In contrast with quantum theory, classical field theory can be formulated in a strict mathematical way by treating classical fields and dynamic variables as sections of smooth fiber bundles. This book aims to compile the relevant material on fiber bundles, jet manifolds, connections, graded manifolds and Lagrangian theory. The book is based on the graduate and post graduate courses of lectures given at the Department of Theoretical Physics of Moscow State University (Russia). It addresses to a wide audience of mathematicians, mathematical physicists and theoreticians.

Gennadi Sardanashvily

Gennadi Sardanashvily, theoretician and mathematical physicist, graduated from Moscow State University in 1973, Ph.D. (1980) and D.Sc. (1998), Principal research scientist at the Department of Theoretical Physics of Moscow State University, published more than 300 scientific works and 23 text books and monographs.

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Sardanashvily

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Advanced Differential Geometry for Theoreticians

Fiber Bundles, Jet Manifolds and Lagrangian Theory

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Contents

Introduction

In contrast with quantum field theory, classical field theory can be formulated in a strict mathematical way by treating classical fields as sections of smooth fibre bundles [9, 17, 21, 24]. This also is the case of time-dependent non-relativistic mechanics on fibre bundles over $\mathbb{R}$ [10, 16, 19].

This book aim to compile the relevant material on fibre bundles, jet manifolds, connections, graded manifolds and Lagrangian theory [9, 17, 22].

The book is based on the graduate and post graduate courses of lectures given at the Department of Theoretical Physics of Moscow State University (Russia). It addresses to a wide audience of mathematicians, mathematical physicists and theoreticians. It is tacitly assumed that the reader has some familiarity with the basics of differential geometry [11, 13, 26].

Throughout the book, all morphisms are smooth (i.e. of class C^∞) and manifolds are smooth real and finite-dimensional. A smooth real *manifold* is customarily assumed to be Hausdorff and second-countable (i.e., it has a countable base for topology). Consequently, it is a locally compact space which is a union of a countable number of compact subsets, a separable space (i.e., it has a countable dense subset), a paracompact and completely regular space. Being paracompact, a smooth manifold admits a partition of unity by smooth real functions. Unless otherwise stated, manifolds are assumed to be connected (and, consequently, arcwise connected). We follow the notion of a manifold without boundary.

Chapter 1

Geometry of fibre bundles

Throughout the book, fibre bundles are smooth finite-dimensional and locally-trivial.

1.1 Fibre bundles

Let Z be a manifold. By

$$\pi_Z : TZ \to Z, \qquad \pi_Z^* : T^*Z \to Z$$

are denoted its tangent and cotangent bundles, respectively. Given coordinates (z^α) on Z, they are equipped with the *holonomic coordinates*

$$(z^\lambda, \dot z^\lambda), \qquad \dot z'^\lambda = \frac{\partial z'^\lambda}{\partial z^\mu}\dot z^\mu,$$
$$(z^\lambda, \dot z_\lambda), \qquad \dot z'_\lambda = \frac{\partial z'^\mu}{\partial z^\lambda}\dot z_\mu,$$

with respect to the *holonomic frames* $\{\partial_\lambda\}$ and *coframes* $\{dz^\lambda\}$ in the tangent and cotangent spaces to Z, respectively. Any manifold morphism $f : Z \to Z'$ yields the *tangent morphism*

$$Tf : TZ \to TZ', \qquad \dot z'^\lambda \circ Tf = \frac{\partial f^\lambda}{\partial x^\mu}\dot z^\mu.$$

Let us consider manifold morphisms of maximal rank. They are immersions (in particular, imbeddings) and submersions. An injective immersion is a submanifold, and a surjective submersion is a fibred manifold (in particular, a fibre bundle).

Given manifolds M and N, by the *rank of a morphism* $f : M \to N$ at a point $p \in M$ is meant the rank of the linear morphism

$$T_p f : T_p M \to T_{f(p)} N.$$

For instance, if f is of maximal rank at $p \in M$, then $T_p f$ is injective when $\dim M \leq \dim N$ and surjective when $\dim N \leq \dim M$. In this case, f is called an *immersion* and a *submersion* at a point $p \in M$, respectively.

Since $p \to \mathrm{rank}_p f$ is a lower semicontinuous function, then the morphism $T_p f$ is of maximal rank on an open neighbourhood of p, too. It follows from the inverse function theorem that:

- if f is an immersion at p, then it is locally injective around p.
- if f is a submersion at p, it is locally surjective around p.

If f is both an immersion and a submersion, it is called a *local diffeomorphism* at p. In this case, there exists an open neighbourhood U of p such that $f : U \to f(U)$ is a diffeomorphism onto an open set $f(U) \subset N$.

A manifold morphism f is called the immersion (resp. submersion) if it is an immersion (resp. submersion) at all points of M. A submersion is necessarily an *open map*, i.e., it sends open subsets of M onto open subsets of N. If an immersion f is open (i.e., f is a homeomorphism onto $f(M)$ equipped with the relative topology from N), it is called the *imbedding*.

A pair (M, f) is called a *submanifold* of N if f is an injective immersion. A submanifold (M, f) is an *imbedded submanifold* if f is an imbedding. For the sake of simplicity, we usually identify (M, f) with $f(M)$. If $M \subset N$, its natural injection is denoted by $i_M : M \to N$.

There are the following criteria for a submanifold to be imbedded.

Theorem 1.1.1: Let (M, f) be a submanifold of N.

(i) The map f is an imbedding iff, for each point $p \in M$, there exists a (cubic) coordinate chart (V, ψ) of N centered at $f(p)$ so that $f(M) \cap V$ consists of all points of V with coordinates $(x^1, \ldots, x^m, 0, \ldots, 0)$.

(ii) Suppose that $f : M \to N$ is a *proper map*, i.e., the pre-images of compact sets are compact. Then (M, f) is a closed imbedded submanifold of N. In particular, this occurs if M is a compact manifold.

(iii) If $\dim M = \dim N$, then (M, f) is an open imbedded submanifold of N. $\Box$

A triple

$$\pi : Y \to X, \qquad \dim X = n > 0, \tag{1.1.1}$$

is called a *fibred manifold* if a manifold morphism π is a surjective submersion, i.e., the tangent morphism $T\pi : TY \to TX$ is a surjection. One says that Y is a *total space* of a fibred manifold (1.1.1), X is its *base*, π is a *fibration*, and $Y_x = \pi^{-1}(x)$ is a *fibre* over $x \in X$.

Any fibre is an imbedded submanifold of Y of dimension $\dim Y - \dim X$.

Theorem 1.1.2: A surjection (1.1.1) is a fired manifold iff a manifold Y admits an atlas of coordinate charts $(U_Y; x^\lambda, y^i)$ such that (x^λ) are coordinates on $\pi(U_Y) \subset X$ and coordinate transition functions read

$$x'^\lambda = f^\lambda(x^\mu), \qquad y'^i = f^i(x^\mu, y^j).$$

These coordinates are called *fibred coordinates* compatible with a fibration π. $\Box$

By a *local section* of a surjection (1.1.1) is meant an injection $s : U \to Y$ of an open subset $U \subset X$ such that $\pi \circ s = \mathrm{Id}\, U$, i.e., a section sends any point $x \in X$ into the fibre Y_x over this point. A local section also is defined over any subset $N \in X$ as the restriction to N of a local section over an open set containing N. If $U = X$, one calls s the *global section*. Hereafter, by a *section* is meant both a global section and a local section (over an open subset).

Theorem 1.1.3: A surjection π (1.1.1) is a fibred manifold iff, for each point $y \in Y$, there exists a local section s of $\pi : Y \to X$ passing through y. $\Box$

The range $s(U)$ of a local section $s : U \to Y$ of a fibred manifold $Y \to X$ is an imbedded submanifold of Y. It also is a *closed map*, which sends closed subsets of U onto closed subsets of Y. If s is a global section, then $s(X)$ is a closed imbedded submanifold of Y. Global sections of a fibred manifold need not exist.

Theorem 1.1.4: Let $Y \to X$ be a fibred manifold whose fibres are diffeomorphic to $\mathbb{R}^m$. Any its section over a closed imbedded submanifold (e.g., a point) of X is extended to a global section. In particular, such a fibred manifold always has a global section. □

Given fibred coordinates $(U_Y; x^\lambda, y^i)$, a section s of a fibred manifold $Y \to X$ is represented by collections of local functions $\{s^i = y^i \circ s\}$ on $\pi(U_Y)$.

A fibred manifold $Y \to X$ is called a *fibre bundle* if admits a fibred coordinate atlas $\{(\pi^{-1}(U_\xi); x^\lambda, y^i)\}$ over a cover $\{\pi^{-1}(U_\iota)\}$ of Y which is the inverse image of a cover $\mathfrak{U} = \{U_\xi\}$ is a cover of X. In this case, there exists a manifold V, called a *typical fibre*, such that Y is locally diffeomorphic to the splittings

$$\psi_\xi : \pi^{-1}(U_\xi) \to U_\xi \times V, \tag{1.1.2}$$

glued together by means of *transition functions*

$$\varrho_{\xi\zeta} = \psi_\xi \circ \psi_\zeta^{-1} : U_\xi \cap U_\zeta \times V \to U_\xi \cap U_\zeta \times V \tag{1.1.3}$$

on overlaps $U_\xi \cap U_\zeta$. Transition functions $\varrho_{\xi\zeta}$ fulfil the *cocycle condition*

$$\varrho_{\xi\zeta} \circ \varrho_{\zeta\iota} = \varrho_{\xi\iota} \tag{1.1.4}$$

on all overlaps $U_\xi \cap U_\zeta \cap U_\iota$. Restricted to a point $x \in X$, *trivialization morphisms* ψ_ξ (1.1.2) and transition functions $\varrho_{\xi\zeta}$ (1.1.3) define diffeomorphisms of fibres

$$\psi_\xi(x) : Y_x \to V, \qquad x \in U_\xi, \tag{1.1.5}$$

$$\varrho_{\xi\zeta}(x) : V \to V, \qquad x \in U_\xi \cap U_\zeta. \tag{1.1.6}$$

Trivialization charts (U_ξ, ψ_ξ) together with transition functions $\varrho_{\xi\zeta}$ (1.1.3) constitute a *bundle atlas*

$$\Psi = \{(U_\xi, \psi_\xi), \varrho_{\xi\zeta}\} \tag{1.1.7}$$

of a fibre bundle $Y \to X$. Two bundle atlases are said to be *equivalent* if their union also is a bundle atlas, i.e., there exist transition functions between trivialization charts of different atlases.

A fibre bundle $Y \to X$ is uniquely defined by a bundle atlas. Given an atlas Ψ (1.1.7), there is a unique manifold structure on Y for which $\pi : Y \to X$ is a fibre bundle with the typical fibre V and the bundle atlas Ψ. All atlases of a fibre bundle are equivalent.

Remark 1.1.1: The notion of a fibre bundle introduced above is the notion of a *smooth locally trivial fibre bundle*. In a general setting, a *continuous fibre bundle* is defined as a continuous surjective submersion of topological spaces $Y \to X$. A continuous map $\pi : Y \to X$ is called a *submersion* if, for any point $y \in Y$, there exists an open neighborhood U of the point $\pi(y)$ and a right inverse $\sigma : U \to Y$ of π such that $\sigma \circ \pi(y) = y$, i.e., there exists a local section of π. The notion of a *locally trivial continuous fibre bundle* is a repetition of that of a smooth fibre bundle, where trivialization morphisms ψ_ξ and transition functions $\varrho_{\xi\zeta}$ are continuous. □

We have the following useful criteria for a fibred manifold to be a fibre bundle.

Theorem 1.1.5: If a fibration $\pi : Y \to X$ is a proper map, then $Y \to X$ is a fibre bundle. In particular, a fibred manifold with a compact total space is a fibre bundle. □

Theorem 1.1.6: A fibred manifold whose fibres are diffeomorphic either to a compact manifold or $\mathbb{R}^r$ is a fibre bundle. □

A comprehensive relation between fibred manifolds and fibre bundles is given in Remark 3.1.2. It involves the notion of an Ehresmann connection.

Unless otherwise stated, we restrict our consideration to fibre bundles. Without a loss of generality, we further assume that a cover $\mathfrak{U}$ for a bundle atlas of $Y \to X$ also is a cover for a manifold atlas of the base X. Then, given a bundle atlas Ψ (1.1.7), a fibre bundle Y is provided with the associated *bundle coordinates*

$$x^\lambda(y) = (x^\lambda \circ \pi)(y), \qquad y^i(y) = (y^i \circ \psi_\xi)(y), \qquad y \in \pi^{-1}(U_\xi),$$

where x^λ are coordinates on $U_\xi \subset X$ and y^i, called *fibre coordinates*, are coordinates on a typical fibre V.

The forthcoming Theorems 1.1.7 – 1.1.9 describe the particular covers which one can choose for a bundle atlas. Throughout the book, only *proper covers* of manifolds are considered, i.e., $U_\xi \neq U_\zeta$ if $\zeta \neq \xi$. Recall that a cover $\mathfrak{U}'$ is a refinement of a cover $\mathfrak{U}$ if, for each $U' \in \mathfrak{U}'$, there exists $U \in \mathfrak{U}$ such that $U' \subset U$. If a fibre bundle $Y \to X$ has a bundle atlas over a cover $\mathfrak{U}$ of X, it admits a bundle atlas over any refinement of $\mathfrak{U}$.

A fibred manifold $Y \to X$ is called trivial if Y is diffeomorphic to the product $X \times V$. Different trivializations of $Y \to X$ differ from each other in surjections $Y \to V$.

Theorem 1.1.7: Any fibre bundle over a contractible base is trivial. □

However, a fibred manifold over a contractible base need not be trivial, even its fibres are mutually diffeomorphic.

It follows from Theorem 1.1.7 that any cover of a base X consisting of *domains* (i.e., contractible open subsets) is a bundle cover.

Theorem 1.1.8: Every fibre bundle $Y \to X$ admits a bundle atlas over a countable cover $\mathfrak{U}$ of X where each member U_ξ of $\mathfrak{U}$ is a domain whose closure $\overline{U}_\xi$ is compact. □

If a base X is compact, there is a bundle atlas of Y over a finite cover of X which obeys the condition of Theorem 1.1.8.

Theorem 1.1.9: Every fibre bundle $Y \to X$ admits a bundle atlas over a finite cover $\mathfrak{U}$ of X, but its members need not be contractible and connected. $\Box$

Morphisms of fibre bundles, by definition, are *fibrewise morphisms*, sending a fibre to a fibre. Namely, a *bundle morphism* of a fibre bundle $\pi : Y \to X$ to a fibre bundle $\pi' : Y' \to X'$ is defined as a pair (Φ, f) of manifold morphisms which form a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\Phi} & Y' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'}, \\ X & \xrightarrow{f} & X' \end{array} \qquad \pi' \circ \Phi = f \circ \pi.$$

Bundle injections and surjections are called *bundle monomorphisms and epimorphisms*, respectively. A bundle diffeomorphism is called a *bundle isomorphism*, or a *bundle automorphism* if it is an isomorphism to itself. For the sake of brevity, a bundle morphism over $f = \mathrm{Id}\, X$ is often said to be a bundle morphism over X, and is denoted by $Y \underset{X}{\longrightarrow} Y'$. In particular, a bundle automorphism over X is called a *vertical automorphism*.

A bundle monomorphism $\Phi : Y \to Y'$ over X is called a *subbundle* of a fibre bundle $Y' \to X$ if $\Phi(Y)$ is a submanifold of Y'. There is the following useful criterion for an image and an inverse image of a bundle morphism to be subbundles.

Theorem 1.1.10: Let $\Phi : Y \to Y'$ be a bundle morphism over X. Given a global section s of the fibre bundle $Y' \to X$ such that $s(X) \subset \Phi(Y)$, by the *kernel of a bundle morphism* Φ with respect to a section s is meant the inverse image

$$\mathrm{Ker}\,_s\Phi = \Phi^{-1}(s(X))$$

of $s(X)$ by Φ. If $\Phi : Y \to Y'$ is a bundle morphism of constant rank over X, then $\Phi(Y)$ and $\mathrm{Ker}\,_s\Phi$ are subbundles of Y' and Y, respectively. □

Let us describe the following standard constructions of new fibre bundles from the old ones.

• Given a fibre bundle $\pi : Y \to X$ and a manifold morphism $f : X' \to X$, the *pull-back* of Y by f is called the manifold

$$f^*Y = \{(x', y) \in X' \times Y \ : \ \pi(y) = f(x')\} \tag{1.1.8}$$

together with the natural projection $(x', y) \to x'$. It is a fibre bundle over X' such that the fibre of f^*Y over a point $x' \in X'$ is that of Y over the point $f(x') \in X$. There is the canonical bundle morphism

$$f_Y : f^*Y \ni (x', y)|_{\pi(y)=f(x')} \underset{f}{\to} y \in Y. \tag{1.1.9}$$

Any section s of a fibre bundle $Y \to X$ yields the *pull-back section*

$$f^*s(x') = (x', s(f(x')))$$

of $f^*Y \to X'$.

• If $X' \subset X$ is a submanifold of X and $i_{X'}$ is the corresponding natural injection, then the pull-back bundle

$$i^*_{X'}Y = Y|_{X'}$$

is called the *restriction* of a fibre bundle Y to the submanifold $X' \subset X$. If X' is an imbedded submanifold, any section of the pull-back bundle

$$Y|_{X'} \to X'$$

is the restriction to X' of some section of $Y \to X$.

• Let $\pi : Y \to X$ and $\pi' : Y' \to X$ be fibre bundles over the same base X. Their *bundle product* $Y \times_X Y'$ over X is defined as the pull-back

$$Y \underset{X}{\times} Y' = \pi^*Y' \quad \text{or} \quad Y \underset{X}{\times} Y' = \pi'^*Y$$

together with its natural surjection onto X. Fibres of the bundle product $Y \times Y'$ are the Cartesian products $Y_x \times Y'_x$ of fibres of fibre bundles Y and Y'.

Let us consider the composition

$$\pi : Y \to \Sigma \to X \tag{1.1.10}$$

of fibre bundles

$$\pi_{Y\Sigma} : Y \to \Sigma, \tag{1.1.11}$$

$$\pi_{\Sigma X} : \Sigma \to X. \tag{1.1.12}$$

One can show that it is a fibre bundle, called the *composite bundle*. It is provided with bundle coordinates $(x^\lambda, \sigma^m, y^i)$, where (x^λ, σ^m) are bundle coordinates on the fibre bundle (1.1.12), i.e., transition functions of coordinates σ^m are independent of coordinates y^i.

Theorem 1.1.11: Given a composite bundle (1.1.10), let h be a global section of the fibre bundle $\Sigma \to X$. Then the restriction

$$Y^h = h^* Y \tag{1.1.13}$$

of the fibre bundle $Y \to \Sigma$ to $h(X) \subset \Sigma$ is a subbundle of the fibre bundle $Y \to X$. □

Theorem 1.1.12: Given a section h of the fibre bundle $\Sigma \to X$ and a section s_Σ of the fibre bundle $Y \to \Sigma$, their composition $s = s_\Sigma \circ h$ is a section of the composite bundle $Y \to X$ (1.1.10). Conversely, every section s of the fibre bundle $Y \to X$ is a composition of the section $h = \pi_{Y\Sigma} \circ s$ of the fibre bundle $\Sigma \to X$ and some section s_Σ of the fibre bundle $Y \to \Sigma$ over the closed imbedded submanifold $h(X) \subset \Sigma$. □

1.2 Vector and affine bundles

A *vector bundle* is a fibre bundle $Y \to X$ such that:

• its typical fibre V and all the fibres $Y_x = \pi^{-1}(x)$, $x \in X$, are real finite-dimensional vector spaces;

• there is a bundle atlas Ψ (1.1.7) of $Y \to X$ whose trivialization morphisms ψ_ξ (1.1.5) and transition functions $\varrho_{\xi\zeta}$ (1.1.6) are linear isomorphisms.

Accordingly, a vector bundle is provided with *linear bundle coordinates* (y^i) possessing linear transition functions $y'^i = A^i_j(x)y^j$. We have

$$y = y^i e_i(\pi(y)) = y^i \psi_\xi(\pi(y))^{-1}(e_i), \qquad \pi(y) \in U_\xi, \tag{1.2.1}$$

where $\{e_i\}$ is a fixed basis for the typical fibre V of Y, and $\{e_i(x)\}$ are the fibre bases (or the *frames*) for the fibres Y_x of Y associated to the bundle atlas Ψ.

By virtue of Theorem 1.1.4, any vector bundle has a global section, e.g., the canonical global *zero-valued section* $\widehat{0}(x) = 0$. Global sections of a vector bundle $Y \to X$ constitute a projective $C^\infty(X)$-module $Y(X)$ of finite rank. It is called the *structure module of a vector bundle*.

Theorem 1.2.1: Let a vector bundle $Y \to X$ admit $m = \dim V$ nowhere vanishing global sections s_i which are linearly independent, i.e., $\overset{m}{\wedge} s_i \neq 0$. Then Y is trivial. □

Remark 1.2.1: Theorem 8.6.3 state the categorial equivalence between the vector bundles over a smooth manifold X and projective $C^\infty(X)$-modules of finite rank. Therefore, the differential calculus (including linear differential operators, linear connections) on vector bundles can be algebraically formulated as the differential calculus on these modules. □

By a morphism of vector bundles is meant a *linear bundle morphism*, which is a fibrewise map whose restriction to each fibre is a linear map.

Given a linear bundle morphism $\Phi : Y' \to Y$ of vector bundles over X, its *kernel* $\mathrm{Ker}\,\Phi$ is defined as the inverse image $\Phi^{-1}(\widehat{0}(X))$ of the canonical zero-valued section $\widehat{0}(X)$ of Y. By virtue of Theorem 1.1.10, if Φ is of constant rank, its kernel and its range are vector subbundles of the vector bundles Y' and Y, respectively. For instance, monomorphisms and epimorphisms of vector bundles fulfil this condition.

There are the following particular constructions of new vector bundles from the old ones.

• Let $Y \to X$ be a vector bundle with a typical fibre V. By $Y^* \to X$ is denoted the *dual vector bundle* with the typical fibre V^* dual of V. The *interior product* of Y and Y^* is defined as a fibred morphism

$$\rfloor : Y \otimes Y^* \underset{X}{\longrightarrow} X \times \mathbb{R}.$$

• Let $Y \to X$ and $Y' \to X$ be vector bundles with typical fibres V and V', respectively. Their *Whitney sum* $Y \underset{X}{\oplus} Y'$ is a vector bundle over X with the typical fibre $V \oplus V'$.

• Let $Y \to X$ and $Y' \to X$ be vector bundles with typical fibres V and V', respectively. Their *tensor product* $Y \underset{X}{\otimes} Y'$ is a vector bundle over X with the typical fibre $V \otimes V'$. Similarly, the *exterior product* of vector bundles $Y \underset{X}{\wedge} Y'$ is defined. The exterior product

$$\wedge Y = X \times \mathbb{R} \underset{X}{\oplus} Y \underset{X}{\oplus} \overset{2}{\wedge} Y \underset{X}{\oplus} \cdots \overset{k}{\wedge} Y, \qquad k = \dim Y - \dim X, \quad (1.2.2)$$

is called the *exterior bundle*.

Remark 1.2.2: Given vector bundles Y and Y' over the same base X, every linear bundle morphism

$$\Phi : Y_x \ni \{e_i(x)\} \to \{\Phi_i^k(x) e'_k(x)\} \in Y'_x$$

over X defines a global section

$$\Phi : x \to \Phi_i^k(x) e^i(x) \otimes e'_k(x)$$

of the tensor product $Y \otimes Y'^*$, and *vice versa*. □

A sequence

$$Y' \xrightarrow{i} Y \xrightarrow{j} Y''$$

of vector bundles over the same base X is called *exact* at Y if $\mathrm{Ker}\, j = \mathrm{Im}\, i$. A sequence of vector bundles

$$0 \to Y' \xrightarrow{i} Y \xrightarrow{j} Y'' \to 0 \tag{1.2.3}$$

over X is said to be a *short exact sequence* if it is exact at all terms Y', Y, and Y''. This means that i is a bundle monomorphism, j is a bundle epimorphism, and $\mathrm{Ker}\, j = \mathrm{Im}\, i$. Then Y'' is the *factor bundle* Y/Y' whose structure module is the quotient $Y(X)/Y'(X)$ of the structure modules of Y and Y'. Given an exact sequence of vector bundles (1.2.3), there is the exact sequence of their duals

$$0 \to Y''^* \xrightarrow{j^*} Y^* \xrightarrow{i^*} Y'^* \to 0.$$

One says that an exact sequence (1.2.3) is *split* if there exists a bundle monomorphism $s : Y'' \to Y$ such that $j \circ s = \mathrm{Id}\, Y''$ or, equivalently,

$$Y = i(Y') \oplus s(Y'') = Y' \oplus Y''.$$

Theorem 1.2.2: Every exact sequence of vector bundles (1.2.3) is split. $\Box$

The tangent bundle TZ and the cotangent bundle T^*Z of a manifold Z exemplify vector bundles.

Remark 1.2.3: Given an atlas $\Psi_Z = \{(U_\iota, \phi_\iota)\}$ of a manifold Z, the tangent bundle is provided with the *holonomic bundle atlas*

$$\Psi_T = \{(U_\iota, \psi_\iota = T\phi_\iota)\}, \tag{1.2.4}$$

where $T\phi_\iota$ is the tangent morphism to ϕ_ι. The associated linear bundle coordinates are holonomic (or *induced*) coordinates $(\dot z^\lambda)$ with respect to the holonomic frames $\{\partial_\lambda\}$ in tangent spaces T_zZ. $\Box$

The tensor product of tangent and cotangent bundles

$$T = (\overset{m}{\otimes} TZ) \otimes (\overset{k}{\otimes} T^*Z), \qquad m, k \in \mathbb{N}, \tag{1.2.5}$$

is called a *tensor bundle*, provided with holonomic bundle coordinates $\dot{x}^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_k}$ possessing transition functions

$$\dot{x}'^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_k} = \frac{\partial x'^{\alpha_1}}{\partial x^{\mu_1}}\cdots\frac{\partial x'^{\alpha_m}}{\partial x^{\mu_m}}\frac{\partial x^{\nu_1}}{\partial x'^{\beta_1}}\cdots\frac{\partial x^{\nu_k}}{\partial x'^{\beta_k}}\dot{x}^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_k}.$$

Let $\pi_Y : TY \to Y$ be the tangent bundle of a fibre bundle $\pi : Y \to X$. Given bundle coordinates (x^λ, y^i) on Y, it is equipped with the holonomic coordinates $(x^\lambda, y^i, \dot{x}^\lambda, \dot{y}^i)$. The tangent bundle $TY \to Y$ has the subbundle $VY = \mathrm{Ker}\,(T\pi)$, which consists of the vectors tangent to fibres of Y. It is called the *vertical tangent bundle* of Y and is provided with the holonomic coordinates $(x^\lambda, y^i, \dot{y}^i)$ with respect to the vertical frames $\{\partial_i\}$. Every bundle morphism $\Phi : Y \to Y'$ yields the linear bundle morphism over Φ of the vertical tangent bundles

$$V\Phi : VY \to VY', \qquad \dot{y}'^i \circ V\Phi = \frac{\partial \Phi^i}{\partial y^j}\dot{y}^j. \tag{1.2.6}$$

It is called the *vertical tangent morphism*.

In many important cases, the vertical tangent bundle $VY \to Y$ of a fibre bundle $Y \to X$ is trivial, and is isomorphic to the bundle product

$$VY = Y \underset{X}{\times} \overline{Y} \tag{1.2.7}$$

where $\overline{Y} \to X$ is some vector bundle. It follows that VY can be provided with bundle coordinates $(x^\lambda, y^i, \overline{y}^i)$ such that transition functions of coordinates $\overline{y}^i$ are independent of coordinates y^i. One calls (1.2.7) the *vertical splitting*. For instance, every vector bundle $Y \to X$ admits the *canonical vertical splitting*

$$VY = Y \underset{X}{\oplus} Y. \tag{1.2.8}$$

The *vertical cotangent bundle* $V^*Y \to Y$ of a fibre bundle $Y \to X$ is defined as the dual of the vertical tangent bundle $VY \to Y$. It is not a subbundle of the cotangent bundle T^*Y, but there is the canonical surjection

$$\zeta : T^*Y \ni \dot{x}_\lambda dx^\lambda + \dot{y}_i dy^i \to \dot{y}_i \overline{d}y^i \in V^*Y, \tag{1.2.9}$$

where $\{\overline{d}y^i\}$, possessing transition functions

$$\overline{d}y'^i = \frac{\partial y'^i}{\partial y^j}\overline{d}y^j,$$

are the duals of the holonomic frames $\{\partial_i\}$ of VY.

For any fibre bundle Y, there exist the exact sequences of vector bundles

$$0 \to VY \longrightarrow TY \xrightarrow{\pi_T} Y \underset{X}{\times} TX \to 0, \tag{1.2.10}$$

$$0 \to Y \underset{X}{\times} T^*X \to T^*Y \to V^*Y \to 0. \tag{1.2.11}$$

Their splitting, by definition, is a connection on $Y \to X$.

For the sake of simplicity, we agree to denote the pull-backs

$$Y \underset{X}{\times} TX, \qquad Y \underset{X}{\times} T^*X$$

by TX and T^*X, respectively.

Let $\overline{\pi} : \overline{Y} \to X$ be a vector bundle with a typical fibre $\overline{V}$. An *affine bundle* modelled over the vector bundle $\overline{Y} \to X$ is a fibre bundle $\pi : Y \to X$ whose typical fibre V is an affine space modelled over $\overline{V}$ such that the following conditions hold.

• All the fibres Y_x of Y are affine spaces modelled over the corresponding fibres $\overline{Y}_x$ of the vector bundle $\overline{Y}$.

• There is an affine bundle atlas

$$\Psi = \{(U_\alpha, \psi_\chi), \varrho_{\chi\zeta}\}$$

of $Y \to X$ whose local trivializations morphisms ψ_χ (1.1.5) and transition functions $\varrho_{\chi\zeta}$ (1.1.6) are affine isomorphisms.

Dealing with affine bundles, we use only *affine bundle coordinates* (y^i) associated to an affine bundle atlas Ψ. There are the bundle morphisms

$$Y \underset{X}{\times} \overline{Y} \underset{X}{\longrightarrow} Y, \qquad (y^i, \overline{y}^i) \to y^i + \overline{y}^i,$$

$$Y \underset{X}{\times} Y \underset{X}{\longrightarrow} \overline{Y}, \qquad (y^i, y'^i) \to y^i - y'^i,$$

where $(\overline{y}^i)$ are linear coordinates on the vector bundle $\overline{Y}$.

By virtue of Theorem 1.1.4, affine bundles have global sections, but in contrast with vector bundles, there is no canonical global section of an affine bundle. Let $\pi : Y \to X$ be an affine bundle. Every global section s of an affine bundle $Y \to X$ modelled over a vector bundle $\overline{Y} \to X$ yields the bundle morphisms

$$Y \ni y \to y - s(\pi(y)) \in \overline{Y}, \tag{1.2.12}$$

$$\overline{Y} \ni \overline{y} \to s(\pi(y)) + \overline{y} \in Y. \tag{1.2.13}$$

In particular, every vector bundle Y has a natural structure of an affine bundle due to the morphisms (1.2.13) where $s = \widehat{0}$ is the canonical zero-valued section of Y. For instance, the tangent bundle TX of a manifold X is naturally an affine bundle ATX called the *affine tangent bundle*.

Theorem 1.2.3: Any affine bundle $Y \to X$ admits bundle coordinates $(x^\lambda, \widetilde{y}^i)$ with linear transition functions $\widetilde{y}'^i = A^i_j(x)\widetilde{y}^j$ (see Example 4.8.2). $\square$

By a morphism of affine bundles is meant a bundle morphism $\Phi : Y \to Y'$ whose restriction to each fibre of Y is an affine map. It is called an *affine bundle morphism*. Every affine bundle morphism $\Phi : Y \to Y'$ of an affine bundle Y modelled over a vector bundle $\overline{Y}$ to an affine bundle Y' modelled over a vector bundle $\overline{Y}'$ yields an unique linear bundle morphism

$$\overline{\Phi} : \overline{Y} \to \overline{Y}', \qquad \overline{y}'^i \circ \overline{\Phi} = \frac{\partial \Phi^i}{\partial y^j}\overline{y}^j, \tag{1.2.14}$$

called the *linear derivative* of Φ.

Similarly to vector bundles, if $\Phi : Y \to Y'$ is an affine morphism of affine bundles of constant rank, then $\Phi(Y)$ and $\operatorname{Ker}\Phi$ are affine subbundles of Y' and Y, respectively.

Every affine bundle $Y \to X$ modelled over a vector bundle $\overline{Y} \to X$ admits the *canonical vertical splitting*

$$VY = Y \underset{X}{\times} \overline{Y}. \tag{1.2.15}$$

Note that Theorems 1.1.8 and 1.1.9 on a particular cover for bundle atlases remain true in the case of linear and affine atlases of vector and affine bundles.

1.3 Vector fields

Vector fields on a manifold Z are global sections of the tangent bundle $TZ \to Z$.

The set $\mathcal{T}(Z)$ of vector fields on Z is both a $C^\infty(Z)$-module and a real Lie algebra with respect to the *Lie bracket*

$$\begin{aligned}
&u = u^\lambda \partial_\lambda, \qquad v = v^\lambda \partial_\lambda, \\
&[v, u] = (v^\lambda \partial_\lambda u^\mu - u^\lambda \partial_\lambda v^\mu)\partial_\mu.
\end{aligned}$$

Given a vector field u on X, a *curve*

$$c : \mathbb{R} \supset (,) \to Z$$

in Z is said to be an *integral curve* of u if $Tc = u(c)$. Every vector field u on a manifold Z can be seen as an *infinitesimal generator* of a local one-parameter group of diffeomorphisms (a *flow*), and *vice versa*. One-dimensional orbits of this group are integral curves of u. A vector field is called *complete* if its flow is a one-parameter group of diffeomorphisms of Z. For instance, every vector field on a compact manifold is complete.

A vector field u on a fibre bundle $Y \to X$ is called *projectable* if it projects onto a vector field on X, i.e., there exists a vector field τ on X such that

$$\tau \circ \pi = T\pi \circ u.$$

A projectable vector field takes the coordinate form

$$u = u^\lambda(x^\mu)\partial_\lambda + u^i(x^\mu, y^j)\partial_i, \qquad \tau = u^\lambda \partial_\lambda. \tag{1.3.1}$$

Its flow is a local one-parameter group of automorphisms of $Y \to X$ over a local one-parameter group of diffeomorphisms of X whose generator is τ. A projectable vector field is called *vertical* if its projection onto X vanishes, i.e., if it lives in the vertical tangent bundle VY.

A vector field $\tau = \tau^\lambda\partial_\lambda$ on a base X of a fibre bundle $Y \to X$ gives rise to a vector field on Y by means of a connection on this fibre bundle (see the formula (3.1.6)). Nevertheless, every tensor bundle (1.2.5) admits the *canonical lift* of vector fields

$$\widetilde{\tau} = \tau^\mu\partial_\mu + [\partial_\nu\tau^{\alpha_1}\dot{x}^{\nu\alpha_2\cdots\alpha_m}_{\beta_1\cdots\beta_k} + \ldots - \partial_{\beta_1}\tau^\nu\dot{x}^{\alpha_1\cdots\alpha_m}_{\nu\beta_2\cdots\beta_k} - \ldots]\dot{\partial}^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_m}, \quad (1.3.2)$$

where we employ the compact notation

$$\dot{\partial}_\lambda = \frac{\partial}{\partial\dot{x}^\lambda}. \quad (1.3.3)$$

This lift is functorial, i.e., it is an $\mathbb{R}$-linear monomorphism of the Lie algebra $\mathcal{T}(X)$ of vector fields on X to the Lie algebra $\mathcal{T}(Y)$ of vector fields on Y (see Section 5.1). In particular, we have the functorial lift

$$\widetilde{\tau} = \tau^\mu\partial_\mu + \partial_\nu\tau^\alpha\dot{x}^\nu\frac{\partial}{\partial\dot{x}^\alpha} \quad (1.3.4)$$

of vector fields on X onto the tangent bundle TX and their functorial lift

$$\widetilde{\tau} = \tau^\mu\partial_\mu - \partial_\beta\tau^\nu\dot{x}_\nu\frac{\partial}{\partial\dot{x}_\beta} \quad (1.3.5)$$

onto the cotangent bundle T^*X.

A fibre bundle admitting functorial lift of vector fields on its base is called the natural bundle (see Chapter 5).

A subbundle $\mathbf{T}$ of the tangent bundle TZ of a manifold Z is called a regular *distribution* (or, simply, a distribution). A vector field u on Z is said to be *subordinate* to a distribution $\mathbf{T}$ if it lives in $\mathbf{T}$. A distribution $\mathbf{T}$ is called *involutive* if the Lie bracket of $\mathbf{T}$-subordinate vector fields also is subordinate to $\mathbf{T}$.

A subbundle of the cotangent bundle T^*Z of Z is called a *codistribution* $\mathbf{T}^*$ on a manifold Z. For instance, the *annihilator* $\mathrm{Ann}\,\mathbf{T}$ of a distribution $\mathbf{T}$ is a codistribution whose fibre over $z \in Z$ consists of covectors $w \in T^*_z$ such that $v\rfloor w = 0$ for all $v \in \mathbf{T}_z$.

Theorem 1.3.1: Let $\mathbf{T}$ be a distribution and $\mathrm{Ann}\,\mathbf{T}$ its annihilator. Let $\wedge\mathrm{Ann}\,\mathbf{T}(Z)$ be the ideal of the exterior algebra $\mathcal{O}^*(Z)$ which is generated by sections of $\mathrm{Ann}\,\mathbf{T} \to Z$. A distribution $\mathbf{T}$ is involutive iff the ideal $\wedge\mathrm{Ann}\,\mathbf{T}(Z)$ is a *differential ideal*, i.e.,

$$d(\wedge\mathrm{Ann}\,\mathbf{T}(Z)) \subset \wedge\mathrm{Ann}\,\mathbf{T}(Z).$$

□

The following local coordinates can be associated to an involutive distribution.

Theorem 1.3.2: Let $\mathbf{T}$ be an involutive r-dimensional distribution on a manifold Z, $\dim Z = k$. Every point $z \in Z$ has an open neighborhood U which is a domain of an adapted coordinate chart $(z^1, \ldots, z^k)$ such that, restricted to U, the distribution $\mathbf{T}$ and its annihilator $\mathrm{Ann}\,\mathbf{T}$ are spanned by the local vector fields $\partial/\partial z^1, \cdots, \partial/\partial z^r$ and the one-forms $dz^{r+1}, \ldots, dz^k$, respectively. □

A connected submanifold N of a manifold Z is called an *integral manifold* of a distribution $\mathbf{T}$ on Z if $TN \subset \mathbf{T}$. Unless otherwise stated, by an integral manifold is meant an integral manifold of dimension of $\mathbf{T}$. An integral manifold is called *maximal* if no other integral manifold contains it. The following is the classical theorem of Frobenius.

Theorem 1.3.3: Let $\mathbf{T}$ be an involutive distribution on a manifold Z. For any $z \in Z$, there exists a unique maximal integral manifold of $\mathbf{T}$ through z, and any integral manifold through z is its open subset. □

Maximal integral manifolds of an involutive distribution on a manifold Z are assembled into a regular foliation $\mathcal{F}$ of Z. A regular r-dimensional *foliation* (or, simply, a foliation) $\mathcal{F}$ of a k-dimensional manifold Z is defined as a partition of Z into connected r-dimensional submanifolds (the *leaves* of a foliation) F_ι, $\iota \in I$, which possesses the following properties. A foliated manifold $(Z, \mathcal{F})$ admits an adapted coordinate atlas

$$\{(U_\xi; z^\lambda; z^i)\}, \quad \lambda = 1, \ldots, n-r, \qquad i = 1, \ldots, r, \tag{1.3.6}$$

such that transition functions of coordinates z^λ are independent of the remaining coordinates z^i and, for each leaf F of a foliation $\mathcal{F}$, the connected components of $F \cap U_\xi$ are given by the equations z^λ =const. These connected components and coordinates (z^i) on them make up a coordinate atlas of a leaf F.

It should be emphasized that leaves of a foliation need not be closed or imbedded submanifolds. Every leaf has an open *tubular neighborhood* U, i.e., if $z \in U$, then a leaf through z also belongs to U.

A pair $(Z, \mathcal{F})$ where $\mathcal{F}$ is a foliation of Z is called a *foliated manifold*. For instance, any submersion $f : Z \to M$ yields a foliation

$$\mathcal{F} = \{F_p = f^{-1}(p)\}_{p \in f(Z)}$$

of Z indexed by elements of $f(Z)$, which is an open submanifold of M, i.e., $Z \to f(Z)$ is a fibred manifold. Leaves of this foliation are closed imbedded submanifolds. Such a foliation is called *simple*. It is a fibred manifold over $f(Z)$. Any (regular) foliation is locally simple.

1.4 Exterior and tangent-valued forms

An *exterior r-form* on a manifold Z is a section

$$\phi = \frac{1}{r!} \phi_{\lambda_1 \ldots \lambda_r} dz^{\lambda_1} \wedge \cdots \wedge dz^{\lambda_r}$$

of the exterior product $\overset{r}{\wedge} T^*Z \to Z$, where

$$dz^{\lambda_1} \wedge \cdots \wedge dz^{\lambda_r} = \frac{1}{r!} \epsilon^{\lambda_1 \ldots \lambda_r}{}_{\mu_1 \ldots \mu_r} dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_r},$$
$$\epsilon^{\ldots \lambda_i \ldots \lambda_j \ldots}{}_{\ldots \mu_p \ldots \mu_k \ldots} = -\epsilon^{\ldots \lambda_j \ldots \lambda_i \ldots}{}_{\ldots \mu_p \ldots \mu_k \ldots} = -\epsilon^{\ldots \lambda_i \ldots \lambda_j \ldots}{}_{\ldots \mu_k \ldots \mu_p \ldots},$$
$$\epsilon^{\lambda_1 \ldots \lambda_r}{}_{\lambda_1 \ldots \lambda_r} = 1.$$

Let $\mathcal{O}^r(Z)$ denote the vector space of exterior r-forms on a manifold Z. By definition, $\mathcal{O}^0(Z) = C^\infty(Z)$ is the ring of smooth real functions on Z. All exterior forms on Z constitute the $\mathbb{N}$-graded commutative algebra

$\mathcal{O}^*(Z)$ of global sections of the exterior bundle $\wedge T^*Z$ (1.2.2) endowed with the *exterior product*

$$\phi = \frac{1}{r!}\phi_{\lambda_1\ldots\lambda_r}dz^{\lambda_1}\wedge\cdots\wedge dz^{\lambda_r}, \qquad \sigma = \frac{1}{s!}\sigma_{\mu_1\ldots\mu_s}dz^{\mu_1}\wedge\cdots\wedge dz^{\mu_s},$$
$$\phi\wedge\sigma = \frac{1}{r!s!}\phi_{\nu_1\ldots\nu_r}\sigma_{\nu_{r+1}\ldots\nu_{r+s}}dz^{\nu_1}\wedge\cdots\wedge dz^{\nu_{r+s}} =$$
$$\frac{1}{r!s!(r+s)!}\epsilon^{\nu_1\ldots\nu_{r+s}}{}_{\alpha_1\ldots\alpha_{r+s}}\phi_{\nu_1\ldots\nu_r}\sigma_{\nu_{r+1}\ldots\nu_{r+s}}dz^{\alpha_1}\wedge\cdots\wedge dz^{\alpha_{r+s}},$$

such that

$$\phi\wedge\sigma = (-1)^{|\phi||\sigma|}\sigma\wedge\phi,$$

where the symbol $|\phi|$ stands for the form degree. An algebra $\mathcal{O}^*(Z)$ also is provided with the *exterior differential*

$$d\phi = dz^\mu\wedge\partial_\mu\phi = \frac{1}{r!}\partial_\mu\phi_{\lambda_1\ldots\lambda_r}dz^\mu\wedge dz^{\lambda_1}\wedge\cdots\wedge dz^{\lambda_r}$$

which obeys the relations

$$d\circ d = 0, \qquad d(\phi\wedge\sigma) = d(\phi)\wedge\sigma + (-1)^{|\phi|}\phi\wedge d(\sigma).$$

The exterior differential d makes $\mathcal{O}^*(Z)$ into a differential graded algebra (henceforth DGA) which is the minimal Chevalley–Eilenberg differential calculus $\mathcal{O}^*\mathcal{A}$ over the real ring $\mathcal{A} = C^\infty(Z)$. Its de Rham complex is (8.6.5).

Given a manifold morphism $f: Z\to Z'$, any exterior k-form ϕ on Z' yields the *pull-back exterior form* $f^*\phi$ on Z given by the condition

$$f^*\phi(v^1,\ldots,v^k)(z) = \phi(Tf(v^1),\ldots,Tf(v^k))(f(z))$$

for an arbitrary collection of tangent vectors $v^1,\cdots,v^k\in T_zZ$. We have the relations

$$f^*(\phi\wedge\sigma) = f^*\phi\wedge f^*\sigma, \qquad df^*\phi = f^*(d\phi).$$

In particular, given a fibre bundle $\pi: Y\to X$, the pull-back onto Y of exterior forms on X by π provides the monomorphism of graded commutative algebras $\mathcal{O}^*(X) \to \mathcal{O}^*(Y)$. Elements of its range $\pi^*\mathcal{O}^*(X)$ are called *basic forms*. Exterior forms

$$\phi : Y \to \overset{r}{\wedge} T^*X, \qquad \phi = \frac{1}{r!}\phi_{\lambda_1\ldots\lambda_r} dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r},$$

on Y such that $u\rfloor\phi = 0$ for an arbitrary vertical vector field u on Y are said to be *horizontal forms*. Horizontal forms of degree $n = \dim X$ are called *densities*. We use for them the compact notation

$$\begin{aligned}
&L = \frac{1}{n!}L_{\mu_1\ldots\mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} = \mathcal{L}\omega, \qquad \mathcal{L} = L_{1\ldots n},\\
&\omega = dx^1 \wedge \cdots \wedge dx^n = \frac{1}{n!}\epsilon_{\mu_1\ldots\mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}, \qquad (1.4.1)\\
&\omega_\lambda = \partial_\lambda\rfloor\omega, \qquad \omega_{\mu\lambda} = \partial_\mu\rfloor\partial_\lambda\rfloor\omega,
\end{aligned}$$

where ϵ is the skew-symmetric *Levi–Civita symbol* with the component $\epsilon_{\mu_1\ldots\mu_n} = 1$.

The *interior product* (or *contraction*) of a vector field u and an exterior r-form ϕ on a manifold Z is given by the coordinate expression

$$\begin{aligned}
u\rfloor\phi = \sum_{k=1}^{r} \frac{(-1)^{k-1}}{r!} u^{\lambda_k}\phi_{\lambda_1\ldots\lambda_k\ldots\lambda_r} dz^{\lambda_1} \wedge \cdots \wedge \widehat{dz}^{\lambda_k} \wedge \cdots \wedge dz^{\lambda_r} =\\
\frac{1}{(r-1)!} u^\mu \phi_{\mu\alpha_2\ldots\alpha_r} dz^{\alpha_2} \wedge \cdots \wedge dz^{\alpha_r},
\end{aligned}$$

where the caret $\widehat{\ }$ denotes omission. It obeys the relations

$$\begin{aligned}
&\phi(u_1,\ldots,u_r) = u_r\rfloor\cdots u_1\rfloor\phi,\\
&u\rfloor(\phi\wedge\sigma) = u\rfloor\phi\wedge\sigma + (-1)^{|\phi|}\phi\wedge u\rfloor\sigma.
\end{aligned}$$

The *Lie derivative* of an exterior form ϕ along a vector field u is

$$\begin{aligned}
&\mathbf{L}_u\phi = u\rfloor d\phi + d(u\rfloor\phi),\\
&\mathbf{L}_u(\phi\wedge\sigma) = \mathbf{L}_u\phi\wedge\sigma + \phi\wedge\mathbf{L}_u\sigma.
\end{aligned}$$

It is a derivation of the graded algebra $\mathcal{O}^*(Z)$ such that

$$\mathbf{L}_u \circ \mathbf{L}_{u'} - \mathbf{L}_{u'} \circ \mathbf{L}_u = \mathbf{L}_{[u,u']}.$$

In particular, if f is a function, then

$$\mathbf{L}_u f = u(f) = u\rfloor df.$$

An exterior form ϕ is invariant under a local one-parameter group of diffeomorphisms $G(t)$ of Z (i.e., $G(t)^*\phi = \phi$) iff its Lie derivative along the infinitesimal generator u of this group vanishes, i.e., $\mathbf{L}_u\phi = 0$.

