Universal grammar

by

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There is in my opinion no important theoretical difference between natural languages and the artificial languages of logicians; indeed, I consider it possible to comprehend the syntax and semantics of both kinds of languages within a single natural and mathematically precise theory. On this point I differ from a number of philosophers, but agree, I believe, with Chomsky and his associates. It is clear, however, that no adequate and comprehensive semantical theory has yet been constructed,[1] and arguable that no comprehensive and semantically significant syntactical theory yet exists.[2]

[1] Or even a reasonable semantics for a reasonably comprehensive fragment of any natural language, with the single exception of the treatment in Montague [4] of a fragment of English. There is, however, a significant difference between that treatment and the treatment below of an overlapping fragment. The novelty lies in the interpretation of singular terms and verbs, and is introduced in order to provide (for the first time, I believe, in the literature; the proposals in question were first made in my talks before the Southern California Logic Colloquium and the Association for Symbolic Logic in April and May of 1969) a reasonable semantics for discourse involving intensional verbs. (Another approach is also possible, more along the lines of Montague [4]; it remains to be seen which of the two is preferable.)

It should be pointed out that the treatment of English in Montague [4] is fully compatible with the present *general* theory, and indeed, like the conflicting treatment below, can be represented as a special case of it. I should like, however, to withdraw my emphasis in Montague [4] on the possibility of doing without a distinction between sense and denotation. While such a distinction can be avoided in special cases, it remains necessary for the general theory, and probably provides the clearest approach even to the special cases in question.

[2] The basic aim of semantics is to characterize the notions of a true sentence (under a given interpretation) and of entailment, while that of syntax is to characterize the various syntactical categories, especially the set of declarative sentences. It is to be expected, then, that the aim of syntax could be realized

The aim of the present work is to fill this gap, that is, to develop a universal syntax and semantics. I shall also consider the resulting notions in connection with two examples—the first a rather important artificial language and the second a fragment of ordinary English. This merely illustrative fragment is intentionally circumscribed in the interests of simplicity but is perhaps sufficiently rich to indicate the manner in which various more extensive portions of natural language may be subsumed within the general framework. The intensional logic which constitutes the first example below has some derivative importance apart from whatever intrinsic interest it may possess. Very extensive portions of natural languages can, like the illustrative fragment considered in this paper, be adequately interpreted by way of translation (in the precise general sense analyzed here) into that system of intensional logic.

For the sake of brevity I shall content myself with the mere statement of definitions, omitting all theorems apart from a few (called Remarks) directly related to comprehension, and avoiding almost all discussion and intuitive amplification. The resulting exposition will, I realize, be cryptic and unsatisfactory, but a more extended development must be deferred to a book, Montague [5].[3]

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in many different ways, only some of which would provide a suitable basis for semantics. It appears to me that the syntactical analyses of particular fragmentary languages that have been suggested by transformational grammarians, even if successful in correctly characterizing the declarative sentences of those languages, will prove to lack semantic relevance; and I fail to see any great interest in syntax except as a preliminary to semantics. (One could also object to existing syntactical efforts by Chomsky and his associates on grounds of adequacy, mathematical precision, and elegance; but such criticism should perhaps await more definitive and intelligible expositions than are yet available. In particular, I believe the transformational grammarians should be expected to produce a rigorous definition, complete in all details, of the set of declarative sentences of some reasonably rich fragment of English—at least as rich as the fragments treated below or in Montague [4]—before their work can be seriously evaluated.)

[3] The present paper was delivered at a joint symposium of the Association for Symbolic Logic and the American Philosophical Association in December, 1969, and before the U.C.L.A. Philosophy Colloquium in February, 1970; its prepara-

1. *Background notions*

In sections 1—5 I use 'α', 'β', 'ξ', 'η' to refer to ordinal numbers. A *β-place relation* (among members of a set A) is a set of β-place sequences (of members of A). A *β-place operation* (on A) is a $(\beta+1)$-place relation F (among members of A) such that whenever $\langle a_\xi \rangle_{\xi<\beta}$ is a β-place sequence (of members of A) there is exactly one object x (in A) such that the concatenation of $\langle a_\xi \rangle_{\xi<\beta}$ with the 1-place sequence $\langle x \rangle$ is in F; we let $F(\langle a_\xi \rangle_{\xi<\beta}) = x$. F is an *operation* (on A) if and only if F is a β-place operation (on A), for some ordinal number β. A *function* is a 1-place operation; if f is a function, we let $f(x) = f(\langle x \rangle)$.

An *algebra* is a system $\langle A, F_\gamma \rangle_{\gamma \in \Gamma}$, where A is a nonempty set, Γ is a set of any sort, and each F_γ (for $\gamma \in \Gamma$) is an operation on A. If A is any set, x is any object, and $\alpha < \beta$, then $\mathrm{I}_{\alpha,\beta,A}$ (or the αth *β-place identity operation on* A) and $\mathrm{C}_{x,\beta,A}$ (or the *β-place constant operation on A with value* x) are those β-place operations on A such that $I_{\alpha,\beta,A}(a) = a_\alpha$ and $\mathrm{C}_{x,\beta,A}(a) = x$ for every β-place sequence a of members of A. If G is an α-place operation on a nonempty set A and $\langle H_\xi \rangle_{\xi<\alpha}$ is an α-place sequence of β-place operations on A, then $\mathrm{G}^A\langle H_\xi \rangle_{\xi<\alpha}$ (or the *composition*, relative to A, of the operation G with the sequence $\langle H_\xi \rangle_{\xi<\alpha}$ of operations) is that β-place operation on A such that $\mathrm{G}^A\langle H_\xi \rangle_{\xi<\alpha}(a) = \mathrm{G}(\langle H_\xi(a) \rangle_{\xi<\alpha})$ for every β-place sequence a of members of A. If $\langle A, F_\gamma \rangle_{\gamma \in \Gamma}$ is an algebra, then the class of *polynomial operations* over $\langle A, F_\gamma \rangle_{\gamma \in \Gamma}$ is the smallest class K such that (1) $F_\gamma \in K$ for all $\gamma \in \Gamma$, (2) $I_{\alpha,\beta,A} \in K$ for all ordinals α, β such that $\alpha < \beta$, (3) $\mathrm{C}_{x,\beta,A} \in K$ whenever $x \in A$ and β is an ordinal, and (4) for all ordinals α and β, all α-place operations G on A, and all α-place sequences $\langle H_\xi \rangle_{\xi<\alpha}$ of β-place operations on A, if $\mathrm{G} \in K$ and, for all $\xi < \alpha$, $H_\xi \in K$, then $\mathrm{G}^A\langle H_\xi \rangle_{\xi<\alpha} \in K$.[4]

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[4] For those interested in set-theoretic technicalities I might point out that in this characterization the word 'class' is used deliberately rather than 'set'. In

If $\langle A, F_\gamma\rangle_{\gamma\in\Gamma}$ and $\langle B, G_\gamma\rangle_{\gamma\in\Delta}$ are algebras, then h is a *homomorphism from* $\langle A, F_\gamma\rangle_{\gamma\in\Gamma}$ *into* $\langle B, G_\gamma\rangle_{\gamma\in\Delta}$ if and only if (1) $\langle A, F_\gamma\rangle_{\gamma\in\Gamma}$ and $\langle B, G_\gamma\rangle_{\gamma\in\Delta}$ are *similar* (in the sense that $\Gamma=\Delta$ and, for each $\gamma\in\Gamma$, F_γ and G_γ are operations of the same number of places), (2) h is a function with domain A and range included in B, and (3) whenever $\gamma\in\Gamma$ and $\langle a_\xi\rangle_{\xi<\beta}$ is a sequence in the domain of F_γ, $h(F_\gamma(\langle a_\xi\rangle_{\xi<\beta}))=G_\gamma(\langle h(a_\xi)\rangle_{\xi<\beta})$. We say that h is a *homomorphism from* $\langle A, F_\gamma\rangle_{\gamma\in\Gamma}$ *to* $\langle B, G_\gamma\rangle_{\gamma\in\Delta}$ if, in addition, B coincides with the range of h.

REMARK. If $\langle A, F_\gamma\rangle_{\gamma\in\Gamma}$ is an algebra, h is a homomorphism from $\langle A, F_\gamma\rangle_{\gamma\in\Gamma}$ to some algebra, and, for each $\gamma\in\Delta$, G_γ is a polynomial operation over $\langle A, F_\gamma\rangle_{\gamma\in\Gamma}$, then there is exactly one algebra $\langle B, H_\gamma\rangle_{\gamma\in\Delta}$ such that h is a homomorphism from $\langle A, G_\gamma\rangle_{\gamma\in\Delta}$ to $\langle B, H_\gamma\rangle_{\gamma\in\Delta}$.

