p = p_s + p_u$ of the stable manifold $p_s = (p_1, p_2, \dots, p_k, 0, \dots, 0)^T$,

$$p_u = (0, \dots, 0, \psi_{k+1}(p_1, \dots, p_k), \dots, \psi_n(p_1, \dots, p_k))^T,$$

and the latter has to be chosen in such a way that the solution starting from p should converge to the origin, i.e. $\lim_{t \rightarrow \infty} x_u(t) = 0$ holds. According to equation (4.5), this can hold if $p_u + \int_0^\infty e^{-J_u \tau} a(x(\tau)) d\tau = 0$, since the eigenvalues of matrix C have positive real part, hence the norm of the matrix $e^{J_u t}$ tends to infinity. If p_u is chosen this way, then

$$x_u(t) = -e^{J_u t} \int_t^\infty e^{-J_u \tau} a(x(\tau)) d\tau.$$

Substituting this expression into equation $x = x_s + x_u$ we get

$$x(t) = e^{J_s t} p_s + \int_0^t e^{J_s(t-\tau)} a(x(\tau)) d\tau - \int_t^\infty e^{J_u(t-\tau)} a(x(\tau)) d\tau. \quad (4.6)$$

It will be shown in STEP 3 that there is suitable neighbourhood U of the origin, such that for all $p_s \in E_s \cap U$ the above equation has a solution x . Then let $\psi_j(p_1, p_2, \dots, p_k) := x_j(0)$, $j = k+1, \dots, n$. The stable manifold W_s^{loc} defined by these functions has the properties listed in the theorem, because an estimate to be verified also in STEP 3, implies that for all $p \in W_s^{loc}$ the function x given by (4.6) satisfies $\lim_{t \rightarrow \infty} x(t) = 0$, that is $\lim_{t \rightarrow \infty} \varphi(t, p) = 0$. On the other hand, the above derivation for (4.6) shows that in the case $p \notin W_s^{loc}$ the limit condition $\lim_{t \rightarrow \infty} \varphi(t, p) = 0$ cannot hold, therefore W_s^{loc} is positively invariant. Namely, if a solution starting from a point $p \in W_s^{loc}$ left the manifold W_s^{loc} , then it would not tend to zero as $t \rightarrow \infty$.

STEP 3

Here we prove that if $p_s \in E_s$ is close enough to the origin, then there exists a function x satisfying equation (4.6) and for which $\lim_{t \rightarrow \infty} x(t) = 0$ holds. The existence of x will be proved by successive approximation. Consider the space $X = C_b([0, +\infty), \mathbb{R}^n)$ of bounded and continuous functions endowed with the norm $\|x\| = \sup_{[0, +\infty)} |x|$. Introduce the operator $T : X \rightarrow X$ based on the right hand side of equation (4.6) as

$$(T(x))(t) = e^{J_s t} p_s + \int_0^t e^{J_s(t-\tau)} a(x(\tau)) d\tau - \int_t^\infty e^{J_u(t-\tau)} a(x(\tau)) d\tau.$$

It is obvious that T is defined on the whole space X , and it is easy to prove that it maps to X , i.e. it maps the space X into itself. Our aim is to show that it has a fixed point x that will be the solution of equation (4.6). Let $x_0 \in X$, $x_0 \equiv 0$ and define the sequence of functions $(x_n) \subset X$ recursively by $x_{n+1} = T(x_n)$. It can be shown by induction that

$$|x_{n+1}(t) - x_n(t)| \leq \frac{K|p_s|e^{-\alpha t}}{2^n},$$

where $K > 0$ and $\alpha > 0$ are constants, for which the estimates

$$\|e^{J_s t}\| \leq K e^{-(\alpha+\sigma)t}, \quad \|e^{-J_u t}\| \leq K e^{-\sigma t} \quad \square$$

hold for all $t \geq 0$ with some $\sigma > 0$. Hence $(x_n) \subset X$ is a Cauchy sequence, therefore it converges to a point $x \in X$, because X is a complete normed space. It can be shown that T is continuous, hence taking $n \rightarrow \infty$ in the recursion $x_{n+1} = T(x_n)$ we get $x = T(x)$ yielding the desired fixed point. Moreover, it can be shown also by induction that $\|x_n(t)\| \leq C e^{-\alpha t}$ for all $t \geq 0$, hence taking again the limit $n \rightarrow \infty$ we get the same estimate also for x , proving that x tends to zero as $t \rightarrow \infty$.

4.1.2 Global stable and unstable manifolds

The global stable and unstable manifolds can be defined by using the local manifolds W_s^{loc} and W_u^{loc} . The global manifolds typically cannot be given as graphs of suitable functions. The stable manifold is defined as the set of points from which the solution tends to the origin. Since these trajectories lie in the local stable manifold when they are close to the origin, it is reasonable to define the global stable manifold as follows

$$W_s := \bigcup_{t \leq 0} \varphi(t, W_s^{loc}).$$

The unstable manifold is the set of points from which the solution tends to the origin as $t \rightarrow -\infty$. These trajectories lie in the local unstable manifold when they are close to the origin, hence the global unstable manifold is defined as follows

$$W_u := \bigcup_{t \geq 0} \varphi(t, W_u^{loc}).$$

The global stable and unstable manifolds may have common points, this situation is shown in Figure 4.2. The system, the phase portrait of which is shown in the Figure, has a homoclinic orbit that is the intersection of the stable and unstable manifolds.

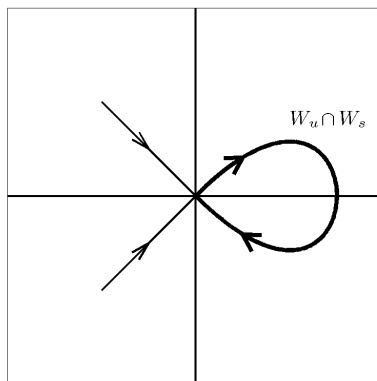


Figure 4.2: Homoclinic orbit as the intersection of the stable and unstable manifolds.

4.2 Center manifold theorem

Before formulating the center manifold theorem let us consider the following motivating example.

Example 4.3. *Consider the system*

$$\begin{aligned}\dot{x} &= xy + x^3 \\ \dot{y} &= -y - 2x^2\end{aligned}$$

and investigate its local phase portrait in the neighbourhood of the origin. The Jacobian obtained by linearisation is $\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$, its eigenvalues are 0 and -1 . Thus the origin is not a hyperbolic equilibrium, therefore the linearisation does not determine the local phase portrait. Since there is negative eigenvalue, the system has a one dimensional stable manifold, along which the trajectories tend to the origin. At the end of this section it will be shown that there is an invariant center manifold that is tangential to the center subspace belonging to the eigenvalue 0, and the behaviour of the trajectories in this manifold can easily be determined by investigating the phase portrait of a one-dimensional system.

4.2.1 General approach

Let us consider again a general autonomous system $\dot{x} = f(x)$, and assume that the origin is an equilibrium, that is $f(0) = 0$, and the Jacobian is written in the form $f'(0) = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$, where the eigenvalues of B have zero real part, the eigenvalues of C have non-zero real part. Thus using the notation $x = \begin{pmatrix} y \\ z \end{pmatrix}$ the system takes the form

$$\dot{y} = By + g(y, z) \quad (4.7)$$

$$\dot{z} = Cz + h(y, z), \quad (4.8)$$

where it is assumed that the derivatives of g and h are zero at the origin.

Theorem 4.2. (Center manifold). *There is a neighbourhood U of the origin and there exists a differentiable map $\psi : E_c \cap U \rightarrow E_s \oplus E_u$ that satisfy the following conditions.*

1. $\psi(0) = 0, \psi'(0) = 0$.
2. *The local center manifold $W_c^{loc} = \{(p_c, \psi(p_c)) : p_c \in E_c \cap U\}$ is locally invariant, i.e. if $p \in W_c^{loc}$ and $\varphi(t, p) \in U$, then $\varphi(t, p) \in W_c^{loc}$.*

Proof. The technical details make the proof of the theorem considerably long, hence we present here only the main ideas of the proof based on the book by Chow and Hale [8].

Consider a solution $\begin{pmatrix} y(t) \\ z(t) \end{pmatrix}$ starting from the point $(p_c, \psi(p_c))$. The invariance of the manifold requires

$$\dot{y} = By + g(y, \psi(y)) \quad (4.9)$$

$$\dot{z} = Cz + h(y, \psi(y)). \quad (4.10)$$

Applying the variation of constants formula to the second equation with a starting point $t = \tau$

$$z(t) = e^{C(t-\tau)} \cdot z(\tau) + \int_{\tau}^t e^{C(t-s)} h(y(s), \psi(y(s))) ds.$$

Let $\tau < 0$, then substituting $t = 0$ into this equation

$$\psi(p_c) = z(0) = e^{-C\tau} \cdot z(\tau) + \int_{\tau}^0 e^{-Cs} h(y(s), \psi(y(s))) ds. \quad (4.11)$$

For simplicity, let us consider only the case when the eigenvalues of C have negative real part. (The general case can be dealt with in a similar way.) For the solutions in the center manifold $z(t)$ cannot tend to infinity as $t \rightarrow -\infty$, therefore

$$\lim_{\tau \rightarrow -\infty} e^{-C\tau} z(\tau) = 0,$$

hence equation (4.11) implies

$$\psi(p_c) = \int_{-\infty}^0 e^{-Cs} h(y(s), \psi(y(s))) ds.$$

Thus the procedure is as follows. For a given function ψ solve differential equation (4.9) subject to the initial condition $y(0) = p_c$, and define the operator

$$(T(\psi))(p_c) = \int_{-\infty}^0 e^{-Cs} h(y(s), \psi(y(s))) ds.$$

Choosing an appropriate Banach space this operator is a contraction, hence by Banach's fixed point theorem it has a fixed point. This fixed point yields the function ψ determining the local center manifold. \square

The following corollary of the center manifold theorem, called the center manifold reduction theorem, enables us to determine the local phase portrait in the neighbourhood of a non-hyperbolic equilibrium. The proof of this theorem can be found in the book by Carr [5].

Theorem 4.3. (Center manifold reduction). *Consider system (4.7)-(4.8) and assume that the above assumptions hold. Let ψ be the function determining the local center manifold. By using the so-called center manifold reduction introduce the system*

$$\dot{u} = Bu + g(u, \psi(u)) \tag{4.12}$$

$$\dot{v} = Cv, \tag{4.13}$$

in which the linearisation is used in the hyperbolic part. Then system (4.7)-(4.8) and system (4.12)-(4.13) are locally topologically equivalent in the origin.

Remark 4.1. *This theorem can be considered as the generalisation of the Hartman–Grobman theorem to the case of non-hyperbolic linear part. The theorem enables us to reduce the dimension, because the phase portrait of the linear system $\dot{v} = Cv$ can easily be determined, hence in order to characterise the full phase portrait it is enough to determine the phase portrait of the lower dimensional non-linear system $\dot{u} = Bu + g(u, \psi(u))$.*

In the next subsection it is shown how center manifold reduction can be used to characterise the local phase portrait.

4.2.2 Approximation of the center manifold

In order to apply center manifold reduction the function ψ determining the center manifold has to be determined. In most of the cases it is not possible to calculate this function

explicitly. In this subsection it is shown that it is enough to calculate an approximation of the center manifold and even this approximation enables us to use the center manifold reduction.

Assume that the solution $x(t) = \begin{pmatrix} y(t) \\ z(t) \end{pmatrix}$ lies in the center manifold. The invariance of the manifold implies $z(t) = \psi(y(t))$. Differentiating this equation we get $\dot{z} = \psi'(y) \cdot \dot{y}$, therefore

$$\dot{z} = Cy + h(y, \psi(y)) = \psi'(y) \cdot (By + g(y, \psi(y))), \quad (4.14)$$

which can be considered as an equation determining the function ψ . This equation does not enable us to use Banach's fixed point theorem to prove the existence of the center manifold, however, it is useful in calculating the coefficients of the power series expansion of the function ψ . According to the following theorem, if this equation holds for the terms in the power series of a function $\tilde{\psi}$ up to degree r , then this function approximates ψ in order r .

Theorem 4.4. (Approximation of the center manifold). *Let $\tilde{\psi} : E_c \cap U \rightarrow E_s \oplus E_u$ be a function, for which*

1. $\tilde{\psi}(0) = 0, \tilde{\psi}'(0) = 0$.
2. (4.14) holds in order r .

Then $|\psi(y) - \tilde{\psi}(y)| = O(|y|^r)$, that is the power series expansions of ψ and $\tilde{\psi}$ coincide up to degree r .

As an application of the above results let us consider again the example shown at the beginning of this section.

Example 4.4.

$$\dot{x} = xy + x^3 \quad (4.15)$$

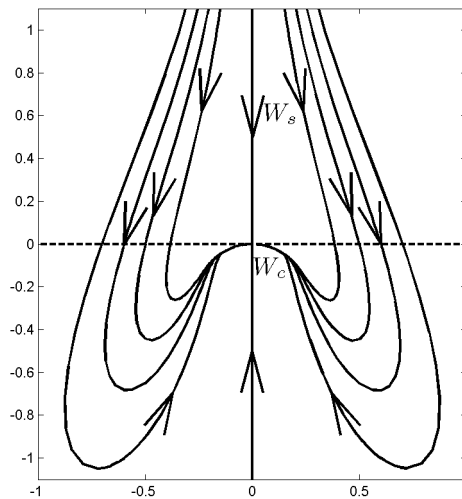
$$\dot{y} = -y - 2x^2. \quad (4.16)$$

The Jacobian is $f'(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$, the stable subspace E_s is determined by $(0,1)$, the center subspace E_c is given by $(1,0)$. The approximation of the function ψ determining the center manifold can be given in the form $\psi(x) = a_2x^2 + a_3x^3 + \dots$, because the center manifold is tangential to the center subspace at the origin, that is ψ can be given as a function of x and $\psi(0) = 0 = \psi'(0)$. The invariance of the manifold implies $y(t) = \psi(x(t))$, the derivative of which yields $\dot{y}(t) = \psi'(x(t)) \cdot \dot{x}(t)$. Substituting the derivatives of x and y from the differential equations to this equation one obtains $-\psi(x) - 2x^2 = \psi'(x)(x \cdot \psi(x) + x^3)$. Using the power series expansion of ψ the coefficients of the corresponding terms are equal in the left and right hand sides. Using the coefficients of

the quadratic term x^2 on both sides we get $-a_2 - 2 = 0$ yielding $a_2 = -2$. Hence the function defining the center manifold can be given as $\psi(x) = -2x^2 + O(x^3)$. Thus the approximation of the center manifold up to second degree can be given as $\tilde{\psi}(x) = -2x^2$. Substituting $\psi(x) = -2x^2 + O(x^3)$ into the first equation of the reduced system

$$\dot{x} = x(-2x^2 + a_3x^3 + \dots) + x^3 = -x^3 + O(x^4)$$

The local phase portrait at the origin does not depend on the terms of $O(x^4)$, because they do not have influence on the direction field. In this one-dimensional system the trajectories tend to the origin, hence along the center manifold the trajectories tend to the origin. Since the other eigenvalue of the system is negative, all solutions in a neighbourhood of the origin tend to the origin, thus according to the center manifold reduction the origin is asymptotically stable. The phase portrait is shown in Figure 4.4.



The phase portrait of system (4.15)-(4.16).

4.3 Exercises

1. Determine the steady states and the local phase portraits at those points in the following two-dimensional systems.

(a) $\dot{x} = y, \dot{y} = -\sin x - 3y$

Answer: The equilibria are $(k\pi, 0)$ for all $k \in \mathbb{Z}$. The Jacobian of the system is

$$J = \begin{pmatrix} 0 & 1 \\ -\cos x & -3 \end{pmatrix}.$$

The trace of the Jacobian is $Tr(J) = -3$, its determinant is $Det(J) = \cos x$. Using these we can get the sign of the real parts of the eigenvalues. For even values of k the steady state is a stable node, for odd values of k it is a saddle.

(b) $\dot{x} = y^2 - 1, \dot{y} = x^2 + y^2 - 2$

Answer: Starting from the first equation the second coordinates of the equilibria are ± 1 , then the second equation yields that the first coordinates are also ± 1 independently of the first coordinate. Hence the steady states are $(1, 1), (1, -1), (-1, 1), (-1, -1)$. The Jacobian of the system is

$$J = \begin{pmatrix} 0 & 2y \\ 2x & 2y \end{pmatrix}$$

hence $Tr(J) = 2y, Det(J) = -4xy, Tr^2(J) - 4Det(J) = 4y(y + 4x)$. At the points $(1, 1)$ and $(-1, -1)$ we have $Det(J) < 0$, hence these are saddle points. At the point $(1, -1)$ we have $Tr(J) < 0, Det(J) > 0$ and $Tr^2(J) - 4Det(J) < 0$, hence this is a stable focus. Finally, at the point $(-1, 1)$ we have $Tr(J) > 0, Det(J) > 0$ and $Tr^2(J) - 4Det(J) < 0$, hence this is an unstable focus.

(c) $\dot{x} = x^2 + y^2 - 25, \dot{y} = xy - 12$

Answer: The second equation yields $y = 12/x$. Substituting this into the first equation $x^4 - 25x^2 + 12^2 = 0$, implying $x = \pm 3$ or $x = \pm 4$. The corresponding values of y are given by $y = 12/x$. Hence the steady states are $(3, 4), (-3, -4), (4, 3), (-4, -3)$. The Jacobian of the system is

$$J = \begin{pmatrix} 2x & 2y \\ y & x \end{pmatrix},$$

hence $Tr(J) = 3x, Det(J) = 2(x^2 - y^2), Tr^2(J) - 4Det(J) = x^2 + 8y^2 > 0$. At the points $(3, 4)$ and $(-3, -4)$ we have $Det(J) < 0$, hence these are saddle points. At the point $(4, 3)$ we have $Tr(J) > 0, Det(J) > 0$ and $Tr^2(J) - 4Det(J) > 0$, hence it is a stable node. Finally, at the point $(-4, -3)$ we have $Tr(J) < 0, Det(J) > 0$ and $Tr^2(J) - 4Det(J) > 0$, hence it is a stable node.

(d) $\dot{x} = -y, \dot{y} = x^3 - x + xy$

Answer: The first equation yields $y = 0$, then from the second $x = 0, x = 1$ or $x = -1$ follows. Hence the steady states are $(0, 0), (1, 0), (-1, 0)$. The Jacobian of the system is

$$J = \begin{pmatrix} 0 & -1 \\ 3x^2 - 1 + y & x \end{pmatrix},$$

hence $Tr(J) = x, Det(J) = 3x^2 - 1 + y, Tr^2(J) - 4Det(J) = 4 - 11x^2 - 4y$. At the point $(0, 0)$ we have $Det(J) < 0$, hence it is a saddle. At the point $(1, 0)$

we have $Tr(J) > 0$, $Det(J) > 0$ and $Tr^2(J) - 4Det(J) < 0$, hence it is an unstable focus. Finally, at the point $(-1, 0)$ we have $Tr(J) < 0$, $Det(J) > 0$ and $Tr^2(J) - 4Det(J) < 0$, hence it is a stable focus.

(e) $\dot{x} = y - x^2 - x$, $\dot{y} = 3x - x^2 - y$

Answer: Adding the two equations we get $2y - 4x = 0$, that is $y = 2x$. Substituting this into the first equation $x - x^2 = 0$, yielding $x = 0$ or $x = 1$. Hence the steady states are $(0, 0)$ and $(1, 2)$. The Jacobian of the system is

$$J = \begin{pmatrix} -2x - 1 & 1 \\ 3 - 2x & -1 \end{pmatrix}.$$

At the point $(0, 0)$ we have $Tr(J(0, 0)) = -2$, $Det(J(0, 0)) = -2$, hence the point $(0, 0)$ is a saddle. At the point $(1, 2)$ we have $Tr(J(1, 2)) = -4$, $Det(J(1, 2)) = 2$, $Tr^2(J(1, 2)) - 4Det(J(1, 2)) > 0$, hence the point $(1, 2)$ is a stable node.

2. Determine the steady states, and the dimension of the stable unstable and center subspaces at the steady states in the following three-dimensional systems.

(a) $\dot{x} = y$, $\dot{y} = z$, $\dot{z} = x^2 - yz - 1$

Answer: The first two equations yield $y = 0$ and $z = 0$, hence from the third one we get $x = 1$ or $x = -1$. Hence the steady states are $(1, 0, 0)$ and $(-1, 0, 0)$. The Jacobian of the system is

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2x & -z & -y \end{pmatrix},$$

its characteristic equation at the points $(\pm 1, 0, 0)$ are $\lambda^3 = \pm 2$. At the point $(1, 0, 0)$ the eigenvalues of the Jacobian (i.e. the solutions of equation $\lambda^3 = 2$) are $\lambda_1 = \sqrt[3]{2}$, $\lambda_{2,3} = \sqrt[3]{2}(\cos(2\pi/3) \pm i \sin(2\pi/3))$. Therefore $\dim(E_s) = 2$ and $\dim(E_u) = 1$. At the point $(-1, 0, 0)$ the eigenvalues of the Jacobian (i.e. the solutions of equation $\lambda^3 = -2$) are $\lambda_1 = -\sqrt[3]{2}$, $\lambda_{2,3} = -\sqrt[3]{2}(\cos(2\pi/3) \pm i \sin(2\pi/3))$. Therefore $\dim(E_s) = 1$ and $\dim(E_u) = 2$.

(b) $\dot{x} = y + z$, $\dot{y} = x^2 - 2y$, $\dot{z} = x + y$

Answer: The third equation yields $y = -x$, then the second yields $x^2 + 2x = 0$, hence $x = 0$ or $x = -2$. Then $y = 0$ and $y = 2$, and from the first equation $z = 0$ and $z = -2$. Hence the steady states are $(0, 0, 0)$ and $(-2, 2, -2)$. The Jacobian of the system is

$$J = \begin{pmatrix} 0 & 1 & 1 \\ 2x & -2 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Its characteristic equation is $\lambda^3 + 2\lambda^2 - \lambda(1 + 2x) - 2(1 + x) = 0$. At the point $(0, 0, 0)$ the characteristic equation is $\lambda^3 + 2\lambda^2 - \lambda - 2 = 0$. Observe that one of its roots is $\lambda_1 = 1$, hence $\lambda^3 + 2\lambda^2 - \lambda - 2 = (\lambda - 1)(\lambda^2 + 3\lambda + 2)$. The roots of the quadratic term are $\lambda_2 = 1, \lambda_3 = 2$. Hence at the point $(0, 0, 0)$ we have $\dim(E_u) = 3$. At the points $(-2, 2, -2)$ the characteristic equation is $\lambda^3 + 2\lambda^2 + 3\lambda + 2 = 0$. Observe that one of its roots is $\lambda_1 = -1$, hence $\lambda^3 + 2\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda^2 + \lambda + 2)$. The roots of the quadratic term have negative real part, therefore at the point $(-2, 2, -2)$ we have $\dim(E_s) = 3$.

3. In the Lorenz system $\dot{x} = \sigma(y - x), \dot{y} = \rho x - y - xz, \dot{z} = -\beta z + xy$ determine the dimension of the stable unstable and center subspaces at the origin for different values of the parameters $\sigma, \rho, \beta > 0$.

Answer: The Jacobian at the origin is

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}.$$

Its characteristic equation is $\lambda^3 + \lambda^2(\beta + \sigma + 1) + \lambda(\beta(\sigma + 1) + \sigma(1 - \rho)) + \beta\sigma(1 - \rho) = 0$. One of its roots is β , hence the other two roots can be obtained as a solution of a quadratic equation as

$$\lambda_{1,2} = \frac{1}{2}(-1 - \sigma \pm \sqrt{(1 + \sigma)^2 - 4\sigma(1 - \rho)})$$

It is easy to see, that for $\rho < 1$ all the three roots are negative, that is the stable subspace is three dimensional. If $\rho = 1$, then one of the eigenvalues is zero, the other two eigenvalues are negative, hence the stable subspace is two dimensional, and the dimension of the center subspace is 1. If $\rho > 1$, then $\lambda_1 > 0, \lambda_2, \lambda_3 < 0$, hence the unstable subspace is one dimensional, and the stable subspace is two dimensional.

4. In the Lorenz system $\dot{x} = \sigma(y - x), \dot{y} = \rho x - y - xz, \dot{z} = -\beta z + xy$ determine the steady states and their stability for different values of the parameters $\sigma, \rho, \beta > 0$.

Answer: The first equation yields $y = x$, then the last implies $z = x^2/\beta$. Substituting these into the second equation $x(\rho - 1)\beta = x^3$. Then $x_1 = 0$ is a solution for any values of the parameters, and for $\rho > 1$ $x_{2,3} = \pm\sqrt{(\rho - 1)\beta}$ are also solutions. Thus for $\rho \leq 1$ the only steady state is $(0, 0, 0)$, while for $\rho > 1$ there are two more equilibria $(x_2, x_2, \rho - 1)$ and $(x_3, x_3, \rho - 1)$. The Jacobian of the system is

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -\beta \end{pmatrix}.$$

The characteristic equation of the Jacobian at the origin is $\lambda^3 + \lambda^2(\beta + \sigma + 1) + \lambda(\beta(\sigma + 1) + \sigma(1 - \rho)) + \beta\sigma(1 - \rho) = 0$. Let us apply now the Routh–Hurwitz criterion to decide the stability of the origin. From the coefficients of the characteristic polynomial one can build up the Routh–Hurwitz matrix as follows.

$$\begin{pmatrix} \beta + \sigma + 1 & 1 & 0 \\ \beta\sigma(1 - \rho) & \beta(\sigma + 1) + \sigma(1 - \rho) & \beta + \sigma + 1 \\ 0 & 0 & \beta\sigma(1 - \rho) \end{pmatrix}$$

All the eigenvalues have negative real part if and only if this matrix is positive definite, that is when the leading principal minors Δ_1 , Δ_2 , Δ_3 are all positive. Since $\Delta_1 = \beta + \sigma + 1 > 0$, and $\Delta_3 = \Delta_2\beta\sigma(1 - \rho)$, the point can be stable when $\Delta_2 > 0$ and $\rho < 1$. For Δ_2 we have

$$\begin{aligned} \Delta_2 &= (\beta + \sigma + 1)(\beta(\sigma + 1) + \sigma(1 - \rho)) - \beta\sigma(1 - \rho) \\ &= (\beta^2 + \beta(\sigma + 1) + \sigma(1 - \rho))(\sigma + 1) \end{aligned}$$

which is positive, if $\rho < 1$. Hence the origin is asymptotically stable for $\rho < 1$ and unstable for $\rho > 1$. The Jacobian at the point $(x_2, x_2, \rho - 1)$ is

$$\begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{\beta(\rho - 1)} \\ \sqrt{\beta(\rho - 1)} & \sqrt{\beta(\rho - 1)} & -\beta \end{pmatrix}.$$

Its characteristic equation is

$$\lambda^3 + \lambda^2(\beta + \sigma + 1) + \lambda\beta(\sigma + \rho) + 2\beta\sigma(\rho - 1) = 0.$$

Let us apply again the Routh–Hurwitz criterion to decide the stability of the point $(x_2, x_2, \rho - 1)$. From the coefficients of the characteristic polynomial one can build up the Routh–Hurwitz matrix as follows.

$$\begin{pmatrix} \beta + \sigma + 1 & 1 & 0 \\ 2\beta\sigma(\rho - 1) & \beta(\sigma + \rho) & \beta + \sigma + 1 \\ 0 & 0 & 2\beta\sigma(\rho - 1) \end{pmatrix}$$

All the eigenvalues have negative real part if and only if this matrix is positive definite, that is when the leading principal minors Δ_1 , Δ_2 , Δ_3 are all positive. Since $\Delta_1 = \beta + \sigma + 1 > 0$ and $\Delta_3 = 2\beta\sigma(\rho - 1)\Delta_2$, the point can be stable when $\Delta_2 > 0$ and $\rho > 1$. The determinant

$$\begin{aligned} \Delta_2 &= \beta(\beta + \sigma + 1)(\sigma + \rho) + 2\beta\sigma(1 - \rho) \\ &= \beta(\sigma(\beta + \sigma + 3) - \rho(\sigma - \beta - 1)) \end{aligned}$$

is positive for any $\rho > 1$, if $\sigma - \beta - 1 < 0$, and for $1 < \rho < \rho_H := \sigma \frac{\sigma + \beta + 3}{\sigma - \beta - 1}$, if $\sigma - \beta - 1 > 0$. In this case the point $(x_2, x_2, \rho - 1)$ is asymptotically stable. We note that in the case $\Delta_2 < 0$ the Jacobian has two complex eigenvalues with positive real part. By using the symmetry of the system we get the same condition on the stability of the point $(x_3, x_3, \rho - 1)$ as for the point $(x_2, x_2, \rho - 1)$.

Chapter 5

Global phase portrait, periodic orbits, index of a vector field

5.1 Investigating the global phase portrait by using the local ones

In the previous sections we studied how can the local phase portrait of a differential equation $\dot{x} = f(x)$ be determined in the neighbourhood of a given point. The results presented can be summarised briefly as follows.

- If the given point p is not an equilibrium, then the flow-box theorem can be applied, hence the local phase portrait at the point p is given by straight lines parallel to the vector $f(p)$.
- If the given point p is an equilibrium, then the local phase portrait can be characterised by using the linearisation $f'(p)$.
 - If the linear part is hyperbolic, that is the eigenvalues of $f'(p)$ have non-zero real part, then according to the Hartman-Grobman theorem the phase portrait is locally conjugate to that of the linearised system.
 - If the linear part is not hyperbolic, then higher order terms play role in determining the phase portrait. The effect of these terms can be studied by using the following tools.
 - * Finding the normal form of the system.
 - * Using center manifold reduction.

The aim of this section is to show methods for studying the global phase portrait by using the local phase portraits determined at different points. In the case of one dimensional systems we can achieve full classification by using this approach. This will be shown in the first subsection.

5.1.1 Global phase portraits of one dimensional systems

The phase portrait of a one dimensional system can be simply obtained based on the sign of the right hand side. This is illustrated in the case of a simple example.

Example 5.1. Consider the differential equation $\dot{x} = x - x^3$. The zeros of the function $f(x) = x - x^3$ are -1 , 0 , and 1 . The sign of the function can simply be determined in the segments determined by these points. In the intervals $(-\infty, -1)$ and $(0, 1)$ the function is positive, while in the intervals $(-1, 0)$ and $(1, +\infty)$ it is negative. The graph of the function is shown in Figure 5.1. Thus the equation has three equilibria -1 , 0 , and 1 . There are two orbits directed positively, these are $(-\infty, -1)$ and $(0, 1)$, and there are two negatively directed orbits $(-1, 0)$ and $(1, +\infty)$. Hence by determining the sign of the function one can get the global phase portrait shown in Figure 5.1.

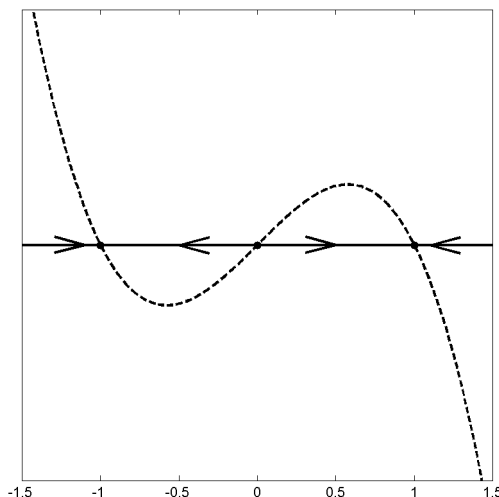


Figure 5.1: The phase portrait of the differential equation $\dot{x} = x - x^3$.

Let us turn now to the classification of the phase portraits of the one dimensional dynamical systems of the form $\dot{x} = f(x)$, where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuously differentiable function. The classification is based on the number of zeros of the function f as it is suggested by the above example.

- If the function f has no zeros, then there is no steady state. Then the phase portrait is equivalent to that of the equation $\dot{x} = 1$ that is shown in Figure 5.2. (We note that the phase portrait is equivalent to this one also in the case when f is negative.)

- If the function f has one zero, then there is a unique equilibrium. Then the dynamical system is equivalent to one of the three equations below, their phase portrait are shown in Figure 5.3.

◇ $\dot{x} = -x$

◇ $\dot{x} = x$

◇ $\dot{x} = x^2$

- If the function f has two zeros, then there are two equilibria. Each of them can be stable, unstable or neutral (stable from one side and unstable from the other side). Then there are four combinations of the two steady states: 1. a stable and an unstable point, 2. a stable and a neutral point, 3. an unstable and a neutral point, 4. two neutral points. The phase portraits corresponding to these cases are shown in Figure 5.4.

Equations having n equilibria can be classified similarly. We note that there can be infinitely many steady states, for example in the case of $f(x) = \sin x$, moreover, the equilibria may accumulate as for the function $f(x) = x^2 \cdot \sin \frac{1}{x}$. These cases are not dealt with here.

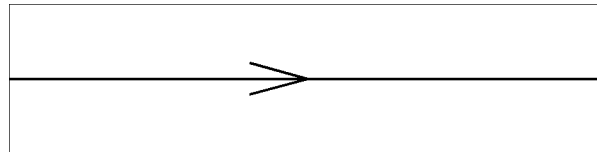


Figure 5.2: The phase portrait of $\dot{x} = f(x)$ when the function f has no zeros.

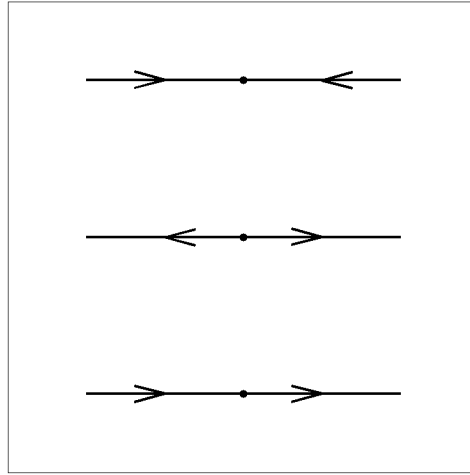


Figure 5.3: The three possible phase portraits of $\dot{x} = f(x)$ when the function f has one zero.

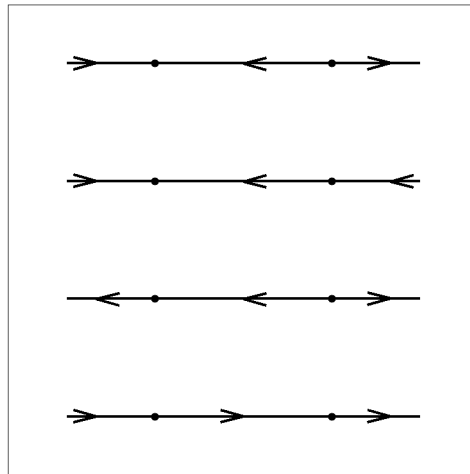


Figure 5.4: The four possible phase portraits of $\dot{x} = f(x)$ when the function f has two zeros.

5.1.2 Global phase portraits of two dimensional systems

Consider the system

$$\begin{aligned}\dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y)\end{aligned}$$

of two differential equations, where $P, Q : \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuously differentiable functions. The goal here is to characterise the global phase portrait. The following methods will be presented.

- Determining the direction field and the nullclines.
- Transforming the system to polar coordinates or to a complex variable.
- Finding a first integral or a Lyapunov function.
- Exploiting the symmetry of the vector field (P, Q) .

Beyond these methods it is always useful to solve the differential equations numerically from suitably chosen initial conditions. In the next subsections we deal with these methods and show by examples how they can be applied.

Direction field and nullclines

The direction field of the differential equation is the function $(P, Q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that associates a two dimensional vector to each point of the phase plane. This vector is the tangent of the trajectory at the given point. In order to characterise the phase portrait it is often enough to know if the vectors of the direction field point up or down, and to the left or to the right. In order to see this the nullclines

$$N_1 := \{p \in \mathbb{R}^2 : P(p) = 0\} \quad N_2 := \{p \in \mathbb{R}^2 : Q(p) = 0\}$$

can help. The null cline N_1 divides the phase plane into two parts (these are not necessarily connected sets). In one of these, in which $P > 0$, trajectories move to the right (since $\dot{x} > 0$ there), in the other part, in which $P < 0$, trajectories move to the left (since $\dot{x} < 0$ there). Similarly, the null cline N_2 divides the phase plane into two parts (these are not necessarily connected sets). In one of these, in which $Q > 0$, trajectories move up (since $\dot{y} > 0$ there), in the other part, in which $Q < 0$, trajectories move down (since $\dot{y} < 0$ there). Thus the nullclines N_1 and N_2 divide the phase plane into four parts, in each of them it can be decided if the trajectories move up or down, and to the left or to the right. (We use the terminology that "the trajectory moves", in fact the point $\varphi(t, p)$ moves along the trajectory as t is varied.)

The intersection points of the nullclines are the equilibria where both P and Q are equal to zero. In the neighbourhood of the steady states the phase portraits can be determined by linearisation. In order to get the global picture it is useful to determine the behaviour of the separatrices of the saddle points, i.e. the trajectories converging to saddle points as $t \rightarrow +\infty$ or as $t \rightarrow -\infty$. We illustrate this approach by the following examples.

