# Chapter 16 Gyroscopes and Angular Momentum

## 16.1 Gyroscopes

So far, most of the examples and applications we have considered concerned the rotation of rigid bodies about a fixed axis, or a moving axis the direction of which does not change. However, there are many examples of rigid bodies that rotate about an axis that is changing its direction. A gyroscope, a turning bicycle wheel, the earth's precession about its axis (not a part of Example 15.6.1), a spinning top, and a coin rolling on a table are all examples of this type of motion. These motions can be very complex and difficult to analyze. We shall make a simplifying assumption called the *gyroscopic approximation* that will enable us to study a few simple cases.

A toy gyroscope consists of a spinning flywheel mounted in a suspension frame that allows the flywheel's axle to point in any direction. One end of the axle is supported on a pylon (Figure 16.1) a distance $b$ from the center of mass of the gyroscope.

The flywheel is spinning about its axis with a radial component of angular velocity, $\left(\vec{\boldsymbol{\omega}}^{\text{spin}}\right)_r = \omega_s$. The center of mass rotates about a vertical axis that passes through the contact point $S$ of the axle with the pylon with a precessional angular velocity $\vec{\boldsymbol{\Omega}} = \Omega\,\hat{\mathbf{k}}$ where, as indicated in Figures 16.1 and 16.2, the $z$-direction, which is the direction of $\hat{\mathbf{k}}$, is vertically upwards.The spin angular velocity is then

$$\vec{\boldsymbol{\omega}}^{\text{spin}} = \omega_s\,\hat{\mathbf{r}} = \omega_s\,\hat{\mathbf{r}} \qquad (16.1.1)$$

The total angular velocity with respect to the contact point $S$ is the vector sum

$$\vec{\boldsymbol{\omega}}^{\text{total}} = \vec{\boldsymbol{\Omega}} + \vec{\boldsymbol{\omega}}^{\text{spin}} = \vec{\boldsymbol{\Omega}} + \omega_s\,\hat{\mathbf{r}}\,. \qquad (16.1.2)$$

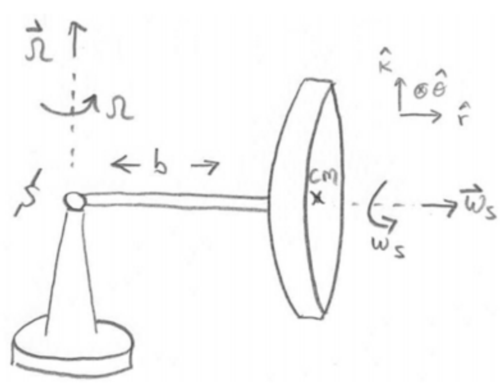



**Figure 16.1** A toy gyroscope.

We shall study the special case when the precessional component $\Omega$ of the angular velocity is much less than the spin component $\omega_{\text{spin}}$ of the spin angular velocity, $\Omega << \omega_{\text{spin}}$, so that $\omega_s \cong \omega_{\text{spin}}$, and that $\Omega$ and $\omega_{\text{spin}}$ are nearly constant. These assumptions are collectively called the gyroscopic approximation.

The precessional motion of the gyroscope seems almost magical. Our expectation is that the gyroscope flywheel should fall downward due to the torque that the gravitational force exerts about the contact point $S$.

The force diagram for the gyroscope is shown in Figure 16.2.