2. *Syntax*

A *disambiguated language* is a system $\langle A, F_\gamma, X_\delta, S, \delta_0\rangle_{\gamma\in\Gamma,\delta\in\Delta}$ such that (1) $\langle A, F_\gamma\rangle_{\gamma\in\Gamma}$ is an algebra, (2) for all $\delta\in\Delta$, X_δ is a subset of A, (3) A is the smallest set including as subsets all the sets X_δ (for $\delta\in\Delta$) and closed under all the operations F_γ (for $\gamma\in\Gamma$), (4) X_δ and the range of F_γ are disjoint whenever $\delta\in\Delta$ and $\gamma\in\Gamma$, (5) for all $\gamma, \gamma'\in\Gamma$, all sequences a in the domain of F_γ, and all sequences a' in the domain of $F_{\gamma'}$, if $F_\gamma(a)=F_{\gamma'}(a')$, then $\gamma=\gamma'$ and $a=a'$, (6) S is a set of sequences of the form $\langle F_\gamma, \langle\delta_\xi\rangle_{\xi<\beta}, \varepsilon\rangle$, where $\gamma\in\Gamma$, β is the number of places of the operation F_γ, $\delta_\xi\in\Delta$ for all $\xi<\beta$, and $\varepsilon\in\Delta$, and (7) $\delta_0\in\Delta$. (Here the sets X_δ are regarded as the categories of basic expressions of the disambiguated language, the operations F_γ as its structural operations, the set A as the set of all its proper expressions (that is, expressions obtainable from basic expressions by repeated application of structural operations), δ_0 as the index of its category of declarative sentences, and S as

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any of the usual axiomatic formulations of set theory one can prove that there is no *set* K satisfying (1)—(4), but in those formulations that recognize proper classes in addition to sets one can prove that there *is* a proper class satisfying those conditions.

the set of its syntactic rules; these play a role that will be clarified by the next definition. It is clear that if conditions (1)—(5) are satisfied, then $\langle A, F_\gamma\rangle_{\gamma\in\Gamma}$ is what is known as a *free algebra* generated by $\bigcup_{\delta\in\Delta} X_\delta$ (that is, the union of the sets X_δ for $\delta\in\Delta$).)

If $\mathfrak{A}=\langle A, F_\gamma, X_\delta, S, \delta_0\rangle_{\gamma\in\Gamma,\ \delta\in\Delta}$, then $\mathfrak{A}$ *generates* the family C of syntactic categories if and only if (1) C is a family, indexed by Δ, of subsets of A, (2) $X_\delta\subseteq C_\delta$ for all $\delta\in\Delta$, (3) whenever $\langle F, \langle\delta_\xi\rangle_{\xi<\beta}, \varepsilon\rangle\in S$ and $a_\xi\in C_{\delta_\xi}$ for all $\xi<\beta$, $F(\langle a_\xi\rangle_{\xi<\beta})\in C_\varepsilon$, and (4) whenever C′ satisfies (1)—(3), $C_\delta\subseteq C'_\delta$ for all $\delta\in\Delta$.

REMARK. If $\mathfrak{A}$ is any disambiguated language, then $\mathfrak{A}$ generates exactly one family of syntactic categories.

A *language* is a pair $\langle\langle A, F_\gamma, X_\delta, S, \delta_0\rangle_{\gamma\in\Gamma,\ \delta\in\Delta}, R\rangle$ such that $\langle A, F_\gamma, X_\delta, S, \delta_0\rangle_{\gamma\in\Gamma,\ \delta\in\Delta}$ is a disambiguated language and R is a binary relation with domain included in A. Suppose that $\mathfrak{A}=\langle A, F_\gamma, X_\delta, S, \delta_0\rangle_{\gamma\in\Gamma,\ \delta\in\Delta}$ and $L=\langle\mathfrak{A}, R\rangle$. Then PE_L (or the set of *proper expressions* of L) is the range of R; OI_L (or the set of *operation indices* of L) is Γ; CI_L (or the set of *category indices* of L) is Δ; SR_L (or the set of *syntactical rules* of L) is S; $BS_{\delta,L}$ (or the δth *basic set* of L) is the set of objects ζ such that $\zeta' R\zeta$ for some $\zeta'\in X_\delta$; $\mathrm{Cat}_{\delta,L}$ (or the δth *syntactic category* of L) is the set of objects ζ such that $\zeta' R\zeta$ for some $\zeta'\in C_\delta$, where C is the family of syntactic categories generated by $\mathfrak{A}$; ME_L (or the set of *meaningful expressions* of L) is $\bigcup_{\delta\in\Delta}\mathrm{Cat}_{\delta,L}$; DS_L (or the set of *declarative sentences* of L) is $\mathrm{Cat}_{\delta_0,L}$; and the class of *derived syntactical rules* of L is the smallest class K such that (1) $S\subseteq K$, (2) whenever α, β are ordinals, $\alpha<\beta$, and $\langle\delta_\xi\rangle_{\xi<\beta}$ is a β-place sequence of members of Δ, the triple $\langle I_{\alpha,\beta,\Delta}, \langle\delta_\xi\rangle_{\xi<\beta}, \delta_\alpha\rangle\in K$, (3) whenever β is an ordinal, $\langle\delta_\xi\rangle_{\xi<\beta}$ is a β-place sequence of members of Δ, $\varepsilon\in\Delta$, and $x\in X_\varepsilon$, the triple $\langle C_{x,\beta,\Delta}, \langle\delta_\xi\rangle_{\xi<\beta}, \varepsilon\rangle\in K$, and (4) whenever α, β are ordinals, $\langle G, \langle\delta_\xi\rangle_{\xi<\alpha}, \varepsilon\rangle\in K$, and $\langle H_\xi\rangle_{\xi<\alpha}$ is a sequence such that $\langle H_\xi, \langle\gamma_\eta\rangle_{\eta<\beta}, \delta_\xi\rangle\in K$ for all $\xi<\alpha$, then $\langle G^{\Delta}\langle H_\xi\rangle_{\xi<\alpha}, \langle\gamma_\eta\rangle_{\eta<\beta}, \varepsilon\rangle\in K$. If $\zeta\in ME_L$, then ζ is *syntactically ambiguous in* L if and only if there are at least two objects $\zeta'\in\bigcup_{\delta\in\Delta}C_\delta$ such that $\zeta' R\zeta$, where C is the family of syntactic categories generated by $\mathfrak{A}$. The *language* L is *syntactically ambiguous* if and only if there is a meaningful expression of L that is syntactically ambiguous in L.

REMARK. If L is a language, $L = \langle \mathfrak{A}, R \rangle$, $\langle H, \langle \delta_\xi \rangle_{\xi < \beta}, \varepsilon \rangle$ is a derived syntactical rule of L, C is the family of syntactic categories generated by L, and $a_\xi \in \mathrm{C}_{\delta_\xi}$ for all $\xi < \beta$, then $H(\langle a_\xi \rangle_{\xi < \beta}) \in \mathrm{C}_\varepsilon$.

3. *Semantics: theory of meaning*

Suppose that $\mathfrak{A} = \langle A, F_\gamma, X_\delta, S, \delta_0 \rangle_{\gamma \in \Gamma, \delta \in \Delta}$, $L = \langle \mathfrak{A}, R \rangle$, and L is a language. An *interpretation* for L is a system $\langle B, G_\gamma, f \rangle_{\gamma \in \Gamma}$ such that $\langle B, G_\gamma \rangle_{\gamma \in \Gamma}$ is an algebra similar to $\langle A, F_\gamma \rangle_{\gamma \in \Gamma}$ and f is a function from $\bigcup_{\delta \in \Delta} X_\delta$ into B. (Here B is regarded as the set of meanings prescribed by the interpretation, G_γ is the semantic operation corresponding to the structural operation F_γ, and f assigns meanings to the basic expressions of the language.) Suppose in addition that $\mathfrak{B} = \langle B, G_\gamma, f \rangle_{\gamma \in \Gamma}$. Then the *meaning assignment* for L determined by $\mathfrak{B}$ is the unique homomorphism g from $\langle A, F_\gamma \rangle_{\gamma \in \Gamma}$ into $\langle B, G_\gamma \rangle_{\gamma \in \Gamma}$ such that $f \subseteq g$. Further, if $\zeta \in ME_L$, then ζ *means* b *in* L according to $\mathfrak{B}$ if and only if there exists $\zeta' \in \bigcup_{\delta \in \Delta} \mathrm{C}_\delta$ such that $\zeta' R \zeta$ and $g(\zeta') = b$, where C is the family of syntactic categories generated by $\mathfrak{A}$ and g is the meaning assignment for L determined by $\mathfrak{B}$. Also, ζ is *semantically ambiguous* in L according to $\mathfrak{B}$ if and only if ζ means at least two different things in L according to $\mathfrak{B}$; ζ is *strongly synonymous* with θ in L according to $\mathfrak{B}$ if and only if $\zeta, \theta \in ME_L$ and, for every $\delta \in \Delta$, $\{g(\zeta') : \zeta' \in \mathrm{C}_\delta \text{ and } \zeta' R \zeta\} = \{g(\theta') : \theta' \in \mathrm{C}_\delta \text{ and } \theta' R \theta\}$, where C and g are as above; and ζ is *weakly synonymous* with θ in L according to $\mathfrak{B}$ if and only if $\zeta, \theta \in ME_L$ and $\{b : \zeta \text{ means } b \text{ in } L \text{ according to } \mathfrak{B}\} = \{b : \theta \text{ means } b \text{ in } L \text{ according to } \mathfrak{B}\}$. Suppose, in addition to the assumptions above, that L' is also a language and $\mathfrak{B}'$ is an interpretation for L'. Then ζ is *interlinguistically synonymous* with ζ' (with respect to $L, \mathfrak{B}, L', \mathfrak{B}'$) if and only if $\zeta \in ME_L$, $\zeta' \in ME_{L'}$, and $\{b : \zeta \text{ means } b \text{ in } L \text{ according to } \mathfrak{B}\} = \{b : \zeta' \text{ means } b \text{ in } L' \text{ according to } \mathfrak{B}'\}$.