A *tangent-valued r-form* on a manifold Z is a section

$$\phi = \frac{1}{r!}\phi^\mu_{\lambda_1\ldots\lambda_r} dz^{\lambda_1}\wedge\cdots\wedge dz^{\lambda_r}\otimes\partial_\mu \tag{1.4.2}$$

of the tensor bundle

$$\overset{r}{\wedge} T^*Z\otimes TZ\to Z.$$

Remark 1.4.1: There is one-to-one correspondence between the tangent-valued one-forms ϕ on a manifold Z and the linear bundle endomorphisms

$$\widehat\phi: TZ\to TZ, \quad \widehat\phi: T_zZ\ni v\to v\rfloor\phi(z)\in T_zZ, \tag{1.4.3}$$

$$\widehat\phi^*: T^*Z\to T^*Z, \quad \widehat\phi^*: T^*_zZ\ni v^*\to \phi(z)\rfloor v^*\in T^*_zZ, \tag{1.4.4}$$

over Z (see Remark 1.2.2). For instance, the *canonical tangent-valued one-form*

$$\theta_Z = dz^\lambda\otimes\partial_\lambda \tag{1.4.5}$$

on Z corresponds to the identity morphisms (1.4.3) and (1.4.4). □

Remark 1.4.2: Let $Z = TX$, and let TTX be the tangent bundle of TX. There is the bundle endomorphism

$$J(\partial_\lambda) = \dot\partial_\lambda, \qquad J(\dot\partial_\lambda) = 0 \tag{1.4.6}$$

of TTX over X. It corresponds to the canonical tangent-valued form

$$\theta_J = dx^\lambda\otimes\dot\partial_\lambda \tag{1.4.7}$$

on the tangent bundle TX. It is readily observed that $J\circ J = 0$. □

The space $\mathcal{O}^*(Z)\otimes\mathcal{T}(Z)$ of tangent-valued forms is provided with the *Frölicher–Nijenhuis bracket*

$$\begin{aligned}
&[\,,\,]_{\rm FN}:\mathcal{O}^r(Z)\otimes\mathcal{T}(Z)\times\mathcal{O}^s(Z)\otimes\mathcal{T}(Z)\to\mathcal{O}^{r+s}(Z)\otimes\mathcal{T}(Z),\\
&[\alpha\otimes u,\,\beta\otimes v]_{\rm FN}=(\alpha\wedge\beta)\otimes[u,v]+(\alpha\wedge\mathbf{L}_u\beta)\otimes v- \qquad (1.4.8)\\
&\qquad(\mathbf{L}_v\alpha\wedge\beta)\otimes u+(-1)^r(d\alpha\wedge u\rfloor\beta)\otimes v+(-1)^r(v\rfloor\alpha\wedge d\beta)\otimes u,\\
&\alpha\in\mathcal{O}^r(Z),\qquad \beta\in\mathcal{O}^s(Z),\qquad u,v\in\mathcal{T}(Z).
\end{aligned}$$

Its coordinate expression is

$$\begin{aligned}
&[\phi,\sigma]_{\rm FN}=\frac{1}{r!s!}(\phi^\nu_{\lambda_1\ldots\lambda_r}\partial_\nu\sigma^\mu_{\lambda_{r+1}\ldots\lambda_{r+s}}-\sigma^\nu_{\lambda_{r+1}\ldots\lambda_{r+s}}\partial_\nu\phi^\mu_{\lambda_1\ldots\lambda_r}-\\
&\qquad r\phi^\mu_{\lambda_1\ldots\lambda_{r-1}\nu}\partial_{\lambda_r}\sigma^\nu_{\lambda_{r+1}\ldots\lambda_{r+s}}+s\sigma^\mu_{\nu\lambda_{r+2}\ldots\lambda_{r+s}}\partial_{\lambda_{r+1}}\phi^\nu_{\lambda_1\ldots\lambda_r})\\
&\qquad dz^{\lambda_1}\wedge\cdots\wedge dz^{\lambda_{r+s}}\otimes\partial_\mu,\\
&\phi\in\mathcal{O}^r(Z)\otimes\mathcal{T}(Z),\qquad \sigma\in\mathcal{O}^s(Z)\otimes\mathcal{T}(Z).
\end{aligned}$$

There are the relations

$$[\phi,\sigma]_{\rm FN}=(-1)^{|\phi||\psi|+1}[\sigma,\phi]_{\rm FN}, \qquad (1.4.9)$$

$$[\phi,[\sigma,\theta]_{\rm FN}]_{\rm FN}=[[\phi,\sigma]_{\rm FN},\theta]_{\rm FN}+(-1)^{|\phi||\sigma|}[\sigma,[\phi,\theta]_{\rm FN}]_{\rm FN}, \qquad (1.4.10)$$

$$\phi,\sigma,\theta\in\mathcal{O}^*(Z)\otimes\mathcal{T}(Z).$$

Given a tangent-valued form θ, the *Nijenhuis differential* on $\mathcal{O}^*(Z)\otimes\mathcal{T}(Z)$ is defined as the morphism

$$d_\theta:\psi\to d_\theta\psi=[\theta,\psi]_{\rm FN},\qquad \psi\in\mathcal{O}^*(Z)\otimes\mathcal{T}(Z).$$

By virtue of (1.4.10), it has the property

$$d_\phi[\psi,\theta]_{\rm FN}=[d_\phi\psi,\theta]_{\rm FN}+(-1)^{|\phi||\psi|}[\psi,d_\phi\theta]_{\rm FN}.$$

In particular, if $\theta=u$ is a vector field, the Nijenhuis differential is the *Lie derivative of tangent-valued forms*

$$\begin{aligned}
&\mathbf{L}_u\sigma=d_u\sigma=[u,\sigma]_{\rm FN}=\frac{1}{s!}(u^\nu\partial_\nu\sigma^\mu_{\lambda_1\ldots\lambda_s}-\sigma^\nu_{\lambda_1\ldots\lambda_s}\partial_\nu u^\mu+\\
&\qquad s\sigma^\mu_{\nu\lambda_2\ldots\lambda_s}\partial_{\lambda_1}u^\nu)dx^{\lambda_1}\wedge\cdots\wedge dx^{\lambda_s}\otimes\partial_\mu,\qquad \sigma\in\mathcal{O}^s(Z)\otimes\mathcal{T}(Z).
\end{aligned}$$

Let $Y \to X$ be a fibre bundle. We consider the following subspaces of the space $\mathcal{O}^*(Y) \otimes \mathcal{T}(Y)$ of tangent-valued forms on Y:

- *horizontal tangent-valued forms*

$$\phi : Y \to \overset{r}{\wedge} T^*X \underset{Y}{\otimes} TY,$$
$$\phi = dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r} \otimes \frac{1}{r!}[\phi^\mu_{\lambda_1 \dots \lambda_r}(y)\partial_\mu + \phi^i_{\lambda_1 \dots \lambda_r}(y)\partial_i],$$

- *projectable horizontal tangent-valued forms*

$$\phi = dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r} \otimes \frac{1}{r!}[\phi^\mu_{\lambda_1 \dots \lambda_r}(x)\partial_\mu + \phi^i_{\lambda_1 \dots \lambda_r}(y)\partial_i],$$

- *vertical-valued form*

$$\phi : Y \to \overset{r}{\wedge} T^*X \underset{Y}{\otimes} VY,$$
$$\phi = \frac{1}{r!}\phi^i_{\lambda_1 \dots \lambda_r}(y) dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r} \otimes \partial_i,$$

- vertical-valued one-forms, called *soldering forms*,

$$\sigma = \sigma^i_\lambda(y) dx^\lambda \otimes \partial_i,$$

- *basic soldering forms*

$$\sigma = \sigma^i_\lambda(x) dx^\lambda \otimes \partial_i.$$

Remark 1.4.3: The tangent bundle TX is provided with the canonical soldering form θ_J (1.4.7). Due to the canonical vertical splitting

$$VTX = TX \underset{X}{\times} TX, \tag{1.4.11}$$

the canonical soldering form (1.4.7) on TX defines the canonical tangent-valued form θ_X (1.4.5) on X. By this reason, tangent-valued one-forms on a manifold X also are called soldering forms. □

Remark 1.4.4: Let $Y \to X$ be a fibre bundle, $f : X' \to X$ a manifold morphism, $f^*Y \to X'$ the pull-back of Y by f, and

$$f_Y : f^*Y \to Y$$

the corresponding bundle morphism (1.1.9). Since

$$Vf^*Y = f^*VY = f_Y^*VY, \qquad V_{y'}Y' = V_{f_Y(y')}Y,$$

one can define the *pull-back* $f^*\phi$ onto f^*Y of any vertical-valued form ϕ on Y in accordance with the relation

$$f^*\phi(v^1,\ldots,v^r)(y') = \phi(Tf_Y(v^1),\ldots,Tf_Y(v^r))(f_Y(y')).$$

□

We also mention the TX-valued forms

$$\phi : Y \to \overset{r}{\wedge} T^*X \underset{Y}{\otimes} TX, \qquad (1.4.12)$$
$$\phi = \frac{1}{r!}\phi^\mu_{\lambda_1\ldots\lambda_r} dx^{\lambda_1}\wedge\cdots\wedge dx^{\lambda_r}\otimes\partial_\mu,$$

and V^*Y-valued forms

$$\phi : Y \to \overset{r}{\wedge} T^*X \underset{Y}{\otimes} V^*Y, \qquad (1.4.13)$$
$$\phi = \frac{1}{r!}\phi_{\lambda_1\ldots\lambda_r i} dx^{\lambda_1}\wedge\cdots\wedge dx^{\lambda_r}\otimes\overline{d}y^i.$$

It should be emphasized that (1.4.12) are not tangent-valued forms, while (1.4.13) are not exterior forms. They exemplify *vector-valued forms*. Given a vector bundle $E \to X$, by a E-valued k-form on X, is meant a section of the fibre bundle

$$(\overset{k}{\wedge} T^*X) \underset{X}{\otimes} E^* \to X.$$

Chapter 2

Jet manifolds

There are different notions of jets. Here we are concerned with jets of sections of fibre bundles. They are the particular jets of maps and the jets of submanifolds. Let us also mention the jets of modules over a commutative ring. In particular, given a smooth manifold X, the jets of a projective $C^\infty(X)$-module P of finite rank are exactly the jets of sections of the vector bundle over X whose module of sections is P in accordance with the Serre–Swan Theorem 8.6.3. The notion of jets is extended to modules over graded commutative rings. However, the jets of modules over a noncommutative ring can not be defined.

2.1 First order jet manifolds

Given a fibre bundle $Y \to X$ with bundle coordinates (x^λ, y^i), let us consider the equivalence classes $j^1_x s$ of its sections s, which are identified by their values $s^i(x)$ and the values of their partial derivatives $\partial_\mu s^i(x)$ at a point $x \in X$. They are called the *first order jets* of sections at x. One can justify that the definition of jets is coordinate-independent. The key point is that the set J^1Y of first order jets $j^1_x s$, $x \in X$, is a smooth manifold with respect to the adapted coordinates $(x^\lambda, y^i, y^i_\lambda)$ such that

$$y^i_\lambda(j^1_x s) = \partial_\lambda s^i(x), \qquad {y'}^i_\lambda = \frac{\partial x^\mu}{\partial {x'}^\lambda}(\partial_\mu + y^j_\mu \partial_j){y'}^i. \tag{2.1.1}$$

It is called the first order *jet manifold* of a fibre bundle $Y \to X$. We call (y^i_λ) the *jet coordinates*.

The jet manifold J^1Y admits the natural fibrations

$$\pi^1 : J^1Y \ni j^1_x s \to x \in X, \tag{2.1.2}$$
$$\pi^1_0 : J^1Y \ni j^1_x s \to s(x) \in Y. \tag{2.1.3}$$

A glance at the transformation law (2.1.1) shows that π^1_0 is an affine bundle modelled over the vector bundle

$$T^*X \underset{Y}{\otimes} VY \to Y. \tag{2.1.4}$$

It is convenient to call π^1 (2.1.2) the *jet bundle*, while π^1_0 (2.1.3) is said to be the *affine jet bundle*.

Let us note that, if $Y \to X$ is a vector or an affine bundle, the jet bundle π_1 (2.1.2) is so.

Jets can be expressed in terms of familiar tangent-valued forms as follows. There are the canonical imbeddings

$$\lambda_{(1)} : J^1Y \underset{Y}{\to} T^*X \underset{Y}{\otimes} TY,$$
$$\lambda_{(1)} = dx^\lambda \otimes (\partial_\lambda + y^i_\lambda \partial_i) = dx^\lambda \otimes d_\lambda, \tag{2.1.5}$$
$$\theta_{(1)} : J^1Y \underset{Y}{\to} T^*Y \underset{Y}{\otimes} VY,$$
$$\theta_{(1)} = (dy^i - y^i_\lambda dx^\lambda) \otimes \partial_i = \theta^i \otimes \partial_i, \tag{2.1.6}$$

where d_λ are called *total derivatives*, and θ^i are *local contact forms*.

Remark 2.1.1: We further identify the jet manifold J^1Y with its images under the canonical morphisms (2.1.5) and (2.1.6), and represent the jets $j^1_x s = (x^\lambda, y^i, y^i_\mu)$ by the tangent-valued forms $\lambda_{(1)}$ (2.1.5) and $\theta_{(1)}$ (2.1.6). □

Any section s of $Y \to X$ has the *jet prolongation* to the section

$$(J^1 s)(x) = j^1_x s, \qquad y^i_\lambda \circ J^1 s = \partial_\lambda s^i(x),$$

of the jet bundle $J^1Y \to X$. A *section of the jet bundle* $J^1Y \to X$ is called *integrable* if it is the jet prolongation of some section of $Y \to X$.

Remark 2.1.2: By virtue of Theorem 1.1.4, the affine jet bundle $J^1Y \to Y$ admits global sections. If Y is trivial, there is the canonical zero section $\widehat{0}(Y)$ of $J^1Y \to Y$ taking its values into centers of its affine fibres. □

Any bundle morphism $\Phi : Y \to Y'$ over a diffeomorphism f admits a *jet prolongation* to a bundle morphism of affine jet bundles

$$J^1\Phi : J^1Y \underset{\Phi}{\longrightarrow} J^1Y', \qquad {y'}^i_\lambda \circ J^1\Phi = \frac{\partial (f^{-1})^\mu}{\partial {x'}^\lambda} d_\mu \Phi^i. \tag{2.1.7}$$

Any projectable vector field u (1.3.1) on a fibre bundle $Y \to X$ has a *jet prolongation* to the projectable vector field

$$\begin{aligned} &J^1u = r_1 \circ J^1u : J^1Y \to J^1TY \to TJ^1Y, \\ &J^1u = u^\lambda \partial_\lambda + u^i \partial_i + (d_\lambda u^i - y^i_\mu \partial_\lambda u^\mu)\partial^\lambda_i, \end{aligned} \tag{2.1.8}$$

on the jet manifold J^1Y. To obtain (2.1.8), the canonical bundle morphism

$$r_1 : J^1TY \to TJ^1Y, \qquad \dot{y}^i_\lambda \circ r_1 = (\dot{y}^i)_\lambda - y^i_\mu \dot{x}^\mu_\lambda$$

is used. In particular, there is the canonical isomorphism

$$VJ^1Y = J^1VY, \qquad \dot{y}^i_\lambda = (\dot{y}^i)_\lambda. \tag{2.1.9}$$

2.2 Higher order jet manifolds

The notion of first jets of sections of a fibre bundle is naturally extended to higher order jets.

Let $Y \to X$ be a fibre bundle. Given its bundle coordinates (x^λ, y^i), a *multi-index* Λ of the length $|\Lambda| = k$ throughout denotes a collection of indices $(\lambda_1 ... \lambda_k)$ modulo permutations. By $\Lambda + \Sigma$ is meant a multi-index $(\lambda_1 \ldots \lambda_k \sigma_1 \ldots \sigma_r)$. For instance $\lambda + \Lambda = (\lambda\lambda_1 ... \lambda_r)$. By $\Lambda\Sigma$ is denoted the union of collections $(\lambda_1 \ldots \lambda_k; \sigma_1 \ldots \sigma_r)$ where the indices λ_i and σ_j

are not permitted. Summation over a multi-index Λ means separate summation over each its index λ_i. We use the compact notation

$$\partial_\Lambda = \partial_{\lambda_k} \circ \cdots \circ \partial_{\lambda_1}, \qquad \Lambda = (\lambda_1 ... \lambda_k).$$

The *r-order jet manifold* J^rY of sections of a bundle $Y \to X$ is defined as the disjoint union of equivalence classes $j^r_x s$ of sections s of $Y \to X$ such that sections s and s' belong to the same equivalence class $j^r_x s$ iff

$$s^i(x) = s'^i(x), \qquad \partial_\Lambda s^i(x) = \partial_\Lambda s'^i(x), \qquad 0 < |\Lambda| \leq r.$$

In brief, one can say that sections of $Y \to X$ are identified by the $r+1$ terms of their Taylor series at points of X. The particular choice of coordinates does not matter for this definition. The equivalence classes $j^r_x s$ are called the *r-order jets* of sections. Their set J^rY is endowed with an atlas of the adapted coordinates

$$(x^\lambda, y^i_\Lambda), \qquad y^i_\Lambda \circ s = \partial_\Lambda s^i(x), \qquad 0 \leq |\Lambda| \leq r, \tag{2.2.1}$$

$$y'^i_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial' x^\lambda} d_\mu y'^i_\Lambda, \tag{2.2.2}$$

where the symbol d_λ stands for the *higher order total derivative*

$$d_\lambda = \partial_\lambda + \sum_{0 \leq |\Lambda| \leq r-1} y^i_{\Lambda+\lambda} \partial^\Lambda_i, \qquad d'_\lambda = \frac{\partial x^\mu}{\partial x'^\lambda} d_\mu. \tag{2.2.3}$$

These derivatives act on exterior forms on J^rY and obey the relations

$$[d_\lambda, d_\mu] = 0, \qquad d_\lambda \circ d = d \circ d_\lambda,$$
$$d_\lambda(\phi \wedge \sigma) = d_\lambda(\phi) \wedge \sigma + \phi \wedge d_\lambda(\sigma), \qquad d_\lambda(d\phi) = d(d_\lambda(\phi)),$$
$$d_\lambda(dx^\mu) = 0, \qquad d_\lambda(dy^i_\Lambda) = dy^i_{\lambda+\Lambda}.$$

We use the compact notation

$$d_\Lambda = d_{\lambda_r} \circ \cdots \circ d_{\lambda_1}, \qquad \Lambda = (\lambda_r ... \lambda_1).$$

The coordinates (2.2.1) bring the set J^rY into a finite-dimensional manifold. The coordinates (2.2.1) are compatible with the natural surjections

$$\pi^r_k : J^rY \to J^kY, \quad r > k,$$

which form the composite bundle

$$\pi^r : J^rY \xrightarrow{\pi^r_{r-1}} J^{r-1}Y \xrightarrow{\pi^{r-1}_{r-2}} \cdots \xrightarrow{\pi^1_0} Y \xrightarrow{\pi} X$$

with the properties

$$\pi^k_s \circ \pi^r_k = \pi^r_s, \qquad \pi^s \circ \pi^r_s = \pi^r.$$

A glance at the transition functions (2.2.2) shows that the fibration

$$\pi^r_{r-1} : J^rY \to J^{r-1}Y$$

is an affine bundle modelled over the vector bundle

$$\overset{r}{\vee} T^*X \underset{J^{r-1}Y}{\otimes} VY \to J^{r-1}Y. \tag{2.2.4}$$

Remark 2.2.1: Let us recall that a base of any affine bundle is a strong deformation retract of its total space. Consequently, Y is a strong deformation retract of J^1Y, which in turn is a strong deformation retract of J^2Y, and so on. It follows that a fibre bundle Y is a strong deformation retract of any finite order jet manifold J^rY. Therefore, by virtue of the Vietoris–Begle theorem, there is an isomorphism

$$H^*(J^rY;\mathbb{R}) = H^*(Y;\mathbb{R}) \tag{2.2.5}$$

of cohomology of J^rY and Y with coefficients in the constant sheaf $\mathbb{R}$. □

Remark 2.2.2: To introduce higher order jet manifolds, one can use the construction of repeated jet manifolds. Let us consider the r-order jet manifold J^rJ^kY of a jet bundle $J^kY \to X$. It is coordinated by $(x^\mu, y^i_{\Sigma\Lambda})$, $|\Lambda| \leq k$, $|\Sigma| \leq r$. There is a canonical monomorphism

$$\sigma_{rk} : J^{r+k}Y \to J^rJ^kY, \qquad y^i_{\Sigma\Lambda} \circ \sigma_{rk} = y^i_{\Sigma+\Lambda}.$$

□

In the calculus in higher order jets, we have the r-order *jet prolongation functor* such that, given fibre bundles Y and Y' over X, every bundle morphism $\Phi : Y \to Y'$ over a diffeomorphism f of X admits the r-order *jet prolongation* to a morphism of r-order jet manifolds

$$J^r\Phi : J^rY \ni j^r_x s \to j^r_{f(x)}(\Phi \circ s \circ f^{-1}) \in J^rY'. \tag{2.2.6}$$

The jet prolongation functor is exact. If Φ is an injection or a surjection, so is $J^r\Phi$. It also preserves an algebraic structure. In particular, if $Y \to X$ is a vector bundle, $J^rY \to X$ is well. If $Y \to X$ is an affine bundle modelled over the vector bundle $\overline{Y} \to X$, then $J^rY \to X$ is an affine bundle modelled over the vector bundle $J^r\overline{Y} \to X$.

Every section s of a fibre bundle $Y \to X$ admits the r-order *jet prolongation* to the *integrable section* $(J^rs)(x) = j^r_x s$ of the jet bundle $J^rY \to X$.

Let $\mathcal{O}^*_k = \mathcal{O}^*(J^kY)$ be the DGA of exterior forms on a jet manifold J^kY. Every exterior form ϕ on a jet manifold J^kY gives rise to the pull-back form $\pi^{k+i}_k{}^*\phi$ on a jet manifold $J^{k+i}Y$. We have the direct sequence of $C^\infty(X)$-algebras

$$\mathcal{O}^*(X) \xrightarrow{\pi^*} \mathcal{O}^*(Y) \xrightarrow{\pi^1_0{}^*} \mathcal{O}^*_1 \xrightarrow{\pi^2_1{}^*} \cdots \xrightarrow{\pi^r_{r-1}{}^*} \mathcal{O}^*_r.$$

Remark 2.2.3: By virtue of de Rham Theorem 8.6.4, the cohomology of the de Rham complex of $\mathcal{O}^*_k$ equals the cohomology $H^*(J^kY;\mathbb{R})$ of J^kY with coefficients in the constant sheaf $\mathbb{R}$. The latter in turn coincides with the sheaf cohomology $H^*(Y;\mathbb{R})$ of Y (see Remark 2.2.1) and, thus, it equals the de Rham cohomology $H^*_{\rm DR}(Y)$ of Y. $\Box$

Given a k-order jet manifold J^kY of $Y \to X$, there exists the canonical bundle morphism

$$r_{(k)} : J^kTY \to TJ^kY$$

over a surjection

$$J^kY \underset{X}{\times} J^kTX \to J^kY \underset{X}{\times} TX$$

whose coordinate expression is

$$\dot{y}^i_\Lambda \circ r_{(k)} = (\dot{y}^i)_\Lambda - \sum (\dot{y}^i)_{\mu+\Sigma}(\dot{x}^\mu)_\Xi, \qquad 0 \leq |\Lambda| \leq k,$$

where the sum is taken over all partitions $\Sigma + \Xi = \Lambda$ and $0 < |\Xi|$. In particular, we have the canonical isomorphism over J^kY

$$r_{(k)} : J^kVY \to VJ^kY, \qquad (\dot{y}^i)_\Lambda = \dot{y}^i_\Lambda \circ r_{(k)}. \tag{2.2.7}$$

As a consequence, every projectable vector field u (1.3.1) on a fibre bundle $Y \to X$ has the following k-order *jet prolongation* to a vector field on J^kY:

$$\begin{aligned} J^ku &= r_{(k)} \circ J^ku : J^kY \to TJ^kY, \\ J^ku &= u^\lambda \partial_\lambda + u^i \partial_i + \sum_{0<|\Lambda|\leq k} (d_\Lambda(u^i - y^i_\mu u^\mu) + y^i_{\mu+\Lambda} u^\mu)\partial_i^\Lambda, \end{aligned} \tag{2.2.8}$$

(cf. (2.1.8) for $k = 1$). In particular, the k-order jet prolongation (2.2.8) of a vertical vector field $u = u^i\partial_i$ on $Y \to X$ is a vertical vector field

$$J^ku = u^i\partial_i + \sum_{0<|\Lambda|\leq k} d_\Lambda u^i \partial_i^\Lambda \tag{2.2.9}$$

on $J^kY \to X$ due to the isomorphism (2.2.7).

A vector field u_r on an r-order jet manifold J^rY is called projectable if, for any $k < r$, there exists a projectable vector field u_k on J^kY such that

$$u_k \circ \pi^r_k = T\pi^r_k \circ u_r.$$

A *projectable vector field* u_k on J^kY has the coordinate expression

$$u_k = u^\lambda \partial_\lambda + \sum_{0\leq|\Lambda|\leq k} u^i_\Lambda \partial_i^\Lambda$$

such that u_λ depends only on coordinates x^μ and every component u^i_Λ is independent of coordinates y^i_Ξ, $|\Xi| > |\Lambda|$. In particular, the k-order jet prolongation J^ku (2.2.8) of a projectable vector field on Y is a projectable vector field on J^kY. It is called an *integrable vector field*.

Let $\mathcal{P}^k$ denote a vector space of projectable vector fields on a jet manifold J^kY. It is easily seen that $\mathcal{P}^r$ is a real Lie algebra and that the morphisms $T\pi^r_k$, $k<r$, constitute the inverse system

$$\mathcal{P}^0 \xleftarrow{T\pi^1_0} \mathcal{P}^1 \xleftarrow{T\pi^2_1} \cdots \xleftarrow{T\pi^{r-1}_{r-2}} \mathcal{P}^{r-1} \xleftarrow{T\pi^r_{r-1}} \mathcal{P}^r \tag{2.2.10}$$

of these Lie algebras. One can show the following.

Theorem 2.2.1: The k-order jet prolongation (2.2.8) is a Lie algebra monomorphism of the Lie algebra $\mathcal{P}^0$ of projectable vector fields on $Y \to X$ to the Lie algebra $\mathcal{P}^k$ of projectable vector fields on J^kY such that

$$T\pi^r_k(J^ru) = J^ku \circ \pi^r_k. \tag{2.2.11}$$

□

Every projectable vector field u_k on J^kY is decomposed into the sum

$$u_k = J^k(T\pi^k_0(u_k)) + v_k \tag{2.2.12}$$

of the integrable vector field $J^k(T\pi^k_0(u_k))$ and a projectable vector field v_k which is vertical with respect to a fibration $J^kY \to Y$.

Similarly to the canonical monomorphisms (2.1.5) – (2.1.6), there are the canonical bundle monomorphisms over J^kY:

$$\begin{aligned} &\lambda_{(k)} : J^{k+1}Y \longrightarrow T^*X \underset{J^kY}{\otimes} TJ^kY, \\ &\lambda_{(k)} = dx^\lambda \otimes d_\lambda, \end{aligned} \tag{2.2.13}$$

$$\begin{aligned} &\theta_{(k)} : J^{k+1}Y \longrightarrow T^*J^kY \underset{J^kY}{\otimes} VJ^kY, \\ &\theta_{(k)} = \sum_{|\Lambda|\le k} (dy^i_\Lambda - y^i_{\lambda+\Lambda}dx^\lambda) \otimes \partial^\Lambda_i. \end{aligned} \tag{2.2.14}$$

The one-forms

$$\theta^i_\Lambda = dy^i_\Lambda - y^i_{\lambda+\Lambda}dx^\lambda \tag{2.2.15}$$

are called the *local contact forms*. Monomorphisms (2.2.13) – (2.2.14) yield the bundle monomorphisms over $J^{k+1}Y$:

$$\widehat\lambda_{(k)} : TX \underset{X}{\times} J^{k+1}Y \longrightarrow TJ^kY \underset{J^kY}{\times} J^{k+1}Y,$$

$$\widehat\theta_{(k)} : V^*J^kY \underset{J^kY}{\times} \longrightarrow T^*J^kY \underset{J^kY}{\times} J^{k+1}Y$$

(cf. (3.2.1) – (3.2.2) for $k = 1$). These monomorphisms in turn define the *canonical horizontal splittings* of the pull-back bundles

$$\pi^{k+1*}_k TJ^kY = \widehat{\lambda}_{(k)}(TX \underset{X}{\times} J^{k+1}Y) \underset{J^{k+1}Y}{\oplus} VJ^kY, \tag{2.2.16}$$
$$\dot{x}^\lambda\partial_\lambda + \sum_{|\Lambda|\leq k} \dot{y}^i_\Lambda \partial^\Lambda_i = \dot{x}^\lambda d_\lambda + \sum_{|\Lambda|\leq k} (\dot{y}^i_\Lambda - \dot{x}^\lambda y^i_{\lambda+\Lambda})\partial^\Lambda_i,$$

$$\pi^{k+1*}_k T^*J^kY = T^*X \underset{J^{k+1}Y}{\oplus} \widehat{\theta}_{(k)}(V^*J^kY \underset{J^kY}{\times} J^{k+1}Y), \tag{2.2.17}$$
$$\dot{x}_\lambda dx^\lambda + \sum_{|\Lambda|\leq k} \dot{y}^\Lambda_i dy^i_\Lambda = (\dot{x}_\lambda + \sum_{|\Lambda|\leq k} \dot{y}^\Lambda_i y^i_{\lambda+\Lambda})dx^\lambda + \sum \dot{y}^\Lambda_i \theta^i_\Lambda.$$

For instance, it follows from the canonical horizontal splitting (2.2.16) that any vector field u_k on J^kY admits the canonical decomposition

$$u_k = u_H + u_V = (u^\lambda\partial_\lambda + \sum_{|\Lambda|\leq k} y^i_{\lambda+\Lambda}\partial^\Lambda_i) + \sum_{|\Lambda|\leq k} (u^i_\Lambda - u^\lambda y^i_{\lambda+\Lambda})\partial^\Lambda_i \tag{2.2.18}$$

over $J^{k+1}Y$ into the horizontal and vertical parts.

By virtue of the canonical horizontal splitting (2.2.17), every exterior one-form ϕ on J^kY admits the canonical splitting of its pull-back onto $J^{k+1}Y$ into the horizontal and vertical parts:

$$\pi^{k+1*}_k \phi = \phi_H + \phi_V = h_0\phi + (\phi - h_0(\phi)), \tag{2.2.19}$$

where h_0 is the *horizontal projection*

$$h_0(dx^\lambda) = dx^\lambda, \qquad h_0(dy^i_{\lambda_1\cdots\lambda_k}) = y^i_{\mu\lambda_1\ldots\lambda_k}dx^\mu.$$

The vertical part of the splitting is called a *contact one-form* on $J^{k+1}Y$.

Let us consider an ideal of the algebra $\mathcal{O}^*_k$ of exterior forms on J^kY which is generated by the contact one-forms on J^kY. This ideal, called the *ideal of contact forms*, is locally generated by the contact forms θ^i_Λ (2.2.15). One can show that an exterior form ϕ on the a manifold J^kY is a contact form iff its pull-back $\overline{s}^*\phi$ onto a base X by means of any integrable section $\overline{s}$ of $J^kY \to X$ vanishes.

2.3 Differential operators and equations

Jet manifolds provides the conventional language of theory of differential equations and differential operators if they need not be linear.

Definition 2.3.1: A system of k-order partial differential equations on a fibre bundle $Y \to X$ is defined as a closed subbundle $\mathfrak{E}$ of a jet bundle $J^kY \to X$. For the sake of brevity, we agree to call $\mathfrak{E}$ a *differential equation.* $\Box$

By a *classical solution* of a differential equation $\mathfrak{E}$ on $Y \to X$ is meant a section s of $Y \to X$ such that its k-order jet prolongation J^ks lives in $\mathfrak{E}$.

Let J^kY be provided with the adapted coordinates (x^λ, y^i_Λ). There exists a local coordinate system (z^A), $A = 1, \ldots, \mathrm{codim}\mathfrak{E}$, on J^kY such that $\mathfrak{E}$ is locally given (in the sense of item (i) of Theorem 1.1.1) by equations

$$\mathcal{E}^A(x^\lambda, y^i_\Lambda) = 0, \qquad A = 1, \ldots, \mathrm{codim}\mathfrak{E}. \tag{2.3.1}$$

Differential equations are often associated to differential operators. There are several equivalent definitions of differential operators.

Definition 2.3.2: Let $Y \to X$ and $E \to X$ be fibre bundles, which are assumed to have global sections. A *k-order E-valued differential operator* on a fibre bundle $Y \to X$ is defined as a section $\mathcal{E}$ of the pull-back bundle

$$\mathrm{pr}_1 : E^k_Y = J^kY \underset{X}{\times} E \to J^kY. \tag{2.3.2}$$

$\Box$

Given bundle coordinates (x^λ, y^i) on Y and (x^λ, χ^a) on E, the pull-back (2.3.2) is provided with coordinates $(x^\lambda, y^j_\Sigma, \chi^a)$, $0 \leq |\Sigma| \leq k$. With respect to these coordinates, a differential operator $\mathcal{E}$ seen as a closed imbedded submanifold $\mathcal{E} \subset E^k_Y$ is given by the equalities

$$\chi^a = \mathcal{E}^a(x^\lambda, y^j_\Sigma). \tag{2.3.3}$$

There is obvious one-to-one correspondence between the sections $\mathcal{E}$ (2.3.3) of the fibre bundle (2.3.2) and the bundle morphisms

$$\Phi : J^kY \underset{X}{\longrightarrow} E, \qquad (2.3.4)$$
$$\Phi = \mathrm{pr}_2 \circ \mathcal{E} \iff \mathcal{E} = (\mathrm{Id}\, J^kY, \Phi).$$

Therefore, we come to the following equivalent definition of differential operators on $Y \to X$.

Definition 2.3.3: Let $Y \to X$ and $E \to X$ be fibre bundles. A bundle morphism $J^kY \to E$ over X is called a *E-valued k-order differential operator* on $Y \to X$. □

It is readily observed that the differential operator Φ (2.3.4) sends each section s of $Y \to X$ onto the section $\Phi \circ J^ks$ of $E \to X$. The mapping

$$\Delta_\Phi : \mathcal{S}(Y) \to \mathcal{S}(E),$$
$$\Delta_\Phi : s \to \Phi \circ J^ks, \qquad \chi^a(x) = \mathcal{E}^a(x^\lambda, \partial_\Sigma s^j(x)),$$

is called the *standard form of a differential operator*.

Let e be a global section of a fibre bundle $E \to X$, the *kernel of a E-valued differential operator* Φ is defined as the kernel

$$\mathrm{Ker}\,_e\Phi = \Phi^{-1}(e(X)) \qquad (2.3.5)$$

of the bundle morphism Φ (2.3.4). If it is a closed subbundle of the jet bundle $J^kY \to X$, one says that $\mathrm{Ker}\,_e\Phi$ (2.3.5) is a *differential equation associated to the differential operator* Φ. By virtue of Theorem 1.1.10, this condition holds if Φ is a bundle morphism of constant rank.

If $E \to X$ is a vector bundle, by the kernel of a E-valued differential operator is usually meant its kernel with respect to the canonical zero-valued section $\widehat{0}$ of $E \to X$.

In the framework of Lagrangian formalism, we deal with differential operators of the following type. Let

$$F \to Y \to X, \qquad E \to Y \to X$$

be composite bundles where $E \to Y$ is a vector bundle. By a k-order differential operator on $F \to X$ taking its values into $E \to X$ is meant a bundle morphism

$$\Phi : J^k F \underset{Y}{\longrightarrow} E, \tag{2.3.6}$$

which certainly is a bundle morphism over X in accordance with Definition 2.3.3. Its kernel $\operatorname{Ker}\Phi$ is defined as the inverse image of the canonical zero-valued section of $E \to Y$. In an equivalent way, the differential operator (2.3.6) is represented by a section $\mathcal{E}_\Phi$ of the vector bundle

$$J^k F \underset{Y}{\times} E \to J^k F.$$

Given bundle coordinates (x^λ, y^i, w^r) on F and (x^λ, y^i, c^A) on E with respect to the fibre basis $\{e_A\}$ for $E \to Y$, this section reads

$$\mathcal{E}_\Phi = \mathcal{E}^A(x^\lambda, y^i_\Lambda, w^r_\Lambda) e_A, \qquad 0 \leq |\Lambda| \leq k. \tag{2.3.7}$$

Then the differential operator (2.3.6) also is represented by a function

$$\mathcal{E}_\Phi = \mathcal{E}^A(x^\lambda, y^i_\Lambda, w^r_\Lambda) c_A \in C^\infty(F \underset{Y}{\times} E^*) \tag{2.3.8}$$

on the product $F \times_Y E^*$, where $E^* \to Y$ is the dual of $E \to Y$ coordinated by (x^λ, y^i, c_A).

If $F \to Y$ is a vector bundle, a differential operator Φ (2.3.6) on the composite bundle $F \to Y \to X$ is called linear if it is linear on the fibres of the vector bundle $J^k F \to J^k Y$. In this case, its representations (2.3.7) and (2.3.8) take the form

$$\mathcal{E}_\Phi = \sum_{0 \leq |\Xi| \leq k} \mathcal{E}^{A,\Xi}_r(x^\lambda, y^i_\Lambda) w^r_\Xi e_A, \qquad 0 \leq |\Lambda| \leq k, \tag{2.3.9}$$

$$\mathcal{E}_\Phi = \sum_{0 \leq |\Xi| \leq k} \mathcal{E}^{A,\Xi}_r(x^\lambda, y^i_\Lambda) w^r_\Xi c_A, \qquad 0 \leq |\Lambda| \leq k. \tag{2.3.10}$$

2.4 Infinite order jet formalism

The finite order jet manifolds J^kY of a fibre bundle $Y \to X$ form the inverse sequence

$$Y \xleftarrow{\pi} J^1Y \longleftarrow \cdots J^{r-1}Y \xleftarrow{\pi^r_{r-1}} J^rY \longleftarrow \cdots, \qquad (2.4.1)$$

where π^r_{r-1} are affine bundles modelled over the vector bundles (2.2.4). Its inductive limit $J^\infty Y$ is defined as a minimal set such that there exist surjections

$$\pi^\infty : J^\infty Y \to X, \quad \pi^\infty_0 : J^\infty Y \to Y, \qquad \pi^\infty_k : J^\infty Y \to J^kY, \quad (2.4.2)$$

obeying the relations $\pi^\infty_r = \pi^k_r \circ \pi^\infty_k$ for all admissible k and $r < k$. A projective limit of the inverse system (2.4.1) always exists. It consists of those elements

$$(\ldots, z_r, \ldots, z_k, \ldots), \qquad z_r \in J^rY, \qquad z_k \in J^kY,$$

of the Cartesian product $\prod\limits_k J^kY$ which satisfy the relations $z_r = \pi^k_r(z_k)$ for all $k > r$. One can think of elements of $J^\infty Y$ as being *infinite order jets* of sections of $Y \to X$ identified by their Taylor series at points of X.

The set $J^\infty Y$ is provided with the projective limit topology. This is the coarsest topology such that the surjections π^∞_r (2.4.2) are continuous. Its base consists of inverse images of open subsets of J^rY, $r = 0, \ldots$, under the maps π^∞_r. With this topology, $J^\infty Y$ is a paracompact Fréchet (complete metrizable) manifold modelled over a locally convex vector space of number series $\{a^\lambda, a^i, a^i_\lambda, \cdots\}$. It is called the *infinite order jet manifold*. One can show that the surjections π^∞_r are open maps admitting local sections, i.e., $J^\infty Y \to J^rY$ are continuous bundles. A bundle coordinate atlas $\{U_Y, (x^\lambda, y^i)\}$ of $Y \to X$ provides $J^\infty Y$ with the manifold coordinate atlas

$$\{(\pi^\infty_0)^{-1}(U_Y), (x^\lambda, y^i_\Lambda)\}_{0\leq|\Lambda|}, \qquad {y'}^i_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial x'^\lambda} d_\mu y'^i_\Lambda. \qquad (2.4.3)$$

Theorem 2.4.1: A fibre bundle Y is a strong deformation retract of the infinite order jet manifold $J^\infty Y$. $\Box$

Corollary 2.4.2: There is an isomorphism

$$H^*(J^\infty Y;\mathbb{R}) = H^*(Y;\mathbb{R}) \tag{2.4.4}$$

between cohomology of $J^\infty Y$ and Y with coefficients in the sheaf $\mathbb{R}$. $\Box$

The inverse sequence (2.4.1) of jet manifolds yields the direct sequence of DGAs $\mathcal{O}^*_r$ of exterior forms on finite order jet manifolds

$$\mathcal{O}^*(X) \xrightarrow{\pi^*} \mathcal{O}^*(Y) \xrightarrow{\pi^1_0{}^*} \mathcal{O}^*_1 \longrightarrow \cdots \mathcal{O}^*_{r-1} \xrightarrow{\pi^r_{r-1}{}^*} \mathcal{O}^*_r \longrightarrow \cdots, \tag{2.4.5}$$

where $\pi^r_{r-1}{}^*$ are the pull-back monomorphisms. Its direct limit

$$\mathcal{O}^*_\infty Y = \overrightarrow{\lim}\, \mathcal{O}^*_r \tag{2.4.6}$$

exists and consists of all exterior forms on finite order jet manifolds modulo the pull-back identification. In accordance with Theorem 8.1.5, $\mathcal{O}^*_\infty Y$ is a DGA which inherits the operations of the exterior differential d and exterior product $\wedge$ of exterior algebras $\mathcal{O}^*_r$. If there is no danger of confusion, we denote $\mathcal{O}^*_\infty = \mathcal{O}^*_\infty Y$.

Theorem 2.4.3: The cohomology $H^*(\mathcal{O}^*_\infty)$ of the de Rham complex

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{O}^0_\infty \xrightarrow{d} \mathcal{O}^1_\infty \xrightarrow{d} \cdots \tag{2.4.7}$$

of the DGA $\mathcal{O}^*_\infty$ equals the de Rham cohomology $H^*_{\rm DR}(Y)$ of Y. $\Box$

Corollary 2.4.4: Any closed form $\phi \in \mathcal{O}^*_\infty$ is decomposed into the sum $\phi = \sigma + d\xi$, where σ is a closed form on Y. $\Box$

One can think of elements of $\mathcal{O}^*_\infty$ as being *differential forms* on the infinite order jet manifold $J^\infty Y$ as follows. Let $\mathfrak{O}^*_r$ be a sheaf of germs of exterior forms on J^rY and $\overline{\mathfrak{O}}^*_r$ the canonical presheaf of local sections of $\mathfrak{O}^*_r$. Since π^r_{r-1} are open maps, there is the direct sequence of presheaves

$$\overline{\mathfrak{O}}^*_0 \xrightarrow{\pi^1_0{}^*} \overline{\mathfrak{O}}^*_1 \cdots \xrightarrow{\pi^r_{r-1}{}^*} \overline{\mathfrak{O}}^*_r \longrightarrow \cdots.$$

Its direct limit $\overline{\mathfrak{O}}^*_\infty$ is a presheaf of DGAs on $J^\infty Y$. Let $\mathfrak{Q}^*_\infty$ be the sheaf of DGAs of germs of $\overline{\mathfrak{O}}^*_\infty$ on $J^\infty Y$. The structure module

$$\mathcal{Q}^*_\infty = \Gamma(\mathfrak{Q}^*_\infty) \tag{2.4.8}$$

of global sections of $\mathfrak{Q}^*_\infty$ is a DGA such that, given an element $\phi \in \mathcal{Q}^*_\infty$ and a point $z \in J^\infty Y$, there exist an open neighbourhood U of z and an exterior form $\phi^{(k)}$ on some finite order jet manifold $J^k Y$ so that

$$\phi|_U = \pi^{\infty *}_k \phi^{(k)}|_U.$$

Therefore, one can regard $\mathcal{Q}^*_\infty$ as an algebra of locally exterior forms on finite order jet manifolds. There is a monomorphism $\mathcal{O}^*_\infty \to \mathcal{Q}^*_\infty$.

Theorem 2.4.5: The paracompact space $J^\infty Y$ admits a partition of unity by elements of the ring $\mathcal{Q}^0_\infty$. □

Since elements of the DGA $\mathcal{Q}^*_\infty$ are locally exterior forms on finite order jet manifolds, the following Poincaré lemma holds.

Lemma 2.4.6: Given a closed element $\phi \in \mathcal{Q}^*_\infty$, there exists a neighbourhood U of each point $z \in J^\infty Y$ such that $\phi|_U$ is exact. □

Theorem 2.4.7: The cohomology $H^*(\mathcal{Q}^*_\infty)$ of the de Rham complex

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{Q}^0_\infty \xrightarrow{d} \mathcal{Q}^1_\infty \xrightarrow{d} \cdots . \tag{2.4.9}$$

of the DGA $\mathcal{Q}^*_\infty$ equals the de Rham cohomology of a fibre bundle Y. □

Due to a monomorphism $\mathcal{O}^*_\infty \to \mathcal{Q}^*_\infty$, one can restrict $\mathcal{O}^*_\infty$ to the coordinate chart (2.4.3) where horizontal forms dx^λ and contact one-forms

$$\theta^i_\Lambda = dy^i_\Lambda - y^i_{\lambda+\Lambda} dx^\lambda$$

make up a local basis for the $\mathcal{O}^0_\infty$-algebra $\mathcal{O}^*_\infty$. Though $J^\infty Y$ is not a smooth manifold, elements of $\mathcal{O}^*_\infty$ are exterior forms on finite order jet manifolds and, therefore, their coordinate transformations are smooth. Moreover, there is the canonical decomposition

$$\mathcal{O}^*_\infty = \oplus \mathcal{O}^{k,m}_\infty$$

of $\mathcal{O}^*_\infty$ into $\mathcal{O}^0_\infty$-modules $\mathcal{O}^{k,m}_\infty$ of k-contact and m-horizontal forms together with the corresponding projectors

$$h_k : \mathcal{O}^*_\infty \to \mathcal{O}^{k,*}_\infty, \qquad h^m : \mathcal{O}^*_\infty \to \mathcal{O}^{*,m}_\infty.$$

Accordingly, the exterior differential on $\mathcal{O}^*_\infty$ is decomposed into the sum

$$d = d_V + d_H$$

of the *vertical differential*

$$d_V \circ h^m = h^m \circ d \circ h^m, \qquad d_V(\phi) = \theta^i_\Lambda \wedge \partial^\Lambda_i \phi, \qquad \phi \in \mathcal{O}^*_\infty,$$

and the *total differential*

$$d_H \circ h_k = h_k \circ d \circ h_k, \quad d_H \circ h_0 = h_0 \circ d, \quad d_H(\phi) = dx^\lambda \wedge d_\lambda(\phi),$$

where

$$d_\lambda = \partial_\lambda + y^i_\lambda \partial_i + \sum_{0<|\Lambda|} y^i_{\lambda+\Lambda} \partial^\Lambda_i \tag{2.4.10}$$

are the *infinite order total derivatives*. They obey the nilpotent conditions

$$d_H \circ d_H = 0, \qquad d_V \circ d_V = 0, \qquad d_H \circ d_V + d_V \circ d_H = 0, \tag{2.4.11}$$

and make $\mathcal{O}^{*,*}_\infty$ into a bicomplex.

Let us consider the $\mathcal{O}^0_\infty$-module $\mathfrak{d}\mathcal{O}^0_\infty$ of derivations of the real ring $\mathcal{O}^0_\infty$.

Theorem 2.4.8: The derivation module $\mathfrak{d}\mathcal{O}^0_\infty$ is isomorphic to the $\mathcal{O}^0_\infty$-dual $(\mathcal{O}^1_\infty)^*$ of the module of one-forms $\mathcal{O}^1_\infty$. $\Box$

One can say something more. The DGA $\mathcal{O}^*_\infty$ is a minimal Chevalley–Eilenberg differential calculus $\mathcal{O}^*\mathcal{A}$ over the real ring $\mathcal{A} = \mathcal{O}^0_\infty$ of smooth real functions on finite order jet manifolds of $Y \to X$. Let $\vartheta\rfloor\phi$, $\vartheta \in \mathfrak{d}\mathcal{O}^0_\infty$, $\phi \in \mathcal{O}^1_\infty$, denote the interior product. Extended to the DGA $\mathcal{O}^*_\infty$, the interior product $\rfloor$ obeys the rule

$$\vartheta\rfloor(\phi \wedge \sigma) = (\vartheta\rfloor\phi) \wedge \sigma + (-1)^{|\phi|}\phi \wedge (\vartheta\rfloor\sigma).$$

Restricted to a coordinate chart (2.4.3), $\mathcal{O}^1_\infty$ is a free $\mathcal{O}^0_\infty$-module generated by one-forms dx^λ, θ^i_Λ. Since $\mathfrak{d}\mathcal{O}^0_\infty = (\mathcal{O}^1_\infty)^*$, any derivation of the real ring $\mathcal{O}^0_\infty$ takes the coordinate form

$$\vartheta = \vartheta^\lambda\partial_\lambda + \vartheta^i\partial_i + \sum_{0<|\Lambda|}\vartheta^i_\Lambda\partial^\Lambda_i, \tag{2.4.12}$$

$$\partial^\Lambda_i(y^j_\Sigma) = \partial^\Lambda_i\rfloor dy^j_\Sigma = \delta^j_i\delta^\Lambda_\Sigma,$$

$$\vartheta'^\lambda = \frac{\partial x'^\lambda}{\partial x^\mu}\vartheta^\mu, \qquad \vartheta'^i = \frac{\partial y'^i}{\partial y^j}\vartheta^j + \frac{\partial y'^i}{\partial x^\mu}\vartheta^\mu,$$

$$\vartheta'^i_\Lambda = \sum_{|\Sigma|\leq|\Lambda|}\frac{\partial y'^i_\Lambda}{\partial y^j_\Sigma}\vartheta^j_\Sigma + \frac{\partial y'^i_\Lambda}{\partial x^\mu}\vartheta^\mu. \tag{2.4.13}$$

Any derivation ϑ (2.4.12) of the ring $\mathcal{O}^0_\infty$ yields a derivation (called the Lie derivative) $\mathbf{L}_\vartheta$ of the DGA $\mathcal{O}^*_\infty$ given by the relations

$$\mathbf{L}_\vartheta\phi = \vartheta\rfloor d\phi + d(\vartheta\rfloor\phi),$$

$$\mathbf{L}_\vartheta(\phi\wedge\phi') = \mathbf{L}_\vartheta(\phi)\wedge\phi' + \phi\wedge\mathbf{L}_\vartheta(\phi').$$

Remark 2.4.1: In particular, the total derivatives (2.4.10) are defined as the local derivations of $\mathcal{O}^0_\infty$ and the corresponding Lie derivatives

$$d_\lambda\phi = \mathbf{L}_{d_\lambda}\phi$$

of $\mathcal{O}^*_\infty$. Moreover, the $C^\infty(X)$-ring $\mathcal{O}^0_\infty$ possesses the canonical connection

$$\nabla = dx^\lambda\otimes d_\lambda \tag{2.4.14}$$

in the sense of Definition 8.2.4. □

Chapter 3

Connections on fibre bundles

There are several equivalent definitions of a connection on a fibre bundle. We start with the traditional notion of a connection as a splitting of the exact sequences (1.2.10) – (1.2.11), but then follow its definition as a global section of an affine jet bundle. In the case of vector bundles, there is an equivalent definition (8.6.3) of a linear connection on their structure modules.

3.1 Connections as tangent-valued forms

A *connection* on a fibre bundle $Y \to X$ is defined traditionally as a linear bundle monomorphism

$$\begin{aligned} &\Gamma : Y \underset{X}{\times} TX \to TY, \\ &\Gamma : \dot{x}^\lambda \partial_\lambda \to \dot{x}^\lambda(\partial_\lambda + \Gamma^i_\lambda \partial_i), \end{aligned} \tag{3.1.1}$$

over Y which splits the exact sequence (1.2.10), i.e.,

$$\pi_T \circ \Gamma = \mathrm{Id}\,(Y \underset{X}{\times} TX).$$

This is a definition of connections on fibred manifolds (see Remark 3.1.2).

By virtue of Theorem 1.2.2, a connection always exists. The local functions $\Gamma^i_\lambda(y)$ in (3.1.1) are said to be *components* of the connection Γ with respect to the bundle coordinates (x^λ, y^i) on $Y \to X$.

The image of $Y \times TX$ by the connection Γ defines the *horizontal distribution* $HY \subset TY$ which splits the tangent bundle TY as follows:

$$\begin{aligned} & TY = HY \underset{Y}{\oplus} VY, \\ & \dot{x}^\lambda \partial_\lambda + \dot{y}^i \partial_i = \dot{x}^\lambda (\partial_\lambda + \Gamma^i_\lambda \partial_i) + (\dot{y}^i - \dot{x}^\lambda \Gamma^i_\lambda) \partial_i. \end{aligned} \tag{3.1.2}$$

Its annihilator is locally generated by the one-forms $dy^i - \Gamma^i_\lambda dx^\lambda$.

Given the *horizontal splitting* (3.1.2), the surjection

$$\Gamma : TY \underset{Y}{\to} VY, \qquad \dot{y}^i \circ \Gamma = \dot{y}^i - \Gamma^i_\lambda \dot{x}^\lambda, \tag{3.1.3}$$

defines a connection on $Y \to X$ in an equivalent way.

The linear morphism Γ over Y (3.1.1) yields uniquely the horizontal tangent-valued one-form

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i) \tag{3.1.4}$$

on Y which projects onto the canonical tangent-valued form θ_X (1.4.5) on X. With this form called the *connection form*, the morphism (3.1.1) reads

$$\Gamma : \partial_\lambda \to \partial_\lambda \rfloor \Gamma = \partial_\lambda + \Gamma^i_\lambda \partial_i.$$

Given a connection Γ and the corresponding horizontal distribution (3.1.2), a vector field u on a fibre bundle $Y \to X$ is called *horizontal* if it lives in HY. A horizontal vector field takes the form

$$u = u^\lambda(y)(\partial_\lambda + \Gamma^i_\lambda \partial_i). \tag{3.1.5}$$

In particular, let τ be a vector field on the base X. By means of the connection form Γ (3.1.4), we obtain the projectable horizontal vector field

$$\Gamma\tau = \tau \rfloor \Gamma = \tau^\lambda (\partial_\lambda + \Gamma^i_\lambda \partial_i) \tag{3.1.6}$$

on Y, called the *horizontal lift* of τ by means of a connection Γ. Conversely, any projectable horizontal vector field u on Y is the horizontal lift $\Gamma\tau$ of its projection τ on X. Moreover, the horizontal distribution HY is generated by the horizontal lifts $\Gamma\tau$ (3.1.6) of vector fields τ on X. The horizontal lift

$$\mathcal{T}(X) \ni \tau \to \Gamma\tau \in \mathcal{T}(Y) \tag{3.1.7}$$

is a $C^\infty(X)$-linear module morphism.

Given the splitting (3.1.1), the dual splitting of the exact sequence (1.2.11) is

$$\Gamma : V^*Y \to T^*Y, \qquad \Gamma : \overline{d}y^i \to dy^i - \Gamma^i_\lambda dx^\lambda. \tag{3.1.8}$$

Hence, a connection Γ on $Y \to X$ is represented by the vertical-valued form

$$\Gamma = (dy^i - \Gamma^i_\lambda dx^\lambda) \otimes \partial_i \tag{3.1.9}$$

such that the morphism (3.1.8) reads

$$\Gamma : \overline{d}y^i \to \Gamma\rfloor\overline{d}y^i = dy^i - \Gamma^i_\lambda dx^\lambda.$$

We call Γ (3.1.9) the *vertical connection form*. The corresponding horizontal splitting of the cotangent bundle T^*Y takes the form

$$\begin{aligned} &T^*Y = T^*X \underset{Y}{\oplus} \Gamma(V^*Y), \\ &\dot{x}_\lambda dx^\lambda + \dot{y}_i dy^i = (\dot{x}_\lambda + \dot{y}_i\Gamma^i_\lambda)dx^\lambda + \dot{y}_i(dy^i - \Gamma^i_\lambda dx^\lambda). \end{aligned} \tag{3.1.10}$$

Then we have the surjection

$$\Gamma = \mathrm{pr}_1 : T^*Y \to T^*X, \qquad \dot{x}_\lambda \circ \Gamma = \dot{x}_\lambda + \dot{y}_i\Gamma^i_\lambda, \tag{3.1.11}$$

which also defines a connection on a fibre bundle $Y \to X$.