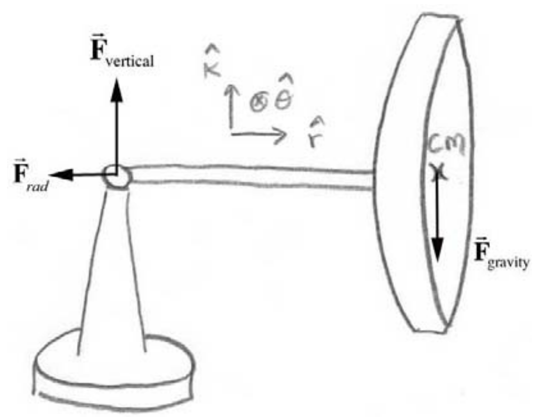



**Figure 16.2** Force diagram for the gyroscope.

The gravitational force acts at the center of the mass and is directed downward, $\vec{\mathbf{F}}_{\text{gravity}} = -mg\,\hat{\mathbf{k}}$. There is also a contact force between the end of the axle and the pylon. It may seem that the contact force has only an upward component, $\vec{\mathbf{F}}_{\text{vertical}} = F_{\text{vertical}}\,\hat{\mathbf{k}}$, but there is necessarily also a radially inward component to the contact force, $\vec{\mathbf{F}}_{\text{rad}} = -F_{\text{rad}}\,\hat{\mathbf{r}}$. As the gyroscope flywheel spins and its axis of rotation precesses, we can decompose the vectors in Newton's Second Law, $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, into vertical and radial directions:

$$\text{Vertical direction:} \quad F_{\text{vertical}} - m\,g = 0, \quad \text{(no vertical acceleration)} \quad (16.1.3)$$

$$\text{Inward direction:} \quad F_{\text{rad}} = m\,b\,\Omega^2, \quad \text{(circular motion)}. \quad (16.1.4)$$

What about the torque about the contact point $S$? The contact forces are acting at the point $S$ so they do not contribute to the torque about $S$; only the gravitational force contributes to the torque about $S$. As in Figures 16.1 and 16.2, choose coordinates so that the axle of the gyroscope flywheel is initially aligned along the $r$-axis and the vertical axis is the $z$-axis. The direction of the torque about $S$ is in the positive $\theta$-direction,

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,\text{cm}} \times \vec{\mathbf{F}}_{\text{gravity}} = b\,\hat{\mathbf{r}} \times mg\left(-\hat{\mathbf{k}}\right) = b\,mg\left(+\hat{\boldsymbol{\theta}}\right). \tag{16.1.5}$$

Because the torque is clearly non-zero, angular momentum is changing in time. The angular momentum is a vector, and has magnitude and direction. How does this vector change in time?

In general, the magnitude or direction of a vector quantity, or both, could change. In the gyroscopic approximation, as described above, the magnitude of the angular momentum is taken to be a constant. For the purposes of the current discussion, then, we will only consider the change in the direction of the vector angular momentum.

We have already encountered a physical situation in which the direction of a constant length vector changes, in our study of circular motion. In these cases, we considered a point object of mass $m$ moving in a circle of radius $r$. When we choose a coordinate system with an origin at the center of the circle, the position vector $\vec{\mathbf{r}}$ is directed radially outward. As the mass moves in a circle, the position vector has a constant magnitude but changes in direction. The velocity vector

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} \tag{16.1.6}$$

is in a direction tangent to the circle and the magnitude of the velocity is

$$v_{\tan} = r\frac{d\theta}{dt}; \tag{16.1.7}$$

the component of the tangential velocity is equal to the product of the magnitude $r$ of the position vector the angular velocity (Figure 16.3(a)).

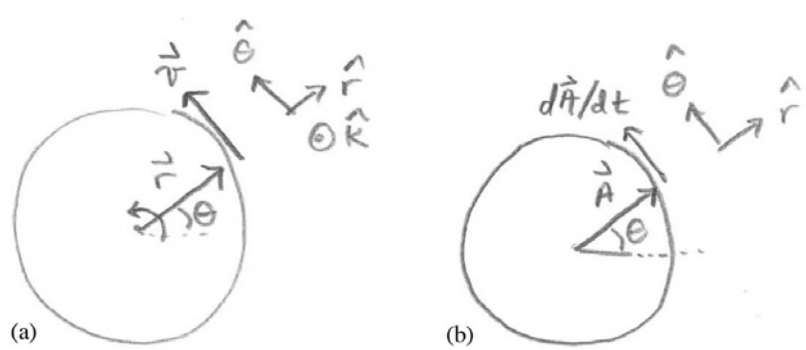



**Figure 16.3** (a) and (b): Rotating vector of constant magnitude directed tangent to the circle traced out by the tip of the vector