4. *Semantics: theory of reference*

Let e, t, s be the respective numbers 0, 1, 2. (The precise choice of these objects is unimportant; the only requirements are that they be distinct and that none of them be an ordered pair.) By T, or the set of *types*, is understood the smallest set such that (1) e and t (which are regarded as the type of entities and the type of truth values respectively) are in T, (2) whenever $\sigma, \tau \in T$, the ordered pair $\langle \sigma, \tau \rangle$ (which is regarded as the type of functions from objects of type σ to objects of type τ) is in T, and (3) whenever $\tau \in T$, the pair $\langle s, \tau \rangle$ (which is regarded as the type of senses corresponding to objects of type τ) is in T. In connection with any sets E and I and any $\tau \in T$, we characterize $D_{\tau, E, I}$, or the set of *possible denotations of type* τ based on the set E of entities (or possible individuals) and the set I of possible worlds, as follows: $D_{e, E, I} = E$; $D_{t, E, I} = \{\Lambda, \{\Lambda\}\}$ (where Λ is as usual the empty set, and Λ, $\{\Lambda\}$ are identified with falsehood and truth respectively); if $\sigma, \tau \in T$, then $D_{\langle \sigma, \tau \rangle, E, I} = D_{\tau, E, I}{}^{D_{\sigma, E, I}}$ (where in general A^B is the set of functions with domain B and range included in A); if $\tau \in T$, then $D_{\langle s, \tau \rangle, E, I} = D_{\tau, E, I}{}^{I}$. If J is also a set, then $M_{\tau, E, I, J}$, or the set of possible *meanings* of type τ based on the set E of entities, the set I of possible worlds, and the set J of contexts of use, is $D_{\tau, E, I}{}^{I \times J}$. (By $I \times J$ is understood as usual the set of ordered pairs $\langle i, j \rangle$ such that $i \in I$ and $j \in J$. Thus *meanings* are functions of two arguments—a possible world and a context of use. The second argument is introduced in order to permit a treatment, in the manner of Montague [2], of such indexical locutions as demonstratives, first- and second-person singular pronouns, and free variables (which are treated in § 6 below as a kind of demonstrative). *Senses* on the other hand—that is, members of $D_{\langle s, \tau \rangle, E, I}$ for some τ—are functions of only one argument, regarded as a possible world. The intuitive distinction is this: meanings are those entities that serve as interpretations of expressions (and hence, if the interpretation of a compound is always to be a function of the interpretations of its components, cannot be identified with functions of possible worlds alone), while senses are those intensional entities that are sometimes *denoted* by expressions. No such distinction was necessary in Frege [1], because there consideration of indexical locutions was deliberately avoided. It is a slight oversimplification to call the members of I possible worlds. In connection with tensed languages, for instance, it is convenient to take I

as the set of all ordered *pairs* consisting of a possible world and a moment of time, and J as the set of all complexes of remaining relevant features of possible contexts of use.

Suppose that L is a language and $L = \langle\langle A, F_\gamma, X_\delta, S, \delta_0\rangle_{\gamma\in\Gamma, \delta\in\Delta}, R\rangle$. A *type assignment* for L is a function σ from Δ into T such that $\sigma(\delta_0) = t$. A *Fregean interpretation* for L is an interpretation $\langle B, G_\gamma, f\rangle_{\gamma\in\Gamma}$ for L such that, for some nonempty sets E, I, J, and some type assignment σ for L, (1) $B \subseteq \bigcup_{\tau\in T} M_{\tau, E, I, J}$, (2) whenever $\delta \in \Delta$ and $\zeta \in X_\delta$, $f(\zeta) \in M_{\sigma(\delta), E, I, J}$, and (3) whenever $\langle F_\gamma, \langle\delta_\xi\rangle_{\xi<\beta}, \varepsilon\rangle \in S$ and $b_\xi \in M_{\sigma(\delta_\xi), E, I, J}$ for all $\xi < \beta$, then $G_\gamma(\langle b_\xi\rangle_{\xi<\beta}) \in M_{\sigma(\varepsilon), E, I, J}$. Here $I \times J$ is uniquely determined and is called the set of *points of reference* of the Fregean interpretation. By a *Fregean interpretation* for L *connected with* E, I, J and σ is understood an interpretation $\langle B, G_\gamma, f\rangle_{\gamma\in\Gamma}$ for L such that conditions (1)—(3) above are satisfied. A *model* for L is a pair $\langle \mathfrak{B}, \langle i, j\rangle\rangle$ such that $\mathfrak{B}$ is a Fregean interpretation for L and $\langle i, j\rangle$ is a point of reference of $\mathfrak{B}$. (Here i and j are respectively regarded as the actual world and the actual context of use specified by the model.) Suppose that $\langle \mathfrak{B}, \langle i, j\rangle\rangle$ is a model for L. Then the *denotation assignment* for L determined by $\langle \mathfrak{B}, \langle i, j\rangle\rangle$ is that function h with domain A such that, for all $\zeta \in A$, $h(\zeta) = g(\zeta)\,(i, j)$, where g is the meaning assignment for L determined by $\mathfrak{B}$. Further, η *has x as a denotation* according to L and $\langle \mathfrak{B}, \langle i, j\rangle\rangle$ if and only if there exists b such that η means b in L according to $\mathfrak{B}$, and $b(i, j) = x$. Also, φ is a *true sentence of L with respect to* $\langle \mathfrak{B}, \langle i, j\rangle\rangle$ *and the analysis* φ' if and only if $\varphi' \in C_{\delta_0}$, $\varphi' R \varphi$, and $h(\varphi') = \{\Lambda\}$, where C is the family of syntactic categories generated by $\langle A, F_\gamma, X_\delta, S, \delta_0\rangle_{\gamma\in\Gamma, \delta\in\Delta}$ and h is the denotation assignment for L determined by $\langle \mathfrak{B}, \langle i, j\rangle\rangle$.

For simplicity, let us now suppose, in addition to the assumptions made in the last paragraph, that L is a syntactically unambiguous language. Then the relativization to an analysis may of course be removed from the characterization of truth: φ is a *true sentence of L with respect to the model* $\langle \mathfrak{B}, \langle i, j\rangle\rangle$ if and only if $\varphi \in DS_L$ and φ is a true sentence of L with respect to $\langle \mathfrak{B}, \langle i, j\rangle\rangle$ and the analysis φ', where φ' is the unique member of C_{δ_0} such that $\varphi' R \varphi$, and C is as in the last paragraph. Let us add the assumption that K is a class of models for L. (The most important cases are those in which K is regarded as the class of logically possible models for L; among the conditions characterizing K might then appear the requirement that the "logical operations" and "logical words" of L receive their usual interpretations.) Then φ is *K-valid* in L if and only if φ is a true sentence of L with respect to every member of K. (In case K is understood in the way just indicated, the present notion amounts to logical validity.) If $\zeta, \eta \in ME_L$, then ζ is *K-equivalent* to η in L if and only if (1) $\zeta, \eta \in \mathrm{Cat}_{\delta,L}$ for some $\delta \in CI_L$, and (2) whenever $\langle \mathfrak{B}, \langle i, j\rangle\rangle \in K$, the denotation of ζ according to L and $\langle \mathfrak{B}, \langle i, j\rangle\rangle$ is the same as the denotation of η according to L and $\langle \mathfrak{B}, \langle i, j\rangle\rangle$. By a *token* in L is understood a pair $\langle \zeta, p\rangle$ such that $\zeta \in PE_L$ and p is any ordered pair. (Here we regard p as a possible point of reference. The useful idea of construing a token as a pair consisting of a type and a point of reference originates with Bar-Hillel.) It is usual to regard entailment (or logical consequence) as a relation between sentence types; but when indexical locutions may come into consideration, it is desirable to consider two relations, one between sentence types and one between sentence tokens.[5] (It is the latter notion that is involved when we say that 'I am hungry', when said by Jones to Smith, entails 'thou art hungry', when said on the same occasion by Smith to Jones. More precisely, let us suppose that the language in question contains, as its only indexical features, the pronouns 'I' and 'thou'. Then a context of use could reasonably be construed as an ordered pair of persons, regarded as the speaker and the person addressed respectively; and the situation under consideration can be described by saying that, for every i, the token ⟨'I am hungry', ⟨i, ⟨Jones, Smith⟩⟩⟩ entails the token ⟨'thou art hungry', ⟨i, ⟨Smith, Jones⟩⟩⟩.) The precise characterizations are the following. If $\langle \varphi, p\rangle$ and $\langle \psi, q\rangle$ are tokens in L, then $\langle \varphi, p\rangle$ *K-entails* $\langle \psi, q\rangle$ in L if and only if $\varphi, \psi \in DS_L$ and, for every Fregean interpretation $\mathfrak{B}$ for L, if $\langle \mathfrak{B}, p\rangle$ is in K and φ is a

[5] For simplicity, I explicitly define entailment only between two sentences or sentence tokens. The more general and useful notion of entailment between a *set* of sentences or sentence tokens and a sentence or sentence token can be characterized in a completely analogous fashion.

true sentence of L with respect to $\langle \mathfrak{B}, p\rangle$, then $\langle \mathfrak{B}, q\rangle$ is in K and ψ is a true sentence of L with respect to $\langle \mathfrak{B}, q\rangle$. If $\varphi, \psi \in DS_L$, then the sentence *type* φ *K-entails* the sentence *type* ψ in L if and only if $\langle \varphi, p\rangle$ K-entails $\langle \psi, p\rangle$ for every ordered pair p. (It is clear then φ is K-equivalent to ψ in L if and only if each of φ and ψ K-entails the other in L.)