Example 5.2. Consider the system

$$\dot{x} = x - xy, \quad \dot{y} = x^2 - y.$$

The equation of the null cline N_1 is $x(1 - y) = 0$. That is this null cline consists of two lines, the line $\{(x, y) \in \mathbb{R}^2 : x = 0\}$ and the line $\{(x, y) \in \mathbb{R}^2 : y = 1\}$. These lines divide the phase plane into four domains. In the upper right and lower left domain the trajectories move to the left, while in the other two domains they move to the right. Since the line $x = 0$ is contained in the null cline, it is an invariant line (this can also be easily seen from the fact that $x = 0$ implies $\dot{x} = 0$). The equation of the null cline N_2 is $y = x^2$, hence this is a parabola. Above the parabola $\dot{y} < 0$, hence trajectories are moving down, while below the parabola $\dot{y} > 0$, hence trajectories are moving up. The nullclines divide the phase plane into 8 parts. In Figure 5.2 an arrow represents the direction field in each region. The equilibria and their types can simply be determined. The point $(0, 0)$ is a saddle, the steady states $(1, 1)$ and $(-1, 1)$ are stable foci. The stable manifold of the saddle is the invariant line $x = 0$. Using the arrows of the direction field it can be shown that the two trajectories in the unstable manifold of the saddle point tend to the focus points. In a similar way, it can be shown that the trajectories starting in the right half plane tend to the stable focus $(1, 1)$ as $t \rightarrow +\infty$, and those starting in the left half plane tend to the stable focus $(-1, 1)$ as $t \rightarrow +\infty$. Hence we get the phase portrait shown in Figure 5.2.

The phase portrait of the system in Example 5.2 is determined by using the direction field.

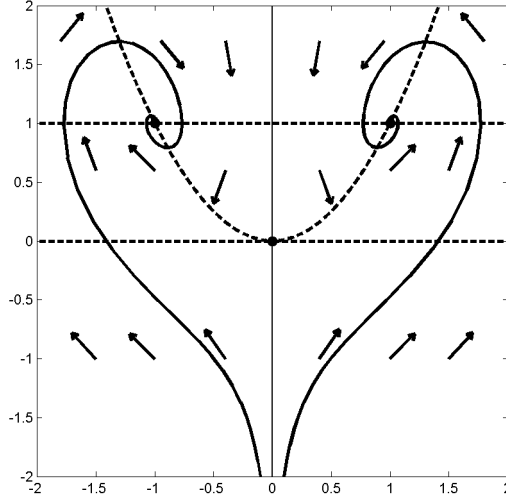
Example 5.3. Consider the system

$$\dot{x} = 2x + y^2 - 1, \quad \dot{y} = 6x - y^2 + 1.$$

First, let us determine the equilibria. Adding the two equations $8x = 0$, then from the first one $y = \pm 1$. Therefore the equilibria are $(0, 1)$ and $(0, -1)$. The Jacobian of the system is

$$J = \begin{pmatrix} 2 & 2y \\ 6 & -2y \end{pmatrix}.$$

At the steady state $(0, 1)$ the Jacobian is $J(0, 1) = \begin{pmatrix} 2 & 2 \\ 6 & -2 \end{pmatrix}$, for which $\text{Tr}(J(0, 1)) = 0$, $\text{Det}(J(0, 1)) = -16$, hence $(0, 1)$ is a saddle point. At the steady state $(0, -1)$ the



Jacobian is $J(0, -1) = \begin{pmatrix} 2 & -2 \\ 6 & 2 \end{pmatrix}$, for which $\text{Tr}(J(0, -1)) = 4$, $\text{Det}(J(0, -1)) = 16$, $\text{Tr}^2(J(0, -1)) - 4\text{Det}(J(0, -1)) < 0$, hence $(0, -1)$ is an unstable focus. The equation of the null cline N_1 is $x = (1 - y^2)/2$, hence this determines a parabola dividing the plane into two regions. In the domain on the right hand side the trajectories move to the right, and in the domain on the left hand side the trajectories move to the left. The equation of the null cline N_2 is $x = (y^2 - 1)/6$, which is also a parabola dividing the plane into two regions. In the domain on the right hand side the trajectories move up, and in the domain on the left hand side the trajectories move down. The two nullclines divide the plane into five parts. The trajectories in the unstable manifold of the saddle point tend to infinity. The trajectories in the stable manifold of the saddle tend to the unstable focus and to infinity as $t \rightarrow -\infty$. (These are shown in red in Figure 5.3.) Computing some trajectories numerically we get the phase portrait shown in Figure 5.3.

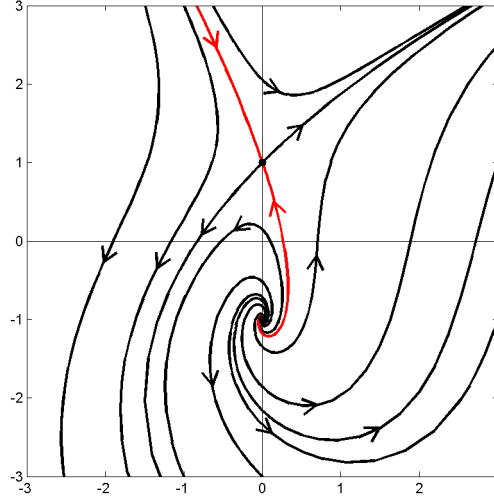
The phase portrait of the system in Example 5.3 is determined by using the direction field.

Example 5.4. Consider the system

$$\dot{x} = xy - 4, \quad \dot{y} = (x - 4)(y - x).$$

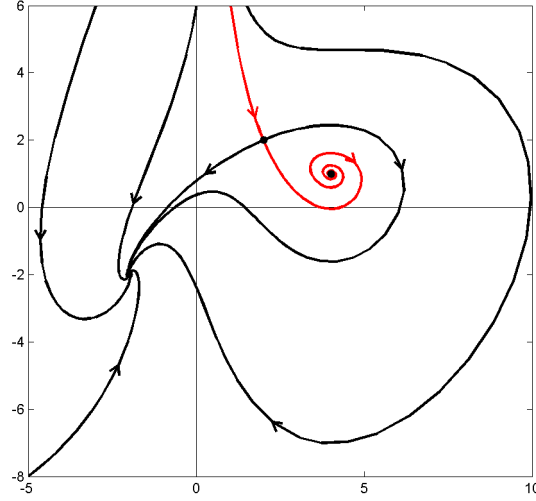
First, let us determine the equilibria. The second equation yields $x = 4$ or $y = x$, then the first one implies $y = 1$ and $x = \pm 2$. Therefore the equilibria are $(4, 1)$, $(2, 2)$ and $(-2, -2)$. The Jacobian of the system is

$$J = \begin{pmatrix} y & x \\ y - 2x + 4 & x - 4 \end{pmatrix}.$$



At the steady state $(4, 1)$ the Jacobian is $J(4, 1) = \begin{pmatrix} 1 & 4 \\ -3 & 0 \end{pmatrix}$, for which $\text{Tr}(J(4, 1)) = 1$, $\text{Det}(J(4, 1)) = 9$, $\text{Tr}^2(J(4, 1)) - 4\text{Det}(J(4, 1)) < 0$, hence $(4, 1)$ is an unstable focus. At the steady state $(2, 2)$ the Jacobian is $J(2, 2) = \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix}$, for which $\text{Tr}(J(2, 2)) = 0$, $\text{Det}(J(2, 2)) = -8$, hence $(2, 2)$ is a saddle. At the steady state $(-2, -2)$ the Jacobian is $J(-2, -2) = \begin{pmatrix} -2 & -2 \\ 6 & -6 \end{pmatrix}$, for which $\text{Tr}(J(-2, -2)) = -8$, $\text{Det}(J(-2, -2)) = 24$, $\text{Tr}^2(J(-2, -2)) - 4\text{Det}(J(-2, -2)) < 0$, hence $(-2, -2)$ is a stable focus. The equation of the null cline N_1 is $y = 4/x$, which is a hyperbola dividing the plane into three regions. In the middle domain (between the two branches) the trajectories move to the left, while in the other two domains they move to the right. The null cline N_2 consists of the lines $x = 4$ and $x = y$ that divide the plane into four parts. In the upper left and lower right domains the trajectories move down, while in the upper right and lower left domains they move up. The two nullclines divide the plane into 9 parts. The trajectories in the unstable manifold of the saddle point tend to the stable focus. The trajectories in the stable manifold of the saddle tend to the unstable focus and to infinity as $t \rightarrow -\infty$. All other trajectories tend to the stable focus as $t \rightarrow +\infty$. Computing some trajectories numerically we get the phase portrait shown in Figure 5.4.

The phase portrait of the system in Example 5.4 is determined by using the direction field.



Transforming the system to polar coordinates or to a complex variable

Introduce the functions r and ϕ to system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ by the transformation formulas $x(t) = r(t) \cos(\phi(t))$, $y(t) = r(t) \sin(\phi(t))$. These yield

$$\dot{x} = \dot{r} \cos(\phi) - r \dot{\phi} \sin(\phi), \quad \dot{y} = \dot{r} \sin(\phi) + r \dot{\phi} \cos(\phi).$$

Multiplying the first equation by $\cos(\phi)$ and the second one by $\sin(\phi)$, then adding the two equations and using the differential equations one obtains

$$\dot{r} = P(r \cos(\phi), r \sin(\phi)) \cos(\phi) + Q(r \cos(\phi), r \sin(\phi)) \sin(\phi).$$

In a similar way, multiplying the first equation by $\sin(\phi)$ and the second one by $\cos(\phi)$, then subtracting the two equations and using the differential equations one obtains

$$\dot{\phi} = Q(r \cos(\phi), r \sin(\phi)) \cos(\phi) - P(r \cos(\phi), r \sin(\phi)) \sin(\phi).$$

In certain cases the differential equations obtained for r and ϕ are simple enough to determine the phase portrait belonging to them. The simplest example is the center when the equations are $\dot{x} = -y$ and $\dot{y} = x$. Then the equations for the polar coordinates take the form $\dot{r} = 0$ and $\dot{\phi} = 1$, showing that the orbits are circles centered at the origin.

Another useful transformation can be $z(t) = x(t) + iy(t)$. For this new function the differential equation takes the form

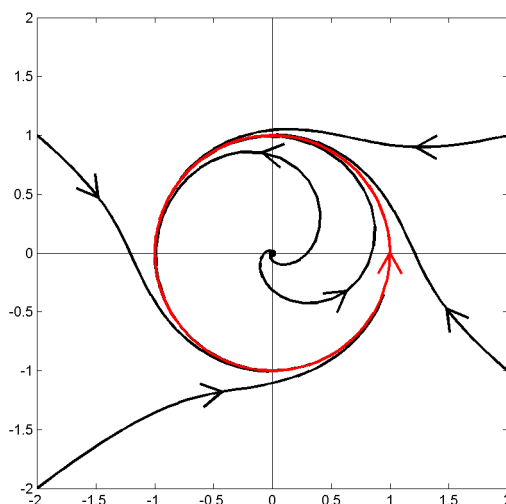
$$\dot{z} = \dot{x} + i\dot{y} = P(\operatorname{Re}(z), \operatorname{Im}(z)) + iQ(\operatorname{Re}(z), \operatorname{Im}(z)).$$

The transformation is effective if the right hand side can be expressed in terms of z without using the real and imaginary parts. The simplest example is the center when the equations are $\dot{x} = -y$ and $\dot{y} = x$ yielding to $\dot{z} = iz$. The solution of this equation is $z(t) = \exp(it)$, that gives explicitly the time dependence of x and y , moreover, it implies simply that $|z(t)|$ is constant in time, hence the orbits are circles centered at the origin.

Example 5.5. Consider the system

$$\dot{x} = x(1 - x^2 - y^2) - y, \quad \dot{y} = y(1 - x^2 - y^2) + x.$$

Following the above approach introduce r and ϕ by using the transformation formulas $x(t) = r(t) \cos(\phi(t))$, $y(t) = r(t) \sin(\phi(t))$. Then for the polar coordinates we get the differential equations $\dot{r} = r(1 - r^2)$ and $\dot{\phi} = 1$. Since $r = 1$ implies $\dot{r} = 0$, the circle $r = 1$ is invariant and $\dot{\phi} = 1$ yields that the trajectory along the circle rotate counterclockwise. If $r < 1$, then $\dot{r} > 0$, thus the radius is increasing inside the circle that is the trajectories tend to the circle. If $r > 1$, then $\dot{r} < 0$, thus the radius is decreasing outside the circle that is these trajectories also tend to the circle. Therefore the phase portrait looks like as it is shown in Figure 5.5.



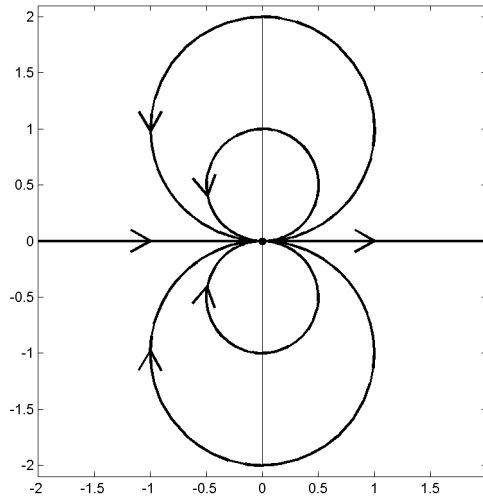
The phase portrait of the system in Example 5.5 is determined by using polar coordinate transformation.

The next example shows an application of using complex coordinates.

Example 5.6. Consider the system

$$\dot{x} = x^2 - y^2, \quad \dot{y} = 2xy.$$

Introducing the complex variable $z(t) = x(t) + iy(t)$ the differential equation takes the form $\dot{z} = \dot{x} + i\dot{y} = P(\operatorname{Re}(z), \operatorname{Im}(z)) + iQ(\operatorname{Re}(z), \operatorname{Im}(z))$, that is $\dot{z} = z^2$. This differential equation can be solved by the method of separating the variables. Divide the equation by z^2 and then integrate. The solution is $z(t) = \frac{1}{c-t}$, where $c \in \mathbb{C}$ is an arbitrary complex constant. Let us determine the trajectories starting from the vertical coordinate axis, i.e. satisfying the initial condition $z(0) = Ki$. In this case $c = 1/Ki$, where $K \in \mathbb{R}$ is an arbitrary real constant. Thus $z(t) = \frac{Ki}{1-tKi} = \frac{Ki-tK^2}{1+t^2K^2}$. Since x and y are the real and imaginary parts of z , we get $x(t) = \frac{-tK^2}{1+t^2K^2}$ and $y(t) = \frac{K}{1+t^2K^2}$. Eliminating t from these two equations (expressing t in terms of y and then substituting this expression into the equation of x) we arrive to $x^2 = (K - y)y$. This is equivalent to $x^2 + (y - K)^2 = K^2$, which is the equation of a circle with radius K and centered at the point $(0, K)$. Hence the equilibrium is the origin, the horizontal axis is an invariant line, along which trajectories move to the right, and the other trajectories are homoclinic orbits lying on circles centered on the vertical axis and containing the origin. These circles in the upper half plane are oriented counterclockwise, while those in the lower half plane are oriented clockwise. The phase portrait is shown in Figure 5.6.



The phase portrait of the system in Example 5.6 is determined transforming the system to complex variable.

First integral and Lyapunov function

The function $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a first integral to system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$, if $P\partial_1 V + Q\partial_2 V = 0$. Using a first integral the phase portrait can easily be obtained,

because trajectories lie on the level curves of the first integral. As an example let us consider the Lotka–Volterra system

$$\dot{x} = x - xy, \quad \dot{y} = xy - y.$$

Dividing the two equations by each other and assuming that y can be expressed in terms of x (at least in a suitable domain) we get the following differential equation for the function $x \mapsto y(x)$: $\frac{dy}{dx} = \frac{y(x-1)}{x(1-y)}$. This can be solved by using the method of separating the variables. Taking the terms containing y to the left hand side $\frac{1-y}{y} \frac{dy}{dx} = \frac{x-1}{x}$. Integrating the equation we get $\ln(|y|) - y = x - \ln(|x|) + K$, where K is an arbitrary constant. Hence the function $V(x, y) = \ln(|x|) - x + \ln(|y|) - y$ is a first integral, that can be checked easily by differentiation (if the above derivation does not seem to be reliable). Namely, $L_f V(x, y) = (1/x - 1)x(1 - y) + (1/y - 1)y(x - 1) = 0$, where $L_f V$ is the Lie-derivative of V along the vector field given by f . We note that searching for a first integral in the form $V(x, y) = F(x) + G(y)$ we arrive to the same function V . It can be shown that the level curves of this function V in the positive quadrant are closed curves, hence the orbits with positive coordinates are periodic as it is shown in Figure 5.5.

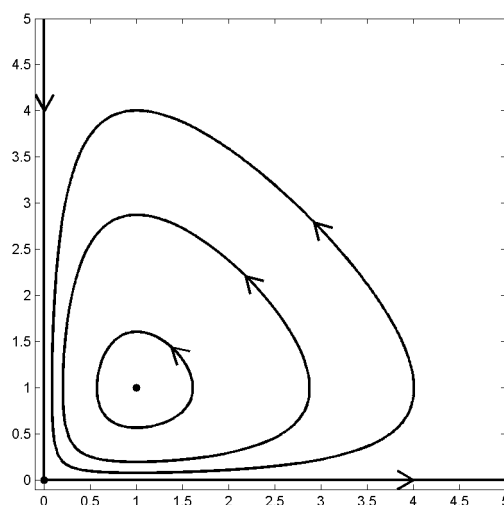


Figure 5.5: The phase portrait of the Lotka–Volterra system.

There is no general method for finding a first integral, however, for an important class of dynamical systems, namely for Hamiltonian systems the first integral can be explicitly given.

The two dimensional system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ is called Hamiltonian if there exists a differentiable function $H : \mathbb{R}^2 \rightarrow \mathbb{R}$, for which $P = \partial_2 H$ and $Q = -\partial_1 H$. Apply-

ing the theorem about the necessary condition of the existence of primitive functions we get that if the system is Hamiltonian, then $\partial_1 P = -\partial_2 Q$, that is $\partial_1 P + \partial_2 Q = 0$, yielding that the divergence of the right hand side is zero. The Jacobian of a Hamiltonian system takes the form

$$J = \begin{pmatrix} \partial_{12}H & \partial_{22}H \\ -\partial_{11}H & -\partial_{21}H \end{pmatrix}$$

The trace of this matrix is $Tr(J) = 0$, hence linearisation shows that an equilibria is either a saddle or a center. On the other hand, $Det(J) = Det(H''(x, y))$, hence the equilibrium is a saddle if $Det(H''(x, y)) < 0$, i.e. when $H''(x, y)$ is indefinite. If $Det(H''(x, y)) > 0$, that is (x, y) is an extremum of H , then the equilibrium is a center, because in a neighbourhood of an extremum the level curves of H are closed curves.

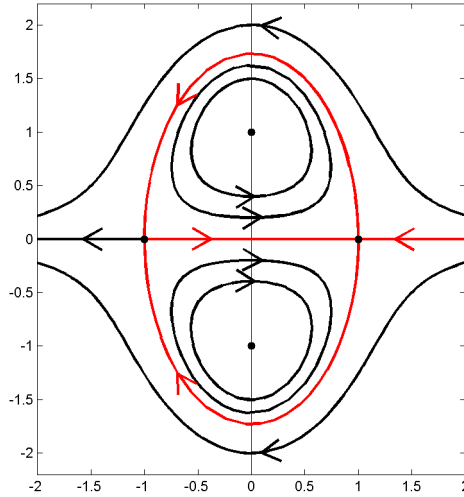
Example 5.7. Consider the system

$$\dot{x} = 1 - x^2 - y^2, \quad \dot{y} = 2xy.$$

Since $\partial_1 P(x, y) + \partial_2 Q(x, y) = -2x + 2x = 0$, the system is Hamiltonian. Integrating the function P with respect to y the Hamiltonian function can be given in the form $H(x, y) = y - x^2y - y^3/3 + C(x)$, where C is an arbitrary differentiable function. Differentiating this function with respect to x and using the relation $Q = -\partial_1 H$ we get $C'(x) = 0$, hence the function C is a constant function, which can be assumed to be the zero function without loss of generality. Therefore the Hamiltonian is $H(x, y) = y - x^2y - y^3/3$. Before determining the level curves of H it is useful to find the steady states and draw the direction field. For the equilibria the second equation yields $x = 0$ or $y = 0$. Substituting these values into the first equation one obtains $y = \pm 1$ or $x = \pm 1$. Therefore the equilibria are $(0, 1)$, $(0, -1)$, $(1, 0)$ and $(-1, 0)$. The type of the steady states is determined by the sign of $Det(H''(x, y))$. In this case $Det(H''(x, y)) = -4x^2 + 4y^2$. Hence $(1, 0)$ and $(-1, 0)$ are saddle points, $(0, 1)$ and $(0, -1)$ are centers. The level curve $H = 0$ consists of two parts. On one hand it contains the x coordinate axis and on the other hand, the ellipse $1 = x^2 + y^2/3$ connecting the two saddle points in the upper and lower half plane. The level curves corresponding to negative values of H are closed curves around the centers and lying inside the ellipse. The level curves corresponding to positive values of H tend to $+\infty$ and to $-\infty$ along the x axis and pass round the ellipse in the upper and lower half plane. The direction of the trajectories can easily be obtained from the second equation according to the sign of \dot{y} . Computing some trajectories numerically we get the phase portrait shown in Figure 5.7.

The phase portrait of the Hamiltonian system in Example 5.7.

An important special case of Hamiltonian systems is $\ddot{x} + U'(x) = 0$ describing a mechanical system with one degree of freedom. The corresponding first order system takes the form $\dot{x} = y$, $\dot{y} = -U'(x)$. The first integral of this system can be given as $V(x, y) = y^2/2 + U(x)$, namely, its Lie derivative along the vector field is $L_f V(x, y) =$



$-yU'(x) + yU'(x) = 0$. Thus the trajectories lie on the level curves of the function V . Thus in order to determine the phase portrait one has to find the level curves of V and then determine the direction of the trajectories based on the sign of \dot{x} and \dot{y} . The level curves of V can be found by using the graph of the function U in the plane (x, U) . Then taking the y axis orthogonal to this plane we plot the parabola $y^2/2$ in the plane (y, U) . Finally, moving this parabola along the graph of the function U we get a surface, which is the graph of the function V . Then the level curves of V can be simply obtained by cutting this surface by horizontal planes. The Jacobian of the system is

$$J = \begin{pmatrix} 0 & 1 \\ -U''(x) & 0 \end{pmatrix}.$$

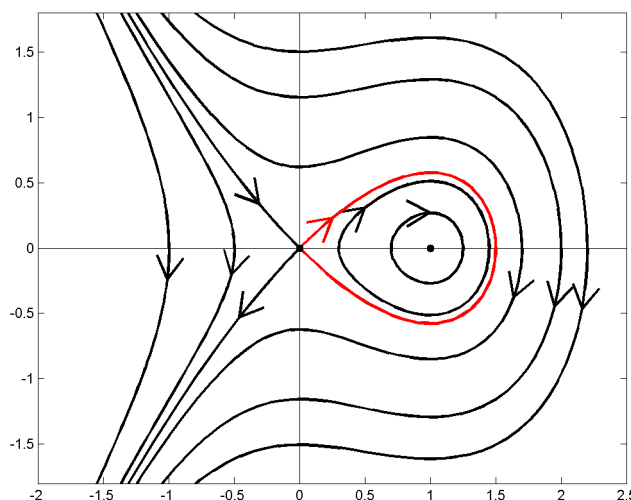
Its trace is $Tr(J) = 0$, and determinant is $Det(J) = U''(x)$. Therefore if $U''(x) < 0$, then the steady state $(x, 0)$ is a saddle, while in the case $U''(x) > 0$ the point $(x, 0)$ is a minimum point of V , hence it is a center.

Example 5.8. Consider the system

$$\dot{x} = y, \quad \dot{y} = x - x^2.$$

The steady states are $(0, 0)$ and $(1, 0)$. In this case $U(x) = x^3/3 - x^2/2$, hence the first integral is $V(x, y) = y^2/2 + x^3/3 - x^2/2$. The function U has a maximum at 0 and a minimum at 1, therefore $(0, 0)$ is a saddle and $(1, 0)$ center. Draw the graph of the function U , then following the above approach determine the level curves of V . The level curve corresponding to $V = 0$ contains the origin and a loop in the right half plane. The

level curves corresponding to negative values of V are closed curves inside the loop, at the point $(1,0)$ there is a local minimum of V . The curves in the left half plane also correspond to negative values of V . The positive level curves are connected sets tending to infinity and passing round the loop. The direction of the trajectories can be obtained easily from the first differential equation. Namely, in the upper half plane, where $y > 0$ the trajectories move to the right, since $\dot{x} > 0$ there. Similarly, in the lower half plane the trajectories move to the left. The phase portrait is shown in Figure 5.8.



The phase portrait of the system in Example 5.8.

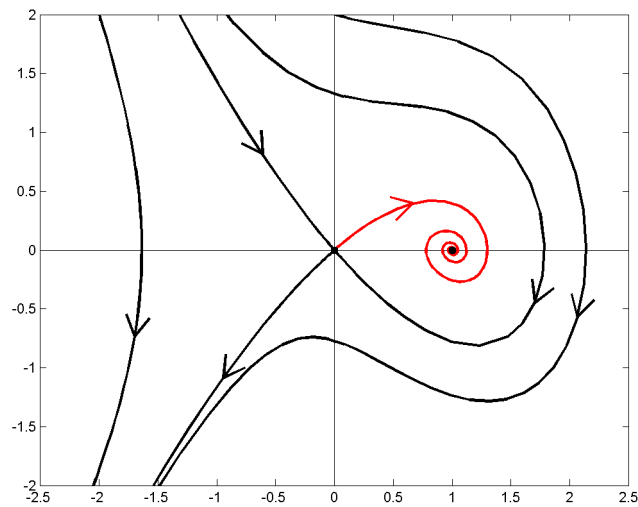
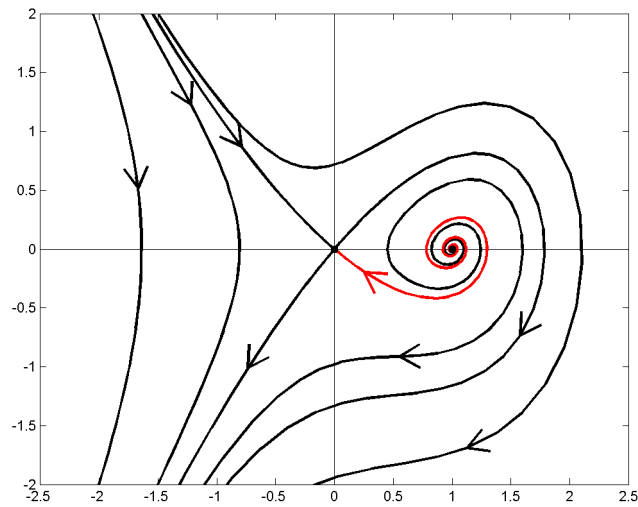
Example 5.9. Consider the following perturbation of the system in the previous example

$$\dot{x} = y, \quad \dot{y} = x - x^2 + cy.$$

The steady states are $(0,0)$ and $(1,0)$. Let us calculate the Lie derivative of the first integral $V(x,y) = y^2/2 + x^3/3 - x^2/2$ in Example 5.8. A simple calculation shows that $L_f V(x,y) = cy^2$, the sign of which is the same as the sign of c . Hence in the case $c < 0$ the value of V decreases along the trajectories. Therefore the orbit in the unstable manifold of the saddle tends to the stable focus $(1,0)$. The phase portrait is shown in Figure 5.9. In the case $c > 0$ the value of V increases along the trajectories. Therefore the orbit in the stable manifold of the saddle tends to the unstable focus $(1,0)$ as $t \rightarrow -\infty$. The phase portrait is shown in Figure 5.9.

The phase portrait of the system in Example 5.9 for $c < 0$.

The phase portrait of the system in Example 5.9 for $c > 0$.



Phase portrait of a vector field with symmetry

The information obtained from the direction field can be supplemented by the symmetry of the vector field. One of the simplest examples is the non-linear center, the existence of which cannot be deduced from the direction field or from linearisation. In this case it may help to know that the orbits are symmetric to an axis through an equilibrium. This fact can be verified from the differential equations without knowing the solutions.

Namely, let us consider a general (possibly n -dimensional) system $\dot{x}(t) = f(x(t))$. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a transformation taking the orbits into each other. If the mapping T maps the positive part ($t > 0$) of an orbit starting from the point $p \in \mathbb{R}^n$ onto the negative part of the orbit starting from $T(p)$, then

$$\varphi(-t, T(p)) = T(\varphi(t, p))$$

holds for all t . Differentiating this relation with respect to t we get $-\dot{\varphi}(-t, T(p)) = T'(\varphi(t, p))\dot{\varphi}(t, p)$. Then substituting $t = 0$ yields

$$-f(T(p)) = T'(p)f(p).$$

In order to check this relation the solutions are not needed.

Let us consider two important 2-dimensional consequences of this relation. Namely, let us formulate the condition ensuring that the trajectories are symmetric to one of the coordinate axes. The reflection to the vertical axis can be given as $T(x, y) = (-x, y)$. Its derivative is the matrix $T' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence using the above formula, the trajectories of the system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ are symmetric to the vertical axis, if $-(P(-x, y), Q(-x, y)) = T'(P(x, y), Q(x, y))$, that is

$$P(-x, y) = P(x, y), \quad -Q(-x, y) = Q(x, y).$$

The reflection to the horizontal axis can be given as $T(x, y) = (x, -y)$. Its derivative is the matrix $T' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Hence using the above formula, the trajectories of the system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ are symmetric to the horizontal axis, if $-(P(x, -y), Q(x, -y)) = T'(P(x, y), Q(x, y))$, that is

$$-P(x, -y) = P(x, y), \quad Q(x, -y) = Q(x, y).$$

Example 5.10. Consider the system

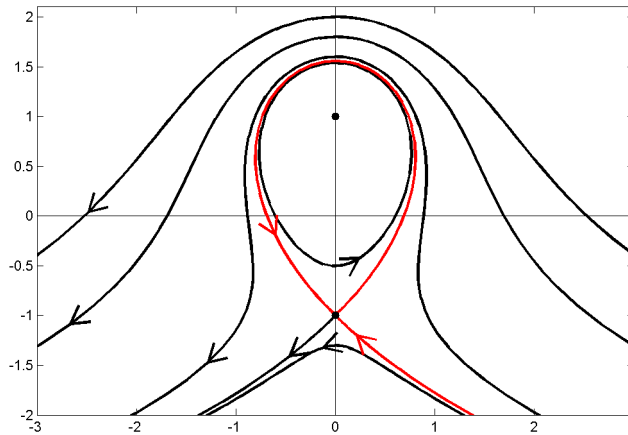
$$\dot{x} = 1 - x^2 - y^2, \quad \dot{y} = 2x.$$

Let us find first the equilibria. From the second equation $x = 0$, then the first one implies $y = \pm 1$. Therefore the equilibria are $(0, 1)$ and $(0, -1)$. The Jacobian of the system is

$$J = \begin{pmatrix} -2x & -2y \\ 2 & 0 \end{pmatrix}.$$

At the point $(0, 1)$ the Jacobian is $J(0, 1) = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$, for which $\text{Tr}(J(0, 1)) = 0$, $\text{Det}(J(0, 1)) = 4$, $\text{Tr}^2(J(0, 1)) - 4\text{Det}(J(0, 1)) < 0$, hence the type of the steady state

$(0, 1)$ cannot be decided by using linearisation (the eigenvalues are pure imaginary). At the point $(0, -1)$ the Jacobian takes the form $J(0, -1) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$, for which $\text{Tr}(J(0, -1)) = 0$, $\text{Det}(J(0, -1)) = -4$, hence $(0, -1)$ is a saddle point. The equation of the nullcline N_1 is $x^2 + y^2 = 1$, hence this nullcline is a circle centered at the origin, dividing the plane into two parts. In the outer part the trajectories move to the left, while inside the circle they move to the right. The nullcline N_2 is the vertical axis, dividing the plane into two parts. In the left half plane $\dot{y} < 0$, hence trajectories are moving down there, in the right half plane $\dot{y} > 0$, hence trajectories are moving up there. The nullclines divide the phase plane into four parts. In Figure 5.10 an arrow shows the direction of the trajectories in each part. One of the trajectories in the unstable manifold of the saddle point passes round the point $(0, 1)$, the other one tends to $-\infty$. One of the trajectories in the stable manifold of the saddle point passes round the point $(0, 1)$, the other one comes from $-\infty$. For the full characterization of the behaviour of the trajectories observe that the phase portrait is symmetric to the vertical axis. This is shown by the fact that $P(-x, y) = P(x, y)$ and $-Q(-x, y) = Q(x, y)$ hold. Hence the equilibrium $(0, 1)$ is surrounded by periodic orbits, i.e. it is a center and a trajectory in the stable and in the unstable manifold of the saddle point forms a homoclinic orbit. Computing some trajectories numerically we get the phase portrait shown in Figure 5.10.



The phase portrait of the system in Example 5.10.

The above examples show that for two dimensional systems the global phase portrait cannot be typically obtained from the local phase portraits. Namely, there global structures may occur. In two dimensional systems these can be

- periodic orbits
- homoclinic orbits
- heteroclinic orbits.

In the next section the first of them will be dealt with.

5.2 Periodic orbits

Consider the system of differential equations

$$\dot{x}(t) = f(x(t)),$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuously differentiable function. The solution starting from the point p is denoted by $\varphi(t, p)$.

Definition 5.1.. *The point p is called a periodic point if there exists $T > 0$, for which $\varphi(T, p) = p$ and p is not an equilibrium. The orbit of p is called a periodic orbit, the smallest value of T is called the period of the orbit.*

The two most important questions concerning periodic orbits are the following.

1. How can be verified the existence or non-existence of the periodic orbit?
2. How does the local phase portrait look like in a neighbourhood of a periodic orbit?

We note that these questions were addressed concerning the steady states. In that case the first question leads to solving the system of algebraic equations $f(p) = 0$, or to studying the existence of its solutions. The second question can be answered with the help of the Jacobian matrix at the equilibria. The corresponding question concerning periodic orbits is more difficult because of the non-local nature of periodic orbits. The most well-known results will be dealt with in the next two subsections together with an important open problem for periodic orbits.

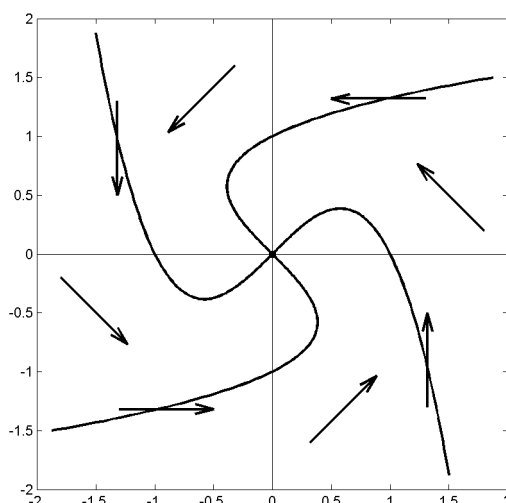
5.2.1 Existence of periodic orbits

The existence of periodic orbits in two dimensional systems can be verified by using the Poincaré-Bendixson theorem. Before stating the theorem a motivating example is shown to illustrate the key idea of the proof.

Example 5.11. Consider the system

$$\begin{aligned}\dot{x} &= x - y - x^3 \\ \dot{y} &= x + y - y^3.\end{aligned}$$

Plotting the nullclines $y = x - x^3$ and $x = y^3 - y$ one can see that the trajectories rotate around the origin, however, it cannot be directly seen if they are tending to the origin or to infinity, see Figure 5.11.



The nullclines and the direction field of the system in Example 5.11.

In order to investigate this question introduce the Lyapunov function $V(x, y) := x^2 + y^2$. For an arbitrary solution $(x(t), y(t))$ let $V^*(t) = x^2(t) + y^2(t)$. Then using the differential equations $\dot{V}^*(t) = 2(x^2(t) + y^2(t) - x^4(t) - y^4(t))$. The sign of this expression can help in understanding the behaviour of the trajectories. The following two statements can be proved by elementary calculations. If $x^2 + y^2 < 1$, then $x^2 + y^2 - x^4 - y^4 > 0$, and $x^2 + y^2 > 2$ implies $x^2 + y^2 - x^4 - y^4 < 0$. This means that the trajectories are spiraling away from the origin inside the unit disc, while they are spiraling inward outside the circle of radius $\sqrt{2}$. Hence the annular domain bordered by the circles of radius 1 and $\sqrt{2}$ is positively invariant, i.e. trajectories cannot leave this region. Consider the segment of the horizontal axis with end points 1 and $\sqrt{2}$ and define the Poincaré map $P : [1, \sqrt{2}] \rightarrow [1, \sqrt{2}]$ in this set as follows. For a given $q \in [1, \sqrt{2}]$ let $P(q) \in [1, \sqrt{2}]$ be the point to which the trajectory starting from q returns after one rotation, see the schematic Figure 5.6.

It can be proved that the Poincaré map is continuous, hence it maps the interval $[1, \sqrt{2}]$ into itself continuously, thus it has a fixed point $p \in [1, \sqrt{2}]$, for which $P(p) = p$

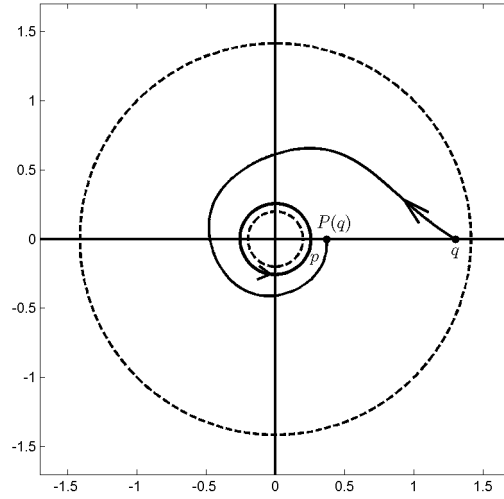


Figure 5.6: The Poincaré map and a periodic orbit starting from p .

holds. This point is a periodic point, because the trajectory starting from this point returns to the same point.