This result is true for any vector $\vec{\mathbf{A}}$ that is constant in magnitude but changes direction. The tangential component of the time derivative of the vector $\vec{\mathbf{A}}$ is given by

$$\left(\frac{d\vec{\mathbf{A}}}{dt}\right)_{\tan} = \left|\vec{\mathbf{A}}\right|\frac{d\theta}{dt}\,\hat{\boldsymbol{\theta}} \qquad (16.1.8)$$

where $\hat{\boldsymbol{\theta}}$ is the unit vector in a direction tangential to the circle that is swept out by the tip of the vector $\vec{\mathbf{A}}$ (Figure 16.3 (b)).

When the flywheel is spinning, the spin angular momentum about the center of mass of the flywheel points along the axle; the spin angular momentum is directed radially outward (Figure 16.4). With our choice of coordinates,

$$\vec{\mathbf{L}}_{\mathrm{cm}}^{\mathrm{spin}} = I_{\mathrm{cm}}\,\omega_{\mathrm{s}}\,\hat{\mathbf{r}}\,. \qquad (16.1.9)$$

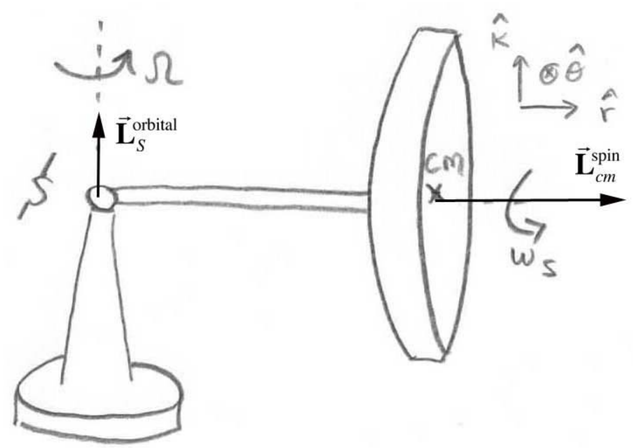



**Figure 16.4** Spin and orbital angular momentum.

In Equation (16.1.9), the moment of inertia $I_{\mathrm{cm}}$ is with respect to the flywheel axis.
Recall that in the gyroscopic approximation, we assume that the spin angular velocity $\omega_{\mathrm{s}}$ is constant. In this approximation, the magnitude of the spin angular momentum is constant, but as the flywheel precesses, the spin angular momentum changes its direction according to

$$\frac{d}{dt}\vec{\mathbf{L}}_{\mathrm{cm}}^{\mathrm{spin}} = \left|\vec{\mathbf{L}}_{\mathrm{cm}}^{\mathrm{spin}}\right|\frac{d\theta}{dt}\,\hat{\boldsymbol{\theta}}\,. \qquad (16.1.10)$$

From our definition of the orbital angular velocity $\vec{\boldsymbol{\Omega}} = \Omega\,\hat{\mathbf{k}}$, the component $\Omega$ of the orbital angular velocity of the flywheel about the vertical axis is

$$\Omega \equiv \frac{d\theta}{dt}. \qquad (16.1.11)$$

The rate of change of the spin angular momentum is then

$$\frac{d}{dt}\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} = \left|\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}\right| \Omega\, \hat{\boldsymbol{\theta}} = I_{\text{cm}}\omega_{\text{s}}\, \Omega\, \hat{\boldsymbol{\theta}}. \qquad (16.1.12)$$

The angular momentum about $S$ consists of two pieces,

$$\vec{\mathbf{L}}_{S}^{\text{total}} = \vec{\mathbf{L}}_{S}^{\text{orbital}} + \vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}. \qquad (16.1.13)$$