Now synonymy with respect to all logically possible interpretations implies logical equivalence, but not conversely. To be a little more exact, if (i) the assumptions of the last two paragraphs are satisfied, (ii) ζ, η are meaningful expressions of L belonging to a common syntactic category of L, and (iii) ζ is weakly synonymous with η in L according to every interpretation $\mathfrak{B}$ such that, for some i and j, $\langle \mathfrak{B}, \langle i, j\rangle\rangle \in K$, then (iv) ζ is K-equivalent to η in L; but there are instances in which (i), (ii), and (iv) hold, but (iii) fails. The reason is roughly that the logical equivalence of two expressions depends on their extensions only at *designated* points of reference of logically possible models, while synonymy of those expressions depends on their extensions at *all* points of reference. (And it might for instance happen that "logical words" and "logical operations" receive their usual extensions at all designated points of reference but not at certain other, "unactualizable" points of reference.) Similarly, there will be cases in which two logically equivalent expressions will not be interchangeable in a sentence without changing its truth value, although synonymous expressions always may be so interchanged. This is because the extension of a compound expression may depend on the full meanings of certain components, that is, their extensions at all points of reference, and not simply their extensions at designated points of reference. In particular, it is possible to provide within the present framework a natural treatment of belief contexts that lacks the controversial property of always permitting interchange on the basis of logical equivalence. Previous model-theoretic treatments of belief contexts (for instance, the one in Montague [1]) had always possessed that property, and so does the treatment proposed in section 7 below. But even to those who, like myself, believe that the best and most elegant approach is to permit unrestricted interchange on the basis of logical equivalence it may be of some interest to learn that this approach has genuine alternatives and is not forced upon us.

5. *Theory of translation*

There appears to be no natural theory of definitions which will apply to arbitrary languages. But instead of generalizing the notion of a definition, we may rather consider the translation functions from one language into another that are induced by systems of definitions, and attempt to develop suitable general notions on this basis.

Assume throughout this section that L, L' are languages, $L = \langle \mathfrak{A}, R\rangle$, $L' = \langle \mathfrak{A}', R'\rangle$, $\mathfrak{A} = \langle A, F_\gamma, X_\delta, S, \delta_0\rangle_{\gamma \in \Gamma, \delta \in \Delta}$, and $\mathfrak{A}' = \langle A', F'_\gamma, X'_\delta, S', \delta_0'\rangle_{\gamma \in \Gamma', \delta \in \Delta'}$. By a *translation base* from L into L' is understood a system $\langle g, H_\gamma, j\rangle_{\gamma \in \Gamma}$ such that (1) g is a function from Δ into Δ', (2) j is a function with domain $\bigcup_{\delta \in \Delta} X_\delta$, (3) whenever $\delta \in \Delta$ and $\zeta \in X_\delta$, $j(\zeta) \in C'_{g(\delta)}$, where C' is the family of syntactic categories generated by $\mathfrak{A}'$, (4) for all $\gamma \in \Gamma$, H_γ is a polynomial operation, of the same number of places as F_γ, over the algebra $\langle A', F'_\gamma\rangle_{\gamma \in \Gamma'}$, (5) whenever $\langle F_\gamma, \langle \delta_\xi\rangle_{\xi < \beta}, \varepsilon\rangle \in S$, $\langle H_\gamma, \langle g(\delta_\xi)\rangle_{\xi < \beta}, g(\varepsilon)\rangle$ is a derived syntactical rule of L', and (6) $g(\delta_0) = \delta_0'$. If $\mathfrak{T}$ $(= \langle g, H_\gamma, j\rangle_{\gamma \in \Gamma})$ is such a translation base, then the *translation function* from L into L' determined by $\mathfrak{T}$ is the unique homomorphism k from $\langle A, F_\gamma\rangle_{\gamma \in \Gamma}$ into $\langle A', H_\gamma\rangle_{\gamma \in \Gamma}$ such that $j \subseteq k$; and ζ' *is a translation* of ζ from L into L' on the basis of $\mathfrak{T}$ if and only if there are η, η' such that $\eta R \zeta$, $\eta' R' \zeta'$, $\eta \in \bigcup_{\delta \in \Delta} C_\delta$, $\eta' \in \bigcup_{\delta \in \Delta'} C'_\delta$, and $k(\eta) = \eta'$, where C, C' are the families of syntactic categories generated by $\mathfrak{A}$ and $\mathfrak{A}'$ respectively, and k is the translation function from L into L' determined by $\mathfrak{T}$.

The principal use of translations is the semantical one of inducing interpretations. Indeed, if we are given a translation base from L into L', together with an interpretation for the "already known" language L', then an interpretation for L is determined in the natural manner prescribed below.

Assume for the remainder of this section that $\mathfrak{T}(= \langle g, H_\gamma, j\rangle_{\gamma \in \Gamma})$ is a translation base from L into L', and that $\mathfrak{B}'(= \langle B', G'_\gamma, f'\rangle_{\gamma \in \Gamma'})$ is an interpretation for L'. Then the *interpretation for L induced by*

L', $\mathcal{B}'$, and $\mathcal{J}$ is the interpretation $\langle B, G_\gamma, f\rangle_{\gamma\in\Gamma}$ for L such that (1) $\langle B, G_\gamma\rangle_{\gamma\in\Gamma}$ is the unique algebra such that h' is a homomorphism from $\langle A', H_\gamma\rangle_{\gamma\in\Gamma}$ to $\langle B, G_\gamma\rangle_{\gamma\in\Gamma}$, where h' is the meaning assignment for L' determined by $\mathcal{B}'$, and (2) for all $\zeta\in\bigcup_{\delta\in\Delta}X_\delta$, $f(\zeta)=h'(j(\zeta))$, where h' is as in (1).

It is in order to insure the existence of an algebra satisfying condition (1) that we require in the definition of a translation base that the operations H_γ be polynomial operations over $\langle A', F'_\gamma\rangle_{\gamma\in\Gamma'}$; compare the Remark at the end of § 1.

REMARK. Suppose that $\mathcal{B}$ is the interpretation for L induced by L', $\mathcal{B}'$, and $\mathcal{J}$. Then (1) if $\mathcal{B}'$ is a Fregean interpretation for L', then $\mathcal{B}$ is a Fregean interpretation for L; (2) if h is the meaning assignment for L determined by $\mathcal{B}$, h' is the meaning assignment for L' determined by $\mathcal{B}'$, and k is the translation function from L into L' determined by $\mathcal{J}$, then h is the relative product of k and h'.

6. *Intensional logic*

I wish now to illustrate the application of the general notions of this paper to artificial, "symbolic" languages. To this end I shall construct within the present framework the syntax and semantics of a rather rich system of intensional logic.[6]

The *letters of intensional logic* are to be [,], $\equiv$, $\check{}$, $\hat{}$, together with symbols λ_τ, $v_{n,\tau}$, and $c_{n,\tau}$ for each natural number n and each $\tau\in T$. (We regard $v_{n,\tau}$ as the nth *variable of type* τ and $c_{n,\tau}$ as the nth *constant of type* τ. Thus, for definiteness, we employ only denumerably many constants; but a relaxation of this restriction, according to which the constants of any given type could be indexed by an arbitrary initial segment of the ordinals, would involve no important change in our considerations.) We assume that all letters are 1-place sequences; but apart from this requirement and normal distinctness conditions, the precise nature of the letters

[6] This system has not appeared previously in the literature, but has been presented in talks before the Southern California Logic Colloquium in April 1969 and the Association for Symbolic Logic in May 1969. It comprehends as a part the intensional logic of Montague [1] and [3].

need not concern us. An *expression of intensional logic* is a finite concatenation of letters of intensional logic.

Let $J_0, \ldots, J_3, J_{\langle 4, \tau\rangle}$ (for $\tau \in T$) be those operations, of 2,2,1,1,2 places respectively, on the set of expressions of intensional logic such that whenever ζ, η are such expressions and $\tau \in T$,

$$J_0(\zeta, \eta) = [\zeta\eta],$$
$$J_1(\zeta, \eta) = [\zeta \equiv \lambda],$$
$$J_2(\zeta) = {}^{\vee}\zeta,$$
$$J_3(\zeta) = {}^{\wedge}\zeta,$$
$$J_{\langle 4, \tau\rangle}(\zeta, \eta) = [\lambda_\tau \zeta\eta].$$

(We indicate concatenation by juxtaposition.) Let us understand by Var_τ the set of all expressions $v_{n,\tau}$ for n a natural number, and by Con_τ the set of all expressions $c_{n,\tau}$ for n a natural number.