Generalising the idea used in this example, the following theorem can be proved.

Theorem 5.2. (Poincaré–Bendixson (weak form)). *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a continuously differentiable function and consider the two dimensional system $\dot{x}(t) = f(x(t))$. If $K \subset \mathbb{R}^2$ is a positively invariant, compact set that contains no equilibria, then there is a periodic orbit in it.*

We do not present the proof here, it can be found in [19]. We note that the role of K is played by the annular domain bordered by the circles of radius 1 and $\sqrt{2}$ in the above example.

The Poincaré–Bendixson theorem gives a sufficient condition for the existence of a periodic orbit in two dimensional systems. The applicability of the theorem relies on the construction of the set K that may be extremely difficult in some cases. Therefore to have a condition for the non-existence of periodic orbits is often very useful. The non-existence of periodic orbits can be proved by Bendixson’s criterion and by the Bendixson–Dulac criterion, that will be dealt with now.

Both criteria can be proved by using the following simple version of Stokes’s theorem.

Lemma 5.3.. *Let $D \subset \mathbb{R}^2$ be a simply connected open set and $G : D \rightarrow \mathbb{R}^2$ be a differentiable function. Then*

$$\int_D (\partial_1 G_1 + \partial_2 G_2) = \int_{\partial D} (-G_2, G_1).$$

Using this lemma we prove first Bendixson's criterion.

Theorem 5.4. (Bendixson's criterion). *Assume that $H \subset \mathbb{R}^2$ is a simply connected open domain, in which the divergence $\operatorname{div} f = \partial_1 f_1 + \partial_2 f_2$ has a given sign and it is zero at most in the points of a curve. Then system $\dot{x}(t) = f(x(t))$ has no periodic orbit lying completely in H .*

Proof. Assume that there is a periodic orbit Γ in the set H . Denote by D the interior of Γ . Applying Stokes's theorem for the function f in the domain D we get a contradiction. Namely, the left hand side has a given sign (according to the sign of the divergence), while the right hand side is zero that can be proved as follows. The vector $(-f_2, f_1)$ is orthogonal to the curve ∂D , because this curve is the periodic orbit itself. Hence denoting the periodic orbit by $\gamma(t)$ the right hand side in Stokes's theorem takes the form

$$\begin{aligned} \int_{\Gamma} (-f_2, f_1) &= \int_0^T \left(-f_2(\gamma(t)) \cdot \dot{\gamma}_1(t) + f_1(\gamma(t)) \cdot \dot{\gamma}_2(t) \right) dt \\ &= \int_0^T (-\dot{\gamma}_2 \dot{\gamma}_1 + \dot{\gamma}_1 \dot{\gamma}_2) dt = 0. \end{aligned} \quad \square$$

This theorem can be generalised to derive the Bendixson–Dulac criterion.

Theorem 5.5. (Bendixson–Dulac criterion). *Let $H \subset \mathbb{R}^2$ be a simply connected open domain and let $B : H \rightarrow \mathbb{R}$ be a differentiable function, for which $\operatorname{div}(Bf)$ has a given sign and it is zero at most in the points of a curve. Then system $\dot{x}(t) = f(x(t))$ has no periodic orbit lying completely in H .*

Proof. The proof is similar to the previous one, however, instead of f the reasoning is applied to the function Bf . \square

Now, examples are shown for the application of the non-existence theorems.

Example 5.12. *Consider the differential equation*

$$\begin{aligned} \dot{x} &= x - xy^2 + y^3 \\ \dot{y} &= 3y - x^2y + x^3. \end{aligned}$$

Then $\partial_1 f_1(x, y) + \partial_2 f_2(x, y) = 1 - y^2 + 3 - x^2 = 4 - (x^2 + y^2)$, which is positive inside the circle of radius 2 and centered at the origin. Hence in this disk the system has no periodic orbit.

Example 5.13. *The system*

$$\dot{x} = x + y^2 + x^3 \quad \dot{y} = -x + y + x^2y$$

has no periodic orbit according to Bendixson's criterion. Namely, $\text{div}f(x, y) = 2 + 4x^2 > 0$ at each point of the phase plane.

All the results presented in this section correspond to the two dimensional case. In higher dimension these theorems have no counter part, because of topological reasons. The investigation of the existence of periodic orbits in higher dimensional phase spaces is beyond the scope of this lecture notes. The difficulty of this problem is also illustrated by Hilbert's 16th problem that has remained open for more than 110 years.

Hilbert's 16th problem

Let P and Q be polynomials of degree n . How many limit cycles can be at most in the following system?

$$\begin{aligned}\dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y)\end{aligned}$$

In the case $n = 1$, i.e. when the system is linear, there is no limit cycle (periodic orbits may exist only in the case of a center, however these periodic orbits are not isolated). In the case $n = 2$ Shi, Chen and Wang proved in 1979 that there can be four limit cycles. It is conjectured that this is the maximal number of limit cycles in the quadratic case, however it has not been proved yet. The partial result can be found in review articles and a good summary can be read in the book by Perko [19].

5.2.2 Local phase portrait in a neighbourhood of a periodic orbit

In this subsection it is shown how can the local phase portrait be investigated in a neighbourhood of a periodic orbit. The stability of periodic orbits and their stable, unstable and center manifolds will be dealt with. The most important tool of the investigation is the Poincaré map, by the use of which the problem can be reduced to the study of the local phase portrait of discrete time dynamical systems at steady states.

Let $M \subset \mathbb{R}^n$ be a domain and let $\varphi : \mathbb{R} \times M \rightarrow M$ be a dynamical system. Let $p \in M$ be a periodic point with period T , that is $\varphi(T, p) = p$. Introduce the notations $\gamma(t) := \varphi(t, p)$ and $\Gamma := \{\gamma(t) : t \in [0, T]\}$.

Definition 5.6.. *Let $L \subset \mathbb{R}^n$ be an $n - 1$ dimensional hyperplane with normal ν . A connected subset $\Sigma \subset L$ is called a transversal section, if $\langle \nu, \partial_t \varphi(0, q) \rangle \neq 0$ for all $q \in \Sigma$.*

Let Σ be a transversal section containing p in its interior. Using the implicit function theorem the following statement can be shown.

Proposition 5.1. *(Existence of the Poincaré map) The point p has a neighbourhood $U \subset \Sigma$ and there exists a continuously differentiable function $\Theta : U \rightarrow \mathbb{R}$, for which $\Theta(p) = T$ and $\varphi(\Theta(q), q) \in \Sigma$, for all $q \in U$.*

Definition 5.7.. *The function $P : U \rightarrow \Sigma$, $P(q) = \varphi(\Theta(q), q)$ is called the Poincaré map (belonging to the transversal section Σ).*

The Poincaré map depends on the choice of the point p and that of the transversal section Σ . The next proposition shows that different Poincaré maps can be transformed to each other by a suitable coordinate transformation.

Proposition 5.2. *Let $p_1, p_2 \in \Gamma$, and let Σ_1, Σ_2 be transversal sections containing the corresponding points. Let $U_i \subset \Sigma_i$ ($i = 1, 2$) be open sets, in which the Poincaré maps $P_i : U_i \rightarrow \Sigma_i$ ($i = 1, 2$) are defined. Then there exists a neighbourhood $V \subset U_1$ of the point p_1 and there exists a differentiable function $S : V \rightarrow U_2$, for which $P_2(S(p)) = S((P_1(p)))$ holds for all $p \in V$.*

Corollary 5.8.. *The eigenvalues of the matrices $P'_1(p_1)$ and $P'_2(p_2)$ coincide.*

If $P(U) \subset U$, then the Poincaré map P defines a discrete time dynamical system (more precisely a semi-dynamical system) $\psi : \mathbb{N} \times U \rightarrow U$ in the following way. Let $\psi(n, q) = P^n(q)$, where $P^n = P \circ P \circ \dots \circ P$ denotes the composition with n terms. (The notion semi-dynamical system is used, because it is not defined for negative values of n .) Since $P(p) = p$, the point p is a steady state of the dynamical system ψ . It will be shown that its stability determines that of the periodic orbit Γ . Hence we will recall the definition of stability for a fixed point of a discrete time dynamical and the results about linearisation at a fixed point. These will be dealt with in detail in Section 8.

Let $G \subset \mathbb{R}^k$ be an open set and let $g : G \rightarrow G$ be a diffeomorphism. Then g defines a discrete time dynamical system $\psi : \mathbb{Z} \times G \rightarrow G$, by the definition $\psi(n, q) = g^n(q)$, where $g^n = g \circ g \circ \dots \circ g$ is a composition of n terms. If $n < 0$, then let $g^n = g^{-1} \circ g^{-1} \circ \dots \circ g^{-1}$ be a composition of $-n$ terms.

Definition 5.9.. *The point $p \in G$ is an equilibrium or fixed point of the dynamical system ψ , if $g(p) = p$ (that is $\psi(n, p) = p$, for all $n \in \mathbb{Z}$).*

Definition 5.10.. *The fixed point p is called stable, if for any $\varepsilon > 0$ there exists $\delta > 0$, such that $|q - p| < \delta$ and $n \in \mathbb{N}$ imply $|g^n(q) - p| < \varepsilon$. The fixed point p is called unstable, if it is not stable.*

The fixed point p is called asymptotically stable, if it is stable and $|q - p| < \delta$ implies $\lim_{n \rightarrow \infty} g^n(q) = p$.

The stability of the fixed point can be determined by linearisation as it was shown for continuous time dynamical systems.

Theorem 5.11.. *(Stability of the fixed point) If $|\lambda| < 1$ for all eigenvalues of the matrix $g'(p)$, then the fixed point p is asymptotically stable.*

Let us return now to the investigation of the Poincaré map $P : U \rightarrow \Sigma$. It will be studied how the stability of a fixed point of the Poincaré map determines the stability of the periodic orbit Γ containing the given fixed point. A first, and important observation is that the periodic orbit cannot be asymptotically stable, because the distance between two solutions starting from two different points of the periodic orbit does not tend to zero. Despite of this fact, trajectories may converge to the periodic orbit, hence it can be stable as a set, in a suitable sense. For example, it can be an ω limit set or an attractor. Therefore in order to characterise its stability the notion of orbital stability will be introduced. The following usual definition for the distance of a set and a point is used.

$$d(q, \Gamma) = \inf\{|q - \gamma(t)| : t \in [0, T]\}.$$

Definition 5.12.. *The periodic solution γ is called orbitally stable, if for all $\varepsilon > 0$ there exists $\delta > 0$, for which $d(q, \Gamma) < \delta$ and $t \geq 0$ imply $d(\varphi(t, q), \Gamma) < \varepsilon$.*

The periodic solution γ is called orbitally asymptotically stable, if it is orbitally stable and

$$\lim_{t \rightarrow \infty} d(\varphi(t, q), \Gamma) = 0.$$

The periodic orbit Γ is a limit cycle, if there is a point $q \notin \Gamma$, for which $\Gamma \subset \omega(q)$ or $\Gamma \subset \alpha(q)$.

The periodic orbit Γ is a stable limit cycle, if γ orbitally asymptotically stable.

Let Σ be a transversal section containing the point p , let $U \subset \Sigma$ be a neighbourhood of p and let $P : U \rightarrow \Sigma$ be the Poincaré map.

Theorem 5.13.. *If p is a (asymptotically) stable fixed point of the Poincaré map P , then the periodic solution γ is orbitally (asymptotically) stable.*

The proof is technical, the key ideas of the proof can be found in the books [19, 26]. Thus the stability of the periodic orbit is determined by the eigenvalues of the derivative of the Poincaré map at its fixed point.

Definition 5.14.. *The eigenvalues of the $((n-1) \times (n-1))$ matrix $P'(p)$ are called the characteristic multipliers of the periodic orbit Γ .*

According to Corollary 5.8. the characteristic multipliers do not depend on the choice of p , Σ and the Poincaré map. On the other hand, Theorem 5.11. implies that they characterise the stability of the fixed point of the Poincaré map. Therefore we get the following theorem.

Theorem 5.15. (Andronov–Witt). *If the absolute values of all characteristic multipliers of Γ are less than 1, then Γ is a stable limit cycle.*

For two dimensional systems there is a single characteristic multiplier $P'(p)$. It can be shown (see Theorem 2 in Section 3.4 in Perko’s book [19]), that this can be given in the form

$$P'(p) = \exp \left(\int_0^T \operatorname{div} f(\gamma(t)) \, dt \right),$$

where $\operatorname{div} f = \partial_1 f_1 + \partial_2 f_2$ denotes the divergence of f . Thus, if the divergence is negative along the periodic orbit, then it is a stable limit cycle, while if it is positive along the periodic orbit, then it is an unstable limit cycle.

In the two dimensional case the periodic orbit has only one characteristic multiplier and an invariant manifold that is either stable or unstable. In higher dimension it may happen that the periodic orbit is attracting in one direction and repelling in another direction, hence the local phase portrait has a saddle-like structure. In that case the local behaviour of trajectories in a neighbourhood of the periodic orbit can be characterised by the dimension of the stable, unstable and center manifolds. The introduction of invariant manifolds of periodic orbits is illustrated first with the following example.

Example 5.14. *Consider the following three variable system.*

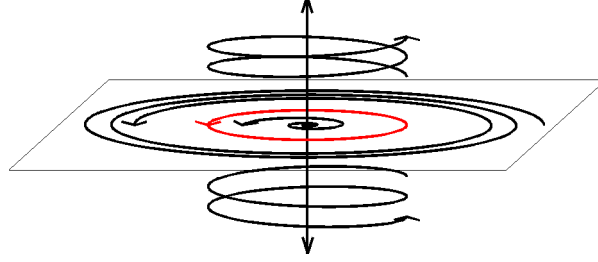
$$\begin{aligned}\dot{x} &= -y + x(1 - x^2 - y^2) \\ \dot{y} &= x + y(1 - x^2 - y^2) \\ \dot{z} &= z\end{aligned}$$

It can be easily seen that the horizontal (x, y) plane and the vertical z axis are invariant, that is trajectories do not leave them. The unit circle centered at the origin and lying in the horizontal plane is a periodic orbit. The trajectories starting in the horizontal plane tend to this limit cycle. The vertical cylinder containing this circle is also invariant, however, the trajectories moving in this cylinder spiral towards infinity, because in the upper half space z is increasing and in the lower half space it is decreasing. Therefore the stable manifold of the periodic orbit is the horizontal plane and its unstable manifold is the vertical cylinder as it is shown in Figure 5.14.

Invariant manifolds for the differential equation in Example 5.14.

Let us turn now to the definition of the stable, unstable and center manifolds of a periodic orbit.

Let $M \subset \mathbb{R}^n$ be a connected open set and let $\varphi : \mathbb{R} \times M \rightarrow M$ be a dynamical system. Let $p \in M$ be a periodic point with period T , that is $\varphi(T, p) = p$. The notations $\gamma(t) := \varphi(t, p)$ and $\Gamma := \{\gamma(t) : t \in [0, T]\}$ are used again. The characteristic multipliers of the periodic orbit Γ are denoted by $\mu_1, \mu_2, \dots, \mu_{n-1}$.



Theorem 5.16. (Stable and unstable manifold theorem). *Assume that there are k characteristic multipliers of Γ within the unit disk (that is having absolute value less than one) and outside the unit disk there are $n - 1 - k$ (having absolute value greater than one). Then for some $\delta > 0$ let $W_s^{loc}(\Gamma)$ be the set of those points $q \in \mathbb{R}^n$, for which*

1. $d(\varphi(t, q), \Gamma) \rightarrow 0$ as $t \rightarrow \infty$,
2. $d(\varphi(t, q), \Gamma) < \delta$, if $t \geq 0$.

Moreover, let $W_u^{loc}(\Gamma)$ be the set of those points $q \in \mathbb{R}^n$, for which

1. $d(\varphi(t, q), \Gamma) \rightarrow 0$ as $t \rightarrow -\infty$,
2. $d(\varphi(t, q), \Gamma) < \delta$, if $t \leq 0$.

Then there exist $\delta > 0$, such that $W_s^{loc}(\Gamma)$ is a $k + 1$ dimensional positively invariant differentiable manifold and $W_u^{loc}(\Gamma)$ is an $n - k$ dimensional negatively invariant differentiable manifold, that are called the stable and unstable manifolds of the periodic orbit Γ . These manifolds intersect transversally at the points of Γ .

The global stable and unstable manifolds can be defined similarly to the case of equilibria, using the local stable and unstable manifolds $W_s^{loc}(\Gamma)$ and $W_u^{loc}(\Gamma)$. The stable manifold is defined in such a way that the trajectories starting in that manifold tend to the periodic orbit Γ . Since these trajectories are in the local stable manifold in a neighbourhood of the periodic orbit, the global stable manifold can be defined by

$$W_s(\Gamma) := \bigcup_{t \leq 0} \varphi(t, W_s^{loc}(\Gamma)).$$

The unstable manifold is defined in such a way that the trajectories starting from it tend to Γ as $t \rightarrow -\infty$. Since these trajectories are in the local unstable manifold in a neighbourhood of the periodic orbit, the global unstable manifold can be defined by

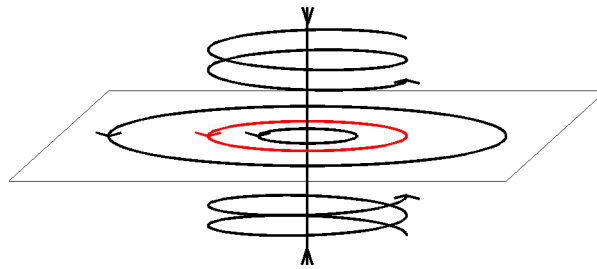
$$W_u(\Gamma) := \bigcup_{t \geq 0} \varphi(t, W_u^{loc}(\Gamma)).$$

If there are characteristic multipliers with absolute value 1, then besides these invariant manifolds the periodic orbit has a center manifold denoted by $W_c(\Gamma)$. Its dimension is $m + 1$ if there are m characteristic multipliers along the unit circle.

Example 5.15. Consider the following three variable system.

$$\begin{aligned}\dot{x} &= -y \\ \dot{y} &= x \\ \dot{z} &= -z\end{aligned}$$

It can be easily seen that the horizontal plane (x, y) and the vertical axis z are invariant, that is trajectories do not leave them. In the horizontal plane the orbits are circles centered at the origin. Consider the unit circle as a periodic orbit. Its center manifold is the horizontal plane, and the stable manifold is the vertical cylinder containing the circle. The later statement is proved by the fact that the cylinder is invariant and for positive values of z its time derivative is negative, while for negative values the time derivative is positive. These invariant manifolds are shown in Figure 5.15.



Invariant manifolds for the differential equation in Example 5.15.

5.3 Index theory for two dimensional systems

An important tool for investigating the global phase portrait in two dimension is the index of the vector field. The index is a topological invariant that can be defined also in higher dimension, however it is dealt with here only for two dimensional systems. Consider the system

$$\begin{aligned}\dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y)\end{aligned}$$

and take a continuous simple closed curve $\gamma : [a, b] \rightarrow \mathbb{R}^2$ (it is not assumed to be an orbit of the differential equation). The index of γ , denoted by $\text{ind}(\gamma)$ is an integer that gives the number of rotations of the vector field $(P(\gamma(s)), Q(\gamma(s)))$ while s moves in the interval $[a, b]$. It can be seen easily that the index of a curve making a round around a (stable or unstable) node is 1, while that around a saddle point the index is -1. These statements can be proved also formally once the index is defined by a formula. This is what will be shown now.

Denote by $\Theta(x, y)$ the angle of the vector $(P(x, y), Q(x, y))$ with the x coordinate axis at a given point (x, y) . Given a curve $\gamma : [a, b] \rightarrow \mathbb{R}^2$ let $\Theta^*(s) = \Theta(\gamma(s))$ for $s \in [a, b]$. Then

$$\tan \Theta^*(s) = \frac{Q(\gamma(s))}{P(\gamma(s))}.$$

Differentiating with respect to s one obtains

$$\begin{aligned}\frac{1}{\cos^2 \Theta^*(s)} \cdot \dot{\Theta}^*(s) &= \frac{\left(\partial_1 Q(\gamma(s)) \cdot \dot{\gamma}_1(s) + \partial_2 Q(\gamma(s)) \cdot \dot{\gamma}_2(s) \right) P(\gamma(s))}{P^2(\gamma(s))} - \\ &\quad - \frac{\left(\partial_1 P(\gamma(s)) \cdot \dot{\gamma}_1(s) + \partial_2 P(\gamma(s)) \cdot \dot{\gamma}_2(s) \right) Q(\gamma(s))}{P^2(\gamma(s))}.\end{aligned}$$

Using that

$$\frac{1}{\cos^2 \Theta^*} = \text{tg}^2 \Theta^* + 1 = \frac{Q^2}{P^2} + 1 = \frac{Q^2 + P^2}{P^2}$$

the derivative $\dot{\Theta}^*$ can be expressed as

$$\dot{\Theta}^*(s) = \frac{P(\partial_1 Q \dot{\gamma}_1 + \partial_2 Q \dot{\gamma}_2) - Q(\partial_1 P \dot{\gamma}_1 + \partial_2 P \dot{\gamma}_2)}{P^2 + Q^2}.$$

The rotation of the vector $(P(\gamma(s)), Q(\gamma(s)))$ can be given by the integral

$$\frac{1}{2\pi} \int_a^b \dot{\Theta}^*(s) ds$$

hence the index of the curve can be defined as follows.

Definition 5.17.. Let $\gamma : [a, b] \rightarrow \mathbb{R}^2$ be a continuous simple closed curve, that do not pass through any equilibrium. Then its index with respect to the system $\dot{x} = P(x, y)$ $\dot{y} = Q(x, y)$ is

$$\text{ind}(\gamma) = \frac{1}{2\pi} \int_a^b \frac{P(\partial_1 Q \dot{\gamma}_1 + \partial_2 Q \dot{\gamma}_2) - Q(\partial_1 P \dot{\gamma}_1 + \partial_2 P \dot{\gamma}_2)}{P^2 + Q^2}.$$

Example 5.16. Compute the index of a curve encircling an unstable node. The origin is an unstable node of the system

$$\begin{aligned}\dot{x} &= x \\ \dot{y} &= y.\end{aligned}$$

Let $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$ be the parametrisation of the unit circle centered at the origin. Then the vector field's direction is radial and pointing outward at each point of the curve as it is shown in Figure ??, therefore its rotation is 2π , hence the index of the curve is 1. This can be computed also by the formal definition as follows. In this case

$$\begin{aligned}P(x, y) &= x, \quad \partial_1 P = 1, \quad \partial_2 P = 0 \\ Q(x, y) &= y, \quad \partial_1 Q = 0, \quad \partial_2 Q = 1\end{aligned}$$

and $\dot{\gamma}_1 = -\sin t$, $\dot{\gamma}_2 = \cos t$. Hence the definition yields

$$\text{ind}(\gamma) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos t(\cos t) - \sin t(-\sin t)}{\cos^2 t + \sin^2 t} dt = 1.$$

Example 5.17. Compute the index of a curve encircling a stable node. The origin is a stable node of the system

$$\begin{aligned}\dot{x} &= -x \\ \dot{y} &= -y.\end{aligned}$$

Let $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$ be the parametrisation of the unit circle centered at the origin. Then the vector field's direction is radial and pointing inward at each point of the curve as it is shown in Figure ??, therefore its rotation is 2π , hence the index of the curve is 1. This can be computed also by the formal definition as follows. In this case

$$\begin{aligned}P(x, y) &= -x, \quad \partial_1 P = -1, \quad \partial_2 P = 0 \\ Q(x, y) &= -y, \quad \partial_1 Q = 0, \quad \partial_2 Q = -1\end{aligned}$$

and $\dot{\gamma}_1 = -\sin t$, $\dot{\gamma}_2 = \cos t$. Hence the definition yields

$$\text{ind}(\gamma) = \frac{1}{2\pi} \int_0^{2\pi} \frac{-\cos t(-\cos t) + \sin t(\sin t)}{\cos^2 t + \sin^2 t} dt = 1.$$

Example 5.18. Compute the index of a curve encircling a saddle. The origin is a saddle point of the system

$$\begin{aligned}\dot{x} &= x \\ \dot{y} &= -y.\end{aligned}$$

Let $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$ be the parametrisation of the unit circle centered at the origin. Then the vector field's direction is shown in Figure ??, therefore its rotation is -2π , hence the index of the curve is -1. This can be computed also by the formal definition as follows. In this case

$$\begin{aligned}P(x, y) &= x, \quad \partial_1 P = 1, \quad \partial_2 P = 0 \\ Q(x, y) &= -y, \quad \partial_1 Q = 0, \quad \partial_2 Q = -1\end{aligned}$$

and $\dot{\gamma}_1 = -\sin t$, $\dot{\gamma}_2 = \cos t$. Hence the definition yields

$$\text{ind}(\gamma) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos t(-\cos t) + \sin t(-\sin t)}{\cos^2 t + \sin^2 t} dt = -1.$$

The following proposition makes the computation of the index easier and it is also important from the theoretical point of view. We do not prove it rigorously, however, geometrically it is easy to see.

Proposition 5.3. For an arbitrary curve γ and for an arbitrary vector field (P, Q) the following statements hold.

1. The index $\text{ind}(\gamma)$ depends continuously on the curve γ , if it does not pass through an equilibria.
2. The index $\text{ind}(\gamma)$ depends continuously on the functions P and Q , if the curve does not pass through an equilibria.

Since the index is an integer value, the continuous dependence implies the following.

Corollary 5.18.. For an arbitrary curve γ and vector field (P, Q) the index $\text{ind}(\gamma)$ is constant as the curve or the vector field is changed, if the curve does not pass through an equilibria of the vector field.

This corollary enables us to prove global results about the phase portrait.

Proposition 5.4. If there is no equilibrium inside γ , then $\text{ind}(\gamma) = 0$.

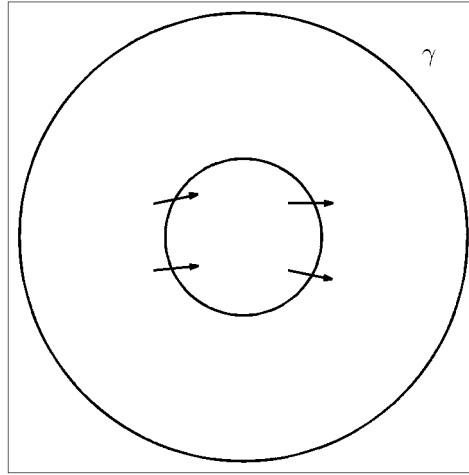


Figure 5.7: The index of a curve is zero, if it does not contain any equilibrium in its interior.

Proof. If there is no equilibrium inside γ , then it can be shrunk to a single point without crossing an equilibrium. In Figure 5.7 one can see that if γ is shrunk to a certain small size then the rotation of the vector field along that curve is 0, hence the index of the curve is 0. Since the index is not changed as the curve is shrunk, the index of γ is also 0. \square

The corollary above also enables us to define the index of an equilibrium.

Definition 5.19.. *Let (x_0, y_0) be an isolated equilibrium of system $\dot{x} = P(x, y)$ $\dot{y} = Q(x, y)$. Then the index of this steady state is defined as the index of a curve encircling (x_0, y_0) but not containing any other equilibrium in its interior. (According to the corollary this is well-defined.)*

Based on the examples above we have the following proposition about the indices of different equilibria.

Proposition 5.5. *The index of a saddle point is -1 , while the index of a node or a focus is 1 .*

The following proposition can also be proved by varying the curve continuously.

Proposition 5.6. *Let (x_i, y_i) , $i = 1, 2, \dots, k$ be the equilibria in the interior of the curve γ . Then the index of the curve is equal to the sum of the indices of the equilibria, that is*

$$\text{ind}(\gamma) = \sum_{i=1}^k \text{ind}(x_i, y_i).$$

It can be seen in Figure 5.8 that in the case when γ is a periodic orbit, the rotation of the vector field along gamma is 2π , yielding the following.

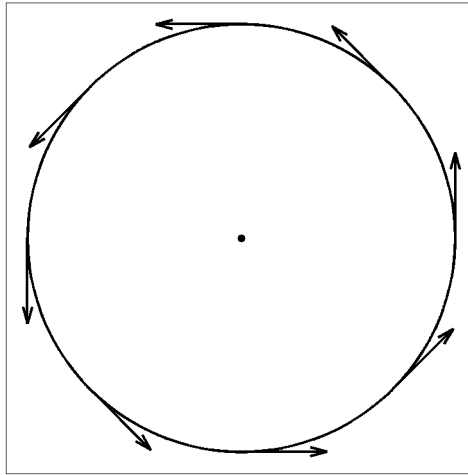


Figure 5.8: Computing the index of a periodic orbit.

Proposition 5.7. *If γ is a periodic orbit, then $\text{ind}(\gamma) = 1$.*

This proposition together with the previous one we immediately get the following.

Corollary 5.20.. *If γ is a periodic orbit, then it contains at least one equilibrium in its interior. Moreover, the sum of the indices of these equilibria is 1.*

We note that the index can be defined also for differential equations on other two dimensional manifolds, for example, on the sphere, or torus. The Poincaré's index theorem states that the sum of the indices on a compact manifold is equal to the Euler characteristic of the manifold, which is 2 for the sphere and 0 for the torus. Thus the sum of the indices of equilibria on the sphere is 2.

5.4 Behaviour of trajectories at infinity

One of the difficulties of classifying two dimensional phase portraits is caused by non-compactness of the phase plane, that makes it difficult to catch the behaviour of trajectories at infinity. This problem can be handled by projecting the phase plane to a compact manifold. Here the Poincaré center projection is presented that maps the plane to the upper hemisphere and takes the points at infinity to the equator of the sphere. By using that the structure of the phase portrait at infinity can be studied in detail.

Let us consider the unit sphere centered at the origin and a plane tangential to the sphere at the point $(0, 0, 1)$ (i.e. at the north pole). Let the origin of the plane be at the tangent point, the coordinates in the plane are denoted by (x, y) . The space coordinates are denoted by X, Y, Z . The points of the plane are projected to the upper hemisphere as follows. Take a point (x, y) in the plane, connect to the origin of the space and associate to (x, y) the intersection point of the ray (from the origin to (x, y)) and the upper hemisphere. In Figure 5.9 there are similar triangles leading to the coordinate transformation $x = X/Z$.

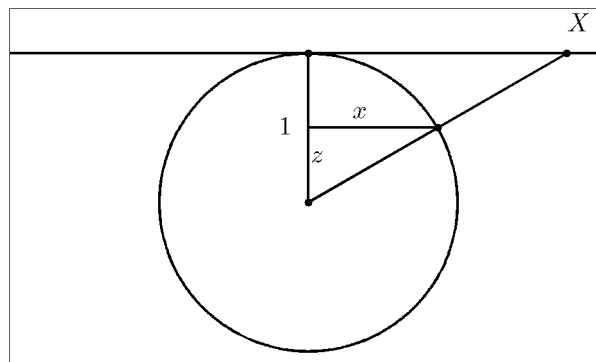


Figure 5.9: The projection of the plane to the upper hemisphere.

Thus let us introduce the space coordinates X, Y, Z in terms of the (x, y) coordinates

of the plane as follows

$$\begin{aligned} X &= xZ \\ Y &= yZ \\ Z &= \frac{1}{\sqrt{x^2 + y^2 + 1}}. \end{aligned}$$

We will now investigate how this coordinate transformation effects the phase portrait of the differential equations

$$\dot{x} = P(x, y) \tag{5.1}$$

$$\dot{y} = Q(x, y). \tag{5.2}$$

Differentiating the identities defining the projection leads to the system of differential equations

$$\begin{aligned} \dot{X} &= \dot{x}Z + x\dot{Z} \\ \dot{Y} &= \dot{y}Z + y\dot{Z} \\ \dot{Z} &= -Z^3(x\dot{x} + y\dot{y}). \end{aligned}$$

Substituting the differential equations (5.1)-(5.2) into these equations and using the equations of the transformation one obtains

$$\dot{X} = Z \cdot P\left(\frac{X}{Z}, \frac{Y}{Z}\right) - Z \cdot X \cdot \left(XP\left(\frac{X}{Z}, \frac{Y}{Z}\right) + YQ\left(\frac{X}{Z}, \frac{Y}{Z}\right) \right).$$

The differential equations for Y and Z can be obtained similarly. For the sake of simplicity we write P instead of $P\left(\frac{X}{Z}, \frac{Y}{Z}\right)$, and similarly for Q . Then the differential equations for the new variables take the form

$$\dot{X} = ZP - ZX(XP + YQ) \tag{5.3}$$

$$\dot{Y} = ZQ - ZY(XP + YQ) \tag{5.4}$$

$$\dot{Z} = -Z^3(XP + YQ). \tag{5.5}$$

It can easily be checked that the unit sphere is invariant for this system. The phase portrait on the sphere is equivalent to that on the plane (x, y) . However, we can see the trajectories at infinity on the sphere, these are the trajectories moving along the equator, i.e. in the plane $Z = 0$.

It will be shown for some linear systems how the known phase portrait can be extended to infinity.

Example 5.19. Consider the linear system

$$\begin{aligned}\dot{x} &= x \\ \dot{y} &= -y\end{aligned}$$

with a saddle point. Then $P(x, y) = x$, $Q(x, y) = -y$, hence $ZP = X$, $ZQ = -Y$ and $Z(XP + YQ) = X^2 - Y^2$. Substituting these into system (5.3)-(5.5), one obtains the system

$$\begin{aligned}\dot{X} &= X(1 - X^2 + Y^2) \\ \dot{Y} &= Y(-1 - X^2 + Y^2) \\ \dot{Z} &= -Z(X^2 - Y^2)\end{aligned}$$

for the new variables. Let us investigate the phase portrait on the sphere. The finite equilibria are $(0, 0, 1)$ and $(0, 0, -1)$. These are saddle points, because the original point in the plane is also a saddle. Let us turn now to the equilibria at infinity. The third coordinate of these is $Z = 0$, hence these are given by equation $X^2 + Y^2 = 1$. This equation yields $X = 0$ and $Y = \pm 1$ or $X = \pm 1$ and $Y = 0$. Hence there are four equilibria at infinity, namely $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$. Their type can be obtained as follows. In the point $(1, 0, 0)$ project the equation to the plane $X = 1$, then we get the two variable system

$$\begin{aligned}\dot{Y} &= Y(-2 + Y^2) \\ \dot{Z} &= -Z(1 - Y^2).\end{aligned}$$

Its steady state is $(0, 0)$ (corresponding to $(1, 0, 0)$). The Jacobian of the system at $(0, 0)$ is

$$\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix},$$

hence $(0, 0)$ is a stable node, implying that $(1, 0, 0)$ is a stable node. The remaining three steady states can be studied in a similar way. The point $(-1, 0, 0)$ is also a stable node, while the points $(0, 1, 0)$ and $(0, -1, 0)$ are unstable nodes. Therefore we get the phase portrait in the upper hemisphere as it is shown in Figure 5.10. (The phase portrait is projected from the upper hemisphere to the plane of the equator vertically.) One can observe that there are 6 equilibria in the whole sphere all together. Two of them are saddles (with index -1) and the remaining four are nodes (with index 1). Hence the sum of the indices is $2(-1) + 4 = 2 = \chi(S^2)$ yielding the Euler characteristic of the sphere, as it is stated by Poincaré's index theorem.

Example 5.20. Consider the linear system

$$\begin{aligned}\dot{x} &= x - y \\ \dot{y} &= x + y\end{aligned}$$

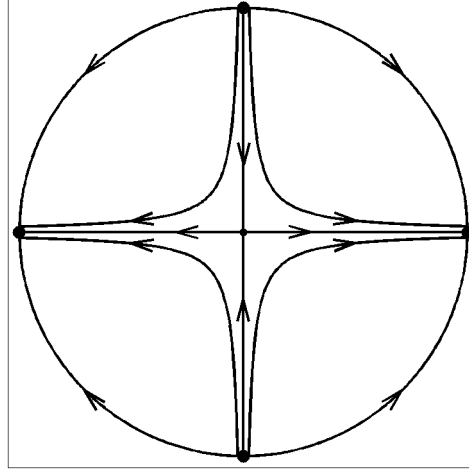


Figure 5.10: The complete phase portrait for a saddle.

with an unstable focus. Then $P(x, y) = x - y$, $Q(x, y) = x + y$, yielding $ZP = X - Y$, $ZQ = X + Y$ and $Z(XP + YQ) = X^2 + Y^2$. Substituting these equations into system (5.3)-(5.5) leads to

$$\begin{aligned}\dot{X} &= X - Y - X(X^2 + Y^2) \\ \dot{Y} &= X + Y - Y(X^2 + Y^2) \\ \dot{Z} &= -Z^2(X^2 + Y^2).\end{aligned}$$

Let us investigate the phase portrait on the sphere. The finite equilibria are $(0, 0, 1)$ and $(0, 0, -1)$. These are unstable foci, because the original point in the plane is also an unstable focus. Let us turn now to the equilibria at infinity. The third coordinate of these is $Z = 0$, hence these are given by equation $X^2 + Y^2 = 1$. Substituting this equation into the first two equations one can see that there are no solutions, that is the system has no equilibrium point at infinity. Therefore we get the phase portrait in the upper hemisphere as it is shown in Figure 5.11. (The phase portrait is projected from the upper hemisphere to the plane of the equator vertically.) One can observe that there are two equilibria in the whole sphere all together, both of them are unstable foci (with index 1). Hence the sum of the indices is $2 = \chi(S^2)$ yielding the Euler characteristic of the sphere, as it is stated by Poincaré's index theorem.