Remark 3.1.1: Treating a connection as the vertical-valued form (3.1.9), we come to the following important construction. Given a fibre bundle $Y \to X$, let $f : X' \to X$ be a morphism and $f^*Y \to X'$ the pull-back of Y by f. Any connection Γ (3.1.9) on $Y \to X$ induces the *pull-back connection*

$$f^*\Gamma = \left(dy^i - (\Gamma \circ f_Y)^i_\lambda \frac{\partial f^\lambda}{\partial x'^\mu} dx'^\mu\right) \otimes \partial_i \tag{3.1.12}$$

on $f^*Y \to X'$ (see Remark 1.4.4). $\Box$

Remark 3.1.2: Let $\pi : Y \to X$ be a fibred manifold. Any connection Γ on $Y \to X$ yields a horizontal lift of a vector field on X onto Y, but need not defines the similar lift of a *path* in X into Y. Let

$$\mathbb{R} \supset [,] \ni t \to x(t) \in X, \qquad \mathbb{R} \ni t \to y(t) \in Y,$$

be smooth paths in X and Y, respectively. Then $t \to y(t)$ is called a *horizontal lift* of $x(t)$ if

$$\pi(y(t)) = x(t), \qquad \dot y(t) \in H_{y(t)}Y, \qquad t \in \mathbb{R},$$

where $HY \subset TY$ is the horizontal subbundle associated to the connection Γ. If, for each path $x(t)$ ($t_0 \le t \le t_1$) and for any $y_0 \in \pi^{-1}(x(t_0))$, there exists a horizontal lift $y(t)$ ($t_0 \le t \le t_1$) such that $y(t_0) = y_0$, then Γ is called the *Ehresmann connection*. A fibred manifold is a fibre bundle iff it admits an Ehresmann connection. $\Box$

3.2 Connections as jet bundle sections

Throughout the book, we follow the equivalent definition of connections on a fibre bundle $Y \to X$ as sections of the affine jet bundle $J^1Y \to Y$.

Let $Y \to X$ be a fibre bundle, and J^1Y its first order jet manifold. Given the canonical morphisms (2.1.5) and (2.1.6), we have the corresponding morphisms

$$\widehat\lambda_{(1)} : J^1Y \underset{X}{\times} TX \ni \partial_\lambda \to d_\lambda = \partial_\lambda \rfloor \lambda_{(1)} \in J^1Y \underset{Y}{\times} TY, \tag{3.2.1}$$
$$\widehat\theta_{(1)} : J^1Y \underset{Y}{\times} V^*Y \ni \overline{d}y^i \to \theta^i = \theta_{(1)} \rfloor dy^i \in J^1Y \underset{Y}{\times} T^*Y \tag{3.2.2}$$

(see Remark 1.2.2). These morphisms yield the *canonical horizontal splittings* of the pull-back bundles

$$J^1Y \underset{Y}{\times} TY = \widehat\lambda_{(1)}(TX) \underset{J^1Y}{\oplus} VY, \tag{3.2.3}$$
$$\dot x^\lambda \partial_\lambda + \dot y^i \partial_i = \dot x^\lambda(\partial_\lambda + y^i_\lambda \partial_i) + (\dot y^i - \dot x^\lambda y^i_\lambda)\partial_i,$$

$$J^1Y \underset{Y}{\times} T^*Y = T^*X \underset{J^1Y}{\oplus} \widehat{\theta}_{(1)}(V^*Y), \tag{3.2.4}$$
$$\dot{x}_\lambda dx^\lambda + \dot{y}_i dy^i = (\dot{x}_\lambda + \dot{y}_i y^i_\lambda) dx^\lambda + \dot{y}_i (dy^i - y^i_\lambda dx^\lambda).$$

Let Γ be a global section of $J^1Y \to Y$. Substituting the tangent-valued form

$$\lambda_{(1)} \circ \Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i)$$

in the canonical splitting (3.2.3), we obtain the familiar horizontal splitting (3.1.2) of TY by means of a connection Γ on $Y \to X$. Accordingly, substitution of the tangent-valued form

$$\theta_{(1)} \circ \Gamma = (dy^i - \Gamma^i_\lambda dx^\lambda) \otimes \partial_i$$

in the canonical splitting (3.2.4) leads to the dual splitting (3.1.10) of T^*Y by means of a connection Γ.

Theorem 3.2.1: There is one-to-one correspondence between the connections Γ on a fibre bundle $Y \to X$ and the global sections

$$\Gamma : Y \to J^1Y, \qquad (x^\lambda, y^i, y^i_\lambda) \circ \Gamma = (x^\lambda, y^i, \Gamma^i_\lambda), \tag{3.2.5}$$

of the affine jet bundle $J^1Y \to Y$. □

There are the following corollaries of this theorem.

• Since $J^1Y \to Y$ is affine, a connection on a fibre bundle $Y \to X$ exists in accordance with Theorem 1.1.4.

• Connections on a fibre bundle $Y \to X$ make up an affine space modelled over the vector space of soldering forms on $Y \to X$, i.e., sections of the vector bundle (2.1.4).

• Connection components possess the coordinate transformation law

$$\Gamma'^i_\lambda = \frac{\partial x^\mu}{\partial x'^\lambda} (\partial_\mu + \Gamma^j_\mu \partial_j) y'^i.$$

• Every connection Γ (3.2.5) on a fibre bundle $Y \to X$ yields the first order differential operator

$$D_\Gamma : J^1Y \underset{Y}{\to} T^*X \underset{Y}{\otimes} VY, \tag{3.2.6}$$
$$D_\Gamma = \lambda_{(1)} - \Gamma \circ \pi^1_0 = (y^i_\lambda - \Gamma^i_\lambda) dx^\lambda \otimes \partial_i,$$

on Y called the *covariant differential* relative to the connection Γ. If $s : X \to Y$ is a section, from (3.2.6) we obtain its covariant differential

$$\begin{aligned} &\nabla^\Gamma s = D_\Gamma \circ J^1 s : X \to T^*X \otimes VY, \qquad (3.2.7)\\ &\nabla^\Gamma s = (\partial_\lambda s^i - \Gamma^i_\lambda \circ s) dx^\lambda \otimes \partial_i, \end{aligned}$$

and the *covariant derivative*

$$\nabla^\Gamma_\tau = \tau \rfloor \nabla^\Gamma$$

along a vector field τ on X. A section s is said to be an *integral section* of a connection Γ if it belongs to the kernel of the covariant differential D_Γ (3.2.6), i.e.,

$$\nabla^\Gamma s = 0 \quad \text{or} \quad J^1 s = \Gamma \circ s. \qquad (3.2.8)$$

Theorem 3.2.2: For any global section $s : X \to Y$, there always exists a connection Γ such that s is an integral section of Γ. □

Treating connections as jet bundle sections, one comes to the following two constructions.

(i) Let Y and Y' be fibre bundles over the same base X. Given a connection Γ on $Y \to X$ and a connection Γ' on $Y' \to X$, the bundle product $Y \times Y'$ is provided with the *product connection*

$$\begin{aligned} &\Gamma \times \Gamma' : Y \underset{X}{\times} Y' \to J^1(Y \underset{X}{\times} Y') = J^1 Y \underset{X}{\times} J^1 Y',\\ &\Gamma \times \Gamma' = dx^\lambda \otimes \left(\partial_\lambda + \Gamma^i_\lambda \frac{\partial}{\partial y^i} + \Gamma'^j_\lambda \frac{\partial}{\partial y'^j} \right). \qquad (3.2.9) \end{aligned}$$

(ii) Let $i_Y : Y \to Y'$ be a subbundle of a fibre bundle $Y' \to X$ and Γ' a connection on $Y' \to X$. If there exists a connection Γ on $Y \to X$ such that the diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\Gamma'} & J^1 Y \\ {\scriptstyle i_Y}\uparrow & & \uparrow{\scriptstyle J^1 i_Y} \\ Y & \xrightarrow{\Gamma} & J^1 Y' \end{array}$$

is commutative, we say that Γ' is *reducible* to a connection Γ. The following conditions are equivalent:

- Γ' is reducible to Γ;
- $Ti_Y(HY) = HY'|_{i_Y(Y)}$, where $HY \subset TY$ and $HY' \subset TY'$ are the horizontal subbundles determined by Γ and Γ', respectively;
- for every vector field τ on X, the vector fields $\Gamma\tau$ and $\Gamma'\tau$ are related as follows:

$$Ti_Y \circ \Gamma\tau = \Gamma'\tau \circ i_Y. \tag{3.2.10}$$

3.3 Curvature and torsion

Let Γ be a connection on a fibre bundle $Y \to X$. Its *curvature* is defined as the Nijenhuis differential

$$R = \frac{1}{2}d_\Gamma\Gamma = \frac{1}{2}[\Gamma,\Gamma]_{\mathrm{FN}} : Y \to \overset{2}{\wedge} T^*X \otimes VY, \tag{3.3.1}$$

$$R = \frac{1}{2}R^i_{\lambda\mu} dx^\lambda \wedge dx^\mu \otimes \partial_i, \tag{3.3.2}$$

$$R^i_{\lambda\mu} = \partial_\lambda\Gamma^i_\mu - \partial_\mu\Gamma^i_\lambda + \Gamma^j_\lambda\partial_j\Gamma^i_\mu - \Gamma^j_\mu\partial_j\Gamma^i_\lambda.$$

This is a VY-valued horizontal two-form on Y. Given vector fields τ, τ' on X and their horizontal lifts $\Gamma\tau$ and $\Gamma\tau'$ (3.1.6) on Y, we have the relation

$$R(\tau,\tau') = -\Gamma[\tau,\tau'] + [\Gamma\tau,\Gamma\tau'] = \tau^\lambda\tau'^\mu R^i_{\lambda\mu}\partial_i. \tag{3.3.3}$$

The curvature (3.3.1) obeys the identities

$$[R,R]_{\mathrm{FN}} = 0, \tag{3.3.4}$$

$$d_\Gamma R = [\Gamma,R]_{\mathrm{FN}} = 0. \tag{3.3.5}$$

They result from the identity (1.4.9) and the graded Jacobi identity (1.4.10), respectively. The identity (3.3.5) is called the *second Bianchi identity*. It takes the coordinate form

$$\sum_{(\lambda\mu\nu)} (\partial_\lambda R^i_{\mu\nu} + \Gamma^j_\lambda\partial_j R^i_{\mu\nu} - \partial_j\Gamma^i_\lambda R^j_{\mu\nu}) = 0, \tag{3.3.6}$$

where the sum is cyclic over the indices λ, μ and ν.

Given a soldering form σ, one defines the *soldered curvature*

$$\rho = \frac{1}{2}d_\sigma\sigma = \frac{1}{2}[\sigma,\sigma]_{\rm FN} : Y \to \overset{2}{\wedge} T^*X \otimes VY, \tag{3.3.7}$$
$$\rho = \frac{1}{2}\rho^i_{\lambda\mu}dx^\lambda\wedge dx^\mu\otimes\partial_i, \qquad \rho^i_{\lambda\mu} = \sigma^j_\lambda\partial_j\sigma^i_\mu - \sigma^j_\mu\partial_j\sigma^i_\lambda.$$

It fulfills the identities

$$[\rho,\rho]_{\rm FN} = 0, \qquad d_\sigma\rho = [\sigma,\rho]_{\rm FN} = 0,$$

similar to (3.3.4) – (3.3.5).

Given a connection Γ and a soldering form σ, the *torsion form* of Γ with respect to σ is defined as

$$T = d_\Gamma\sigma = d_\sigma\Gamma : Y \to \overset{2}{\wedge} T^*X\otimes VY,$$
$$T = (\partial_\lambda\sigma^i_\mu + \Gamma^j_\lambda\partial_j\sigma^i_\mu - \partial_j\Gamma^i_\lambda\sigma^j_\mu)dx^\lambda\wedge dx^\mu\otimes\partial_i. \tag{3.3.8}$$

It obeys the *first Bianchi identity*

$$d_\Gamma T = d^2_\Gamma\sigma = [R,\sigma]_{\rm FN} = -d_\sigma R. \tag{3.3.9}$$

If $\Gamma' = \Gamma + \sigma$, we have the relations

$$T' = T + 2\rho, \tag{3.3.10}$$
$$R' = R + \rho + T. \tag{3.3.11}$$

3.4 Linear and affine connections

A connection Γ on a vector bundle $Y \to X$ is called the *linear connection* if the section

$$\Gamma : Y \to J^1Y, \qquad \Gamma = dx^\lambda\otimes(\partial_\lambda + \Gamma_\lambda{}^i{}_j(x)y^j\partial_i), \tag{3.4.1}$$

is a linear bundle morphism over X. Note that linear connections are principal connections, and they always exist (see Theorem 4.4.1).

The curvature R (3.3.2) of a linear connection Γ (3.4.1) reads

$$R = \frac{1}{2}R_{\lambda\mu}{}^i{}_j(x)y^j dx^\lambda \wedge dx^\mu \otimes \partial_i,$$
$$R_{\lambda\mu}{}^i{}_j = \partial_\lambda \Gamma_\mu{}^i{}_j - \partial_\mu \Gamma_\lambda{}^i{}_j + \Gamma_\lambda{}^h{}_j \Gamma_\mu{}^i{}_h - \Gamma_\mu{}^h{}_j \Gamma_\lambda{}^i{}_h. \qquad (3.4.2)$$

Due to the vertical splitting (1.2.8), we have the linear morphism

$$R : Y \ni y^i e_i \to \frac{1}{2}R_{\lambda\mu}{}^i{}_j y^j dx^\lambda \wedge dx^\mu \otimes e_i \in \mathcal{O}^2(X) \otimes Y. \qquad (3.4.3)$$

There are the following standard constructions of new linear connections from the old ones.

• Let $Y \to X$ be a vector bundle, coordinated by (x^λ, y^i), and $Y^* \to X$ its dual, coordinated by (x^λ, y_i). Any linear connection Γ (3.4.1) on a vector bundle $Y \to X$ defines the *dual linear connection*

$$\Gamma^* = dx^\lambda \otimes (\partial_\lambda - \Gamma_\lambda{}^j{}_i(x) y_j \partial^i) \qquad (3.4.4)$$

on $Y^* \to X$.

• Let Γ and Γ' be linear connections on vector bundles $Y \to X$ and $Y' \to X$, respectively. The direct sum connection $\Gamma \oplus \Gamma'$ on their Whitney sum $Y \oplus Y'$ is defined as the product connection (3.2.9).

• Let Y coordinated by (x^λ, y^i) and Y' coordinated by (x^λ, y^a) be vector bundles over the same base X. Their tensor product $Y \otimes Y'$ is endowed with the bundle coordinates (x^λ, y^{ia}). Linear connections Γ and Γ' on $Y \to X$ and $Y' \to X$ define the linear *tensor product connection*

$$\Gamma \otimes \Gamma' = dx^\lambda \otimes \left[\partial_\lambda + (\Gamma_\lambda{}^i{}_j y^{ja} + \Gamma'_\lambda{}^a{}_b y^{ib}) \frac{\partial}{\partial y^{ia}}\right] \qquad (3.4.5)$$

on $Y \otimes Y'$.

An important example of linear connections is a linear connection

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma_\lambda{}^\mu{}_\nu \dot{x}^\nu \dot{\partial}_\mu) \qquad (3.4.6)$$

on the tangent bundle TX of a manifold X. We agree to call it a *world connection* on a manifold X. The *dual world connection* (3.4.4) on the cotangent bundle T^*X is

$$\Gamma^* = dx^\lambda \otimes (\partial_\lambda - \Gamma_\lambda{}^\mu{}_\nu \dot{x}_\mu \dot{\partial}^\nu). \qquad (3.4.7)$$

Then, using the construction of the tensor product connection (3.4.5), one can introduce the corresponding linear *world connection on an arbitrary tensor bundle* T (1.2.5).

Remark 3.4.1: It should be emphasized that the expressions (3.4.6) and (3.4.7) for a world connection differ in a minus sign from those usually used in the physical literature. □

The *curvature of a world connection* is defined as the curvature R (3.4.2) of the connection Γ (3.4.6) on the tangent bundle TX. It reads

$$R = \frac{1}{2} R_{\lambda\mu}{}^\alpha{}_\beta \dot x^\beta dx^\lambda \wedge dx^\mu \otimes \dot\partial_\alpha, \tag{3.4.8}$$
$$R_{\lambda\mu}{}^\alpha{}_\beta = \partial_\lambda \Gamma_\mu{}^\alpha{}_\beta - \partial_\mu \Gamma_\lambda{}^\alpha{}_\beta + \Gamma_\lambda{}^\gamma{}_\beta \Gamma_\mu{}^\alpha{}_\gamma - \Gamma_\mu{}^\gamma{}_\beta \Gamma_\lambda{}^\alpha{}_\gamma.$$

By the *torsion of a world connection* is meant the torsion (3.3.8) of the connection Γ (3.4.6) on the tangent bundle TX with respect to the canonical soldering form θ_J (1.4.7):

$$T = \frac{1}{2} T_\mu{}^\nu{}_\lambda dx^\lambda \wedge dx^\mu \otimes \dot\partial_\nu, \qquad T_\mu{}^\nu{}_\lambda = \Gamma_\mu{}^\nu{}_\lambda - \Gamma_\lambda{}^\nu{}_\mu. \tag{3.4.9}$$

A world connection is said to be *symmetric* if its torsion (3.4.9) vanishes, i.e., $\Gamma_\mu{}^\nu{}_\lambda = \Gamma_\lambda{}^\nu{}_\mu$.

Remark 3.4.2: For any vector field τ on a manifold X, there exists a connection Γ on the tangent bundle $TX \to X$ such that τ is an integral section of Γ, but this connection is not necessarily linear. If a vector field τ is non-vanishing at a point $x \in X$, then there exists a local symmetric world connection Γ (3.4.6) around x for which τ is an integral section

$$\partial_\nu \tau^\alpha = \Gamma_\nu{}^\alpha{}_\beta \tau^\beta. \tag{3.4.10}$$

Then the canonical lift $\tilde\tau$ (1.3.4) of τ onto TX can be seen locally as the horizontal lift $\Gamma\tau$ (3.1.6) of τ by means of this connection. □

Remark 3.4.3: Every manifold X can be provided with a non-degenerate fibre metric

$$g \in \overset{2}{\vee} \mathcal{O}^1(X), \qquad g = g_{\lambda\mu} dx^\lambda \otimes dx^\mu,$$

in the tangent bundle TX, and with the corresponding metric

$$g \in \overset{2}{\vee} \mathcal{T}^1(X), \qquad g = g^{\lambda\mu}\partial_\lambda \otimes \partial_\mu,$$

in the cotangent bundle T^*X. We call it a *world metric* on X. For any world metric g, there exists a unique symmetric world connection Γ (3.4.6) with the components

$$\Gamma_\lambda{}^\nu{}_\mu = \{_\lambda{}^\nu{}_\mu\} = -\frac{1}{2}g^{\nu\rho}(\partial_\lambda g_{\rho\mu} + \partial_\mu g_{\rho\lambda} - \partial_\rho g_{\lambda\mu}), \tag{3.4.11}$$

called the *Christoffel symbols*, such that g is an integral section of Γ, i.e.

$$\partial_\lambda g^{\alpha\beta} = g^{\alpha\gamma}\{_\lambda{}^\beta{}_\gamma\} + g^{\beta\gamma}\{_\lambda{}^\alpha{}_\gamma\}.$$

It is called the *Levi–Civita connection* associated to g. □

Let $Y \to X$ be an affine bundle modelled over a vector bundle $\overline{Y} \to X$. A connection Γ on $Y \to X$ is called an *affine connection* if the section $\Gamma : Y \to J^1Y$ (3.2.5) is an affine bundle morphism over X. Associated to principal connections, affine connections always exist (see Theorem 4.4.1).

For any affine connection $\Gamma : Y \to J^1Y$, the corresponding linear derivative $\overline{\Gamma} : \overline{Y} \to J^1\overline{Y}$ (1.2.14) defines a unique linear connection on the vector bundle $\overline{Y} \to X$. Since every vector bundle is an affine bundle, any linear connection on a vector bundle also is an affine connection.

With respect to affine bundle coordinates (x^λ, y^i) on Y, an affine connection Γ on $Y \to X$ reads

$$\Gamma^i_\lambda = \Gamma_\lambda{}^i{}_j(x)y^j + \sigma^i_\lambda(x). \tag{3.4.12}$$

The coordinate expression of the associated linear connection is

$$\overline{\Gamma}^i_\lambda = \Gamma_\lambda{}^i{}_j(x)\overline{y}^j, \tag{3.4.13}$$

where $(x^\lambda, \overline{y}^i)$ are the associated linear bundle coordinates on $\overline{Y}$.

Affine connections on an affine bundle $Y \to X$ constitute an affine space modelled over the soldering forms on $Y \to X$. In view of the vertical splitting (1.2.15), these soldering forms can be seen as global sections of the vector bundle

$$T^*X \underset{X}{\otimes} \overline{Y} \to X.$$

If $Y \to X$ is a vector bundle, both the affine connection Γ (3.4.12) and the associated linear connection $\overline{\Gamma}$ are connections on the same vector bundle $Y \to X$, and their difference is a basic soldering form on Y. Thus, every affine connection on a vector bundle $Y \to X$ is the sum of a linear connection and a basic soldering form on $Y \to X$.

Given an affine connection Γ on a vector bundle $Y \to X$, let R and $\overline{R}$ be the curvatures of a connection Γ and the associated linear connection $\overline{\Gamma}$, respectively. It is readily observed that $R = \overline{R} + T$, where the VY-valued two-form

$$\begin{aligned} & T = d_\Gamma \sigma = d_\sigma \Gamma : X \to \overset{2}{\wedge} T^*X \underset{X}{\otimes} VY, \qquad (3.4.14) \\ & T = \frac{1}{2} T^i_{\lambda\mu} dx^\lambda \wedge dx^\mu \otimes \partial_i, \\ & T^i_{\lambda\mu} = \partial_\lambda \sigma^i_\mu - \partial_\mu \sigma^i_\lambda + \sigma^h_\lambda \Gamma_\mu{}^i{}_h - \sigma^h_\mu \Gamma_\lambda{}^i{}_h, \end{aligned}$$

is the torsion (3.3.8) of Γ with respect to the basic soldering form σ.

In particular, let us consider the tangent bundle TX of a manifold X. We have the canonical soldering form $\sigma = \theta_J = \theta_X$ (1.4.7) on TX. Given an arbitrary world connection Γ (3.4.6) on TX, the corresponding affine connection

$$A = \Gamma + \theta_X, \qquad A^\mu_\lambda = \Gamma_\lambda{}^\mu{}_\nu \dot{x}^\nu + \delta^\mu_\lambda, \qquad (3.4.15)$$

on TX is called the *Cartan connection*. Since the soldered curvature ρ (3.3.7) of θ_J equals zero, the torsion (3.3.10) of the Cartan connection coincides with the torsion T (3.4.9) of the world connection Γ, while its curvature (3.3.11) is the sum $R + T$ of the curvature and the torsion of Γ.

3.5 Flat connections

By a *flat* or *curvature-free* connection is meant a connection which satisfies the following equivalent conditions.

Theorem 3.5.1: Let Γ be a connection on a fibre bundle $Y \to X$. The following assertions are equivalent.

(i) The curvature R of a connection Γ vanishes identically, i.e., $R \equiv 0$.

(ii) The horizontal lift (3.1.7) of vector fields on X onto Y is an $\mathbb{R}$-linear Lie algebra morphism (in accordance with the formula (3.3.3)).

(iii) The horizontal distribution is involutive.

(iv) There exists a local integral section for Γ through any point $y \in Y$. □

By virtue of Theorem 1.3.3 and item (iii) of Theorem 3.5.1, a flat connection Γ on a fibre bundle $Y \to X$ yields a *horizontal foliation* on Y, transversal to the fibration $Y \to X$. The leaf of this foliation through a point $y \in Y$ is defined locally by an integral section s_y for the connection Γ through y. Conversely, let a fibre bundle $Y \to X$ admit a transversal foliation such that, for each point $y \in Y$, the leaf of this foliation through y is locally defined by a section s_y of $Y \to X$ through y. Then the map

$$\Gamma : Y \to J^1Y, \qquad \Gamma(y) = j^1_x s_y, \qquad \pi(y) = x,$$

introduces a flat connection on $Y \to X$. Thus, there is one-to-one correspondence between the flat connections and the transversal foliations of a fibre bundle $Y \to X$.

Given a transversal foliation on a fibre bundle $Y \to X$, there exists the associated atlas of bundle coordinates (x^λ, y^i) of Y such that every leaf of this foliation is locally generated by the equations y^i =const., and the transition functions $y^i \to y'^i(y^j)$ are independent of the base coordinates x^λ. This is called the *atlas of constant local trivializations*. Two such atlases are said to be equivalent if their union also is an atlas of constant local trivializations. They are associated to the same horizontal foliation. Thus, we come to the following assertion.

Theorem 3.5.2: There is one-to-one correspondence between the flat connections Γ on a fibre bundle $Y \to X$ and the equivalence classes of atlases of constant local trivializations of Y such that

$$\Gamma = dx^\lambda \otimes \partial_\lambda$$

relative to these atlases. □

In particular, if $Y \to X$ is a trivial bundle, one associates to each its trivialization a flat connection represented by the global zero section $\widehat{0}(Y)$ of $J^1Y \to Y$ with respect to this trivialization (see Remark 2.1.2).

3.6 Connections on composite bundles

Let $Y \to \Sigma \to X$ be a composite bundle (1.1.10). Let us consider the jet manifolds $J^1\Sigma$, $J^1_\Sigma Y$, and J^1Y of the fibre bundles

$$\Sigma \to X, \qquad Y \to \Sigma, \qquad Y \to X,$$

respectively. They are provided with the adapted coordinates

$$(x^\lambda, \sigma^m, \sigma^m_\lambda), \quad (x^\lambda, \sigma^m, y^i, \widetilde{y}^i_\lambda, y^i_m), \quad (x^\lambda, \sigma^m, y^i, \sigma^m_\lambda, y^i_\lambda).$$

One can show the following.

Theorem 3.6.1: There is the canonical map

$$\varrho : J^1\Sigma \underset{\Sigma}{\times} J^1_\Sigma Y \underset{Y}{\longrightarrow} J^1Y, \qquad y^i_\lambda \circ \varrho = y^i_m \sigma^m_\lambda + \widetilde{y}^i_\lambda. \tag{3.6.1}$$

□

Using the canonical map (3.6.1), we can get the relations between connections on the fibre bundles $Y \to X$, $Y \to \Sigma$ and $\Sigma \to X$. These connections are given by the corresponding connection forms

$$\gamma = dx^\lambda \otimes (\partial_\lambda + \gamma^m_\lambda \partial_m + \gamma^i_\lambda \partial_i), \tag{3.6.2}$$

$$A_\Sigma = dx^\lambda \otimes (\partial_\lambda + A^i_\lambda \partial_i) + d\sigma^m \otimes (\partial_m + A^i_m \partial_i), \tag{3.6.3}$$

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^m_\lambda \partial_m). \tag{3.6.4}$$

A connection γ (3.6.2) on the fibre bundle $Y \to X$ is called *projectable* onto a connection Γ (3.6.4) on the fibre bundle $\Sigma \to X$ if, for any vector field τ on X, its horizontal lift $\gamma\tau$ on Y by means of the connection γ is a projectable vector field over the horizontal lift $\Gamma\tau$ of τ on Σ by means of the connection Γ. This property holds iff $\gamma_\lambda^m = \Gamma_\lambda^m$, i.e., components γ_λ^m of the connection γ (3.6.2) must be independent of the fibre coordinates y^i.

A connection A_Σ (3.6.3) on the fibre bundle $Y \to \Sigma$ and a connection Γ (3.6.4) on the fibre bundle $\Sigma \to X$ define a connection on the composite bundle $Y \to X$ as the composition of bundle morphisms

$$\gamma : Y \underset{X}{\times} TX \xrightarrow{(\mathrm{Id}\,,\Gamma)} Y \underset{\Sigma}{\times} T\Sigma \xrightarrow{A_\Sigma} TY.$$

It is called the *composite connection*. This composite connection reads

$$\gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma_\lambda^m \partial_m + (A_\lambda^i + A_m^i \Gamma_\lambda^m)\partial_i). \tag{3.6.5}$$

It is projectable onto Γ. Moreover, this is a unique connection such that the horizontal lift $\gamma\tau$ on Y of a vector field τ on X by means of the composite connection γ (3.6.5) coincides with the composition $A_\Sigma(\Gamma\tau)$ of horizontal lifts of τ on Σ by means of the connection Γ and then on Y by means of the connection A_Σ. For the sake of brevity, let us write $\gamma = A_\Sigma \circ \Gamma$.

Given a composite bundle Y (1.1.10), there are the exact sequences of vector bundles over Y:

$$0 \to V_\Sigma Y \longrightarrow VY \to Y \underset{\Sigma}{\times} V\Sigma \to 0, \tag{3.6.6}$$

$$0 \to Y \underset{\Sigma}{\times} V^*\Sigma \longrightarrow V^*Y \to V_\Sigma^* Y \to 0, \tag{3.6.7}$$

where $V_\Sigma Y$ and $V_\Sigma^* Y$ are the vertical tangent and the vertical cotangent bundles of $Y \to \Sigma$ which are coordinated by $(x^\lambda, \sigma^m, y^i, \dot y^i)$ and $(x^\lambda, \sigma^m, y^i, \dot y_i)$, respectively. Let us consider a splitting of these exact sequences

$$B : VY \ni \dot y^i \partial_i + \dot\sigma^m \partial_m \to (\dot y^i \partial_i + \dot\sigma^m \partial_m) \rfloor B = \tag{3.6.8}$$

$$(\dot y^i - \dot\sigma^m B^i_m)\partial_i \in V_\Sigma Y,$$
$$B : V^*_\Sigma Y \ni \overline{d}y^i \to B\rfloor \overline{d}y^i = \overline{d}y^i - B^i_m \overline{d}\sigma^m \in V^*Y, \tag{3.6.9}$$

given by the form

$$B = (\overline{d}y^i - B^i_m\overline{d}\sigma^m)\otimes\partial_i. \tag{3.6.10}$$

Then the connection γ (3.6.2) on $Y \to X$ and the splitting B (3.6.8) define the connection

$$A_\Sigma = B\circ\gamma : TY \to VY \to V_\Sigma Y, \tag{3.6.11}$$
$$A_\Sigma = dx^\lambda\otimes(\partial_\lambda + (\gamma^i_\lambda - B^i_m\gamma^m_\lambda)\partial_i) + d\sigma^m\otimes(\partial_m + B^i_m\partial_i),$$

on the fibre bundle $Y \to \Sigma$.

Conversely, every connection A_Σ (3.6.3) on the fibre bundle $Y \to \Sigma$ yields the splitting

$$A_\Sigma : TY \supset VY \ni \dot y^i\partial_i + \dot\sigma^m\partial_m \to (\dot y^i - A^i_m\dot\sigma^m)\partial_i \tag{3.6.12}$$

of the exact sequence (3.6.6). Using this splitting, one can construct a first order differential operator

$$\widetilde D : J^1Y \to T^*X \underset{Y}{\otimes} V_\Sigma Y, \tag{3.6.13}$$
$$\widetilde D = dx^\lambda\otimes(y^i_\lambda - A^i_\lambda - A^i_m\sigma^m_\lambda)\partial_i,$$

on the composite bundle $Y \to X$. It is called the *vertical covariant differential*. This operator also can be defined as the composition

$$\widetilde D = \mathrm{pr}_1\circ D^\gamma : J^1Y \to T^*X \underset{Y}{\otimes} VY \to T^*X \underset{Y}{\otimes} VY_\Sigma,$$

where D^γ is the covariant differential (3.2.6) relative to some composite connection $A_\Sigma\circ\Gamma$ (3.6.5), but $\widetilde D$ does not depend on the choice of a connection Γ on the fibre bundle $\Sigma \to X$.

The vertical covariant differential (3.6.13) possesses the following important property. Let h be a section of the fibre bundle $\Sigma \to X$, and let $Y^h \to X$ be the restriction (1.1.13) of the fibre bundle $Y \to \Sigma$ to $h(X) \subset \Sigma$. This is a subbundle

$$i_h : Y^h \to Y$$

of the fibre bundle $Y \to X$. Every connection A_Σ (3.6.3) induces the pull-back connection

$$A_h = i_h^* A_\Sigma = dx^\lambda \otimes [\partial_\lambda + ((A^i_m \circ h)\partial_\lambda h^m + (A \circ h)^i_\lambda)\partial_i] \tag{3.6.14}$$

on $Y^h \to X$. Then the restriction of $\widetilde{D}$ (3.6.13) to

$$J^1 i_h(J^1 Y^h) \subset J^1 Y$$

coincides with the familiar covariant differential D^{A_h} (3.2.6) on Y^h relative to the pull-back connection A_h (3.6.14).

Remark 3.6.1: Let $\Gamma : Y \to J^1Y$ be a connection on a fibre bundle $Y \to X$. In accordance with the canonical isomorphism $VJ^1Y = J^1VY$ (2.1.9), the vertical tangent map

$$V\Gamma : VY \to VJ^1Y$$

to Γ defines the connection

$$\begin{aligned} &V\Gamma : VY \to J^1VY, \\ &V\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i + \partial_j \Gamma^i_\lambda \dot{y}^j \dot{\partial}_i), \end{aligned} \tag{3.6.15}$$

on the composite vertical tangent bundle

$$VY \to Y \to X.$$

This is called the *vertical connection* to Γ. Of course, the connection $V\Gamma$ projects onto Γ. Moreover, $V\Gamma$ is linear over Γ. Then the dual connection of $V\Gamma$ on the composite vertical cotangent bundle

$$V^*Y \to Y \to X$$

reads

$$\begin{aligned} &V^*\Gamma : V^*Y \to J^1V^*Y, \\ &V^*\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i - \partial_j \Gamma^i_\lambda \dot{y}_i \dot{\partial}^j). \end{aligned} \tag{3.6.16}$$

It is called the *covertical connection* to Γ. If $Y \to X$ is an affine bundle, the connection $V\Gamma$ (3.6.15) can be seen as the composite connection generated by the connection Γ on $Y \to X$ and the linear connection

$$\widetilde{\Gamma} = dx^\lambda \otimes (\partial_\lambda + \partial_j \Gamma^i_\lambda \dot{y}^j \dot{\partial}_i) + dy^i \otimes \partial_i \tag{3.6.17}$$

on the vertical tangent bundle $VY \to Y$. □

Chapter 4

Geometry of principal bundles

Classical gauge theory is adequately formulated as Lagrangian field theory on principal and associated bundles where gauge potentials are identified with principal connections. The main ingredient in this formulation is the bundle of principal connections $C = J^1P/G$ whose sections are principal connections on a principal bundle P with a structure group G.

4.1 Geometry of Lie groups

Let G be a topological group which is not reduced to the unit $\mathbf{1}$. Let V be a topological space. By a *continuous action* of G on V *on the left* is meant a continuous map

$$\zeta : G \times V \to V, \qquad \zeta(g'g, v) = \zeta(g', \zeta(g, v)), \tag{4.1.1}$$

If there is no danger of confusion, we denote $\zeta(g, v) = gv$. One says that a group G acts on V *on the right* if the map (4.1.1) obeys the relations

$$\zeta(g'g, v) = \zeta(g, \zeta(g', v)).$$

In this case, we agree to write $\zeta(g, v) = vg$.

Remark 4.1.1: Strictly speaking, by an action of a group G on V is meant a class of morphisms ζ (4.1.1) which differ from each other in inner automorphisms of G, that is,

$$\zeta'(g,v) = \zeta(g'^{-1}gg',v)$$

for some element $g' \in G$. $\Box$

An action of G on V is called:

• *effective* if there is no $g \neq \mathbf{1}$ such that $\zeta(g,v) = v$ for all $v \in V$,

• *free* if, for any two elements $v, v \in V$, there exists an element $g \in G$ such that $\zeta(g,v) = v'$.

• *transitive* if there is no element $v \in V$ such that $\zeta(g,v) = v$ for all $g \in G$.

Unless otherwise stated, an action of a group is assumed to be effective. If an action ζ (4.1.1) of G on V is transitive, then V is called the *homogeneous space*, homeomorphic to the quotient $V = G/H$ of G with respect to some subgroup $H \subset G$. If an action ζ is both free and transitive, then V is homeomorphic to the group space of G. For instance, this is the case of action of G on itself by left ($\zeta = L_G$) and right ($\zeta = R_G$) multiplications.

Let G be a connected real Lie group of finite dimension $\dim G > 0$. A vector field ξ on G is called *left-invariant* if

$$\xi(g) = TL_g(\xi(\mathbf{1})), \qquad g \in G,$$

where TL_g denotes the tangent morphism to the map

$$L_g : G \to gG.$$

Accordingly, *right-invariant* vector fields ξ on G obey the condition

$$\xi(g) = TR_g(\xi(\mathbf{1})),$$

where TR_g is the tangent morphism to the map

$$T_g : G \to gG.$$

Let $\mathfrak{g}_l$ (resp. $\mathfrak{g}_r$) denote the Lie algebra of left-invariant (resp. right-invariant) vector fields on G. They are called the *left* and *right* Lie algebras of G, respectively. Every left-invariant vector field $\xi_l(g)$ (resp. a right-invariant vector field $\xi_r(g)$) can be associated to the element $v = \xi_l(\mathbf{1})$ (resp. $v = \xi_r(\mathbf{1})$) of the tangent space $T_{\mathbf{1}}G$ at the unit $\mathbf{1}$ of G. Accordingly, this tangent space is provided both with left and right Lie algebra structures. Given $v \in T_{\mathbf{1}}G$, let $v_l(g)$ and $v_r(g)$ be the corresponding left-invariant and right-invariant vector fields on G, respectively. There is the relation

$$v_l(g) = (TL_g \circ TR_g^{-1})(v_r(g)).$$

Let $\{\epsilon_m = \epsilon_m(\mathbf{1})\}$ (resp. $\{\varepsilon_m = \varepsilon_m(\mathbf{1})\}$) denote the basis for the left (resp. right) Lie algebra, and let c^k_{mn} be the *right structure constants*:

$$[\varepsilon_m, \varepsilon_n] = c^k_{mn}\varepsilon_k.$$

The map $g \to g^{-1}$ yields an isomorphism

$$\mathfrak{g}_l \ni \epsilon_m \to \varepsilon_m = -\epsilon_m \in \mathfrak{g}_r$$

of left and right Lie algebras.

The tangent bundle

$$\pi_G : TG \to G \tag{4.1.2}$$

of a Lie group G is trivial because of the isomorphisms

$$\varrho_l : TG \ni q \to (g = \pi_G(q), TL_g^{-1}(q)) \in G \times \mathfrak{g}_l,$$
$$\varrho_r : TG \ni q \to (g = \pi_G(q), TR_g^{-1}(q)) \in G \times \mathfrak{g}_r.$$

Let ζ (4.1.1) be a smooth action of a Lie group G on a smooth manifold V. Let us consider the tangent morphism

$$T\zeta : TG \times TV \to TV \tag{4.1.3}$$

to this action. Given an element $g \in G$, the restriction of $T\zeta$ (4.1.3) to $(g, 0) \times TV$ is the tangent morphism $T\zeta_g$ to the map

$$\zeta_g : g \times V \to V.$$

Therefore, the restriction

$$T\zeta_G : \widehat{0}(G) \times TV \to TV \tag{4.1.4}$$

of the tangent morphism $T\zeta$ (4.1.3) to $\widehat{0}(G)\times TV$ (where $\widehat{0}$ is the canonical zero section of $TG \to G$) is called the *tangent prolongation* of a smooth action of G on V.

In particular, the above mentioned morphisms

$$TL_g = TL_G|_{(g,0)\times TG}, \qquad TR_g = TR_G|_{(g,0)\times TG}$$

are of this type. For instance, the morphism TL_G (resp. TR_G) (4.1.4) defines the adjoint representation $g \to \mathrm{Ad}_g$ (resp. $g \to \mathrm{Ad}_{g^{-1}}$) of a group G in its right Lie algebra $\mathfrak{g}_r$ (resp. left Lie algebra $\mathfrak{g}_l$) and the identity representation in its left (resp. right) one.

Restricting $T\zeta$ (4.1.3) to $T_{\mathbf{1}}G\times \widehat{0}(V)$, one obtains a homomorphism of the right (resp. left) Lie algebra of G to the Lie algebra $\mathcal{T}(V)$ of vector field on V if ζ is a left (resp. right) action. We call this homomorphism a *representation* of the Lie algebra of G in V. For instance, a vector field on a manifold V associated to a local one-parameter group G of diffeomorphisms of V (see Section 1.3) is exactly an image of such a homomorphism of the one-dimensional Lie algebra of G to $\mathcal{T}(V)$.

In particular, the adjoint representation Ad_g of a Lie group G in its right Lie algebra $\mathfrak{g}_r$ yields the corresponding *adjoint representation*

$$\varepsilon' : \varepsilon \to \mathrm{ad}_{\varepsilon'}(\varepsilon) = [\varepsilon', \varepsilon], \qquad \mathrm{ad}_{\varepsilon_m}(\varepsilon_n) = c^k_{mn}\varepsilon_k, \tag{4.1.5}$$

of the right Lie algebra $\mathfrak{g}_r$ in itself. Accordingly, the adjoint representation of the left Lie algebra $\mathfrak{g}_l$ in itself reads

$$\mathrm{ad}_{\epsilon_m}(\epsilon_n) = -c^k_{mn}\epsilon_k,$$

where c^k_{mn} are the right structure constants (4.1.5).

Remark 4.1.2: Let G be a *matrix group*, i.e., a subgroup of the algebra $M(V)$ of endomorphisms of some finite-dimensional vector space

V. Then its Lie algebras are Lie subalgebras of $M(V)$. In this case, the adjoint representation Ad_g of G reads

$$\mathrm{Ad}_g(e) = geg^{-1}, \qquad e \in \mathfrak{g}. \tag{4.1.6}$$

□

An exterior form ϕ on a Lie group G is said to be *left-invariant* (resp. *right-invariant*) if $\phi(\mathbf{1}) = L_g^*(\phi(g))$ (resp. $\phi(\mathbf{1}) = R_g^*(\phi(g))$). The exterior differential of a left-invariant (resp right-invariant) form is left-invariant (resp. right-invariant). In particular, the left-invariant one-forms satisfy the *Maurer–Cartan equation*

$$d\phi(\epsilon, \epsilon') = -\frac{1}{2}\phi([\epsilon, \epsilon']), \qquad \epsilon, \epsilon' \in \mathfrak{g}_l. \tag{4.1.7}$$

There is the *canonical $\mathfrak{g}_l$-valued left-invariant one-form*

$$\theta_l : T_{\mathbf{1}}G \ni \epsilon \to \epsilon \in \mathfrak{g}_l \tag{4.1.8}$$

on a Lie group G. The components θ_l^m of its decomposition $\theta_l = \theta_l^m \epsilon_m$ with respect to the basis for the left Lie algebra $\mathfrak{g}_l$ make up the basis for the space of left-invariant exterior one-forms on G:

$$\epsilon_m \rfloor \theta_l^n = \delta_m^n.$$

The Maurer–Cartan equation (4.1.7), written with respect to this basis, reads

$$d\theta_l^m = \frac{1}{2} c_{nk}^m \theta_l^n \wedge \theta_l^k.$$

4.2 Bundles with structure groups

Principal bundles are particular bundles with a structure group. Since equivalence classes of these bundles are topological invariants (see Theorem 4.2.5), we consider continuous bundles with a structure topological group.

Let G be a topological group. Let $\pi : Y \to X$ be a locally trivial continuous bundle (see Remark 1.1.1) whose typical fibre V is provided with a certain left action (4.1.1) of a topological group G (see Remark 4.1.1). Moreover, let Y admit an atlas

$$\Psi = \{(U_\alpha, \psi_\alpha), \varrho_{\alpha\beta}\}, \qquad \psi_\alpha = \varrho_{\alpha\beta}\psi_\beta, \tag{4.2.1}$$

whose transition functions $\varrho_{\alpha\beta}$ (1.1.3) factorize as

$$\varrho_{\alpha\beta} : U_\alpha \cap U_\beta \times V \longrightarrow U_\alpha \cap U_\beta \times (G \times V) \xrightarrow{\mathrm{Id}\times\zeta} U_\alpha \cap U_\beta \times V \tag{4.2.2}$$

through local continuous G-valued functions

$$\varrho^G_{\alpha\beta} : U_\alpha \cap U_\beta \to G \tag{4.2.3}$$

on X. This means that transition morphisms $\varrho_{\alpha\beta}(x)$ (1.1.6) are elements of G acting on V. Transition functions (4.2.2) are called *G-valued*.

Provided with an atlas (4.2.1) with G-valued transition functions, a locally trivial continuous bundle Y is called the *bundle with a structure group G* or, in brief, a *G-bundle*. Two G-bundles (Y, Ψ) and (Y, Ψ') are called equivalent if their atlases Ψ and Ψ' are equivalent. Atlases Ψ and Ψ' with G-valued transition functions are said to be *equivalent* iff, given a common cover $\{U_i\}$ of X for the union of these atlases, there exists a continuous G-valued function g_i on each U_i such that

$$\psi'_i(x) = g_i(x)\psi_i(x), \qquad x \in U_i. \tag{4.2.4}$$

Let $h(X, G, V)$ denote the set of equivalence classes of continuous bundles over X with a structure group G and a typical fibre V. In order to characterize this set, let us consider the presheaf $G^0_{\{U\}}$ of continuous G-valued functions on a topological space X. Let G^0_X be the sheaf of germs of these functions generated by the canonical presheaf $G^0_{\{U\}}$, and let $H^1(X; G^0_X)$ be the first cohomology of X with coefficients in G^0_X (see Remark 8.5.3). The group functions $\varrho^G_{\alpha\beta}$ (4.2.3) obey the cocycle condition

$$\varrho^G_{\alpha\beta}\varrho^G_{\beta\gamma} = \varrho^G_{\alpha\gamma}$$

on overlaps $U_\alpha \cap U_\beta \cap U_\gamma$ (cf. (8.5.12)) and, consequently, they form a one-cocycle $\{\varrho^G_{\alpha\beta}\}$ of the presheaf $G^0_{\{U\}}$. This cocycle is a representative of some element of the first cohomology $H^1(X; G^0_X)$ of X with coefficients in the sheaf G^0_X.

Thus, any atlas of a G-bundle over X defines an element of the cohomology set $H^1(X; G^0_X)$. Moreover, it follows at once from the condition (4.2.4) that equivalent atlases define the same element of $H^1(X; G^0_X)$. Thus, there is an injection

$$h(X, G, V) \to H^1(X; G^0_X) \tag{4.2.5}$$

of the set of equivalence classes of G-bundles over X with a typical fibre V to the first cohomology $H^1(X; G^0_X)$ of X with coefficients in the sheaf G^0_X. Moreover, the injection (4.2.5) is a bijection as follows.

Theorem 4.2.1: There is one-to-one correspondence between the equivalence classes of G-bundles over X with a typical fibre V and the elements of the cohomology set $H^1(X; G^0_X)$. □

The bijection (4.2.5) holds for G-bundles with any typical fibre V. Two G-bundles (Y, Ψ) and (Y', Ψ') over X with different typical fibres are called *associated* if the cocycles of transition functions of their atlases Ψ and Ψ' are representatives of the same element of the cohomology set $H^1(X; G^0_X)$. Then Theorem 4.2.1 can be reformulated as follows.

Theorem 4.2.2: There is one-to-one correspondence between the classes of associated G-bundles over X and the elements of the cohomology set $H^1(X; G^0_X)$. □

Let $f : X' \to X$ be a continuous map. Every continuous G-bundle $Y \to X$ yields the pull-back bundle $f^*Y \to X'$ (1.1.8) with the same structure group G. Therefore, f induces the map

$$[f] : H^1(X; G^0_X) \to H^1(X'; G^0_{X'}).$$

Theorem 4.2.3: Given a continuous G-bundle Y over a paracompact base X, let f_1 and f_2 be two continuous maps of X' to X. If these maps are homotopic, the pull-back G-bundles f_1^*Y and f_2^*Y over X' are equivalent. □

Let us return to smooth fibre bundles. Let G, $\dim G > 0$, be a real Lie group which acts on a smooth manifold V on the left. A smooth fibre bundle $\pi : Y \to X$ is called a *bundle with a structure group* G if it is a continuous G-bundle possessing a smooth atlas Ψ (4.2.1) whose transition functions factorize as those (4.2.1) through smooth G-valued functions (4.2.3).

Example 4.2.1: Any vector (resp. affine) bundle of fibre dimension $\dim V = m$ is a bundle with a structure group which is the general linear group $GL(m, \mathbb{R})$ (resp. the general affine group $GA(m, \mathbb{R})$). □

Let G_X^∞ be the sheaf of germs of smooth G-valued functions on X and $H^1(X; G_X^\infty)$ the first cohomology of a manifold X with coefficients in the sheaf G_X^∞. The following theorem is analogous to Theorem 4.2.2.

Theorem 4.2.4: There is one-to-one correspondence between the classes of associated smooth G-bundles over X and the elements of the cohomology set $H^1(X; G_X^\infty)$. □

Since a smooth manifold is paracompact, one can show the following.

Theorem 4.2.5: There is a bijection

$$H^1(X; G_X^\infty) = H^1(X; G_X^0), \tag{4.2.6}$$

where a Lie group G is treated as a topological group. □

The bijection (4.2.6) enables one to classify smooth G-bundles as the continuous ones by means of topological invariants.

4.3 Principal bundles

We restrict our consideration to smooth bundles with a structure Lie group of non-zero dimension.

Given a real Lie group G, let

$$\pi_P : P \to X \tag{4.3.1}$$

be a G-bundle whose typical fibre is the group space of G, which a group G acts on by left multiplications. It is called a *principal bundle* with a structure group G. Equivalently, a principal G-bundle is defined as a fibre bundle P (4.3.1) which admits an *action of* G *on* P on the right by a fibrewise morphism

$$\begin{aligned} &R_{GP} : G \underset{X}{\times} P \underset{X}{\longrightarrow} P, \\ &R_{gP} : p \to pg, \qquad \pi_P(p) = \pi_P(pg), \qquad p \in P, \end{aligned} \tag{4.3.2}$$

which is free and transitive on each fibre of P. As a consequence, the quotient of P with respect to the action (4.3.2) of G is diffeomorphic to a base X, i.e., $P/G = X$.

Remark 4.3.1: The definition of a *continuous principal bundle* is a repetition of that of a smooth one, but all morphisms are continuous. □

A principal G-bundle P is equipped with a bundle atlas

$$\Psi_P = \{(U_\alpha, \psi^P_\alpha), \varrho_{\alpha\beta}\} \tag{4.3.3}$$

whose trivialization morphisms

$$\psi^P_\alpha : \pi_P^{-1}(U_\alpha) \to U_\alpha \times G$$

obey the condition

$$\psi^P_\alpha(pg) = g\psi^P_\alpha(p), \qquad g \in G.$$

Due to this property, every trivialization morphism ψ^P_α determines a unique local section $z_\alpha : U_\alpha \to P$ such that

$$(\psi^P_\alpha \circ z_\alpha)(x) = \mathbf{1}, \qquad x \in U_\alpha.$$

The transformation law for z_α reads

$$z_\beta(x) = z_\alpha(x)\varrho_{\alpha\beta}(x), \qquad x \in U_\alpha \cap U_\beta. \tag{4.3.4}$$

Conversely, the family

$$\{(U_\alpha, z_\alpha), \varrho_{\alpha\beta}\} \tag{4.3.5}$$

of local sections of P which obey the transformation law (4.3.4) uniquely determines a bundle atlas Ψ_P of a principal bundle P.