By L_0, or the *language of intensional logic*, is understood the system $\langle\langle A, F_\gamma, X_\delta, S, t\rangle_{\gamma\in\Gamma, \delta\in\Delta}, R\rangle$, where (1) A is the smallest set including all sets Con_τ and Var_τ (for $\tau \in T$) and closed under $J_0, \ldots, J_3, J_{\langle 4, \tau\rangle}$ (for all $\tau \in T$), (2) Γ is the set consisting of the numbers 0, 1, 2, 3, together with all pairs $\langle 4, \tau\rangle$ for $\tau \in T$, (3) for each $\gamma \in \Gamma$, F_γ is J_γ restricted to A, (4) $\Delta = T \cup (\{T\} \times T)$, (5) for each $\tau \in T$, $X_\tau = \mathrm{Con}_\tau \cup \mathrm{Var}_\tau$ and $X_{\langle T, \tau\rangle} = \mathrm{Var}_\tau$, (6) S is the set consisting of all sequences

$$\langle F_0, \langle\sigma, \tau\rangle, \sigma, \tau\rangle,$$
$$\langle F_1, \tau, \tau, t\rangle,$$
$$\langle F_2, \tau, \langle s, \tau\rangle\rangle,$$
$$\langle F_3, \langle s, \tau\rangle, \tau\rangle,$$
$$\langle F_{\langle 4, \sigma\rangle}, \langle T, \sigma\rangle, \tau, \langle\sigma, \tau\rangle\rangle,$$

where $\sigma, \tau \in T$, and (7) R is the identity relation on A.

REMARK. Assume that $\sigma, \tau \in T$.

(1) $\mathrm{Con}_\tau \cup \mathrm{Var}_\tau \subseteq \mathrm{Cat}_{\tau, L_0}$.
(2) If $\zeta \in \mathrm{Cat}_{\langle\sigma, \tau\rangle, L_0}$ and $\eta \in \mathrm{Cat}_{\sigma, L_0}$ then $[\zeta\eta] \in \mathrm{Cat}_{\tau, L_0}$.
(3) If $\zeta, \eta \in \mathrm{Cat}_{\tau, L_0}$, then $[\zeta \equiv \eta] \in \mathrm{Cat}_{t, L_0}$.
(4) If $\zeta \in \mathrm{Cat}_{\tau, L_0}$, then ${}^{\wedge}\zeta \in \mathrm{Cat}_{\langle s, \tau\rangle, L_0}$.

(5) If $\zeta \in \mathrm{Cat}_{\langle s, \tau\rangle, L_0}$, then ${}^{\vee}\zeta \in \mathrm{Cat}_{\tau, L_0}$.
(6) If $\zeta \in \mathrm{Var}_\sigma$ and $\eta \in \mathrm{Cat}_{\tau, L_0}$, then $[\lambda_\sigma \zeta\eta] \in \mathrm{Cat}_{\langle\sigma, \tau\rangle, L_0}$.
(7) L_0 is a syntactically unambiguous language.

If E, I are any sets, then a *value assignment* relative to E and I is a function j having as its domain the set of ordered pairs $\langle n, \tau\rangle$ for which n is a natural number and $\tau \in T$, and such that whenever $\langle n, \tau\rangle$ is such a pair, $j(n, \tau) \in D_{\tau, E, I}$. If j is such a value assignment, then $j_x^{n, \tau}$ is to be that function j' with the same domain as j such that (1) $j'(n, \tau) = x$ and (2) $j'(m, \sigma) = j(m, \sigma)$ for every pair $\langle m, \sigma\rangle$ in the domain of j other than $\langle n, \tau\rangle$.

Let σ_0 be that type assignment for L_0 such that, for all $\tau \in T$, $\sigma_0(\tau) = \sigma_0(\langle T, \tau\rangle) = \tau$. By K_0, or the class of *logically possible models for* L_0, is understood the class of models $\langle\langle B, G_\gamma, f\rangle_{\gamma\in\Gamma}, \langle i_0, j_0\rangle\rangle$ for L_0 such that, for some nonempty sets E, I, J, (1) $\langle B, G_\gamma, f\rangle_{\gamma\in\Gamma}$ is a Fregean interpretation for L_0 connected with E, I, J, and σ_0, (2) J is the set of value assignments relative to E and I, (3) whenever $\alpha \in \bigcup_{\tau\in T}\mathrm{Con}_\tau$, $i \in I$, and $j, j' \in J$, $f(\alpha)(i, j) = f(\alpha)(i, j')$, (4) whenever n is a natural number, $\tau \in T$, $i \in I$, and $j \in J$, $f(v_{n, \tau})(i, j) = j(n, \tau)$, and (5) for all a, $b \in B$, $i \in I$, $j \in J$, σ, $\tau \in T$, and natural number n,

$G_0(a, b)(i, j) = a(i, j)(b(i, j))$ if $a \in M_{\langle\sigma, \tau\rangle, E, I, J}$ and $b \in M_{\sigma, E, I, J}$,
$G_1(a, b)(i, j) = \{\Lambda\}$ if and only if $a(i, j) = b(i, j)$,
$G_2(a)(i, j)$ is that function p on I such that, for all $k \in I$, $p(k) = a(k, j)$,
$G_3(a)(i, j) = a(i, j)(i)$ if $a \in M_{\langle\sigma, \tau\rangle, E, I, J}$, and
if $a = f(v_{n, \tau})$, then $G_{\langle 4, \tau\rangle}(a, b)(i, j)$ is that function p on $D_{\tau, E, I}$ such that, for all $x \in D_{\tau, E, I}$, $p(x) = b(i, j_x^{n, \tau})$.

REMARK. Assume that $\langle \mathcal{B}, \langle i_0, j_0\rangle\rangle \in K_0$; E, I, J are nonempty sets; $\mathcal{B}$ is a Fregean interpretation connected with E, I, J, and σ_0; g is the meaning assignment for L_0 determined by $\mathcal{B}$; h is the denotation assignment for L_0 determined by $\langle \mathcal{B}, \langle i_0, j_0\rangle\rangle$; σ, $\tau \in T$; and n is a natural number. Then:

(1) If $\zeta \in \mathrm{Con}_\tau$, then $h(\zeta) \in D_{\tau, E, I}$, and $g(\zeta)(i, j) = g(\zeta)(i, j')$ for all $i \in I$ and $j, j' \in J$.

(2) If $\zeta \in \mathrm{Var}_\tau$, then $h(\zeta) \in D_{\tau, E, I}$ and $g(\zeta)\ (i, j) = g(\zeta)\ (i', j)$ for all $i, i' \in I$ and $j \in J$.

(3) If $\zeta \in \mathrm{Cat}_{\langle \sigma, \tau \rangle, L0}$ and $\eta \in \mathrm{Cat}_{\sigma, L0}$, then $h([\zeta\eta]) = h(\zeta)\ (h(\eta))$.

(4) If $\zeta, \eta \in \mathrm{Cat}_{\tau, L0}$, then $h([\zeta \equiv \eta]) = \{\Lambda\}$ if and only if $h(\zeta) = h(\eta)$.

(5) If $\zeta \in \mathrm{Cat}_{\tau, L0}$, then $h(\hat{}\zeta)$ is that function p on I such that, for all $i \in I$, $p(i) = g(\zeta)\ (i, j_0)$.

(6) If $\zeta \in \mathrm{Cat}_{\langle s, \tau \rangle, L0}$, then $h(\check{}\zeta) = h(\zeta)\ (i_0)$.

(7) If $\zeta \in \mathrm{Cat}_{\tau, L0}$, then $h(\lambda_\sigma v_{n,\sigma}\zeta)$ is that function p on $D_{\sigma, E, I}$ such that, for all $x \in D_{\sigma, E, I}$, $p(x) = g(\zeta)\ (i_0, j_{0x}^{n, \sigma})$.

It is convenient to introduce a few metamathematical abbreviations designating expressions of L_0. Among them will be found combinations corresponding to all the usual notions of propositional, quantificational, and modal logic; in these cases the expected truth conditions will be satisfied in connection with all models in K_0.[7] In particular, suppose that $\alpha \in \bigcup_{\tau \in T} \mathrm{Var}_\tau$, $\varphi, \psi \in \mathrm{Cat}_{t, L0}$, and $\varrho, \sigma, \tau \in T$. Then we set

$\lambda\alpha\zeta = [\lambda_\pi \alpha\zeta]$, where π is the unique member of T such that $\alpha \in \mathrm{Var}_\pi$;
$\bigwedge\alpha\varphi = [\lambda\alpha\varphi \equiv \lambda\alpha[\alpha \equiv \alpha]]$;
$\neg\varphi = [\varphi \equiv \bigwedge\beta\beta]$, where $\beta = v_{0, t}$;
$\varphi \wedge \psi = \bigwedge\beta[\psi \equiv [[\beta\varphi] \equiv [\beta\psi]]]$, where $\beta = v_{0, \langle t, t \rangle}$;
$\varphi \rightarrow \psi = \neg(\varphi \wedge \neg\psi)$;
$\varphi \vee \psi = \neg\varphi \rightarrow \psi$;
$\bigvee\alpha\varphi = \neg\bigwedge\alpha\neg\varphi$;
$\delta\{\zeta\} = [\check{}\delta\zeta]$, if $\zeta \in \mathrm{Cat}_{\sigma, L0}$ and $\delta \in \mathrm{Cat}_{\langle s, \langle \sigma, \tau \rangle\rangle, L0}$;
$\delta\{\zeta, \eta\} = \delta\{\eta\}\{\zeta\}$, if $\zeta \in \mathrm{Cat}_{\varrho, L0}$, $\eta \in \mathrm{Cat}_{\sigma, L0}$, and $\delta \in \mathrm{Cat}_{\langle s, \langle \sigma, \langle s, \langle \varrho, \tau \rangle\rangle\rangle\rangle, L0}$;
$\Box\varphi = [\hat{}\varphi \equiv \hat{}\bigwedge\beta[\beta \equiv \beta]]$, where $\beta = v_{0, e}$;
$\zeta \equiv \eta = \Box[\zeta \equiv \eta]$, if $\zeta, \eta \in \mathrm{Cat}_{\tau, L0}$;
$\hat{\alpha}\varphi = \hat{}(\lambda\alpha\varphi)$.