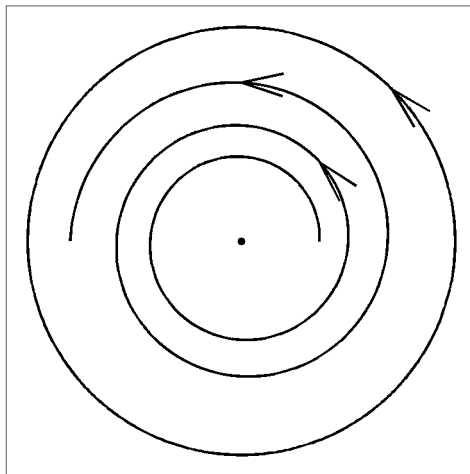


Figure 5.11: The complete phase portrait for an unstable focus.

5.5 Exercises

1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function. If f has three zeros, then how many classes are under topological equivalence for the equations of the form $\dot{x} = f(x)$?

(A) 10, (B) 9, (C) 8

Answer: (A).

2. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function. If f has four zeros, then how many classes are under topological equivalence for the equations of the form $\dot{x} = f(x)$?

(A) 15, (B) 16, (C) 17

Answer: (B).

3. How many equilibria are at infinity for the system

$$\begin{aligned}\dot{x} &= -y \\ \dot{y} &= x\end{aligned}$$

with a center at the origin?

(A) 2, (B) 3, (C) 0

Answer: (C).

4. How many equilibria are at infinity for the system

$$\dot{x} = x$$

$$\dot{y} = y$$

with a node at the origin?

$$(A) \quad 4, \quad (B) \quad 3, \quad (C) \quad 0$$

Answer: (A).

Chapter 6

Introduction to bifurcation theory and structural stability

In the previous chapters several methods were shown for the investigation of the phase portrait of systems of differential equations. Systems occurring in applications typically contain parameters, hence it is a natural question how the phase portrait changes as the values of the parameters are varied. Changing a parameter the solution of the differential equation changes, however, qualitative change in the behaviour of the solutions occurs only at a few (isolated) values of the parameter. As an illustration, consider the differential equation $\dot{x} = ax$, in which $a \in \mathbb{R}$ is a parameter. The solution can be easily given as $x(t) = e^{at}x(0)$ that obviously depends on a , but as the value of a is changed from $a = 1$ to $a = 1.01$, then there is no qualitative change in the solution. However, changing the value of a around zero we can observe qualitative change in the behaviour. For $a = 0$ all solutions are constant functions, for negative values of a the solutions tend to zero, while for positive values of a they tend to infinity. The bifurcation is the qualitative change in the phase portrait. The bifurcation occurs at those parameter values for which the phase portrait is not topologically equivalent to those belonging to nearby parameter values. This is formulated in the following definition.

Consider the equation $\dot{x}(t) = f(x(t), \lambda)$, where $f : \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^n$ is a continuously differentiable function and $\lambda \in \mathbb{R}^k$ is a parameter.

Definition 6.1.. *The parameter value $\lambda_0 \in \mathbb{R}^k$ is called regular, if there exists $\delta > 0$, for which $|\lambda - \lambda_0| < \delta$ implies that the system $f(\cdot, \lambda)$ is topologically equivalent to the system $f(\cdot, \lambda_0)$. At the parameter value $\lambda_0 \in \mathbb{R}^k$ there is a bifurcation if it is not regular.*

In the first section several types of bifurcations will be shown through examples. The most important two bifurcations will be dealt with in detail in the next section, where also sufficient conditions will be given for these bifurcations. In the last section the structural stability is introduced and studied.

6.1 Normal forms of elementary bifurcations

In this section the normal forms of the most frequently occurring bifurcations are presented, that is the simplest systems are shown, where these bifurcations can be observed.

Example 6.1. Consider the differential equation $\dot{x} = \lambda - x$, in which $\lambda \in \mathbb{R}$ is a parameter. For a given value of λ the equilibrium is the point $x = \lambda$. This point is globally asymptotically stable for all values of λ , that is trajectories are tending to this point. The phase portrait for different values of λ can be shown in a coordinate system, where the horizontal axis is for λ and for a given value of λ the corresponding phase portrait is given on the vertical line at λ , as it is shown in Figure 6.1. This Figure shows that the phase portrait is the same for all values of λ , that is all values of λ are regular, i.e. there is no bifurcation. The topological equivalence of the phase portraits corresponding to different values of λ can be formally verified by determining the homeomorphism the orbits to each other. For example, the orbits for $\lambda = 0$ can be taken to those belonging to $\lambda = 1$ by the homeomorphism $h(p) = p - 1$.

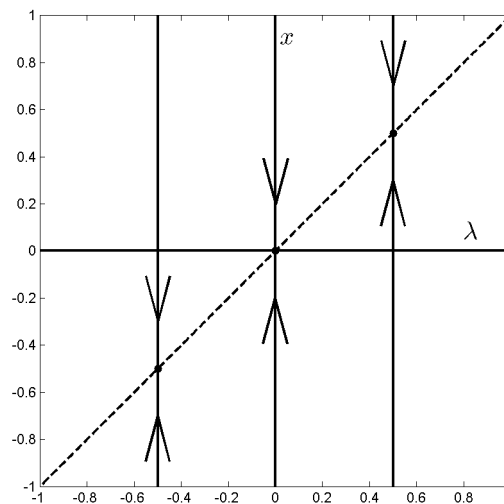


Figure 6.1: The phase portrait of the differential equation $\dot{x} = \lambda - x$ for different values of the parameter λ .

The animation shows how the phase portrait of the differential equation in Example 6.1 changes as λ is varied between -1 and 1 .

Let us now turn to differential equations where bifurcation may occur. The simplest of them is the fold bifurcation that is also called saddle-node bifurcation.

Example 6.2 (fold or saddle-node bifurcation). Consider the differential equation $\dot{x} = \lambda - x^2$, where $\lambda \in \mathbb{R}$ is a parameter. In this case the existence of the equilibrium depends on the parameter λ . If $\lambda < 0$, then there is no equilibrium, for $\lambda = 0$ the origin $x = 0$ is an equilibrium, and for $\lambda > 0$ there are two equilibria $x = \pm\sqrt{\lambda}$. The phase portrait can be shown for different values of λ by using the same method as in the previous example, as it is shown in Figure 6.2. It can be seen in the Figure that the bifurcation is at $\lambda = 0$, since the phase portrait is different for positive and negative values of the parameter. The values $\lambda \neq 0$ are regular, because choosing a positive or negative value of λ the phase portrait does not change as λ is varied in a suitably small neighbourhood. The topological equivalence of the phase portraits corresponding to different non-zero values of λ can be formally verified by determining the homeomorphism taking the orbits to each other. For example, the orbits for $\lambda < 0$ can be taken to each other by the homeomorphism $h(p) = p$. For positive values of λ the orbits for can be taken to each other by a piece-wise linear homeomorphism. The bifurcation in this example is called fold or saddle-node bifurcation. The latter refers to the bifurcation in the two dimensional system $\dot{x} = \lambda - x^2$, $\dot{y} = -y$ that is illustrated by the animation ??.

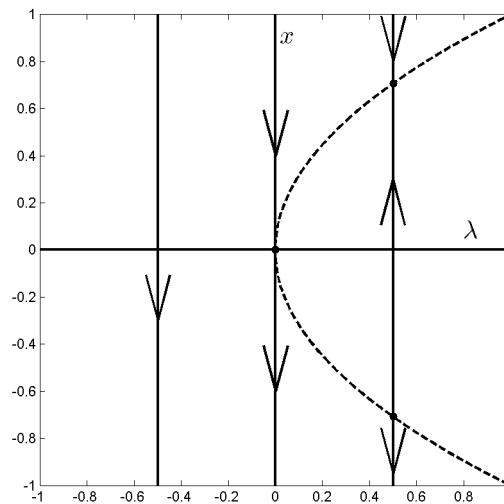


Figure 6.2: Fold or saddle-node bifurcation in the differential equation $\dot{x} = \lambda - x^2$ at $\lambda = 0$.

The animation shows how the phase portrait of the differential equation $\dot{x} = \lambda - x^2$ changes as λ is varied between -1 and 1 .

Saddle-node bifurcation in the two dimensional system $\dot{x} = \lambda - x^2$, $\dot{y} = -y$ at $\lambda = 0$.

Animations belonging to these differential equations can be found at <http://www.cs.elte.hu/~simonp/DinRJegyz/>.

Example 6.3 (Transcritical bifurcation). Consider the differential equation $\dot{x} = \lambda x - x^2$, where $\lambda \in \mathbb{R}$ is a parameter. The point $x = 0$ is an equilibrium for any value of λ . Besides this point $x = \lambda$ is also an equilibrium, therefore in the case $\lambda \neq 0$ there are two equilibria, while for $\lambda = 0$ there is only one. Hence there is a bifurcation at $\lambda = 0$. The phase portrait can be shown for different values of λ by using the same method as in Example 6.1, as it is shown in Figure 6.3. It can be seen in the Figure that the bifurcation is at $\lambda = 0$, since the phase portrait for non-zero values of the parameter is different from that of belonging to $\lambda = 0$. The values $\lambda \neq 0$ are regular, because choosing a positive or negative value of λ the phase portrait does not change as λ is varied in a suitably small neighbourhood. For negative values of λ the point $x = 0$ is stable and $x = \lambda$ is unstable, while for positive λ values it is the other way around. This bifurcation is called transcritical because of the exchange of stability. The topological equivalence of the phase portraits corresponding to different non-zero values of λ can be formally verified by determining the homeomorphism taking the orbits to each other.

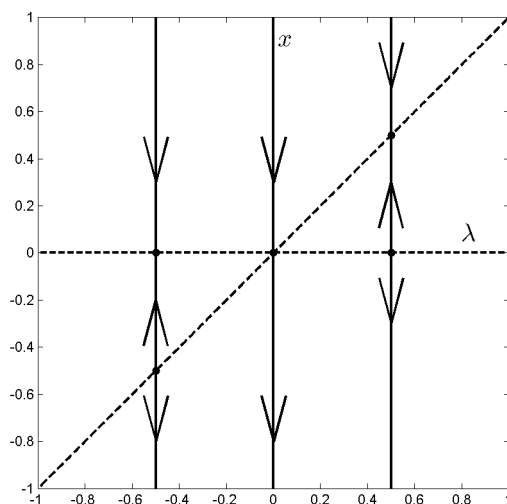


Figure 6.3: Transcritical-bifurcation in the differential equation $\dot{x} = \lambda x - x^2$ at $\lambda = 0$.

The animation shows how the phase portrait of the differential equation $\dot{x} = \lambda x - x^2$ changes as λ is varied between -1 and 1 .

Example 6.4 (Pitchfork bifurcation). Consider the differential equation $\dot{x} = \lambda x - x^3$, where $\lambda \in \mathbb{R}$ is a parameter. The point $x = 0$ is an equilibrium for any value of λ . Besides this point $x = \pm\sqrt{\lambda}$ is also an equilibrium if $\lambda > 0$. Thus for $\lambda < 0$ there is a unique equilibrium, while for $\lambda > 0$ there are 3 equilibria. Hence there is a bifurcation

at $\lambda = 0$. The phase portrait can be shown for different values of λ by using the same method as in Example 6.1, as it is shown in Figure 6.4. It can be seen in the Figure that the bifurcation is at $\lambda = 0$, since the phase portrait for non-zero values of the parameter is different from that of belonging to $\lambda = 0$. The values $\lambda \neq 0$ are regular, because choosing a positive or negative value of λ the phase portrait does not change as λ is varied in a suitably small neighbourhood. For negative values of λ the equilibrium point $x = 0$ is globally stable. For positive values of λ the points $x = \pm\sqrt{\lambda}$ take over stability. This bifurcation is called *pitchfork bifurcation* because of the shape of the bifurcation curve. The topological equivalence of the phase portraits corresponding to different non-zero values of λ can be formally verified by determining the homeomorphism taking the orbits to each other.

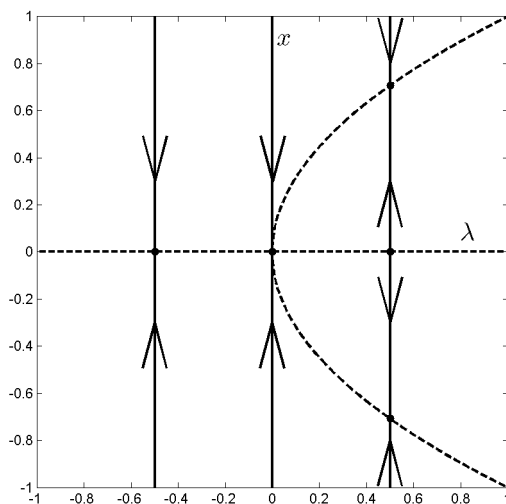


Figure 6.4: Pitchfork bifurcation in the differential equation $\dot{x} = \lambda x - x^3$ at $\lambda = 0$.

The animation shows how the phase portrait of the differential equation $\dot{x} = \lambda x - x^3$ changes as λ is varied between -1 and 1 .

The next example shows how the bifurcation diagram changes by introducing a second parameter.

Example 6.5. Consider the differential equation $\dot{x} = \lambda x - x^3 + \varepsilon$, where $\lambda \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$ are two parameters. First the value of λ is fixed and it is investigated how the phase portrait changes as ε is varied. The equilibria are located along the curve $\varepsilon = x^3 - \lambda x$, the shape of which depends on the sign of λ . If $\lambda < 0$, then it is strictly monotone, while for $\lambda > 0$, it has a local maximum and minimum. The curve is shown in Figure 6.5

for $\lambda = -1$ and for $\lambda = 1$. It can be seen that in the case $\lambda = -1$ all values of ε are regular, because the phase portrait does not change, for all values of ε there is a unique globally attracting equilibrium. In the case of $\lambda = 1$ the phase portrait changes at ε_1 and at ε_2 , at these parameter values fold bifurcation occurs, because two equilibria appear and disappear. Thus changing the value of λ the shape of the bifurcation curve in ε changes as λ crosses zero. The evolution of the bifurcation curve is shown by the animation ?? and in Figure 6.6.

Using the three dimensional bifurcation diagram one can determine how the bifurcation curve $\lambda = x^2 - \varepsilon/x$ (for the parameter λ) looks like when ε is fixed. First, the value $\varepsilon = 0$ is fixed and take the section of the surface shown in Figure 6.6 with the plane $\varepsilon = 0$. Then we get the curve of the pitchfork bifurcation shown in Figure 6.4. Then choosing a negative value of ε , say $\varepsilon = -1$ and taking the section of the surface shown in Figure 6.6 with the plane $\varepsilon = -1$ we get the curve $\lambda = x^2 + 1/x$ shown in the left part of Figure 6.7. Choosing now a positive value of ε , say $\varepsilon = 1$ and taking the section of the surface shown in Figure 6.6 with the plane $\varepsilon = 1$ we get the curve $\lambda = x^2 - 1/x$ shown in the right part of Figure 6.7. This Figure also shows that an arbitrarily small perturbation of the equation of the pitchfork bifurcation destroys this bifurcation and fold bifurcation occurs instead. This means that in one-parameter systems the pitchfork bifurcation is not typical because it contains a degeneracy of codimension two, hence it can be observed in systems with two parameters.

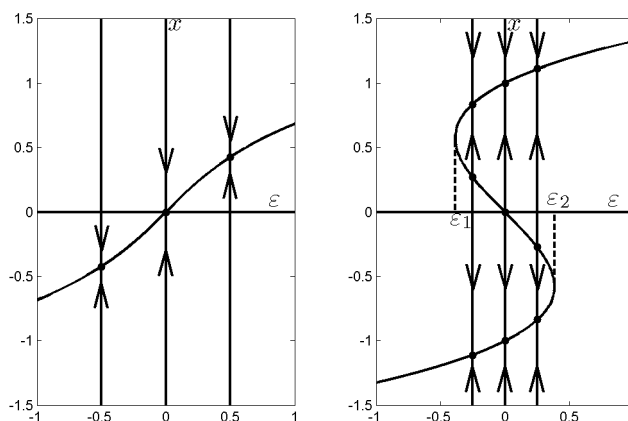


Figure 6.5: The bifurcation diagram with respect to ε in the differential equation $\dot{x} = \lambda x - x^3 + \varepsilon$ for $\lambda = -1$ and for $\lambda = 1$.

The animation shows how the bifurcation diagram with respect to ε of the differential equation $\dot{x} = \lambda x - x^3 + \varepsilon$ changes as λ is varied between -1 and 1 .

Animations belonging to these differential equations can be found at <http://www.cs.elte.hu/~simonp/DinRJegyz/>.

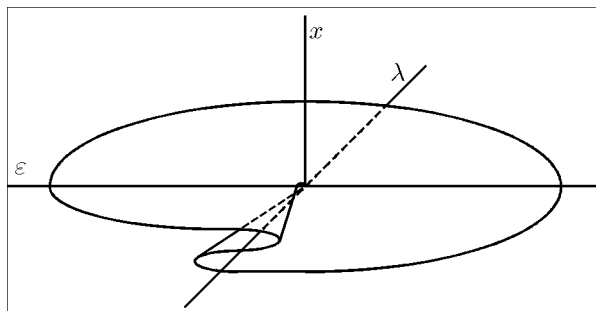


Figure 6.6: The two-parameter bifurcation diagram of the differential equation $\dot{x} = \lambda x - x^3 + \varepsilon$.

The animation shows how the bifurcation diagram with respect to λ of the differential equation $\dot{x} = \lambda x - x^3 + \varepsilon$ changes as ε is varied between -1 and 1 .

Now we turn to bifurcations occurring in at least two dimensional systems.

Example 6.6 (Andronov–Hopf bifurcation). Consider the differential equation $\dot{r} = \lambda r + \sigma r^3$, $\dot{\phi} = 1$ given in polar coordinates, where $\lambda \in \mathbb{R}$ and $\sigma \in \mathbb{R}$ are parameters. First, fix the value $\sigma = -1$ (any $\sigma < 0$ yields the same phenomenon) and see how the phase portrait changes as the value of λ is varied. The origin is an equilibrium for any value of λ and its stability can be easily determined from the differential equation for r . Namely, in the case $\lambda < 0$, we have $\dot{r} = \lambda r - r^3 < 0$, hence r is strictly decreasing and converges to zero, therefore the solutions tend to the origin. However, for $\lambda > 0$ and $r < \sqrt{\lambda}$ we have $\dot{r} = r(\lambda - r^2) > 0$, hence r is strictly increasing, therefore the origin is unstable. Moreover, for $r = \sqrt{\lambda}$ we have $\dot{r} = 0$, that is the circle with radius $\sqrt{\lambda}$ is a periodic orbit that is orbitally asymptotically stable, because inside the circle $\dot{r} > 0$ and outside $\dot{r} < 0$. This phenomenon is illustrated in Figure 6.8. In the Figure the behaviour of r is shown as λ is varied. The bifurcation is at $\lambda = 0$ and the values $\lambda \neq 0$ are regular.

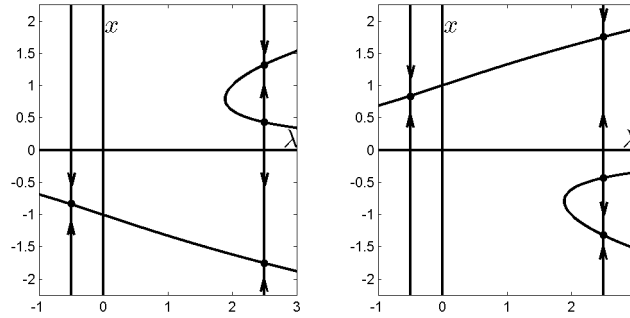


Figure 6.7: The bifurcation diagram with respect to λ in the differential equation $\dot{x} = \lambda x - x^3 + \varepsilon$ for $\varepsilon = -1$ and for $\varepsilon = 1$.

The bifurcation in the two dimensional phase space is shown in Figure 6.9. If $\lambda < 0$, then the origin is globally asymptotically stable, while for $\lambda > 0$ the origin is unstable and the stability is taken over by a stable limit cycle, the size of which is increasing as $\sqrt{\lambda}$. This bifurcation is called *supercritical Andronov–Hopf bifurcation*.

The animation shows how the phase portrait of the differential equation $\dot{r} = \lambda r - r^3$ changes as λ is varied between -0.1 and 0.1 .

Returning to the differential equation $\dot{r} = \lambda r + \sigma r^3$, $\dot{\phi} = 1$ consider the case of positive σ values, say let $\sigma = 1$. The origin is an equilibrium again the stability of which is changed in the same way with λ as before, however, the periodic solution now appears for $\lambda < 0$, and it is unstable. The behaviour of r as λ is varied is shown in Figure 6.10. The origin loses its stability for $\lambda > 0$, however, in this case the periodic orbit does not take over the stability, the trajectories tend to infinity. The bifurcation in the two dimensional phase space is shown in Figure 6.11. If $\lambda < 0$, then the origin is stable but its domain of attraction is only the interior of the periodic orbit. If $\lambda > 0$, then the origin is unstable and the trajectories tend to infinity. This bifurcation is called *subcritical Andronov–Hopf bifurcation*.

The animation shows how the phase portrait of the differential equation $\dot{r} = \lambda r + r^3$ changes as λ is varied between -0.1 and 0.1 .

In the previous examples the bifurcation occurred locally in the phase space, in a neighbourhood of an equilibrium. These kind of bifurcations are called *local bifurcations*.

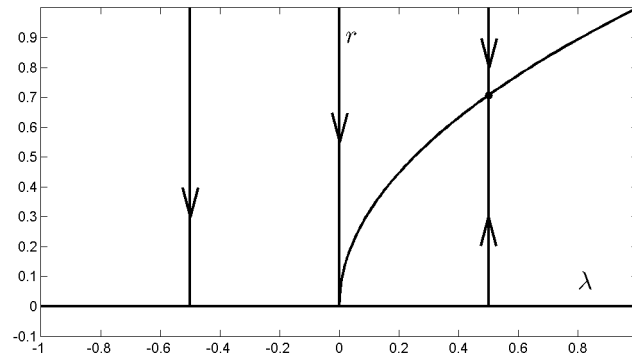


Figure 6.8: Bifurcation of the differential equation $\dot{r} = \lambda r - r^3$ at $\lambda = 0$.

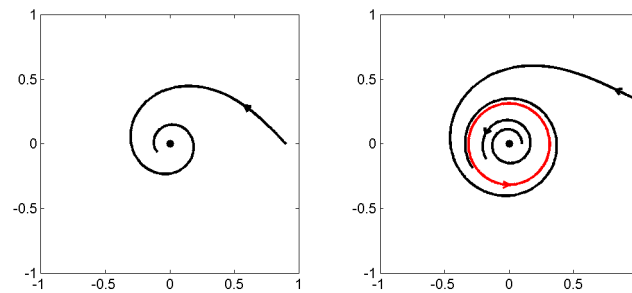


Figure 6.9: Supercritical Andronov–Hopf bifurcation, the origin loses its stability and a stable limit cycle is born.

In the rest of this section global (non-local) bifurcations will be shown, where global structures appear as the parameter is varied. These local structures will be periodic,

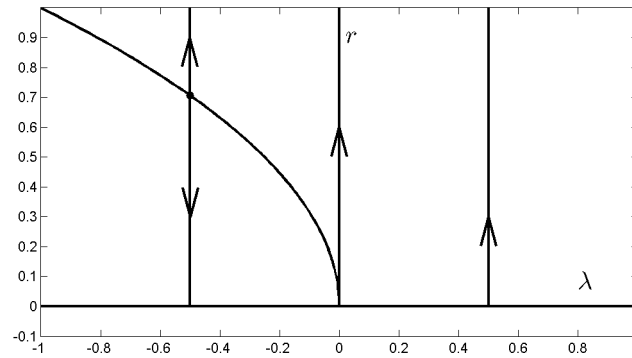


Figure 6.10: Bifurcation of the differential equation $\dot{r} = \lambda r + r^3$ at $\lambda = 0$.

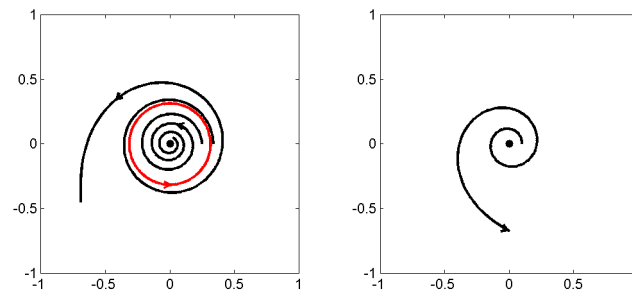


Figure 6.11: Subcritical Andronov–Hopf bifurcation, the origin loses its stability and the unstable limit cycle disappears.

homoclinic and heteroclinic orbits.

Example 6.7 (Fold bifurcation of periodic orbits). *Consider the differential equa-*

tion $\dot{r} = r(\lambda - (r - 1)^2)$, $\dot{\phi} = 1$ given in polar coordinates, where $\lambda \in \mathbb{R}$ is a parameter. Similarly to the previous example, investigate first the behaviour of r as λ is varied. The bifurcation diagram is shown in Figure 6.12. If $\lambda < 0$, then $\dot{r} = r(\lambda - (r - 1)^2) < 0$, hence r is strictly decreasing and converges to zero, therefore the solutions tend to the origin. If λ is positive and close to zero ($\lambda < 1$), then for $\lambda = (r - 1)^2$ we have $\dot{r} = 0$, that is the circles with radius $1 \pm \sqrt{\lambda}$ and centered at the origin are periodic orbits. The inner circle is unstable, while the outer one is stable. Thus the bifurcation occurs at $\lambda = 0$. (We note that the inner cycle disappears at $\lambda = 1$, but this bifurcation is not studied here.) The bifurcation in the two dimensional phase space is shown in Figure 6.13. If $\lambda < 0$, then the origin is globally asymptotically stable. At the bifurcation value $\lambda = 0$ two limit cycles are born with radii close to 1. It is important to note that in the case of Andronov–Hopf bifurcation the radius of the limit cycle born is zero. If $0 < \lambda < 1$, then one of the limit cycles is stable, the other one is unstable. The phenomenon is similar to the fold bifurcation, but in this case instead of equilibria periodic orbits are born. Hence this bifurcation is called fold bifurcation of periodic orbits.

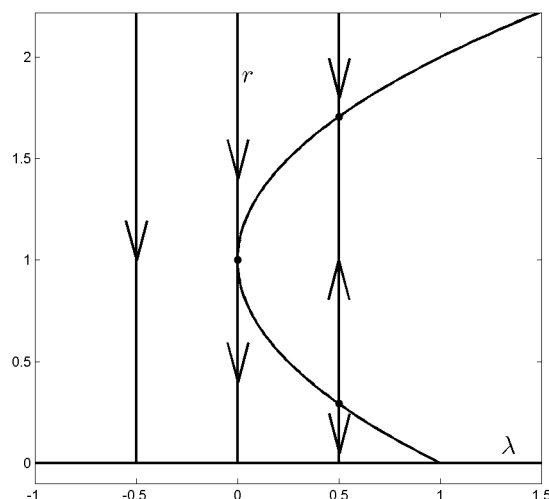


Figure 6.12: Bifurcation of the differential equation $\dot{r} = r(\lambda - (r - 1)^2)$ at $\lambda = 0$.

The animation shows how the phase portrait of the differential equation $\dot{r} = r(\lambda - (r - 1)^2)$ changes as λ is varied between -0.1 and 0.1 .

Example 6.8 (Homoclinic bifurcation). Consider the two dimensional system

$$\dot{x} = y, \quad \dot{y} = x - x^2 + \lambda y,$$

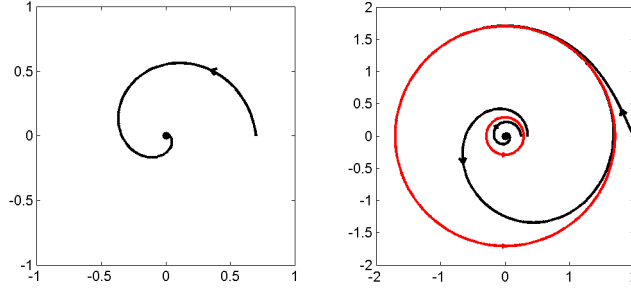


Figure 6.13: Fold-bifurcation for periodic orbits. At $\lambda = 0$ a stable and an unstable periodic orbit are born with radius close to 1.

where $\lambda \in \mathbb{R}$ is a parameter. It is known that for $\lambda = 0$ the function $H(x, y) = \frac{x^3}{3} - \frac{x^2}{2} + \frac{y^2}{2}$ is a first integral. Namely, let $H^*(t) = H(x(t), y(t))$, then

$$\dot{H}^*(t) = x^2(t) \cdot \dot{x}(t) - x(t) \cdot \dot{x}(t) + y(t) \cdot \dot{y}(t) = \lambda y^2(t).$$

Thus for $\lambda = 0$ the value of the function H is constant along trajectories. Moreover, H can also serve as a Lyapunov function. Since for $\lambda > 0$, it is increasing along trajectories and for $\lambda < 0$ it is decreasing along trajectories. The phase portrait is determined first for $\lambda = 0$. Then simply the level curves of H are to be determined, since the orbits lie on the level curves. This way the phase portrait given in Figure 6.14 is obtained. Let us consider now the case $\lambda < 0$. The equilibria remain at $(0, 0)$ and at $(1, 0)$ as for $\lambda = 0$. The origin remains a saddle point, but the point $(1, 0)$ becomes a stable focus. Using that H is decreasing along trajectories, the trajectory along the unstable manifold of the saddle point in the first quadrant tends to the stable focus. Hence also using the direction field one obtains the phase portrait shown in Figure 6.15. The case $\lambda > 0$ can be investigated similarly. The equilibria are again the points $(0, 0)$ and $(1, 0)$, the origin remains a saddle, but the point $(1, 0)$ is an unstable focus. Using that H is now increasing along trajectories, the trajectory along the stable manifold of the saddle point in the right half plane tends to the unstable focus as time goes to $-\infty$. Hence also using the direction field one obtains the phase portrait shown in Figure 6.15. Thus the bifurcation occurs at $\lambda = 0$. In that case the stable and unstable manifolds of the saddle point form a homoclinic orbit, hence the bifurcation is called homoclinic bifurcation.

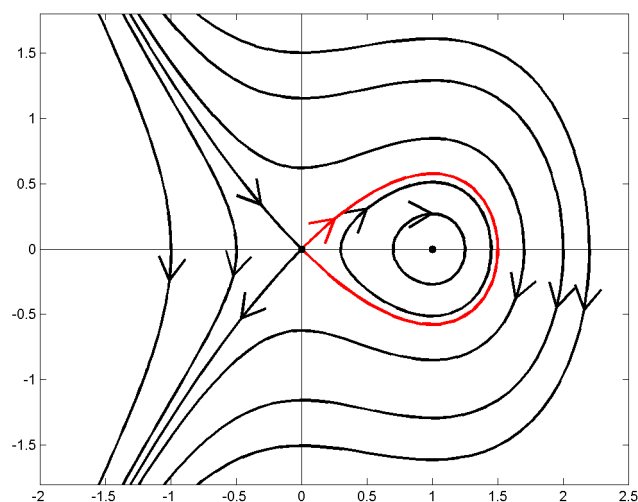


Figure 6.14: The phase portrait of system $\dot{x} = y$, $\dot{y} = x - x^2 + \lambda y$ for $\lambda = 0$.

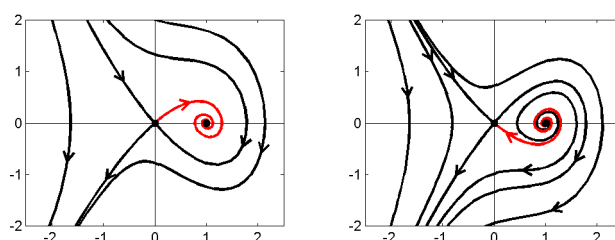


Figure 6.15: The phase portrait of system $\dot{x} = y$, $\dot{y} = x - x^2 + \lambda y$ for $\lambda < 0$ (left) and for $\lambda > 0$ (right).

The animation shows how the phase portrait of the differential equation $\dot{x} = y$, $\dot{y} = x - x^2 + \lambda y$ changes as λ is varied between -0.3 and 0.3 .

Animations belonging to this differential equation can be found at <http://www.cs.elte.hu/~simonp/DinRJegyz/>.

Example 6.9 (Heteroclinic bifurcation). Consider the two dimensional system

$$\dot{x} = 1 - x^2 - \lambda xy, \quad \dot{y} = xy + \lambda(1 - x^2),$$

where $\lambda \in \mathbb{R}$ is a parameter. In order to find the steady states multiply the second equation by λ , then add it to the first equation. This leads to $(1 - x^2)(1 + \lambda^2) = 0$, hence the first coordinate of a steady state can be $x = \pm 1$ and using the second equation $y = 0$. Thus the equilibria are $(\pm 1, 0)$. The phase portrait is determined first for $\lambda = 0$. Using the direction field the phase portrait can be easily given as it is shown in Figure 6.16. The segment of the x coordinate axis between -1 and 1 is a heteroclinic orbit connecting the two saddle points. It will be investigated what happens to this orbit as the value of λ is varied. It is easy to verify that the steady states $(\pm 1, 0)$ remain saddle points as λ is changed. If $\lambda < 0$, then for $y = 0$ and $x \in (-1, 1)$ we have $\dot{x} > 0$ and $\dot{y} = \lambda(1 - x^2) < 0$, hence the trajectories cross the x axis to the right and down. Hence the trajectory on the right part of the unstable manifold of the saddle point $(-1, 0)$ lies in the lower half plane and tends to infinity as it is shown in Figure 6.17. Thus for $\lambda < 0$ the heteroclinic orbit connecting the saddle points does not exist. The situation is similar in the case $\lambda > 0$. In this case for $y = 0$ and $x \in (-1, 1)$ we have $\dot{x} > 0$ and $\dot{y} = \lambda(1 - x^2) > 0$, hence the trajectories cross the x axis to the right and up. Hence the trajectory on the right part of the unstable manifold of the saddle point $(-1, 0)$ lies in the upper half plane and tends to infinity as it is shown in Figure 6.17. Thus varying the value of λ a heteroclinic orbit appears at $\lambda = 0$, hence this bifurcation is called heteroclinic bifurcation

The animation shows how the phase portrait of the differential equation $\dot{x} = 1 - x^2 - \lambda xy$, $\dot{y} = xy + \lambda(1 - x^2)$ changes as λ is varied between -0.5 and 0.5 .

Animations belonging to this differential equation can be found at <http://www.cs.elte.hu/~simonp/DinRJegyz/>.

6.2 Necessary conditions of bifurcations

The examples of the previous section show that local bifurcation may occur at non-hyperbolic equilibria. This statement will be proved generally in this section. The notion of local bifurcation is defined first. Consider a general system of the form $\dot{x}(t) = f(x(t), \lambda)$, where $f : \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^n$ is a continuously differentiable function, and $\lambda \in \mathbb{R}^k$ is a parameter.

Definition 6.2.. The pair (x_0, λ_0) is called locally regular, if there exist a neighbourhood $U \subset \mathbb{R}^n$ of x_0 and $\delta > 0$, such that for $|\lambda - \lambda_0| < \delta$ the systems $f|_U(\cdot, \lambda_0)$ and $f|_U(\cdot, \lambda)$ are topologically equivalent (that is the phase portraits are topologically equivalent in U). There is a local bifurcation at (x_0, λ_0) , if (x_0, λ_0) is not locally regular.

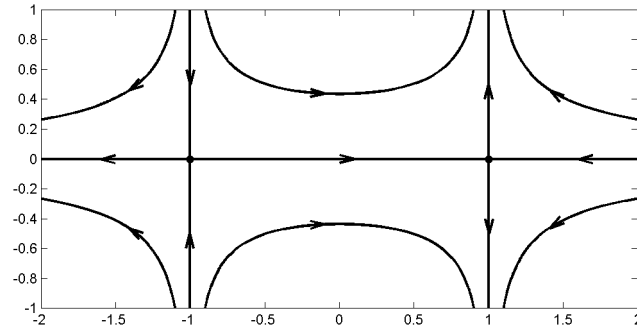


Figure 6.16: The phase portrait of system $\dot{x} = 1 - x^2 - \lambda xy$, $\dot{y} = xy + \lambda(1 - x^2)$ for $\lambda = 0$.

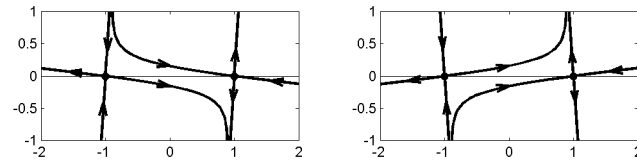


Figure 6.17: The phase portrait of system $\dot{x} = 1 - x^2 - \lambda xy$, $\dot{y} = xy + \lambda(1 - x^2)$ for $\lambda < 0$ (left) and for $\lambda > 0$ (right).

First, it is shown that local bifurcation may occur only at an equilibrium.

Proposition 6.1. *If $f(x_0, \lambda_0) \neq 0$, then (x_0, λ_0) is locally regular.*

Proof. For simplicity, the proof is shown for the case $n = 1$. Without loss of generality one can assume that $f(x_0, \lambda_0) > 0$. Then the continuity of f implies that there exist a neighbourhood $U \subset \mathbb{R}^n$ of x_0 and $\delta > 0$, such that in the set $\bar{U} \times [\lambda_0 - \delta, \lambda_0 + \delta]$ the value of f is positive. Hence in this set the trajectories are segments directed upward, as it is shown in Figure 6.18. Hence the phase portraits are obviously topologically equivalent in U for all $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. (The homeomorphism taking the orbits into each other is the identity.) We note that for $n > 1$ the proof is similar by using the local flow box theorem. \square

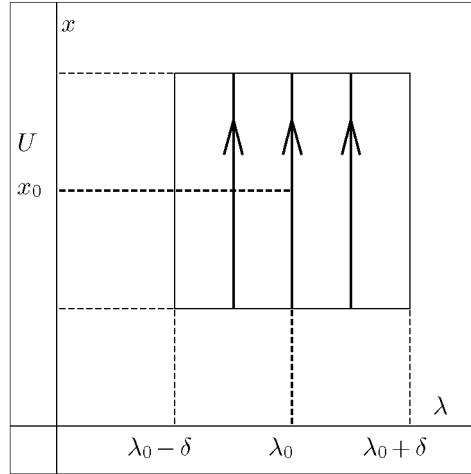


Figure 6.18: The trajectories in a neighbourhood U of a non-equilibrium point are the same for all values of the parameter λ .