Theorem 4.3.1: A principal bundle admits a global section iff it is trivial. □

Example 4.3.2: Let H be a closed subgroup of a real Lie group G. Then H is a Lie group. Let G/H be the quotient of G with respect to an action of H on G by right multiplications. Then

$$\pi_{GH} : G \to G/H \tag{4.3.6}$$

is a principal H-bundle. If H is a maximal compact subgroup of G, then G/H is diffeomorphic to $\mathbb{R}^m$ and, by virtue of Theorem 1.1.7, $G \to G/H$ is a trivial bundle, i.e., G is diffeomorphic to the product $\mathbb{R}^m \times H$. □

Remark 4.3.3: The pull-back f^*P (1.1.8) of a principal bundle also is a principal bundle with the same structure group. □

Remark 4.3.4: Let $P \to X$ and $P' \to X'$ be principal G- and G'-bundles, respectively. A bundle morphism $\Phi : P \to P'$ is a *morphism of principal bundles* if there exists a Lie group homomorphism $\gamma : G \to G'$ such that

$$\Phi(pg) = \Phi(p)\gamma(g).$$

In particular, equivalent principal bundles are isomorphic. □

Any class of associated smooth bundles on X with a structure Lie group G contains a principal bundle. In other words, any smooth bundle with a structure Lie group G is associated with some principal bundle.

Let us consider the tangent morphism

$$TR_{GP} : (G \times \mathfrak{g}_l) \underset{X}{\times} TP \underset{X}{\longrightarrow} TP \tag{4.3.7}$$

to the right action R_{GP} (4.3.2) of G on P. Its restriction to $T_{\mathbf{1}}G \underset{X}{\times} TP$ provides a homomorphism

$$\mathfrak{g}_l \ni \epsilon \to \xi_\epsilon \in \mathcal{T}(P) \tag{4.3.8}$$

of the left Lie algebra $\mathfrak{g}_l$ of G to the Lie algebra $\mathcal{T}(P)$ of vector fields on P. Vector fields ξ_ϵ (4.3.8) are obviously vertical. They are called *fundamental vector fields*. Given a basis $\{\epsilon_r\}$ for $\mathfrak{g}_l$, the corresponding fundamental vector fields $\xi_r = \xi_{\epsilon_r}$ form a family of $m = \dim \mathfrak{g}_l$ nowhere vanishing and linearly independent sections of the vertical tangent bundle VP of $P \to X$. Consequently, this bundle is trivial

$$VP = P \times \mathfrak{g}_l \tag{4.3.9}$$

by virtue of Theorem 1.2.1.

Restricting the tangent morphism TR_{GP} (4.3.7) to

$$TR_{GP} : \widehat{0}(G) \underset{X}{\times} TP \underset{X}{\longrightarrow} TP, \tag{4.3.10}$$

we obtain the *tangent prolongation* of the structure group action R_{GP} (4.3.2). If there is no danger of confusion, it is simply called the *action of G on TP*. Since the action of G (4.3.2) on P is fibrewise, its action (4.3.10) is restricted to the vertical tangent bundle VP of P.

Taking the quotient of the tangent bundle $TP \to P$ and the vertical tangent bundle VP of P by G (4.3.10), we obtain the vector bundles

$$T_GP = TP/G, \qquad V_GP = VP/G \tag{4.3.11}$$

over X. Sections of $T_GP \to X$ are G-invariant vector fields on P. Accordingly, sections of $V_GP \to X$ are G-invariant vertical vector fields on P. Hence, a typical fibre of $V_GP \to X$ is the right Lie algebra $\mathfrak{g}_r$ of G subject to the adjoint representation of a structure group G. Therefore, V_GP (4.3.11) is called the *Lie algebra bundle*.

Given a bundle atlas Ψ_P (4.3.3) of P, there is the corresponding atlas

$$\Psi = \{(U_\alpha, \psi_\alpha), \mathrm{Ad}_{\varrho_{\alpha\beta}}\} \tag{4.3.12}$$

of a Lie algebra fibre bundle V_GP, which is provided with bundle coordinates $(U_\alpha; x^\mu, \chi^m)$ with respect to the fibre frames

$$\{e_m = \psi_\alpha^{-1}(x)(\varepsilon_m)\},$$

where $\{\varepsilon_m\}$ is a basis for the Lie algebra $\mathfrak{g}_r$. These coordinates obey the transformation rule

$$\varrho(\chi^m)\varepsilon_m = \chi^m \mathrm{Ad}_{\varrho^{-1}}(\varepsilon_m). \tag{4.3.13}$$

A glance at this transformation rule shows that V_GP is a bundle with a structure group G. Moreover, it is associated with a principal G-bundle P (see Example 4.7.2).

Accordingly, the vector bundle T_GP (4.3.11) is endowed with bundle coordinates $(x^\mu, \dot{x}^\mu, \chi^m)$ with respect to the fibre frames $\{\partial_\mu, e_m\}$. Their transformation rule is

$$\varrho(\chi^m)\varepsilon_m = \chi^m \mathrm{Ad}_{\varrho^{-1}}(\varepsilon_m) + \dot{x}^\mu R_\mu^m \varepsilon_m. \tag{4.3.14}$$

If G is a matrix group (see Remark 4.1.2), this transformation rule reads

$$\varrho(\chi^m)\varepsilon_m = \chi^m \varrho^{-1}\varepsilon_m \varrho - \dot{x}^\mu \partial_\mu(\varrho^{-1})\varrho. \tag{4.3.15}$$

Since the second term in the right-hand sides of expressions (4.3.14) – (4.3.15) depend on derivatives of a G-valued function ϱ on X, the vector bundle T_GP (4.3.11) fails to be a G-bundle.

The Lie bracket of G-invariant vector fields on P goes to the quotient by G and defines the Lie bracket of sections of the vector bundle $T_GP \to X$. This bracket reads

$$\xi = \xi^\lambda \partial_\lambda + \xi^p e_p, \qquad \eta = \eta^\mu \partial_\mu + \eta^q e_q, \tag{4.3.16}$$

$$[\xi, \eta] = (\xi^\mu \partial_\mu \eta^\lambda - \eta^\mu \partial_\mu \xi^\lambda)\partial_\lambda + \tag{4.3.17}$$
$$(\xi^\lambda \partial_\lambda \eta^r - \eta^\lambda \partial_\lambda \xi^r + c^r_{pq}\xi^p \eta^q) e_r.$$

Putting $\xi^\lambda = 0$ and $\eta^\mu = 0$ in the formulas (4.3.16) – (4.3.17), we obtain the Lie bracket

$$[\xi, \eta] = c^r_{pq}\xi^p \eta^q e_r \tag{4.3.18}$$

of sections of the Lie algebra bundle V_GP. A glance at the expression (4.3.18) shows that sections of V_GP form a finite-dimensional Lie $C^\infty(X)$-algebra, called the *gauge algebra*. Therefore, V_GP also is called the *gauge algebra bundle*.

4.4 Principal connections

Principal connections on a principal bundle P (4.3.1) are connections on P which are equivariant with respect to the right action (4.3.2) of a structure group G on P. In order to describe them, we follow the definition of connections on a fibre bundle $Y \to X$ as global sections of the affine jet bundle $J^1Y \to X$ (Theorem 3.2.1).

Let J^1P be the first order jet manifold of a principal G-bundle $P \to X$ (4.3.1). Then connections on a principal bundle $P \to X$ are global sections

$$A : P \to J^1P \tag{4.4.1}$$

of the affine jet bundle $J^1P \to P$ modelled over the vector bundle

$$T^*X \underset{P}{\otimes} VP = (T^*X \underset{P}{\otimes} \mathfrak{g}_l).$$

In order to define principal connections on $P \to X$, let us consider the jet prolongation

$$J^1R_G : J^1(X \times G) \underset{X}{\times} J^1P \to J^1P$$

of the morphism R_{GP} (4.3.2). Restricting this morphism to

$$J^1R_G : \widehat{0}(G) \underset{X}{\times} J^1P \to J^1P,$$

we obtain the *jet prolongation* of the structure group action R_{GP} (4.3.2) called, simply, the *action of G on J^1P*. It reads

$$G \ni g : j^1_xp \to (j^1_xp)g = j^1_x(pg). \tag{4.4.2}$$

Taking the quotient of the affine jet bundle J^1P by G (4.4.2), we obtain the affine bundle

$$C = J^1P/G \to X \tag{4.4.3}$$

modelled over the vector bundle

$$\overline{C} = T^*X \underset{X}{\otimes} V_GP \to X.$$

Hence, there is the vertical splitting

$$VC = C \underset{X}{\otimes} \overline{C}$$

of the vertical tangent bundle VC of $C \to X$.

Remark 4.4.1: A glance at the expression (4.4.2) shows that the fibre bundle $J^1P \to C$ is a principal bundle with the structure group G. It is canonically isomorphic to the pull-back

$$J^1P = P_C = C \underset{X}{\times} P \to C. \tag{4.4.4}$$

□

Taking the quotient with respect to the action of a structure group G, one can reduce the canonical imbedding (2.1.5) (where $Y = P$) to the bundle monomorphism

$$\begin{aligned} &\lambda_C : C \underset{X}{\longrightarrow} T^*X \underset{X}{\otimes} T_GP, \\ &\lambda_C : dx^\mu \otimes (\partial_\mu + \chi_\mu^m e_m). \end{aligned} \tag{4.4.5}$$

It follows that, given atlases Ψ_P (4.3.3) of P and Ψ (4.3.12) of T_GP, the bundle of principal connections C is provided with bundle coordinates (x^λ, a_μ^m) possessing the transformation rule

$$\varrho(a_\mu^m)\varepsilon_m = (a_\nu^m \mathrm{Ad}_{\varrho^{-1}}(\varepsilon_m) + R_\nu^m \varepsilon_m)\frac{\partial x^\nu}{\partial x'^\mu}. \tag{4.4.6}$$

If G is a matrix group, this transformation rule reads

$$\varrho(a_\mu^m)\varepsilon_m = (a_\nu^m \varrho^{-1}(\varepsilon_m)\varrho - \partial_\mu(\varrho^{-1})\varrho)\frac{\partial x^\nu}{\partial x'^\mu}. \tag{4.4.7}$$

A glance at this expression shows that the bundle of principal connections C as like as the vector bundle T_GP (4.3.11) fails to be a bundle with a structure group G.

As was mentioned above, a connection A (4.4.1) on a principal bundle $P \to X$ is called a *principal connection* if it is *equivariant* under the action (4.4.2) of a structure group G, i.e.,

$$A(pg) = A(p)g \qquad g \in G. \tag{4.4.8}$$

There is obvious one-to-one correspondence between the principal connections on a principal G-bundle P and global sections

$$A : X \to C \tag{4.4.9}$$

of the bundle $C \to X$ (4.4.3), called the *bundle of principal connections*.

Theorem 4.4.1: Since the bundle of principal connections $C \to X$ is affine, principal connections on a principal bundle always exist. □

Due to the bundle monomorphism (4.4.5), any principal connection A (4.4.9) is represented by a T_GP-valued form

$$A = dx^\lambda \otimes (\partial_\lambda + A^q_\lambda e_q). \tag{4.4.10}$$

Taking the quotient with respect to the action of a structure group G, one can reduce the exact sequence (1.2.10) (where $Y = P$) to the exact sequence

$$0 \to V_GP \underset{X}{\longrightarrow} T_GP \longrightarrow TX \to 0. \tag{4.4.11}$$

A principal connection A (4.4.10) defines a splitting of this exact sequence.

Remark 4.4.2: A principal connection A (4.4.1) on a principal bundle $P \to X$ can be represented by the vertical-valued form A (3.1.9) on P which is a $\mathfrak{g}_l$-valued form due to the trivialization (4.3.9). It is the familiar $\mathfrak{g}_l$-valued *connection form* on a principal bundle P. Given a local bundle splitting (U_α, z_α) of P, this form reads

$$\overline{A} = \psi_\alpha^*(\theta_l - \overline{A}_\lambda^q dx^\lambda \otimes \epsilon_q), \tag{4.4.12}$$

where θ_l is the canonical $\mathfrak{g}_l$-valued one-form (4.1.8) on G and A_λ^p are local functions on P such that

$$\overline{A}_\lambda^q(pg)\epsilon_q = \overline{A}_\lambda^q(p)\mathrm{Ad}_{g^{-1}}(\epsilon_q).$$

The pull-back $z_\alpha^*\overline{A}$ of the connection form $\overline{A}$ (4.4.12) onto U_α is the well-known *local connection one-form*

$$A_\alpha = -A_\lambda^q dx^\lambda \otimes \epsilon_q = A_\lambda^q dx^\lambda \otimes \varepsilon_q, \tag{4.4.13}$$

where $A_\lambda^q = \overline{A}_\lambda^q \circ z_\alpha$ are local functions on X. It is readily observed that the coefficients A_λ^q of this form are exactly the coefficients of the form (4.4.10). □

There are both pull-back and push-forward operations of principal connections.

Theorem 4.4.2: Let P be a principal bundle and f^*P (1.1.8) the pull-back principal bundle with the same structure group. Let f_P be the canonical morphism (1.1.9) of f^*P to P. If A is a principal connection on P, then the pull-back connection f^*A (3.1.12) on f^*P is a principal connection. □

Theorem 4.4.3: Let $P' \to X$ and $P \to X$ be principle bundles with structure groups G' and G, respectively. Let $\Phi : P' \to P$ be a principal bundle morphism over X with the corresponding homomorphism $G' \to G$ (see Remark 4.3.4). For every principal connection A' on P', there exists a unique principal connection A on P such that $T\Phi$ sends the horizontal subspaces of TP' A' onto the horizontal subspaces of TP with respect to A. □

Let $P \to X$ be a principal G-bundle. The Frölicher–Nijenhuis bracket (1.4.8) on the space $\mathcal{O}^*(P) \otimes \mathcal{T}(P)$ of tangent-valued forms on P is compatible with the right action R_{GP} (4.3.2). Therefore, it induces the Frölicher–Nijenhuis bracket on the space $\mathcal{O}^*(X)\otimes T_GP(X)$ of T_GP-valued forms on X, where $T_GP(X)$ is the vector space of sections of the vector bundle $T_GP \to X$. Note that, as it follows from the exact sequence (4.4.11), there is an epimorphism

$$T_GP(X) \to \mathcal{T}(X).$$

Let $A \in \mathcal{O}^1(X) \otimes T_GP(X)$ be a principal connection (4.4.10). The associated Nijenhuis differential is

$$d_A : \mathcal{O}^r(X) \otimes T_GP(X) \to \mathcal{O}^{r+1}(X) \otimes V_GP(X),$$
$$d_A\phi = [A, \phi]_{\mathrm{FN}}, \quad \phi \in \mathcal{O}^r(X) \otimes T_GP(X). \tag{4.4.14}$$

The *strength* of a principal connection A (4.4.10) is defined as the V_GP-valued two-form

$$F_A = \frac{1}{2} d_A A = \frac{1}{2}[A, A]_{\mathrm{FN}} \in \mathcal{O}^2(X) \otimes V_GP(X). \tag{4.4.15}$$

Its coordinated expression

$$\begin{aligned}
&F_A = \frac{1}{2} F^r_{\lambda\mu} dx^\lambda \wedge dx^\mu \otimes e_r,\\
&F^r_{\lambda\mu} = [\partial_\lambda + A^p_\lambda e_p, \partial_\mu + A^q_\mu e_q]^r =\\
&\qquad \partial_\lambda A^r_\mu - \partial_\mu A^r_\lambda + c^r_{pq} A^p_\lambda A^q_\mu,
\end{aligned} \tag{4.4.16}$$

results from the bracket (4.3.17).

Remark 4.4.3: It should be emphasized that the strength F_A (4.4.15) is not the standard curvature (3.3.1) of a principal connection because A (4.4.10) is not a tangent-valued form. The *curvature of a principal connection* A (4.4.1) on P is the VP-valued two-form R (3.3.1) on P, which is brought into the $\mathfrak{g}_l$-valued form owing to the canonical isomorphism (4.3.9). □

Remark 4.4.4: Given a principal connection A (4.4.9), let Φ_C be a vertical principal automorphism of the bundle of principal connections C.

The connection $A' = \Phi_C \circ A$ is called *conjugate* to a principal connection A. The strength forms (4.4.15) of conjugate principal connections A and A' coincide with each other, i.e., $F_A = F_{A'}$. □

4.5 Canonical principal connection

Given a principal G-bundle $P \to X$ and its jet manifold J^1P, let us consider the canonical morphism $\theta_{(1)}$ (2.1.5) where $Y = P$. By virtue of Remark 1.2.2, this morphism defines the morphism

$$\theta : J^1P \underset{P}{\times} TP \to VP.$$

Taking its quotient with respect to G, we obtain the morphism

$$\begin{aligned} &C \underset{X}{\times} T_GP \xrightarrow{\theta} V_GP, \\ &\theta(\partial_\lambda) = -a^p_\lambda e_p, \qquad \theta(e_p) = e_p. \end{aligned} \tag{4.5.1}$$

Consequently, the exact sequence (4.4.11) admits the canonical splitting over C.

In view of this fact, let us consider the pull-back principal G-bundle P_C (4.4.4). Since

$$V_G(C \underset{X}{\times} P) = C \underset{X}{\times} V_GP, \qquad T_G(C \underset{X}{\times} P) = TC \underset{X}{\times} T_GP, \tag{4.5.2}$$

the exact sequence (4.4.11) for the principal bundle P_C reads

$$0 \to C \underset{X}{\times} V_GP \underset{C}{\longrightarrow} TC \underset{X}{\times} T_GP \longrightarrow TC \to 0. \tag{4.5.3}$$

The morphism (4.5.1) yields the horizontal splitting (3.1.3):

$$TC \underset{X}{\times} T_GP \longrightarrow C \underset{X}{\times} T_GP \longrightarrow C \underset{X}{\times} V_GP,$$

of the exact sequence (4.5.3). Thus, it defines the principal connection

$$\begin{aligned} &\mathcal{A} : TC \to TC \underset{X}{\times} T_GP, \\ &\mathcal{A} = dx^\lambda \otimes (\partial_\lambda + a^p_\lambda e_p) + da^r_\lambda \otimes \partial^\lambda_r, \\ &\mathcal{A} \in \mathcal{O}^1(C) \otimes T_G(C \underset{X}{\times} P)(X), \end{aligned} \tag{4.5.4}$$

on the principal bundle

$$P_C = C \underset{X}{\times} P \to C. \tag{4.5.5}$$

It follows that the principal bundle P_C admits the *canonical principal connection* (4.5.4).

Following the expression (4.4.15), let us define the strength

$$F_\mathcal{A} = \frac{1}{2} d_\mathcal{A}\mathcal{A} = \frac{1}{2}[\mathcal{A}, \mathcal{A}] \in \mathcal{O}^2(C) \otimes V_G P(X),$$
$$F_\mathcal{A} = (da^r_\mu \wedge dx^\mu + \frac{1}{2} c^r_{pq} a^p_\lambda a^q_\mu dx^\lambda \wedge dx^\mu) \otimes e_r, \tag{4.5.6}$$

of the canonical principal connection $\mathcal{A}$ (4.5.4). It is called the *canonical strength* because, given a principal connection A (4.4.9) on a principal bundle $P \to X$, the pull-back

$$A^* F_\mathcal{A} = F_A \tag{4.5.7}$$

is the strength (4.4.16) of A.

With the $V_G P$-valued two-form $F_\mathcal{A}$ (4.5.6) on C, let us define the $V_G P$-valued horizontal two-form

$$\mathcal{F} = h_0(F_\mathcal{A}) = \frac{1}{2} \mathcal{F}^r_{\lambda\mu} dx^\lambda \wedge dx^\mu \otimes \varepsilon_r,$$
$$\mathcal{F}^r_{\lambda\mu} = a^r_{\lambda\mu} - a^r_{\mu\lambda} + c^r_{pq} a^p_\lambda a^q_\mu, \tag{4.5.8}$$

on $J^1 C$. It is called the *strength form*. For each principal connection A (4.4.9) on P, the pull-back

$$J^1 A^* \mathcal{F} = F_A \tag{4.5.9}$$

is the strength (4.4.16) of A.

The strength form (4.5.8) yields an affine surjection

$$\mathcal{F}/2 : J^1 C \underset{C}{\longrightarrow} C \underset{X}{\times} (\overset{2}{\wedge} T^* X \otimes V_G P) \tag{4.5.10}$$

over C of the affine jet bundle $J^1 C \to C$ to the vector (and, consequently, affine) bundle

$$C \underset{X}{\times} (\overset{2}{\wedge} T^* X \otimes V_G P) \to C.$$

By virtue of Theorem 1.1.10, its kernel $C_+ = \mathrm{Ker}\,\mathcal{F}/2$ is an affine subbundle of $J^1C \to C$. Thus, we have the canonical splitting of the affine jet bundle

$$J^1C = C_+ \underset{C}{\oplus} C_- = C_+ \underset{C}{\oplus} (C \underset{X}{\times} \overset{2}{\wedge} T^*X \otimes V_GP), \tag{4.5.11}$$

$$a^r_{\lambda\mu} = \frac{1}{2}(\mathcal{F}^r_{\lambda\mu} + \mathcal{S}^r_{\lambda\mu}) = \frac{1}{2}(a^r_{\lambda\mu} + a^r_{\mu\lambda} - c^r_{pq}a^p_\lambda a^q_\mu) + \tag{4.5.12}$$
$$\frac{1}{2}(a^r_{\lambda\mu} - a^r_{\mu\lambda} + c^r_{pq}a^p_\lambda a^q_\mu).$$

The jet manifold J^1C plays a role of the configuration space of classical gauge theory on principal bundles.

4.6 Gauge transformations

In classical gauge theory, *gauge transformations* are defined as principal automorphisms of a principal bundle P. In accordance with Remark 4.3.4, an automorphism Φ_P of a principal G-bundle P is called *principal* if it is *equivariant* under the right action (4.3.2) of a structure group G on P, i.e.,

$$\Phi_P(pg) = \Phi_P(p)g, \qquad g \in G, \qquad p \in P. \tag{4.6.1}$$

In particular, every vertical principal automorphism of a principal bundle P is represented as

$$\Phi_P(p) = pf(p), \qquad p \in P, \tag{4.6.2}$$

where f is a G-valued *equivariant function* on P, i.e.,

$$f(pg) = g^{-1}f(p)g, \qquad g \in G. \tag{4.6.3}$$

There is one-to-one correspondence between the equivariant functions f (4.6.3) and the global sections s of the associated *group bundle*

$$\pi_{P^G} : P^G \to X \tag{4.6.4}$$

whose fibres are groups isomorphic to G and whose typical fibre is the group G which acts on itself by the adjoint representation. This one-to-one correspondence is defined by the relation

$$s(\pi_P(p))p = pf(p), \qquad p \in P, \tag{4.6.5}$$

(see Example 4.7.3). The group of vertical principal automorphisms of a principal G-bundle is called the *gauge group*. It is isomorphic to the group $P^G(X)$ of global sections of the group bundle (4.6.4). Its unit element is the canonical global section $\widehat{\mathbf{1}}$ of $P^G \to X$ whose values are unit elements of fibres of P^G.

Remark 4.6.1: Note that transition functions of atlases of a principle bundle P also are represented by local sections of the associated group bundle P^G (4.6.4). $\square$

Let us consider (local) one-parameter groups of principal automorphisms of P. Their infinitesimal generators are G-invariant projectable vector fields ξ on P, and *vice versa*. We call ξ the *principal vector fields* or the *infinitesimal gauge transformations*. They are represented by sections ξ (4.3.16) of the vector bundle T_GP (4.3.11). Principal vector fields constitute a real Lie algebra $T_GP(X)$ with respect to the Lie bracket (4.3.17). *Vertical principal vector fields* are the sections

$$\xi = \xi^p e_p \tag{4.6.6}$$

of the gauge algebra bundle $V_GP \to X$ (4.3.11). They form a finite-dimensional Lie $C^\infty(X)$-algebra $\mathcal{G}(X) = V_GP(X)$ (4.3.18) that has been called the gauge algebra.

Any (local) one-parameter group of principal automorphism Φ_P (4.6.1) of a principal bundle P admits the jet prolongation $J^1\Phi_P$ (2.1.7) to a one-parameter group of G-equivariant automorphism of the jet manifold J^1P which, in turn, yields a one-parameter group of *principal automorphisms* Φ_C of the bundle of principal connections C (4.4.3). Its infinitesimal generator is a vector field on C, called the principal vector field on

C and regarded as an infinitesimal gauge transformation of C. Thus, any principal vector field ξ (4.3.16) on P yields a principal vector field u_ξ on C, which can be defined as follows.

Using the morphism (4.5.1), we obtain the morphism

$$\xi\rfloor\theta : C \underset{X}{\longrightarrow} V_G P,$$

which is a section of of the Lie algebra bundle

$$V_G(C \underset{X}{\times} P) \to C$$

in accordance with the first formula (4.5.2). Then the equation

$$u_\xi\rfloor F_{\mathcal{A}} = d_{\mathcal{A}}(\xi\rfloor\theta)$$

uniquely determines a desired vector field u_ξ on C. A direct computation leads to

$$u_\xi = \xi^\mu\partial_\mu + (\partial_\mu\xi^r + c^r_{pq}a^p_\mu\xi^q - a^r_\nu\partial_\mu\xi^\nu)\partial^\mu_r. \tag{4.6.7}$$

In particular, if ξ is a vertical principal field (4.6.6), we obtain the vertical vector field

$$u_\xi = (\partial_\mu\xi^r + c^r_{pq}a^p_\mu\xi^q)\partial^\mu_r. \tag{4.6.8}$$

Remark 4.6.2: The jet prolongation (2.1.8) of the vector field u_ξ (4.6.7) onto J^1C reads

$$\begin{aligned} J^1u_\xi = u_\xi + (\partial_{\lambda\mu}\xi^r + c^r_{pq}a^p_\mu\partial_\lambda\xi^q + c^r_{pq}a^p_{\lambda\mu}\xi^q - \\ a^r_\nu\partial_{\lambda\mu}\xi^\nu - a^r_{\lambda\nu}\partial_\mu\xi^\nu - a^r_{\nu\mu}\partial_\lambda\xi^\nu)\partial^{\lambda\mu}_r. \end{aligned} \tag{4.6.9}$$

□

Example 4.6.3: Let A (4.4.10) be a principal connection on P. For any vector field τ on X, this connection yields a section

$$\tau\rfloor A = \tau^\lambda\partial_\lambda + A^p_\lambda\tau^\lambda e_p$$

of the vector bundle $T_GP \to X$. It, in turn, defines a principal vector field (4.6.7) on the bundle of principal connection C which reads

$$\tau_A = \tau^\lambda \partial_\lambda + (\partial_\mu(A^r_\nu \tau^\nu) + c^r_{pq} a^p_\mu A^q_\nu \tau^\nu - a^r_\nu \partial_\mu \tau^\nu)\partial^\mu_r, \tag{4.6.10}$$
$$\xi^\lambda = \tau^\lambda, \qquad \xi^p = A^p_\nu \tau^\nu.$$

□

It is readily justified that the monomorphism

$$T_GP(X) \ni \xi \to u_\xi \in \mathcal{T}(C) \tag{4.6.11}$$

obeys the equality

$$u_{[\xi,\eta]} = [u_\xi, u_\eta], \tag{4.6.12}$$

i.e., it is a monomorphism of the real Lie algebra $T_GP(X)$ to the real Lie algebra $\mathcal{T}(C)$. In particular, the image of the gauge algebra $\mathcal{G}(X)$ in $\mathcal{T}(C)$ also is a real Lie algebra, but not the $C^\infty(X)$-one because

$$u_{f\xi} \neq f u_\xi, \qquad f \in C^\infty(X).$$

Remark 4.6.4: A glance at the expression (4.6.7) shows that the monomorphism (4.6.11) is a linear first order differential operator which sends sections of the pull-back bundle

$$C \underset{X}{\times} T_GP \to C$$

onto sections of the tangent bundle $TC \to C$. Refereing to Definition 7.2.10, we therefore can treat principal vector fields (4.6.7) as infinitesimal gauge transformations depending on gauge parameters $\xi \in T_GP(X)$.
□

4.7 Geometry of associated bundles

Given a principal G-bundle P (4.3.1), any associated G-bundle over X with a typical fibre V is equivalent to the following one.

Let us consider the quotient

$$Y = (P \times V)/G \tag{4.7.1}$$

of the product $P \times V$ by identification of elements (p, v) and $(pg, g^{-1}v)$ for all $g \in G$. Let $[p]$ denote the restriction of the canonical surjection

$$P \times V \to (P \times V)/G \tag{4.7.2}$$

to the subset $\{p\} \times V$ so that

$$[p](v) = [pg](g^{-1}v).$$

Then the map

$$Y \ni [p](V) \to \pi_P(p) \in X$$

makes the quotient Y (4.7.1) into a fibre bundle over X. This is a smooth G-bundle with the typical fibre V which is associated with the principal G-bundle P. For short, we call it the *P-associated bundle*.

Remark 4.7.1: The tangent morphism to the morphism (4.7.2) and the jet prolongation of the morphism (4.7.2) lead to the bundle isomorphisms

$$TY = (TP \times TV)/G, \tag{4.7.3}$$

$$J^1Y = (J^1P \times V)/G. \tag{4.7.4}$$

□

The peculiarity of the P-associated bundle Y (4.7.1) is the following.

(i) Every bundle atlas $\Psi_P = \{(U_\alpha, z_\alpha)\}$ (4.3.5) of P defines a unique *associated bundle atlas*

$$\Psi = \{(U_\alpha, \psi_\alpha(x) = [z_\alpha(x)]^{-1})\} \tag{4.7.5}$$

of the quotient Y (4.7.1).

Example 4.7.2: Because of the splitting (4.3.9), the Lie algebra bundle

$$V_G P = (P \times \mathfrak{g}_l)/G,$$

by definition, is of the form (4.7.1). Therefore, it is a P-associated bundle. □

Example 4.7.3: The group bundle $\overline{P}$ (4.6.4) is defined as the quotient

$$P^G = (P \times G)/G, \tag{4.7.6}$$

where the group G which acts on itself by the adjoint representation. There is the following fibre-to-fibre action of the group bundle P^G on any P-associated bundle Y (4.7.1):

$$\begin{array}{l} P^G \underset{X}{\times} Y \underset{X}{\longrightarrow} Y, \\ ((p,g)/G,(p,v)/G) \to (p,gv)/G, \qquad g \in G, \qquad v \in V. \end{array}$$

For instance, the action of P^G on P in the formula (4.6.5) is of this type. □

(ii) Any principal automorphism Φ_P (4.6.1) of P yields a unique *principal automorphism*

$$\Phi_Y : (p,v)/G \to (\Phi_P(p),v)/G, \qquad p \in P, \qquad v \in V, \tag{4.7.7}$$

of the P-associated bundle Y (4.7.1). For the sake of brevity, we agree to write

$$\Phi_Y : (P \times V)/G \to (\Phi_P(P) \times V)/G.$$

Accordingly, any (local) one-parameter group of principal automorphisms of P induces a (local) one-parameter group of automorphisms of the P-associated bundle Y (4.7.1). Passing to infinitesimal generators of these groups, we obtain that any principal vector field ξ (4.3.16) yields a vector field υ_ξ on Y regarded as an infinitesimal gauge transformation of Y. Owing to the bundle isomorphism (4.7.3), we have

$$\begin{array}{l} \upsilon_\xi : X \to (\xi(P) \times TV)/G \subset TY, \\ \upsilon_\xi = \xi^\lambda \partial_\lambda + \xi^p I_p^i \partial_i, \end{array} \tag{4.7.8}$$

where $\{I_p\}$ is a representation of the Lie algebra $\mathfrak{g}_r$ of G in V.

(iii) Any principal connection on $P \to X$ defines a unique connection on the P-associated fibre bundle Y (4.7.1) as follows. Given a principal connection A (4.4.8) on P and the corresponding horizontal distribution $HP \subset TP$, the tangent map to the canonical morphism (4.7.2) defines the horizontal splitting of the tangent bundle TY of Y (4.7.1) and the corresponding connection on $Y \to X$. Owing to the bundle isomorphism (4.7.4), we have

$$A : (P \times V)/G \to (A(P) \times V)/G \subset J^1Y,$$
$$A = dx^\lambda \otimes (\partial_\lambda + A^p_\lambda I^i_p \partial_i), \tag{4.7.9}$$

where $\{I_p\}$ is a representation of the Lie algebra $\mathfrak{g}_r$ of G in V. The connection A (4.7.9) on Y is called the *associated principal connection* or, simply, a principal connection on $Y \to X$. The *curvature* (3.3.2) of this connection takes the form

$$R = \frac{1}{2} F^p_{\lambda\mu} I^i_p dx^\lambda \wedge dx^\mu \otimes \partial_i. \tag{4.7.10}$$

Example 4.7.4: A principal connection A on P yields the associated connection (4.7.9) on the associated Lie algebra bundle $V_G P$ which reads

$$A = dx^\lambda \otimes (\partial_\lambda - c^m_{pq} A^p_\lambda \xi^q e_m). \tag{4.7.11}$$

□

Remark 4.7.5: If an associated principal connection A is linear, one can define its *strength*

$$F_A = \frac{1}{2} F^p_{\lambda\mu} I_p dx^\lambda \wedge dx^\mu, \tag{4.7.12}$$

where I_p are matrices of a representation of the Lie algebra $\mathfrak{g}_r$ in fibres of Y with respect to the fibre bases $\{e_i(x)\}$. They coincide with the matrices of a representation of $\mathfrak{g}_r$ in the typical fibre V of Y with respect to its fixed basis $\{e_i\}$ (see the relation (1.2.1)). It follows that G-valued transition functions act on I_p by the adjoint representation. Note that, because of the canonical splitting (1.2.8), one can identify $e_i(x) = \partial_i$. □

In view of the above mentioned properties, the P-associated bundle Y (4.7.1) is called *canonically associated* to a principal bundle P. Unless otherwise stated, only canonically associated bundles are considered, and we simply call Y (4.7.1) an *associated bundle*.

4.8 Reduced structure

Let H and G be Lie groups and $\phi : H \to G$ a Lie group homomorphism. If $P_H \to X$ is a principal H-bundle, there always exists a principal G-bundle $P_G \to X$ together with the principal bundle morphism

$$\Phi : P_H \underset{X}{\longrightarrow} P_G \tag{4.8.1}$$

over X (see Remark 4.3.4). It is the P_H-associated bundle

$$P_G = (P_H \times G)/H$$

with the typical fibre G on which H acts on the left by the rule $h(g) = \phi(h)g$, while G acts on P_G as

$$G \ni g' : (p, g)/H \to (p, gg')/H.$$

Conversely, if $P_G \to X$ is a principal G-bundle, a problem is to find a principal H-bundle $P_H \to X$ together with a principal bundle morphism (4.8.1). If $H \to G$ is a closed subgroup, we have the *structure group reduction*. If $H \to G$ is a group epimorphism, one says that P_G *lifts* to P_H.

Here, we restrict our consideration to the reduction problem. In this case, the bundle monomorphism (4.8.1) is called a *reduced H-structure*.

Let P (4.3.1) be a principal G-bundle, and let H, $\dim H > 0$, be a closed (and, consequently, Lie) subgroup of G. Then we have the composite bundle

$$P \to P/H \to X, \tag{4.8.2}$$

where

$$P_\Sigma = P \xrightarrow{\pi_{P\Sigma}} P/H \tag{4.8.3}$$

is a principal bundle with a structure group H and

$$\Sigma = P/H \xrightarrow{\pi_{\Sigma X}} X \tag{4.8.4}$$

is a P-associated bundle with the typical fibre G/H on which the structure group G acts on the left (see Example 4.3.2).

One says that a structure Lie group G of a principal bundle P is reduced to its closed subgroup H if the following equivalent conditions hold.

• A principal bundle P admits a bundle atlas Ψ_P (4.3.3) with H-valued transition functions $\varrho_{\alpha\beta}$.

• There exists a principal *reduced subbundle* P_H of P with a structure group H.

Theorem 4.8.1: There is one-to-one correspondence

$$P^h = \pi_{P\Sigma}^{-1}(h(X)) \tag{4.8.5}$$

between the reduced principal H-subbundles $i_h : P^h \to P$ of P and the global sections h of the quotient bundle $P/H \to X$ (4.8.4). $\Box$

Corollary 4.8.2: A glance at the formula (4.8.5) shows that the reduced principal H-bundle P^h is the restriction h^*P_Σ (1.1.13) of the principal H-bundle P_Σ (4.8.3) to $h(X) \subset \Sigma$. $\Box$

In general, there is topological obstruction to reduction of a structure group of a principal bundle to its subgroup.

Theorem 4.8.3: In accordance with Theorem 1.1.4, the structure group G of a principal bundle P is always reducible to its closed subgroup H, if the quotient G/H is diffeomorphic to a Euclidean space $\mathbb{R}^m$. $\Box$

In particular, this is the case of a maximal compact subgroup H of a Lie group G. Then the following is a corollary of Theorem 4.8.3.

Theorem 4.8.4: A structure group G of a principal bundle is always reducible to its maximal compact subgroup H. □

Example 4.8.1: For instance, this is the case of $G = GL(n, \mathbb{C})$, $H = U(n)$ and $G = GL(n, \mathbb{R})$, $H = O(n)$. □

Example 4.8.2: Any affine bundle admits an atlas with linear transition functions. In accordance with Theorem 4.8.3, its structure group $GA(m, \mathbb{R})$ is always reducible to the linear subgroup $GL(m, \mathbb{R})$ because

$$GA(m, \mathbb{R})/GL(m, \mathbb{R}) = \mathbb{R}^m.$$

□

Different principal H-subbundles P^h and $P^{h'}$ of a principal G-bundle P are not isomorphic to each other in general.

Theorem 4.8.5: Let a structure Lie group G of a principal bundle be reducible to its closed subgroup H.

(i) Every vertical principal automorphism Φ of P sends a reduced principal H-subbundle P^h of P onto an isomorphic principal H-subbundle $P^{h'}$.

(ii) Conversely, let two reduced subbundles P^h and $P^{h'}$ of a principal bundle $P \to X$ be isomorphic to each other, and let $\Phi : P^h \to P^{h'}$ be their isomorphism over X. Then Φ is extended to a vertical principal automorphism of P. □

Theorem 4.8.6: If the quotient G/H is homeomorphic to a Euclidean space $\mathbb{R}^m$, all principal H-subbundles of a principal G-bundle P are isomorphic to each other. □

There are the following properties of principal connections compatible with a reduced structure.

Theorem 4.8.7: Since principal connections are equivariant, every principal connection A_h on a reduced principal H-subbundle P^h of a principal G-bundle P gives rise to a principal connection on P. □

Theorem 4.8.8: A principal connection A on a principal G-bundle P is reducible to a principal connection on a reduced principal H-subbundle P^h of P iff the corresponding global section h of the P-associated fibre bundle $P/H \to X$ is an integral section of the associated principal connection A on $P/H \to X$. $\Box$

Theorem 4.8.9: Let the Lie algebra $\mathfrak{g}_l$ of G be the direct sum

$$\mathfrak{g}_l = \mathfrak{h}_l \oplus \mathfrak{m} \tag{4.8.6}$$

of the Lie algebra $\mathfrak{h}_l$ of H and a subspace $\mathfrak{m}$ such that $\mathrm{Ad}_g(\mathfrak{m}) \subset \mathfrak{m}$, $g \in H$ (e.g., H is a Cartan subgroup of G). Let $\overline{A}$ be a $\mathfrak{g}_l$-valued connection form (4.4.12) on P. Then, the pull-back of the $\mathfrak{h}_l$-valued component of $\overline{A}$ onto a reduced principal H-subbundle P^h is a $\mathfrak{h}_l$-valued connection form of a principal connection $\overline{A}_h$ on P^h. $\Box$

The following is a corollary of Theorem 4.4.2.

Theorem 4.8.10: Given the composite bundle (4.8.2), let A_Σ be a principal connection on the principal H-bundle $P \to \Sigma$ (4.8.3). Then, for any reduced principal H-bundle $i_h : P^h \to P$, the pull-back connection $i_h^* A_\Sigma$ (3.6.14) is a principal connection on P^h. $\Box$

Chapter 5

Geometry of natural bundles

Classical gravitation theory is formulated as field theory on natural bundles, exemplified by tensor bundles. Therefore, we agree to call connections on these bundles the world connections.

5.1 Natural bundles

Let $\pi : Y \to X$ be a smooth fibre bundle coordinated by (x^λ, y^i). Any automorphism (Φ, f) of Y, by definition, is projected as

$$\pi \circ \Phi = f \circ \pi$$

onto a diffeomorphism f of its base X. The converse is not true. A diffeomorphism of X need not give rise to an automorphism of Y, unless $Y \to X$ is a trivial bundle.

Given a one-parameter group (Φ_t, f_t) of automorphisms of Y, its infinitesimal generator is a projectable vector field

$$u = u^\lambda(x^\mu)\partial_\lambda + u^i(x^\mu, y^j)\partial_i$$

on Y. This vector field is projected as

$$\tau \circ \pi = T\pi \circ u$$

onto a vector field $\tau = u^\lambda \partial_\lambda$ on X. Its flow is the one-parameter group (f_t) of diffeomorphisms of X which are projections of autmorphisms

(Φ_t, f_t) of Y. Conversely, let $\tau = \tau^\lambda\partial_\lambda$ be a vector field on X. There is a problem of constructing its lift to a projectable vector field

$$u = \tau^\lambda\partial_\lambda + u^i\partial_i$$

on Y projected onto τ. Such a lift always exists, but it need not be canonical. Given a connection Γ on Y, any vector field τ on X gives rise to the horizontal vector field $\Gamma\tau$ (3.1.6) on Y. This horizontal lift $\tau \to \Gamma\tau$ yields a monomorphism of the $C^\infty(X)$-module $\mathcal{T}(X)$ of vector fields on X to the $C^\infty(Y)$-module of vector fields on Y, but this monomorphisms is not a Lie algebra morphism, unless Γ is a flat connection.

Let us address the category of *natural bundles* $T \to X$ which admit the *functorial lift* $\tilde\tau$ onto T of any vector field τ on X such that $\tau \to \overline{\tau}$ is a monomorphism

$$\mathcal{T}(X) \to \mathcal{T}(T), \qquad [\tilde\tau, \tilde\tau'] = \widetilde{[\tau, \tau']},$$

of the real Lie algebra $\mathcal{T}(X)$ of vector fields on X to the real Lie algebra $\mathcal{T}(Y)$ of vector fields on T. One treats the functorial lift $\tilde\tau$ as an *infinitesimal general covariant transformation*, i.e., an infinitesimal generator of a local one-parameter group of general covariant transformations of T.

Remark 5.1.1: It should be emphasized that, in general, there exist diffeomorphisms of X which do not belong to any one-parameter group of diffeomorphisms of X. In a general setting, one therefore considers a monomorphism $f \to \tilde f$ of the group of diffeomorphisms of X to the group of bundle automorphisms of a natural bundle $T \to X$. Automorphisms $\tilde f$ are called *general covariant transformations* of T. No vertical automorphism of T, unless it is the identity morphism, is a general covariant transformation. □

Natural bundles are exemplified by tensor bundles (1.2.5). For instance, the tangent and cotangent bundles TX and T^*X of X are natural bundles. Given a vector field τ on X, its functorial (or canonical) lift onto the tensor bundle T (1.2.5) is given by the formula (1.3.2). In particular, let us recall the functorial lift (1.3.4) and (1.3.5) of τ onto the tangent bundle TX and the cotangent bundle T^*X, respectively.

Remark 5.1.2: Any diffeomorphism f of X gives rise to the tangent automorphisms $\widetilde{f} = Tf$ of TX which is a general covariant transformation of TX as a natural bundle. Accordingly, the general covariant transformation of the cotangent bundle T^*X over a diffeomorphism f of its base X reads

$$\dot{x}'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\dot{x}_\nu.$$

□

Tensor bundles over a manifold X have the structure group

$$GL_n = GL^+(n, \mathbb{R}). \tag{5.1.1}$$

The associated principal bundle is the fibre bundle

$$\pi_{LX} : LX \to X$$

of oriented linear frames in the tangent spaces to a manifold X. It is called the *linear frame bundle*. Its (local) sections are termed *frame fields*.

Given holonomic frames $\{\partial_\mu\}$ in the tangent bundle TX associated with the holonomic atlas Ψ_T (1.2.4), every element $\{H_a\}$ of the linear frame bundle LX takes the form $H_a = H_a^\mu \partial_\mu$, where H_a^μ is a matrix of the natural representation of the group GL_n in $\mathbb{R}^n$. These matrices constitute the bundle coordinates

$$(x^\lambda, H_a^\mu), \qquad H'^\mu_a = \frac{\partial x'^\mu}{\partial x^\lambda} H_a^\lambda,$$

on LX associated to its *holonomic atlas*

$$\Psi_T = \{(U_\iota, z_\iota = \{\partial_\mu\})\} \tag{5.1.2}$$

given by the local frame fields $z_\iota = \{\partial_\mu\}$. With respect to these coordinates, the right action (4.3.2) of GL_n on LX reads

$$R_{gP} : H_a^\mu \to H_b^\mu g^b{}_a, \qquad g \in GL_n.$$

The frame bundle LX admits the canonical $\mathbb{R}^n$-valued one-form

$$\theta_{LX} = H^a_\mu dx^\mu \otimes t_a, \tag{5.1.3}$$

where $\{t_a\}$ is a fixed basis for $\mathbb{R}^n$ and H^a_μ is the inverse matrix of H^μ_a.

The frame bundle $LX \to X$ belongs to the category of natural bundles. Any diffeomorphism f of X gives rise to the principal automorphism

$$\widetilde{f} : (x^\lambda, H^\lambda_a) \to (f^\lambda(x), \partial_\mu f^\lambda H^\mu_a) \tag{5.1.4}$$

of LX which is its general covariant transformation (or a *holonomic automorphism*). For instance, the associated automorphism of TX is the tangent morphism Tf to f.

Given a (local) one-parameter group of diffeomorphisms of X and its infinitesimal generator τ, their lift (5.1.4) results in the functorial lift

$$\widetilde{\tau} = \tau^\mu \partial_\mu + \partial_\nu \tau^\alpha H^\nu_a \frac{\partial}{\partial H^\alpha_a} \tag{5.1.5}$$

of a vector field τ on X onto LX defined by the condition

$$\mathbf{L}_{\widetilde{\tau}} \theta_{LX} = 0.$$

Every LX-associated bundle $Y \to X$ admits a lift of any diffeomorphism f of its base to the principal automorphism $\widetilde{f}_Y$(4.7.7) of Y associated with the principal automorphism $\widetilde{f}$ (5.1.4) of the liner frame bundle LX. Thus, all bundles associated with the linear frame bundle LX are natural bundles. However, there are natural bundles which are not associated with LX.

Remark 5.1.3: In a more general setting, higher order natural bundles and gauge natural bundles are considered. Note that the linear frame bundle LX over a manifold X is the set of first order jets of local diffeomorphisms of the vector space $\mathbb{R}^n$ to X, $n = \dim X$, at the origin of $\mathbb{R}^n$. Accordingly, one considers r-order frame bundles $L^r X$ of r-order jets of local diffeomorphisms of $\mathbb{R}^n$ to X. Furthermore, given a principal bundle

$P \to X$ with a structure group G, the r-order jet bundle $J^1P \to X$ of its sections fails to be a principal bundle. However, the product

$$W^rP = L^rX \times J^rP$$

is a principal bundle with the structure group W^r_nG which is a semidirect product of the group G^r_n of invertible r-order jets of maps $\mathbb{R}^n$ to itself at its origin (e.g., $G^1_n = GL(n,\mathbb{R})$) and the group T^r_nG of r-order jets of morphisms $\mathbb{R}^n \to G$ at the origin of $\mathbb{R}^n$. Moreover, if $Y \to X$ is a P-associated bundle, the jet bundle $J^rY \to X$ is a vector bundle associated with the principal bundle W^rP. It exemplifies *gauge natural bundles* which can be described as fibre bundles associated with principal bundles W^rP. Natural bundles are gauge natural bundles for a trivial group $G = \mathbf{1}$. The bundle of principal connections C (4.4.3) is a first order gauge natural bundle. □

5.2 Linear world connections

Since the tangent bundle TX is associated with the linear frame bundle LX, every world connection (3.4.6):

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma_\lambda{}^\mu{}_\nu \dot x^\nu \dot\partial_\mu), \tag{5.2.1}$$

on a manifold X is associated with a principal connection on LX. We agree to call Γ (5.2.1) the *linear world connection* in order to distinct it from an affine world connection in Section 5.3.

Being principal connections on the linear frame bundle LX, linear world connections are represented by sections of the quotient bundle

$$C_{\rm W} = J^1LX/GL_n, \tag{5.2.2}$$

called the *bundle of world connections*. With respect to the holonomic atlas Ψ_T (5.1.2), this bundle is provided with the coordinates

$$(x^\lambda, k_\lambda{}^\nu{}_\alpha), \qquad k'_\lambda{}^\nu{}_\alpha = \left[\frac{\partial x'^\nu}{\partial x^\gamma}\frac{\partial x^\beta}{\partial x'^\alpha}k_\mu{}^\gamma{}_\beta + \frac{\partial x^\beta}{\partial x'^\alpha}\frac{\partial^2 x'^\nu}{\partial x^\mu \partial x^\beta}\right]\frac{\partial x^\mu}{\partial x'^\lambda},$$

so that, for any section Γ of $C_{\rm W}\to X$,

$$k_\lambda{}^\nu{}_\alpha\circ\Gamma=\Gamma_\lambda{}^\nu{}_\alpha$$

are components of the linear world connection Γ (5.2.1).

Though the bundle of world connections $C_{\rm W}\to X$ (5.2.2) is not LX-associated, it is a natural bundle. It admits the lift

$$\widetilde f_C: J^1LX/GL_n\to J^1\widetilde f(J^1LX)/GL_n$$

of any diffeomorphism f of its base X and, consequently, the functorial lift

$$\widetilde\tau_C=\tau^\mu\partial_\mu+[\partial_\nu\tau^\alpha k_\mu{}^\nu{}_\beta-\partial_\beta\tau^\nu k_\mu{}^\alpha{}_\nu-\partial_\mu\tau^\nu k_\nu{}^\alpha{}_\beta+\partial_{\mu\beta}\tau^\alpha]\frac{\partial}{\partial k_\mu{}^\alpha{}_\beta} \qquad (5.2.3)$$

of any vector field τ on X.

The first order jet manifold $J^1C_{\rm W}$ of the bundle of world connections admits the canonical splitting (4.5.11). In order to obtain its coordinate expression, let us consider the strength (4.7.12) of the linear world connection Γ (5.2.1). It reads

$$F_\Gamma=\frac12F_{\lambda\mu}{}^b{}_aI_b{}^adx^\lambda\wedge dx^\mu=\frac12R_{\mu\nu}{}^\alpha{}_\beta dx^\lambda\wedge dx^\mu,$$

where

$$(I_b{}^a)^\alpha{}_\beta=H^\alpha_bH^a_\beta$$

are generators of the group GL_n (5.1.1) in fibres of TX with respect to the holonomic frames, and

$$R_{\lambda\mu}{}^\alpha{}_\beta=\partial_\lambda\Gamma_\mu{}^\alpha{}_\beta-\partial_\mu\Gamma_\lambda{}^\alpha{}_\beta+\Gamma_\lambda{}^\gamma{}_\beta\Gamma_\mu{}^\alpha{}_\gamma-\Gamma_\mu{}^\gamma{}_\beta\Gamma_\lambda{}^\alpha{}_\gamma \qquad (5.2.4)$$

are components if the curvature (3.4.8) of a linear world connection Γ. Accordingly, the above mentioned canonical splitting (4.5.11) of $J^1C_{\rm W}$ can be written in the form

$$\begin{aligned}k_{\lambda\mu}{}^\alpha{}_\beta&=\frac12(\mathcal R_{\lambda\mu}{}^\alpha{}_\beta+\mathcal S_{\lambda\mu}{}^\alpha{}_\beta)= \qquad (5.2.5)\\ &\frac12(k_{\lambda\mu}{}^\alpha{}_\beta-k_{\mu\lambda}{}^\alpha{}_\beta+k_\lambda{}^\gamma{}_\beta k_\mu{}^\alpha{}_\gamma-k_\mu{}^\gamma{}_\beta k_\lambda{}^\alpha{}_\gamma)+\\ &\frac12(k_{\lambda\mu}{}^\alpha{}_\beta+k_{\mu\lambda}{}^\alpha{}_\beta-k_\lambda{}^\gamma{}_\beta k_\mu{}^\alpha{}_\gamma+k_\mu{}^\gamma{}_\beta k_\lambda{}^\alpha{}_\gamma).\end{aligned}$$

It is readily observed that, if Γ is a section of $C_{\rm W}\to X$, then

$$\mathcal{R}_{\lambda\mu}{}^\alpha{}_\beta\circ J^1\Gamma = R_{\lambda\mu}{}^\alpha{}_\beta.$$

Because of the canonical vertical splitting (1.4.11) of the vertical tangent bundle VTX of TX, the curvature form (3.4.8) of a linear world connection Γ can be represented by the tangent-valued two-form

$$R=\frac12 R_{\lambda\mu}{}^\alpha{}_\beta\dot x^\beta dx^\lambda\wedge dx^\mu\otimes\partial_\alpha \tag{5.2.6}$$

on TX. Due to this representation, the *Ricci tensor*

$$R_c=\frac12 R_{\lambda\mu}{}^\lambda{}_\beta dx^\mu\otimes dx^\beta \tag{5.2.7}$$

of a linear world connection Γ is defined.