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[7] The methods of expressing negation and conjunction are due to Tarski [1]. I am grateful to Dr. Mohammed Amer for suggesting their use in this connection.

If ζ, $\eta \in ME_{L_0}$ and $\alpha \in \bigcup_{\tau \in T} \mathrm{Var}_\tau$, then let us understand by ζ^{α}_{η} the result of replacing all "free occurrences" of α by η in ζ; we do not bother to construct an exact definition here. The following remark indicates the extent to which principles of substitutivity of identity and of universal instantiation—always questionable in the context of modal or intensional logic—hold within the present system.

REMARK. If σ, $\tau \in T$ and $\varphi \in \mathrm{Cat}_{t, L_0}$, then the following expressions are K_0-valid in L_0:

$\bigwedge \alpha\varphi \to \varphi^{\alpha}_{\beta}$, if α, $\beta \in \mathrm{Var}_\tau$ and β is not "bound" in φ;

$\bigwedge \alpha\varphi \to \varphi^{\alpha}_{\zeta}$, if $\alpha \in \mathrm{Var}_\tau$, $\zeta \in \mathrm{Cat}_{\tau, L_0}$, α does not "stand within the scope" of $\hat{}$ in φ, and no variable "free in" ζ is "bound in" φ;

$(\bigwedge \alpha\varphi \wedge \bigvee \alpha(\alpha \equiv \zeta)) \to \varphi^{\alpha}_{\zeta}$, if $\alpha \in \mathrm{Var}_\tau$, $\zeta \in \mathrm{Cat}_{\tau, L_0}$, no variable free in ζ is bound in φ, and α is not free in ζ;

$\bigvee \alpha(\alpha \equiv \hat{}\zeta)$, if $\zeta \in \mathrm{Cat}_{\tau, L_0}$, $\alpha \in \mathrm{Var}_{\langle s, \tau \rangle}$, and α is not free in ζ;

$[\beta \equiv \gamma] \to [\zeta^{\alpha}_{\beta} \equiv \zeta^{\alpha}_{\gamma}]$, if α, β, $\gamma \in \mathrm{Var}_\tau$, $\zeta \in ME_{L_0}$, and neither β nor γ is bound in ζ;

$(\eta \equiv \theta) \to [\zeta^{\alpha}_{\eta} \equiv \zeta^{\alpha}_{\theta}]$, if $\zeta \in ME_{L_0}$, η, $\theta \in \mathrm{Cat}_{\tau, L_0}$, $\alpha \in \mathrm{Var}_\tau$, and no variable free in η or θ is bound in ζ;

$[\eta \equiv \theta] \to [\zeta^{\alpha}_{\eta} \equiv \zeta^{\alpha}_{\theta}]$, if $\zeta \in ME_{L_0}$, η, $\theta \in \mathrm{Cat}_{\tau, L_0}$, $\alpha \in \mathrm{Var}_\tau$, α does not stand within the scope of $\hat{}$ in ζ, and no variable free in η or θ is bound in ζ;

$([\eta \equiv \theta] \wedge \bigvee \alpha(\alpha \equiv \eta) \wedge \bigvee \alpha(\alpha \equiv \theta)) \to [\zeta^{\alpha}_{\eta} \equiv \zeta^{\alpha}_{\theta}]$, if $\zeta \in ME_{L_0}$, η, $\theta \in \mathrm{Cat}_{\tau, L_0}$, $\alpha \in \mathrm{Var}_\tau$, no variable free in η or θ is bound in ζ, and α is not free in η or θ.

7. *A fragment of English*

As our second example we may take a natural language—indeed, a deliberately restricted fragment of English. The letters of this fragment are **a**, …, **z**, the blank, *, $\{$, $\}$, $\{\!\!\!-$, $-\!\!\!\}$, $\{\!\!\!=$, $=\!\!\!\}$, $\{\!\!\!\equiv$, $\equiv\!\!\!\}$, together with symbols v_n for each natural number n. Again we assume normal distinctness conditions, and that all letters are 1-place sequences; and an *expression* is again to be a finite concatenation of letters. We let

BDS (or the set of basic declarative sentences) = Λ,
BPDE (or the set of basic proposition-denoting expressions) = Λ,
BIE (or the set of basic individual expressions) = the set consisting of the symbols v_{2n+1} for n a natural number,
BCNP (or the set of basic common noun phrases) = the set of common count nouns of English,
BST (or the set of basic singular terms) = the set of proper nouns of English that are not in BCNP,
BAP (or the set of basic adjective phrases) = the set of "ordinary" English adjectives that are not in BCNP or BST,
BVPPO (or the set of basic verb phrases taking a propositional object) = {**believe, assert, deny, know, prove**},
BTVP (or the set of basic transitive verb phrases) = the set of transitive verbs of English, including **be**, that are not in BCNP, BST, BAP, or BVPPO,
BIVP (or the set of basic intransitive verb phrases) = the set of intransitive verbs of English that are not in BCNP, BST, BAP, BVPPO, or BTVP.

Let Y be the unique 11-place sequence such that (i) $Y_0, \ldots, Y_8$ are sets of expressions, (ii) Y_9, Y_{10} are binary relations between expressions, (iii) for all ζ, α, φ, δ, β, δ',

(1) BDS $\subseteq Y_0$, BPDE $\subseteq Y_1$, BIE $\subseteq Y_2$, BCNP $\subseteq Y_3$, BST $\subseteq Y_4$, BAP $\subseteq Y_5$, BVPPO $\subseteq Y_6$, BTVP $\subseteq Y_7$, BIVP $\subseteq Y_8$,
(2) if $\zeta \in Y_3$, then **every** ζ, **no** ζ, **the** ζ, **a** $\zeta \in Y_4$,
(3) if $\alpha \in Y_2$, then **he** $\alpha \in Y_4$,
(4) if $\varphi \in Y_0$, then **that** $\varphi \in Y_1$,
(5) if $\delta \in Y_7$, and $\beta \in Y_4$, then $\{\delta\ \beta'\} \in Y_8$, where either (a) for some $\alpha \in$ BIE, $\beta =$ **he** α and $\beta' =$ **him** α, or (b) there is no $\alpha \in$ BIE such that $\beta =$ **he** α, and $\beta' = \beta$,
(6) if $\delta \in Y_6$ and $\beta \in Y_1$, then $\{\delta\ \beta\} \in Y_8$,
(7) if $\alpha \in Y_2$ and $\varphi \in Y_0$, then **such** α **that** $\varphi \in Y_5$,
(8) if $\delta \in Y_5$ and $\zeta \in Y_3$, then $\theta \in Y_3$, where
(a) either δ does not have the form **such** α **that** φ

for $\alpha \in$ BIE and φ an expression, or no member of BCNP properly occurs (that is, as a full word) in ζ, and $\theta = \{\delta\ \zeta\}$, or (b) $\delta =$ **such** α **that** φ, the member of BCNP that properly occurs first in ζ is of

$$\left\{\begin{matrix}\text{masculine}\\ \text{feminine}\\ \text{neuter}\end{matrix}\right\} \text{ gender, and } \theta = \{\zeta \textbf{ such } \alpha \textbf{ that } \varphi'\},$$

where φ' is obtained from φ by replacing each free occurrence of **he** α or **him** α (that is, occurrence that does not stand in a part of φ of the form **such** α **that** χ, where χ is an expression in which all parentheses are matched), by

$$\left\{\begin{matrix}\textbf{he } \alpha\\ \textbf{she } \alpha\\ \textbf{it } \alpha\end{matrix}\right\} \text{ or } \left\{\begin{matrix}\textbf{him } \alpha\\ \textbf{her } \alpha\\ \textbf{it } \alpha\end{matrix}\right\} \text{ respectively,}$$

(9) if $\delta \in$ BVPPO $\cup$ BTVP $\cup$ BIVP, then $\delta Y_9 \delta$ and $*\ Y_{10}\delta$,

(10) if $\delta \in Y_7$, $\beta \in Y_4$, and $\alpha Y_9 \delta$, then $\alpha Y_9 \{\delta\ \beta'\}$, where β' is as in (5);

(11) if $\delta \in Y_7$, $\beta \in Y_4$, and $\delta' Y_{10} \delta$, then $\{\delta'\ \beta'\}\ Y_{10}\ \{\delta\ \beta'\}$, where β' is as in (5),