It will be shown that at a hyperbolic equilibrium local bifurcation cannot occur.

Proposition 6.2. *If $f(x_0, \lambda_0) = 0$ and $\partial_x f(x_0, \lambda_0)$ is hyperbolic, then (x_0, λ_0) is locally regular. (Here $\partial_x f(x_0, \lambda_0)$ denotes the Jacobian matrix of f .)*

Proof. For simplicity, the proof is shown again for the case $n = 1$. Without loss of generality one can assume that $\partial_x f(x_0, \lambda_0) < 0$. Then according to the implicit function theorem there exist $\delta > 0$ and a differentiable function $g : (\lambda_0 - \delta, \lambda_0 + \delta) \rightarrow \mathbb{R}$, for which $g(\lambda_0) = x_0$ and $f(g(\lambda), \lambda) \equiv 0$ for all $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, moreover, there is neighbourhood U of x_0 , such that in other points of the set $U \times (\lambda_0 - \delta, \lambda_0 + \delta)$ the

function f is nonzero. Since f is continuously differentiable, the number δ can be chosen so small that $\partial_x f(g(\lambda), \lambda) < 0$ holds for all $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. Hence for these values of λ there is exactly one stable equilibrium in U and the trajectories tend to this point as it is shown in Figure 6.19. Therefore the phase portraits are obviously topologically equivalent in U for all $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. (The homeomorphism taking the orbits into each other is a translation taking the steady states to each other.) We note that for $n > 1$ the proof is similar by using the Hartman–Grobman theorem. \square

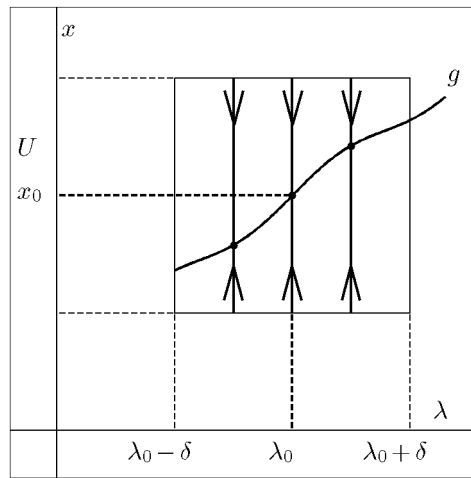


Figure 6.19: The phase portraits in a neighbourhood U of a hyperbolic equilibrium are the same for all values of the parameter λ .

6.3 Structural stability

In the course of studying bifurcations a system of differential equations $\dot{x}(t) = f(x(t))$ was considered to be a member of a k parameter family $\dot{x}(t) = f(x(t), \lambda)$ and it was investigated how the phase portrait is changing as the k dimensional parameter λ is varied. Here a more general approach is shown, where all C^1 perturbations are investigated together with the system $\dot{x}(t) = f(x(t))$, that is the right hand side f is considered as an element of a function space. In the case of bifurcations a value of the parameter was called regular if the corresponding system was topologically equivalent to all other systems belonging to nearby parameter values. The generalisation of this is the structurally

stable system that is topologically equivalent to all other systems that are sufficiently close in the C^1 norm.

For formulating the definition in abstract terms let X be a topological space and let $\sim \subset X \times X$ be an equivalence relation. (If the Reader is not familiar with the notion of topological spaces, he or she may have a metric or normed space in mind.) In our case the topological space will be a suitable function space with the C^1 topology and the equivalence relation will be the topological equivalence.

Definition 6.3.. *An element $x \in X$ is called structurally stable, if it has a neighbourhood $U \subset X$, for which $y \in U$ implies $x \sim y$.*

An element $x \in X$ is called a bifurcation point, if it is not structurally stable.

In other words, we can say that $x \in X$ is structurally stable, if it is an interior point of an equivalence class, and it is bifurcation point if it is a boundary point of an equivalence class. This interpretation enables us to define the co-dimension of a bifurcation. The co-dimension of a bifurcation is the co-dimension of the surface that forms the boundary in a neighbourhood of the given bifurcation point. In Figure 6.20 the point A is structurally stable, point B is a one co-dimensional bifurcation point and point C is a two co-dimensional bifurcation point. This can also be formulated as follows. There is a curve through B that intersects both domains that are separated by the border containing B , while there is no such curve through C . The classes touching C can be reached by a two parameter family, i.e. a surface in the space. This is formulated rigorously in the following definition.

Definition 6.4.. *A bifurcation point $x \in X$ is called k co-dimensional, if there is a continuous function $g : \mathbb{R}^k \rightarrow X$, for which $g(0) = x$ and the point x has a neighbourhood U and it has an open dense subset V , such that for all $y \in V$ there is an $\alpha \in \mathbb{R}^k$ satisfying $g(\alpha) \sim y$, and k is the smallest dimension with these properties.*

Now the abstract definitions are applied in the context of differential equations.

As it was mentioned above the equivalence relation will be the topological equivalence, because two differential equations are considered to be in the same class if they have similar phase portraits. Let us choose now a suitable topological space X .

Consider the differential equation $\dot{x}(t) = f(x(t))$, where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuously differentiable function. The space X can be chosen as the space of continuously differentiable functions with a suitable norm or topology. Let us start from C^0 or supremum topology that will be denoted by $\|f\|_0$ and is given by $\sup |f|$ on a suitably chosen set. (Later it will be investigated how the finiteness of this supremum can be ensured.) A simple one dimensional example shows that this topology is not a suitable choice. Namely, consider the differential equation $\dot{x} = x$ given by the identity $f(x) = x$. To this function f one can find a function g that is arbitrarily close to f in the C^0 topology and has more than one root. Hence the phase portrait of the differential equation $\dot{y} = g(y)$

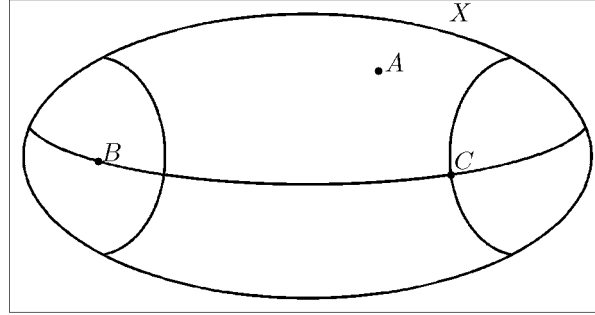


Figure 6.20: Structurally stable (A), one co-dimensional (B) and two co-dimensional (C) bifurcation points in a topological space X .

is not topologically equivalent to the phase portrait of $\dot{x} = x$. This example shows that choosing the C^0 norm on the space X only those systems can be structurally stable that have no equilibrium. This would give a very restrictive definition of structural stability, most of the equivalence classes would not have interior points. A better norm for our purposes is the C^1 norm that is given by $\|f\|_1 = \|f\|_0 + \|f'\|_0$. Namely, let us consider again the identity function $f(x) = x$ and the corresponding differential equation $\dot{x} = x$. If a function g is close to f in C^1 norm, then it has also a unique zero, hence the corresponding differential equation $\dot{y} = g(y)$ is topologically equivalent to the equation $\dot{x} = x$. Therefore the equation $\dot{x} = x$ having a hyperbolic equilibrium is structurally stable, when the C^1 norm is used. The C^1 norm of a continuously differentiable function is not necessarily finite if the domain of the function is not compact. Therefore investigating structural stability is more convenient for dynamical systems with compact state space. If the state space is one dimensional, then the most straightforward choice for the state space is the circle. This case will be dealt with in the next subsection.

6.3.1 Structural stability of one dimensional systems

Let us introduce the space $X = C^1(S^1, \mathbb{R})$ that consists of continuously differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$, which are periodic with period 1, that is $f(x + 1) = f(x)$ for all

$x \in \mathbb{R}$. This space will be endowed with the norm

$$\|f\|_1 = \max_{[0,1]} |f| + \max_{[0,1]} |f'|.$$

It will be shown that those systems are structurally stable, for which all equilibria are hyperbolic. Introduce the following notation for these systems.

$$G = \{f \in X : f(x) = 0 \Rightarrow f'(x) \neq 0\}$$

In the proof of the theorem the following notion and lemma are crucial.

Definition 6.5.. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuously differentiable function. The value y is called a regular value of f , if $f(x) = y$ implies $f'(x) \neq 0$. In the higher dimensional case when $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ the assumption is that $f(x) = y$ implies $\det f'(x) \neq 0$. If y is not a regular value, then it is called a critical value of f .

Lemma 6.6. (Sard). If $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuously differentiable function, then the set of its critical values has measure zero.

The lemma is not proved here, because its proof is beyond the topics of this lecture notes. Using the lemma the following proposition can be proved.

Proposition 6.3. The above set G of dynamical systems having only hyperbolic equilibria is dense in the space $X = C^1(S^1, \mathbb{R})$.

Proof. A function is in the set G , if and only if 0 is its regular value. Let $f \in X$ and $\varepsilon > 0$ be arbitrary. It has to be shown that there exists $g \in G$, for which $\|f - g\|_1 < \varepsilon$. If 0 is a regular value of f , then $g = f$ is a suitable choice, since then $f \in G$. If 0 is not a regular value, then choose a positive regular value $c < \varepsilon$. The existence of this c is guaranteed by Sard's lemma. Then let $g = f - c$, therefore $\|f - g\|_1 = c < \varepsilon$ and $g(x) = 0$ implies $f(x) = c$, hence the regularity of c yields $f'(x) \neq 0$, which directly gives $g'(x) \neq 0$. Thus 0 is a regular value of g , that is $g \in G$. \square

In the proof of the theorem we use the fact that if a function has a degenerate zero, then a small C^1 perturbation makes it constant zero in a neighbourhood of the zero. This statement can easily be seen geometrically, in the next proposition we give a rigorous proof.

Proposition 6.4. Let $f \in X$ and assume that for some $x \in (0, 1)$ we have $f(x) = 0 = f'(x)$. Then for any $\varepsilon > 0$ and $\alpha > 0$ there exists a function $g \in X$, for which the following statements hold.

1. $f(y) = g(y)$ for all $y \notin (x - \alpha, x + \alpha)$,

2. g is constant 0 in a neighbourhood of x ,

3. $\|f - g\|_1 < \varepsilon$.

Proof. Let $\eta : \mathbb{R} \rightarrow [0, 1]$ be a C^1 function (in fact it can be chosen as a C^∞ function), that is constant zero outside the interval $[-1, 1]$ and constant 1 in the interval $(-1/2, 1/2)$. The maximum of $|\eta'|$ is denoted by M . The assumption on f implies that there exists a positive number $\beta < \alpha$, for which

$$|f(y)| < \frac{\varepsilon}{4M}|y - x| \quad (6.1)$$

holds for all $y \in (x - \beta, x + \beta)$. Let $\delta < \beta$ be a positive number, for which

$$\max_{[x-\delta, x+\delta]} |f| < \frac{\varepsilon}{2} \quad \text{and} \quad \max_{[x-\delta, x+\delta]} |f'| < \frac{\varepsilon}{4}. \quad (6.2)$$

Then let $g \in X$ be given as follows

$$g(y) = f(y) \left(1 - \eta \left(\frac{y - x}{\delta} \right) \right).$$

Now it is checked that g satisfies the conditions. If $|y - x| \geq \alpha$, then $|y - x| > \delta$ yielding $\eta \left(\frac{y - x}{\delta} \right) = 0$, hence the first condition holds, i.e. $g(y) = f(y)$. If $|y - x| < \delta/2$, then $\eta \left(\frac{y - x}{\delta} \right) = 1$, hence the second condition holds, i.e. g is constant zero in a neighbourhood of x . In order to check the last condition we use that

$$f(y) - g(y) = f(y) \eta \left(\frac{y - x}{\delta} \right). \quad (6.3)$$

Therefore $f - g$ is zero in $[x - \delta, x + \delta]$, hence it is enough to prove that for all $y \in [x - \delta, x + \delta]$ the following holds

$$|f(y) - g(y)| < \frac{\varepsilon}{2} \quad \text{and} \quad |f'(y) - g'(y)| < \frac{\varepsilon}{2}. \quad (6.4)$$

For proving both inequalities one can use (6.3). This yields for $y \in [x - \delta, x + \delta]$ that $|f(y) - g(y)| < |f(y)| < \frac{\varepsilon}{2}$, where the first inequality of (6.2) was used. Differentiating (6.3)

$$f'(y) - g'(y) = f'(y) \eta \left(\frac{y - x}{\delta} \right) + f(y) \eta' \left(\frac{y - x}{\delta} \right) \frac{1}{\delta}.$$

Applying the second equation in (6.2), the inequality (6.1) and that M is the maximum of $|\eta'|$ leads to

$$|f'(y) - g'(y)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4M}|y - x| \frac{M}{\delta}.$$

Since $y \in [x - \delta, x + \delta]$, we have $|y - x| \leq \delta$, hence the previous estimate can be continued as

$$|f'(y) - g'(y)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4M} \delta \frac{M}{\delta} = \frac{\varepsilon}{2}.$$

Hence the desired estimates in (6.4) are proved. \square

The next statement can be verified in an elementary way.

Proposition 6.5. *If all the equilibria of the differential equation $\dot{x} = f(x)$ are hyperbolic, then there are at most finitely many of them in $[0, 1]$.*

Proof. Assume that there are infinitely many equilibria in $[0, 1]$. Then one can choose a convergent sequence of equilibria tending to a point $x \in [0, 1]$. Then since f is continuously differentiable we have $f(x) = 0$ and $f'(x) = 0$, that is x is not a hyperbolic equilibria, which is a contradiction. \square

Now we turn to the characterisation of one dimensional structurally stable systems.

Theorem 6.7.. *The dynamical system belonging to the function $f \in X$ is structurally stable, if and only if all equilibria of f are hyperbolic, that is $f \in G$. Moreover, the set G of structurally stable systems is open and dense in the space X .*

Proof. Assume first that f is structurally stable and prove $f \in G$. Since f is equivalent to the systems in a neighbourhood and G is dense, there exists in this neighbourhood a function $g \in G$. Hence all the roots of g are hyperbolic, implying that there are finitely many of them, therefore the equivalence of f and g implies that f has finitely many roots. We show that all of them are hyperbolic. Since the roots are isolated, if one of them were not be hyperbolic, then according to Proposition 6.4 an arbitrarily small C^1 perturbation would make it constant zero. That would mean that arbitrarily close to f there is a function, which is zero in an interval, hence it is not equivalent to f contradicting to the fact that f is structurally stable. This proves the first implication.

Assume now that $f \in G$ and prove that f is structurally stable. Proposition 6.5 yields that f has finitely many roots. If it has no zeros at all, then the functions close to f in the C^1 norm cannot have zeros, hence they are equivalent to f . If f has zeros, then it can be easily seen that functions close to f has the same number of zeros and the sign changes at the zeros are the same as those for f . This implies that their phase portraits are equivalent to that belonging to f . \square

6.3.2 Structural stability of higher dimensional systems

In the previous section it was shown that in a one dimensional system the phase portrait can change only in a neighbourhood of non-hyperbolic equilibrium. In two dimensional

systems there are bifurcations that are not related to equilibria, namely, the fold bifurcation of periodic orbits, the homoclinic and heteroclinic bifurcation.

Briefly, it can be said that those two dimensional systems are structurally stable, in which these cannot occur, that is all equilibria and periodic orbits are hyperbolic and there are no trajectories connecting saddle points. This was proved first by Andronov and Pontryagin in 1937 for two dimensional systems defined in a positively invariant bounded domain. The exact formulation of this statement is Theorem 2.5 in Kuznetsov's book [17]. The analogous statement for compact two dimensional manifolds was proved by Peixoto in 1962. This statement will be formulated below. Let S^2 denote the two dimensional sphere and introduce the space $X = C^1(S^2, \mathbb{R}^2)$ as the space of vector fields on the sphere. A vector field, as the right hand side of a differential equation determines a phase portrait on the sphere S^2 . Defining their topological equivalence as above, Definition 6.3. gives the structurally stable systems in X . These are characterised by the following theorem.

Theorem 6.8. (Peixoto). *The dynamical system given by the vector field $f \in X$ is structurally stable if and only if*

- *there are finitely many equilibria and all of them are hyperbolic,*
- *there are finitely many periodic orbits and all of them are hyperbolic,*
- *there are no trajectories connecting saddle points (heteroclinic or homoclinic orbits).*

Moreover, the set of structurally stable systems is open and dense in the space X .

We note that the theorem was proved in a more general way for compact two dimensional manifolds. For this general case an extra assumption is needed for structural stability. Even in the case of a torus one can give a structurally unstable system that do not violates any of the assumption of the theorem. In the general case it has to be assumed that the non-wandering points can only be equilibria or periodic points. The general formulation of the theorem can be found in Perko's book [19] and in the book by Wiggins [27].

Based on the cases of one and two dimensional systems one can formulate a conjecture about the structural stability of systems with arbitrary dimension. This conjecture motivated the following definition of Morse–Smale systems.

Definition 6.9.. *A dynamical systems is called a Morse–Smale system, if*

- *there are finitely many equilibria and all of them are hyperbolic,*
- *there are finitely many periodic orbits and all of them are hyperbolic,*
- *their stable and unstable manifolds intersect each other transversally,*
- *the non-wandering points can only be equilibria or periodic points.*

It can be shown that Morse–Smale systems are structurally stable, however, the opposite is not true for more than two dimensional systems, as it was shown by Smale in 1966. If the phase space is at least three dimensional, then there are structurally stable systems with strange attractors as non-wandering sets, containing chaotic orbits. Moreover, it can be proved that in the case of at least three dimensional systems the space of C^1 systems contains open sets containing only structurally unstable systems. That is the structurally stable systems do not form a dense and open subset, moreover, the set of them cannot be given as the intersection of open dense sets, that is the structurally stable systems are not generic among three or higher dimensional systems. This means that topological equivalence does not divide the space of C^1 systems into open sets as it is shown in Figure 6.20. The investigation of structural stability in higher dimensional systems is based on a two dimensional map introduced by Smale and named as Smale horseshoe. This map is dealt with in detail in the book by Guckenheimer and Holmes [11] and in Wiggins’s monograph [27].

Chapter 7

One co-dimensional bifurcations, the saddle-node and the Andronov-Hopf bifurcations

It was shown in Section 6.2 that local bifurcations may occur only at non-hyperbolic equilibria. In this chapter it is shown that at these points either saddle-node or Andronov-Hopf bifurcation occurs typically. The type of bifurcation depends on the degeneracy of the linear part, namely, if there is a zero eigenvalue, then saddle-node bifurcation may occur, while having pure imaginary eigenvalues may indicate Andronov-Hopf bifurcation. The sufficient conditions of these bifurcations involve the higher order terms of the right hand side. The detailed presentation of these two bifurcations aims at presenting the approach of bifurcation theory that consists of the following steps. The bifurcation is observed first in a simple system that is typically the normal form of the bifurcation. Then the change in the phase portrait at the bifurcation is formulated in geometrical terms (e.g. the steady state loses its stability and a stable limit cycle is born). Finally, a sufficient condition is established that ensures that the given change occurs in the phase portrait. This program is carried out for the saddle-node and for the Andronov-Hopf bifurcation. In the second part of the chapter it is shown how bifurcation curves can be analytically determined in two-parameter systems by using the parametric representation method, and how the Takens–Bogdanov bifurcation points can be given as the common points of the saddle-node and Andronov-Hopf bifurcation curves.

7.1 Saddle-node bifurcation

Recall that in the differential equation $\dot{x} = \lambda - x^2$ saddle-node or fold bifurcation occurs at $\lambda = 0$. This means that for $\lambda < 0$, there is no equilibrium, if $\lambda = 0$, then there is a degenerate equilibrium at $x = 0$ and for $\lambda > 0$ there are two equilibria at $x = \pm\sqrt{\lambda}$, one

of them is stable, the other one is unstable. That is the first step, the identification of the bifurcation in a simple case.

The second step is to define what kind of change in the phase portrait will be called a fold or saddle-node bifurcation. Consider the differential equation $\dot{x}(t) = f(x(t), \lambda)$, where $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuously differentiable function and $\lambda \in \mathbb{R}$ is a parameter.

Definition 7.1.. *There is a fold or saddle-node bifurcation at the point $(x_0, \lambda_0) \in \mathbb{R}^2$, if the point x_0 has a neighbourhood $U \subset \mathbb{R}$ and there exists $\delta > 0$, such that*

- *for $\lambda_0 - \delta < \lambda < \lambda_0$ there is no equilibrium (there are 2 equilibria) in U ;*
- *for $\lambda = \lambda_0$ there is a single equilibrium in U ;*
- *for $\lambda_0 < \lambda < \lambda_0 + \delta$ there are 2 equilibria (there is no equilibrium) in U .*

That is, there is no equilibrium before λ_0 and there are two of them after λ_0 or vice versa, as it is shown in Figure 7.1.

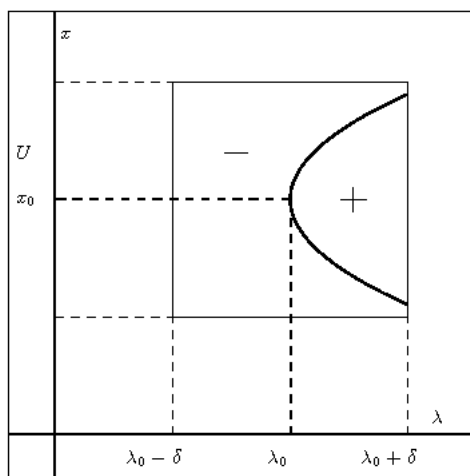


Figure 7.1: Saddle-node or fold bifurcation at the point $(x_0, \lambda_0) \in \mathbb{R}^2$.

We note that under some natural assumptions on the smoothness of the function f the above conditions mean that the curve $\{(x, \lambda) \in \mathbb{R}^2 : f(x, \lambda) = 0\}$ in the rectangle $(\lambda_0 - \delta, \lambda_0 + \delta) \times U$ lies on one side of the line $\lambda = \lambda_0$ and divides the rectangle into two parts according to the sign of f .

Using Figure 7.1 the definition can be reformulated as follows. (This definition is introduced in the book by Kuznetsov [17].) Consider the differential equation $\dot{x} = \lambda - x^2$, as the normal form of the bifurcation, in the domain $(-1, 1) \times (-1, 1)$. (In this case $\lambda_0 = 0$ and $x_0 = 0$.) There is a fold or saddle-node bifurcation in the differential equation $\dot{x}(t) = f(x(t), \lambda)$ at $(x_0, \lambda_0) \in \mathbb{R}^2$, if the point x_0 has a neighbourhood $U \subset \mathbb{R}$ and there exists $\delta > 0$, for which a homeomorphism $p : (-1, 1) \rightarrow (\lambda_0 - \delta, \lambda_0 + \delta)$ can be given, such that the phase portrait of $\dot{x} = \lambda - x^2$ in the interval $(-1, 1)$ belonging to λ is topologically equivalent to the phase portrait of $\dot{x}(t) = f(x(t), p(\lambda))$ in U . That is the map p relates the parameter values in the two systems that give the same phase portraits.

Consider the following simple example, where fold bifurcation occurs.

Example 7.1. *The equilibria of the differential equation $\dot{x} = \lambda - x^2 + x^3$ are given by the curve $\lambda = x^2 - x^3$ that touches the vertical line $\lambda = 0$ at the point $(0, 0)$. Then choosing a small enough number $\delta > 0$ and a suitable interval U containing zero the conditions in the definition are fulfilled as it is shown in Figure 7.2.*

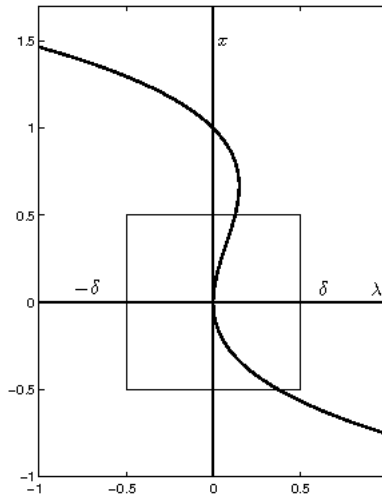


Figure 7.2: Fold bifurcation at the point $(0, 0)$ in the differential equation $\dot{x} = \lambda - x^2 + x^3$.

The next theorem yields a sufficient condition in terms of the derivatives of f at the point (x_0, λ_0) for the appearance of fold bifurcation in the differential equation $\dot{x}(t) = f(x(t), \lambda)$.

Theorem 7.2.. *Let $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a twice continuously differentiable function and consider the differential equation $\dot{x}(t) = f(x(t), \lambda)$. Assume that $f(x_0, \lambda_0) = 0$ and $\partial_x f(x_0, \lambda_0) = 0$. (These are the necessary conditions for a local bifurcation.) If besides these the conditions*

$$\partial_\lambda f(x_0, \lambda_0) \neq 0 \quad \text{and} \quad \partial_{xx} f(x_0, \lambda_0) \neq 0$$

hold, then fold bifurcation occurs at the point (x_0, λ_0) .

Proof. The implicit function theorem is used to express λ in terms of x from equation $f(x, \lambda) = 0$. This can be done because $\partial_\lambda f(x_0, \lambda_0) \neq 0$, that is the condition of the implicit function theorem is fulfilled. According to the theorem the point x_0 has a neighbourhood U and there is a twice continuously differentiable function $g : U \rightarrow \mathbb{R}$, for which $g(x_0) = \lambda_0$ and $f(x, g(x)) = 0$ for all $x \in U$, moreover, there exists $\delta > 0$, such that there is no other solution of equation $f(x, \lambda) = 0$ in $(\lambda_0 - \delta, \lambda_0 + \delta) \times U$. That is the solution is $\lambda = g(x)$. As it was remarked above, it is enough to prove that the graph of function g lies on one side of the line $\lambda = \lambda_0$ locally. In order to prove that it is sufficient to show that $g'(x_0) = 0$ and $g''(x_0) \neq 0$. Differentiating equation $f(x, g(x)) = 0$

$$\partial_1 f(x, g(x)) + \partial_2 f(x, g(x))g'(x) = 0, \tag{7.1}$$

substituting $x = x_0$ and using $g(x_0) = \lambda_0$ leads to

$$\partial_x f(x_0, \lambda_0) + \partial_\lambda f(x_0, \lambda_0)g'(x_0) = 0.$$

The assumptions of the theorem yield $\partial_x f(x_0, \lambda_0) = 0$ and $\partial_\lambda f(x_0, \lambda_0) \neq 0$, therefore $g'(x_0) = 0$.

Differentiating equation (7.1) substituting $x = x_0$ and using $g'(x_0) = 0$ yield

$$\partial_{xx} f(x, g(x)) + \partial_\lambda f(x, g(x))g''(x) = 0,$$

leading to $g''(x_0) \neq 0$ by using again the assumptions of the theorem. □

We note that a similar but technically more complicated sufficient condition can be formulated in the case of higher dimensional phase space. The n dimensional generalisation of the theorem was proved by Sotomayor in 1976. This can be found in Perko's book [19] as Theorem 1 in Chapter 4.

7.2 Andronov–Hopf bifurcation

The simplest system where Andronov–Hopf bifurcation occurs is $\dot{r} = \lambda r + \sigma r^3$, $\dot{\phi} = -1$. The bifurcation occurs as λ crosses zero. There are two types of this bifurcation. In the

supercritical case when $\sigma < 0$ the phase portrait changes as follows. If $\lambda < 0$, then the origin is globally asymptotically stable, while in the case $\lambda > 0$ the origin is unstable and a stable limit cycle appears around the origin, the size of which increases with λ . In the subcritical case when $\sigma > 0$, the limit cycle is unstable and appears for $\lambda < 0$.

The system given in polar coordinates takes the following form in cartesian coordinates.

$$\dot{x}_1 = \lambda x_1 + x_2 + \sigma x_1(x_1^2 + x_2^2), \quad (7.2)$$

$$\dot{x}_2 = -x_1 + \lambda x_2 + \sigma x_2(x_1^2 + x_2^2) \quad (7.3)$$

A sufficient condition will be given for this bifurcation in a general system of the form $\dot{x}(t) = f(x(t), \lambda)$ at the parameter value λ_0 in the point x_0 , where $f : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$ is a smooth enough function. Since x_0 is an equilibrium, we have $f(x_0, \lambda_0) = 0$ and according to Proposition 6.2 it cannot be hyperbolic, hence the eigenvalues of the Jacobian matrix $\partial_x f(x_0, \lambda_0)$ have zero real part. In the previous section it was shown that in the case when 0 is an eigenvalue, saddle-node bifurcation occurs. The above example shows that at the bifurcation the Jacobian matrix has pure imaginary eigenvalues. Therefore it is assumed that the eigenvalues of the matrix $\partial_x f(x_0, \lambda_0)$ are $\pm i\omega$. It will be shown that under suitable transversality conditions Andronov–Hopf bifurcation occurs in the point x_0 at λ_0 . The bifurcation is defined by using its geometrical characterisation.

Definition 7.3.. *At the parameter value λ_0 supercritical Andronov–Hopf bifurcation occurs in the point x_0 , if there exists $\delta > 0$ and a neighbourhood U of x_0 , for which*

- $\lambda_0 - \delta < \lambda \leq \lambda_0$ (or $\lambda_0 \leq \lambda < \lambda_0 + \delta$) *implies that there is a stable equilibrium and no limit cycle in U ;*
- $\lambda_0 < \lambda < \lambda_0 + \delta$ (or $\lambda_0 - \delta < \lambda < \lambda_0$) *implies that there is an unstable equilibrium and a stable limit cycle in U .*

Definition 7.4.. *At the parameter value λ_0 subcritical Andronov–Hopf bifurcation occurs in the point x_0 , if there exists $\delta > 0$ and a neighbourhood U of x_0 , for which*

- $\lambda_0 - \delta < \lambda \leq \lambda_0$ (or $\lambda_0 \leq \lambda < \lambda_0 + \delta$) *implies that there is a stable equilibrium and an unstable limit cycle in U ;*
- $\lambda_0 < \lambda < \lambda_0 + \delta$ (or $\lambda_0 - \delta < \lambda < \lambda_0$) *implies that there is an unstable equilibrium and no limit cycle in U .*

We note that this definition can also be formulated in a slightly more general form as it is presented in Kuznetsov’s book [17] as follows. Consider system (7.2)–(7.3), the normal form of the Andronov–Hopf bifurcation, in the set $B(0, 1) \times (-1, 1)$, where $B(0, 1)$ is the open unit disk centered at the origin in the phase plane. (In this case $\lambda_0 = 0$ and

$x_0 = 0$.) According to the alternative definition there is Andronov–Hopf bifurcation in the system $\dot{x}(t) = f(x(t), \lambda)$ at $(x_0, \lambda_0) \in \mathbb{R}^2$, if the point x_0 has a neighbourhood $U \subset \mathbb{R}^2$ and there exists $\delta > 0$, for which there exist a homeomorphism $p : (-1, 1) \rightarrow (\lambda_0 - \delta, \lambda_0 + \delta)$, such that the phase portrait of system (7.2)–(7.3) in $B(0, 1)$ and belonging to λ is topologically equivalent to the phase portrait of system $\dot{x}(t) = f(x(t), p(\lambda))$ in U . Thus the homeomorphism p relates the parameter values yielding the same phase portrait.

After transforming system (7.2)–(7.3) to polar coordinates it is obvious that Andronov–Hopf bifurcation occurs according to the definition. The goal of this section is to give a sufficient condition for the Andronov–Hopf bifurcation in a general system (when the periodic orbit is not a circle, hence polar coordinates cannot help). The sufficient condition is proved first for simpler systems and finally for the most general case. This way the ideas of the proof can be understood more easily.

There are two different ways of proving the theorem about the sufficient condition. One of them is based on application of the Poincaré map, the other one uses a Lyapunov function to construct a positively invariant domain containing a periodic orbit according to the Poincaré–Bendixson theorem. The relation of the two methods is dealt with in Chicone’s book [7]. Here the second method will be applied, the Lyapunov function will be constructed to the system at the bifurcation value.

7.2.1 Construction of the Lyapunov function

The following notations will be used. Let H_k denote the space of homogeneous polynomials of degree k in the form

$$P(x_1, x_2) = p_0 x_1^k + p_1 x_1^{k-1} x_2 + p_2 x_1^{k-2} x_2^2 + \dots + p_k x_2^k.$$

The following simple propositions will be used to estimate the values of these polynomials.

Proposition 7.1. *Let $P \in H_2$, then*

$$|P(x_1, x_2)| \leq K(x_1^2 + x_2^2),$$

where $K = |p_0| + |p_1| + |p_2|$.

Proposition 7.2. *Let $P \in H_3$, then there exists a continuous function $r : \mathbb{R}^2 \rightarrow \mathbb{R}$, for which $r(0, 0) = 0$ and*

$$|P(x_1, x_2)| \leq r(x_1, x_2)(x_1^2 + x_2^2).$$

The second proposition follows easily from the first one, which can be proved by using the simple inequality $2x_1^2 x_2^2 \leq (x_1^2 + x_2^2)^2$.

The Lyapunov function is constructed to the system

$$\dot{x}_1 = x_2 + A_2(x_1, x_2) + A_3(x_1, x_2) + A_4(x_1, x_2), \quad (7.4)$$

$$\dot{x}_2 = -x_1 + B_2(x_1, x_2) + B_3(x_1, x_2) + B_4(x_1, x_2), \quad (7.5)$$

where $A_2, B_2 \in H_2$ are homogeneous polynomials of degree 2, $A_3, B_3 \in H_3$ are homogeneous polynomials of degree 3, and the functions A_4, B_4 contain the higher order terms that means that there are constants $c_1, c_2 \in \mathbb{R}$, such that

$$|A_4(x_1, x_2)| \leq c_1(x_1^2 + x_2^2)^2, \quad |B_4(x_1, x_2)| \leq c_2(x_1^2 + x_2^2)^2. \quad (7.6)$$

Let us look for the Lyapunov function in the form

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) + P_3(x_1, x_2) + P_4(x_1, x_2),$$

where $P_3 \in H_3$ are homogeneous polynomials of degree 3 and $P_4 \in H_4$ are homogeneous polynomials of degree 4. Then the Lie derivative of the function V is

$$\begin{aligned} L_f V &= \partial_1 V f_1 + \partial_2 V f_2 \\ &= (x_1 + \partial_1 P_3 + \partial_1 P_4)(x_2 + A_2 + A_3 + A_4) \\ &\quad + (x_2 + \partial_2 P_3 + \partial_2 P_4)(-x_1 + B_2 + B_3 + B_4) \\ &= Q_3 + Q_4 + Q_5, \end{aligned}$$

where the arguments of the functions are omitted for the sake of simplicity and

$$\begin{aligned} Q_3 &= x_1 A_2 + x_2 B_2 + x_2 \partial_1 P_3 - x_1 \partial_2 P_3, \\ Q_4 &= x_1 A_3 + x_2 B_3 + A_2 \partial_1 P_3 + B_2 \partial_2 P_3 + x_2 \partial_1 P_4 - x_1 \partial_2 P_4, \end{aligned}$$

and Q_5 contains the terms of degree at least five, that is there exists a continuous function $r : \mathbb{R}^2 \rightarrow \mathbb{R}$, for which $r(0, 0) = 0$ and

$$|Q_5(x_1, x_2)| \leq r(x_1, x_2)(x_1^2 + x_2^2)^2. \quad (7.7)$$

In order to prove that V is a suitable Lyapunov function we will prove the following statements.

1. The polynomial P_3 can be chosen in such a way that $Q_3 = 0$ holds.
2. The polynomial P_4 can be chosen in such a way that Q_4 is positive or negative definite. (Under a certain transversality condition.)
3. The term Q_5 can be estimated in such a way that $Q_4 + Q_5$ is also definite in a neighbourhood of the origin.

The linear mapping $T_k : H_k \rightarrow H_k$

$$T_k(P) = x_2 \partial_1 P - x_1 \partial_2 P$$

plays an important role in the proof. The above formulas can be written in the following form by using the mapping T_k .

$$\begin{aligned} Q_3 &= x_1 A_2 + x_2 B_2 + T_3(P_3), \\ Q_4 &= x_1 A_3 + x_2 B_3 + A_2 \partial_1 P_3 + B_2 \partial_2 P_3 + T_4(P_4). \end{aligned}$$

The space H_k can be identified with \mathbb{R}^{k+1} , therefore there is a $(k+1) \times (k+1)$ matrix corresponding to the linear mapping T_k . Using the basis $\{x_1^k, x_1^{k-1}x_2, \dots, x_2^k\}$ the matrices in the cases $k=3$ and $k=4$ are

$$T_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 3 & 0 & -2 & 0 \\ 0 & 2 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad T_4 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 4 & 0 & -2 & 0 & 0 \\ 0 & 3 & 0 & -3 & 0 \\ 0 & 0 & 2 & 0 & -4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

It can be checked easily that $\det T_3 \neq 0$, hence T_3 is an isomorphism, thus there exists a polynomial $P_3 \in H_3$, such that

$$T_3(P_3) = -x_1 A_2 - x_2 B_2, \quad (7.8)$$

yielding $Q_3 = 0$. This proves statement 1 above.