Owing to the above mentioned vertical splitting (1.4.11) of VTX, the torsion form T (3.4.9) of Γ can be written as the tangent-valued two-form

$$\begin{aligned} &T=\frac12 T_\mu{}^\nu{}_\lambda dx^\lambda\wedge dx^\mu\otimes\partial_\nu, \\ &T_\mu{}^\nu{}_\lambda=\Gamma_\mu{}^\nu{}_\lambda-\Gamma_\lambda{}^\nu{}_\mu, \end{aligned} \tag{5.2.8}$$

on X. The *soldering torsion form*

$$T=T_\mu{}^\nu{}_\lambda\dot x^\lambda dx^\mu\otimes\dot\partial_\nu \tag{5.2.9}$$

on TX is also defined. Then one can show the following.

- Given a linear world connection Γ (5.2.1) and its soldering torsion form T (5.2.9), the sum $\Gamma+cT$, $c\in\mathbb{R}$, is a linear world connection.
- Every linear world connection Γ defines a unique symmetric world connection

$$\Gamma'=\Gamma-\frac12 T. \tag{5.2.10}$$

- If Γ and Γ' are linear world connections, then

$$c\Gamma+(1-c)\Gamma'$$

is so for any $c\in\mathbb{R}$.

A manifold X is said to be *flat* if it admits a flat linear world connection Γ. By virtue of Theorem 3.5.2, there exists an atlas of local constant trivializations of TX such that

$$\Gamma = dx^\lambda \otimes \partial_\lambda$$

relative to this atlas. As a consequence, the curvature form R (5.2.6) of this connection equals zero. However, such an atlas is not holonomic in general. Relative to this atlas, the canonical soldering form (1.4.7) on TX reads

$$\theta_J = H^a_\mu dx^\mu \dot\partial_a,$$

and the torsion form T (3.4.9) of Γ defined as the Nijenhuis differential $d_\Gamma \theta_J$ (3.3.8) need not vanish.

A manifold X is called *parallelizable* if the tangent bundle $TX \to X$ is trivial. By virtue of Theorem 3.5.2, a parallelizable manifold is flat. Conversely, a flat manifold is parallelizable if it is simply connected.

Every linear world connection Γ (5.2.1) yields the horizontal lift

$$\Gamma\tau = \tau^\lambda(\partial_\lambda + \Gamma_\lambda{}^\beta{}_\alpha \dot x^\alpha \dot\partial_\beta) \tag{5.2.11}$$

of a vector field τ on X onto the tangent bundle TX. A vector field τ on X is said to be *parallel* relative to a connection Γ if it is an integral section of Γ. Its integral curve is called the *autoparallel* of a world connection Γ.

Remark 5.2.1: By virtue of Theorem 3.2.2, any vector field on X is an integral section of some linear world connection. If $\tau(x) \neq 0$ at a point $x \in X$, there exists a coordinate system (q^i) on some neighbourhood U of x such that $\tau^i(x)$ =const. on U. Then τ on U is an integral section of the local symmetric linear world connection

$$\Gamma_\tau(x) = dq^i \otimes \partial_i, \qquad x \in U, \tag{5.2.12}$$

on U. The functorial lift $\tilde\tau$ (1.3.4) can be obtained at each point $x \in X$ as the horizontal lift of τ by means of the local symmetric connection (5.2.12). $\Box$

The horizontal lift of a vector field τ on X onto the linear frame bundle LX by means of a world connection K reads

$$\Gamma\tau = \tau^\lambda \left(\partial_\lambda + \Gamma_\lambda{}^\nu{}_\alpha H^\alpha_a \frac{\partial}{\partial H^\nu_a}\right). \tag{5.2.13}$$

It is called *standard* if the morphism

$$u\rfloor\theta_{LX} : LX \to \mathbb{R}^n$$

is constant on LX. It is readily observed that every standard horizontal vector field on LX takes the form

$$u_v = H^\lambda_b v^b \left(\partial_\lambda + \Gamma_\lambda{}^\nu{}_\alpha H^\alpha_a \frac{\partial}{\partial H^\nu_a}\right) \tag{5.2.14}$$

where $v = v^b t_b \in \mathbb{R}^n$. A glance at this expression shows that a standard horizontal vector field is not projectable.

Since TX is an LX-associated fibre bundle, we have the canonical morphism

$$LX \times \mathbb{R}^n \to TX, \qquad (H^\mu_a, v^a) \to \dot x^\mu = H^\mu_a v^a.$$

The tangent map to this morphism sends every standard horizontal vector field (5.2.14) on LX to the horizontal vector field

$$u = \dot x^\lambda(\partial_\lambda + \Gamma_\lambda{}^\nu{}_\alpha \dot x^\alpha \dot\partial_\nu) \tag{5.2.15}$$

on TX. Such a vector field on TX is called *holonomic*. Given holonomic coordinates $(x^\mu, \dot x^\mu, \dot{\mathrm{x}}^\mu, \ddot x^\mu)$ on the double tangent bundle TTX, the holonomic vector field (5.2.15) defines the second order dynamic equation

$$\ddot x^\nu = \Gamma_\lambda{}^\nu{}_\alpha \dot x^\lambda \dot x^\alpha \tag{5.2.16}$$

on X which is called the *geodesic equation* with respect to a linear world connection Γ. Solutions of the geodesic equation (5.2.16), called the *geodesics* of Γ, are the projection of integral curves of the vector field (5.2.15) in TX onto X. Moreover, one can show the following.

Theorem 5.2.1: The projection of an integral curve of any standard horizontal vector field (5.2.14) on LX onto X is a geodesic in X. Conversely, any geodesic in X is of this type. □

It is readily observed that, if linear world connections Γ and Γ' differ from each other only in the torsion, they define the same holonomic vector field (5.2.15) and the same geodesic equation (5.2.16).

Let τ be an integral vector field of a linear world connection Γ, i.e., $\nabla^\Gamma_\mu \tau = 0$. Consequently, it obeys the equation $\tau^\mu \nabla^\Gamma_\mu \tau = 0$. Any autoparallel of a linear world connection Γ is its geodesic and, conversely, a geodesic of Γ is an autoparallel of its symmetric part (5.2.10).

5.3 Affine world connections

The tangent bundle TX of a manifold X as like as any vector bundle possesses a natural structure of an affine bundle (see Section 1.2). Therefore, one can consider affine connections on TX, called *affine world connections*. Here we study them as principal connections.

Let $Y \to X$ be an affine bundle with an k-dimensional typical fibre V. It is associated with a principal bundle AY of affine frames in Y, whose structure group is the general affine group $GA(k, \mathbb{R})$. Then any affine connection on $Y \to X$ can be seen as an associated with a principal connection on $AY \to X$. These connections are represented by global sections of the affine bundle

$$J^1P/GA(k, \mathbb{R}) \to X.$$

They always exist.

As was mentioned in Section 1.3.5, every affine connection Γ (3.4.12) on $Y \to X$ defines a unique associated linear connection $\overline{\Gamma}$ (3.4.13) on the underlying vector bundle $\overline{Y} \to X$. This connection $\overline{\Gamma}$ is associated with a linear principal connection on the principal bundle $L\overline{Y}$ of linear frames in $\overline{Y}$ whose structure group is the general linear group $GL(k,\mathbb{R})$. We have the exact sequence of groups

$$0 \to T_k \to GA(k,\mathbb{R}) \to GL(k,\mathbb{R}) \to \mathbf{1}, \tag{5.3.1}$$

where T_k is the group of translations in $\mathbb{R}^k$. It is readily observed that there is the corresponding principal bundle morphism $AY \to L\overline{Y}$ over X, and the principal connection $\overline{\Gamma}$ on $L\overline{Y}$ is the image of the principal connection Γ on $AY \to X$ under this morphism in accordance with Theorem 4.4.3.

The exact sequence (5.3.1) admits a splitting

$$GL(k,\mathbb{R}) \to GA(k,\mathbb{R}),$$

but this splitting is not canonical. It depends on the morphism

$$V \ni v \to v - v_0 \in \overline{V},$$

i.e., on the choice of an origin v_0 of the affine space V. Given v_0, the image of the corresponding monomorphism

$$GL(k,\mathbb{R}) \to GA(k,\mathbb{R})$$

is a stabilizer

$$G(v_0) \subset GA(k,\mathbb{R})$$

of v_0. Different subgroups $G(v_0)$ and $G(v'_0)$ are related to each other as follows:

$$G(v'_0) = T(v'_0 - v_0)G(v_0)T^{-1}(v'_0 - v_0),$$

where $T(v'_0 - v_0)$ is the translation along the vector $(v'_0 - v_0) \in \overline{V}$.

Remark 5.3.1: The well-known morphism of a k-dimensional affine space V onto a hypersurface $\overline{y}^{k+1} = 1$ in $\mathbb{R}^{k+1}$ and the corresponding representation of elements of $GA(k,\mathbb{R})$ by particular $(k+1)\times(k+1)$-matrices also fail to be canonical. They depend on a point $v_0 \in V$ sent to vector $(0,\ldots,0,1) \in \mathbb{R}^{k+1}$. $\Box$

One can say something more if $Y \to X$ is a vector bundle provided with the natural structure of an affine bundle whose origin is the canonical zero section $\widehat{0}$. In this case, we have the canonical splitting of the exact sequence (5.3.1) such that $GL(k,\mathbb{R})$ is a subgroup of $GA(k,\mathbb{R})$ and $GA(k,\mathbb{R})$ is the semidirect product of $GL(k,\mathbb{R})$ and the group $T(k,\mathbb{R})$ of translations in $\mathbb{R}^k$. Given a $GA(k,\mathbb{R})$-principal bundle $AY \to X$, its affine structure group $GA(k,\mathbb{R})$ is always reducible to the linear subgroup since the quotient $GA(k,\mathbb{R})/GL(k,\mathbb{R})$ is a vector space $\mathbb{R}^k$ provided with the natural affine structure (see Example 4.8.2). The corresponding quotient bundle is isomorphic to the vector bundle $Y \to X$. There is the canonical injection of the linear frame bundle $LY \to AY$ onto the reduced $GL(k,\mathbb{R})$-principal subbundle of AY which corresponds to the zero section $\widehat{0}$ of $Y \to X$. In this case, every principal connection on the linear frame bundle LY gives rise to a principal connection on the affine frame bundle in accordance with Theorem 4.8.7. This is equivalent to the fact that any affine connection Γ on a vector bundle $Y \to X$ defines a linear connection $\overline{\Gamma}$ on $Y \to X$ and that every linear connection on $Y \to X$ can be seen as an affine one. Then any affine connection Γ on the vector bundle $Y \to X$ is represented by the sum of the associated linear connection $\overline{\Gamma}$ and a basic soldering form σ on $Y \to X$. Due to the vertical splitting (1.2.8), this soldering form is represented by a global section of the tensor product $T^*X \otimes Y$.

Let now $Y \to X$ be the tangent bundle $TX \to X$ considered as an affine bundle. Then the relationship between affine and linear world connections on TX is the repetition of that we have said in the case of an arbitrary vector bundle $Y \to X$. In particular, any affine world connection

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma_\lambda{}^\alpha{}_\mu(x)\dot{x}^\mu + \sigma^\alpha_\lambda(x))\partial_\alpha \tag{5.3.2}$$

on $TX \to X$ is represented by the sum of the associated linear world connection

$$\overline{\Gamma} = \Gamma_\lambda{}^\alpha{}_\mu(x)\dot{x}^\mu dx^\lambda \otimes \partial_\alpha \tag{5.3.3}$$

on $TX \to X$ and a basic soldering form

$$\sigma = \sigma^\alpha_\lambda(x) dx^\lambda \otimes \partial_\alpha \tag{5.3.4}$$

on $Y \to X$, which is the $(1,1)$-tensor field on X. For instance, if $\sigma = \theta_X$ (1.4.5), we have the Cartan connection (3.4.15).

It is readily observed that the soldered curvature (3.3.7) of any soldering form (5.3.4) equals zero. Then we obtain from (3.3.10) that the torsion (3.4.14) of the affine connection Γ (5.3.2) with respect to σ (5.3.4) coincides with that of the associated linear connection $\overline{\Gamma}$ (5.3.3) and reads

$$\begin{aligned} & T = \frac{1}{2} T^i_{\lambda\mu} dx^\mu \wedge dx^\lambda \otimes \partial_i, \\ & T_\lambda{}^\lambda{}_\mu = \Gamma_\lambda{}^\alpha{}_\nu \sigma^\nu_\mu - \Gamma_\mu{}^\alpha{}_\nu \sigma^\nu_\lambda. \end{aligned} \tag{5.3.5}$$

The relation between the curvatures of an affine world connection Γ (5.3.2) and the associated linear connection $\overline{\Gamma}$ (5.3.3) is given by the general expression (3.3.11) where $\rho = 0$ and T is (5.3.5).

Chapter 6

Geometry of graded manifolds

In classical field theory, there are different descriptions of odd fields on graded manifolds and supermanifolds. Both graded manifolds and supermanifolds are phrased in terms of sheaves of graded commutative algebras. However, graded manifolds are characterized by sheaves on smooth manifolds, while supermanifolds are constructed by gluing of sheaves on supervector spaces. Treating odd fields on a manifold X, we follow the Serre–Swan theorem generalized to graded manifolds (Theorem 6.3.2). It states that, if a Grassmann algebra is an exterior algebra of some projective $C^\infty(X)$-module of finite rank, it is isomorphic to the algebra of graded functions on a graded manifold whose body is X. By virtue of this theorem, odd fields on an arbitrary manifold X are described as generating elements of the structure ring of a graded manifold whose body is X [9, 24].

6.1 Grassmann-graded algebraic calculus

Throughout the book, by the Grassmann gradation is meant $\mathbb{Z}_2$-gradation. Hereafter, the symbol [.] stands for the Grassmann parity. In the literature, a $\mathbb{Z}_2$-graded structure is simply called the graded structure if there is no danger of confusion. Let us summarize the relevant notions of the Grassmann-graded algebraic calculus.

An algebra $\mathcal{A}$ is called *graded* if it is endowed with a *grading automorphism* γ such that $\gamma^2 = \mathrm{Id}$. A graded algebra falls into the direct sum $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ of $\mathbb{Z}$-modules $\mathcal{A}_0$ and $\mathcal{A}_1$ of *even* and *odd* elements such that

$$\gamma(a) = (-1)^i a, \qquad a \in \mathcal{A}_i, \qquad i = 0, 1,$$
$$[aa'] = ([a] + [a'])\bmod 2, \qquad a \in \mathcal{A}_{[a]}, \qquad a' \in \mathcal{A}_{[a']}.$$

One calls $\mathcal{A}_0$ and $\mathcal{A}_1$ the even and odd parts of $\mathcal{A}$, respectively. The even part $\mathcal{A}_0$ is a subalgebra of $\mathcal{A}$ and the odd one $\mathcal{A}_1$ is an $\mathcal{A}_0$-module. If $\mathcal{A}$ is a *graded ring*, then $[\mathbf{1}] = 0$.

A graded algebra $\mathcal{A}$ is called *graded commutative* if

$$aa' = (-1)^{[a][a']} a'a,$$

where a and a' are *graded-homogeneous elements* of $\mathcal{A}$.

Given a graded algebra $\mathcal{A}$, a left *graded $\mathcal{A}$-module* Q is defined as a left $\mathcal{A}$-module provided with the grading automorphism γ such that

$$\gamma(aq) = \gamma(a)\gamma(q), \qquad a \in \mathcal{A}, \qquad q \in Q,$$
$$[aq] = ([a] + [q])\bmod 2.$$

A graded module Q is split into the direct sum $Q = Q_0 \oplus Q_1$ of two $\mathcal{A}_0$-modules Q_0 and Q_1 of even and odd elements.

If $\mathcal{K}$ is a graded commutative ring, a graded $\mathcal{K}$-module can be provided with a graded *$\mathcal{K}$-bimodule* structure by letting

$$qa = (-1)^{[a][q]} aq, \qquad a \in \mathcal{K}, \qquad q \in Q.$$

A graded module is called *free* if it has a basis generated by graded-homogeneous elements. This basis is said to be of type (n, m) if it contains n even and m odd elements.

In particular, by a real *graded vector space* $B = B_0 \oplus B_1$ is meant a graded $\mathbb{R}$-module. A real graded vector space is said to be (n, m)-dimensional if $B_0 = \mathbb{R}^n$ and $B_1 = \mathbb{R}^m$.

Given a graded commutative ring $\mathcal{K}$, the following are standard constructions of new graded modules from old ones.

• The direct sum of graded modules and a graded factor module are defined just as those of modules over a commutative ring.

• The *tensor product* $P \otimes Q$ of graded $\mathcal{K}$-modules P and Q is an additive group generated by elements $p \otimes q$, $p \in P$, $q \in Q$, obeying the relations

$$\begin{aligned}
&(p + p') \otimes q = p \otimes q + p' \otimes q,\\
&p \otimes (q + q') = p \otimes q + p \otimes q',\\
&ap \otimes q = (-1)^{[p][a]} pa \otimes q = (-1)^{[p][a]} p \otimes aq, \qquad a \in \mathcal{K}.
\end{aligned}$$

In particular, the tensor algebra $\otimes P$ of a graded $\mathcal{K}$-module P is defined as that (8.1.5) of a module over a commutative ring. Its quotient $\wedge P$ with respect to the ideal generated by elements

$$p \otimes p' + (-1)^{[p][p']} p' \otimes p, \qquad p, p' \in P,$$

is the *bigraded exterior algebra* of a graded module P with respect to the *graded exterior product*

$$p \wedge p' = -(-1)^{[p][p']} p' \wedge p.$$

• A morphism $\Phi : P \to Q$ of graded $\mathcal{K}$-modules seen as additive groups is said to be *even graded morphism* (resp. *odd graded morphism*) if Φ preserves (resp. change) the Grassmann parity of all graded-homogeneous elements of P and obeys the relations

$$\Phi(ap) = (-1)^{[\Phi][a]} a\Phi(p), \qquad p \in P, \qquad a \in \mathcal{K}.$$

A morphism $\Phi : P \to Q$ of graded $\mathcal{K}$-modules as additive groups is called a *graded $\mathcal{K}$-module morphism* if it is represented by a sum of even and odd graded morphisms. The set $\mathrm{Hom}_{\mathcal{K}}(P, Q)$ of graded morphisms of a graded $\mathcal{K}$-module P to a graded $\mathcal{K}$-module Q is naturally a graded $\mathcal{K}$-module. The graded $\mathcal{K}$-module $P^* = \mathrm{Hom}_{\mathcal{K}}(P, \mathcal{K})$ is called the *dual* of a graded $\mathcal{K}$-module P.

A *graded commutative* $\mathcal{K}$*-ring* $\mathcal{A}$ is a graded commutative ring which also is a graded $\mathcal{K}$-module. A *real graded commutative ring* is said to be of rank N if it is a free algebra generated by the unit $\mathbf{1}$ and N odd elements. A *graded commutative Banach ring* $\mathcal{A}$ is a real graded commutative ring which is a real Banach algebra whose norm obeys the condition

$$\|a_0 + a_1\| = \|a_0\| + \|a_1\|, \qquad a_0 \in \mathcal{A}_0, \quad a_1 \in \mathcal{A}_1.$$

Let V be a real vector space, and let $\Lambda = \wedge V$ be its exterior algebra endowed with the Grassmann gradation

$$\Lambda = \Lambda_0 \oplus \Lambda_1, \qquad \Lambda_0 = \mathbb{R} \bigoplus_{k=1} \overset{2k}{\wedge} V, \qquad \Lambda_1 = \bigoplus_{k=1} \overset{2k-1}{\wedge} V. \tag{6.1.1}$$

It is a real graded commutative ring, called the *Grassmann algebra*. A Grassmann algebra, seen as an additive group, admits the decomposition

$$\Lambda = \mathbb{R} \oplus R = \mathbb{R} \oplus R_0 \oplus R_1 = \mathbb{R} \oplus (\Lambda_1)^2 \oplus \Lambda_1, \tag{6.1.2}$$

where R is the *ideal of nilpotents* of Λ. The corresponding projections $\sigma : \Lambda \to \mathbb{R}$ and $s : \Lambda \to R$ are called the *body* and *soul* maps, respectively.

Hereafter, we restrict our consideration to Grassmann algebras of finite rank. Given a basis $\{c^i\}$ for the vector space V, the elements of the Grassmann algebra Λ (6.1.1) take the form

$$a = \sum_{k=0,1,\ldots} \sum_{(i_1 \cdots i_k)} a_{i_1 \cdots i_k} c^{i_1} \cdots c^{i_k}, \tag{6.1.3}$$

where the second sum runs through all the tuples $(i_1 \cdots i_k)$ such that no two of them are permutations of each other. The Grassmann algebra Λ becomes a graded commutative Banach ring with respect to the norm

$$\|a\| = \sum_{k=0} \sum_{(i_1 \cdots i_k)} |a_{i_1 \cdots i_k}|.$$

Let B be a graded vector space. Given a Grassmann algebra Λ, it can be brought into a graded Λ-module

$$\Lambda B = \Lambda B_0 \oplus \Lambda B_1 = (\Lambda_0 \otimes B_0 \oplus \Lambda_1 \otimes B_1) \oplus (\Lambda_1 \otimes B_0 \oplus \Lambda_0 \otimes B_1),$$

called a *superspace*. The superspace

$$B^{n|m} = [(\overset{n}{\oplus}\Lambda_0) \oplus (\overset{m}{\oplus}\Lambda_1)] \oplus [(\overset{n}{\oplus}\Lambda_1) \oplus (\overset{m}{\oplus}\Lambda_0)] \tag{6.1.4}$$

is said to be (n,m)-dimensional. The graded Λ_0-module

$$B^{n,m} = (\overset{n}{\oplus}\Lambda_0) \oplus (\overset{m}{\oplus}\Lambda_1)$$

is called an (n,m)-dimensional *supervector space*. Whenever referring to a topology on a supervector space $B^{n,m}$, we will mean the Euclidean topology on a $2^{N-1}[n+m]$-dimensional real vector space.

Let $\mathcal{K}$ be a graded commutative ring. A graded commutative (non-associative) $\mathcal{K}$-algebra $\mathfrak{g}$ is called a *Lie $\mathcal{K}$-superalgebra* if its product $[.,.]$, called the *Lie superbracket*, obeys the relations

$$[\varepsilon,\varepsilon'] = -(-1)^{[\varepsilon][\varepsilon']}[\varepsilon',\varepsilon],$$
$$(-1)^{[\varepsilon][\varepsilon'']}[\varepsilon,[\varepsilon',\varepsilon'']] + (-1)^{[\varepsilon'][\varepsilon]}[\varepsilon',[\varepsilon'',\varepsilon]] + (-1)^{[\varepsilon''][\varepsilon']}[\varepsilon'',[\varepsilon,\varepsilon']] = 0.$$

The even part $\mathfrak{g}_0$ of a Lie $\mathcal{K}$-superalgebra $\mathfrak{g}$ is a Lie $\mathcal{K}_0$-algebra. A graded $\mathcal{K}$-module P is called a *$\mathfrak{g}$-module* if it is provided with a $\mathcal{K}$-bilinear map

$$\mathfrak{g}\times P \ni (\varepsilon,p) \to \varepsilon p \in P, \qquad [\varepsilon p] = ([\varepsilon]+[p]) \bmod 2,$$
$$[\varepsilon,\varepsilon']p = (\varepsilon\circ\varepsilon' - (-1)^{[\varepsilon][\varepsilon']}\varepsilon'\circ\varepsilon)p.$$

6.2 Grassmann-graded differential calculus

Linear differential operators on graded modules over a graded commutative ring are defined similarly to those in commutative geometry (Section 8.2).

Let $\mathcal{K}$ be a graded commutative ring and $\mathcal{A}$ a graded commutative $\mathcal{K}$-ring. Let P and Q be graded $\mathcal{A}$-modules. The graded $\mathcal{K}$-module $\mathrm{Hom}_{\mathcal{K}}(P,Q)$ of graded $\mathcal{K}$-module homomorphisms $\Phi: P\to Q$ can be endowed with the two graded $\mathcal{A}$-module structures

$$(a\Phi)(p) = a\Phi(p), \quad (\Phi\bullet a)(p) = \Phi(ap), \quad a\in\mathcal{A}, \quad p\in P, \tag{6.2.1}$$

called $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures, respectively. Let us put

$$\delta_a\Phi = a\Phi - (-1)^{[a][\Phi]}\Phi \bullet a, \qquad a \in \mathcal{A}. \tag{6.2.2}$$

An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P, Q)$ is said to be a Q-valued *graded differential operator* of order s on P if

$$\delta_{a_0} \circ \cdots \circ \delta_{a_s}\Delta = 0$$

for any tuple of $s+1$ elements $a_0, \ldots, a_s$ of $\mathcal{A}$. The set $\mathrm{Diff}_s(P, Q)$ of these operators inherits the graded module structures (6.2.1).

In particular, zero order graded differential operators obey the condition

$$\delta_a\Delta(p) = a\Delta(p) - (-1)^{[a][\Delta]}\Delta(ap) = 0, \qquad a \in \mathcal{A}, \qquad p \in P,$$

i.e., they coincide with graded $\mathcal{A}$-module morphisms $P \to Q$. A first order graded differential operator Δ satisfies the relation

$$\delta_a \circ \delta_b\,\Delta(p) = ab\Delta(p) - (-1)^{([b]+[\Delta])[a]}b\Delta(ap) - (-1)^{[b][\Delta]}a\Delta(bp) +$$
$$(-1)^{[b][\Delta]+([\Delta]+[b])[a]} = 0, \qquad a, b \in \mathcal{A}, \quad p \in P.$$

For instance, let $P = \mathcal{A}$. Any zero order Q-valued graded differential operator Δ on $\mathcal{A}$ is defined by its value $\Delta(\mathbf{1})$. Then there is a graded $\mathcal{A}$-module isomorphism

$$\mathrm{Diff}_0(\mathcal{A}, Q) = Q, \qquad Q \ni q \to \Delta_q \in \mathrm{Diff}_0(\mathcal{A}, Q),$$

where Δ_q is given by the equality $\Delta_q(\mathbf{1}) = q$. A first order Q-valued graded differential operator Δ on $\mathcal{A}$ fulfils the condition

$$\Delta(ab) = \Delta(a)b + (-1)^{[a][\Delta]}a\Delta(b) - (-1)^{([b]+[a])[\Delta]}ab\Delta(\mathbf{1}), \quad a, b \in \mathcal{A}.$$

It is called a Q-valued *graded derivation* of $\mathcal{A}$ if $\Delta(\mathbf{1}) = 0$, i.e., the *Grassmann-graded Leibniz rule*

$$\Delta(ab) = \Delta(a)b + (-1)^{[a][\Delta]}a\Delta(b), \quad a, b \in \mathcal{A}, \tag{6.2.3}$$

holds. One obtains at once that any first order graded differential operator on $\mathcal{A}$ falls into the sum

$$\Delta(a) = \Delta(\mathbf{1})a + [\Delta(a) - \Delta(\mathbf{1})a]$$

of a zero order graded differential operator $\Delta(\mathbf{1})a$ and a graded derivation $\Delta(a) - \Delta(\mathbf{1})a$. If ∂ is a graded derivation of $\mathcal{A}$, then $a\partial$ is so for any $a \in \mathcal{A}$. Hence, graded derivations of $\mathcal{A}$ constitute a graded $\mathcal{A}$-module $\mathfrak{d}(\mathcal{A}, Q)$, called the *graded derivation module*.

If $Q = \mathcal{A}$, the graded derivation module $\mathfrak{d}\mathcal{A}$ also is a Lie superalgebra over the graded commutative ring $\mathcal{K}$ with respect to the superbracket

$$[u, u'] = u \circ u' - (-1)^{[u][u']} u' \circ u, \qquad u, u' \in \mathcal{A}. \tag{6.2.4}$$

We have the graded $\mathcal{A}$-module decomposition

$$\mathrm{Diff}\,_1(\mathcal{A}) = \mathcal{A} \oplus \mathfrak{d}\mathcal{A}. \tag{6.2.5}$$

Since $\mathfrak{d}\mathcal{A}$ is a Lie $\mathcal{K}$-superalgebra, let us consider the Chevalley–Eilenberg complex $C^*[\mathfrak{d}\mathcal{A}; \mathcal{A}]$ where the graded commutative ring $\mathcal{A}$ is a regarded as a $\mathfrak{d}\mathcal{A}$-module. It is the complex

$$0 \to \mathcal{A} \xrightarrow{d} C^1[\mathfrak{d}\mathcal{A}; \mathcal{A}] \xrightarrow{d} \cdots C^k[\mathfrak{d}\mathcal{A}; \mathcal{A}] \xrightarrow{d} \cdots \tag{6.2.6}$$

where

$$C^k[\mathfrak{d}\mathcal{A}; \mathcal{A}] = \mathrm{Hom}\,_{\mathcal{K}}(\overset{k}{\wedge} \mathfrak{d}\mathcal{A}, \mathcal{A})$$

are $\mathfrak{d}\mathcal{A}$-modules of $\mathcal{K}$-linear graded morphisms of the graded exterior products $\overset{k}{\wedge} \mathfrak{d}\mathcal{A}$ of the $\mathcal{K}$-module $\mathfrak{d}\mathcal{A}$ to $\mathcal{A}$. Let us bring homogeneous elements of $\overset{k}{\wedge} \mathfrak{d}\mathcal{A}$ into the form

$$\varepsilon_1 \wedge \cdots \varepsilon_r \wedge \epsilon_{r+1} \wedge \cdots \wedge \epsilon_k, \qquad \varepsilon_i \in \mathfrak{d}\mathcal{A}_0, \quad \epsilon_j \in \mathfrak{d}\mathcal{A}_1.$$

Then the even coboundary operator d of the complex (6.2.6) is given by the expression

$$dc(\varepsilon_1 \wedge \cdots \wedge \varepsilon_r \wedge \epsilon_1 \wedge \cdots \wedge \epsilon_s) = \tag{6.2.7}$$

$$\sum_{i=1}^{r}(-1)^{i-1}\varepsilon_i c(\varepsilon_1\wedge\cdots\widehat{\varepsilon}_i\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\epsilon_s)+$$
$$\sum_{j=1}^{s}(-1)^{r}\varepsilon_i c(\varepsilon_1\wedge\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\widehat{\epsilon}_j\cdots\wedge\epsilon_s)+$$
$$\sum_{1\leq i<j\leq r}(-1)^{i+j}c([\varepsilon_i,\varepsilon_j]\wedge\varepsilon_1\wedge\cdots\widehat{\varepsilon}_i\cdots\widehat{\varepsilon}_j\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\wedge\epsilon_s)+$$
$$\sum_{1\leq i<j\leq s}c([\epsilon_i,\epsilon_j]\wedge\varepsilon_1\wedge\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\widehat{\epsilon}_i\cdots\widehat{\epsilon}_j\cdots\wedge\epsilon_s)+$$
$$\sum_{1\leq i<r,1\leq j\leq s}(-1)^{i+r+1}c([\varepsilon_i,\epsilon_j]\wedge\varepsilon_1\cdots\widehat{\varepsilon}_i\cdots\wedge\varepsilon_r\wedge\epsilon_1\cdots\widehat{\epsilon}_j\cdots\wedge\epsilon_s),$$

where the caret $\widehat{}$ denotes omission. This operator is called the *graded Chevalley–Eilenberg coboundary operator*.

Let us consider the extended Chevalley–Eilenberg complex

$$0\to\mathcal{K}\xrightarrow{\text{in}}C^*[\mathfrak{d}\mathcal{A};\mathcal{A}].$$

This complex contains a subcomplex $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ of $\mathcal{A}$-linear graded morphisms. The $\mathbb{N}$-graded module $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ is provided with the structure of a bigraded $\mathcal{A}$-algebra with respect to the graded exterior product

$$\begin{aligned}&\phi\wedge\phi'(u_1,...,u_{r+s})= \qquad (6.2.8)\\ &\qquad\sum_{i_1<\cdots<i_r;j_1<\cdots<j_s}\mathrm{Sgn}^{i_1\cdots i_rj_1\cdots j_s}_{1\cdots r+s}\phi(u_{i_1},\ldots,u_{i_r})\phi'(u_{j_1},\ldots,u_{j_s}),\\ &\phi\in\mathcal{O}^r[\mathfrak{d}\mathcal{A}],\qquad \phi'\in\mathcal{O}^s[\mathfrak{d}\mathcal{A}],\qquad u_k\in\mathfrak{d}\mathcal{A},\end{aligned}$$

where $u_1,\ldots,u_{r+s}$ are graded-homogeneous elements of $\mathfrak{d}\mathcal{A}$ and

$$u_1\wedge\cdots\wedge u_{r+s}=\mathrm{Sgn}^{i_1\cdots i_rj_1\cdots j_s}_{1\cdots r+s}u_{i_1}\wedge\cdots\wedge u_{i_r}\wedge u_{j_1}\wedge\cdots\wedge u_{j_s}.$$

The graded Chevalley–Eilenberg coboundary operator d (6.2.7) and the graded exterior product $\wedge$ (6.2.8) bring $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ into a *differential bigraded algebra* (henceforth DBGA) whose elements obey the relations

$$\phi\wedge\phi'=(-1)^{|\phi||\phi'|+[\phi][\phi']}\phi'\wedge\phi, \qquad (6.2.9)$$
$$d(\phi\wedge\phi')=d\phi\wedge\phi'+(-1)^{|\phi|}\phi\wedge d\phi'. \qquad (6.2.10)$$

It is called the *graded Chevalley–Eilenberg differential calculus* over a graded commutative $\mathcal{K}$-ring $\mathcal{A}$. In particular, we have

$$\mathcal{O}^1[\mathfrak{d}\mathcal{A}]=\mathrm{Hom}_{\mathcal{A}}(\mathfrak{d}\mathcal{A},\mathcal{A})=\mathfrak{d}\mathcal{A}^*. \qquad (6.2.11)$$

One can extend this duality relation to the *graded interior product* of $u \in \mathfrak{d}\mathcal{A}$ with any element $\phi \in \mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ by the rules

$$u\rfloor(bda) = (-1)^{[u][b]}u(a), \qquad a, b \in \mathcal{A},$$
$$u\rfloor(\phi \wedge \phi') = (u\rfloor\phi) \wedge \phi' + (-1)^{|\phi|+[\phi][u]}\phi \wedge (u\rfloor\phi'). \qquad (6.2.12)$$

As a consequence, any graded derivation $u \in \mathfrak{d}\mathcal{A}$ of $\mathcal{A}$ yields a derivation

$$\mathbf{L}_u\phi = u\rfloor d\phi + d(u\rfloor\phi), \qquad \phi \in \mathcal{O}^*, \qquad u \in \mathfrak{d}\mathcal{A}, \qquad (6.2.13)$$
$$\mathbf{L}_u(\phi \wedge \phi') = \mathbf{L}_u(\phi) \wedge \phi' + (-1)^{[u][\phi]}\phi \wedge \mathbf{L}_u(\phi'),$$

called the *graded Lie derivative* of the DBGA $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$.

The minimal graded Chevalley–Eilenberg differential calculus $\mathcal{O}^*\mathcal{A} \subset \mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ over a graded commutative ring $\mathcal{A}$ consists of the monomials

$$a_0da_1 \wedge \cdots \wedge da_k, \qquad a_i \in \mathcal{A}.$$

The corresponding complex

$$0 \to \mathcal{K} \longrightarrow \mathcal{A} \xrightarrow{d} \mathcal{O}^1\mathcal{A} \xrightarrow{d} \cdots \mathcal{O}^k\mathcal{A} \xrightarrow{d} \cdots \qquad (6.2.14)$$

is called the *bigraded de Rham complex* of $\mathcal{A}$.

Following the construction of a connection in commutative geometry (see Section 8.2), one comes to the notion of a connection on modules over a real graded commutative ring $\mathcal{A}$. The following are the straightforward counterparts of Definitions 8.2.3 and 8.2.4.

Definition 6.2.1: A *connection* on a graded $\mathcal{A}$-module P is a graded $\mathcal{A}$-module morphism

$$\mathfrak{d}\mathcal{A} \ni u \to \nabla_u \in \mathrm{Diff}\,_1(P, P) \qquad (6.2.15)$$

such that the first order differential operators ∇_u obey the *Grassmann-graded Leibniz rule*

$$\nabla_u(ap) = u(a)p + (-1)^{[a][u]}a\nabla_u(p), \quad a \in \mathcal{A}, \quad p \in P. \qquad (6.2.16)$$

□

Definition 6.2.2: Let P in Definition 6.2.1 be a graded commutative $\mathcal{A}$-ring and $\mathfrak{d}P$ the derivation module of P as a graded commutative $\mathcal{K}$-ring. A *connection* on a graded commutative $\mathcal{A}$-ring P is a graded $\mathcal{A}$-module morphism

$$\mathfrak{d}\mathcal{A} \ni u \to \nabla_u \in \mathfrak{d}P, \tag{6.2.17}$$

which is a connection on P as an $\mathcal{A}$-module, i.e., it obeys the graded Leibniz rule (6.2.16). □

6.3 Graded manifolds

A *graded manifold* of dimension (n,m) is defined as a local-ringed space $(Z,\mathfrak{A})$ where Z is an n-dimensional smooth manifold Z and $\mathfrak{A} = \mathfrak{A}_0 \oplus \mathfrak{A}_1$ is a sheaf of graded commutative algebras of rank m such that:

• there is the exact sequence of sheaves

$$0 \to \mathcal{R} \to \mathfrak{A} \xrightarrow{\sigma} C^\infty_Z \to 0, \qquad \mathcal{R} = \mathfrak{A}_1 + (\mathfrak{A}_1)^2, \tag{6.3.1}$$

where C^∞_Z is the sheaf of smooth real functions on Z;

• $\mathcal{R}/\mathcal{R}^2$ is a locally free sheaf of C^∞_Z-modules of finite rank (with respect to pointwise operations), and the sheaf $\mathfrak{A}$ is locally isomorphic to the exterior product $\wedge_{C^\infty_Z}(\mathcal{R}/\mathcal{R}^2)$.

The sheaf $\mathfrak{A}$ is called a *structure sheaf* of a graded manifold $(Z,\mathfrak{A})$, and a manifold Z is said to be the *body* of $(Z,\mathfrak{A})$. Sections of the sheaf $\mathfrak{A}$ are called *graded functions* on a graded manifold $(Z,\mathfrak{A})$. They make up a graded commutative $C^\infty(Z)$-ring $\mathfrak{A}(Z)$ called the *structure ring* of $(Z,\mathfrak{A})$.

A graded manifold $(Z,\mathfrak{A})$ possesses the following local structure. Given a point $z \in Z$, there exists its open neighborhood U, called a *splitting domain*, such that

$$\mathfrak{A}(U) = C^\infty(U) \otimes \wedge\mathbb{R}^m. \tag{6.3.2}$$

This means that the restriction $\mathfrak{A}|_U$ of the structure sheaf $\mathfrak{A}$ to U is isomorphic to the sheaf $C^\infty_U \otimes \wedge\mathbb{R}^m$ of sections of some exterior bundle

$$\wedge E^*_U = U \times \wedge\mathbb{R}^m \to U.$$

The well-known *Batchelor theorem* states that such a structure of a graded manifold is global as follows.

Theorem 6.3.1: Let $(Z, \mathfrak{A})$ be a graded manifold. There exists a vector bundle $E \to Z$ with an m-dimensional typical fibre V such that the structure sheaf $\mathfrak{A}$ of $(Z, \mathfrak{A})$ is isomorphic to the structure sheaf $\mathfrak{A}_E = S_{\wedge E^*}$ of germs of sections of the exterior bundle $\wedge E^*$ (1.2.2), whose typical fibre is the Grassmann algebra $\wedge V^*$. $\Box$

Note that Batchelor's isomorphism in Theorem 6.3.1 fails to be canonical. In field models, it however is fixed from the beginning. Therefore, we restrict our consideration to graded manifolds $(Z, \mathfrak{A}_E)$ whose structure sheaf is the sheaf of germs of sections of some exterior bundle $\wedge E^*$. We agree to call $(Z, \mathfrak{A}_E)$ a *simple graded manifold* modelled over a vector bundle $E \to Z$, called its *characteristic vector bundle*. Accordingly, the structure ring $\mathcal{A}_E$ of a simple graded manifold $(Z, \mathfrak{A}_E)$ is the structure module

$$\mathcal{A}_E = \mathfrak{A}_E(Z) = \wedge E^*(Z) \tag{6.3.3}$$

of sections of the exterior bundle $\wedge E^*$. Automorphisms of a simple graded manifold $(Z, \mathfrak{A}_E)$ are restricted to those induced by automorphisms of its characteristic vector bundles $E \to Z$ (see Remark 6.3.2).

Combining Batchelor Theorem 6.3.1 and classical Serre–Swan Theorem 8.6.3, we come to the following *Serre–Swan theorem for graded manifolds*.

Theorem 6.3.2: Let Z be a smooth manifold. A graded commutative $C^\infty(Z)$-algebra $\mathcal{A}$ is isomorphic to the structure ring of a graded manifold with a body Z iff it is the exterior algebra of some projective $C^\infty(Z)$-module of finite rank. $\Box$

Given a graded manifold $(Z, \mathfrak{A}_E)$, every trivialization chart $(U; z^A, y^a)$ of the vector bundle $E \to Z$ yields a splitting domain $(U; z^A, c^a)$ of $(Z, \mathfrak{A}_E)$. Graded functions on such a chart are Λ-valued functions

$$f = \sum_{k=0}^{m} \frac{1}{k!} f_{a_1 \ldots a_k}(z) c^{a_1} \cdots c^{a_k}, \tag{6.3.4}$$

where $f_{a_1 \cdots a_k}(z)$ are smooth functions on U and $\{c^a\}$ is the fibre basis for E^*. In particular, the sheaf epimorphism σ in (6.3.1) is induced by the body map of Λ. One calls $\{z^A, c^a\}$ the local *basis for the graded manifold* $(Z, \mathfrak{A}_E)$. Transition functions $y'^a = \rho^a_b(z^A) y^b$ of bundle coordinates on $E \to Z$ induce the corresponding transformation

$$c'^a = \rho^a_b(z^A) c^b \tag{6.3.5}$$

of the associated local basis for the graded manifold $(Z, \mathfrak{A}_E)$ and the according coordinate transformation law of graded functions (6.3.4).

Remark 6.3.1: Strictly speaking, elements c^a of the local basis for a graded manifold are locally constant sections c^a of $E^* \to X$ such that $y_b \circ c^a = \delta^a_b$. Therefore, graded functions are locally represented by Λ-valued functions (6.3.4), but they are not Λ-valued functions on a manifold Z because of the transformation law (6.3.5). □

Remark 6.3.2: In general, automorphisms of a graded manifold read

$$c'^a = \rho^a(z^A, c^b). \tag{6.3.6}$$

Considering a simple graded manifold $(Z, \mathfrak{A}_E)$, we restrict the class of graded manifold transformations (6.3.6) to the linear ones (6.3.5), compatible with given Batchelor's isomorphism. □

Let $E \to Z$ and $E' \to Z$ be vector bundles and $\Phi : E \to E'$ their bundle morphism over a morphism $\varphi : Z \to Z'$. Then every section s^* of the dual bundle $E'^* \to Z'$ defines the pull-back section $\Phi^* s^*$ of the dual bundle $E^* \to Z$ by the law

$$v_z \rfloor \Phi^* s^*(z) = \Phi(v_z) \rfloor s^*(\varphi(z)), \qquad v_z \in E_z.$$

It follows that the bundle morphism (Φ, φ) yields a *morphism of simple graded manifolds*

$$\widehat{\Phi} : (Z, \mathfrak{A}_E) \to (Z', \mathfrak{A}_{E'}) \tag{6.3.7}$$

as local-ringed spaces. This is a pair $(\varphi, \varphi_* \circ \Phi^*)$ of a morphism φ of body manifolds and the composition $\varphi_* \circ \Phi^*$ of the pull-back

$$\mathcal{A}_{E'} \ni f \to \Phi^* f \in \mathcal{A}_E$$

of graded functions and the direct image φ_* of the sheaf $\mathfrak{A}_E$ onto Z'. Relative to local bases (z^A, c^a) and (z'^A, c'^a) for $(Z, \mathfrak{A}_E)$ and $(Z', \mathfrak{A}_{E'})$, the morphism (6.3.7) of graded manifolds reads

$$\widehat{\Phi}(z) = \varphi(z), \qquad \widehat{\Phi}(c'^a) = \Phi^a_b(z) c^b.$$

Given a graded manifold $(Z, \mathfrak{A})$, by the *sheaf* $\mathfrak{d}\mathfrak{A}$ *of graded derivations* of $\mathfrak{A}$ is meant a subsheaf of endomorphisms of the structure sheaf $\mathfrak{A}$ such that any section $u \in \mathfrak{d}\mathfrak{A}(U)$ of $\mathfrak{d}\mathfrak{A}$ over an open subset $U \subset Z$ is a graded derivation of the real graded commutative algebra $\mathfrak{A}(U)$, i.e., $u \in \mathfrak{d}(\mathfrak{A}(U))$. Conversely, one can show that, given open sets $U' \subset U$, there is a surjection of the graded derivation modules

$$\mathfrak{d}(\mathfrak{A}(U)) \to \mathfrak{d}(\mathfrak{A}(U')).$$

It follows that any graded derivation of the local graded algebra $\mathfrak{A}(U)$ also is a local section over U of the sheaf $\mathfrak{d}\mathfrak{A}$. Global sections of $\mathfrak{d}\mathfrak{A}$ are called *graded vector fields* on the graded manifold $(Z, \mathfrak{A})$. They make up the graded derivation module $\mathfrak{d}\mathfrak{A}(Z)$ of the real graded commutative ring $\mathfrak{A}(Z)$. This module is a real Lie superalgebra with the superbracket (6.2.4).

A key point is that graded vector fields $u \in \mathfrak{d}\mathcal{A}_E$ on a simple graded manifold $(Z, \mathfrak{A}_E)$ can be represented by sections of some vector bundle as follows. Due to the canonical splitting $VE = E \times E$, the vertical tangent bundle VE of $E \to Z$ can be provided with the fibre bases

$\{\partial/\partial c^a\}$, which are the duals of the bases $\{c^a\}$. Then graded vector fields on a splitting domain $(U; z^A, c^a)$ of $(Z, \mathfrak{A}_E)$ read

$$u = u^A \partial_A + u^a \frac{\partial}{\partial c^a}, \tag{6.3.8}$$

where u^λ, u^a are local graded functions on U. In particular,

$$\frac{\partial}{\partial c^a} \circ \frac{\partial}{\partial c^b} = -\frac{\partial}{\partial c^b} \circ \frac{\partial}{\partial c^a}, \qquad \partial_A \circ \frac{\partial}{\partial c^a} = \frac{\partial}{\partial c^a} \circ \partial_A.$$

Graded derivations (6.3.8) act on graded functions f (6.3.4) by the rule

$$u(f_{a\ldots b} c^a \cdots c^b) = u^A \partial_A(f_{a\ldots b}) c^a \cdots c^b + u^k f_{a\ldots b} \frac{\partial}{\partial c^k} \rfloor (c^a \cdots c^b). \tag{6.3.9}$$

This rule implies the corresponding coordinate transformation law

$$u'^A = u^A, \qquad u'^a = \rho^a_j u^j + u^A \partial_A(\rho^a_j) c^j$$

of graded vector fields. It follows that graded vector fields (6.3.8) can be represented by sections of the following vector bundle $\mathcal{V}_E \to Z$. This vector bundle is locally isomorphic to the vector bundle

$$\mathcal{V}_E|_U \approx \wedge E^* \underset{Z}{\otimes} (E \underset{Z}{\oplus} TZ)|_U, \tag{6.3.10}$$

and is characterized by an atlas of bundle coordinates

$$(z^A, z^A_{a_1\ldots a_k}, v^i_{b_1\ldots b_k}), \qquad k = 0, \ldots, m,$$

possessing the transition functions

$$z'^A_{i_1\ldots i_k} = \rho^{-1}{}^{a_1}_{i_1} \cdots \rho^{-1}{}^{a_k}_{i_k} z^A_{a_1\ldots a_k},$$
$$v'^i_{j_1\ldots j_k} = \rho^{-1}{}^{b_1}_{j_1} \cdots \rho^{-1}{}^{b_k}_{j_k} \left[\rho^i_j v^j_{b_1\ldots b_k} + \frac{k!}{(k-1)!} z^A_{b_1\ldots b_{k-1}} \partial_A \rho^i_{b_k} \right],$$

which fulfil the cocycle condition (1.1.4). Thus, the graded derivation module $\mathfrak{d}\mathcal{A}_E$ is isomorphic to the structure module $\mathcal{V}_E(Z)$ of global sections of the vector bundle $\mathcal{V}_E \to Z$.

There is the exact sequence

$$0 \to \wedge E^* \underset{Z}{\otimes} E \to \mathcal{V}_E \to \wedge E^* \underset{Z}{\otimes} TZ \to 0 \tag{6.3.11}$$

of vector bundles over Z. Its splitting

$$\tilde{\gamma}: \dot{z}^A\partial_A \to \dot{z}^A\left(\partial_A + \tilde{\gamma}_A^a\frac{\partial}{\partial c^a}\right) \tag{6.3.12}$$

transforms every vector field τ on Z into the graded vector field

$$\tau = \tau^A\partial_A \to \nabla_\tau = \tau^A\left(\partial_A + \tilde{\gamma}_A^a\frac{\partial}{\partial c^a}\right), \tag{6.3.13}$$

which is a graded derivation of the real graded commutative ring $\mathcal{A}_E$ (6.3.3) satisfying the Leibniz rule

$$\nabla_\tau(sf) = (\tau\rfloor ds)f + s\nabla_\tau(f), \quad f\in\mathcal{A}_E, \quad s\in C^\infty(Z).$$

It follows that the splitting (6.3.12) of the exact sequence (6.3.11) yields a connection on the graded commutative $C^\infty(Z)$-ring $\mathcal{A}_E$ in accordance with Definition 6.2.2. It is called a *graded connection* on the simple graded manifold $(Z,\mathfrak{A}_E)$. In particular, this connection provides the corresponding horizontal splitting

$$u = u^A\partial_A + u^a\frac{\partial}{\partial c^a} = u^A\left(\partial_A + \tilde{\gamma}_A^a\frac{\partial}{\partial c^a}\right) + (u^a - u^A\tilde{\gamma}_A^a)\frac{\partial}{\partial c^a}$$

of graded vector fields. In accordance with Theorem 1.2.2, a graded connection (6.3.12) always exists.