How does this angular momentum vector change in time? The first term in Equation (16.1.13),

$$\vec{\mathbf{L}}_{S}^{\text{orbital}} = \vec{\mathbf{r}}_{S,\text{cm}} \times \vec{\mathbf{p}}^{\,\text{total}}, \qquad (16.1.14)$$

is the orbital angular momentum about $S$. The total linear momentum vector $\vec{\mathbf{p}}^{\,\text{total}}$ is in the tangential direction,

$$\vec{\mathbf{p}}^{\,\text{total}} = m v_{\text{tan}}\, \hat{\boldsymbol{\theta}} = m b \Omega\, \hat{\boldsymbol{\theta}}, \qquad (16.1.15)$$

where Equations (16.1.7) and (16.1.11) have been combined to give the magnitude $v_{\text{tan}}$ of tangential velocity in terms of $r = b$ and $\Omega$.

The position vector from $S$ to the center of mass is $\vec{\mathbf{r}}_{S,\text{cm}} = b\,\hat{\mathbf{r}}$ and so the orbital angular momentum about $S$ is

$$\vec{\mathbf{L}}_{S}^{\text{orbital}} = \left(b\,\hat{\mathbf{r}}\right) \times \left(m b \Omega\, \hat{\boldsymbol{\theta}}\right) = m b^2 \Omega\, \hat{\mathbf{k}}. \qquad (16.1.16)$$

The angular momentum vector as given in Equation (16.1.16) is a constant since we have assumed in the gyroscopic approximation that $\Omega$ is constant. Thus

$$\frac{d}{dt}\vec{\mathbf{L}}_{S}^{\text{orbital}} = \vec{\mathbf{0}}. \qquad (16.1.17)$$

The spin angular momentum $\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}$ as given in Equation (16.1.9) is only valid in the gyroscopic approximation. If this approximation is not made, $\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}$ would also have a component in the $+\hat{\mathbf{k}}$ direction since the flywheel is rotating about its vertical axis as the center of mass rotates in a circle about the vertical axis. This component of the spin angular momentum would be $\left(\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}\right)_z = I'_{\text{cm}}\,\Omega$, where $I'_{\text{cm}}$ is the moment of inertia of the flywheel with respect to an axis parallel to the $z$-axis passing through the center of mass of the flywheel. In the gyroscopic approximation, the ratio of the magnitudes

$$\frac{\left|\left(\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}\right)_z\right|}{\left|\left(\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}\right)_r\right|} \approx \frac{\Omega}{\omega_{\text{s}}} << 1 \qquad (16.1.18)$$

and so $\left(\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}\right)_z$ need not be considered. In any event, $\left(\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}\right)_z = I'_{\text{cm}}\,\Omega$ is constant in time.

Thus in the gyroscopic approximation the only change to the angular momentum about $S$ is the change in direction of the spin angular momentum about the axis passing through the center of mass perpendicular to the wheel,

$$\frac{d}{dt}\vec{\mathbf{L}}_S^{\text{total}} = \frac{d}{dt}\left(\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}\right)_r \hat{\mathbf{r}}\,. \qquad (16.1.19)$$

We can now return to our idea that the net torque about the point $S$ causes the angular momentum about $S$ to change, Equation (15.1.3),

$$\vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_S}{dt}\,. \qquad (16.1.20)$$

Because only the spin angular momentum changes, use of Equations (16.1.5) and (16.1.12) in Equation (16.1.20) yields

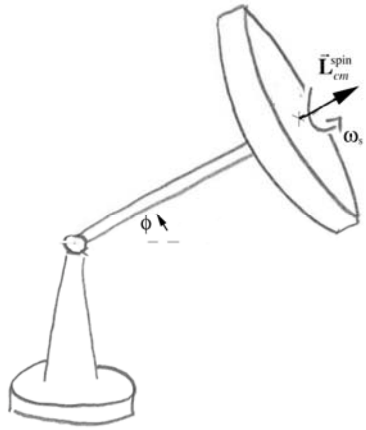
$$b\,mg = \left|\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}\right|_r \Omega\,. \qquad (16.1.21)$$

Solving Equation (16.1.21) for the precessional frequency of the gyroscope yields

$$\Omega = \frac{b\,mg}{\left|\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}\right|_r} = \frac{b\,mg}{I_{\text{cm}}\,\omega_{\text{s}}}\,. \qquad (16.1.22)$$