In order to prove statement 2, introduce c_0, c_1, \dots, c_4 in such a way that

$$c_0 x_1^4 + c_1 x_1^3 x_2 + c_2 x_1^2 x_2^2 + c_3 x_1 x_2^3 + c_4 x_2^4 = x_1 A_3 + x_2 B_3 + A_2 \partial_1 P_3 + B_2 \partial_2 P_3 \quad (7.9)$$

holds. It will be shown that with a suitable choice of P_4 one can guarantee

$$Q_4(x_1, x_2) = K(x_1^2 + x_2^2)^2$$

with some constant K . It is enough to verify that

$$K(1, 0, 2, 0, 1)^T - (c_0, c_1, c_2, c_3, c_4) \in \text{Im} T_4. \quad (7.10)$$

The range of the linear mapping T_4 consists of those vectors that are orthogonal to the kernel of its transpose. Using the matrix of T_4 it is easy to see that the kernel of its transpose is spanned by the vector $(3, 0, 1, 0, 3)^T$ (since $(3, 0, 1, 0, 3)T_4$ is the null vector). Hence (7.10) holds if

$$0 = 3(K - c_0) + 2K - c_2 + 3(K - c_4) = 8K - (3c_0 + c_2 + 3c_4),$$

that is $K = \frac{3c_0+c_2+3c_4}{8}$.

In order to get the coefficients c_i the coefficients of P_3 are needed. Determine the polynomial in the form

$$P_3(x_1, x_2) = q_0x_1^3 + q_1x_1^2x_2 + q_2x_1x_2^2 + q_3x_2^3$$

and for the coefficients of the polynomials A_2 and B_2 introduce the notations

$$A_2(x_1, x_2) = a_{20}x_1^2 + a_{11}x_1x_2 + a_{02}x_2^2, \quad B_2(x_1, x_2) = b_{20}x_1^2 + b_{11}x_1x_2 + b_{02}x_2^2. \quad (7.11)$$

Then equation (7.8) determining the polynomial P_3 takes the form

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 3 & 0 & -2 & 0 \\ 0 & 2 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = - \begin{pmatrix} a_{20} \\ a_{11} \\ a_{02} \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ b_{20} \\ b_{11} \\ b_{02} \end{pmatrix}$$

for the unknown coefficients q_i . This can be solved as

$$\begin{aligned} q_0 &= -\frac{1}{3}(2b_{02} + a_{11} + b_{20}), \quad q_1 = a_{20} \\ q_2 &= -b_{02}, \quad q_3 = \frac{1}{3}(2a_{20} + b_{11} + a_{02}). \end{aligned}$$

The coefficients of A_3 and B_3 are denoted by

$$A_3(x_1, x_2) = a_{30}x_1^3 + a_{21}x_1^2x_2 + a_{12}x_1x_2^2 + a_{03}x_2^3, \quad (7.12)$$

$$B_3(x_1, x_2) = b_{30}x_1^3 + b_{21}x_1^2x_2 + b_{12}x_1x_2^2 + b_{03}x_2^3. \quad (7.13)$$

Then (7.9) determining the coefficients c_i yields

$$\begin{aligned} c_0 &= 3q_0a_{20} + q_1b_{20} + a_{30}, \\ c_4 &= 3q_3b_{02} + q_2a_{02} + b_{03}, \\ c_2 &= 3q_0a_{02} + 3q_3b_{20} + 2q_1a_{11} + 2q_2b_{11} + q_2a_{20} + q_1b_{02} + a_{12} + b_{21}. \end{aligned}$$

Substituting the formulas above for the coefficients q_i into these equations simple algebra shows that $3c_0 + c_2 + 3c_4 = L$, where

$$L = 3a_{30} + 3b_{03} + a_{12} + b_{21} + 2a_{20}b_{20} - 2a_{02}b_{02} - a_{11}(a_{20} + a_{02}) + b_{11}(b_{20} + b_{02}) \quad (7.14)$$

is the so-called Lyapunov coefficient. This proves statement 2, because $Q_4(x_1, x_2) = \frac{L}{8}(x_1^2 + x_2^2)^2$ is definite once $L \neq 0$.

Let us now turn to the proof of statement 3 in the case $L \neq 0$. Inequality (7.7) will be used. Let us choose a number $\varepsilon > 0$, for which $|x| < \varepsilon$ implies $|r(x)| < \frac{|L|}{16}$. If $L > 0$, then $|x| < \varepsilon$, $x \neq 0$ yields

$$L_f V(x) = Q_4(x) + Q_5(x) \geq \frac{L}{8}(x_1^2 + x_2^2)^2 - \frac{L}{16}(x_1^2 + x_2^2)^2 = \frac{L}{16}(x_1^2 + x_2^2)^2 > 0$$

proving statement 3. The statement can be verified similarly in the case $L < 0$. Summarising the statements above, we proved the following theorem.

Theorem 7.5.. *Consider system (7.4)-(7.5), where the polynomials A_2, B_2 are given in (7.11) the polynomials A_3, B_3 are given in (7.12) and (7.13) and the functions A_4, B_4 satisfy the estimates (7.6). Let the Lyapunov coefficient L be given by (7.14). Then in the case $L < 0$ the origin is asymptotically stable, while for $L > 0$ it is unstable.*

The proof of the theorem now is only the simple application of Lyapunov's theorem about the stability and unstability of an equilibrium by using the above Lyapunov function.

We note that in the case $L = 0$ further Lyapunov coefficients can be introduced by using higher order terms. The stability of the origin is determined by the first non-zero Lyapunov coefficient. Moreover, Lyapunov also proved that if all coefficients are zero in an analytic system, then the origin is a center. This theorem is called Lyapunov's center theorem, see e.g. in the book by Chicone [7].

7.2.2 Andronov–Hopf bifurcation for linear parameter dependence

Extend system (7.4)-(7.5) with the simplest parameter dependence that appears also in the normal form (7.2)-(7.3) of the bifurcation, that is consider the system

$$\dot{x}_1 = \lambda x_1 + x_2 + A_2(x_1, x_2) + A_3(x_1, x_2) + A_4(x_1, x_2), \quad (7.15)$$

$$\dot{x}_2 = -x_1 + \lambda x_2 + B_2(x_1, x_2) + B_3(x_1, x_2) + B_4(x_1, x_2), \quad (7.16)$$

where the polynomials A_2, B_2 are given in (7.11) the polynomials A_3, B_3 are given in (7.12) and (7.13) and the functions A_4, B_4 satisfy the estimates (7.6). Apply the Lyapunov function

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) + P_3(x_1, x_2) + P_4(x_1, x_2)$$

introduced above. Then the Lie derivative of the function V is

$$\begin{aligned} L_f V &= \partial_1 V f_1 + \partial_2 V f_2 \\ &= (x_1 + \partial_1 P_3 + \partial_1 P_4)(\lambda x_1 + x_2 + A_2 + A_3 + A_4) \\ &\quad + (x_2 + \partial_2 P_3 + \partial_2 P_4)(-x_1 + \lambda x_2 + B_2 + B_3 + B_4) \\ &= \lambda D + Q_4 + Q_5, \end{aligned}$$

where the term Q_3 does not present, because it is zero by the suitable choice of P_3 and

$$D(x_1, x_2) = x_1^2 + x_2^2 + x_1(\partial_1 P_3 + \partial_1 P_4) + x_2(\partial_2 P_3 + \partial_2 P_4).$$

Since the second degree part of D is a positive definite quadratic form, there exists $\varepsilon > 0$, for which $|x| < 2\varepsilon$ implies $D(x_1, x_2) > 0$. We can assume that ε is chosen so small that $Q_4 + Q_5$ is also definite in the ball $|x| < 2\varepsilon$. Consider the case $L < 0$. Then in the ball $|x| < 2\varepsilon$ we have $Q_4(x) + Q_5(x) < 0$, hence in the case $\lambda < 0$

$$L_f V(x) = \lambda D(x) + Q_4(x) + Q_5(x) < 0,$$

for $|x| < 2\varepsilon$, that is the origin is asymptotically stable. If $\lambda > 0$, then linearisation shows that the origin is unstable. We prove that there exists $\lambda_0 > 0$, such that for all $\lambda \in (0, \lambda_0)$ there is a periodic orbit. Let $r > 0$ be a number, for which $V(x) \leq r$ implies $|x| < 2\varepsilon$ and let

$$M = \max_{V(x) \leq r} D(x) > 0, \quad \text{and} \quad m = \max_{V(x)=r} Q_4(x) + Q_5(x) < 0,$$

and let $\lambda_0 = -\frac{m}{M}$. Then $V(x) = r$ and $\lambda \in (0, \lambda_0)$ imply $L_f V(x) < \lambda M + m < 0$. Hence the set given by $V(x) \leq r$ is positively invariant and it contains a single equilibrium the origin that is unstable. Hence the Poincaré-Bendixson theorem implies that there is a stable limit cycle in this set. This proves that supercritical Andronov–Hopf bifurcation occurs at $\lambda = 0$. It can be proved similarly that for $L > 0$ the bifurcation is subcritical. Summarising, the following theorem is proved.

Theorem 7.6.. *Consider system (7.15)–(7.16), where the polynomials A_2, B_2 are given in (7.11), the polynomials A_3, B_3 are given in (7.12) and (7.13) and the functions A_4, B_4 satisfy the estimates (7.6). Let the Lyapunov coefficient L be given by (7.14). Then in the case $L < 0$ supercritical Andronov–Hopf bifurcation occurs at $\lambda = 0$, while for $L > 0$ subcritical Andronov–Hopf bifurcation occurs at $\lambda = 0$.*

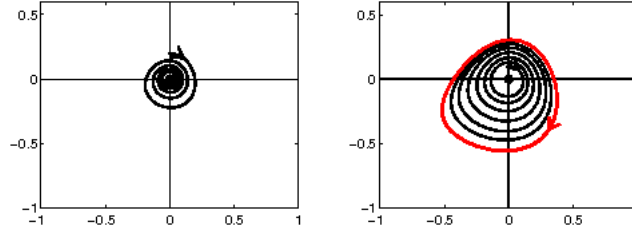
The application of the theorem is illustrated by the following example.

Example 7.2. *Investigate Andronov–Hopf bifurcation in the system*

$$\dot{x}_1 = \lambda x_1 + x_2 + x_1^2, \tag{7.17}$$

$$\dot{x}_2 = -x_1 + \lambda x_2 - x_1^2 \tag{7.18}$$

and determine the type of the bifurcation. According to the theorem the sign of the Lyapunov coefficient L is needed using formula (7.14). In our case $A_2(x_1, x_2) = B_2(x_1, x_2) = x_1^2$, $A_3(x_1, x_2) = B_3(x_1, x_2) = 0$ and $A_4(x_1, x_2) = B_4(x_1, x_2) = 0$. Therefore (7.11) yields $a_{20} = b_{20} = 1$, and according to (7.12) and (7.13) the other coefficients are zero. Hence $L = 2a_{20}b_{20} = -2$, thus at $\lambda = 0$ supercritical Andronov–Hopf bifurcation occurs, that



is for $\lambda < 0$ the origin is asymptotically stable, and in the case $\lambda > 0$ it is unstable surrounded by a stable limit cycle as it is shown in Figure 7.2.

Andronov–Hopf bifurcation in the differential equation of Example 7.2 at $\lambda = 0$. The figure in the left hand side shows the phase portrait for $\lambda < 0$, the figure in the right hand side shows the phase portrait for $\lambda > 0$.

7.2.3 Andronov–Hopf bifurcation with general parameter dependence and with linear part in Jordan canonical form

In this subsection system (7.15)–(7.16) is generalised in such a way that the coefficients may depend on the parameter λ arbitrarily, but the linear part is still in Jordan canonical form. That is system

$$\dot{x}_1 = \alpha(\lambda)x_1 + \beta(\lambda)x_2 + A_2(x_1, x_2) + A_3(x_1, x_2) + A_4(x_1, x_2), \quad (7.19)$$

$$\dot{x}_2 = -\beta(\lambda)x_1 + \alpha(\lambda)x_2 + B_2(x_1, x_2) + B_3(x_1, x_2) + B_4(x_1, x_2) \quad (7.20)$$

is considered, where the polynomials A_2, B_2 are given again in (7.11), the polynomials A_3, B_3 are given in (7.12) and (7.13) and the functions A_4, B_4 satisfy the estimates (7.6). In order to have Andronov-Hopf bifurcation at $\lambda = 0$, we assume about the parameter dependence in the linear part that

$$\alpha(0) = 0, \quad \alpha'(0) \neq 0, \quad \beta(0) \neq 0. \quad (7.21)$$

Namely, the eigenvalues of the linear part are $\alpha(\lambda) + i\beta(\lambda)$, hence as λ crosses zero, the pair of eigenvalues crosses the imaginary axis in the complex plane. Thus for example

in the case $\alpha'(0) > 0$, the origin is asymptotically stable for $\lambda < 0$ and it is unstable for $\lambda > 0$. Using the Lyapunov function introduced above, it will be shown that depending on the sign of the Lyapunov coefficient L a periodic orbit appears for negative or positive values of λ .

The Lyapunov function takes again the form

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) + P_3(x_1, x_2) + P_4(x_1, x_2).$$

The Lie derivative of the this function is

$$\begin{aligned} L_f V &= \partial_1 V f_1 + \partial_2 V f_2 \\ &= (x_1 + \partial_1 P_3 + \partial_1 P_4)(\alpha(\lambda)x_1 + \beta(\lambda)x_2 + A_2 + A_3 + A_4) \\ &\quad + (x_2 + \partial_2 P_3 + \partial_2 P_4)(-\beta(\lambda)x_1 + \alpha(\lambda)x_2 + B_2 + B_3 + B_4) \\ &= \alpha(\lambda)D + Q_3 + Q_4 + Q_5, \end{aligned}$$

where

$$\begin{aligned} D(x_1, x_2) &= x_1^2 + x_2^2 + x_1(\partial_1 P_3 + \partial_1 P_4) + x_2(\partial_2 P_3 + \partial_2 P_4), \\ Q_3 &= x_1 A_2 + x_2 B_2 + \beta(\lambda)T_3(P_3), \\ Q_4 &= x_1 A_3 + x_2 B_3 + A_2 \partial_1 P_3 + B_2 \partial_2 P_3 + \beta(\lambda)T_4(P_4), \end{aligned}$$

and Q_5 contains the higher order terms that satisfy the inequality (7.7). Since the second degree part of D is a positive definite quadratic form, there exists $\varepsilon > 0$, for which $|x| < 2\varepsilon$ implies $D(x_1, x_2) > 0$. We can assume that ε is chosen so small that $|r(x)| < \frac{|L|}{16}$ in the ball $|x| < 2\varepsilon$. Hence

$$|Q_5(x_1, x_2)| \leq \frac{L}{16}(x_1^2 + x_2^2)^2. \quad (7.22)$$

In this case the value $\beta(\lambda)$ changes as λ is varied, hence it cannot be assumed that $Q_3 = 0$ for all λ . Instead, P_3 can be chosen in such a way that $\beta(0)T_3(P_3) = -(x_1 A_2 + x_2 B_2)$ holds. Then

$$Q_3 = (\beta(\lambda) - \beta(0))T_3(P_3). \quad (7.23)$$

Similarly, one can chose a polynomial P_4 , for which

$$x_1 A_3 + x_2 B_3 + A_2 \partial_1 P_3 + B_2 \partial_2 P_3 + \beta(0)T_4(P_4) = \frac{L}{8}(x_1^2 + x_2^2)^2$$

holds. Then

$$Q_4 = (\beta(\lambda) - \beta(0))T_4(P_4) + \frac{L}{8}(x_1^2 + x_2^2)^2. \quad (7.24)$$

It will be proved now that in the case $L < 0$ and $\alpha'(0) > 0$ there exists $\lambda_0 > 0$, such that for all $\lambda \in (0, \lambda_0)$ there is a periodic orbit around the origin. Let $r > 0$ be a number, for which $V(x) \leq r$ implies $|x| < 2\varepsilon$ and let

$$M = \max_{V(x) \leq r} D(x) > 0, \text{ and } m = \max_{V(x)=r} |T_3(P_3)(x) + T_4(P_4)(x)|.$$

Then (7.23) and (7.24) imply that for $V(x) = r$ and $\lambda > 0$ we have

$$\begin{aligned} L_f V(x) &= \alpha(\lambda)D(x) + Q_3(x) + Q_4(x) + Q_5(x) \\ &= \alpha(\lambda)D(x) + (\beta(\lambda) - \beta(0))(T_3(P_3)(x) + T_4(P_4)(x)) + \frac{L}{8}(x_1^2 + x_2^2)^2 + Q_5(x) \\ &\leq \alpha(\lambda)M + |\beta(\lambda) - \beta(0)|m + \frac{L}{16}(x_1^2 + x_2^2)^2, \end{aligned}$$

where inequality (7.22) was used in the last step. The continuity of the functions α and β imply that there exist λ_0 , for which $\lambda \in (0, \lambda_0)$ implies

$$\alpha(\lambda)M + |\beta(\lambda) - \beta(0)|m < \frac{L}{32}\varepsilon^4.$$

Hence $V(x) = r$ and $\lambda \in (0, \lambda_0)$ imply

$$L_f V(x) < \frac{L}{32}\varepsilon^4 < 0.$$

Therefore the set given by $V(x) \leq r$ is positively invariant and it contains a single equilibrium the origin that is unstable. Hence the Poincaré-Bendixson theorem implies that there is a stable limit cycle in this set. This proves that supercritical Andronov–Hopf bifurcation occurs at $\lambda = 0$. It can be proved similarly that for $L > 0$ the bifurcation is subcritical. Summarising, the following theorem is proved.

Theorem 7.7.. *Consider system (7.19)–(7.20), where the polynomials A_2, B_2 are given in (7.11), the polynomials A_3, B_3 are given in (7.12) and (7.13) and the functions A_4, B_4 satisfy the estimates (7.6). Assume that the function α is differentiable, β is continuous and the conditions (7.21) hold. Let the Lyapunov coefficient L be given by (7.14). Then in the case $L < 0$ supercritical Andronov–Hopf bifurcation occurs at $\lambda = 0$, while for $L > 0$ subcritical Andronov–Hopf bifurcation occurs at $\lambda = 0$.*

Remark 7.1. *We note that the theorem can be generalised to the case when the functions A_i and B_i depend continuously also on the parameter λ . Namely, the Lyapunov function is constructed to the parameter value $\lambda = 0$ and the continuous dependence on the parameter implies that the sign of $L_f V$ is the same for small λ values as for $\lambda = 0$.*

7.2.4 Andronov–Hopf bifurcation for arbitrary parameter dependence

Consider now a general two dimensional system $\dot{x} = f(x, \lambda)$, where $\lambda \in \mathbb{R}$ is a parameter and $x(t) \in \mathbb{R}^2$. Assume that for the parameter value λ_0 x_0 is an equilibrium and the eigenvalues of the Jacobian are purely imaginary, that is $f(x_0, \lambda_0) = 0$ and the eigenvalues of $\partial_x f(x_0, \lambda_0)$ are $\pm i\omega$. Then the implicit function theorem implies that there exists $\delta > 0$ and a differentiable function $g : (\lambda_0 - \delta, \lambda_0 + \delta) \rightarrow \mathbb{R}^2$, such that $g(\lambda_0) = x_0$ and $f(g(\lambda), \lambda) = 0$ for all $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. Introducing the new variable $\tilde{x} = x - g(\lambda)$, a new parameter $\tilde{\lambda} = \lambda - \lambda_0$ and $\tilde{f}(\tilde{x}, \tilde{\lambda}) = f(\tilde{x} + g(\lambda), \tilde{\lambda} + \lambda_0)$ one obtains

$$\dot{\tilde{x}} = \dot{x} = f(x, \lambda) = \tilde{f}(\tilde{x}, \tilde{\lambda})$$

and

$$\partial_x f(x_0, \lambda_0) = \partial_{\tilde{x}} \tilde{f}(0, 0).$$

Moreover, $\tilde{f}(0, \tilde{\lambda}) = 0$ for all $|\tilde{\lambda}| < \delta$.

It can be easily seen that in the system $\dot{x} = f(x, \lambda)$ Andronov–Hopf bifurcation occurs at λ in the point $g(\lambda)$, if and only if in the system $\dot{\tilde{x}} = \tilde{f}(\tilde{x}, \tilde{\lambda})$ Andronov–Hopf bifurcation occurs at $\tilde{\lambda} = 0$ in the origin. Therefore it is enough to investigate systems, where the origin remains an equilibrium as the parameter is varied and the bifurcation is at zero. That is it can be assume without loss of generality that $f(0, \lambda) = 0$ for all λ and the eigenvalues of the matrix $\partial_x f(0, 0)$ have zero real part. Thus it is enough to investigate the bifurcation in systems of the form

$$\dot{x} = B(\lambda)x + h(x, \lambda), \tag{7.25}$$

where $h(0) = 0$ and $h'(0) = 0$.

This system can be reduced further by transforming the linear part to Jordan canonical form. Let the eigenvalues of $B(\lambda)$ be $\alpha(\lambda) \pm i\beta(\lambda)$ and the eigenvectors be $s(\lambda) = r(\lambda) \pm iq(\lambda)$. Introducing the matrix $P = (r(\lambda), k(\lambda)) \in \mathbb{R}^{2 \times 2}$ let $X = P^{-1} \cdot x$ be the new unknown function. For this function the differential equation takes the form

$$\dot{X} = P^{-1} \cdot \dot{x} = P^{-1} \cdot f(x, \lambda) = P^{-1} \cdot f(PX, \lambda).$$

Then

$$P^{-1} \cdot f(PX, \lambda) = P^{-1}B(\lambda)PX + P^{-1}h(PX, \lambda).$$

Let

$$A(\lambda) = P^{-1}B(\lambda)P, \quad \text{and} \quad F(X, \lambda) = P^{-1}h(PX, \lambda),$$

where the definition of P yields

$$A(\lambda) = \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ -\beta(\lambda) & \alpha(\lambda) \end{pmatrix}$$

and $F : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable with $F(0, \lambda) = 0$, $\partial_x F(0, \lambda) = 0 \in \mathbb{R}^{2 \times 2}$. Then for the function X the differential equation takes the form

$$\dot{X} = A(\lambda)X + F(X, \lambda).$$

That is the coordinates of X are solutions of system (7.19)-(7.20) with a possible dependence of A_i and B_i on the parameter λ . Since this system was obtained from system (7.25) by a linear transformation, Andronov–Hopf bifurcation occurs in this system, if and only if it occurs in system (7.25). The sufficient condition of the bifurcation for this system was given in Theorem 7.7.. The Lyapunov coefficient can be expressed in terms of the coordinates of the function F , because $F_1 = A_2 + A_3 + A_4$ and $F_2 = B_2 + B_3 + B_4$. Then the Lyapunov coefficient is

$$L = \partial_1^3 F_1 + \partial_1 \partial_2^2 F_1 + \partial_1^2 \partial_2 F_2 + \partial_2^3 F_2 \\ + \partial_{12} F_1 \cdot (\partial_1^2 F_1 + \partial_2^2 F_1) - \partial_{12} F_2 \cdot (\partial_1^2 F_2 + \partial_2^2 F_2) - \partial_1^2 F_1 \cdot \partial_1^2 F_2 + \partial_2^2 F_1 \cdot \partial_2^2 F_2,$$

where the partial derivatives are taken at $\lambda = 0$ and at the origin, i.e. for example, $\partial_1^3 F_1$ stands for $\partial_1^3 F_1(0, 0)$. Therefore Theorem 7.7. yields the sufficient condition for the Andronov–Hopf bifurcation in system (7.25).

Theorem 7.8.. *Consider system (7.25), where the eigenvalues of $B(\lambda)$ are $\alpha(\lambda) \pm i\beta(\lambda)$ and its eigenvectors are $s(\lambda) = r(\lambda) \pm iq(\lambda)$. Assume that the function α is differentiable, β is continuous and the conditions (7.21) hold. Denoting by P the matrix, the columns of which are the vectors $r(\lambda)$ and $q(\lambda)$ introduce the function $F(X, \lambda) = P^{-1}h(PX, \lambda)$, and using its partial derivatives at $(0, 0)$ compute the above Lyapunov coefficient L . Then in the case $L < 0$ supercritical Andronov–Hopf bifurcation occurs at $\lambda = 0$ in the origin, while for $L > 0$ the Andronov–Hopf bifurcation at $\lambda = 0$ is subcritical.*

This theorem enables us to find Andronov–Hopf bifurcation a general two dimensional system. The theorem can be generalised for arbitrary dimensional phase spaces by using center manifold reduction. This general formulation can be found in Kuznetsov’ book [17].

7.2.5 Case study for finding the Andronov–Hopf bifurcation

It was shown in the previous subsection that in order to find the Andronov–Hopf bifurcation several transformations of the system are needed and the Lyapunov coefficient can only be determined after these transformations. In this subsection we illustrate the whole process by an example.

Consider the following system of differential equations

$$\dot{x} = x(x - x^2 - y), \tag{7.26}$$

$$\dot{y} = y(x - a), \tag{7.27}$$

where $a \in \mathbb{R}$ is a positive parameter. We note that the equation originates from a chemical reaction (called Lotka–Volterra-autocatalator reaction). The equilibria are $(0, 0)$, $(1, 0)$ and $(a, a - a^2)$. The Jacobian matrix takes the form

$$\begin{pmatrix} 2x - 3x^2 - y & -x \\ y & x - a \end{pmatrix}.$$

The point $(0, 0)$ is degenerate, and the equilibrium $(1, 0)$ is a saddle when $a > 1$ and it is a stable node when $a < 1$. Investigate the system at the point $(a, a - a^2)$. At this point the Jacobian matrix is

$$\begin{pmatrix} a - 2a^2 & -a \\ a - a^2 & 0 \end{pmatrix}.$$

If $\frac{1}{2} < a < 1$, then this point is stable, while for $0 < a < \frac{1}{2}$, it is unstable. Hence at $a = \frac{1}{2}$ Andronov–Hopf bifurcation may occur. Now it will be shown that supercritical Andronov–Hopf bifurcation occurs at $a = \frac{1}{2}$.

First, the equilibrium is transformed to the origin and the bifurcation value of the parameter is transformed to zero. That is let $\xi = x - a$, $\eta = y - a + a^2$ and $\lambda = \frac{1}{2} - a$. Then $x = \xi + a$, $y = \eta + a - a^2$ and $a = \frac{1}{2} - \lambda$. The differential equations yield

$$\dot{x} = \dot{\xi} = \xi \cdot \lambda(1 - 2\lambda) - \eta(\frac{1}{2} - \lambda) - \xi \cdot \eta + \xi^2 \cdot (3\lambda - \frac{1}{2}) - \xi^3$$

and

$$\dot{y} = \dot{\eta} = \xi(\eta + \frac{1}{4} - \lambda^2).$$

Thus the system is transformed to the form given in (7.25), i.e. the point $(0, 0)$ is an equilibrium for all λ and the linear part is

$$B(\lambda) = \begin{pmatrix} \lambda(1 - 2\lambda) & \lambda - \frac{1}{2} \\ \frac{1}{4} - \lambda^2 & 0 \end{pmatrix}.$$

The trace of the matrix is $Tr = \lambda(1 - 2\lambda)$, that changes sign at $\lambda = 0$, hence the bifurcation may occur at this parameter value. In order to apply Theorem 7.8. we need the eigenvalues and eigenvectors of this matrix. The characteristic equation is

$$\mu(\mu - \lambda(1 - 2\lambda)) + (-\lambda + \frac{1}{2})(\frac{1}{4} - \lambda^2) = 0,$$

hence the eigenvalues are

$$\mu_{1,2} = \frac{\lambda(1 - 2\lambda) \pm \sqrt{\lambda^2(1 - 2\lambda)^2 - 4(\frac{1}{2} - \lambda)(\frac{1}{4} - \lambda^2)}}{2}.$$

Their real part is $\alpha(\lambda) = \lambda(\frac{1}{2} - \lambda)$, hence the conditions $\alpha(0) = 0$ and $\alpha'(0) = \frac{1}{2} \neq 0$ are fulfilled. The imaginary part of the eigenvalues are

$$\beta(\lambda) = \frac{\sqrt{\lambda^2(1 - 2\lambda)^2 - 4(\frac{1}{2} - \lambda)(\frac{1}{4} - \lambda^2)}}{2},$$

hence $\beta(0) = \frac{1}{2\sqrt{2}} \neq 0$. Thus the conditions in (7.21) hold. The matrix belonging to $\lambda = 0$ is

$$\begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{4} & 0 \end{pmatrix},$$

its eigenvalues are $\pm \frac{1}{2\sqrt{2}}$ and its eigenvectors take the form

$$\begin{pmatrix} 1 \\ -i\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Introduce the new coordinates u, v by the linear transformation

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

The system at $\lambda = 0$ takes the form

$$\begin{aligned} \dot{\xi} &= -\frac{1}{2}\eta - \xi\eta - \frac{1}{2}\eta^2 - \xi^3, \\ \dot{\eta} &= \frac{1}{4}\xi + \xi\eta, \end{aligned}$$

In the new coordinates u, v it is

$$\begin{aligned} \dot{u} &= \frac{1}{\sqrt{8}}v + \frac{1}{\sqrt{2}}uv - \frac{1}{2}u^2 - u^3, \\ \dot{v} &= -\frac{1}{\sqrt{8}}u + uv. \end{aligned}$$

Here the linear part is in Jordan canonical form and the non-linear part F is

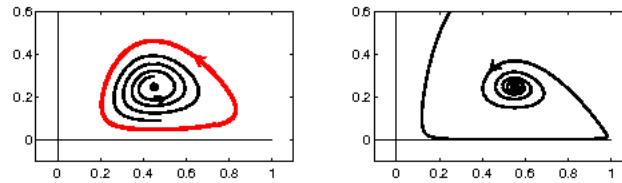
$$F(u, v) = \begin{pmatrix} \frac{1}{\sqrt{2}}uv - \frac{1}{2}u^2 - u^3 \\ uv \end{pmatrix}$$

The following partial derivatives are needed to compute the Lyapunov coefficient.

$$\partial_1 F_1 = \frac{1}{\sqrt{2}}v - u - 3u^2, \quad \partial_1^2 F_1 = -1 - 6u, \quad \partial_1^3 F_1 = -6,$$

$$\partial_2 F_1 = \frac{1}{\sqrt{2}}, \quad \partial_2^2 F_1 = 0, \quad \partial_{12} F_1 = \frac{1}{\sqrt{2}}, \quad \partial_1 F_2 = v, \quad \partial_2 F_2 = u.$$

Taking these partial derivatives at the point $(u, v) = (0, 0)$ and substituting them into the formula of L we get $L = -1/4$. Therefore supercritical Andronov–Hopf bifurcation occurs. In the original system (7.26)–(7.27) the bifurcation is at $a = 1/2$. Figure 7.2.5 shows that for $a > 1/2$ the equilibrium $(a, a - a^2)$ is stable, while for $a < 1/2$ it is unstable and there is a stable limit cycle.



Andronov–Hopf bifurcation in the system (7.26)–(7.27) at $a = 1/2$. The phase portraits are shown for $a < 1/2$ (left) and for $a > 1/2$ (right).

7.3 Computing bifurcation curves in two-parameter systems by using the parametric representation method

7.3.1 The parametric representation method

The computation of one co-dimensional bifurcation curves in two-parameter systems often leads to the equation

$$f(x, u_1, u_2) = 0,$$

where x is the unknown and u_1, u_2 are given parameters. In this subsection it is shown that the parametric representation method is a useful tool to compute the bifurcation curve when the parameters are involved linearly in the above equation (which is often the case in several applications). Let us assume that

$$f(x, u_1, u_2) = f_0(x) + f_1(x)u_1 + f_2(x)u_2,$$

where $f_i : \mathbb{R} \rightarrow \mathbb{R}$ are continuously differentiable functions.

Definition 7.9.. For a given $u \in \mathbb{R}^2$ denote by $N(u)$ the number of solutions of $f(x, u_1, u_2) = 0$ (that can also be equal to infinity), that is

$$N(u) := |\{x \in \mathbb{R} : f(x, u) = 0\}|.$$

A pair of parameters will be called bifurcational if the number of solutions is not constant in its small neighbourhood.

Definition 7.10.. The parameter $u_0 \in \mathbb{R}^2$ is called regular, if it has a neighbourhood $U \subset \mathbb{R}^2$, for which $u \in U$ implies $N(u) = N(u_0)$, that is N is constant in the neighbourhood.

The parameter $u_0 \in \mathbb{R}^2$ is called bifurcational, if it is not regular.

The set of bifurcational parameter values is called bifurcation set that is denoted by B .

According to the implicit function theorem x can be expressed from equation $f(x, u_1, u_2) = 0$ in terms of the parameters when $\partial_x f$ is not zero. Hence the number of solutions may change only in the case when $\partial_x f(x, u_1, u_2) = 0$ holds together with $f(x, u_1, u_2) = 0$. The points satisfying both conditions form the singularity set.

Definition 7.11.. The singularity set belonging to equation $f(x, u_1, u_2) = 0$ is

$$S := \{u \in \mathbb{R}^2 : \exists x \in \mathbb{R} : f(x, u) = 0 \text{ and } \partial_x f(x, u) = 0\}.$$

The goal of the further investigation is to determine the number and value of the solutions for a given parameter pair $u \in \mathbb{R}^2$, and to reveal the relation of the bifurcation set B and the singularity set S .

When f depends linearly on the parameters u_1 and u_2 , then the singularity set is given by the system of equations

$$\begin{aligned} f_0(x) + f_1(x)u_1 + f_2(x)u_2 &= 0, \\ f'_0(x) + f'_1(x)u_1 + f'_2(x)u_2 &= 0. \end{aligned}$$

The main idea of the parametric representation method is that in this system x can be considered as a parameter in terms of which the original parameters u_1, u_2 can be

expressed by simply solving a system of linear equations. For sake of brevity, the method will be presented only in the special case of $f_1(x) = 1$, $f_2(x) = x$. The function f_0 will then be denoted by g . The singularity is given then by

$$\begin{aligned}u_1 + u_2x + g(x) &= 0, \\u_2 + g'(x) &= 0.\end{aligned}$$

Expressing the parameters u_1, u_2 in terms of x we get

$$\begin{aligned}u_1 &= xg'(x) - g(x), \\u_2 &= -g'(x).\end{aligned}$$

Introduce the curve $D : \mathbb{R} \rightarrow \mathbb{R}^2$ that associates the pair (u_1, u_2) to x , that is

$$D_1(x) = xg'(x) - g(x), \quad D_2(x) = -g'(x).$$

This curve, that yields the points of the singularity set as a curve parametrised by x , is called discriminant curve.

Our original goal was to find x in terms of the parameters u_1 and u_2 . Hence introduce the set

$$M(x) = \{(u_1, u_2) \in \mathbb{R}^2 : f_0(x) + f_1(x)u_1 + f_2(x)u_2 = 0\}$$

that contains those parameter pairs, for which x is a solution of the equation. Because of the linear parameter dependence, this is a line in the parameter plane. One of the most important results of the parametric representation method is the following tangential property.

Proposition 7.3 (Tangential property). *The line $M(x)$ is tangential to the discriminant curve at the point $D(x)$.*

Proof. Since the pair $(u_1, u_2) = D(x)$ is a solution of equation $f(x, u_1, u_2) = 0$, we have $D(x) \in M(x)$, i.e. the point $D(x)$ lies on the line $M(x)$. Thus it is enough to prove that the tangent vector $D'(x)$ is orthogonal to the normal of the line $M(x)$, which is the vector $(f_1(x), f_2(x))$. This can easily be shown for any differentiable functions f_i . Namely, the pair $(u_1, u_2) = D(x)$ is a solution of $f(x, u_1, u_2) = 0$, i.e.

$$f_0(x) + f_1(x)D_1(x) + f_2(x)D_2(x) = 0$$

holds for all x . Hence differentiating this equation leads to

$$f'_0(x) + f'_1(x)D_1(x) + f'_2(x)D_2(x) + f_1(x)D'_1(x) + f_2(x)D'_2(x) = 0.$$

Here $f'_0(x) + f'_1(x)D_1(x) + f'_2(x)D_2(x) = 0$, because the pair $(u_1, u_2) = D(x)$ is a solution of the equation $\partial_x f(x, u_1, u_2) = 0$ too, hence

$$f_1(x)D'_1(x) + f_2(x)D'_2(x) = 0,$$

that proves the desired orthogonality. □

Remark 7.2. We note that the tangential property expresses the well-known fact that the envelope of the solution curves $M(x)$ of the equation $f(x, u_1, u_2) = 0$ is the singularity set.

In the case of linear parameter dependence the tangential property yields a simple geometrical method for finding the solutions x of equation $f_0(x) + f_1(x)u_1 + f_2(x)u_2 = 0$ for given values of the parameters u_1 and u_2 . Namely, for a given pair (u_1, u_2) the number x is a solution of equation $f_0(x) + f_1(x)u_1 + f_2(x)u_2 = 0$, if and only if (u_1, u_2) lies on the tangent of the D curve drawn at the point corresponding to x , i.e. at $D(x)$.

The application of the method is illustrated by the following examples.

Example 7.3. Let $f_1(x) = 1$, $f_2(x) = x$, $f_0(x) = g(x) = x^2$, i.e. consider the quadratic equation $x^2 + u_2x + u_1 = 0$. Then $D_1(x) = x^2$, $D_2(x) = -2x$ and note that $D_1 = D_2^2/4$, hence the discriminant curve D is a parabola in the parameter plane (u_1, u_2) . Its equation is $4u_1 = u_2^2$, that is exactly the discriminant of the quadratic equation. For a given pair (u_1, u_2) the number x is a solution of equation $x^2 + u_2x + u_1 = 0$, if and only if (u_1, u_2) lies on the tangent of the D curve drawn at the point corresponding to x . This is shown in Figure 7.3, where also the direction of the parametrisation of the D curve is shown by an arrow. The negative values of x belong to the upper branch of the parabola, while the positive x values correspond to the lower branch. It is easy to see from the figure that from each point of the domain on the left hand side of the parabola two tangents can be drawn to the parabola. From the points lying in the right hand side no tangents can be drawn. This way we arrived in a geometric way to the simple algebraic fact, that in the case $4u_1 < u_2^2$ the quadratic equation has two solutions, while for $4u_1 > u_2^2$ it has no solution. Moreover, the tangent points yield information about the value of the solutions. For example, choosing a parameter pair from the left half plane we get one tangent point on the upper branch and another one on the lower branch. This proves that in the case $u_1 < 0$ the equation has one positive and one negative solution. Another simple geometric fact is that from the quadrant $u_1, u_2 > 0$ tangents can only be drawn to the upper branch, proving that for $u_1, u_2 > 0$ the equation can have only negative solutions.