Remark 6.3.3: By virtue of the isomorphism (6.3.2), any connection $\tilde{\gamma}$ on a graded manifold $(Z,\mathfrak{A})$, restricted to a splitting domain U, takes the form (6.3.12). Given two splitting domains U and U' of $(Z,\mathfrak{A})$ with the transition functions (6.3.6), the connection components $\tilde{\gamma}_A^a$ obey the transformation law

$$\tilde{\gamma}'^a_A = \tilde{\gamma}_A^b\frac{\partial}{\partial c^b}\rho^a + \partial_A\rho^a. \tag{6.3.14}$$

If U and U' are the trivialization charts of the same vector bundle E in Theorem 6.3.1 together with the transition functions (6.3.5), the transformation law (6.3.14) takes the form

$$\tilde{\gamma}'^a_A = \rho^a_b(z)\tilde{\gamma}_A^b + \partial_A\rho^a_b(z)c^b. \tag{6.3.15}$$

□

Remark 6.3.4: Every linear connection

$$\gamma = dz^A \otimes (\partial_A + \gamma_A{}^a{}_b y^b \partial_a)$$

on a vector bundle $E \to Z$ yields the graded connection

$$\gamma_S = dz^A \otimes \left(\partial_A + \gamma_A{}^a{}_b c^b \frac{\partial}{\partial c^a}\right) \tag{6.3.16}$$

on the simple graded manifold $(Z, \mathfrak{A}_E)$ modelled over E. In view of Remark 6.3.3, γ_S also is a graded connection on the graded manifold $(Z, \mathfrak{A}) \cong (Z, \mathfrak{A}_E)$, but its linear form (6.3.16) is not maintained under the transformation law (6.3.14). □

6.4 Graded differential forms

Given the structure ring $\mathcal{A}_E$ of graded functions on a simple graded manifold $(Z, \mathfrak{A}_E)$ and the real Lie superalgebra $\mathfrak{d}\mathcal{A}_E$ of its graded derivations, let us consider the graded Chevalley–Eilenberg differential calculus

$$\mathcal{S}^*[E; Z] = \mathcal{O}^*[\mathfrak{d}\mathcal{A}_E] \tag{6.4.1}$$

over $\mathcal{A}_E$. Since the graded derivation module $\mathfrak{d}\mathcal{A}_E$ is isomorphic to the structure module of sections of the vector bundle $\mathcal{V}_E \to Z$, elements of $\mathcal{S}^*[E; Z]$ are sections of the exterior bundle $\wedge\overline{\mathcal{V}}_E$ of the $\mathcal{A}_E$-dual $\overline{\mathcal{V}}_E \to Z$ of $\mathcal{V}_E$. The bundle $\overline{\mathcal{V}}_E$ is locally isomorphic to the vector bundle

$$\overline{\mathcal{V}}_E|_U \approx (E^* \underset{Z}{\oplus} T^*Z)|_U. \tag{6.4.2}$$

With respect to the dual fibre bases $\{dz^A\}$ for T^*Z and $\{dc^b\}$ for E^*, sections of $\overline{\mathcal{V}}_E$ take the coordinate form

$$\phi = \phi_A dz^A + \phi_a dc^a,$$

together with transition functions

$$\phi'_a = \rho^{-1b}{}_a \phi_b, \qquad \phi'_A = \phi_A + \rho^{-1b}{}_a \partial_A(\rho^a_j) \phi_b c^j.$$

The duality isomorphism $\mathcal{S}^1[E;Z] = \mathfrak{d}\mathcal{A}_E^*$ (6.2.11) is given by the graded interior product

$$u\rfloor\phi = u^A\phi_A + (-1)^{[\phi_a]}u^a\phi_a. \tag{6.4.3}$$

Elements of $\mathcal{S}^*[E;Z]$ are called *graded exterior forms* on the graded manifold $(Z,\mathfrak{A}_E)$.

Seen as an $\mathcal{A}_E$-algebra, the DBGA $\mathcal{S}^*[E;Z]$ (6.4.1) on a splitting domain $(U;z^A,c^a)$ is locally generated by the graded one-forms dz^A, dc^i such that

$$dz^A\wedge dc^i = -dc^i\wedge dz^A, \qquad dc^i\wedge dc^j = dc^j\wedge dc^i. \tag{6.4.4}$$

Accordingly a graded Chevalley–Eilenberg coboundary operator d (6.2.7), called the *graded exterior differential*, reads

$$d\phi = dz^A\wedge\partial_A\phi + dc^a\wedge\frac{\partial}{\partial c^a}\phi,$$

where the derivatives ∂_λ, $\partial/\partial c^a$ act on coefficients of graded exterior forms by the formula (6.3.9), and they are graded commutative with the graded forms dz^A and dc^a. The formulas (6.2.9) – (6.2.13) hold.

Theorem 6.4.1: The DBGA $\mathcal{S}^*[E;Z]$ (6.4.1) is a minimal differential calculus over $\mathcal{A}_E$, i.e., it is generated by elements df, $f\in\mathcal{A}_E$. □

The bigraded de Rham complex (6.2.14) of the minimal Chevalley–Eilenberg differential calculus $\mathcal{S}^*[E;Z]$ reads

$$0\to\mathbb{R}\to\mathcal{A}_E \xrightarrow{d} \mathcal{S}^1[E;Z] \xrightarrow{d} \cdots \mathcal{S}^k[E;Z] \xrightarrow{d} \cdots. \tag{6.4.5}$$

Its cohomology $H^*(\mathcal{A}_E)$ is called the *de Rham cohomology of a simple graded manifold* $(Z,\mathfrak{A}_E)$.

In particular, given the DGA $\mathcal{O}^*(Z)$ of exterior forms on Z, there exist the canonical monomorphism

$$\mathcal{O}^*(Z)\to\mathcal{S}^*[E;Z] \tag{6.4.6}$$

and the body epimorphism $\mathcal{S}^*[E;Z]\to\mathcal{O}^*(Z)$ which are cochain morphisms of the de Rham complexes (6.4.5) and (8.6.5).

Theorem 6.4.2: The de Rham cohomology of a simple graded manifold $(Z, \mathfrak{A}_E)$ equals the de Rham cohomology of its body Z. $\Box$

Corollary 6.4.3: Any closed graded exterior form is decomposed into a sum $\phi = \sigma + d\xi$ where σ is a closed exterior form on Z. $\Box$

Chapter 7

Lagrangian theory

Lagrangian theory on fibre bundles is algebraically formulated in terms of the variational bicomplex without appealing to the calculus of variations. This formulation is extended to Lagrangian theory on graded manifolds [9, 24].

7.1 Variational bicomplex

Let $Y \to X$ be a fibre bundle. The DGA $\mathcal{O}^*_\infty$ (2.4.6), decomposed into the variational bicomplex, describes finite order Lagrangian theories on $Y \to X$. One also considers the variational bicomplex of the DGA $\mathcal{Q}^*_\infty$ (2.4.8) and different variants of the variational sequence of finite jet order.

In order to transform the bicomplex $\mathcal{O}^{*,*}_\infty$ into the variational one, let us consider the following two operators acting on $\mathcal{O}^{*,n}_\infty$.

(i) There exists an $\mathbb{R}$-module endomorphism

$$\varrho = \sum_{k>0} \frac{1}{k}\overline{\varrho} \circ h_k \circ h^n : \mathcal{O}^{*>0,n}_\infty \to \mathcal{O}^{*>0,n}_\infty, \tag{7.1.1}$$
$$\overline{\varrho}(\phi) = \sum_{0\leq|\Lambda|} (-1)^{|\Lambda|}\theta^i \wedge [d_\Lambda(\partial^\Lambda_i \rfloor \phi)], \qquad \phi \in \mathcal{O}^{>0,n}_\infty,$$

possessing the following properties.

Lemma 7.1.1: For any $\phi \in \mathcal{O}^{>0,n}_\infty$, the form $\phi - \varrho(\phi)$ is locally d_H-exact on each coordinate chart (2.4.3). □

Lemma 7.1.2: The operator ϱ obeys the relation

$$(\varrho \circ d_H)(\psi) = 0, \qquad \psi \in \mathcal{O}_\infty^{>0,n-1}. \tag{7.1.2}$$

□

It follows from Lemmas 7.1.1 and 7.1.2 that ϱ (7.1.1) is a projector.

(ii) One defines the *variational operator*

$$\delta = \varrho \circ d : \mathcal{O}_\infty^{*,n} \to \mathcal{O}_\infty^{*+1,n}, \tag{7.1.3}$$

which is nilpotent, i.e., $\delta \circ \delta = 0$, and obeys the relation $\delta \circ \varrho = \delta$.

Let us denote $\mathbf{E}_k = \varrho(\mathcal{O}_\infty^{k,n})$. Provided with the operators d_H, d_V, ϱ and δ, the DGA $\mathcal{O}_\infty^*$ is decomposed into the *variational bicomplex*

$$\begin{array}{cccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & \\
 & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle -\delta}\uparrow & \\
 & 0 \to & \mathcal{O}_\infty^{1,0} & \xrightarrow{d_H} & \mathcal{O}_\infty^{1,1} & \xrightarrow{d_H} \cdots & \mathcal{O}_\infty^{1,n} & \xrightarrow{\varrho} & \mathbf{E}_1 \to 0 & \\
 & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle -\delta}\uparrow & \\
0 \to \mathbb{R} \to & & \mathcal{O}_\infty^{0} & \xrightarrow{d_H} & \mathcal{O}_\infty^{0,1} & \xrightarrow{d_H} \cdots & \mathcal{O}_\infty^{0,n} & \equiv & \mathcal{O}_\infty^{0,n} & \\
 & & \uparrow & & \uparrow & & \uparrow & & & \\
0 \to \mathbb{R} \to & & \mathcal{O}^0(X) & \xrightarrow{d} & \mathcal{O}^1(X) & \xrightarrow{d} \cdots & \mathcal{O}^n(X) & \xrightarrow{d} & 0 & \\
 & & \uparrow & & \uparrow & & \uparrow & & & \\
 & & 0 & & 0 & & 0 & & &
\end{array} \tag{7.1.4}$$

It possesses the following cohomology [20, 24].

Theorem 7.1.3: The second row from the bottom and the last column of the variational bicomplex (7.1.4) make up the *variational complex*

$$0 \to \mathbb{R} \to \mathcal{O}_\infty^0 \xrightarrow{d_H} \mathcal{O}_\infty^{0,1} \cdots \xrightarrow{d_H} \mathcal{O}_\infty^{0,n} \xrightarrow{\delta} \mathbf{E}_1 \xrightarrow{\delta} \mathbf{E}_2 \longrightarrow \cdots . \tag{7.1.5}$$

Its cohomology is isomorphic to the de Rham cohomology of a fibre bundle Y, namely,

$$H^{k<n}(d_H; \mathcal{O}_\infty^*) = H^{k<n}_{\mathrm{DR}}(Y), \qquad H^{k\geq n}(\delta; \mathcal{O}_\infty^*) = H^{k\geq n}_{\mathrm{DR}}(Y). \tag{7.1.6}$$

□

Theorem 7.1.4: The rows of contact forms of the variational bicomplex (7.1.4) are exact sequences. □

The cohomology isomorphism (7.1.6) gives something more. Due to the relations $d_H \circ h_0 = h_0 \circ d$ and $\delta \circ \varrho = \delta$, we have the cochain morphism of the de Rham complex (2.4.7) of the DGA $\mathcal{O}_\infty^*$ to its variational complex (7.1.5). By virtue of Theorems 2.4.3 and 7.1.3, the corresponding homomorphism of their cohomology groups is an isomorphism. Then the splitting of a closed form $\phi \in \mathcal{O}_\infty^*$ in Corollary 2.4.4 leads to the following decompositions.

Theorem 7.1.5: Any d_H-closed form $\phi \in \mathcal{O}^{0,m}$, $m < n$, is represented by a sum

$$\phi = h_0\sigma + d_H\xi, \qquad \xi \in \mathcal{O}_\infty^{m-1}, \tag{7.1.7}$$

where σ is a closed m-form on Y. Any δ-closed form $\phi \in \mathcal{O}^{k,n}$ is split into

$$\phi = h_0\sigma + d_H\xi, \qquad k = 0, \qquad \xi \in \mathcal{O}_\infty^{0,n-1}, \tag{7.1.8}$$

$$\phi = \varrho(\sigma) + \delta(\xi), \qquad k = 1, \qquad \xi \in \mathcal{O}_\infty^{0,n}, \tag{7.1.9}$$

$$\phi = \varrho(\sigma) + \delta(\xi), \qquad k > 1, \qquad \xi \in \mathbf{E}_{k-1}, \tag{7.1.10}$$

where σ is a closed $(n+k)$-form on Y. □

7.2 Lagrangian theory on fibre bundles

In Lagrangian formalism on fibre bundles, a finite order *Lagrangian* and its *Euler–Lagrange operator* are defined as elements

$$L = \mathcal{L}\omega \in \mathcal{O}_\infty^{0,n}, \tag{7.2.1}$$

$$\delta L = \mathcal{E}_L = \mathcal{E}_i\theta^i \wedge \omega \in \mathbf{E}_1, \tag{7.2.2}$$

$$\mathcal{E}_i = \sum_{0\leq|\Lambda|} (-1)^{|\Lambda|} d_\Lambda(\partial_i^\Lambda \mathcal{L}), \tag{7.2.3}$$

of the variational complex (7.1.5) (see the notation (1.4.1)). Components $\mathcal{E}_i$ (7.2.3) of the Euler–Lagrange operator (7.2.2) are called the *variational derivatives*. Elements of $\mathbf{E}_1$ are called the *Euler–Lagrange-type operators*.

Hereafter, we call a pair $(\mathcal{O}^*_\infty, L)$ the *Lagrangian system*. The following are corollaries of Theorem 7.1.5.

Corollary 7.2.1: A finite order Lagrangian L (7.2.1) is *variationally trivial*, i.e., $\delta(L) = 0$ iff

$$L = h_0\sigma + d_H\xi, \qquad \xi \in \mathcal{O}^{0,n-1}_\infty, \tag{7.2.4}$$

where σ is a closed n-form on Y. $\Box$

Corollary 7.2.2: A finite order Euler–Lagrange-type operator $\mathcal{E} \in \mathbf{E}_1$ satisfies the *Helmholtz condition* $\delta(\mathcal{E}) = 0$ iff

$$\mathcal{E} = \delta L + \varrho(\sigma), \qquad L \in \mathcal{O}^{0,n}_\infty,$$

where σ is a closed $(n+1)$-form on Y. $\Box$

A glance at the expression (7.2.2) shows that, if a Lagrangian L (7.2.1) is of r-order, its Euler–Lagrange operator $\mathcal{E}_L$ is of $2r$-order. Its kernel is called the *Euler–Lagrange equation*. Euler–Lagrange equations traditionally came from the *variational formula*

$$dL = \delta L - d_H\Xi_L \tag{7.2.5}$$

of the calculus of variations. In formalism of the variational bicomplex, this formula is a corollary of Theorem 7.1.4.

Corollary 7.2.3: The exactness of the row of one-contact forms of the variational bicomplex (7.1.4) at the term $\mathcal{O}^{1,n}_\infty$ relative to the projector ϱ provides the $\mathbb{R}$-module decomposition

$$\mathcal{O}^{1,n}_\infty = \mathbf{E}_1 \oplus d_H(\mathcal{O}^{1,n-1}_\infty).$$

In particular, any Lagrangian L admits the decomposition (7.2.5). $\Box$

Defined up to a d_H-closed term, a form $\Xi_L\in\mathcal{O}^n_\infty$ in the variational formula (7.2.5) reads

$$\Xi_L = L + [(\partial^\lambda_i\mathcal{L} - d_\mu F^{\mu\lambda}_i)\theta^i + \sum_{s=1} F^{\lambda\nu_s\ldots\nu_1}_i\theta^i_{\nu_s\ldots\nu_1}]\wedge\omega_\lambda, \tag{7.2.6}$$
$$F^{\nu_k\ldots\nu_1}_i = \partial^{\nu_k\ldots\nu_1}_i\mathcal{L} - d_\mu F^{\mu\nu_k\ldots\nu_1}_i + \sigma^{\nu_k\ldots\nu_1}_i, \qquad k=2,3,\ldots,$$

where $\sigma^{\nu_k\ldots\nu_1}_i$ are local functions such that

$$\sigma^{(\nu_k\nu_{k-1})\ldots\nu_1}_i = 0.$$

The form Ξ_L (7.2.6) possesses the following properties:

- $h_0(\Xi_L) = L$,
- $h_0(\vartheta\rfloor d\Xi_L) = \vartheta^i\mathcal{E}_i\omega$ for any derivation ϑ (2.4.12).

Consequently, Ξ_L is a Lepage equivalent of a Lagrangian L.

A special interest is concerned with Lagrangian theories on an affine bundle $Y\to X$. Since X is a strong deformation retract of an affine bundle Y, the de Rham cohomology of Y equals that of X. In this case, the cohomology (7.1.6) of the variational complex (7.1.5) equals the de Rham cohomology of X, namely,

$$\begin{aligned}
&H^{<n}(d_H;\mathcal{O}^*_\infty) = H^{<n}_{\rm DR}(X),\\
&H^n(\delta;\mathcal{O}^*_\infty) = H^n_{\rm DR}(X),\\
&H^{>n}(\delta;\mathcal{O}^*_\infty) = 0.
\end{aligned} \tag{7.2.7}$$

It follows that every d_H-closed form $\phi\in\mathcal{O}^{0,m<n}_\infty$ is represented by the sum

$$\phi = \sigma + d_H\xi, \qquad \xi\in\mathcal{O}^{0,m-1}_\infty, \tag{7.2.8}$$

where σ is a closed m-form on X. Similarly, any variationally trivial Lagrangian takes the form

$$L = \sigma + d_H\xi, \qquad \xi\in\mathcal{O}^{0,n-1}_\infty, \tag{7.2.9}$$

where σ is an n-form on X.

In view of the cohomology isomorphism (7.2.7), if $Y \to X$ is an affine bundle, let us restrict our consideration to the *short variational complex*

$$0 \to \mathbb{R} \to \mathcal{O}^0_\infty \xrightarrow{d_H} \mathcal{O}^{0,1}_\infty \cdots \xrightarrow{d_H} \mathcal{O}^{0,n}_\infty \xrightarrow{\delta} \mathbf{E}_1, \qquad (7.2.10)$$

whose non-trivial cohomology equals that of the variational complex (7.1.5). Let us consider a DGA $\mathcal{P}^*_\infty \subset \mathcal{O}^*_\infty$ of exterior forms whose coefficients are polynomials in jet coordinates y^i_Λ, $0 \leq |\Lambda|$, of the continuous bundle $J^\infty Y \to X$.

Theorem 7.2.4: The cohomology of the short variational complex

$$0 \to \mathbb{R} \to \mathcal{P}^0_\infty \xrightarrow{d_H} \mathcal{P}^{0,1}_\infty \cdots \xrightarrow{d_H} \mathcal{P}^{0,n}_\infty \xrightarrow{\delta} 0 \qquad (7.2.11)$$

of the polynomial algebra $\mathcal{P}^*_\infty$ equals that of the complex (7.2.10), i.e., the de Rham cohomology of X. □

Given a Lagrangian system $(\mathcal{O}^*_\infty, L)$, its *infinitesimal transformations* are defined to be contact derivations of the ring $\mathcal{O}^0_\infty$.

A derivation $\vartheta \in \mathfrak{d}\mathcal{O}^0_\infty$ (2.4.12) is called *contact* if the Lie derivative $\mathbf{L}_\upsilon$ preserves the ideal of contact forms of the DGA $\mathcal{O}^*_\infty$, i.e., the Lie derivative $\mathbf{L}_\upsilon$ of a contact form is a contact form.

Lemma 7.2.5: A derivation ϑ (2.4.12) is contact iff it takes the form

$$\vartheta = \upsilon^\lambda\partial_\lambda + \upsilon^i\partial_i + \sum_{0<|\Lambda|}[d_\Lambda(\upsilon^i - y^i_\mu\upsilon^\mu) + y^i_{\mu+\Lambda}\upsilon^\mu]\partial^\Lambda_i. \qquad (7.2.12)$$

□

The expression (2.2.8) enables one to regard a contact derivation ϑ (7.2.12) as an infinite order jet prolongation $\vartheta = J^\infty\upsilon$ of its restriction

$$\upsilon = \upsilon^\lambda\partial_\lambda + \upsilon^i\partial_i \qquad (7.2.13)$$

to the ring $C^\infty(Y)$. Since coefficients υ^λ and υ^i of υ (7.2.13) depend generally on jet coordinates y^i_Λ, $0 < |\Lambda|$, one calls υ (7.2.13) a *generalized vector field*. It can be represented as a section of some pull-back bundle

$$J^rY \underset{Y}{\times} TY \to J^rY.$$

A contact derivation ϑ (7.2.12) is called *projectable*, if the generalized vector field υ (7.2.13) projects onto a vector field $\upsilon^\lambda\partial_\lambda$ on X.

Any contact derivation ϑ (7.2.12) admits the horizontal splitting

$$\vartheta = \vartheta_H + \vartheta_V = \upsilon^\lambda d_\lambda + [\upsilon^i_V\partial_i + \sum_{0<|\Lambda|} d_\Lambda \upsilon^i_V \partial_i^\Lambda], \tag{7.2.14}$$

$$\upsilon = \upsilon_H + \upsilon_V = \upsilon^\lambda d_\lambda + (\upsilon^i - y^i_\mu \upsilon^\mu)\partial_i, \tag{7.2.15}$$

relative to the canonical connection ∇ (2.4.14) on the $C^\infty(X)$-ring $\mathcal{O}^0_\infty$.

Lemma 7.2.6: Any *vertical contact derivation*

$$\vartheta = \upsilon^i\partial_i + \sum_{0<|\Lambda|} d_\Lambda \upsilon^i \partial_i^\Lambda \tag{7.2.16}$$

obeys the relations

$$\vartheta\rfloor d_H\phi = -d_H(\vartheta\rfloor\phi), \tag{7.2.17}$$

$$\mathbf{L}_\vartheta(d_H\phi) = d_H(\mathbf{L}_\vartheta\phi), \qquad \phi \in \mathcal{O}^*_\infty. \tag{7.2.18}$$

□

The global decomposition (7.2.5) leads to the following first variational formula (Theorem 7.2.7) and the first Noether theorem (Theorem 7.2.9).

Theorem 7.2.7: Given a Lagrangian $L \in \mathcal{O}^{0,n}_\infty$, its Lie derivative $\mathbf{L}_\upsilon L$ along a contact derivation υ (7.2.14) fulfils the *first variational formula*

$$\mathbf{L}_\vartheta L = \upsilon_V\rfloor\delta L + d_H(h_0(\vartheta\rfloor\Xi_L)) + \mathcal{L}d_V(\upsilon_H\rfloor\omega), \tag{7.2.19}$$

where Ξ_L is the Lepage equivalent (7.2.6) of L. □

A contact derivation ϑ (7.2.12) is called a *variational symmetry* of a Lagrangian L if the Lie derivative $\mathbf{L}_\vartheta L$ is d_H-exact, i.e.,

$$\mathbf{L}_\vartheta L = d_H\sigma. \tag{7.2.20}$$

Lemma 7.2.8: A glance at the expression (7.2.19) shows the following.

(i) A contact derivation ϑ is a variational symmetry only if it is projectable.

(ii) Any projectable contact derivation is a variational symmetry of a variationally trivial Lagrangian. It follows that, if ϑ is a variational symmetry of a Lagrangian L, it also is a variational symmetry of a Lagrangian $L + L_0$, where L_0 is a variationally trivial Lagrangian.

(iii) A projectable contact derivations ϑ is a variational symmetry iff its vertical part υ_V (7.2.14) is well.

(iv) A projectable contact derivations ϑ is a variational symmetry iff the density $\upsilon_V \rfloor \delta L$ is d_H-exact. $\square$

It is readily observed that variational symmetries of a Lagrangian L constitute a real vector subspace $\mathcal{G}_L$ of the derivation module $\mathfrak{d}\mathcal{O}^0_\infty$. By virtue of item (ii) of Lemma 7.2.8, the Lie bracket

$$\mathbf{L}_{[\vartheta,\vartheta']} = [\mathbf{L}_\vartheta, \mathbf{L}_{\vartheta'}]$$

of variational symmetries is a variational symmetry and, therefore, their vector space $\mathcal{G}_L$ is a real Lie algebra. The following is the *first Noether theorem*.

Theorem 7.2.9: If a contact derivation ϑ (7.2.12) is a variational symmetry (7.2.20) of a Lagrangian L, the first variational formula (7.2.19) restricted to the kernel of the Euler–Lagrange operator $\operatorname{Ker}\mathcal{E}_L$ leads to the *weak conservation law*

$$0 \approx d_H(h_0(\vartheta \rfloor \Xi_L) - \sigma) \tag{7.2.21}$$

on the shell $\delta L = 0$. $\square$

A variational symmetry ϑ of a Lagrangian L is called its *exact symmetry* or, simply, a *symmetry* if

$$\mathbf{L}_\vartheta L = 0. \tag{7.2.22}$$

Symmetries of a Lagrangian L constitute a real vector space, which is a real Lie algebra. Its vertical symmetries υ (7.2.16) obey the relation

$$\mathbf{L}_\upsilon L = \upsilon \rfloor dL$$

and, therefore, make up a $\mathcal{O}^0_\infty$-module which is a Lie $C^\infty(X)$-algebra.

If ϑ is an exact symmetry of a Lagrangian L, the weak conservation law (7.2.21) takes the form

$$0 \approx d_H(h_0(\vartheta\rfloor\Xi_L)) = -d_H\mathcal{J}_\upsilon, \tag{7.2.23}$$

where

$$\mathcal{J}_\upsilon = \mathcal{J}^\mu_\upsilon\omega_\mu = -h_0(\vartheta\rfloor\Xi_L) \tag{7.2.24}$$

is called the *symmetry current*. Of course, the symmetry current (7.2.24) is defined with the accuracy of a d_H-closed term.

Let ϑ be an exact symmetry of a Lagrangian L. Whenever L_0 is a variationally trivial Lagrangian, ϑ is a variational symmetry of the Lagrangian $L + L_0$ such that the weak conservation law (7.2.21) for this Lagrangian is reduced to the weak conservation law (7.2.23) for a Lagrangian L as follows:

$$\mathbf{L}_\vartheta(L + L_0) = d_H\sigma \approx d_H\sigma - d_H\mathcal{J}_\upsilon.$$

Remark 7.2.1: In accordance with standard terminology, variational and exact symmetries generated by generalized vector fields (7.2.13) are called *generalized symmetries* because they depend on derivatives of variables. Accordingly, by variational symmetries and symmetries one means only those generated by vector fields u on Y. We agree to call them *classical symmetries*. $\Box$

Let ϑ be a classical variational symmetry of a Lagrangian L, i.e., ϑ (7.2.12) is the jet prolongation of a vector field u on Y. Then the relation

$$\mathbf{L}_\vartheta\mathcal{E}_L = \delta(\mathbf{L}_\vartheta L) \tag{7.2.25}$$

holds. It follows that ϑ also is a symmetry of the Euler–Lagrange operator $\mathcal{E}_L$ of L, i.e., $\mathbf{L}_\vartheta\mathcal{E}_L = 0$. However, the equality (7.2.25) fails to be true in the case of generalized symmetries.

Definition 7.2.10: Let $E \to X$ be a vector bundle and $E(X)$ the $C^\infty(X)$ module $E(X)$ of sections of $E \to X$. Let ζ be a linear differential operator on $E(X)$ taking values into the vector space $\mathcal{G}_L$ of variational symmetries of a Lagrangian L (see Definition 8.2.1). Elements

$$u_\xi = \zeta(\xi) \tag{7.2.26}$$

of $\operatorname{Im}\zeta$ are called the *gauge symmetry* of a Lagrangian L parameterized by sections ξ of $E \to X$. They are called the *gauge parameters*. □

Remark 7.2.2: The differential operator ζ in Definition 7.2.10 takes its values into the vector space $\mathcal{G}_L$ as a subspace of the $C^\infty(X)$-module $\mathfrak{d}\mathcal{O}^0_\infty$, but it sends the $C^\infty(X)$-module $E(X)$ into the real vector space $\mathcal{G}_L \subset \mathfrak{d}\mathcal{O}^0_\infty$. The differential operator ζ is assumed to be at least of first order. □

Equivalently, the gauge symmetry (7.2.26) is given by a section $\widetilde\zeta$ of the fibre bundle

$$(J^rY \underset{Y}{\times} J^mE) \underset{Y}{\times} TY \to J^rY \underset{Y}{\times} J^mE$$

(see Definition 2.3.2) such that

$$u_\xi = \zeta(\xi) = \widetilde\zeta \circ \xi$$

for any section ξ of $E \to X$. Hence, it is a generalized vector field u_ζ on the product $Y \times E$ represented by a section of the pull-back bundle

$$J^k(Y \underset{X}{\times} E) \underset{Y}{\times} T(Y \underset{X}{\times} E) \to J^k(Y \underset{X}{\times} E), \qquad k = \max(r, m),$$

which lives in $TY \subset T(Y \times E)$. This generalized vector field yields a contact derivation $J^\infty u_\zeta$ (7.2.12) of the real ring $\mathcal{O}^0_\infty[Y \times E]$ which obeys the following condition.

Condition 7.2.11: Given a Lagrangian

$$L \in \mathcal{O}^{0,n}_\infty E \subset \mathcal{O}^{0,n}_\infty[Y \times E],$$

let us consider its Lie derivative

$$\mathbf{L}_{J^\infty u_\zeta} L = J^\infty u_\zeta \rfloor dL + d(J^\infty u_\zeta \rfloor L) \tag{7.2.27}$$

where d is the exterior differential of $\mathcal{O}^0_\infty[Y \times E]$. Then, for any section ξ of $E \to X$, the pull-back $\xi^* \mathbf{L}_\vartheta$ is d_H-exact. □

It follows from the first variational formula (7.2.19) for the Lie derivative (7.2.27) that Condition 7.2.11 holds only if u_ζ projects onto a generalized vector field on E and, in this case, iff the density $(u_\zeta)_V \rfloor \mathcal{E}$ is d_H-exact. Thus, we come to the following equivalent definition of gauge symmetries.

Definition 7.2.12: Let $E \to X$ be a vector bundle. A gauge symmetry of a Lagrangian L parameterized by sections ξ of $E \to X$ is defined as a contact derivation $\vartheta = J^\infty u$ of the real ring $\mathcal{O}^0_\infty[Y \times E]$ such that:

(i) it vanishes on the subring $\mathcal{O}^0_\infty E$,

(ii) the generalized vector field u is linear in coordinates χ^a_Λ on $J^\infty E$, and it projects onto a generalized vector field on E, i.e., it takes the form

$$u = \left(\sum_{0 \leq |\Lambda| \leq m} u_a^{\lambda\Lambda}(x^\mu)\chi^a_\Lambda\right)\partial_\lambda + \left(\sum_{0 \leq |\Lambda| \leq m} u_a^{i\Lambda}(x^\mu, y^j_\Sigma)\chi^a_\Lambda\right)\partial_i, \tag{7.2.28}$$

(iii) the vertical part of u (7.2.28) obeys the equality

$$u_V \rfloor \mathcal{E} = d_H \sigma. \tag{7.2.29}$$

□

For the sake of convenience, we also call a generalized vector field (7.2.28) the gauge symmetry. By virtue of item (iii) of Definition 7.2.12, u (7.2.28) is a gauge symmetry iff its vertical part is so.

Gauge symmetries possess the following particular properties.

(i) Let $E' \to X$ be a vector bundle and ζ' a linear $E(X)$-valued differential operator on the $C^\infty(X)$-module $E'(X)$ of sections of $E' \to X$. Then

$$u_{\zeta'(\xi')} = (\zeta \circ \zeta')(\xi')$$

also is a gauge symmetry of L parameterized by sections ξ' of $E' \to X$. It factorizes through the gauge symmetries u_ϕ (7.2.26).

(ii) If a gauge symmetry is an exact Lagrangian symmetry, the corresponding conserved symmetry current $\mathcal{J}_u$ (7.2.24) is reduced to a superpotential (see Theorem 7.5.4).

(iii) The *direct second Noether theorem* associates to a gauge symmetry of a Lagrangian L the Noether identities of its Euler–Lagrange operator δL.

Theorem 7.2.13: Let u (7.2.28) be a gauge symmetry of a Lagrangian L, then its Euler–Lagrange operator δL obeys the Noether identities

$$\begin{aligned}\mathcal{E}_a = \sum_{0\leq|\Lambda|} (-1)^{|\Lambda|} d_\Lambda[(u_a^{i\Lambda} - y^i_\lambda u_a^{\lambda\Lambda})\mathcal{E}_i] = \\ \sum_{0\leq|\Lambda|} \eta(u^i_a - y^i_\lambda u^\lambda_a)^\Lambda d_\Lambda \mathcal{E}_i = 0\end{aligned} \tag{7.2.30}$$

(see Notation 7.5.2). □

It follows from direct second Noether Theorem 7.2.13 that gauge symmetries of Lagrangian field theory characterize its degeneracy. A problem is that any Lagrangian possesses gauge symmetries and, therefore, one must separate them into the trivial and non-trivial ones. Moreover, gauge symmetries can be *reducible*, i.e., $\mathrm{Ker}\,\zeta \neq 0$. To solve these problems, we follow a different definition of gauge symmetries as those associated to non-trivial Noether identities by means of inverse second Noether Theorem 7.5.3.

7.3 Grassmann-graded Lagrangian theory

We start with the following definition of jets of odd variables. Let us consider a vector bundle $F \to X$ and the simple graded manifolds $(X, \mathcal{A}_{J^rF})$ modelled over the vector bundles $J^rF \to X$. There is the direct system of the corresponding DBGA

$$\mathcal{S}^*[F;X] \longrightarrow \mathcal{S}^*[J^1F;X] \longrightarrow \cdots \mathcal{S}^*[J^rF;X] \longrightarrow \cdots$$

of graded exterior forms on graded manifolds $(X, \mathcal{A}_{J^rF})$. Its direct limit $\mathcal{S}^*_\infty[F;X]$ is the Grassmann-graded counterpart of the DGA $\mathcal{P}^*_\infty$.

In order to describe Lagrangian theories both of even and odd variables, let us consider a composite bundle

$$F \to Y \to X \tag{7.3.1}$$

where $F \to Y$ is a vector bundle provided with bundle coordinates (x^λ, y^i, q^a). We call the simple graded manifold $(Y, \mathfrak{A}_F)$ modelled over $F \to Y$ the *composite graded manifold*. Let us associate to this graded manifold the following DBGA $\mathcal{S}^*_\infty[F;Y]$.

It is readily observed that the jet manifold J^rF of $F \to X$ is a vector bundle $J^rF \to J^rY$ coordinated by $(x^\lambda, y^i_\Lambda, q^a_\Lambda)$, $0 \le |\Lambda| \le r$. Let $(J^rY, \mathfrak{A}_r)$ be a simple graded manifold modelled over this vector bundle. Its local basis is $(x^\lambda, y^i_\Lambda, c^a_\Lambda)$, $0 \le |\Lambda| \le r$. Let

$$\mathcal{S}^*_r[F;Y] = \mathcal{S}^*_r[J^rF; J^rY] \tag{7.3.2}$$

denote the DBGA of graded exterior forms on the simple graded manifold $(J^rY, \mathfrak{A}_r)$. In particular, there is a cochain monomorphism

$$\mathcal{O}^*_r = \mathcal{O}^*(J^rY) \to \mathcal{S}^*_r[F;Y]. \tag{7.3.3}$$

The surjection

$$\pi^{r+1}_r : J^{r+1}Y \to J^rY$$

yields an epimorphism of graded manifolds

$$(\pi^{r+1}_r, \widehat{\pi}^{r+1}_r) : (J^{r+1}Y, \mathfrak{A}_{r+1}) \to (J^rY, \mathfrak{A}_r),$$

including the sheaf monomorphism

$$\widehat{\pi}^{r+1}_r : \pi^{r+1*}_r \mathfrak{A}_r \to \mathfrak{A}_{r+1},$$

where $\pi^{r+1*}_r \mathfrak{A}_r$ is the pull-back onto $J^{r+1}Y$ of the continuous fibre bundle $\mathfrak{A}_r \to J^rY$. This sheaf monomorphism induces the monomorphism of the canonical presheaves $\overline{\mathfrak{A}}_r \to \overline{\mathfrak{A}}_{r+1}$, which associates to each open subset

$U \subset J^{r+1}Y$ the ring of sections of $\mathfrak{A}_r$ over $\pi^{r+1}_r(U)$. Accordingly, there is a monomorphism of the structure rings

$$\pi^{r+1*}_r : \mathcal{S}^0_r[F;Y] \to \mathcal{S}^0_{r+1}[F;Y] \tag{7.3.4}$$

of graded functions on graded manifolds $(J^rY, \mathfrak{A}_r)$ and $(J^{r+1}Y, \mathfrak{A}_{r+1})$. By virtue of Lemma 6.4.1, the differential calculus $\mathcal{S}^*_r[F;Y]$ and $\mathcal{S}^*_{r+1}[F;Y]$ are minimal. Therefore, the monomorphism (7.3.4) yields that of the DBGA

$$\pi^{r+1*}_r : \mathcal{S}^*_r[F;Y] \to \mathcal{S}^*_{r+1}[F;Y]. \tag{7.3.5}$$

As a consequence, we have the direct system of DBGAs

$$\begin{aligned} &\mathcal{S}^*[F;Y] \xrightarrow{\pi^*} \mathcal{S}^*_1[F;Y] \longrightarrow \cdots \mathcal{S}^*_{r-1}[F;Y] \xrightarrow{\pi^{r*}_{r-1}} \\ &\qquad \mathcal{S}^*_r[F;Y] \longrightarrow \cdots. \end{aligned} \tag{7.3.6}$$

The DBGA $\mathcal{S}^*_\infty[F;Y]$ that we associate to the composite graded manifold $(Y, \mathfrak{A}_F)$ is defined as the direct limit

$$\mathcal{S}^*_\infty[F;Y] = \overrightarrow{\lim} \mathcal{S}^*_r[F;Y] \tag{7.3.7}$$

of the direct system (7.3.6). It consists of all graded exterior forms $\phi \in \mathcal{S}^*[F_r; J^rY]$ on graded manifolds $(J^rY, \mathfrak{A}_r)$ modulo the monomorphisms (7.3.5). Its elements obey the relations (6.2.9) – (6.2.10).

Cochain monomorphisms $\mathcal{O}^*_r \to \mathcal{S}^*_r[F;Y]$ (7.3.3) provide a monomorphism of the direct system (2.4.5) to the direct system (7.3.6) and, consequently, the monomorphism

$$\mathcal{O}^*_\infty \to \mathcal{S}^*_\infty[F;Y] \tag{7.3.8}$$

of their direct limits. In particular, $\mathcal{S}^*_\infty[F;Y]$ is an $\mathcal{O}^0_\infty$-algebra. Accordingly, the body epimorphisms $\mathcal{S}^*_r[F;Y] \to \mathcal{O}^*_r$ yield the epimorphism of $\mathcal{O}^0_\infty$-algebras

$$\mathcal{S}^*_\infty[F;Y] \to \mathcal{O}^*_\infty. \tag{7.3.9}$$

It is readily observed that the morphisms (7.3.8) and (7.3.9) are cochain morphisms between the de Rham complex (2.4.7) of the DGA $\mathcal{O}_\infty^*$ (2.4.6) and the de Rham complex

$$0 \to \mathbb{R} \to \mathcal{S}_\infty^0[F;Y] \xrightarrow{d} \mathcal{S}_\infty^1[F;Y] \cdots \xrightarrow{d} \mathcal{S}_\infty^k[F;Y] \longrightarrow \cdots \quad (7.3.10)$$

of the DBGA $\mathcal{S}_\infty^0[F;Y]$. Moreover, the corresponding homomorphisms of cohomology groups of these complexes are isomorphisms as follows.

Theorem 7.3.1: There is an isomorphism

$$H^*(\mathcal{S}_\infty^*[F;Y]) = H_{DR}^*(Y) \quad (7.3.11)$$

of the cohomology $H^*(\mathcal{S}_\infty^*[F;Y])$ of the de Rham complex (7.3.10) to the de Rham cohomology $H_{DR}^*(Y)$ of Y. □

Corollary 7.3.2: Any closed graded form $\phi \in \mathcal{S}_\infty^*[F;Y]$ is decomposed into the sum $\phi = \sigma + d\xi$ where σ is a closed exterior form on Y. □

Similarly to the DGA $\mathcal{O}_\infty^*$ (2.4.6), one thinks of elements of $\mathcal{S}_\infty^*[F;Y]$ as being *graded differential forms* on the infinite order jet manifold $J^\infty Y$. We can restrict $\mathcal{S}_\infty^*[F;Y]$ to the coordinate chart (2.4.3) of $J^\infty Y$ and say that $\mathcal{S}_\infty^*[F;Y]$ as an $\mathcal{O}_\infty^0$-algebra is locally generated by the elements

$$(c_\Lambda^a, dx^\lambda, \theta_\Lambda^a = dc_\Lambda^a - c_{\lambda+\Lambda}^a dx^\lambda, \theta_\Lambda^i = dy_\Lambda^i - y_{\lambda+\Lambda}^i dx^\lambda), \qquad 0 \leq |\Lambda|,$$

where c_Λ^a, θ_Λ^a are odd and dx^λ, θ_Λ^i are even. We agree to call (y^i, c^a) the local *generating basis* for $\mathcal{S}_\infty^*[F;Y]$. Let the collective symbol s^A stand for its elements. Accordingly, the notation s_Λ^A and

$$\theta_\Lambda^A = ds_\Lambda^A - s_{\lambda+\Lambda}^A dx^\lambda$$

is introduced. For the sake of simplicity, we further denote $[A] = [s^A]$.

The DBGA $\mathcal{S}_\infty^*[F;Y]$ is split into $\mathcal{S}_\infty^0[F;Y]$-modules $\mathcal{S}_\infty^{k,r}[F;Y]$ of *k-contact and r-horizontal graded forms* together with the corresponding projections

$$h_k : \mathcal{S}_\infty^*[F;Y] \to \mathcal{S}_\infty^{k,*}[F;Y], \qquad h^m : \mathcal{S}_\infty^*[F;Y] \to \mathcal{S}_\infty^{*,m}[F;Y].$$

Accordingly, the graded exterior differential d on $\mathcal{S}^*_\infty[F;Y]$ falls into the sum $d = d_V + d_H$ of the *vertical graded differential*

$$d_V \circ h^m = h^m \circ d \circ h^m, \qquad d_V(\phi) = \theta^A_\Lambda \wedge \partial^\Lambda_A \phi, \qquad \phi \in \mathcal{S}^*_\infty[F;Y],$$

and the *total graded differential*

$$d_H \circ h_k = h_k \circ d \circ h_k, \quad d_H \circ h_0 = h_0 \circ d, \quad d_H(\phi) = dx^\lambda \wedge d_\lambda(\phi),$$

where

$$d_\lambda = \partial_\lambda + \sum_{0\leq|\Lambda|} s^A_{\lambda+\Lambda}\partial^\Lambda_A$$

are the *graded total derivatives*. These differentials obey the nilpotent relations (2.4.11).

Similarly to the DGA $\mathcal{O}^*_\infty$, the DBGA $\mathcal{S}^*_\infty[F;Y]$ is provided with the graded projection endomorphism

$$\begin{aligned}
&\varrho = \sum_{k>0} \frac{1}{k}\overline{\varrho} \circ h_k \circ h^n : \mathcal{S}^{*>0,n}_\infty[F;Y] \to \mathcal{S}^{*>0,n}_\infty[F;Y],\\
&\overline{\varrho}(\phi) = \sum_{0\leq|\Lambda|} (-1)^{|\Lambda|}\theta^A \wedge [d_\Lambda(\partial^\Lambda_A \rfloor \phi)], \qquad \phi \in \mathcal{S}^{>0,n}_\infty[F;Y],
\end{aligned}$$

such that $\varrho \circ d_H = 0$, and with the nilpotent graded variational operator

$$\delta = \varrho \circ d\mathcal{S}^{*,n}_\infty[F;Y] \to \mathcal{S}^{*+1,n}_\infty[F;Y]. \tag{7.3.12}$$

With these operators the DBGA $\mathcal{S}^{*,}_\infty[F;Y]$ is split into the *Grassmann-graded variational bicomplex*. We restrict our consideration to its short *variational subcomplex*

$$\begin{aligned}
&0 \to \mathbb{R} \to \mathcal{S}^0_\infty[F;Y] \xrightarrow{d_H} \mathcal{S}^{0,1}_\infty[F;Y] \cdots \xrightarrow{d_H}\\
&\qquad \mathcal{S}^{0,n}_\infty[F;Y] \xrightarrow{\delta} \mathbf{E}_1, \qquad \mathbf{E}_1 = \varrho(\mathcal{S}^{1,n}_\infty[F;Y]),
\end{aligned} \tag{7.3.13}$$

and the subcomplex of one-contact graded forms

$$\begin{aligned}
&0 \to \mathcal{S}^{1,0}_\infty[F;Y] \xrightarrow{d_H} \mathcal{S}^{1,1}_\infty[F;Y] \cdots \xrightarrow{d_H} \mathcal{S}^{1,n}_\infty[F;Y]\\
&\qquad \xrightarrow{\varrho} \mathbf{E}_1 \to 0.
\end{aligned} \tag{7.3.14}$$

Theorem 7.3.3: Cohomology of the complex (7.3.13) equals the de Rham cohomology $H^*_{DR}(Y)$ of Y. $\Box$

Theorem 7.3.4: The complex (7.3.14) is exact. $\Box$

Decomposed into the variational bicomplex, the DBGA $\mathcal{S}^*_\infty[F;Y]$ describes Grassmann-graded Lagrangian theory on the composite graded manifold $(Y,\mathfrak{A}_F)$. Its *graded Lagrangian* is defined as an element

$$L = \mathcal{L}\omega \in \mathcal{S}^{0,n}_\infty[F;Y] \tag{7.3.15}$$

of the graded variational complex (7.3.13), while the graded exterior form

$$\delta L = \theta^A \wedge \mathcal{E}_A\omega = \sum_{0\leq|\Lambda|}(-1)^{|\Lambda|}\theta^A \wedge d_\Lambda(\partial^\Lambda_A L)\omega \in \mathbf{E}_1 \tag{7.3.16}$$

is said to be its *graded Euler–Lagrange operator*. We agree to call a pair $(\mathcal{S}^{0,n}_\infty[F;Y],L)$ the *Grassmann-graded Lagrangian system*.

The following is a corollary of Theorem 7.3.3.

Theorem 7.3.5: Every d_H-closed graded form $\phi \in \mathcal{S}^{0,m<n}_\infty[F;Y]$ falls into the sum

$$\phi = h_0\sigma + d_H\xi, \qquad \xi \in \mathcal{S}^{0,m-1}_\infty[F;Y], \tag{7.3.17}$$

where σ is a closed m-form on Y. Any δ-closed (i.e., variationally trivial) Grassmann-graded Lagrangian $L \in \mathcal{S}^{0,n}_\infty[F;Y]$ is the sum

$$L = h_0\sigma + d_H\xi, \qquad \xi \in \mathcal{S}^{0,n-1}_\infty[F;Y], \tag{7.3.18}$$

where σ is a closed n-form on Y. $\Box$

Corollary 7.3.6: Any variationally trivial odd Lagrangian is d_H-exact. $\Box$

The exactness of the complex (7.3.14) at the term $\mathcal{S}^{1,n}_\infty[F;Y]$ results in the following.

Theorem 7.3.7: Given a graded Lagrangian L, there is the decomposition

$$dL = \delta L - d_H \Xi_L, \qquad \Xi \in \mathcal{S}^{n-1}_\infty[F;Y], \tag{7.3.19}$$

$$\Xi_L = L + \sum_{s=0} \theta^A_{\nu_s\ldots\nu_1} \wedge F^{\lambda\nu_s\ldots\nu_1}_A \omega_\lambda, \tag{7.3.20}$$

$$F^{\nu_k\ldots\nu_1}_A = \partial^{\nu_k\ldots\nu_1}_A \mathcal{L} - d_\lambda F^{\lambda\nu_k\ldots\nu_1}_A + \sigma^{\nu_k\ldots\nu_1}_A, \qquad k = 1, 2, \ldots,$$

where local graded functions σ obey the relations

$$\sigma^\nu_A = 0, \qquad \sigma^{(\nu_k\nu_{k-1})\ldots\nu_1}_A = 0.$$

□

The form Ξ_L (7.3.20) provides a global Lepage equivalent of a graded Lagrangian L.

Given a Grassmann-graded Lagrangian system $(\mathcal{S}^*_\infty[F;Y], L)$, by its *infinitesimal transformations* are meant contact graded derivations of the real graded commutative ring $\mathcal{S}^0_\infty[F;Y]$. They constitute a $\mathcal{S}^0_\infty[F;Y]$-module $\mathfrak{d}\mathcal{S}^0_\infty[F;Y]$ which is a real Lie superalgebra with the Lie superbracket (6.2.4).

Theorem 7.3.8: The derivation module $\mathfrak{d}\mathcal{S}^0_\infty[F;Y]$ is isomorphic to the $\mathcal{S}^0_\infty[F;Y]$-dual $(\mathcal{S}^1_\infty[F;Y])^*$ of the module of graded one-forms $\mathcal{S}^1_\infty[F;Y]$. It follows that the DBGA $\mathcal{S}^*_\infty[F;Y]$ is minimal differential calculus over the real graded commutative ring $\mathcal{S}^0_\infty[F;Y]$. □

Let $\vartheta\rfloor\phi$, $\vartheta \in \mathfrak{d}\mathcal{S}^0_\infty[F;Y]$, $\phi \in \mathcal{S}^1_\infty[F;Y]$, denote the corresponding interior product. Extended to the DBGA $\mathcal{S}^*_\infty[F;Y]$, it obeys the rule

$$\vartheta\rfloor(\phi \wedge \sigma) = (\vartheta\rfloor\phi) \wedge \sigma + (-1)^{|\phi|+[\phi][\vartheta]}\phi \wedge (\vartheta\rfloor\sigma), \quad \phi, \sigma \in \mathcal{S}^*_\infty[F;Y].$$

Restricted to a coordinate chart (2.4.3) of $J^\infty Y$, the algebra $\mathcal{S}^*_\infty[F;Y]$ is a free $\mathcal{S}^0_\infty[F;Y]$-module generated by one-forms dx^λ, θ^A_Λ. Due to the isomorphism stated in Theorem 7.3.8, any graded derivation $\vartheta \in \mathfrak{d}\mathcal{S}^0_\infty[F;Y]$ takes the local form

$$\vartheta = \vartheta^\lambda \partial_\lambda + \vartheta^A \partial_A + \sum_{0<|\Lambda|} \vartheta^A_\Lambda \partial^\Lambda_A, \tag{7.3.21}$$

$$\partial^\Lambda_A \rfloor dy^B_\Sigma = \delta^B_A \delta^\Lambda_\Sigma. \tag{7.3.22}$$

Every graded derivation ϑ (7.3.21) yields the graded Lie derivative

$$\mathbf{L}_\vartheta\phi = \vartheta\rfloor d\phi + d(\vartheta\rfloor\phi), \qquad \phi\in\mathcal{S}^*_\infty[F;Y],$$
$$\mathbf{L}_\vartheta(\phi\wedge\sigma) = \mathbf{L}_\vartheta(\phi)\wedge\sigma + (-1)^{[\vartheta][\phi]}\phi\wedge\mathbf{L}_\vartheta(\sigma),$$

of the DBGA $\mathcal{S}^*_\infty[F;Y]$. A graded derivation ϑ (7.3.21) is called *contact* if the Lie derivative $\mathbf{L}_\vartheta$ preserves the ideal of contact graded forms.

Lemma 7.3.9: With respect to the local generating basis (s^A) for the DBGA $\mathcal{S}^*_\infty[F;Y]$, any its contact graded derivation takes the form

$$\vartheta = \upsilon_H + \upsilon_V = \upsilon^\lambda d_\lambda + [\upsilon^A\partial_A + \sum_{|\Lambda|>0} d_\Lambda(\upsilon^A - s^A_\mu\upsilon^\mu)\partial^\Lambda_A], \qquad (7.3.23)$$

where υ_H and υ_V denotes the horizontal and vertical parts of ϑ. □

A glance at the expression (7.3.23) shows that a contact graded derivation ϑ is an infinite order jet prolongation of its restriction

$$\upsilon = \upsilon^\lambda\partial_\lambda + \upsilon^A\partial_A \qquad (7.3.24)$$

to the graded commutative ring $S^0[F;Y]$. We call υ (7.3.24) the *generalized graded vector field*. It is readily justified the following (see Lemma 7.2.16).

Lemma 7.3.10: Any vertical contact graded derivation

$$\vartheta = \upsilon^A\partial_A + \sum_{|\Lambda|>0} d_\Lambda\upsilon^A\partial^\Lambda_A \qquad (7.3.25)$$

satisfies the relations

$$\vartheta\rfloor d_H\phi = -d_H(\vartheta\rfloor\phi), \qquad (7.3.26)$$
$$\mathbf{L}_\vartheta(d_H\phi) = d_H(\mathbf{L}_\vartheta\phi) \qquad (7.3.27)$$

for all $\phi\in\mathcal{S}^*_\infty[F;Y]$. □

Then the forthcoming assertions are the straightforward generalizations of Theorem 7.2.7, Lemma 7.2.8 and Theorem 7.2.9.

A corollary of the decomposition (7.3.19) is the *first variational formula* for a graded Lagrangian.

Theorem 7.3.11: The Lie derivative of a graded Lagrangian along any contact graded derivation (7.3.23) obeys the first variational formula

$$\mathbf{L}_\vartheta L = \upsilon_V \rfloor \delta L + d_H(h_0(\vartheta\rfloor\Xi_L)) + d_V(\upsilon_H\rfloor\omega)\mathcal{L}, \tag{7.3.28}$$

where Ξ_L is the Lepage equivalent (7.3.20) of L. □

A contact graded derivation ϑ (7.3.23) is called a *variational symmetry* (strictly speaking, a variational *supersymmetry*) of a graded Lagrangian L if the Lie derivative $\mathbf{L}_\vartheta L$ is d_H-exact, i.e.,

$$\mathbf{L}_\vartheta L = d_H\sigma. \tag{7.3.29}$$

Lemma 7.3.12: A glance at the expression (7.3.28) shows the following.