(12) if $\delta \in Y_6$, $\beta \in Y_1$, and $\alpha Y_9 \delta$, then $\alpha Y_9 \{\delta\ \beta\}$,

(13) if $\delta \in Y_6$, $\beta \in Y_1$, and $\delta' Y_{10} \delta$, then $\{\delta'\ \beta\}\ Y_{10}\ \{\delta\ \beta\}$,

(14) if $\alpha \in Y_4$ and $\delta \in Y_8$, then $\{\alpha\ \delta''\} \in Y_0$, where either (a) there exist δ', β, β' such that $\delta' Y_{10} \delta$, $\beta Y_9 \delta$, β' is the third person singular of β, and δ'' is the result of substituting β' for $*$ in δ', or (b) there do not exist δ', β, β' such that $\delta' Y_{10} \delta$, $\beta Y_9 \delta$, and β' is the third person singular of β, and $\delta'' = \delta$,

(15) if $\alpha \in Y_4$ and $\delta \in Y_8$, then $\{\alpha\ \delta''\} \in Y_0$, where either (a) there exist δ', β such that $\delta' Y_{10} \delta$, $\beta Y_9 \delta$, $\beta \neq$ **be**, and δ'' is the result of substituting **does not** β for $*$ in δ', or (b) there exists δ' such that $\delta' Y_{10} \delta$, **be** $Y_9 \delta$, and δ'' is the result of substituting **is not** for $*$ in δ', or (c) there do not exist δ', β such that $\delta' Y_{10} \delta$ and $\beta Y_9 \delta$, and $\delta'' = \delta$,

and (iv) for every 11-place sequence Z, if (i)—(iii) hold for Z, then $Y_0 \subseteq Z_0, \ldots, Y_{10} \subseteq Z_{10}$. (It is a consequence of a simple theorem on simultaneous recursion, due to Dr. Perry Smith and me, that there is exactly one sequence satisfying these conditions. We regard '$\alpha Y_9 \delta$' and '$\delta' Y_{10} \delta$' as meaning 'α is the main verb of the verb phrase δ' and 'δ' is the main verb location in the verb phrase δ' respectively.)

Let $K_0, \ldots, K_{10}$ be those operations on the set of expressions, of 1,1,1,1,1,1,2,2,2,2,2 places respectively, such that, for all expressions α, β, δ, ζ, φ,

$K_0(\zeta) =$ **every** ζ,
$K_1(\zeta) =$ **no** ζ,
$K_2(\zeta) =$ **the** ζ,
$K_3(\zeta) =$ **a** ζ,
$K_4(\zeta) =$ **he** ζ,
$K_5(\varphi) =$ **that** φ,
$K_6(\delta, \beta) = \{\delta\, \beta'\}$, where β' is as in (5),
$K_7(\alpha, \varphi) =$ **such** α **that** φ,
$K_8(\delta, \zeta) = \theta$, where θ is as in (8),
$K_9(\alpha, \delta) = \{\!\{\alpha\, \delta''\}\!\}$, where δ'' is as in (14),
$K_{10}(\alpha, \delta) = \{\!\{\alpha\, \delta''\}\!\}$, where δ'' is as in (15).

(In the definitions above some terms from traditional grammar—for instance, 'common count noun' and 'the third person singular of the verb __________'—have been employed without explicit analysis. These terms are admittedly vague but can cause no problem. Unlike certain other traditional grammatical terms, for example, 'declarative sentence', they all have a finite range of application and could therefore be replaced by precise terms exactly characterized by simple enumeration (in case no shorter and more elegant procedure should come to hand).)

By L_1 let us understand the system $\langle\langle A, F_\gamma, X_\delta, S, 0\rangle_{\gamma \in \Gamma, \delta \in \Delta}, R\rangle$, where (1) $\Delta = \{0, \ldots, 8\}$, (2) $X_0 =$ BDS, $X_1 =$ BPDE, $X_2 =$ BIE, $X_3 =$ BCNP, $X_4 =$ BST, $X_5 =$ BAP, $X_6 =$ BVPPO, $X_7 =$ BTVP, $X_8 =$ BIVP, (3) A is the smallest set including all the sets X_δ (for $\delta \in \Delta$) and closed under the operations $K_0, \ldots, K_{10}$, (4) $\Gamma = \{0, \ldots, 10\}$,

(5) for each $\gamma \in \Gamma$, F_γ is K_γ restricted to A, (6) S is the set consisting of the sequences

$$\begin{array}{l}\langle F_0, 3, 4\rangle,\\ \langle F_1, 3, 4\rangle,\\ \langle F_2, 3, 4\rangle,\\ \langle F_3, 3, 4\rangle,\\ \langle F_4, 2, 4\rangle,\\ \langle F_5, 0, 1\rangle,\\ \langle F_6, 7, 4, 8\rangle,\\ \langle F_6, 6, 1, 8\rangle,\\ \langle F_7, 2, 0, 5\rangle,\\ \langle F_8, 5, 3, 3\rangle,\\ \langle F_9, 4, 8, 0\rangle,\\ \langle F_{10}, 4, 8, 0\rangle,\end{array}$$

and (7) R is that function with domain A such that, for all $\zeta \in A$, $R(\zeta)$ is the result of deleting all parentheses and members of BIE from ζ.

REMARK. (1) L_1 is a syntactically ambiguous language. (2) If $L_1 = \langle \mathfrak{A}, R\rangle$ and C is the family of syntactic categories generated by $\mathfrak{A}$, then $C_i = Y_i$ for $i = 0, \ldots, 8$.

Suppose that L_1 has, as above, the form $\langle\langle A, F_\gamma, X_\delta, S, 0\rangle_{\gamma \in \Gamma, \delta \in \Delta}, R\rangle$. By $\mathfrak{T}_0$, or the *standard translation base* from L_1 into L_0, is understood the system $\langle g, H_\gamma, j\rangle_{\gamma \in \Gamma}$ such that (1) g is that function with domain $\{0, \ldots, 8\}$ such that

$$\begin{array}{l}g(0) = t,\\ g(1) = \langle s, t\rangle,\\ g(2) = \langle T, e\rangle,\\ g(3) = \langle e, t\rangle,\\ g(4) = \langle s, \langle\langle s, g(3)\rangle, t\rangle\rangle,\\ g(5) = \langle\langle s, g(3)\rangle, g(3)\rangle,\\ g(6) = \langle g(1), \langle g(4), t\rangle\rangle,\\ g(7) = \langle g(4), \langle g(4), t\rangle\rangle,\\ g(8) = \langle g(4), t\rangle,\end{array}$$

(2) $H_0, \ldots, H_{10}$ are those operations over PE_{L_0}, of 1,1,1,1,1,1,2,2,2,2,2 places respectively, such that, for all ζ, α, φ, δ, $\beta \in PE_{L_0}$,

$$\begin{aligned}
&H_0(\zeta) = \hat{P}\Lambda u([\zeta u] \rightarrow P\{u\}),\\
&H_1(\zeta) = \hat{P}\neg \vee u([\zeta u] \wedge P\{u\}),\\
&H_2(\zeta) = \hat{P}\vee u(\Lambda v[[v \equiv u] \equiv [\zeta v]] \wedge P\{u\}),\\
&H_3(\zeta) = \hat{P}\vee u([\zeta u] \wedge P\{u\}),\\
&H_4(\zeta) = \hat{P}\, P\{\zeta\},\\
&H_5(\varphi) = {}^\wedge\varphi,\\
&H_6(\delta, \beta) = [\delta\beta],\\
&H_7(\alpha, \zeta) = \lambda P[\lambda_e \alpha(P\{\alpha\} \wedge \zeta)],\\
&H_8(\delta, \zeta) = [\delta{}^\wedge\zeta],\\
&H_9(\alpha, \delta) = [\delta\alpha],\\
&H_{10}(\alpha, \delta) = \neg[\delta\alpha],
\end{aligned}$$

where u, v are $v_{0,e}$, $v_{2,e}$ respectively and P is $v_{0,\langle s,\langle e,t\rangle\rangle}$, (3) j is a function with $\bigcup_{\delta \in \Delta} X_\delta$ as its domain, (4) for every natural number n, $j(v_n) = v_{n,e}$, and (5) for every $\delta \in \{3, \ldots, 8\}$, every natural number n, and every ζ, if ζ is the nth member of X_δ (in, let us say, the standard lexicographic ordering of expressions of L_1), then $j(\zeta) = c_{n,\sigma(\delta)}$.

For the remainder of this section let us assume that $\mathcal{T}_0$ has, as above, the form $\langle g, H_\gamma, j\rangle_{\gamma \in \Gamma}$.