Example 7.4. Let $f_1(x) = 1$, $f_2(x) = x$, $f_0(x) = g(x) = x^3$, i.e. consider the cubic equation $x^3 + u_2x + u_1 = 0$. Then $D_1(x) = 2x^3$, $D_2(x) = -3x^2$ and note that $27D_1^2 = -4D_2^3$, hence the discriminant curve D lies in the lower half plane, it touches the vertical axis at the origin, where it has a cusp point. Its equation is $27u_1^2 + 4u_2^3 = 0$, that is exactly the discriminant of the cubic equation. For a given pair (u_1, u_2) the number x is a solution of equation $x^3 + u_2x + u_1 = 0$, if and only if (u_1, u_2) lies on the tangent of the D curve drawn at the point corresponding to x . The D curve is shown in Figure 7.4, where also the direction of the parametrisation of the curve is shown by an arrow. The negative values of x belong to the left branch of the curve, while the positive x values correspond to the right branch. It is easy to see from the figure that from each point between the two branches three tangents can be drawn to the curve. From the points

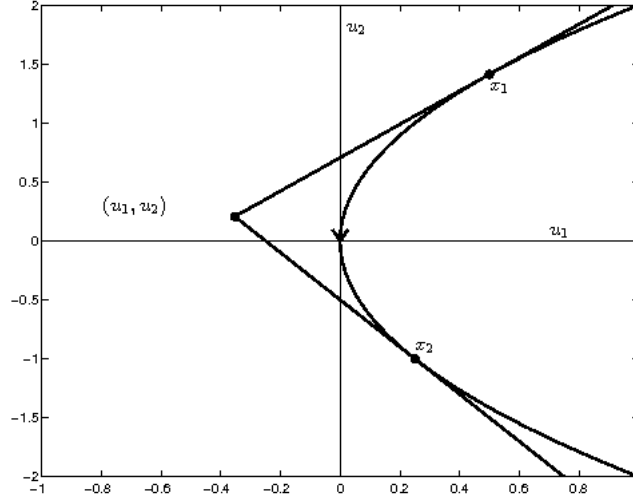


Figure 7.3: The two tangent lines drawn from the point (u_1, u_2) to the discriminant curve belonging to the quadratic equation $x^2 + u_2x + u_1 = 0$. They touch the curve at the points corresponding to x_1 and x_2 . These are the solutions of the quadratic equation for the given value of the parameters.

lying outside the cusp one tangent can be drawn. The numbers in the figure indicate the number of solutions for parameter pairs chosen from the given domain. Moreover, the tangent points yield information about the value of the solutions. Here we can again observe the simple geometric fact that from the quadrant $u_1, u_2 > 0$ tangents can only be drawn to the left branch, proving that for $u_1, u_2 > 0$ the equation can have only negative solutions.

We note, that it can be easily shown that the discriminant curve consists of convex arcs. The number of tangents that can be drawn to a convex arc can be easily determined as it is shown in Figure 7.5. This geometrical result is used when the number of solutions of a quartic equation is studied in the next example.

Example 7.5. Let $f_1(x) = 1$, $f_2(x) = x$, $f_0(x) = g(x) = -x^2 + x^4$, i.e. consider the quartic equation $u_1 + u_2x - x^2 + x^4 = 0$. Then $D_1(x) = 3x^4 - x^2$, $D_2(x) = 2x - 4x^3$. This curve can be plotted in the plane by using the graphs of the coordinate function D_1 and D_2 . The curve has two cusp points as it is shown in Figure 7.6. This figure shows that from the points of the inner part of the so-called "swallow-tail" four tangents can be drawn to the curve. From the points in the outer domain two tangents can be drawn and from the remaining part of the plane no tangents can be drawn. The numbers in the

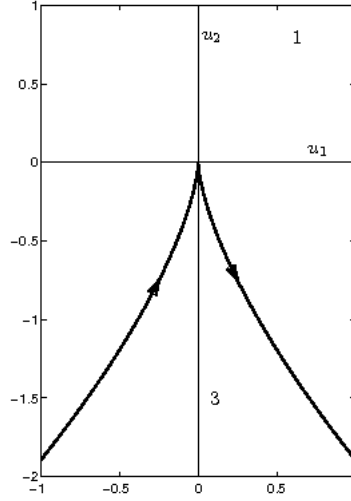


Figure 7.4: The discriminant curve of the cubic equation $x^3 + u_2x + u_1 = 0$ divides the parameter plane into two parts according to the number of solutions. In the domain between the two branches there are three solutions, while outside the cusp there is only one.

figure indicate the number of tangents, i.e. the number of solutions for parameter pairs chosen from the given domain.

7.3.2 Bifurcation curves in two-parameter systems

In this subsection it is shown how the saddle-node and Andronov-Hopf bifurcation curves can be determined in systems of the form $\dot{x}(t) = f(x(t), u_1, u_2)$, where the phase space is two dimensional, i.e. $x(t) \in \mathbb{R}^2$ and there are two parameters u_1 and u_2 . The goal is to divide the parameter plane (u_1, u_2) according to the number and type of equilibria. The bifurcation curves will be determined by using the parametric representation method presented in the previous subsection. By plotting these two bifurcation curves one can see that when they have a common tangent at a point, then this point is Takens–Bogdanov bifurcation point, at which the linear part has a double zero eigenvalue.

Consider first the following two dimensional system.

$$\dot{x} = y, \tag{7.28}$$

$$\dot{y} = u_1 + u_2 \cdot x + x^2 + x \cdot y. \tag{7.29}$$

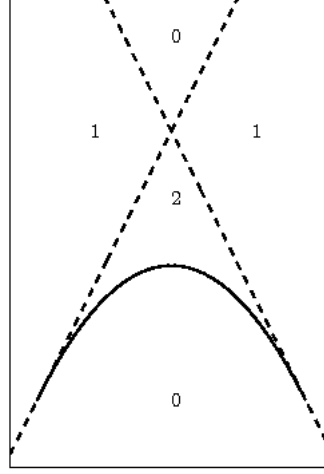


Figure 7.5: The number of tangents that can be drawn to a convex arc from different points of the plane.

The equilibria are determined by the equations

$$\begin{aligned} 0 &= y, \\ 0 &= u_1 + u_2x + x^2 + x \cdot 0. \end{aligned}$$

Hence they are determined by a single quadratic equation $f(x, u) := u_1 + u_2x + x^2 = 0$. The number of solutions of equations of this type can be determined by using the parametric representation method. The singularity set is given by the equations $f(x, u) = 0$ and $\partial_x f(x, u) = 0$, that is

$$\begin{aligned} u_1 + u_2x + x^2 &= 0 \\ u_2 + 2x &= 0. \end{aligned}$$

The parameters u_1, u_2 can be expressed from these equations in terms of x , this way we get the discriminant curve, that is the parabola

$$\begin{aligned} u_2 &= D_2(x) = -2x, \\ u_1 &= D_1(x) = x^2, \end{aligned}$$

in our case, as it is shown in Figure 7.3.2.

The saddle-node and Andronov-Hopf bifurcation curves in the system (7.28)-(7.29).

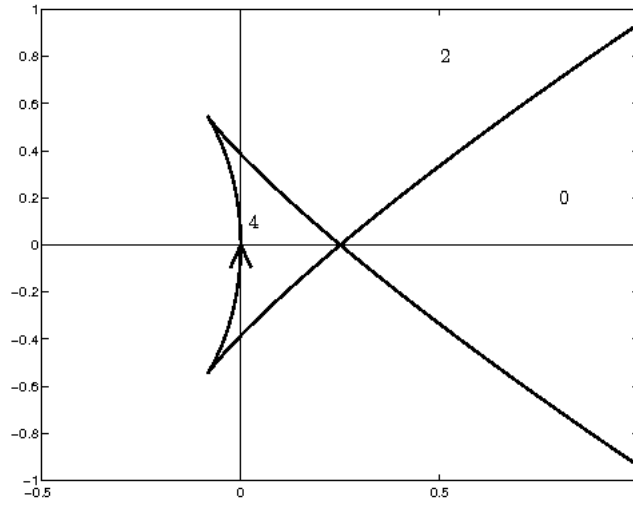
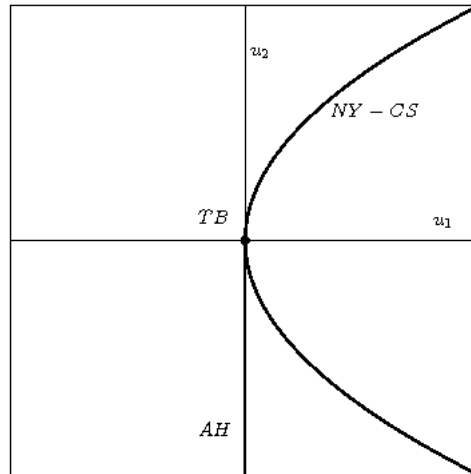


Figure 7.6: The discriminant curve of the quartic equation $u_1 + u_2x - x^2 + x^4 = 0$ divides the parameter plane into three parts according to the number of solutions. The numbers in the figure indicate the number of solutions for parameter pairs chosen from the given domain.



The number of solutions for a given parameter pair (u_1, u_2) is equal to the number of tangents that can be drawn to the discriminant curve from the parameter point (u_1, u_2) .

Thus choosing a parameter pair in the left hand side of the parabola, there are two equilibria, while for parameter pairs in the right hand side there is no equilibria. That is the discriminant curve is the saddle-node bifurcation curve. Crossing this curve with the parameters the number of solutions changes by two.

Let us now determine the Andronov-Hopf bifurcation curve. The Jacobian matrix at a point (x, y) is

$$\begin{pmatrix} 0 & 1 \\ u_2 + 2x + y & x \end{pmatrix}.$$

Its trace is $Tr = x$ and its determinant is $Det = -(u_2 + 2x + y)$. Let $x \in \mathbb{R}$ be an arbitrary number. Then for the Jacobian at the equilibrium $(x, 0)$ we have $Tr = x$, and $Det = -(u_2 + 2x)$. It is easy to see that the necessary condition of the Andronov-Hopf bifurcation is $Tr = 0$ and $Det > 0$, because in this case the Jacobian has pure imaginary eigenvalues. The curve determined by these conditions is called now the Andronov-Hopf bifurcation curve, despite of the fact that at some isolated points of the curve the Lyapunov coefficient may be zero, hence the sufficient condition may not hold. The Andronov-Hopf bifurcation consists of those parameter pairs (u_1, u_2) , for which there exists an equilibrium $(x, 0)$, at which $Tr = x = 0$ and $Det = -(u_2 + 2x) > 0$. That is the following should hold

$$u_1 + u_2x + x^2 = 0, \quad x = 0, \quad u_2 + 2x < 0.$$

These imply $u_1 = 0$ and $u_2 < 0$, thus the Andronov-Hopf bifurcation curve is the negative part of the vertical coordinate axis. This half line touches the parabola of the saddle-node bifurcation, hence the origin is the Takens–Bogdanov bifurcation point, see Figure 7.3.2.

The next example is the normal form of the Takens–Bogdanov bifurcation.

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= u_1 + u_2 \cdot y + x^2 + x \cdot y. \end{aligned}$$

The equilibria are determined by the equations

$$\begin{aligned} 0 &= y, \\ 0 &= u_1 + x^2. \end{aligned}$$

Hence they are determined by a single quadratic equation $u_1 + x^2 = 0$. The singularity set is given by the equations $f(x, u) = 0$ and $\partial_x f(x, u) = 0$, that is

$$\begin{aligned} u_1 + x^2 &= 0, \\ 2x &= 0. \end{aligned}$$

It can be seen that the parametric representation method cannot be applied, because this system cannot be solved for (u_1, u_2) . However, $x = 0$ implies $u_1 = 0$, hence the singularity set, that is the saddle-node bifurcation curve is the vertical axis $u_1 = 0$

Let us now determine the Andronov-Hopf bifurcation curve. The Jacobian matrix at a point (x, y) is

$$\begin{pmatrix} 0 & 1 \\ 2x + y & u_2 + x \end{pmatrix}.$$

Its trace at an equilibrium $(x, 0)$ is $Tr = u_2 + x = u_2 + \sqrt{-u_1}$, by using $x^2 = -u_1$, and its determinant is $Det = -2x$. Thus the Andronov-Hopf bifurcation curve is the parabola $u_2 + \sqrt{-u_1} = 0$. This touches the vertical axis, i.e. the saddle-node bifurcation curve, at the origin, hence the origin is the Takens–Bogdanov bifurcation point.

Exercise

Investigate how the saddle-node and Andronov-Hopf bifurcation curves are changing in the (u_1, u_2) parameter plane as the parameter $a \in \mathbb{R}$ is varied in system

$$\begin{aligned}\dot{x} &= x - y, \\ \dot{y} &= u_1 + u_2 \cdot y + a \cdot x \cdot y^2.\end{aligned}$$

Chapter 8

Discrete dynamical systems, symbolic dynamics, chaotic behaviour

8.1 Discrete dynamical systems

Let $M \subset \mathbb{R}^k$ be a domain and $f : M \rightarrow M$ be a diffeomorphism (i.e. a differentiable bijection with differentiable inverse). Let $f^n = f \circ f \circ \dots \circ f$ be a composition of n terms, that is $f^n(p) = f(f(\dots f(p)\dots))$. If $n < 0$, then let $f^n = (f^{-1})^{-n}$, where the inverse of f is applied in the composition $(-n)$ times. Then the function $\varphi(n, p) = f^n(p)$, $\varphi : \mathbb{Z} \times M \rightarrow M$ defines a discrete time dynamical system. The discrete dynamical system can also be given in the form of a recursion

$$x_{n+1} = f(x_n), \quad x_0 = p,$$

then $\varphi(n, p) = x_n$. The orbit starting from the point p is the discrete set of points $\{\varphi(n, p) : n \in \mathbb{Z}\}$ or a bi-infinite sequence.

Definition 8.1.. *The point $p \in M$ is called a fixed point of the dynamical system, if $f(p) = p$, that is $\varphi(n, p) = p$ for all $n \in \mathbb{Z}$. In this case the orbit of p consists of a single point.*

The stability of a fixed point can be defined analogously to the continuous time case.

Definition 8.2.. *The fixed point $p \in M$ is called stable, if for any $\varepsilon > 0$ there exists a positive number δ , such that $|q - p| < \delta$ implies $|\varphi(n, q) - p| < \varepsilon$ for all $n \in \mathbb{N}$.*

The fixed point $p \in M$ is called asymptotically stable, if it is stable and $\lim_{n \rightarrow \infty} \varphi(n, p) = p$.

The fixed point $p \in M$ is called unstable, if it is not stable.

These notions are illustrated by one dimensional linear examples.

Example 8.1. For a given $a \in \mathbb{R}$ let $f(x) = ax$ and $x_0 = p$. If $a = 2$, i.e. $f(x) = 2x$, then $\varphi(n, p) = 2^n p$ that tends to infinity as $n \rightarrow \infty$. Hence the fixed point 0 is unstable.

If $a = -1$, i.e. $f(x) = -x$, then $\varphi(n, p) = (-1)^n p$, hence the orbits consist of two points (since $x_{n+2} = x_n$). These orbits will be referred to as 2-periodic orbits. In this case the fixed point 0 is stable but not asymptotically stable.

If $a = 1/2$, i.e. $f(x) = x/2$, then $\varphi(n, p) = (1/2)^n p$, hence the orbits tend to zero as $n \rightarrow \infty$. Therefore the fixed point 0 is asymptotically stable.

In general, 0 is a fixed point of the dynamical system corresponding to the map $f(x) = a \cdot x$. This point is asymptotically stable, if $|a| < 1$, because $\varphi(n, p) = a^n \cdot p \rightarrow 0$, if $|a| < 1$. The fixed point is unstable, if $|a| > 1$, and stable but not asymptotically stable, if $|a| = 1$.

The stability of a hyperbolic fixed point can be investigated by linearisation, also analogously to the continuous time case. This is formulated for one dimensional systems below.

Theorem 8.3.. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function, and let p be a fixed point of the dynamical system defined by the recursion $x_{n+1} = f(x_n)$, that is $f(p) = p$. If $|f'(p)| < 1$, then p is asymptotically stable and in the case $|f'(p)| > 1$, p is unstable.

Proof. Since $\lim_{x \rightarrow p} \frac{f(x) - f(p)}{x - p} = f'(p)$, there exists $\varepsilon > 0$ and $a < 1$, such that for all $x \in (p - \varepsilon, p + \varepsilon)$ one has $|\frac{f(x) - f(p)}{x - p}| < a$, implying $|f(x) - f(p)| < a \cdot |x - p|$, that is $|f(x) - p| < a \cdot |x - p|$. Therefore $x \in (p - \varepsilon, p + \varepsilon)$ implies $f(x) \in (p - \varepsilon, p + \varepsilon)$ and $|f^2(x) - f^2(p)| < a \cdot |f(x) - f(p)| < a^2 \cdot |x - p|$. By induction we obtain

$$|f^k(x) - f^k(p)| < a^k \cdot |x - p| \rightarrow 0,$$

as $k \rightarrow \infty$. This means that solutions starting in a neighbourhood of the fixed point remain close to it and converge to it as $k \rightarrow \infty$. Hence the fixed point is asymptotically stable. The instability in the case $|f'(p)| > 1$ can be verified similarly. \square

Definition 8.4.. The point $p \in M$ is called a periodic point, if there exist an integer $k > 1$, such that $f^k(p) = p$ and p is not a fixed point. The smallest integer $k > 1$ with this property is called the period of the point p . The orbit $\{p, f(p), f^2(p), \dots, f^{k-1}(p)\}$ is called a k -periodic orbit.

Notice that $f^k(p) = p$ means that p is a fixed point of the map f^k . Based on this observation one can define the stability of a periodic orbit as follows.

Definition 8.5.. The k -periodic orbit starting at p is called (asymptotically) stable, if p is an (asymptotically) stable fixed point of the map f^k . The instability of a periodic orbit can be defined similarly.

According to the above theorem the stability of a periodic orbit is determined by $(f^k)'$. Notice that the chain rule implies $(f^2)'(p) = f'(f(p))f'(p)$. Similarly one obtains $(f^3)'(p) = f'(f^2(p)) \cdot f'(f(p)) \cdot f'(p)$. Introducing the notation $p_1 = p, p_2 = f(p_1), \dots, p_k = f^{k-1}(p_1)$ for the points of a k periodic orbit leads to

$$(f^k)'(p) = f'(p_k) \cdot f'(p_{k-1}) \cdot \dots \cdot f'(p_1).$$

Hence the above theorem yields the following one for the stability of periodic orbits.

Theorem 8.6.. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function, and let p be a k -periodic point of the dynamical system defined by the recursion $x_{n+1} = f(x_n)$, that is $f^k(p) = p$. The k -periodic orbit $\{p_1, p_2, \dots, p_k\}$ is asymptotically stable if $|f'(p_1) \cdot \dots \cdot f'(p_k)| < 1$.*

8.1.1 Logistic map

The notions introduced above will now be illustrated in the case of the logistic map. Moreover, this dynamical system will lead us to the definition of the chaotic orbit.

The logistic map is the function $f(x) = ax(1 - x)$, that will be considered in the interval $[0, 1]$. It will be assumed about the parameter $a \in \mathbb{R}$ that $0 < a \leq 4$, namely in this case the function maps the interval $[0, 1]$ into itself.

The fixed points of the map are given by the equation $ax(1 - x) = x$ as $x = 0$ and $x = 1 - \frac{1}{a}$. The latter lies in the interval $[0, 1]$ only for $a > 1$.

The goal of our investigation is to understand how the behaviour of the orbits changes as the parameter $a \in [0, 4]$ is varied.

Let us determine first the stability of the fixed points. The derivative of the map is $f'(x) = a - 2ax$, hence $f'(0) = a$ and $f'(\frac{1}{a}) = 2 - a$. The stability of the fixed point is guaranteed by the condition $|f'(p)| < 1$, hence in the case $0 < a < 1$ the fixed point 0 is asymptotically stable, and for $1 < a < 3$ the fixed point $\frac{1}{a}$ becomes asymptotically stable. It can be proved that in these cases they are globally asymptotically stable in the interval $[0, 1]$, i.e. we have the following proposition.

Proposition 8.1. *1. If $0 < a < 1$, then for any initial condition $x_0 \in [0, 1]$ the solution converges to the fixed point 0, that is for the sequence $x_{n+1} = ax_n(1 - x_n)$ we have $x_n \rightarrow 0$ as $n \rightarrow \infty$.*

2. If $1 < a < 3$, then for any initial condition $x_0 \in (0, 1)$ the solution converges to the fixed point $1 - \frac{1}{a}$, that is for the sequence $x_{n+1} = ax_n(1 - x_n)$ we have $x_n \rightarrow 1 - \frac{1}{a}$ as $n \rightarrow \infty$.

It is a natural question, what happens as the value of a is increased further, that is for $a > 3$ what can be said about the asymptotic behaviour of the sequence. Numerical experiments show that a 2-periodic orbit appears and takes over attractivity. Let us

compute this 2-periodic orbit. Denoting its points by x_0 and x_1 we arrive to the equations $x_1 = ax_0(1 - x_0)$ and $x_0 = ax_1(1 - x_1)$ leading to

$$x_0 = a^2 x_0 (1 - x_0) (1 - ax_0 (1 - x_0)).$$

It is easy to see that $x_0 = 0$ and $x_0 = 1 - \frac{1}{a}$ are solutions of this quartic equation, because fixed points are also 2-periodic. Factoring out these roots we arrive to a quadratic equation, the roots of which can easily be computed. It turns out that these roots are real if and only if $a > 3$, that is the 2-periodic orbit is born exactly when the fixed point $x_0 = 1 - \frac{1}{a}$ loses its stability. The stability of the 2-periodic orbit can also be determined analytically. Denoting by x_0 and x_1 the points of the periodic orbit, the stability requires $|f'(x_0)f'(x_1)| < 1$. A straightforward but tiresome calculation shows that this holds if $3 < a < 1 + \sqrt{6}$. Thus we have proved the following.

Proposition 8.2. *1. If $a > 3$, then there exists a 2-periodic orbit, that is there exist numbers $x_0, x_1 \in [0, 1]$, satisfying the equations $x_1 = ax_0(1 - x_0)$ and $x_0 = ax_1(1 - x_1)$.*

2. If $3 < a < 1 + \sqrt{6}$, then the 2-periodic orbit is asymptotically stable.

The next question is that what happens as the value of a is increased further, that is for $a > 1 + \sqrt{6}$ what can be said about the asymptotic behaviour of the sequence. Numerical experiments show that a 4-periodic orbit appears and takes over attractivity. However, this cannot be determined analytically. The numerical investigation shows that for a narrow range of a values it is stable and when it loses its stability an 8-periodic orbit appears. This also loses its stability soon as a is increased and a 16-periodic orbit appears. This process is continued with periodic orbits of period 2^k , that is called period doubling. This is a sequence of bifurcations that is illustrated in Figure 8.1. In this Figure a is varied along the horizontal axis, and the asymptotic state of the sequence is plotted along the vertical direction. When there is an asymptotically stable fixed point, then this is a single point. If there is an asymptotically stable periodic orbit, then it consists of its points. The figure can be constructed only numerically by applying the recursion until the orbit gets close enough to the asymptotic state.

Denote by a_n the value of a , where the 2^n -periodic orbit appears. Thus $a_1 = 3$, $a_2 = 1 + \sqrt{6}$, ... The further values cannot be determined analytically, but numerical computations show that the sequence a_k has a limit and $\lim a_k \approx 3.569946$. There is no theoretical formula yielding this limit, however, for the relative change in the sequence the following nice result was obtained by Feigenbaum in 1975.

$$\lim_{n \rightarrow \infty} \frac{a_n - a_{n-1}}{a_{n+1} - a_n} = 4,669201609 \dots$$

This is important from the theoretical point of view, because period doubling occurs also for other dynamical systems, such as $f(x) = a - x^2$ or $f(x) = a \sin x$ and introducing the

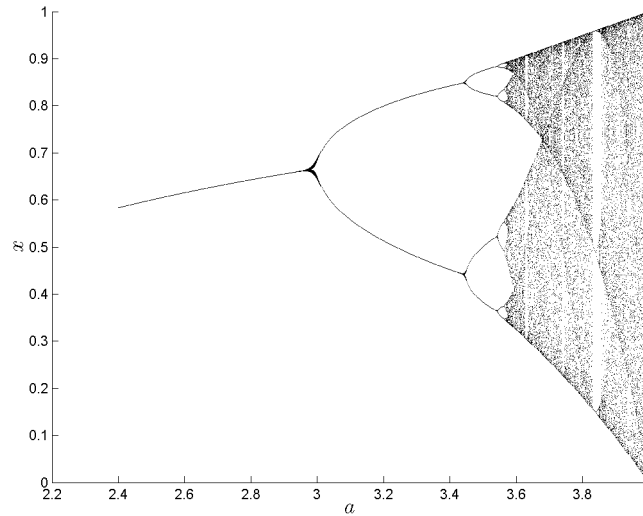


Figure 8.1: Period doubling for the logistic map. The change in the asymptotic state as the parameter a is varied.

sequence a_n in the same way, the limit above has always the same value, that is called Feigenbaum number.

8.1.2 Chaotic orbits

The detailed study of chaos is beyond the topic of this lecture notes here only a definition of chaotic orbits is introduced and it is shown that in two simple systems there are chaotic orbits. The most important elements of chaotic behaviour are the following.

- Sensitive dependence on initial conditions in a bounded set.
- Aperiodicity.

The definitions are formulated for one dimensional systems for the sake of simplicity, obviously all of them can be extended to higher dimensional systems.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuously differentiable function and consider the dynamical system given by the recursion

$$x_{n+1} = f(x_n).$$

First the sensitive dependence on initial conditions is defined.

Definition 8.7.. *At a point $p \in \mathbb{R}$ the solution depends sensitively on the initial condition, if there exists a number $\delta > 0$, such that for all $\varepsilon > 0$ there exists $q \in (p - \varepsilon, p + \varepsilon)$*

and $k \in \mathbb{N}$, for which $|f^k(q) - f^k(p)| \geq \delta$, that is the distance between the solutions starting from q and p becomes greater than a given number.

We note that sensitive dependence occurs in an unstable linear system in a trivial way. Namely, for the function $f(x) = 2x$ the orbit is given by $f^k(q) = 2^k q$, hence the distance between the points of two orbits increases exponentially. The sensitive dependence is related to chaos when the orbits are bounded.

The local measure for the change of the distance between orbits at a point can be given by the derivative of the function, because

$$f(q) - f(p) = f'(p)(q - p) + o(q - p).$$

Hence in the case $|f'(p)| < 1$, the map is a local contraction around the point p , while for $|f'(p)| > 1$, the function is a local expansion in a neighbourhood of p . Following the orbit of a point p at the point $f^k(p)$ the function can be a local expansion or a local contraction depending on k . Computing the average along the orbit of p one obtains if there is sensitive dependence on initial conditions at the point p . To measure this average the Lyapunov number and Lyapunov exponent were introduced.

Definition 8.8.. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuously differentiable function. The Lyapunov number belonging to a point $p \in \mathbb{R}$ is

$$L(p) = \lim_{n \rightarrow \infty} (|f'(p)| \cdot |f'(f(p))| \cdot \dots \cdot |f'(f^{n-1}(p))|)^{\frac{1}{n}}.$$

The Lyapunov exponent belonging to a point p is its logarithm $h(p) = \ln L(p)$, that is

$$h(p) = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot [\ln |f'(p)| + \ln |f'(f(p))| + \dots + \ln |f'(f^{n-1}(p))|].$$

If $L(p) > 1$, or in other words $h(p) > 0$, then the solution depends sensitively on initial conditions at the point p .

Let us turn to the other characteristic of chaos, the aperiodicity.

Definition 8.9.. A point $p \in \mathbb{R}$ is called asymptotically periodic, if there is a periodic point q , for which

$$\lim_{n \rightarrow \infty} |\varphi(n, p) - \varphi(n, q)| = 0$$

(that is the orbit starting at p converges to a periodic orbit).

Based on these definitions one can define the notion of a chaotic orbit.

Definition 8.10.. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuously differentiable function. The orbit starting at a point $p \in \mathbb{R}$ is called chaotic, if it is

1. *bounded,*
2. *not asymptotically periodic,*
3. *$h(p) > 0$, that is it depends sensitively on initial conditions.*

Although the definition of chaos can be relatively simply formulated, it is far from obvious to prove that a given system has chaotic orbits. The numerical investigation of the logistic map revealed that it may have chaotic orbits when the value of the parameter a is close to 4. However, it seems to be difficult to prove this directly. Instead, we introduce first another mapping, the tent map, and prove that it has chaotic orbits, by using symbolic dynamics. Then it will be shown that the logistic map for $a = 4$ and the tent map are topologically conjugate, hence the logistic map has chaotic orbits.

The tent map is defined as $T(x) = 1 - |2x - 1|$. It maps the interval $[0, 1]$ into itself, this will be considered the phase space. Let us check the conditions in the definition of chaotic orbits. Since the interval $[0, 1]$ is positively invariant under T , the orbits are bounded (for positive k values). The definition of the Lyapunov exponent uses the derivative of the function hence it can be only for those orbits that avoid the point $1/2$, where the function T is not differentiable. For these points $|T'(p)| = 2$, hence $h(p) = \ln 2 > 0$. Thus an orbit is chaotic if it is not asymptotically periodic. In the next subsection it will be shown, by using symbolic dynamics, that the points of the interval, with a countable number of exception, are not asymptotically periodic, that implies the following statement.

Proposition 8.3. *The tent map $T(x) = 1 - |2x - 1|$ has infinitely many chaotic orbits.*

In the case of the logistic map the Lyapunov exponent is difficult to determine, instead the conjugacy with the tent map is proved. Consider the homeomorphism $H : [0, 1] \rightarrow [0, 1]$,

$$H(x) = \sin^2\left(\frac{\pi x}{2}\right).$$

We show that $H(T(x)) = f(H(x))$ holds, where f is the logistic map. Namely, on one hand

$$f(H(x)) = 4\sin^2\left(\frac{\pi x}{2}\right)\left(1 - \sin^2\left(\frac{\pi x}{2}\right)\right) = 4\sin^2\left(\frac{\pi x}{2}\right)\cos^2\left(\frac{\pi x}{2}\right) = \sin^2(\pi x),$$

on the other hand $x \in [0, 1/2)$ implies $T(x) = 2x$, hence

$$H(T(x)) = \sin^2\left(\frac{\pi 2x}{2}\right) = \sin^2(\pi x).$$

If $x \in (1/2, 1]$, then $T(x) = 2 - 2x$, and this implies similarly that $H(T(x)) = \sin^2(\pi x)$.

Thus the homeomorphism H takes the orbits of the tent map to the orbits of the logistic map, that proves the following statement.

Proposition 8.4. *The logistic map has infinitely many chaotic orbits.*

8.1.3 Symbolic dynamics

Here the symbolic dynamics is introduced for the tent map, although it can be defined for a wide class of maps in a similar way. Let $T : [0, 1] \rightarrow [0, 1]$, $T(x) = 1 - |2x - 1|$ be the tent map and let

$$\Lambda = \{x \in [0, 1] : T^n(x) \neq \frac{1}{2}, \forall n \in \mathbb{N}\}$$

be the set of those points, from which the orbit avoids the point $1/2$. The tent map will be considered in this set. The interval $[0, 1]$ will be divided into a left and a right part denoted by $I_L = [0, \frac{1}{2})$, $I_R = (\frac{1}{2}, 1]$.

Let Σ be the set of sequences of two symbols L and R , i.e. $\Sigma = \{a : \mathbb{N} \rightarrow \{L, R\}\}$, that is, if $a \in \Sigma$, then $a = (a_0 a_1 a_2 \dots)$, where $a_i = L$ or $a_i = R$ for all i . Introduce the mapping $\Phi : \Lambda \rightarrow \Sigma$

$$(\Phi(x))_n = \begin{cases} L & \text{if } f^n(x) \in I_L \\ R & \text{if } f^n(x) \in I_R \end{cases}.$$

That is the n -th term of the sequence $(\Phi(x))$ is L , if the orbit starting from x is in the left part after n steps. Similarly, the n -th term of the sequence $(\Phi(x))$ is R , if the orbit starting from x is in the right part after n steps. The mapping Φ associates L, R sequences to the points of the set $\Lambda \subset [0, 1]$. For example, the sequence associated to the point 0 is L, L, L, \dots , because 0 is a fixed point and the orbit starting from it is always in the left part. Similarly, the sequence associated to the fixed point $2/3$ is R, R, R, \dots , because the orbit starting from this point is always in the right part. The orbit starting from 1 jumps to the fixed point 0 , hence the corresponding sequence is R, L, L, L, \dots . An important property of the map Φ is that it is bijective between the set Λ and the set of L, R sequences. This will be proved first.

Proposition 8.5. *The mapping $\Phi : \Lambda \rightarrow \Sigma$ is injective, that is different points yield different sequences.*

Proof. Let $x, y \in \Lambda$, $x \neq y$. It will be shown that $\Phi(x) \neq \Phi(y)$. If x and y are in the same part of the interval, then $|f(x) - f(y)| = 2|x - y|$, because the slope of the tent map is either 2 or -2 . Assume on contrary that $\Phi(x) = \Phi(y)$. Then $f^n(x)$ and $f^n(y)$ are in the same part of the interval for all $n \in \mathbb{N}$. Hence $|f^2(x) - f^2(y)| = 2|f(x) - f(y)| = 2^2|x - y|$ and by induction $|f^n(x) - f^n(y)| = 2^n|x - y|$, yielding $|f^n(x) - f^n(y)| \rightarrow \infty$, that contradicts to the boundedness of the orbits. \square

Proposition 8.6. *The mapping $\Phi : \Lambda \rightarrow \Sigma$ is surjective, that is to any L, R sequence there is a point in Λ , to which this sequence corresponds.*

Proof. Take a point $x \in \Lambda$ and consider the first few terms of the corresponding sequence. For example, let the first three terms are L, R, L . The first L means that $x \in [0, 1/2)$, the

second symbol R shows that inside this interval x is in the right part, i.e. $x \in (1/4, 1/2)$. Since the third symbol is L , we have that $x \in (3/8, 1/2)$. Each term of the sequence shrinks the interval, where x can be, to the half of the previous one. Hence Cantor's intersection theorem implies that there is a unique point in the interval to which a given sequence belongs. \square

The point of our phase space have been associated to L, R sequences, now a dynamics is defined in the set of sequences that will be conjugate to the tent map.

Proposition 8.7. *Let $\sigma : \Sigma \rightarrow \Sigma$, be the shift map defined by $\sigma(a_0a_1a_2 \dots) = (a_1a_2a_3 \dots)$. Then $\Phi \circ T = \sigma \circ \Phi$, that is T and σ are conjugate.*

Proof. Let $\Phi(x) = (a_0a_1a_2 \dots)$, then $x \in I_{a_0}$, $T(x) \in I_{a_1}$, $T^2(x) \in I_{a_2}, \dots$. This means that $\Phi(T(x)) = (a_1a_2 \dots)$, because $T(x) \in I_{a_1}$, $T^2(x) = T(T(x)) \in I_{a_2}, \dots$. Hence $\sigma(\Phi(x)) = (a_1a_2 \dots) = \Phi(T(x))$, that we wanted to prove. \square

Corollary 8.11.. *The sequence $a \in \Sigma$ is a k -periodic point of the shift map σ , if and only if $x = \Phi^{-1}(a) \in \Lambda$ is a k -periodic point of the tent map T .*

This is a very useful corollary, because the periodic orbits of the shift map are extremely easy to determine, while it is difficult to do the same for the tent map. For example, the sequence L, R, L, R, L, R, \dots is a 2-periodic point of σ , for shifting this sequence twice we get back itself. To construct a three periodic orbit there are the following possibilities. The following triples are to be repeated:

$$LLR, LRL, RLL, \quad \text{or} \quad RRL, RLR, LRR.$$

Based on these examples one can easily derive the following statement.

Proposition 8.8. *For any $k \in \mathbb{N}$ the shift map σ has finitely many k -periodic points, hence the same is true for the tent map, because their conjugacy.*

For proving the existence of chaotic orbits the asymptotically periodic orbits of the tent map T are to be investigated. Observe, that all periodic orbits of the tent map are unstable, because $|T'| = 2$ and the theorem about the stability of periodic orbits applies. Hence a point p is asymptotically periodic if it is periodic itself or there exist $n \in \mathbb{N}$, such that $f^n(p)$ is periodic. These are called eventually periodic points. The conjugacy of T and σ implies that $x \in \Lambda$ is an eventually periodic point of T , if and only if $\Phi(x)$ is an eventually periodic point of σ . The eventually periodic points of σ are easy to generate. Namely, taking a periodic sequence an arbitrary finite sequence can be put at the beginning. For example the sequence $L, L, L, L, R, L, R, L, R, \dots$ is eventually 2-periodic, because after three shifts it yields a 2-periodic sequence L, R, L, R, L, R, \dots . This construction shows that there are countably many eventually k -periodic sequences, hence in total there are countably many eventually periodic sequences. This proves the following statement.