(i) A contact graded derivation ϑ is a variational symmetry only if it is projected onto X.

(ii) Any projectable contact graded derivation is a variational symmetry of a variationally trivial graded Lagrangian. It follows that, if ϑ is a variational symmetry of a graded Lagrangian L, it also is a variational symmetry of a Lagrangian $L + L_0$, where L_0 is a variationally trivial graded Lagrangian.

(iii) A contact graded derivations ϑ is a variational symmetry iff its vertical part υ_V (7.3.23) is well.

(iv) It is a variational symmetry iff the graded density $\upsilon_V\rfloor\delta L$ is d_H-exact. □

Variational symmetries of a graded Lagrangian L constitute a real vector subspace $\mathcal{G}_L$ of the graded derivation module $\mathfrak{d}\mathcal{S}^0_\infty[F;Y]$. By virtue of item (ii) of Lemma 7.3.12, the Lie superbracket

$$\mathbf{L}_{[\vartheta,\vartheta']} = [\mathbf{L}_\vartheta, \mathbf{L}_{\vartheta'}]$$

of variational symmetries is a variational symmetry and, therefore, their vector space $\mathcal{G}_L$ is a real Lie superalgebra.

A corollary of the first variational formula (7.3.28) is the *first Noether theorem* for graded Lagrangians.

Theorem 7.3.13: If a contact graded derivation ϑ (7.3.23) is a variational symmetry (7.3.29) of a graded Lagrangian L, the first variational formula (7.3.28) restricted to $\mathrm{Ker}\,\delta L$ leads to the weak conservation law

$$0 \approx d_H(h_0(\vartheta\rfloor\Xi_L) - \sigma). \tag{7.3.30}$$

□

A vertical contact graded derivation ϑ (7.3.25) is said to be *nilpotent* if

$$\mathbf{L}_\vartheta(\mathbf{L}_\vartheta\phi) = \sum_{0\leq|\Sigma|,0\leq|\Lambda|} (\upsilon^B_\Sigma\partial^\Sigma_B(\upsilon^A_\Lambda)\partial^\Lambda_A + \tag{7.3.31}$$
$$(-1)^{[s^B][\upsilon^A]}\upsilon^B_\Sigma\upsilon^A_\Lambda\partial^\Sigma_B\partial^\Lambda_A)\phi = 0$$

for any horizontal graded form $\phi\in S^{0,*}_\infty$.

Lemma 7.3.14: A vertical contact graded derivation (7.3.25) is nilpotent only if it is odd and iff the equality

$$\mathbf{L}_\vartheta(\upsilon^A) = \sum_{0\leq|\Sigma|}\upsilon^B_\Sigma\partial^\Sigma_B(\upsilon^A) = 0$$

holds for all υ^A. □

For the sake of brevity, the common symbol υ further stands for a generalized graded vector field υ, the contact graded derivation ϑ determined by υ, and the Lie derivative $\mathbf{L}_\vartheta$. We agree to call all these operators, simply, a *graded derivation of a field system algebra.*

Remark 7.3.1: For the sake of convenience, *right derivations*

$$\overleftarrow{\upsilon} = \overleftarrow{\partial}_A\upsilon^A \tag{7.3.32}$$

also are considered. They act on graded functions and differential forms ϕ on the right by the rules

$$\overleftarrow{\upsilon}(\phi) = d\phi\lfloor\overleftarrow{\upsilon} + d(\phi\lfloor\overleftarrow{\upsilon}),$$
$$\overleftarrow{\upsilon}(\phi\wedge\phi') = (-1)^{[\phi']}\overleftarrow{\upsilon}(\phi)\wedge\phi' + \phi\wedge\overleftarrow{\upsilon}(\phi'),$$
$$\theta_{\Lambda A}\lfloor\overleftarrow{\partial}^{\Sigma B} = \delta^A_B\delta^\Sigma_\Lambda.$$

One associates to any graded right derivation $\overleftarrow{\upsilon}$ (7.3.32) the left one

$$\upsilon^l = (-1)^{[\upsilon][A]}\upsilon^A\partial_A, \qquad (7.3.33)$$
$$\upsilon^l(f) = (-1)^{[\upsilon][f]}\upsilon(f), \qquad f \in \mathcal{S}^0_\infty[F;Y].$$

□

7.4 Noether identities

The degeneracy of Lagrangian theory is characterized by a set of non-trivial reducible Noether identities. Any Euler–Lagrange operator satisfies Noether identities (henceforth NI) which therefore must be separated into the trivial and non-trivial ones. These NI can obey first-stage NI, which in turn are subject to the second-stage ones, and so on. Thus, there is a hierarchy of higher-stage NI which also are separated into the trivial and non-trivial ones. If certain conditions hold, one can associate to a Grassmann-graded Lagrangian system the exact Koszul–Tate complex possessing the boundary operator whose nilpotentness is equivalent to all non-trivial NI and higher-stage NI. The inverse second Noether theorem formulated in homology terms associates to this Koszul–Tate complex the cochain sequence of ghosts with the ascent operator, called the gauge operator, whose components are non-trivial gauge and higher-stage gauge symmetries of Lagrangian theory.

Let $(\mathcal{S}^*_\infty[F;Y], L)$ be a Grassmann-graded Lagrangian system. Without a lose of generality, let a Lagrangian L be even. Its Euler–Lagrange operator δL (7.3.16) is assumed to be at least of order 1 in order to guarantee that transition functions of Y do not vanish on-shell. This Euler–Lagrange operator $\delta L \in \mathbf{E}_1$ takes its values into the graded vector bundle

$$\overline{VF} = V^*F \underset{F}{\otimes} \overset{n}{\wedge} T^*X \to F, \qquad (7.4.1)$$

where V^*F is the vertical cotangent bundle of $F \to X$. It however is not a vector bundle over Y. Therefore, we restrict our consideration to the case of a pull-back composite bundle F (7.3.1), that is,

$$F = Y \underset{X}{\times} F^1 \to Y \to X, \tag{7.4.2}$$

where $F^1 \to X$ is a vector bundle. Let us introduce the following notation.

Notation 7.4.1: Given the vertical tangent bundle VE of a fibre bundle $E \to X$, by its *density-dual bundle* is meant the fibre bundle

$$\overline{VE} = V^*E \underset{E}{\otimes} \overset{n}{\wedge} T^*X. \tag{7.4.3}$$

If $E \to X$ is a vector bundle, we have

$$\overline{VE} = \overline{E} \underset{X}{\times} E, \qquad \overline{E} = E^* \underset{X}{\otimes} \overset{n}{\wedge} T^*X, \tag{7.4.4}$$

where $\overline{E}$ is called the *density-dual* of E. Let

$$E = E^0 \underset{X}{\oplus} E^1$$

be a *graded vector bundle* over X. Its *graded density-dual* is defined to be

$$\overline{E} = \overline{E}^1 \underset{X}{\oplus} \overline{E}^0.$$

In these terms, we treat the composite bundle F (7.3.1) as a graded vector bundle over Y possessing only odd part. The density-dual $\overline{VF}$ (7.4.3) of the vertical tangent bundle VF of $F \to X$ is $\overline{VF}$ (7.4.1). If F (7.3.1) is the pull-back bundle (7.4.2), then

$$\overline{VF} = ((\overline{F}^1 \underset{Y}{\oplus} V^*Y) \underset{Y}{\otimes} \overset{n}{\wedge} T^*X) \underset{Y}{\oplus} F^1 \tag{7.4.5}$$

is a graded vector bundle over Y. Given a graded vector bundle

$$E = E^0 \underset{Y}{\oplus} E^1 \to Y,$$

we consider the composite bundle $E \to E^0 \to X$ and the DBGA (7.3.7):

$$\mathcal{P}^*_\infty[E; Y] = \mathcal{S}^*_\infty[E; E^0]. \tag{7.4.6}$$

□

Let us consider the density-dual $\overline{VF}$ (7.4.5) of the vertical tangent bundle $VF \to F$, and let us enlarge the DBGA $\mathcal{S}^*_\infty[F;Y]$ to the DBGA $\mathcal{P}^*_\infty[\overline{VF};Y]$ (7.4.6) with the local generating basis $(s^A, \overline{s}_A)$, $\qquad [\overline{s}_A] = ([A]+1)\mathrm{mod}\,2$. Following the physical terminology, we agree to call its elements $\overline{s}_A$ the *antifields* of antifield number $\mathrm{Ant}[\overline{s}_A] = 1$. The DBGA $\mathcal{P}^*_\infty[\overline{VF};Y]$ is endowed with the nilpotent right graded derivation $\overline{\delta} = \overleftarrow{\partial}^A \mathcal{E}_A$, where $\mathcal{E}_A$ are the variational derivatives (7.3.16). Then we have the chain complex

$$0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{VF};Y]_1 \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{VF};Y]_2 \qquad (7.4.7)$$

of graded densities of antifield number ≤ 2. Its one-boundaries $\overline{\delta}\Phi$, $\Phi \in \mathcal{P}^{0,n}_\infty[\overline{VF};Y]_2$, by very definition, vanish on-shell.

Lemma 7.4.2: One can associate to any Grassmann-graded Lagrangian system $(\mathcal{S}^*_\infty[F;Y], L)$ the chain complex (7.4.7). □

Any one-cycle

$$\Phi = \sum_{0\leq|\Lambda|} \Phi^{A,\Lambda}\overline{s}_{\Lambda A}\omega \in \mathcal{P}^{0,n}_\infty[\overline{VF};Y]_1 \qquad (7.4.8)$$

of the complex (7.4.7) is a differential operator on the bundle $\overline{VF}$ such that it is linear on fibres of $\overline{VF} \to F$ and its kernel contains the graded Euler–Lagrange operator δL (7.3.16), i.e.,

$$\overline{\delta}\Phi = 0, \qquad \sum_{0\leq|\Lambda|} \Phi^{A,\Lambda} d_\Lambda \mathcal{E}_A \omega = 0. \qquad (7.4.9)$$

Thus, the one-cycles (7.4.8) define the NI (7.4.9) of the Euler–Lagrange operator δL, which we call *Noether identities* (NI) of the Grassmann-graded Lagrangian system $(\mathcal{S}^*_\infty[F;Y], L)$.

In particular, one-chains Φ (7.4.8) are necessarily NI if they are boundaries. Accordingly, *non-trivial NI* modulo the trivial ones are associated to elements of the first homology $H_1(\overline{\delta})$ of the complex (7.4.7). A Lagrangian L is called *degenerate* if there are non-trivial NI.

Non-trivial NI obey first-stage NI. To describe them, let us assume that the module $H_1(\overline{\delta})$ is finitely generated. Namely, there exists a graded projective $C^\infty(X)$-module $\mathcal{C}_{(0)} \subset H_1(\overline{\delta})$ of finite rank with a local basis $\{\Delta_r\omega\}$:

$$\Delta_r\omega = \sum_{0\leq|\Lambda|} \Delta_r^{A,\Lambda}\overline{s}_{\Lambda A}\omega, \qquad \Delta_r^{A,\Lambda} \in \mathcal{S}^0_\infty[F;Y], \tag{7.4.10}$$

such that any element $\Phi \in H_1(\overline{\delta})$ factorizes as

$$\Phi = \sum_{0\leq|\Xi|} \Phi^{r,\Xi} d_\Xi \Delta_r\omega, \qquad \Phi^{r,\Xi} \in \mathcal{S}^0_\infty[F;Y], \tag{7.4.11}$$

through elements (7.4.10) of $\mathcal{C}_{(0)}$. Thus, all non-trivial NI (7.4.9) result from the NI

$$\overline{\delta}\Delta_r = \sum_{0\leq|\Lambda|} \Delta_r^{A,\Lambda} d_\Lambda \mathcal{E}_A = 0, \tag{7.4.12}$$

called the *complete NI*. Clearly, the factorization (7.4.11) is independent of specification of a local basis $\{\Delta_r\omega\}$.

A Lagrangian system whose non-trivial NI are finitely generated is called *finitely degenerate*. Hereafter, *degenerate Lagrangian systems* only of this type are considered.

By virtue of Serre–Swan Theorem 6.3.2, the graded module $\mathcal{C}_{(0)}$ is isomorphic to a module of sections of the density-dual $\overline{E}_0$ of some graded vector bundle $E_0 \to X$. Let us enlarge $\mathcal{P}^*_\infty[\overline{VF};Y]$ to the DBGA

$$\overline{\mathcal{P}}^*_\infty\{0\} = \mathcal{P}^*_\infty[\overline{VF} \underset{Y}{\oplus} \overline{E}_0;Y] \tag{7.4.13}$$

possessing the local generating basis $(s^A, \overline{s}_A, \overline{c}_r)$ where $\overline{c}_r$ are *Noether antifields* of Grassmann parity $[\overline{c}_r] = ([\Delta_r]+1) \bmod 2$ and antifield number $\mathrm{Ant}[\overline{c}_r] = 2$. The DBGA (7.4.13) is provided with the odd right graded derivation $\delta_0 = \overline{\delta} + \overleftarrow{\partial}{}^r\Delta_r$ which is nilpotent iff the complete NI (7.4.12) hold. Then δ_0 is a boundary operator of the chain complex

$$0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{VF};Y]_1 \xleftarrow{\delta_0} \overline{\mathcal{P}}^{0,n}_\infty\{0\}_2 \xleftarrow{\delta_0} \overline{\mathcal{P}}^{0,n}_\infty\{0\}_3 \tag{7.4.14}$$

of graded densities of antifield number ≤ 3. Let $H_*(\delta_0)$ denote its homology. We have

$$H_0(\delta_0) = H_0(\overline{\delta}) = 0.$$

Furthermore, any one-cycle Φ up to a boundary takes the form (7.4.11) and, therefore, it is a δ_0-boundary. Hence, $H_1(\delta_0) = 0$, i.e., the complex (7.4.14) is one-exact.

Lemma 7.4.3: If the homology $H_1(\overline{\delta})$ of the complex (7.4.7) is finitely generated in the above mentioned sense, this complex can be extended to the one-exact chain complex (7.4.14) with a boundary operator whose nilpotency conditions are equivalent to the complete NI (7.4.12). $\Box$

Let us consider the second homology $H_2(\delta_0)$ of the complex (7.4.14). Its two-chains read

$$\Phi = G + H = \sum_{0\leq|\Lambda|} G^{r,\Lambda}\overline{c}_{\Lambda r}\omega + \sum_{0\leq|\Lambda|,|\Sigma|} H^{(A,\Lambda)(B,\Sigma)}\overline{s}_{\Lambda A}\overline{s}_{\Sigma B}\omega. \qquad (7.4.15)$$

Its two-cycles define the *first-stage NI*

$$\delta_0\Phi = 0, \qquad \sum_{0\leq|\Lambda|} G^{r,\Lambda} d_\Lambda \Delta_r\omega = -\overline{\delta}H. \qquad (7.4.16)$$

The first-stage NI (7.4.16) are *trivial* either if a two-cycle Φ (7.4.15) is a δ_0-boundary or its summand G vanishes on-shell. Therefore, non-trivial first-stage NI fails to exhaust the second homology $H_2(\delta_0)$ the complex (7.4.14) in general.

Lemma 7.4.4: *Non-trivial first-stage NI* modulo the trivial ones are identified with elements of the homology $H_2(\delta_0)$ iff any $\overline{\delta}$-cycle $\phi \in \overline{\mathcal{P}}^{0,n}_\infty\{0\}_2$ is a δ_0-boundary. $\Box$

A degenerate Lagrangian system is called *reducible* (resp. *irreducible*) if it admits (resp. does not admit) non-trivial first stage NI.

If the condition of Lemma 7.4.4 is satisfied, let us assume that non-trivial first-stage NI are finitely generated as follows. There exists a graded projective $C^\infty(X)$-module $\mathcal{C}_{(1)} \subset H_2(\delta_0)$ of finite rank with a local basis $\{\Delta_{r_1}\omega\}$:

$$\Delta_{r_1}\omega = \sum_{0\leq|\Lambda|} \Delta^{r,\Lambda}_{r_1}\overline{c}_{\Lambda r}\omega + h_{r_1}\omega, \tag{7.4.17}$$

such that any element $\Phi \in H_2(\delta_0)$ factorizes as

$$\Phi = \sum_{0\leq|\Xi|} \Phi^{r_1,\Xi} d_\Xi \Delta_{r_1}\omega, \qquad \Phi^{r_1,\Xi} \in \mathcal{S}^0_\infty[F;Y], \tag{7.4.18}$$

through elements (7.4.17) of $\mathcal{C}_{(1)}$. Thus, all non-trivial first-stage NI (7.4.16) result from the equalities

$$\sum_{0\leq|\Lambda|} \Delta^{r,\Lambda}_{r_1} d_\Lambda \Delta_r + \overline{\delta} h_{r_1} = 0, \tag{7.4.19}$$

called the *complete first-stage NI*.

The complete first-stage NI obey second-stage NI, and so on. Iterating the arguments, one comes to the following.

A degenerate Grassmann-graded Lagrangian system $(\mathcal{S}^*_\infty[F;Y], L)$ is called *N-stage reducible* if it admits finitely generated non-trivial N-stage NI, but no non-trivial $(N+1)$-stage ones. It is characterized as follows.

• There are graded vector bundles $E_0, \ldots, E_N$ over X and a DBGA $\mathcal{P}^*_\infty[\overline{VF};Y]$ is enlarged to the DBGA

$$\overline{\mathcal{P}}^*_\infty\{N\} = \mathcal{P}^*_\infty[\overline{VF} \underset{Y}{\oplus} \overline{E}_0 \underset{Y}{\oplus} \cdots \underset{Y}{\oplus} \overline{E}_N; Y] \tag{7.4.20}$$

with the local generating basis

$$(s^A, \overline{s}_A, \overline{c}_r, \overline{c}_{r_1}, \ldots, \overline{c}_{r_N})$$

where $\overline{c}_{r_k}$ are *Noether k-stage antifields* of antifield number $\mathrm{Ant}[\overline{c}_{r_k}] = k+2$.

• The DBGA (7.4.20) admits with the nilpotent right graded derivation

$$\delta_{\mathrm{KT}} = \delta_N = \overline{\delta} + \sum_{0\leq|\Lambda|} \overleftarrow{\partial}^r \Delta^{A,\Lambda}_r \overline{s}_{\Lambda A} + \sum_{1\leq k\leq N} \overleftarrow{\partial}^{r_k} \Delta_{r_k}, \tag{7.4.21}$$

$$\Delta_{r_k}\omega = \sum_{0\leq|\Lambda|} \Delta^{r_{k-1},\Lambda}_{r_k}\overline{c}_{\Lambda r_{k-1}}\omega + \qquad (7.4.22)$$
$$\sum_{0\leq|\Sigma|,|\Xi|}(h^{(r_{k-2},\Sigma)(A,\Xi)}_{r_k}\overline{c}_{\Sigma r_{k-2}}\overline{s}_{\Xi A} + ...)\omega \in \overline{\mathcal{P}}^{0,n}_\infty\{k-1\}_{k+1},$$

of antifield number -1. The index $k=-1$ here stands for $\overline{s}_A$. The nilpotent derivation $\delta_{\rm KT}$ (7.4.21) is called the *Koszul–Tate operator*.

• With this graded derivation, the module $\overline{\mathcal{P}}^{0,n}_\infty\{N\}_{\leq N+3}$ of densities of antifield number $\leq (N+3)$ is decomposed into the exact *Koszul–Tate chain complex*

$$0\leftarrow {\rm Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{VF};Y]_1 \xleftarrow{\delta_0} \overline{\mathcal{P}}^{0,n}_\infty\{0\}_2 \xleftarrow{\delta_1} \overline{\mathcal{P}}^{0,n}_\infty\{1\}_3\cdots \qquad (7.4.23)$$
$$\xleftarrow{\delta_{N-1}} \overline{\mathcal{P}}^{0,n}_\infty\{N-1\}_{N+1} \xleftarrow{\delta_{\rm KT}} \overline{\mathcal{P}}^{0,n}_\infty\{N\}_{N+2} \xleftarrow{\delta_{\rm KT}} \overline{\mathcal{P}}^{0,n}_\infty\{N\}_{N+3}$$

which satisfies the following *homology regularity condition*.

Condition 7.4.5: Any $\delta_{k<N}$-cycle

$$\phi\in \overline{\mathcal{P}}^{0,n}_\infty\{k\}_{k+3}\subset \overline{\mathcal{P}}^{0,n}_\infty\{k+1\}_{k+3}$$

is a δ_{k+1}-boundary. □

• The nilpotentness $\delta^2_{\rm KT}=0$ of the Koszul–Tate operator (7.4.21) is equivalent to the complete non-trivial NI (7.4.12) and the *complete non-trivial* $(k\leq N)$*-stage NI*

$$\sum_{0\leq|\Lambda|} \Delta^{r_{k-1},\Lambda}_{r_k} d_\Lambda\left(\sum_{0\leq|\Sigma|}\Delta^{r_{k-2},\Sigma}_{r_{k-1}}\overline{c}_{\Sigma r_{k-2}}\right) = \qquad (7.4.24)$$
$$-\overline{\delta}\left(\sum_{0\leq|\Sigma|,|\Xi|} h^{(r_{k-2},\Sigma)(A,\Xi)}_{r_k}\overline{c}_{\Sigma r_{k-2}}\overline{s}_{\Xi A}\right).$$

This item means the following.

Theorem 7.4.6: Any δ_k-cocycle $\Phi\in\mathcal{P}^{0,n}_\infty\{k\}_{k+2}$ is a k-stage NI, and *vice versa*. □

Theorem 7.4.7: Any trivial k-stage NI is a δ_k-boundary $\Phi\in\mathcal{P}^{0,n}_\infty\{k\}_{k+2}$. □

Theorem 7.4.8: All non-trivial k-stage NI, by assumption, factorize as

$$\Phi = \sum_{0\leq|\Xi|} \Phi^{r_k,\Xi} d_\Xi \Delta_{r_k}\omega, \qquad \Phi^{r_1,\Xi} \in \mathcal{S}^0_\infty[F;Y],$$

through the complete ones (7.4.24). □

It may happen that a Grassmann-graded Lagrangian field system possesses non-trivial NI of any stage. However, we restrict our consideration to N-reducible Lagrangian systems for a finite integer N.

7.5 Gauge symmetries

Different variants of the second Noether theorem have been suggested in order to relate reducible NI and gauge symmetries. The *inverse second Noether theorem* (Theorem 7.5.3), that we formulate in homology terms, associates to the Koszul–Tate complex (7.4.23) of non-trivial NI the cochain sequence (7.5.7) with the ascent operator $\mathbf{u}$ (7.5.8) whose components are non-trivial gauge and higher-stage gauge symmetries of Lagrangian system. Let us start with the following notation.

Notation 7.5.1: Given the DBGA $\overline{\mathcal{P}}^*_\infty\{N\}$ (7.4.20), we consider the DBGA

$$\mathcal{P}^*_\infty\{N\} = \mathcal{P}^*_\infty[F \underset{Y}{\oplus} E_0 \underset{Y}{\oplus} \cdots \underset{Y}{\oplus} E_N;Y], \tag{7.5.1}$$

possessing the local generating basis

$$(s^A, c^r, c^{r_1}, \ldots, c^{r_N}), \qquad [c^{r_k}] = ([\overline{c}_{r_k}]+1)\mathrm{mod}\,2,$$

and the DBGA

$$P^*_\infty\{N\} = \mathcal{P}^*_\infty[\overline{VF} \underset{Y}{\oplus} E_0 \oplus \cdots \underset{Y}{\oplus} E_N \underset{Y}{\oplus} \overline{E}_0 \underset{Y}{\oplus} \cdots \underset{Y}{\oplus} \overline{E}_N;Y] \tag{7.5.2}$$

with the local generating basis

$$(s^A, \overline{s}_A, c^r, c^{r_1}, \ldots, c^{r_N}, \overline{c}_r, \overline{c}_{r_1}, \ldots, \overline{c}_{r_N}).$$

Their elements c^{r_k} are called k-stage *ghosts* of ghost number $\mathrm{gh}[c^{r_k}] = k+1$ and antifield number

$$\mathrm{Ant}[c^{r_k}] = -(k+1).$$

The $C^\infty(X)$-module $\mathcal{C}^{(k)}$ of k-stage ghosts is the density-dual of the module $\mathcal{C}_{(k+1)}$ of $(k+1)$-stage antifields. The DBGAs $\overline{\mathcal{P}}^*_\infty\{N\}$ (7.4.20) and $\mathcal{P}^*_\infty\{N\}$ (7.5.1) are subalgebras of $P^*_\infty\{N\}$ (7.5.2). The Koszul–Tate operator δ_{KT} (7.4.21) is naturally extended to a graded derivation of the DBGA $P^*_\infty\{N\}$. $\Box$

Notation 7.5.2: Any graded differential form $\phi \in \mathcal{S}^*_\infty[F;Y]$ and any finite tuple (f^Λ), $0 \leq |\Lambda| \leq k$, of local graded functions $f^\Lambda \in \mathcal{S}^0_\infty[F;Y]$ obey the following relations:

$$\sum_{0\leq|\Lambda|\leq k} f^\Lambda d_\Lambda \phi \wedge \omega = \sum_{0\leq|\Lambda|} (-1)^{|\Lambda|} d_\Lambda(f^\Lambda)\phi \wedge \omega + d_H\sigma, \tag{7.5.3}$$

$$\sum_{0\leq|\Lambda|\leq k} (-1)^{|\Lambda|} d_\Lambda(f^\Lambda\phi) = \sum_{0\leq|\Lambda|\leq k} \eta(f)^\Lambda d_\Lambda \phi, \tag{7.5.4}$$

$$\eta(f)^\Lambda = \sum_{0\leq|\Sigma|\leq k-|\Lambda|} (-1)^{|\Sigma+\Lambda|} \frac{(|\Sigma+\Lambda|)!}{|\Sigma|!|\Lambda|!} d_\Sigma f^{\Sigma+\Lambda}, \tag{7.5.5}$$

$$\eta(\eta(f))^\Lambda = f^\Lambda. \tag{7.5.6}$$

$\Box$

Theorem 7.5.3: Given the Koszul–Tate complex (7.4.23), the module of graded densities $\mathcal{P}^{0,n}_\infty\{N\}$ is decomposed into the cochain sequence

$$0 \to \mathcal{S}^{0,n}_\infty[F;Y] \xrightarrow{\mathbf{u}} \mathcal{P}^{0,n}_\infty\{N\}^1 \xrightarrow{\mathbf{u}} \mathcal{P}^{0,n}_\infty\{N\}^2 \xrightarrow{\mathbf{u}} \cdots, \tag{7.5.7}$$

$$\mathbf{u} = u + u^{(1)} + \cdots + u^{(N)} = \tag{7.5.8}$$
$$u^A\frac{\partial}{\partial s^A} + u^r\frac{\partial}{\partial c^r} + \cdots + u^{r_{N-1}}\frac{\partial}{\partial c^{r_{N-1}}},$$

graded in ghost number. Its ascent operator $\mathbf{u}$ (7.5.8) is an odd graded derivation of ghost number 1 where

$$u = u^A\frac{\partial}{\partial s^A}, \qquad u^A = \sum_{0\leq|\Lambda|} c^r_\Lambda \eta(\Delta^A_r)^\Lambda, \tag{7.5.9}$$

is a variational symmetry of a graded Lagrangian L and the graded derivations

$$u_{(k)} = u^{r_{k-1}} \frac{\partial}{\partial c^{r_{k-1}}} = \sum_{0\leq|\Lambda|} c^{r_k}_\Lambda \eta(\Delta^{r_{k-1}}_{r_k})^\Lambda \frac{\partial}{\partial c^{r_{k-1}}}, \quad k=1,\ldots,N, \tag{7.5.10}$$

obey the relations

$$\begin{aligned} & \sum_{0\leq|\Sigma|} d_\Sigma u^{r_{k-1}} \frac{\partial}{\partial c^{r_{k-1}}_\Sigma} u^{r_{k-2}} = \overline{\delta}(\alpha^{r_{k-2}}), \\ & \alpha^{r_{k-2}} = -\sum_{0\leq|\Sigma|} \eta(h^{(r_{k-2})(A,\Xi)}_{r_k})^\Sigma d_\Sigma(c^{r_k}\overline{s}_{\Xi A}). \end{aligned} \tag{7.5.11}$$

□

A glance at the expression (7.5.9) shows that the variational symmetry u is a linear differential operator on the $C^\infty(X)$-module $\mathcal{C}^{(0)}$ of ghosts with values into the real space $\mathfrak{g}_L$ of variational symmetries. Following Definition 7.2.10 extended to Lagrangian theories of odd variables, we call u (7.5.9) the *gauge symmetry of a graded Lagrangian* L which is associated to the complete NI (7.4.12).

Remark 7.5.1: In contrast with Definitions 7.2.10 and 7.2.12, gauge symmetries u (7.5.9) are parameterized by ghosts, but not gauge parameters. Given a gauge symmetry u (7.2.28) defined as a derivation of the real ring $\mathcal{O}^0_\infty[Y\times E]$, one can associate to it the gauge symmetry

$$u = \left(\sum_{0\leq|\Lambda|\leq m} u^{\lambda\Lambda}_a(x^\mu)c^a_\Lambda\right)\partial_\lambda + \left(\sum_{0\leq|\Lambda|\leq m} u^{i\Lambda}_a(x^\mu, y^j_\Sigma)c^a_\Lambda\right)\partial_i, \tag{7.5.12}$$

which is an odd graded derivation of the real ring $\mathcal{S}^0_\infty[E;Y]$, and *vice versa*. □

Turn now to the relation (7.5.11). For $k=1$, it takes the form

$$\sum_{0\leq|\Sigma|} d_\Sigma u^r \frac{\partial}{\partial c^r_\Sigma} u^A = \overline{\delta}(\alpha^A)$$

of a first-stage gauge symmetry condition on-shell which the non-trivial gauge symmetry u (7.5.9) satisfies. Therefore, one can treat the odd graded derivation

$$u^{(1)} = u^r \frac{\partial}{\partial c^r}, \qquad u^r = \sum_{0\leq|\Lambda|} c^{r_1}_\Lambda \eta(\Delta^r_{r_1})^\Lambda,$$

as a *first-stage gauge symmetry* associated to the complete first-stage NI

$$\sum_{0\leq|\Lambda|} \Delta^{r,\Lambda}_{r_1} d_\Lambda \left(\sum_{0\leq|\Sigma|} \Delta^{A,\Sigma}_r \overline{s}_{\Sigma A} \right) = -\overline{\delta} \left(\sum_{0\leq|\Sigma|,|\Xi|} h^{(B,\Sigma)(A,\Xi)}_{r_1} \overline{s}_{\Sigma B} \overline{s}_{\Xi A} \right).$$

Iterating the arguments, one comes to the relation (7.5.11) which provides a k-stage gauge symmetry condition which is associated to the complete k-stage NI (7.4.24). The odd graded derivation $u_{(k)}$ (7.5.10) is called the *k-stage gauge symmetry*.

In accordance with Theorem 7.5.3, components of the ascent operator $\mathbf{u}$ (7.5.8) are complete non-trivial gauge and higher-stage gauge symmetries. Therefore, we agree to call this operator the *gauge operator*.

Being a variational symmetry, a gauge symmetry u (7.5.9) defines the weak conservation law (7.3.30). Let u be an exact Lagrangian symmetry. In this case, the associated symmetry current

$$\mathcal{J}_u = -h_0(u\rfloor \Xi_L) \tag{7.5.13}$$

is conserved. The peculiarity of gauge conservation laws always is that the symmetry current (7.5.13) is reduced to a superpotential as follows.

Theorem 7.5.4: If u (7.5.9) is an exact gauge symmetry of a graded Lagrangian L, the corresponding conserved symmetry current $\mathcal{J}_u$ (7.5.13) takes the form

$$\mathcal{J}_u = W + d_H U = (W^\mu + d_\nu U^{\nu\mu})\omega_\mu, \tag{7.5.14}$$

where the term W vanishes on-shell, and U is a horizontal $(n-2)$-form. □

Chapter 8

Topics on commutative geometry

Several relevant topics on commutative geometry and algebraic topology are compiled in this Chapter [9, 17, 23].

8.1 Commutative algebra

An *algebra* $\mathcal{A}$ is an additive group which is additionally provided with distributive multiplication. All algebras throughout the book are associative, unless they are Lie algebras. A *ring* is defined to be a *unital* algebra, i.e., it contains a unit element $\mathbf{1} \neq 0$.

A subset $\mathcal{I}$ of an algebra $\mathcal{A}$ is called a left (resp. right) *ideal* if it is a subgroup of the additive group $\mathcal{A}$ and $ab \in \mathcal{I}$ (resp. $ba \in \mathcal{I}$) for all $a \in \mathcal{A}$, $b \in \mathcal{I}$. If $\mathcal{I}$ is both a left and right ideal, it is called a two-sided ideal. An ideal is a subalgebra, but a *proper ideal* (i.e., $\mathcal{I} \neq \mathcal{A}$) of a ring is not a subring because it does not contain the unit element.

Let $\mathcal{A}$ be a commutative ring. Of course, its ideals are two-sided. Its proper ideal is said to be *maximal* if it does not belong to another proper ideal. A commutative ring $\mathcal{A}$ is called *local* if it has a unique maximal ideal. This ideal consists of all non-invertible elements of $\mathcal{A}$.

Given an ideal $\mathcal{I} \subset \mathcal{A}$, the additive factor group $\mathcal{A}/\mathcal{I}$ is an algebra, called the *factor algebra*. If $\mathcal{A}$ is a ring, then $\mathcal{A}/\mathcal{I}$ is so. If $\mathcal{I}$ is a maximal ideal, the factor ring $\mathcal{A}/\mathcal{I}$ is a field.

Given an algebra $\mathcal{A}$, an additive group P is said to be a left (resp. right) $\mathcal{A}$-*module* if it is provided with distributive multiplication $\mathcal{A}\times P \to P$ by elements of $\mathcal{A}$ such that $(ab)p = a(bp)$ (resp. $(ab)p = b(ap)$) for all $a, b \in \mathcal{A}$ and $p \in P$. If $\mathcal{A}$ is a ring, one additionally assumes that $\mathbf{1}p = p = p\mathbf{1}$ for all $p \in P$. Left and right module structures are usually written by means of left and right multiplications $(a, p) \to ap$ and $(a, p) \to pa$, respectively. If P is both a left module over an algebra $\mathcal{A}$ and a right module over an algebra $\mathcal{A}'$, it is called an $(\mathcal{A} - \mathcal{A}')$-bimodule (an $\mathcal{A}$-bimodule if $\mathcal{A} = \mathcal{A}'$). If $\mathcal{A}$ is a commutative algebra, an $\mathcal{A}$-bimodule P is said to be *commutative* if $ap = pa$ for all $a \in \mathcal{A}$ and $p \in P$. Any left or right module over a commutative algebra $\mathcal{A}$ can be brought into a commutative bimodule. Therefore, unless otherwise stated, any module over a commutative algebra $\mathcal{A}$ is called an $\mathcal{A}$-module. A module over a field is called a *vector space*.

If an algebra $\mathcal{A}$ is a module over a commutative ring $\mathcal{K}$, it is said to be a $\mathcal{K}$-algebra.

Hereafter, all associative algebras are assumed to be commutative.

The following are standard constructions of new modules from old ones.

• The *direct sum* $P_1 \oplus P_2$ of $\mathcal{A}$-modules P_1 and P_2 is the additive group $P_1 \times P_2$ provided with the $\mathcal{A}$-module structure

$$a(p_1, p_2) = (ap_1, ap_2), \qquad p_{1,2} \in P_{1,2}, \qquad a \in \mathcal{A}.$$

Let $\{P_i\}_{i\in I}$ be a set of modules. Their direct sum $\oplus P_i$ consists of elements $(\ldots, p_i, \ldots)$ of the Cartesian product $\prod P_i$ such that $p_i \neq 0$ at most for a finite number of indices $i \in I$.

• Given a submodule Q of an $\mathcal{A}$-module P, the quotient P/Q of the additive group P with respect to its subgroup Q also is provided with an $\mathcal{A}$-module structure. It is called a *factor module*.

• The set $\mathrm{Hom}_{\mathcal{A}}(P, Q)$ of $\mathcal{A}$-linear morphisms of an $\mathcal{A}$-module P to an $\mathcal{A}$-module Q is naturally an $\mathcal{A}$-module. The $\mathcal{A}$-module

$$P^* = \mathrm{Hom}_{\mathcal{A}}(P, \mathcal{A})$$

is called the *dual* of an $\mathcal{A}$-module P. There is a monomorphism $P \to P^{**}$.

• The *tensor product* $P \otimes Q$ of $\mathcal{A}$-modules P and Q is an additive group which is generated by elements $p \otimes q$, $p \in P$, $q \in Q$, obeying the relations

$$\begin{aligned}
&(p + p') \otimes q = p \otimes q + p' \otimes q,\\
&p \otimes (q + q') = p \otimes q + p \otimes q',\\
&pa \otimes q = p \otimes aq, \qquad p \in P, \qquad q \in Q, \qquad a \in \mathcal{A}.
\end{aligned}$$

It is provided with the $\mathcal{A}$-module structure

$$a(p \otimes q) = (ap) \otimes q = p \otimes (qa) = (p \otimes q)a.$$

In particular, we have the following.

(i) If a ring $\mathcal{A}$ is treated as an $\mathcal{A}$-module, the tensor product $\mathcal{A} \otimes_{\mathcal{A}} Q$ is canonically isomorphic to Q via the assignment

$$\mathcal{A} \otimes_{\mathcal{A}} Q \ni a \otimes q \leftrightarrow aq \in Q.$$

(ii) The *tensor product of Abelian groups* G and G' is defined as their tensor product $G \otimes G'$ as $\mathbb{Z}$-modules.

(iii) The *tensor product of commutative algebras* $\mathcal{A}$ and $\mathcal{A}'$ is defined as their tensor product $\mathcal{A} \otimes \mathcal{A}'$ as modules provided with the multiplication

$$(a \otimes a')(b \otimes b') = (aa') \otimes bb'.$$

An $\mathcal{A}$-module P is called *free* if it has a *basis*, i.e., a linearly independent subset $I \subset P$ spanning P such that each element of P has a unique expression as a linear combination of elements of I with a finite number of non-zero coefficients from an algebra $\mathcal{A}$. Any vector space is free. Any module is isomorphic to a quotient of a free module. A module is said to be *finitely generated* (or of *finite rank*) if it is a quotient of a free module with a finite basis.

One says that a module P is *projective* if it is a direct summand of a free module, i.e., there exists a module Q such that $P \oplus Q$ is a free module. A module P is projective iff $P = \mathbf{p}S$ where S is a free module and $\mathbf{p}$ is a projector of S, i.e., $\mathbf{p}^2 = \mathbf{p}$.

Theorem 8.1.1: Any projective module over a local ring is free. $\Box$

Now we focus on exact sequences, direct and inverse limits of modules. A composition of module morphisms

$$P \xrightarrow{i} Q \xrightarrow{j} T$$

is said to be *exact* at Q if $\operatorname{Ker} j = \operatorname{Im} i$. A composition of module morphisms

$$0 \to P \xrightarrow{i} Q \xrightarrow{j} T \to 0 \tag{8.1.1}$$

is called a *short exact sequence* if it is exact at all the terms P, Q, and T. This condition implies that: (i) i is a monomorphism, (ii) $\operatorname{Ker} j = \operatorname{Im} i$, and (iii) j is an epimorphism onto the quotient $T = Q/P$.

Theorem 8.1.2: Given an exact sequence of modules (8.1.1) and another $\mathcal{A}$-module R, the sequence of modules

$$0 \to \operatorname{Hom}_{\mathcal{A}}(T, R) \xrightarrow{j^*} \operatorname{Hom}_{\mathcal{A}}(Q, R) \xrightarrow{i^*} \operatorname{Hom}(P, R)$$

is exact at the first and second terms, i.e., j^* is a monomorphism, but i^* need not be an epimorphism. $\Box$

One says that the exact sequence (8.1.1) is *split* if there exists a monomorphism $s : T \to Q$ such that $j \circ s = \operatorname{Id} T$ or, equivalently,

$$Q = i(P) \oplus s(T) \cong P \oplus T.$$

Theorem 8.1.3: The exact sequence (8.1.1) is always split if T is a projective module. $\Box$

A *directed set* I is a set with an order relation $<$ which satisfies the following three conditions: (i) $i < i$, for all $i \in I$; (ii) if $i < j$ and $j < k$, then $i < k$; (iii) for any $i, j \in I$, there exists $k \in I$ such that $i < k$ and $j < k$. It may happen that $i \neq j$, but $i < j$ and $j < i$ simultaneously.

A family of modules $\{P_i\}_{i\in I}$ (over the same algebra), indexed by a directed set I, is called a *direct system* if, for any pair $i < j$, there exists a morphism $r^i_j : P_i \to P_j$ such that

$$r^i_i = \mathrm{Id}\, P_i, \qquad r^i_j \circ r^j_k = r^i_k, \qquad i < j < k.$$

A direct system of modules admits a *direct limit*. This is a module P_∞ together with morphisms $r^i_\infty : P_i \to P_\infty$ such that $r^i_\infty = r^j_\infty \circ r^i_j$ for all $i < j$. The module P_∞ consists of elements of the direct sum $\oplus_I P_i$ modulo the identification of elements of P_i with their images in P_j for all $i < j$. An example of a direct system is a *direct sequence*

$$P_0 \longrightarrow P_1 \longrightarrow \cdots P_i \xrightarrow{r^i_{i+1}} \cdots, \qquad I = \mathbb{N}. \tag{8.1.2}$$

Note that direct limits also exist in the categories of commutative and graded commutative algebras and rings, but not in categories containing non-Abelian groups.

Theorem 8.1.4: Direct limits commute with direct sums and tensor products of modules. Namely, let $\{P_i\}$ and $\{Q_i\}$ be two direct systems of modules over the same algebra which are indexed by the same directed set I, and let P_∞ and Q_∞ be their direct limits. Then the direct limits of the direct systems $\{P_i \oplus Q_i\}$ and $\{P_i \otimes Q_i\}$ are $P_\infty \oplus Q_\infty$ and $P_\infty \otimes Q_\infty$, respectively. □

Theorem 8.1.5: A morphism of a direct system $\{P_i, r^i_j\}_I$ to a direct system $\{Q_{i'}, \rho^{i'}_{j'}\}_{I'}$ consists of an order preserving map $f : I \to I'$ and morphisms $F_i : P_i \to Q_{f(i)}$ which obey the compatibility conditions

$$\rho^{f(i)}_{f(j)} \circ F_i = F_j \circ r^i_j.$$

If P_∞ and Q_∞ are limits of these direct systems, there exists a unique morphism $F_\infty : P_\infty \to Q_\infty$ such that

$$\rho_\infty^{f(i)} \circ F_i = F_\infty \circ r_\infty^i.$$

□

Theorem 8.1.6: Direct limits preserve monomorphisms and epimorphisms, i.e., if all $F_i : P_i \to Q_{f(i)}$ are monomorphisms or epimorphisms, so is $\Phi_\infty : P_\infty \to Q_\infty$. Let short exact sequences

$$0 \to P_i \xrightarrow{F_i} Q_i \xrightarrow{\Phi_i} T_i \to 0 \qquad (8.1.3)$$

for all $i \in I$ define a short exact sequence of direct systems of modules $\{P_i\}_I$, $\{Q_i\}_I$, and $\{T_i\}_I$ which are indexed by the same directed set I. Then their direct limits form a short exact sequence

$$0 \to P_\infty \xrightarrow{F_\infty} Q_\infty \xrightarrow{\Phi_\infty} T_\infty \to 0. \qquad (8.1.4)$$

□

In particular, the direct limit of factor modules Q_i/P_i is the factor module Q_∞/P_∞. By virtue of Theorem 8.1.4, if all the exact sequences (8.1.3) are split, the exact sequence (8.1.4) is well.

Remark 8.1.1: Let P be an $\mathcal{A}$-module. We denote

$$P^{\otimes k} = \overset{k}{\otimes} P.$$

Let us consider the direct system of $\mathcal{A}$-modules

$$\mathcal{A} \longrightarrow (\mathcal{A} \oplus P) \longrightarrow \cdots (\mathcal{A} \oplus P \oplus \cdots \oplus P^{\otimes k}) \longrightarrow \cdots.$$

Its direct limit

$$\otimes P = \mathcal{A} \oplus P \oplus \cdots \oplus P^{\otimes k} \oplus \cdots \qquad (8.1.5)$$

is an $\mathbb{N}$-graded $\mathcal{A}$-algebra with respect to the tensor product $\otimes$. It is called the *tensor algebra* of a module P. Its quotient with respect to the ideal generated by elements

$$p \otimes p' + p' \otimes p, \qquad p, p' \in P,$$

is an $\mathbb{N}$-graded commutative algebra, called the *exterior algebra* of P. $\square$

Given an *inverse sequences* of modules

$$P^0 \longleftarrow P^1 \longleftarrow \cdots P^i \overset{\pi_i^{i+1}}{\longleftarrow} \cdots, \tag{8.1.6}$$

its *inductive limit* is a module P^∞ together with morphisms $\pi_i^\infty : P^\infty \to P^i$ such that $\pi_i^\infty = \pi_i^j \circ \pi_j^\infty$ for all $i < j$. It consists of elements $(\ldots, p^i, \ldots)$, $p^i \in P^i$, of the Cartesian product $\prod P^i$ such that $p^i = \pi_i^j(p^j)$ for all $i < j$.

Theorem 8.1.7: Inductive limits preserve monomorphisms, but not epimorphisms. Let exact sequences

$$0 \to P^i \xrightarrow{F^i} Q^i \xrightarrow{\Phi^i} T^i, \qquad i \in \mathbb{N},$$

for all $i \in \mathbb{N}$ define an exact sequence of inverse systems of modules $\{P^i\}$, $\{Q^i\}$ and $\{T^i\}$. Then their inductive limits form an exact sequence

$$0 \to P^\infty \xrightarrow{F^\infty} Q^\infty \xrightarrow{\Phi^\infty} T^\infty.$$

$\square$

In contrast with direct limits, the inductive ones exist in the category of groups which are not necessarily commutative.

8.2 Differential operators on modules

This Section addresses the notion of a linear differential operator on a module over a commutative ring.

Let $\mathcal{K}$ be a commutative ring and $\mathcal{A}$ a commutative $\mathcal{K}$-ring. Let P and Q be $\mathcal{A}$-modules. The $\mathcal{K}$-module $\mathrm{Hom}_{\mathcal{K}}(P, Q)$ of $\mathcal{K}$-module homomorphisms $\Phi : P \to Q$ can be endowed with the two different $\mathcal{A}$-module structures

$$(a\Phi)(p) = a\Phi(p), \quad (\Phi \bullet a)(p) = \Phi(ap), \quad a \in \mathcal{A}, \quad p \in P. \tag{8.2.1}$$

For the sake of convenience, we refer to the second one as the $\mathcal{A}^\bullet$-module structure. Let us put

$$\delta_a\Phi = a\Phi - \Phi \bullet a, \qquad a \in \mathcal{A}. \tag{8.2.2}$$

Definition 8.2.1: An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P,Q)$ is called a Q-valued *differential operator* of order s on P if

$$\delta_{a_0} \circ \cdots \circ \delta_{a_s}\Delta = 0$$

for any tuple of $s+1$ elements $a_0, \ldots, a_s$ of $\mathcal{A}$. The set $\mathrm{Diff}_s(P,Q)$ of these operators inherits the $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures (8.2.1). □

In particular, zero order differential operators obey the condition

$$\delta_a\Delta(p) = a\Delta(p) - \Delta(ap) = 0, \qquad a \in \mathcal{A}, \qquad p \in P,$$

and, consequently, they coincide with $\mathcal{A}$-module morphisms $P \to Q$. A first order differential operator Δ satisfies the condition

$$\delta_b \circ \delta_a\,\Delta(p) = ba\Delta(p) - b\Delta(ap) - a\Delta(bp) + \Delta(abp) = 0, \quad a, b \in \mathcal{A}.$$

The following fact reduces the study of Q-valued differential operators on an $\mathcal{A}$-module P to that of Q-valued differential operators on the ring $\mathcal{A}$.

Theorem 8.2.2: Let us consider the $\mathcal{A}$-module morphism

$$h_s : \mathrm{Diff}_s(\mathcal{A},Q) \to Q, \qquad h_s(\Delta) = \Delta(\mathbf{1}). \tag{8.2.3}$$

Any Q-valued s-order differential operator $\Delta \in \mathrm{Diff}_s(P,Q)$ on P uniquely factorizes as

$$\Delta : P \xrightarrow{\mathfrak{f}_\Delta} \mathrm{Diff}_s(\mathcal{A},Q) \xrightarrow{h_s} Q \tag{8.2.4}$$

through the morphism h_s (8.2.3) and some homomorphism

$$\mathfrak{f}_\Delta : P \to \mathrm{Diff}_s(\mathcal{A},Q), \qquad (\mathfrak{f}_\Delta p)(a) = \Delta(ap), \qquad a \in \mathcal{A}, \tag{8.2.5}$$

of the $\mathcal{A}$-module P to the $\mathcal{A}^\bullet$-module $\mathrm{Diff}_s(\mathcal{A}, Q)$. The assignment $\Delta \to \mathfrak{f}_\Delta$ defines the isomorphism

$$\mathrm{Diff}_s(P, Q) = \mathrm{Hom}_{\mathcal{A}-\mathcal{A}^\bullet}(P, \mathrm{Diff}_s(\mathcal{A}, Q)). \tag{8.2.6}$$

□

Let $P = \mathcal{A}$. Any zero order Q-valued differential operator Δ on $\mathcal{A}$ is defined by its value $\Delta(\mathbf{1})$. Then there is an isomorphism

$$\mathrm{Diff}_0(\mathcal{A}, Q) = Q$$

via the association

$$Q \ni q \to \Delta_q \in \mathrm{Diff}_0(\mathcal{A}, Q),$$

where Δ_q is given by the equality $\Delta_q(\mathbf{1}) = q$. A first order Q-valued differential operator Δ on $\mathcal{A}$ fulfils the condition

$$\Delta(ab) = b\Delta(a) + a\Delta(b) - ba\Delta(\mathbf{1}), \qquad a, b \in \mathcal{A}.$$

It is called a Q-valued *derivation* of $\mathcal{A}$ if $\Delta(\mathbf{1}) = 0$, i.e., the *Leibniz rule*

$$\Delta(ab) = \Delta(a)b + a\Delta(b), \qquad a, b \in \mathcal{A}, \tag{8.2.7}$$

holds. One obtains at once that any first order differential operator on $\mathcal{A}$ falls into the sum

$$\Delta(a) = a\Delta(\mathbf{1}) + [\Delta(a) - a\Delta(\mathbf{1})]$$

of the zero order differential operator $a\Delta(\mathbf{1})$ and the derivation $\Delta(a) - a\Delta(\mathbf{1})$. If ∂ is a Q-valued derivation of $\mathcal{A}$, then $a\partial$ is well for any $a \in \mathcal{A}$. Hence, Q-valued derivations of $\mathcal{A}$ constitute an $\mathcal{A}$-module $\mathfrak{d}(\mathcal{A}, Q)$, called the *derivation module*. There is the $\mathcal{A}$-module decomposition

$$\mathrm{Diff}_1(\mathcal{A}, Q) = Q \oplus \mathfrak{d}(\mathcal{A}, Q). \tag{8.2.8}$$

If $P = Q = \mathcal{A}$, the derivation module $\mathfrak{d}\mathcal{A}$ of $\mathcal{A}$ also is a Lie $\mathcal{K}$-algebra with respect to the Lie bracket

$$[u, u'] = u \circ u' - u' \circ u, \qquad u, u' \in \mathcal{A}. \tag{8.2.9}$$

Accordingly, the decomposition (8.2.8) takes the form

$$\mathrm{Diff}_1(\mathcal{A}) = \mathcal{A} \oplus \mathfrak{d}\mathcal{A}. \tag{8.2.10}$$

Definition 8.2.3: A *connection* on an $\mathcal{A}$-module P is an $\mathcal{A}$-module morphism

$$\mathfrak{d}\mathcal{A} \ni u \to \nabla_u \in \mathrm{Diff}_1(P,P) \tag{8.2.11}$$

such that the first order differential operators ∇_u obey the *Leibniz rule*

$$\nabla_u(ap) = u(a)p + a\nabla_u(p), \quad a \in \mathcal{A}, \quad p \in P. \tag{8.2.12}$$

□

Though ∇_u (8.2.11) is called a connection, it in fact is a *covariant differential* on a module P.