**Example 16.1.1 Tilted Toy Gyroscope**

A wheel is at one end of an axle of length $l$. The axle is pivoted at an angle $\phi$ with respect to the horizontal. The wheel is set into motion so that it executes uniform precession; that is, the wheel's center of mass moves with uniform circular motion. The wheel has mass $m$ and moment of inertia $I_{\text{cm}}$ about its center of mass. Its spin angular velocity has magnitude $\omega_{\text{spin}}$ and is directed as shown in the figure below. Neglect the mass of the axle. What is the angular frequency that the gyroscope precesses about the vertical axis? Does the gyroscope rotate clockwise or counterclockwise about the vertical axis (as seen from above)?




**Solution:**

The gravitational force acts at the center of mass and is directed downward, $\vec{\mathbf{F}}_{\text{gravity}} = -mg\,\hat{\mathbf{k}}$. Let $S$ denote the contact point between the pylon and the axle. The contact forces between the pylon and the axle are acting at the point $S$ so they do not contribute to the torque. Only the gravitational force contributes to the torque. Let's choose cylindrical coordinates. The torque about $S$ is

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,\text{cm}} \times \vec{\mathbf{F}}_{\text{gravity}} = (l\cos\phi\,\hat{\mathbf{r}} + l\sin\phi\,\hat{\mathbf{k}}) \times mg(-\hat{\mathbf{k}}) = mgl\cos\phi(+\hat{\boldsymbol{\theta}}) \,, \tag{16.1.23}$$

which is into the page in the above figure.

The spin angular momentum has a vertical and radial component,

$$\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} = \left|\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}\right|\cos\phi\,\hat{\mathbf{r}} + \left|\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}\right|\sin\phi\,\hat{\mathbf{k}}\,. \tag{16.1.24}$$

Assume that the spin angular velocity $\omega_{\text{spin}}$ is constant. Then the magnitude of the spin angular momentum is constant,

$$\left|\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}\right| = I_{\text{cm}}\omega_{\text{spin}}\,. \tag{16.1.25}$$

As the wheel precesses, the time derivative of the spin angular momentum arises from the change in the direction of the radial component of the spin angular momentum,

$$\frac{d}{dt}\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} = \left|\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}\right|_r \frac{d\theta}{dt}\,\hat{\boldsymbol{\theta}} = \left|\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}\right|\cos\phi\frac{d\theta}{dt}\,\hat{\boldsymbol{\theta}}\,. \tag{16.1.26}$$

The component of the angular velocity of the flywheel about the vertical axis is defined to be

$$\Omega \equiv \frac{d\theta}{dt} . \qquad (16.1.27)$$

The rate of change of the spin angular momentum is then

$$\frac{d}{dt}\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} = \left|\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}\right| \cos\phi \, \Omega \, \hat{\boldsymbol{\theta}} = I_{\text{cm}} \omega_S \cos\phi \, \Omega \, \hat{\boldsymbol{\theta}} . \qquad (16.1.28)$$

In the gyroscopic approximation, the torque about the point $S$ induces the spin angular momentum about $S$ to change,

$$\vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}}{dt} . \qquad (16.1.29)$$

Now substitute Equation (16.1.23) for the torque about $S$, and Equation (16.1.12) for the rate of change of the spin angular momentum into Equation (16.1.20),

$$l \cos\phi \, mg = I_{\text{cm}} \omega_{\text{spin}} \cos\phi \, \Omega . \qquad (16.1.30)$$

Solving Equation (16.1.21) for the precessional frequency of the gyroscope yields

$$\Omega = \frac{l \, mg}{I_{\text{cm}} \, \omega_{\text{spin}}} , \qquad (16.1.31)$$

the precessional frequency is independent of the angle $\phi$. Both the torque and the time derivative of the spin angular momentum point in the $\hat{\boldsymbol{\theta}}$-direction, indicating that the gyroscope will precess counterclockwise when seen from above.