Proposition 8.9. *The tent map has countably many asymptotically periodic points.*

Therefore using also the results of the previous subsection we have that most of the orbits are chaotic.

Theorem 8.12.. *In the case of the tent map and the logistic map for $a = 4$ there is a countable set in the interval, such that choosing any point outside this set, the corresponding orbit is chaotic.*

Chapter 9

Reaction-diffusion equations

Reaction-diffusion equations are semilinear parabolic partial differential equations, that can be written in the form

$$\partial_t u = D\Delta u + f(u), \quad (9.1)$$

where $u : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the unknown function, $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a continuously differentiable function and D is a diagonal matrix with positive elements (differentiation with respect to time and space variables is applied coordinate-wise to the function u). The equation may subject to different boundary conditions. For example, in the case of travelling waves the equation is considered in the whole space \mathbb{R}^n , when the boundary conditions contain some conditions about the boundedness of u or its limit at infinity. In the case of a bounded space domain all the three well-known boundary conditions can be used. Besides boundary conditions the equation is also subject to an initial condition, $u(\cdot, 0)$ is given. The name of the equation originates from the application in chemistry, when a system of reactions is considered and $u_k(t, x)$ denotes the concentration of the k -th ($k = 1, 2, \dots, m$) species at time t in the point x . The first term of the right hand side describes diffusion and the second term stands for reactions. However, these type of equations often appear in physics, biology and economics modeling several phenomena as epidemic propagation, population dynamics or pattern formation.

The theory of reaction-diffusion equations originates, besides different applications, from the theory of dynamical systems. Namely, introducing the function $U(t) = u(\cdot, t)$ system (9.1) takes the form

$$\dot{U}(t) = AU(t) + F(U(t)) \quad (9.2)$$

that is an abstract Cauchy problem. It is a natural mathematical question how the results corresponding to the system of ordinary differential equations (ODE) $\dot{x}(t) = f(x(t))$ (a dynamical system with finite state space) can be generalised to the system (9.2) with infinite dimensional state space. It is known that the solutions of the ODE form a dynamical system. In the case of system (9.2) it is much harder to prove, because the

operator A in the right hand side is not bounded. The theory of operator semigroups (developed from the 1950's) enables us to prove the existence and uniqueness of solutions of the Cauchy problem (9.2) that can be found for example in the books by Henry, Pazy or Cazenave and Haraux [6, 14, 18]. Once it is proved that the Cauchy problem (9.2) determines a dynamical system (in fact a semi-dynamical system, where trajectories are defined only for nonnegative time), one can study qualitative properties of solutions. This includes the investigation of the existence, number and stability of stationary, periodic or travelling wave solutions. Besides these solutions of special form, it is important to understand the long time (asymptotic) behaviour of solutions, to prove the existence of attractors and find their basins of attraction. These questions are dealt with in detail in Robinson's book [21].

To present even the most important result of the theory of reaction-diffusion equations a whole book would not be enough, hence in this lecture notes the goal is only to deal with two simple questions, the existence of stationary solutions and the existence and stability of travelling wave solutions in the case of a single equation, i.e. when $m = 1$ in (9.1).

9.1 Stationary solutions of reaction-diffusion equations

In this section the elementary results concerning the number of stationary solutions of a single ($m = 1$) reaction-diffusion equation are dealt with. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, in the one space dimensional case it will be an open interval, and consider the semilinear elliptic problem

$$\Delta u + f(u) = 0$$

subject to Dirichlet boundary condition, that is $u|_{\partial\Omega} = 0$. The goal of our investigation is to determine the exact number of positive solutions. This is an extremely widely studied question, there are thousands of publications investigating the question for different types of domains and for different non-linearities. Here we consider only the one dimensional case when Ω is an interval. It can be assumed without loss of generality that $\Omega = (-R, R)$ with a given number R , because the equation is autonomous, hence the space variable can be transformed to this interval. Thus the boundary value problem

$$u''(x) + f(u(x)) = 0 \tag{9.3}$$

$$u(-R) = u(R) = 0 \tag{9.4}$$

will be investigated. It is easy to prove that positive solutions are even functions, that is $u(-x) = u(x)$, hence the problem can be reduced to the boundary value problem (BVP)

$$u''(x) + f(u(x)) = 0 \quad (9.5)$$

$$u'(0) = 0, \quad u(R) = 0. \quad (9.6)$$

Thus the goal is to determine the number of positive solutions of this problem.

The so-called shooting method, applying the "time-map", will be used. This method reduces the problem to the investigation of an initial value problem, that is the differential equation (9.5) is considered with the initial condition

$$u(0) = c, \quad u'(0) = 0 \quad (9.7)$$

with some $c > 0$. According to the theorem on existence and uniqueness of solutions of the initial value problem (IVP) there exist a unique solution $u(\cdot, c)$ of the IVP to any $c > 0$. Then the solution of the BVP is searched by the shooting method, that is the value of c is changed until the first root of the solution is at R . In order to that introduce the time-map

$$T(c) = \min\{r > 0 : u(r, c) = 0\} ; \quad D(T) = \{c > 0 : \exists r > 0 \, u(r, c) = 0\}. \quad (9.8)$$

Thus the number of positive solutions of the BVP (9.5)-(9.6) is equal to the number of solutions of the equation $T(c) = R$.

Notice that the differential equation has the following first integral

$$H(x) := u'(x)^2 + 2F(u(x)), \quad (9.9)$$

where $F(u) := \int_0^u f$. Differentiation shows that $H' = 0$, that is H is constant along the solutions of the differential equation. The initial condition (9.7) yields that this constant is $2F(c)$, hence for all $x \in [0, T(c)]$ the identity

$$u'(x)^2 + 2F(u(x)) = 2F(c)$$

holds. Since only positive solutions are considered and $u(R) = 0$, it is easy to see that $u' < 0$, hence the above equation yields

$$u'(x) = -\sqrt{2F(c) - 2F(u(x))}.$$

Integration leads to

$$\int_0^{T(c)} \frac{u'(x)}{\sqrt{2F(c) - 2F(u(x))}} dx = -T(c).$$

Substituting $s = u(x)$ yields

$$T(c) = \int_0^c \frac{1}{\sqrt{2F(c) - 2F(s)}} ds,$$

then introducing the new variable $t = s/c$

$$T(c) = \int_0^1 \frac{c}{\sqrt{2F(c) - 2F(ct)}} dt. \quad (9.10)$$

This formula enables us to determine the exact number of solutions of equation $T(c) = R$ for different nonlinearities. This will be illustrated by the following example.

Example 9.1. *Determine the number of positive solutions of the BVP*

$$u''(x) + u(x)(1 - u(x)) = 0 \quad (9.11)$$

$$u(-R) = u(R) = 0 \quad (9.12)$$

for different values of R . As it was shown above, it is enough to determine the number of solutions of equation $T(c) = R$. The integral giving the function T is computed for the nonlinearity $f(u) = u(1 - u)$. Then $F(u) = u^2/2 - u^3/3$, hence from (9.10)

$$T(c) = \int_0^1 \frac{1}{\sqrt{1 - t^2 - \frac{2c}{3}(1 - t^3)}} dt.$$

It can be easily seen that T is defined in the interval $[0, 1)$, and it is strictly increasing, because the variable c is contained in the denominator with negative sign. Moreover,

$$T(0) = \int_0^1 \frac{1}{\sqrt{1 - t^2}} dt = \frac{\pi}{2}.$$

An elementary estimate of the integral shows that $\lim_{c \rightarrow 1} T(c) = \infty$. Thus the range of T is the half line $[\pi/2, \infty)$, and all values in that are assumed by T exactly once, that is T is a bijection between the interval $[0, 1)$ and the half line $[\pi/2, \infty)$. This proves that equation $T(c) = R$, together with the BVP (9.11)-(9.12), have a unique positive solution if $R \geq \pi/2$, and there is no positive solution if $R < \pi/2$.

9.2 Travelling wave solutions of reaction-diffusion equations

In this section we remain in the case when the space domain is one dimensional, that is $n = 1$, however it will be unbounded. Thus consider the reaction-diffusion equation

$$\partial_t u = \partial_{xx} u + f(u), \quad (9.13)$$

where $u : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is the unknown function and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuously differentiable function again.

The solution of the form $u(t, x) = U(x - ct)$, where $U : \mathbb{R} \rightarrow \mathbb{R}$ is called a travelling wave solution propagating with velocity c . Then function U satisfies the differential equation

$$U''(y) + cU'(y) + f(U(y)) = 0, \quad (y = x - ct). \quad (9.14)$$

This second order differential equation requires boundary conditions. For given numbers $U_-, U_+ \in \mathbb{R}$ the solution satisfying the boundary conditions

$$\lim_{-\infty} U = U_-, \quad \lim_{+\infty} U = U_+$$

is called a front, if $U_- \neq U_+$, and it is called a pulse if $U_- = U_+$. If the function U is periodic, then the solution is called a wave train.

9.2.1 Existence of travelling waves

In order to prove the existence of a travelling wave, it has to be shown that there exists $c \in \mathbb{R}$, for which equation (9.14) has a solution satisfying the prescribed boundary condition. The prototype of these proofs is presented below. Consider the differential equation

$$U''(y) + cU'(y) + f(U(y)) = 0 \quad (9.15)$$

subject to boundary conditions

$$\lim_{-\infty} U = 1, \quad \lim_{+\infty} U = 0. \quad (9.16)$$

This problem with the nonlinearity $f(u) = u(1 - u)$ was first studied by Kolmogorov, Petrovskii and Piscunov, hence it is often referred to as KPP equation. Another important nonlinearity is $f(u) = u(1 - u)(u - a)$, with some $a \in (0, 1)$, that originates from the FitzHugh–Nagumo (FHN) equation. These cases are dealt with in detail in Fife's book [10]. There is a significant theoretical difference between the two cases from the

travelling wave point of view. Namely, consider the first order system corresponding to the second order differential equation (9.15).

$$U' = V \quad (9.17)$$

$$V' = -f(U) - cV \quad (9.18)$$

The equilibria of this system are $(0, 0)$ and $(1, 0)$. Based on the boundary conditions it can be proved that the limit of U' at $\pm\infty$ is 0, hence the travelling wave corresponds to a heteroclinic orbit connecting the two equilibria $(0, 0)$ and $(1, 0)$. In the case of the KPP equation if $c > 0$, then one of them is a stable node, the other one is a saddle, and the trajectory in the unstable manifold of the saddle tends to the stable node. Thus for any $c > 0$ there is a travelling wave solution. It can be shown that this solution is positive if $c > 2$, while it changes sign otherwise. It can also be shown that among all travelling waves only that belonging to $c = 2$ is stable. In the case of the FitzHugh–Nagumo type nonlinearity both equilibria are saddles, hence the heteroclinic orbit is not typical. It can be shown that there is a unique value of c when it exists, that is travelling wave exists only for this special value of c .

Given a function f simple phase plane analysis helps to prove the existence of a heteroclinic orbit, i.e. the existence of a travelling wave solution. Fife proved the following statement about travelling waves in [10] Theorem 4.15.

Theorem 9.1.. *Let $f \in C^1(0, 1)$ be a function, for which $f(0) = f(1) = 0$.*

- *If $f > 0$ in the interval $(0, 1)$, then there exists a number c^* , such that in the case $c > c^*$ there is a travelling wave solution of (9.15) satisfying the boundary condition (9.16).*
- *If there exists $a \in (0, 1)$, for which $f < 0$ in the interval $(0, a)$ and $f > 0$ holds in interval $(a, 1)$, then there is a unique value of c , for which there is a travelling wave solution of (9.15) satisfying the boundary condition (9.16).*

9.2.2 Stability of travelling waves

Equation (9.14) determining travelling waves is autonomous, hence if U is a solution, then all of its translations are also solutions, that is $U^*(y) = U(y + y_0)$ is also a solution of (9.14) for any $y_0 \in \mathbb{R}$. Thus the stability of travelling waves can only be defined as orbital stability (similarly to the stability of a periodic orbit). In this case the asymptotic stability is referred to as stability.

Definition 9.2.. *The travelling wave solution U is said to be stable, if for a solution u of (9.13) with $|u(0, x) - U(x)|$ is small enough for all $x \in \mathbb{R}$, there exists $x_0 \in \mathbb{R}$, such that*

$$\sup_{x \in \mathbb{R}} |u(t, x) - U(x_0 + x - ct)| \rightarrow 0$$

as $t \rightarrow \infty$, i.e. starting from a small perturbation of U the solution tends to a shift of U .

In this case of a single equation (9.14) a comparison theorem can be derived from the maximum principle that enables us to prove theorems on the stability of the travelling wave, see [10]. Then stronger statements about the basin of attraction of travelling waves can also be proved. In the case of the KPP equation it is shown in [10] that for a step function as initial condition $u(0, \cdot)$, that is 1 for positive values of x and 0 for negative values of x , there is a function ψ , such that

$$\sup_{x \in \mathbb{R}} |u(t, x) - U(\psi(t) + x - c^*t)| \rightarrow 0 \quad (9.19)$$

as $t \rightarrow \infty$ and $\psi'(t) \rightarrow 0$, where c^* is the minimal velocity given in Theorem 9.1.. For the sign changing, FHN type nonlinearity the following theorem is proved in [10].

Theorem 9.3.. *Let $f \in C^1(0, 1)$ be a function, for which $f(0) = f(1) = 0$ and there exists $a \in (0, 1)$, for which $f < 0$ in the interval $(0, a)$ and $f > 0$ holds in the interval $(a, 1)$. Then the unique travelling wave is stable.*

The local stability of travelling waves can be investigated by linearisation. Substituting $u(t, x) = U(x - ct) + v(t, x - ct)$ into (9.13), using the linear approximation $f(U) + f'(U)v$ for $f(U + v)$ and equation (9.14) for U leads to the following linear parabolic equation for v

$$\partial_t v = \partial_{yy} v + c \partial_y v + f'(U(y))v. \quad (9.20)$$

Then the following two questions arise.

- What is the condition for the stability of the zero solution of the linear equation (9.20)?
- Does the stability of the linear equation imply the stability of the travelling wave?

The first question can be answered by substituting $v(t, y) = \exp(\lambda t)V(y)$ into the linear equation (9.20). Then for the function V we get

$$V''(y) + cV'(y) + f'(U(y))V(y) = \lambda V(y). \quad (9.21)$$

The solution in the above form tend to zero if $\text{Re} \lambda < 0$. Thus it can be expected that the stability of the linear equation (9.20) is determined by the spectrum of the second order differential operator

$$(LV)(y) = V''(y) + cV'(y) + f'(U(y))V(y), \quad (9.22)$$

considered in the space $BUC(\mathbb{R}, \mathbb{C}) \cap C^2(\mathbb{R}, \mathbb{C})$. It is important to note that 0 is an eigenvalue of the operator with eigenfunction U' . (This follows simply by differentiating equation (9.14).) This is related to the fact that all translations of a travelling wave solution are also travelling waves, i.e. the stability of the travelling wave is only orbital stability. Thus the spectrum of L cannot be completely in the left half of the complex plane, 0 is always an eigenvalue. The above questions are answered in Chapter 25 of Smoller's book [25].

Theorem 9.4.. *If 0 is a simple eigenvalue of the operator L , for the other eigenvalues $\operatorname{Re} \lambda < 0$ and there exists $\beta < 0$, such that $\operatorname{Re} \lambda < \beta$ for all $\lambda \in \sigma_{ess}(L)$, then the travelling wave U is stable.*

Here the essential spectrum is understood in the following sense.

$$\sigma_{ess}(L) := \{\lambda \in \sigma(L) \mid \lambda \text{ is not an isolated eigenvalue with finite multiplicity}\}.$$

The spectrum of the linearised operator

In this subsection the spectrum of the operator

$$(LV)(y) = V''(y) + cV'(y) + Q(y)V(y), \quad (9.23)$$

defined in the space $BUC(\mathbb{R}, \mathbb{C}) \cap C^2(\mathbb{R}, \mathbb{C})$ is studied. This operator was derived as the linearisation around a travelling wave, hence $Q(y) = f'(U(y))$. Now it is only assumed about Q that $Q : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and the limits $Q^\pm = \lim_{y \rightarrow \pm\infty} Q(y)$ are finite.

The spectrum cannot be determined explicitly in the general case because a second order linear differential equation cannot be solved explicitly in general. However, in the case of constant coefficients, i.e. when Q is a constant function the spectrum can be determined. This will be carried out in the next example, the result of which will be useful in the general case.

Example 9.2. *Let $c, q \in \mathbb{R}$ be given numbers and consider the operator*

$$LV = V'' + cV' + qV$$

with constant coefficients. The number $\lambda \in \mathbb{C}$ is a regular value of L , if the differential equation

$$V'' + cV' + (q - \lambda)V = W \quad (9.24)$$

has a unique solution $V \in BUC(\mathbb{R}, \mathbb{C})$ that depends continuously on $W \in BUC(\mathbb{R}, \mathbb{C})$. The solution of the homogeneous equation (when $W = 0$) can be given as

$$V(y) = c_1 \exp(\mu_1 y) + c_2 \exp(\mu_2 y),$$

where $\mu_{1,2}$ are the solutions of the quadratic equation $\mu^2 + c\mu + q - \lambda = 0$. We have $V \in BUC(\mathbb{R}, \mathbb{C})$, if and only if $\operatorname{Re}\mu_{1,2} = 0$, that holds when

$$\operatorname{Re}\lambda = q - \left(\frac{\operatorname{Im}\lambda}{c}\right)^2.$$

The spectrum of L consists of those complex values λ that satisfy this equation. The inhomogeneous equation ($W \neq 0$) can be solved by using the variation of constants formula. It can be shown that if λ is not a solution of the above equation, then (9.24) has a unique solution $V \in BUC(\mathbb{R}, \mathbb{C})$ that depends continuously on $W \in BUC(\mathbb{R}, \mathbb{C})$, that is λ is a regular value. Thus the spectrum of L is the parabola

$$\sigma(L) = \sigma_{\text{ess}}(L) = P := \{\lambda_1 + i\lambda_2 \in \mathbb{C} \mid \lambda_1 = q - \left(\frac{\lambda_2}{c}\right)^2\},$$

that is shown in Figure 9.1. If $c = 0$, then the spectrum is the half line $H = \{\lambda_1 < q\}$.

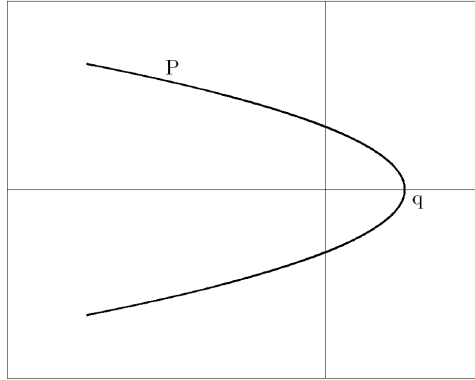


Figure 9.1: The spectrum of the operator $LV = V'' + cV' + qV$.

Invariant subspaces characterising the spectrum

Let us return to the investigation of the general operator (9.23). Introduce the first order system corresponding to the second order equation $LV = \lambda V$. Let $X_1 = V$, $X_2 = V'$, then the first order system takes the form

$$X'(y) = A_\lambda X(y), \tag{9.25}$$

where

$$A_\lambda(y) = \begin{pmatrix} 0 & I \\ \lambda I - Q(y) & -c \end{pmatrix}. \tag{9.26}$$

Since Q has limits at $\pm\infty$, the limits

$$A_\lambda^\pm = \lim_{y \rightarrow \pm\infty} A_\lambda(y)$$

exist. The stable, unstable and center subspaces of the matrices A_λ^\pm will play an important role. Their dimensions for A_λ^+ will be denoted by $n_u^+(\lambda)$, $n_s^+(\lambda)$ and $n_c^+(\lambda)$. The dimensions $n_u^-(\lambda)$, $n_s^-(\lambda)$, $n_c^-(\lambda)$ are defined similarly for the matrix A_λ^- .

In Example 9.2 we can see how these dimensions are related to the spectrum of L . In the example $LV = V'' + cV' + qV$, hence

$$A_\lambda^+ = A_\lambda^- = \begin{pmatrix} 0 & 1 \\ \lambda - q & -c \end{pmatrix}.$$

The characteristic equation of this matrix is $\mu^2 + c\mu + q - \lambda = 0$. It can be easily proved that in the case $c > 0$ the following statements hold.

- If $\operatorname{Re}\lambda < q - \left(\frac{\operatorname{Im}\lambda}{c}\right)^2$, then $\operatorname{Re}\mu_1 < 0$, $\operatorname{Re}\mu_2 < 0$, hence $n_s^\pm(\lambda) = 2$, $n_u^\pm(\lambda) = 0$, $n_c^\pm(\lambda) = 0$.
- If $\operatorname{Re}\lambda = q - \left(\frac{\operatorname{Im}\lambda}{c}\right)^2$, then $\operatorname{Re}\mu_1 = 0$, $\operatorname{Re}\mu_2 < 0$, hence $n_s^\pm(\lambda) = 1$, $n_u^\pm(\lambda) = 0$, $n_c^\pm(\lambda) = 1$.
- If $\operatorname{Re}\lambda > q - \left(\frac{\operatorname{Im}\lambda}{c}\right)^2$, then $\operatorname{Re}\mu_1 > 0$, $\operatorname{Re}\mu_2 < 0$, hence $n_s^\pm(\lambda) = 1$, $n_u^\pm(\lambda) = 1$, $n_c^\pm(\lambda) = 0$.

The dimensions of the invariant subspaces are summarised in Figure 9.2.

In Example 9.2 one can see that the spectrum is the parabola shown in Figure 9.2. This consists of those values of λ , for which $n_c^\pm(\lambda) > 0$. According to the next theorem these numbers belong to the spectrum also in the general case. However, in that case there can be λ values in the spectrum, for which $n_c^\pm(\lambda) = 0$.

Theorem 9.5.. *Assume that at least one of the conditions below hold.*

$$(a) \ n_c^+(\lambda) > 0 \text{ and } \int_0^{+\infty} |A_\lambda(y) - A_\lambda^+| < \infty$$

$$(b) \ n_c^-(\lambda) > 0 \text{ and } \int_{-\infty}^0 |A_\lambda(y) - A_\lambda^-| < \infty.$$

Then $\lambda \in \sigma(L)$.

The theorem is proved in [23].

Thus it is enough to consider the case $n_c^+(\lambda) = 0 = n_c^-(\lambda)$. Then using exponential dichotomies and perturbation theorems the following theorem can be proved, see [9].

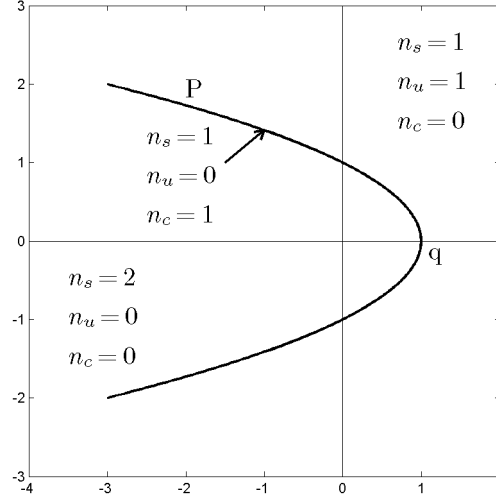


Figure 9.2: The dimensions of the invariant subspaces corresponding to the operator $LV = V'' + cV' + qV$ with constant coefficients.

Theorem 9.6.. Assume that $n_c^+(\lambda) = 0 = n_c^-(\lambda)$.

- There exists an $n_s^+(\lambda)$ dimensional subspace $E_s^+(\lambda) \subset \mathbb{C}^2$, starting from which the solutions of (9.25) tend to zero at infinity.
- There exists an $n_u^-(\lambda)$ dimensional subspace $E_u^-(\lambda) \subset \mathbb{C}^2$, starting from which the solutions of (9.25) tend to zero at $-\infty$.

If a non-zero initial condition is in $E_s^+(\lambda)$ and in $E_u^-(\lambda)$, then the solution starting from this point tends to zero both at $+\infty$ and at $-\infty$. Therefore λ is an eigenvalue if $\dim(E_s^+(\lambda) \cap E_u^-(\lambda)) > 0$. Using exponential dichotomies the following theorem is proved in [23].

Theorem 9.7.. Assume that $n_c^+(\lambda) = 0 = n_c^-(\lambda)$.

- The number λ is an eigenvalue of L , if and only if $\dim(E_s^+(\lambda) \cap E_u^-(\lambda)) > 0$.
- The number λ is a regular value of L , if and only if $E_s^+(\lambda) \oplus E_u^-(\lambda) = \mathbb{C}^2$.

Using that $E_s^+(\lambda) \oplus E_u^-(\lambda) = \mathbb{C}^2$ is equivalent to the conditions $\dim E_s^+(\lambda) + \dim E_u^-(\lambda) = 2$ and $\dim(E_s^+(\lambda) \cap E_u^-(\lambda)) = 0$, the following corollary can easily be verified.

Corollary 9.8.. Assume that $n_c^+(\lambda) = 0 = n_c^-(\lambda)$.

1. If $\dim E_s^+(\lambda) + \dim E_u^-(\lambda) > 2$, then λ is an eigenvalue of L .
2. If $\dim E_s^+(\lambda) + \dim E_u^-(\lambda) < 2$, then $\lambda \in \sigma(L)$.
3. If $\dim E_s^+(\lambda) + \dim E_u^-(\lambda) = 2$ and $\dim(E_s^+(\lambda) \cap E_u^-(\lambda)) = 0$, then λ is a regular value of L .
4. If $\dim E_s^+(\lambda) + \dim E_u^-(\lambda) = 2$ and $\dim(E_s^+(\lambda) \cap E_u^-(\lambda)) > 0$, then λ is an eigenvalue of L .

Stability of the travelling wave

Consider again the travelling wave equation

$$U'' + cU' + f(U) = 0 \quad (9.27)$$

$$U(-\infty) = U_-, \quad U(+\infty) = U_+, \quad (9.28)$$

for the function $U : \mathbb{R} \rightarrow \mathbb{R}$, where $f \in C^1(\mathbb{R}, \mathbb{R})$, $U_-, U_+ \in \mathbb{R}$ and assume that $c \geq 0$ (this can be achieved by changing U_- and U_+). The linear operator determining the stability of U is

$$L(V) = V'' + cV' + f'(U)V. \quad (9.29)$$

The function $q(y) = f'(U(y))$ is continuous and has limits at $\pm\infty$:

$$q^+ = f'(U_+), \quad q^- = f'(U_-). \quad (9.30)$$

If U converges exponentially at $\pm\infty$ to the values U_+ and U_- , then the integrals in Theorem 9.5. are convergent. The eigenvalues $\mu_{1,2}$ of the matrices A_λ^\pm are determined by the characteristic equation

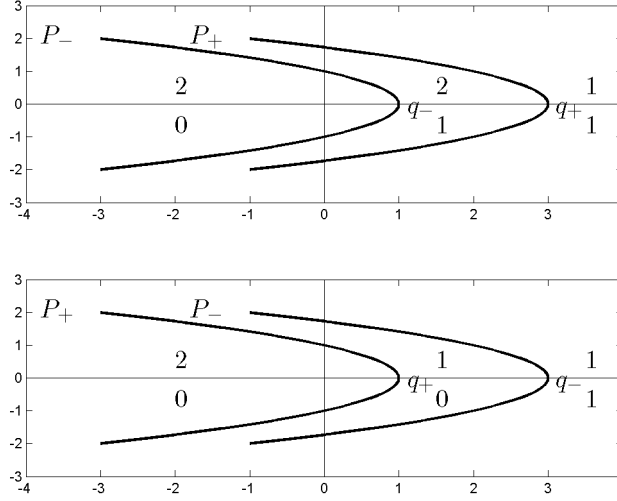
$$\mu^2 + c\mu + q^\pm - \lambda = 0. \quad (9.31)$$

The essential spectrum is determined by those values of λ , for which $n_c^+(\lambda) \geq 1$ and $n_c^-(\lambda) \geq 1$. If $n_c^+(\lambda) \geq 1$, then $\mu = i\omega$ is a solution of (9.31). Substituting this into (9.31) we get that those λ values, for which $n_c^+(\lambda) \geq 1$ lie on the parabola

$$\mathcal{P}^+ = \{\lambda_1 + i\lambda_2 \in \mathbb{C} \mid \lambda_1 = q^+ - \left(\frac{\lambda_2}{c}\right)^2\}. \quad (9.32)$$

It can be easily seen that in the left hand side of the parabola $\dim E_s^+(\lambda) = 2$ holds, while on the right hand side $\dim E_s^+(\lambda) = 1$, as it is shown in Figure 9.2.2. Similarly, those values of λ , for which $n_c^-(\lambda) \geq 1$ lie on the parabola

$$\mathcal{P}^- = \{\lambda_1 + i\lambda_2 \in \mathbb{C} \mid \lambda_1 = q^- - \left(\frac{\lambda_2}{c}\right)^2\}. \quad (9.33)$$



It can be easily seen that in the left hand side of the parabola $\dim E_u^-(\lambda) = 0$ holds, while on the right hand side $\dim E_u^-(\lambda) = 1$, as it is shown also in Figure 9.2.2.

The spectrum of the operator given in (9.29). The parabolas \mathcal{P}^+ and \mathcal{P}^- are determined by (9.32) and (9.33). The numbers in the upper part denote the dimensions of the subspaces $E_s^+(\lambda)$, the numbers in the lower part denote the dimensions of the subspaces $E_u^-(\lambda)$.

Using Theorem 9.5. and Corollary 9.8. we get the following theorem about the spectrum of the operator L .

Theorem 9.9.. • *The essential spectrum of L is formed by the parabolas \mathcal{P}^+ , \mathcal{P}^- and the domain between them. If $q^+ > q^-$, then every point between the parabolas is an eigenvalue, while in the case $q^+ < q^-$ these points are not eigenvalues, see Figure 9.2.2b.*

- *The points in the left hand side of both parabolas are regular values of L .*
- *The points in the right hand side of both parabolas are regular values or isolated eigenvalues of L .*

In this simple case of a single equation it can be determined by analytic (not numerical) calculation if there are isolated eigenvalues of L in the right half of the complex plane.

Let λ be an isolated eigenvalue with eigenfunction V . Then it can be proved that for the function $w(x) = V(x) \exp(cx/2)$ we have $w \in L^2(\mathbb{R})$ and

$$\lim_{\pm\infty} w = \lim_{\pm\infty} w' = 0. \quad (9.34)$$

Differentiation shows that

$$w'' + (q(x) - \frac{c^2}{4})w = \lambda w. \quad (9.35)$$

Multiplying by the conjugate of w and integrating from $-\infty$ to $+\infty$ leads to

$$\int_{-\infty}^{+\infty} (q(x) - \frac{c^2}{4})|w(x)|^2 - |w'(x)|^2 dx = \lambda \int_{-\infty}^{+\infty} |w(x)|^2 dx. \quad (9.36)$$

These imply the following propositions.

Proposition 9.1. *The isolated eigenvalues of L are real and less than $\max |q| - c^2/4$ and the corresponding eigenspace is one dimensional.*

Proof. In equation (9.36) all integrals are real, hence λ is also real. One can assume that $\int_{-\infty}^{+\infty} |w(x)|^2 = 1$, hence (9.36) implies $\lambda < \max |q| - c^2/4$. Let V_1 and V_2 be eigenfunctions belonging to λ . It will be shown that they differ from each other only in a constant factor. Let $w_i(x) = V_i(x) \exp(cx/2)$, $i = 1, 2$. Using equation (9.35)

$$(w'_1 w_2 - w_1 w'_2)' = w''_1 w_2 - w_1 w''_2 = 0.$$

Hence $w'_1 w_2 - w_1 w'_2$ is a constant function. The boundary conditions (9.34) imply $w'_1 w_2 - w_1 w'_2 = 0$, hence in an interval, where w_2 does not vanish we get $(w_1/w_2)' = 0$, that is w_1/w_2 is a constant function. The uniqueness of the solution of (9.35) yields that w_1 and w_2 differ from each other only in a constant factor. Hence this is true also for V_1 and V_2 . \square

Proposition 9.2. *Let λ_1 and λ_2 be isolated eigenvalues of L and V_1, V_2 be the corresponding eigenfunctions.*

- *Let y_1 and y_2 be consecutive roots of V_2 . If V_1 does not vanish in the closed interval $[y_1, y_2]$, then $\lambda_1 > \lambda_2$.*
- *If neither V_1 nor V_2 changes sign in \mathbb{R} , then $\lambda_1 = \lambda_2$ and $V_1 = cV_2$ with some constant.*

Proof. Let $w_i(x) = V_i(x) \exp(cx/2)$, $i = 1, 2$, they satisfy equation (9.35). Multiply the equation of w_1 by w_2 and the equation of w_2 by w_1 and subtract them. Then we get

$$w''_1 w_2 - w_1 w''_2 = (\lambda_1 - \lambda_2) w_1 w_2.$$

Integrate this from y_1 to y_2 . After integrating by parts and using $w_2(y_1) = w_2(y_2) = 0$ yields

$$w_2'(y_1)w_1(y_1) - w_2'(y_2)w_1(y_2) = (\lambda_1 - \lambda_2) \int_{y_1}^{y_2} w_1 w_2. \quad (9.37)$$

One can assume that w_1 and w_2 are positive in the interval (y_1, y_2) . This implies $w_2'(y_1) > 0$ and $w_2'(y_2) < 0$. (The strict inequality follows from the uniqueness of the solution, since $w_2'(y_1) = 0$ together with $w_2(y_1) = 0$ would imply that w_2 is the zero function). Thus the left hand side of (9.37) is positive. Since the integral in the right hand side is positive, $\lambda_1 - \lambda_2 > 0$, that we wanted to prove.

To prove the second part equation (9.37) is used again. Take now $y_1 = -\infty$ and $y_2 = +\infty$. Then the left hand side of (9.37) is zero, hence $\lambda_1 - \lambda_2 = 0$, because the integral is not zero. Proposition 9.1 implies that the eigenspace corresponding to λ_1 is one dimensional, hence there is constant c , for which $V_1 = cV_2$. \square

Proposition 9.3. *The eigenfunction belonging to the greatest eigenvalue of L has no root, hence it can be assumed to be positive.*

Proof. Introduce the functional

$$I(w) = \int_{-\infty}^{+\infty} \left(q(x) - \frac{c^2}{4} \right) w(x)^2 - w'(x)^2 dx, \quad w \in L^2(\mathbb{R}), \quad \int_{-\infty}^{+\infty} |w(x)|^2 = 1.$$

It can be shown that the greatest eigenvalue of L is the maximum of I . Assume that a function w has a root. Then let \tilde{w} be a function that coincides with w outside a neighbourhood of the root and it is zero in a neighbourhood. It can be shown that $I(\tilde{w}) > I(w)$, if the neighbourhood is small enough. Hence, if w is the eigenfunction belonging to the greatest eigenvalue of L , then this yields the maximum of I , hence it has no root. \square

Recall that 0 is an eigenvalue of L with eigenfunction U' . If U' changes sign, then Proposition 9.3 implies that 0 is not the greatest eigenvalue, that is the operator L has at least one positive eigenvalue. If U' does not change sign, then Proposition 9.2 implies that 0 is a simple eigenvalue, and the eigenfunctions that are linearly independent change sign, hence they are negative. Hence the following statement is proved about the eigenvalues of L .

Theorem 9.10.. • *If U' changes sign, then the operator L has at least one positive eigenvalue.*

- *If U' does not change sign, then 0 is a simple eigenvalue of L and the other isolated eigenvalues are negative.*

Thus linearisation yields the following stability results for travelling waves, by using Theorems 9.4., 9.9. and 9.10..

Theorem 9.11.. *If the function U is strictly monotone and the inequalities $f'(U_-) < 0$ and $f'(U_+) < 0$ hold, then the travelling wave is stable. If U is not monotone, then it is unstable.*

Proof. If the function U is strictly monotone, then U' does not change sign, hence according to Theorem 9.10. 0 is a simple eigenvalue of L and the other isolated eigenvalues are negative. The inequalities $f'(U_-) < 0$ and $f'(U_+) < 0$ imply that the parabolas determining the essential spectrum in Theorem 9.9. lie in the left half plane. Thus for the spectrum of L the conditions of Theorem 9.4. hold, therefore the travelling wave is stable.

If U is not monotone, then U' changes sign, hence according to Theorem 9.10. L has at least one positive eigenvalue, therefore U is unstable. \square

Finally, let us investigate the consequences of this theorem in the case of the KPP and FHN equations.

In the FHN case, such as $f(u) = u(1 - u)(u - a)$, it was shown that the travelling wave corresponds to a heteroclinic orbit connecting two saddle points and exist for only a special value of c . Simple phase plane analysis shows that U' does not change sign. i.e. U is strictly monotone. In our case $U_- = 1$ and $U_+ = 0$ hold for the limits, hence the conditions $f'(U_-) < 0$ and $f'(U_+) < 0$ hold. Thus Theorem 9.11. yields that the travelling wave in the FHN case is stable.

In the KPP case, when $f(u) = u(1 - u)$ it was shown that the heteroclinic orbit exist for all $c > 0$ and connects a saddle point to a stable node or focus. If $0 < c < 2$, then it is a focus, hence U is not monotone, thus the travelling wave is unstable. If $c \geq 2$, then the heteroclinic orbit connects a saddle to a stable node and U' does not change sign. i.e. U is strictly monotone. However, $U_- = 1$ and $U_+ = 0$, hence $f'(U_-) < 0$ and $f'(U_+) > 0$, thus Theorem 9.11. cannot be applied to prove stability. In Section 4.5 of Fife's book [10] it is shown that the travelling wave is not stable in the sense of Definition 9.2., however, the travelling wave belonging to $c = 2$ is stable in the sense defined in equation (9.19).

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