Let P be a commutative $\mathcal{A}$-ring and $\mathfrak{d}P$ the derivation module of P as a $\mathcal{K}$-ring. The $\mathfrak{d}P$ is both a P- and $\mathcal{A}$-module. Then Definition 8.2.3 is modified as follows.

Definition 8.2.4: A *connection* on an $\mathcal{A}$-ring P is an $\mathcal{A}$-module morphism

$$\mathfrak{d}\mathcal{A} \ni u \to \nabla_u \in \mathfrak{d}P \subset \mathrm{Diff}_1(P,P), \tag{8.2.13}$$

which is a connection on P as an $\mathcal{A}$-module. □

8.3 Homology and cohomology of complexes

This Section summarizes the relevant basics on complexes of modules over a commutative ring.

Let $\mathcal{K}$ be a commutative ring. A sequence

$$0 \leftarrow B_0 \xleftarrow{\partial_1} B_1 \xleftarrow{\partial_2} \cdots B_p \xleftarrow{\partial_{p+1}} \cdots \tag{8.3.1}$$

of $\mathcal{K}$-modules B_p and homomorphisms ∂_p is said to be a *chain complex* if

$$\partial_p \circ \partial_{p+1} = 0, \qquad p \in \mathbb{N},$$

i.e., $\operatorname{Im}\partial_{p+1} \subset \operatorname{Ker}\partial_p$. Homomorphisms ∂_p are called *boundary operators*. Elements of a module B_p, its submodules $\operatorname{Ker}\partial_p \subset B_p$ and $\operatorname{Im}\partial_{p+1} \subset \operatorname{Ker}\partial_p$ are called *p-chains*, *p-cycles* and *p-boundaries*, respectively. The *p-th homology group* of the chain complex B_* (8.3.1) is the factor module

$$H_p(B_*) = \operatorname{Ker}\partial_p/\operatorname{Im}\partial_{p+1}.$$

It is a $\mathcal{K}$-module. In particular, we have $H_0(B_*) = B_0/\operatorname{Im}\partial_1$. The chain complex (8.3.1) is exact at a term B_p iff $H_p(B_*) = 0$. This complex is said to be *k-exact* if its homology groups $H_{p\leq k}(B_*)$ are trivial. It is called *exact* if all its homology groups are trivial, i.e., it is an exact sequence.

A sequence

$$0 \to B^0 \xrightarrow{\delta^0} B^1 \xrightarrow{\delta^1} \cdots B^p \xrightarrow{\delta^p} \cdots \tag{8.3.2}$$

of modules B^p and their homomorphisms δ^p is said to be a *cochain complex* (or, simply, a *complex*) if

$$\delta^{p+1} \circ \delta^p = 0, \qquad p \in \mathbb{N},$$

i.e., $\operatorname{Im}\delta^p \subset \operatorname{Ker}\delta^{p+1}$. The homomorphisms δ^p are called *coboundary operators*. Elements of a module B^p, its submodules $\operatorname{Ker}\delta^p \subset B^p$ and $\operatorname{Im}\delta^{p-1}$ are called *p-cochains*, *p-cocycles* and *p-coboundaries*, respectively. The *p-th cohomology group* of the complex B^* (8.3.2) is the factor module

$$H^p(B^*) = \operatorname{Ker}\delta^p/\operatorname{Im}\delta^{p-1}.$$

It is a $\mathcal{K}$-module. In particular, $H^0(B^*) = \operatorname{Ker}\delta^0$. The complex (8.3.2) is exact at a term B^p iff $H^p(B^*) = 0$. This complex is an exact sequence if all its cohomology groups are trivial.

A complex (B^*, δ^*) is called *acyclic* if its cohomology groups $H^{0<p}(B^*)$ are trivial. A complex (B^*, δ^*) is said to be a *resolution* of a module B if it is acyclic and $H^0(B^*) = \operatorname{Ker}\delta^0 = B$.

The following are the standard constructions of new complexes from old ones.

• Given complexes (B_1^*, δ_1^*) and (B_2^*, δ_2^*), their *direct sum* $B_1^* \oplus B_2^*$ is a complex of modules

$$(B_1^* \oplus B_2^*)^p = B_1^p \oplus B_2^p$$

with respect to the coboundary operators

$$\delta_\oplus^p(b_1^p + b_2^p) = \delta_1^p b_1^p + \delta_2^p b_2^p.$$

• Given a subcomplex (C^*, δ^*) of a complex (B^*, δ^*), the *factor complex* B^*/C^* is defined as a complex of factor modules B^p/C^p provided with the coboundary operators $\delta^p[b^p] = [\delta^p b^p]$, where $[b^p] \in B^p/C^p$ denotes the coset of an element b^p.

• Given complexes (B_1^*, δ_1^*) and (B_2^*, δ_2^*), their *tensor product* $B_1^* \otimes B_2^*$ is a complex of modules

$$(B_1^* \otimes B_2^*)^p = \bigoplus_{k+r=p} B_1^k \otimes B_2^r$$

with respect to the coboundary operators

$$\delta_\otimes^p(b_1^k \otimes b_2^r) = (\delta_1^k b_1^k) \otimes b_2^r + (-1)^k b_1^k \otimes (\delta_2^r b_2^r).$$

A *cochain morphism* of complexes

$$\gamma : B_1^* \to B_2^* \tag{8.3.3}$$

is defined as a family of degree-preserving homomorphisms

$$\gamma^p : B_1^p \to B_2^p, \qquad p \in \mathbb{N},$$

such that

$$\delta_2^p \circ \gamma^p = \gamma^{p+1} \circ \delta_1^p, \qquad p \in \mathbb{N}.$$

It follows that if $b^p \in B_1^p$ is a cocycle or a coboundary, then $\gamma^p(b^p) \in B_2^p$ is so. Therefore, the cochain morphism of complexes (8.3.3) yields an induced homomorphism of their cohomology groups

$$[\gamma]^* : H^*(B_1^*) \to H^*(B_2^*).$$

Let short exact sequences

$$0 \to C^p \xrightarrow{\gamma_p} B^p \xrightarrow{\zeta_p} F^p \to 0$$

for all $p \in \mathbb{N}$ define a short exact sequence of complexes

$$0 \to C^* \xrightarrow{\gamma} B^* \xrightarrow{\zeta} F^* \to 0, \tag{8.3.4}$$

where γ is a cochain monomorphism and ζ is a cochain epimorphism onto the quotient $F^* = B^*/C^*$.

Theorem 8.3.1: The short exact sequence of complexes (8.3.4) yields the long exact sequence of their cohomology groups

$$\begin{aligned} 0 \to H^0(C^*) \xrightarrow{[\gamma]^0} H^0(B^*) \xrightarrow{[\zeta]^0} H^0(F^*) \xrightarrow{\tau^0} H^1(C^*) \longrightarrow \cdots \\ \longrightarrow H^p(C^*) \xrightarrow{[\gamma]^p} H^p(B^*) \xrightarrow{[\zeta]^p} H^p(F^*) \xrightarrow{\tau^p} H^{p+1}(C^*) \longrightarrow \cdots . \end{aligned} \tag{8.3.5}$$

□

Theorem 8.3.2: A direct sequence of complexes

$$B_0^* \longrightarrow B_1^* \longrightarrow \cdots B_k^* \xrightarrow{\gamma_{k+1}^k} B_{k+1}^* \longrightarrow \cdots \tag{8.3.6}$$

admits a direct limit B_∞^* which is a complex whose cohomology $H^*(B_\infty^*)$ is a direct limit of the direct sequence of cohomology groups

$$H^*(B_0^*) \longrightarrow H^*(B_1^*) \longrightarrow \cdots H^*(B_k^*) \xrightarrow{[\gamma_{k+1}^k]} H^*(B_{k+1}^*) \longrightarrow \cdots .$$

□

8.4 Differential calculus over a commutative ring

Let $\mathfrak{g}$ be a Lie algebra over a commutative ring $\mathcal{K}$. Let $\mathfrak{g}$ act on a $\mathcal{K}$-module P on the left such that

$$[\varepsilon, \varepsilon']p = (\varepsilon \circ \varepsilon' - \varepsilon' \circ \varepsilon)p, \qquad \varepsilon, \varepsilon' \in \mathfrak{g}.$$

Then one calls P the Lie algebra $\mathfrak{g}$-*module*. Let us consider $\mathcal{K}$-multilinear skew-symmetric maps

$$c^k : \overset{k}{\times} \mathfrak{g} \to P.$$

They form a $\mathfrak{g}$-module $C^k[\mathfrak{g}; P]$. Let us put $C^0[\mathfrak{g}; P] = P$. We obtain the cochain complex

$$0 \to P \xrightarrow{\delta^0} C^1[\mathfrak{g}; P] \xrightarrow{\delta^1} \cdots C^k[\mathfrak{g}; P] \xrightarrow{\delta^k} \cdots \tag{8.4.1}$$

with respect to the *Chevalley–Eilenberg coboundary operators*

$$\delta^k c^k(\varepsilon_0, \ldots, \varepsilon_k) = \sum_{i=0}^{k} (-1)^i \varepsilon_i c^k(\varepsilon_0, \ldots, \widehat{\varepsilon}_i, \ldots, \varepsilon_k) + \tag{8.4.2}$$
$$\sum_{1\leq i<j\leq k} (-1)^{i+j} c^k([\varepsilon_i, \varepsilon_j], \varepsilon_0, \ldots, \widehat{\varepsilon}_i, \ldots, \widehat{\varepsilon}_j, \ldots, \varepsilon_k),$$

where the caret $\widehat{}$ denotes omission. For instance, we have

$$\delta^0 p(\varepsilon_0) = \varepsilon_0 p, \tag{8.4.3}$$
$$\delta^1 c^1(\varepsilon_0, \varepsilon_1) = \varepsilon_0 c^1(\varepsilon_1) - \varepsilon_1 c^1(\varepsilon_0) - c^1([\varepsilon_0, \varepsilon_1]). \tag{8.4.4}$$

The complex (8.4.1) is called the *Chevalley–Eilenberg complex*, and its cohomology $H^*(\mathfrak{g}, P)$ is the *Chevalley–Eilenberg cohomology* of a Lie algebra $\mathfrak{g}$ with coefficients in P.

Let $\mathcal{A}$ be a commutative $\mathcal{K}$-ring. Since the derivation module $\mathfrak{d}\mathcal{A}$ of $\mathcal{A}$ is a Lie $\mathcal{K}$-algebra, one can associate to $\mathcal{A}$ the Chevalley–Eilenberg complex $C^*[\mathfrak{d}\mathcal{A}; \mathcal{A}]$. Its subcomplex of $\mathcal{A}$-multilinear maps is a DGA, also called the differential calculus over $\mathcal{A}$. By a gradation throughout this Section is meant the $\mathbb{N}$-gradation.

A *graded algebra* Ω^* over a commutative ring $\mathcal{K}$ is defined as a direct sum $\Omega^* = \bigoplus_k \Omega^k$ of $\mathcal{K}$-modules Ω^k, provided with an associative multiplication law $\alpha \cdot \beta$, $\alpha, \beta \in \Omega^*$, such that $\alpha \cdot \beta \in \Omega^{|\alpha|+|\beta|}$, where $|\alpha|$ denotes the degree of an element $\alpha \in \Omega^{|\alpha|}$. In particular, it follows that Ω^0 is a (non-commutative) $\mathcal{K}$-algebra $\mathcal{A}$, while $\Omega^{k>0}$ are $\mathcal{A}$-bimodules and Ω^* is an $(\mathcal{A}-\mathcal{A})$-algebra. A graded algebra is said to be *graded commutative* if

$$\alpha \cdot \beta = (-1)^{|\alpha||\beta|} \beta \cdot \alpha, \qquad \alpha, \beta \in \Omega^*.$$

A graded algebra Ω^* is called the *differential graded algebra* (DGA) or the *differential calculus* over $\mathcal{A}$ if it is a cochain complex of $\mathcal{K}$-modules

$$0 \to \mathcal{K} \longrightarrow \mathcal{A} \xrightarrow{\delta} \Omega^1 \xrightarrow{\delta} \cdots \Omega^k \xrightarrow{\delta} \cdots \tag{8.4.5}$$

relative to a coboundary operator δ which obeys the *graded Leibniz rule*

$$\delta(\alpha \cdot \beta) = \delta\alpha \cdot \beta + (-1)^{|\alpha|}\alpha \cdot \delta\beta. \tag{8.4.6}$$

In particular, $\delta : \mathcal{A} \to \Omega^1$ is a Ω^1-valued derivation of a $\mathcal{K}$-algebra $\mathcal{A}$. The cochain complex (8.4.5) is said to be the *abstract de Rham complex* of the DGA (Ω^*, δ). Cohomology $H^*(\Omega^*)$ of the complex (8.4.5) is called the *abstract de Rham cohomology*.

A morphism γ between two DGAs (Ω^*, δ) and (Ω'^*, δ') is defined as a cochain morphism, i.e., $\gamma \circ \delta = \gamma \circ \delta'$. It yields the corresponding morphism of the abstract de Rham cohomology groups of these algebras.

One considers the minimal differential graded subalgebra $\Omega^*\mathcal{A}$ of the DGA Ω^* which contains $\mathcal{A}$. Seen as an $(\mathcal{A} - \mathcal{A})$-algebra, it is generated by the elements δa, $a \in \mathcal{A}$, and consists of monomials

$$\alpha = a_0 \delta a_1 \cdots \delta a_k, \qquad a_i \in \mathcal{A},$$

whose product obeys the *juxtaposition rule*

$$(a_0\delta a_1) \cdot (b_0 \delta b_1) = a_0 \delta(a_1 b_0) \cdot \delta b_1 - a_0 a_1 \delta b_0 \cdot \delta b_1$$

in accordance with the equality (8.4.6). The DGA $(\Omega^*\mathcal{A}, \delta)$ is called the *minimal differential calculus* over $\mathcal{A}$.

Let now $\mathcal{A}$ be a commutative $\mathcal{K}$-ring possessing a non-trivial Lie algebra $\mathfrak{d}\mathcal{A}$ of derivations. We consider the extended Chevalley–Eilenberg complex

$$0 \to \mathcal{K} \xrightarrow{\text{in}} C^*[\mathfrak{d}\mathcal{A}; \mathcal{A}]$$

of the Lie algebra $\mathfrak{d}\mathcal{A}$ with coefficients in the ring $\mathcal{A}$, regarded as a $\mathfrak{d}\mathcal{A}$-module. It is easily justified that this complex contains a subcomplex $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ of $\mathcal{A}$-multilinear skew-symmetric maps

$$\phi^k : \overset{k}{\times} \mathfrak{d}\mathcal{A} \to \mathcal{A} \tag{8.4.7}$$

with respect to the Chevalley–Eilenberg coboundary operator

$$d\phi(u_0,\ldots,u_k) = \sum_{i=0}^{k}(-1)^i u_i(\phi(u_0,\ldots,\widehat{u_i},\ldots,u_k)) + \sum_{i<j}(-1)^{i+j}\phi([u_i,u_j],u_0,\ldots,\widehat{u}_i,\ldots,\widehat{u}_j,\ldots,u_k). \tag{8.4.8}$$

In particular, we have

$$(da)(u) = u(a), \qquad a\in\mathcal{A}, \qquad u\in\mathfrak{d}\mathcal{A},$$
$$(d\phi)(u_0,u_1) = u_0(\phi(u_1)) - u_1(\phi(u_0)) - \phi([u_0,u_1]), \quad \phi\in\mathcal{O}^1[\mathfrak{d}\mathcal{A}],$$
$$\mathcal{O}^0[\mathfrak{d}\mathcal{A}] = \mathcal{A},$$
$$\mathcal{O}^1[\mathfrak{d}\mathcal{A}] = \mathrm{Hom}_{\mathcal{A}}(\mathfrak{d}\mathcal{A},\mathcal{A}) = \mathfrak{d}\mathcal{A}^*.$$

It follows that $d(\mathbf{1})=0$ and d is a $\mathcal{O}^1[\mathfrak{d}\mathcal{A}]$-valued derivation of $\mathcal{A}$.

The graded module $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ is provided with the structure of a graded $\mathcal{A}$-algebra with respect to the exterior product

$$\phi\wedge\phi'(u_1,...,u_{r+s}) = \sum_{i_1<\cdots<i_r;j_1<\cdots<j_s}\mathrm{sgn}^{i_1\cdots i_r j_1\cdots j_s}_{1\cdots r+s}\phi(u_{i_1},\ldots,u_{i_r})\phi'(u_{j_1},\ldots,u_{j_s}), \tag{8.4.9}$$
$$\phi\in\mathcal{O}^r[\mathfrak{d}\mathcal{A}], \qquad \phi'\in\mathcal{O}^s[\mathfrak{d}\mathcal{A}], \qquad u_k\in\mathfrak{d}\mathcal{A},$$

where $\mathrm{sgn}^{\cdots}_{\cdots}$ is the sign of a permutation. This product obeys the relations

$$d(\phi\wedge\phi') = d(\phi)\wedge\phi' + (-1)^{|\phi|}\phi\wedge d(\phi'), \quad \phi,\phi'\in\mathcal{O}^*[\mathfrak{d}\mathcal{A}],$$
$$\phi\wedge\phi' = (-1)^{|\phi||\phi'|}\phi'\wedge\phi. \tag{8.4.10}$$

By virtue of the first one, $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ is a differential graded $\mathcal{K}$-algebra, called the *Chevalley–Eilenberg differential calculus* over a $\mathcal{K}$-ring $\mathcal{A}$. The relation (8.4.10) shows that $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ is a graded commutative algebra.

The *minimal Chevalley–Eilenberg differential calculus* $\mathcal{O}^*\mathcal{A}$ over a ring $\mathcal{A}$ consists of the monomials

$$a_0da_1\wedge\cdots\wedge da_k, \qquad a_i\in\mathcal{A}.$$

Its complex

$$0\to\mathcal{K}\longrightarrow\mathcal{A}\xrightarrow{d}\mathcal{O}^1\mathcal{A}\xrightarrow{d}\cdots\mathcal{O}^k\mathcal{A}\xrightarrow{d}\cdots \tag{8.4.11}$$

is said to be the *de Rham complex* of a $\mathcal{K}$-ring $\mathcal{A}$, and its cohomology $H^*(\mathcal{A})$ is called the *de Rham cohomology* of $\mathcal{A}$.

8.5 Sheaf cohomology

A *sheaf* on a topological space X is a continuous fibre bundle $\pi : S \to X$ in modules over a commutative ring $\mathcal{K}$, where the surjection π is a local homeomorphism and fibres S_x, $x \in X$, called the *stalks*, are provided with the discrete topology. Global sections of a sheaf S make up a $\mathcal{K}$-module $S(X)$, called the *structure module* of S.

Any sheaf is generated by a presheaf. A *presheaf* $S_{\{U\}}$ on a topological space X is defined if a module S_U over a commutative ring $\mathcal{K}$ is assigned to every open subset $U \subset X$ ($S_\emptyset = 0$) and if, for any pair of open subsets $V \subset U$, there exists the restriction morphism $r^U_V : S_U \to S_V$ such that

$$r^U_U = \mathrm{Id}\, S_U, \qquad r^U_W = r^V_W r^U_V, \qquad W \subset V \subset U.$$

Every presheaf $S_{\{U\}}$ on a topological space X yields a sheaf on X whose stalk S_x at a point $x \in X$ is the direct limit of the modules S_U, $x \in U$, with respect to the restriction morphisms r^U_V. It means that, for each open neighborhood U of a point x, every element $s \in S_U$ determines an element $s_x \in S_x$, called the *germ* of s at x. Two elements $s \in S_U$ and $s' \in S_V$ belong to the same germ at x iff there exists an open neighborhood $W \subset U \cap V$ of x such that $r^U_W s = r^V_W s'$.

Example 8.5.1: Let $C^0_{\{U\}}$ be the presheaf of continuous real functions on a topological space X. Two such functions s and s' define the same germ s_x if they coincide on an open neighborhood of x. Hence, we obtain the *sheaf C^0_X of continuous functions* on X. Similarly, the *sheaf C^∞_X of smooth functions* on a smooth manifold X is defined. Let us also mention the presheaf of real functions which are constant on connected open subsets of X. It generates the *constant sheaf* on X denoted by $\mathbb{R}$. □

Different presheaves may generate the same sheaf. Conversely, every sheaf S defines a presheaf $S(\{U\})$ of modules $S(U)$ of its local sections. It is called the *canonical presheaf* of the sheaf S. If a sheaf S is constructed from a presheaf $S_{\{U\}}$, there are natural module morphisms

$$S_U \ni s \to s(U) \in S(U), \qquad s(x) = s_x, \quad x \in U,$$

which are neither monomorphisms nor epimorphisms in general. For instance, it may happen that a non-zero presheaf defines a zero sheaf. The sheaf generated by the canonical presheaf of a sheaf S coincides with S.

A direct sum and a tensor product of presheaves (as families of modules) and sheaves (as fibre bundles in modules) are naturally defined. By virtue of Theorem 8.1.4, a direct sum (resp. a tensor product) of presheaves generates a direct sum (resp. a tensor product) of the corresponding sheaves.

Remark 8.5.2: In a different terminology, a sheaf is introduced as a presheaf which satisfies the following additional axioms.

(S1) Suppose that $U \subset X$ is an open subset and $\{U_\alpha\}$ is its open cover. If $s, s' \in S_U$ obey the condition

$$r^U_{U_\alpha}(s) = r^U_{U_\alpha}(s')$$

for all U_α, then $s = s'$.

(S2) Let U and $\{U_\alpha\}$ be as in previous item. Suppose that we are given a family of presheaf elements $\{s_\alpha \in S_{U_\alpha}\}$ such that

$$r^{U_\alpha}_{U_\alpha\cap U_\lambda}(s_\alpha) = r^{U_\lambda}_{U_\alpha\cap U_\lambda}(s_\lambda)$$

for all U_α, U_λ. Then there exists a presheaf element $s \in S_U$ such that $s_\alpha = r^U_{U_\alpha}(s)$.

Canonical presheaves are in one-to-one correspondence with presheaves obeying these axioms. For instance, presheaves of continuous, smooth and locally constant functions in Example 8.5.1 satisfy the axioms (S1) – (S2). $\Box$

Remark 8.5.3: The notion of a sheaf can be extended to sets, but not to non-commutative groups. One can consider a presheaf of such groups, but it generates a sheaf of sets because a direct limit of non-commutative groups need not be a group. The first (but not higher) cohomology of X with coefficients in this sheaf is defined. □

A *morphism of a presheaf* $S_{\{U\}}$ to a presheaf $S'_{\{U\}}$ on the same topological space X is defined as a set of module morphisms $\gamma_U : S_U \to S'_U$ which commute with restriction morphisms. A morphism of presheaves yields a *morphism of sheaves* generated by these presheaves. This is a bundle morphism over X such that $\gamma_x : S_x \to S'_x$ is the direct limit of morphisms γ_U, $x \in U$. Conversely, any morphism of sheaves $S \to S'$ on a topological space X yields a morphism of canonical presheaves of local sections of these sheaves. Let $\mathrm{Hom}\,(S|_U, S'|_U)$ be the commutative group of sheaf morphisms $S|_U \to S'|_U$ for any open subset $U \subset X$. These groups are assembled into a presheaf, and define the sheaf $\mathrm{Hom}\,(S, S')$ on X. There is a monomorphism

$$\mathrm{Hom}\,(S, S')(U) \to \mathrm{Hom}\,(S(U), S'(U)), \tag{8.5.1}$$

which need not be an isomorphism.

By virtue of Theorem 8.1.6, if a presheaf morphism is a monomorphism or an epimorphism, so is the corresponding sheaf morphism. Furthermore, the following holds.

Theorem 8.5.1: A short exact sequence

$$0 \to S'_{\{U\}} \to S_{\{U\}} \to S''_{\{U\}} \to 0 \tag{8.5.2}$$

of presheaves on the same topological space yields the short exact sequence of sheaves generated by these presheaves

$$0 \to S' \to S \to S'' \to 0, \tag{8.5.3}$$

where the *factor sheaf* $S'' = S/S'$ is isomorphic to that generated by the factor presheaf $S''_{\{U\}} = S_{\{U\}}/S'_{\{U\}}$. If the exact sequence of presheaves (8.5.2) is split, i.e.,

$$S_{\{U\}} \cong S'_{\{U\}} \oplus S''_{\{U\}},$$

the corresponding splitting

$$S \cong S' \oplus S''$$

of the exact sequence of sheaves (8.5.3) holds. □

The converse is more intricate. A sheaf morphism induces a morphism of the corresponding canonical presheaves. If $S \to S'$ is a monomorphism,

$$S(\{U\}) \to S'(\{U\})$$

also is a monomorphism. However, if $S \to S'$ is an epimorphism,

$$S(\{U\}) \to S'(\{U\})$$

need not be so. Therefore, the short exact sequence (8.5.3) of sheaves yields the exact sequence of the canonical presheaves

$$0 \to S'(\{U\}) \to S(\{U\}) \to S''(\{U\}), \tag{8.5.4}$$

where $S(\{U\}) \to S''(\{U\})$ is not necessarily an epimorphism. At the same time, there is the short exact sequence of presheaves

$$0 \to S'(\{U\}) \to S(\{U\}) \to S''_{\{U\}} \to 0, \tag{8.5.5}$$

where the factor presheaf

$$S''_{\{U\}} = S(\{U\})/S'(\{U\})$$

generates the factor sheaf $S'' = S/S'$, but need not be its canonical presheaf.

Let us turn now to sheaf cohomology. Note that only proper covers are considered.

Let $S_{\{U\}}$ be a presheaf of modules on a topological space X, and let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open cover of X. One constructs a cochain complex where a p-cochain is defined as a function s^p which associates an element

$$s^p(i_0, \ldots, i_p) \in S_{U_{i_0}\cap\cdots\cap U_{i_p}} \tag{8.5.6}$$

to each $(p+1)$-tuple $(i_0,\ldots,i_p)$ of indices in I. These p-cochains are assembled into a module $C^p(\mathfrak{U},S_{\{U\}})$. Let us introduce the coboundary operator

$$\begin{aligned}
&\delta^p : C^p(\mathfrak{U},S_{\{U\}}) \to C^{p+1}(\mathfrak{U},S_{\{U\}}),\\
&\delta^p s^p(i_0,\ldots,i_{p+1}) = \sum_{k=0}^{p+1}(-1)^k r_W^{W_k} s^p(i_0,\ldots,\widehat{i}_k,\ldots,i_{p+1}), \qquad (8.5.7)\\
&W = U_{i_0}\cap\ldots\cap U_{i_{p+1}}, \qquad W_k = U_{i_0}\cap\cdots\cap\widehat{U}_{i_k}\cap\cdots\cap U_{i_{p+1}}.
\end{aligned}$$

One can easily check that $\delta^{p+1}\circ\delta^p = 0$. Thus, we obtain the cochain complex of modules

$$0\to C^0(\mathfrak{U},S_{\{U\}}) \xrightarrow{\delta^0} \cdots C^p(\mathfrak{U},S_{\{U\}}) \xrightarrow{\delta^p} C^{p+1}(\mathfrak{U},S_{\{U\}})\cdots. \qquad (8.5.8)$$

Its cohomology groups

$$H^p(\mathfrak{U};S_{\{U\}}) = \operatorname{Ker}\delta^p/\operatorname{Im}\delta^{p-1}$$

are modules. Of course, they depend on an open cover $\mathfrak{U}$ of X.

Let $\mathfrak{U}'$ be a refinement of the cover $\mathfrak{U}$. Then there is a morphism of cohomology groups

$$H^*(\mathfrak{U};S_{\{U\}}) \to H^*(\mathfrak{U}';S_{\{U\}}). \qquad (8.5.9)$$

Let us take the direct limit of cohomology groups $H^*(\mathfrak{U};S_{\{U\}})$ relative to these morphisms, where $\mathfrak{U}$ runs through all open covers of X. This limit $H^*(X;S_{\{U\}})$ is called the cohomology of X with coefficients in the presheaf $S_{\{U\}}$.

Let S be a sheaf on a topological space X. *Cohomology of X with coefficients in S* or, simply, *sheaf cohomology* of X is defined as cohomology

$$H^*(X;S) = H^*(X;S(\{U\}))$$

with coefficients in the canonical presheaf $S(\{U\})$ of the sheaf S.

In this case, a p-cochain $s^p\in C^p(\mathfrak{U},S(\{U\}))$ is a collection

$$s^p = \{s^p(i_0,\ldots,i_p)\}$$

of local sections $s^p(i_0,\ldots,i_p)$ of the sheaf S over $U_{i_0}\cap\cdots\cap U_{i_p}$ for each $(p+1)$-tuple $(U_{i_0},\ldots,U_{i_p})$ of elements of the cover $\mathfrak{U}$. The coboundary operator (8.5.7) reads

$$\delta^p s^p(i_0,\ldots,i_{p+1})=\sum_{k=0}^{p+1}(-1)^k s^p(i_0,\ldots,\widehat{i}_k,\ldots,i_{p+1})|_{U_{i_0}\cap\cdots\cap U_{i_{p+1}}}.$$

For instance, we have

$$\delta^0 s^0(i,j)=[s^0(j)-s^0(i)]|_{U_i\cap U_j}, \tag{8.5.10}$$

$$\delta^1 s^1(i,j,k)=[s^1(j,k)-s^1(i,k)+s^1(i,j)]|_{U_i\cap U_j\cap U_k}. \tag{8.5.11}$$

A glance at the expression (8.5.10) shows that a zero-cocycle is a collection $s=\{s(i)\}_I$ of local sections of the sheaf S over $U_i\in\mathfrak{U}$ such that $s(i)=s(j)$ on $U_i\cap U_j$. It follows from the axiom (S2) in Remark 8.5.2 that s is a global section of the sheaf S, while each $s(i)$ is its restriction $s|_{U_i}$ to U_i. Consequently, the cohomology group $H^0(\mathfrak{U};S(\{U\}))$ is isomorphic to the structure module $S(X)$ of global sections of the sheaf S. A one-cocycle is a collection $\{s(i,j)\}$ of local sections of the sheaf S over overlaps $U_i\cap U_j$ which satisfy the *cocycle condition*

$$[s(j,k)-s(i,k)+s(i,j)]|_{U_i\cap U_j\cap U_k}=0. \tag{8.5.12}$$

If X is a paracompact space, the study of its sheaf cohomology is essentially simplified due to the following fact.

Theorem 8.5.2: Cohomology of a paracompact space X with coefficients in a sheaf S coincides with cohomology of X with coefficients in any presheaf generating the sheaf S. $\square$

Remark 8.5.4: We follow the definition of a *paracompact topological space* as a Hausdorff space such that any its open cover admits a *locally finite* open refinement, i.e., any point has an open neighborhood which intersects only a finite number of elements of this refinement. A topological space X is paracompact iff any cover $\{U_\xi\}$ of X admits a subordinate *partition of unity* $\{f_\xi\}$, i.e.:

(i) f_ξ are real positive continuous functions on X;

(ii) $\operatorname{supp} f_\xi \subset U_\xi$;

(iii) each point $x \in X$ has an open neighborhood which intersects only a finite number of the sets $\operatorname{supp} f_\xi$;

(iv) $\sum_\xi f_\xi(x) = 1$ for all $x \in X$. □

The key point of the analysis of sheaf cohomology is that short exact sequences of sheaves yield long exact sequences of their cohomology groups.

Let $S_{\{U\}}$ and $S'_{\{U\}}$ be presheaves on the same topological space X. It is readily observed that, given an open cover $\mathfrak{U}$ of X, any morphism $S_{\{U\}} \to S'_{\{U\}}$ yields a cochain morphism of complexes

$$C^*(\mathfrak{U}, S_{\{U\}}) \to C^*(\mathfrak{U}, S'_{\{U\}})$$

and the corresponding morphism

$$H^*(\mathfrak{U}; S_{\{U\}}) \to H^*(\mathfrak{U}; S'_{\{U\}})$$

of cohomology groups of these complexes. Passing to the direct limit through all refinements of $\mathfrak{U}$, we come to a morphism of cohomology groups

$$H^*(X; S_{\{U\}}) \to H^*(X; S'_{\{U\}})$$

of X with coefficients in the presheaves $S_{\{U\}}$ and $S'_{\{U\}}$. In particular, any sheaf morphism $S \to S'$ yields a morphism of canonical presheaves

$$S(\{U\}) \to S'(\{U\})$$

and the corresponding cohomology morphism

$$H^*(X; S) \to H^*(X; S').$$

By virtue of Theorems 8.3.1 and 8.3.2, every short exact sequence

$$0 \to S'_{\{U\}} \longrightarrow S_{\{U\}} \longrightarrow S''_{\{U\}} \to 0 \tag{8.5.13}$$

of presheaves on the same topological space X and the corresponding exact sequence of complexes (8.5.8) yield the long exact sequence

$$\begin{aligned}0 \to H^0(X;S'_{\{U\}}) &\longrightarrow H^0(X;S_{\{U\}}) \longrightarrow H^0(X;S''_{\{U\}}) \longrightarrow \\ H^1(X;S'_{\{U\}}) &\longrightarrow \cdots H^p(X;S'_{\{U\}}) \longrightarrow H^p(X;S_{\{U\}}) \longrightarrow \\ H^p(X;S''_{\{U\}}) &\longrightarrow H^{p+1}(X;S'_{\{U\}}) \longrightarrow \cdots\end{aligned} \tag{8.5.14}$$

of the cohomology groups of X with coefficients in these presheaves. This result is extended to the exact sequence of sheaves. Let

$$0 \to S' \longrightarrow S \longrightarrow S'' \to 0 \tag{8.5.15}$$

be a short exact sequence of sheaves on X. It yields the short exact sequence of presheaves (8.5.5) where the presheaf $S''_{\{U\}}$ generates the sheaf S''. If X is paracompact,

$$H^*(X;S''_{\{U\}}) = H^*(X;S'')$$

in accordance with Theorem 8.5.2, and we have the exact sequence of sheaf cohomology

$$\begin{aligned}0 \to H^0(X;S') &\longrightarrow H^0(X;S) \longrightarrow H^0(X;S'') \longrightarrow \\ H^1(X;S') &\longrightarrow \cdots H^p(X;S') \longrightarrow H^p(X;S) \longrightarrow \\ H^p(X;S'') &\longrightarrow H^{p+1}(X;S') \longrightarrow \cdots .\end{aligned} \tag{8.5.16}$$

Let us turn now to the abstract de Rham theorem which provides a powerful tool of studying algebraic systems on paracompact spaces.

Let us consider an exact sequence of sheaves

$$0 \to S \xrightarrow{h} S_0 \xrightarrow{h^0} S_1 \xrightarrow{h^1} \cdots S_p \xrightarrow{h^p} \cdots . \tag{8.5.17}$$

It is said to be a *resolution of the sheaf* S if each sheaf $S_{p\geq 0}$ is *acyclic*, i.e., its cohomology groups $H^{k>0}(X;S_p)$ vanish.

Any exact sequence of sheaves (8.5.17) yields the sequence of their structure modules

$$0 \to S(X) \xrightarrow{h_*} S_0(X) \xrightarrow{h^0_*} S_1(X) \xrightarrow{h^1_*} \cdots S_p(X) \xrightarrow{h^p_*} \cdots \tag{8.5.18}$$

which is always exact at terms $S(X)$ and $S_0(X)$ (see the exact sequence (8.5.4)). The sequence (8.5.18) is a cochain complex because

$$h_*^{p+1} \circ h_*^p = 0.$$

If X is a paracompact space and the exact sequence (8.5.17) is a resolution of S, the forthcoming *abstract de Rham theorem* establishes an isomorphism of cohomology of the complex (8.5.18) to cohomology of X with coefficients in the sheaf S.

Theorem 8.5.3: Given a resolution (8.5.17) of a sheaf S on a paracompact topological space X and the induced complex (8.5.18), there are isomorphisms

$$H^0(X; S) = \operatorname{Ker} h_*^0, \quad H^q(X; S) = \operatorname{Ker} h_*^q / \operatorname{Im} h_*^{q-1}, \quad q > 0. \tag{8.5.19}$$

□

A sheaf S on a paracompact space X is called *fine* if, for each locally finite open cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of X, there exists a system $\{h_i\}$ of endomorphisms $h_i : S \to S$ such that:

(i) there is a closed subset $V_i \subset U_i$ and $h_i(S_x) = 0$ if $x \notin V_i$,

(ii) $\sum\limits_{i \in I} h_i$ is the identity map of S.

Theorem 8.5.4: A fine sheaf on a paracompact space is acyclic. □

There is the following important example of fine sheaves.

Theorem 8.5.5: Let X be a paracompact topological space which admits a partition of unity performed by elements of the structure module $\mathfrak{A}(X)$ of some sheaf $\mathfrak{A}$ of real functions on X. Then any sheaf S of $\mathfrak{A}$-modules on X, including $\mathfrak{A}$ itself, is fine. □

In particular, the sheaf C_X^0 of continuous functions on a paracompact topological space is fine, and so is any sheaf of C_X^0-modules.

8.6 Local-ringed spaces

Local-ringed spaces are sheafs of local rings. For instance, smooth manifolds, represented by sheaves of real smooth functions, make up a subcategory of the category of local-ringed spaces.

A sheaf $\mathfrak{R}$ on a topological space X is said to be a *ringed space* if its stalk $\mathfrak{R}_x$ at each point $x \in X$ is a real commutative ring. A ringed space is often denoted by a pair $(X, \mathfrak{R})$ of a topological space X and a sheaf $\mathfrak{R}$ of rings on X. They are called the *body* and the *structure sheaf* of a ringed space, respectively.

A ringed space is said to be a *local-ringed space* (a *geometric space*) if it is a sheaf of local rings.

For instance, the sheaf C^0_X of continuous real functions on a topological space X is a local-ringed space. Its stalk C^0_x, $x \in X$, contains the unique maximal ideal of germs of functions vanishing at x.

Morphisms of local-ringed spaces are defined as those of sheaves on different topological spaces as follows.

Let $\varphi : X \to X'$ be a continuous map. Given a sheaf S on X, its *direct image* $\varphi_* S$ on X' is generated by the presheaf of assignments

$$X' \supset U' \to S(\varphi^{-1}(U'))$$

for any open subset $U' \subset X'$. Conversely, given a sheaf S' on X', its *inverse image* $\varphi^* S'$ on X is defined as the pull-back onto X of the continuous fibre bundle S' over X', i.e., $\varphi^* S'_x = S_{\varphi(x)}$. This sheaf is generated by the presheaf which associates to any open $V \subset X$ the direct limit of modules $S'(U)$ over all open subsets $U \subset X'$ such that $V \subset f^{-1}(U)$.

Remark 8.6.1: Let $i : X \to X'$ be a closed subspace of X'. Then $i_* S$ is a unique sheaf on X' such that

$$i_* S|_X = S, \qquad i_* S|_{X' \setminus X} = 0.$$

Indeed, if $x' \in X \subset X'$, then

$$i_* S(U') = S(U' \cap X)$$

for any open neighborhood U of this point. If $x' \notin X$, there exists its neighborhood U' such that $U' \cap X$ is empty, i.e., $i_*S(U') = 0$. The sheaf i_*S is called the *trivial extension* of the sheaf S. $\Box$

By a *morphism of ringed spaces*

$$(X, \mathfrak{R}) \to (X', \mathfrak{R}')$$

is meant a pair $(\varphi, \widehat{\varphi})$ of a continuous map $\varphi : X \to X'$ and a sheaf morphism $\widehat{\varphi} : \mathfrak{R}' \to \varphi_* \mathfrak{R}$ or, equivalently, a sheaf morphisms $\varphi^* \mathfrak{R}' \to \mathfrak{R}$. Restricted to each stalk, a sheaf morphism Φ is assumed to be a ring homomorphism. A morphism of ringed spaces is said to be:

- a monomorphism if φ is an injection and Φ is an epimorphism,
- an epimorphism if φ is a surjection, while Φ is a monomorphism.

Let $(X, \mathfrak{R})$ be a local-ringed space. By a *sheaf* $\mathfrak{d}\mathfrak{R}$ *of derivations* of the sheaf $\mathfrak{R}$ is meant a subsheaf of endomorphisms of $\mathfrak{R}$ such that any section u of $\mathfrak{d}\mathfrak{R}$ over an open subset $U \subset X$ is a derivation of the real ring $\mathfrak{R}(U)$. It should be emphasized that, since the monomorphism (8.5.1) is not necessarily an isomorphism, a derivation of the ring $\mathfrak{R}(U)$ need not be a section of the sheaf $\mathfrak{d}\mathfrak{R}|_U$. Namely, it may happen that, given open sets $U' \subset U$, there is no restriction morphism

$$\mathfrak{d}(\mathfrak{R}(U)) \to \mathfrak{d}(\mathfrak{R}(U')).$$

Given a local-ringed space $(X, \mathfrak{R})$, a sheaf P on X is called a *sheaf of* $\mathfrak{R}$*-modules* if every stalk P_x, $x \in X$, is an $\mathfrak{R}_x$-module or, equivalently, if $P(U)$ is an $\mathfrak{R}(U)$-module for any open subset $U \subset X$. A sheaf of $\mathfrak{R}$-modules P is said to be *locally free* if there exists an open neighborhood U of every point $x \in X$ such that $P(U)$ is a free $\mathfrak{R}(U)$-module. If all these free modules are of finite rank (resp. of the same finite rank), one says that P is of *finite type* (resp. of constant rank). The structure module of a locally free sheaf is called a *locally free module*.

The following is a generalization of Theorem 8.5.5.

Theorem 8.6.1: Let X be a paracompact space which admits a partition of unity by elements of the structure module $S(X)$ of some sheaf S of real functions on X. Let P be a sheaf of S-modules. Then P is fine and, consequently, acyclic. $\Box$

Assumed to be paracompact, a smooth manifold X admits a partition of unity performed by smooth real functions. It follows that the sheaf C^∞_X of smooth real functions on X is fine, and so is any sheaf of C^∞_X-modules, e.g., the sheaves of sections of smooth vector bundles over X.

Similarly to the sheaf C^0_X of continuous functions, the sheaf C^∞_X of smooth real functions on a smooth manifold X is a local-ringed spaces. Its stalk C^∞_x at a point $x \in X$ has a unique maximal ideal μ_x of germs of smooth functions vanishing at x. Though the sheaf C^∞_X is defined on a topological space X, it fixes a unique smooth manifold structure on X as follows.

Theorem 8.6.2: Let X be a paracompact topological space and $(X, \mathfrak{R})$ a local-ringed space. Let X admit an open cover $\{U_i\}$ such that the sheaf $\mathfrak{R}$ restricted to each U_i is isomorphic to the local-ringed space $(\mathbb{R}^n, C^\infty_{R^n})$. Then X is an n-dimensional smooth manifold together with a natural isomorphism of local-ringed spaces $(X, \mathfrak{R})$ and (X, C^∞_X). $\Box$

One can think of this result as being an alternative definition of smooth real manifolds in terms of local-ringed spaces. In particular, there is one-to-one correspondence between smooth manifold morphisms $X \to X'$ and the $\mathbb{R}$-ring morphisms $C^\infty(X') \to C^\infty(X)$.

For instance, let $Y \to X$ be a smooth vector bundle. The germs of its sections make up a sheaf of C^∞_X-modules, called the *structure sheaf* S_Y of a vector bundle $Y \to X$. The sheaf S_Y is fine. The structure module of this sheaf coincides with the structure module $Y(X)$ of global sections of a vector bundle $Y \to X$. The following *Serre–Swan theorem* shows that these modules exhaust all projective modules of finite rank over $C^\infty(X)$. Originally proved for bundles over a compact base X, this theorem has been extended to an arbitrary X.

Theorem 8.6.3: Let X be a smooth manifold. A $C^\infty(X)$-module P is isomorphic to the structure module of a smooth vector bundle over X iff it is a projective module of finite rank. $\Box$

This theorem states the categorial equivalence between the vector bundles over a smooth manifold X and projective modules of finite rank over the ring $C^\infty(X)$ of smooth real functions on X. The following are corollaries of this equivalence

• The structure module $Y^*(X)$ of the dual $Y^* \to X$ of a vector bundle $Y \to X$ is the $C^\infty(X)$-dual $Y(X)^*$ of the structure module $Y(X)$ of $Y \to X$.

• Any exact sequence of vector bundles

$$0 \to Y \longrightarrow Y' \longrightarrow Y'' \to 0 \tag{8.6.1}$$

over the same base X yields the exact sequence

$$0 \to Y(X) \longrightarrow Y'(X) \longrightarrow Y''(X) \to 0 \tag{8.6.2}$$

of their structure modules, and *vice versa*. In accordance with Theorem 1.2.2, the exact sequence (8.6.1) is always split. Every its splitting defines that of the exact sequence (8.6.2), and *vice versa*.

• The derivation module of the real ring $C^\infty(X)$ coincides with the $C^\infty(X)$-module $\mathcal{T}(X)$ of vector fields on X, i.e., with the structure module of the tangent bundle TX of X. Hence, it is a projective $C^\infty(X)$-module of finite rank. It is the $C^\infty(X)$-dual $\mathcal{T}(X) = \mathcal{O}^1(X)^*$ of the structure module $\mathcal{O}^1(X)$ of the cotangent bundle T^*X of X which is the module of differential one-forms on X and, conversely,

$$\mathcal{O}^1(X) = \mathcal{T}(X)^*.$$

• Therefore, if P is a $C^\infty(X)$-module, one can reformulate Definition 8.2.3 of a connection on P as follows. A connection on a $C^\infty(X)$-module P is a $C^\infty(X)$-module morphism

$$\nabla : P \to \mathcal{O}^1(X) \otimes P, \tag{8.6.3}$$

which satisfies the Leibniz rule

$$\nabla(fp) = df \otimes p + f\nabla(p), \qquad f \in C^\infty(X), \qquad p \in P.$$

It associates to any vector field $\tau \in \mathcal{T}(X)$ on X a first order differential operator ∇_τ on P which obeys the Leibniz rule

$$\nabla_\tau(fp) = (\tau\rfloor df)p + f\nabla_\tau p. \tag{8.6.4}$$

In particular, let $Y \to X$ be a vector bundle and $Y(X)$ its structure module. The notion of a connection on the structure module $Y(X)$ is equivalent to the standard geometric notion of a connection on a vector bundle $Y \to X$.

Since the derivation module of the real ring $C^\infty(X)$ is the $C^\infty(X)$-module $\mathcal{T}(X)$ of vector fields on X and

$$\mathcal{O}^1(X) = \mathcal{T}(X)^*,$$

the Chevalley–Eilenberg differential calculus over the real ring $C^\infty(X)$ is exactly the DGA $(\mathcal{O}^*(X), d)$ of exterior forms on X, where the Chevalley–Eilenberg coboundary operator d (8.4.8) coincides with the exterior differential. Moreover, one can show that $(\mathcal{O}^*(X), d)$ is a minimal differential calculus, i.e., the $C^\infty(X)$-module $\mathcal{O}^1(X)$ is generated by elements df, $f \in C^\infty(X)$. Therefore, the de Rham complex (8.4.11) of the real ring $C^\infty(X)$ is the *de Rham complex*

$$0 \to \mathbb{R} \longrightarrow C^\infty(X) \xrightarrow{d} \mathcal{O}^1(X) \xrightarrow{d} \cdots \mathcal{O}^k(X) \xrightarrow{d} \cdots \tag{8.6.5}$$

of exterior forms on a manifold X.

The de Rham cohomology of the complex (8.6.5) is called the *de Rham cohomology* $H^*_{\rm DR}(X)$ of X. To describe them, let us consider the *de Rham complex*

$$0 \to \mathbb{R} \longrightarrow C^\infty_X \xrightarrow{d} \mathcal{O}^1_X \xrightarrow{d} \cdots \mathcal{O}^k_X \xrightarrow{d} \cdots \tag{8.6.6}$$

of sheaves $\mathcal{O}^k_X$, $k \in \mathbb{N}_+$, of germs of exterior forms on X. These sheaves are fine. Due to the *Poincaré lemma*, the complex (8.6.6) is exact and, thereby, is a fine resolution of the constant sheaf $\mathbb{R}$ on a manifold X. Then a corollary of Theorem 8.5.3 is the classical *de Rham theorem*.

Theorem 8.6.4: There is an isomorphism

$$H^k_{\mathrm{DR}}(X) = H^k(X;\mathbb{R}) \tag{8.6.7}$$

of the de Rham cohomology $H^*_{\mathrm{DR}}(X)$ of a manifold X to cohomology of X with coefficients in the constant sheaf $\mathbb{R}$. $\square$

Bibliography

[1] Anderson, I. (1992). Introduction to the variational bicomplex, *Contemp. Math.* **132**, 51.

[2] Barnich, G., Brandt, F. and Henneaux, M. (2000). Local BRST cohomology in gauge theories, *Phys. Rep.* **338**, 439.

[3] Bartocci, C., Bruzzo, U. and Hernández Ruipérez, D.(1991). *The Geometry of Supermanifolds* (Kluwer, Dordrecht).

[4] Bashkirov, D., Giachetta, G., Mangiarotti, L. and Sardanashvily, G. (2008). The KT-BRST complex of degenerate Lagrangian systems, *Lett. Math. Phys.* **83**, 237.

[5] Bryant, R., Chern, S., Gardner, R., Goldschmidt, H. and Griffiths, P. (1991). *Exterior Differential Systems* (Springer-Verlag, Berlin).

[6] Fatibene, L. and Francaviglia, M. (2003). *Natural and Gauge Natural Formalism for Classical Field Theories. A Geometric Perspective Including Spinors and Gauge Theories* (Kluwer, Dordrecht).

[7] Giachetta, G., Mangiarotti, L. and Sardanashvily, G. (2005). Lagrangian supersymmetries depending on derivatives. Global analysis and cohomology, *Commun. Math. Phys.* **259**, 103.

[8] Giachetta, G., Mangiarotti, L. and Sardanashvily, G. (2009). On the notion of gauge symmetries of generic Lagrangian field theory, *J. Math. Phys.* **50**, 012903.

[9] Giachetta, G., Mangiarotti, L. and Sardanashvily, G. (2009). *Advanced Classical Field Theory* (World Scientific, Singapore).

[10] Giachetta, G., Mangiarotti, L. and Sardanashvily, G. (2010). *Geometric Formulation of Classical and Quantum Mechanics* (World Scientific, Singapore).

[11] Greub, W., Halperin, S. and Vanstone, R. (1972). *Connections, Curvature and Cohomology* (Academic Press, New York).

[12] Hirzebruch, F. (1966) *Topological Methods in Algebraic Geometry* (Springer-Verlag, Berlin).

[13] Kobayashi, S. and Nomizu, K. (1963) *Foundations of Differential Geometry* (John Wiley, New York - Singapore).

[14] Kolář, I., Michor, P. and Slovák, J. (1993) *Natural Operations in Differential Geometry* (Springer-Verlag, Berlin).

[15] Krasil'shchik, I., Lychagin, V. and Vinogradov, A. (1985). *Geometry of Jet Spaces and Nonlinear Partial Differential Equations* (Gordon and Breach, Glasgow).

[16] Mangiarotti, L. and Sardanashvily, G. (1998). *Gauge Mechanics* (World Scientific, Singapore).

[17] Mangiarotti, L. and Sardanashvily, G. (2000). *Connections in Classical and Quantum Field Theory* (World Scientific, Singapore).

[18] Massey, W. (1978). *Homology and Cohomology Theory* (Marcel Dekker, Inc., New York).

[19] Sardanashvily, G. (1998). Hamiltonian time-dependent mechanics, *J. Math. Phys.* **39**, 2714.

[20] Sardanashvily, G. (2007). Graded infinite order jet manifolds, *Int. J. Geom. Methods Mod. Phys.* **4**, 1335.

[21] Sardanashvily, G. (2008). Classical field theory. Advanced mathematical formulation, *Int. J. Geom. Methods Mod. Phys.* **5**, 1163.

[22] Sardanashvily, G. (2008). Fibre bundles, jet manifolds and Lagrangian theory. Lectures for theoreticians,*arXiv*: 0908.1886.

[23] Sardanashvily, G. (2012). *Lectures on Differential Geometry of Modules and Rings. Application to Quantum Theory* (Lambert Academic Publishing, Saarbrucken).

[24] Sardanashvily, G. (2013). Graded Lagrangian formalism, *Int. J. Geom. Methods Mod. Phys.* **10**, N5 1350016.

[25] Saunders, D. (1989). *The Geometry of Jet Bundles* (Cambridge Univ. Press, Cambridge).

[26] Steenrod, N. (1972). *The Topology of Fibre Bundles* (Princeton Univ. Press, Princeton).

[27] Takens, F. (1979) A global version of the inverse problem of the calculus of variations, *J. Diff. Geom.* **14**, 543.

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