We could, as in Montague [4], characterize directly the logically possible models for L_1; but a somewhat more perspicuous method is to proceed by way of translation into L_0. Indeed, we understand by K_1, or the class of *logically possible models for* L_1, the class of pairs $\langle \mathfrak{B}, \langle i_0, j_0\rangle\rangle$ such that, for some $\mathfrak{B}'$, (1) $\langle \mathfrak{B}', \langle i_0, j_0\rangle\rangle \in K_0$, (2) $\mathfrak{B}$ is the interpretation for L_1 induced by L_0, $\mathfrak{B}'$, and $\mathcal{T}_0$, and (3) for all $\alpha \in$ BST, the expressions

$$\begin{aligned}
&j(\textbf{entity}) \equiv \lambda u[u \equiv u],\\
&j(\textbf{be}) \equiv \lambda Q \lambda \mathcal{P}\, \mathcal{P}\{\hat{u} Q\{\hat{v}[u \equiv v]\}\},\\
&\vee u(j(\alpha) \equiv \hat{P}\, P\{u\})
\end{aligned}$$

are true sentences of L_0 with respect to $\langle \mathfrak{B}', \langle i_0, j_0\rangle\rangle$, where u, v, P, $\mathcal{P}$, Q are $v_{0,e}$, $v_{1,e}$, $v_{0,\langle s,\langle e,t\rangle\rangle}$, $v_{0,\sigma(4)}$, $v_{1,\sigma(4)}$ respectively.

EXAMPLES.[8] (1) The expression **every man is a man** is K_1-valid in L_1.

(2) The expression **every man such that he loves a woman is a man** is K_1-valid in L_1.

(3) The expression **every alleged murderer is a murderer** is in DS_{L_1} but is not K_1-valid in L_1.

(4) The expression **every tall murderer is a murderer** is also in DS_{L_1} but is not K_1-valid in L_1.

(5) The expression **every big midget is a big entity** is in DS_{L_1} but is not K_1-valid in L_1.

(6) The expression **every unmarried midget is a(n) unmarried entity** is in DS_{L_1} but is not K_1-valid in L_1.

(7) **Jones seeks a horse such that it speaks** and **a horse such that it speaks is a(n) entity such that Jones seeks it** are in DS_{L_1}, but neither K_1-entails the other in L_1.

(8) **Jones finds a horse such that it speaks** and **a horse such that it speaks is a(n) entity such that Jones finds it** are in DS_{L_1}, but neither K_1-entails the other in L_1.

Parts (3) and (5) show that our treatment of adjectives—which is essentially due to unpublished work of J. A.W. Kamp and Terence Parsons—is capable of accommodating so-called *nonintersective* adjectives; and part (7) that the present treatment of verbs can accommodate *intensional* verbs. The analogues (4), (6), and (8) may, however, seem strange. The sentences mentioned in (4) and (6) are certainly true in the standard or intended model for L_1—indeed, *necessarily* true in that model, in the sense of being true in every model like it except in the choice of a designated point of reference; and the two sentences mentioned in (8) are synonymous according to that model. One may wonder, however, whether natural notions of logical truth and logical equivalence could be found according to which the sentences in (4) and (6) would be logically true and those

[8] In these examples the notions of K_1-validity and K_1-entailment are applied to expressions of the *ambiguous* language L_1, while they were defined above only in connection with *unambiguous* languages. The extension of the notions to the ambiguous case (involving relativization to analyses) is, however, routine. Further, the examples given here involve no important ambiguities, in the sense that each has only one *natural* analysis; and it is with respect to this analysis that the assertions are meant to hold.

in (8) logically equivalent. As far as the sentences in (8) are concerned—and more generally sentences whose logical properties depend on the extensionality of certain verbs—the solution is provided by the notion of K_1'-equivalence, where K_1' is as characterized below. Adjectives can be dealt with in a related but simpler and more obvious way; indications may be found in Montague [4] and in unpublished work of Parsons.

Suppose that $\mathcal{M}$ is a model for intensional logic (L_0). If $\zeta \in \mathrm{Cat}_{g(8), L_0}$, then ζ is said to be (*first-order*) *reducible* in $\mathcal{M}$ if and only if the expression

$$\bigvee R \bigwedge \mathcal{P}([\zeta \mathcal{P}] \equiv \mathcal{P}\{R\})$$

is a true sentence of L_0 with respect to $\mathcal{M}$, where R, $\mathcal{P}$ are the first variables of types $\langle s, \langle e, t\rangle\rangle$ and $g(4)$ respectively which do not occur in ζ. An expression $\zeta \in \mathrm{Cat}_{g(7), L_0}$ is said to be *first-order reducible with respect to its subject*, or simply *subject-reducible*, in $\mathcal{M}$ if and only if the expression

$$\bigwedge Q \bigvee R \bigwedge \mathcal{P}([[\zeta Q]\mathcal{P}] \equiv \mathcal{P}\{R\})$$

is a true sentence of L_0 with respect to $\mathcal{M}$, and *fully* (*first-order*) *reducible* in $\mathcal{M}$ if and only if the expression

$$\bigvee S \bigwedge \mathcal{P} \bigwedge Q([[\zeta Q]\mathcal{P}] \equiv \mathcal{P}\{\hat{u} Q\{\hat{v} S\{u, v\}\}\})$$

is a true sentence of L_0 with respect to $\mathcal{M}$, where R, S, $\mathcal{P}$, Q are the first variables of types $\langle s, \langle e, t\rangle\rangle$, $\langle s, \langle e, \langle s, \langle e, t\rangle\rangle\rangle\rangle$, $g(4)$, and $g(4)$ respectively which do not occur in ζ.

We now distinguish a certain subset EIV, or the set of *extensional intransitive verbs*, of the set BIVP. (The other members of BIVP might be called *intensional intransitive verbs*.) In view of the finitude of BIVP, membership in EIV could be determined by simple enumeration of the positive or negative cases. (To be sure, one would be hard pressed to find *any* intransitive verb of English that should clearly qualify as intensional.) In a similar way we distinguish sets SETV and FETV such that FETV ⊆ SETV ⊆ BTVP. The members of SETV should be those verbs that one wishes to regard as *subject-extensional transitive verbs*, and the members of FETV those that one wishes to regard as *fully extensional transitive* *verbs*; for example, **love** and **find** are to be in FETV, and **seek, worship, conceive** and **see** (in the nonveridical sense, in which some men have seen dragons) in SETV − FETV.

By K_1', or the class of *strongly logically possible models for* L_1, is understood the class of pairs $\langle \mathcal{B}, \langle i_0, j_0 \rangle \rangle$ such that, for some $\mathcal{B}'$, conditions (1)—(3) of the definition above of K_1 are satisfied, and in addition (4) for all $\alpha \in$ EIV, $j(\alpha)$ is first-order reducible in $\langle \mathcal{B}', \langle i_0, j_0 \rangle \rangle$, (5) for all $\alpha \in$ SETV, $j(\alpha)$ is subject-reducible in $\langle \mathcal{B}', \langle i_0, j_0 \rangle \rangle$, and (6) for all $\alpha \in$ FETV, $j(\alpha)$ is fully reducible in $\langle \mathcal{B}', \langle i_0, j_0 \rangle \rangle$.

REMARK. (1) **Jones finds a horse such that it speaks** and **a horse such that it speaks is a(n) entity such that Jones finds it** are K_1'-equivalent in L_1.

(2) Neither of the expressions **Jones seeks a horse such that it speaks** and **a horse such that it speaks is a(n) entity such that Jones seeks it** K_1'-entails the other in L_1.

Quantification on multiple occurrences of variables is expressible in L_1 by **such that** locutions. Consider, for instance, the sentence **every man loves a woman such that she loves him.** Ordinary usage would endow this sentence with two readings. According to one, which is the only reading allowed in L_1, multiple reference does not occur. The pronoun **him** "dangles"; it has no antecedent within the sentence itself but refers to an object specified by the linguistic or extralinguistic context of utterance. According to a second and more natural reading, multiple reference occurs and **him** has **man** "as its antecedent". This assertion can be expressed in L_1, not by the original sentence, but by **every man is a(n) entity such that it loves a woman such that she loves it.**

Notice that the reduction of multiple reference to **such that** locutions has the consequence, in my opinion correct, that *multiple reference often necessitates transparency*. Thus, although **Jones seeks a unicorn** could be true even though there are no unicorns, the more natural reading of **a man such that he seeks a unicorn loves a woman such that she seeks it** (according to which **it** does not dangle but has **unicorn** as its antecedent) could not be;

this reading would have to be expressed not by the original sentence but by **a unicorn**[9] **is a(n) entity such that a man such that he seeks it loves a woman such that she seeks it.**

The qualification 'often' appears in the dictum above in order to allow for such at least apparent exceptions as **Jones seeks a unicorn such that Robinson seeks it,** which has an interpretation in L_1 that involves neither a dangling pronoun nor the existence of unicorns.

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[9] That is, **some unicorn.** I have made no attempt to capture the ambiguity, felt strongly in this sentence, according to which the indefinite article **a** may sometimes have the force of universal, as well as the more usual existential, quantification. By the way, no significance should be attached to the failure of L_1 to reflect the ambiguity of such sentences as **every man loves a woman such that she loves him** and **a man such that he seeks a unicorn loves a woman such that she seeks it.** It would be possible to represent this ambiguity formally at the expense of complicating the characterization of L_1—for instance, by altering the constituent R of L_1 so that each of the sentences above would be related by R to the disambiguated sentences underlying *both* (and not just one) of its paraphrases in L_1. The general characterization of R would then present some difficulty. It would have to account, for instance, for the "multiple reference" reading of **every man such that he seeks a unicorn loves a woman such that she seeks it,** which is **every unicorn is a(n) entity such that every man such that he seeks it loves a woman such that she seeks it.** (In this case the singular term **a unicorn** *must* be treated universally and not existentially.)

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