Some learning objects to explain Kepler's laws[a]

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Abstract

In this paper, we present some learning objects for the study of Kepler's laws that graphically show the orbits and the movements of various planets. One of them shows the orbit of a planet from the point of view of a fixed planet, showing that the orbit is quite involved. No differentials equations are required, but only elementary vector calculus. The learning objects have been implemented in Matlab.

Keywords: learning object, Kepler's laws, Matlab.

1. Introduction

Nowadays, we have at our disposal many computer programs to explain Mathematics. We believe that the teaching of this branch of Science will improve if we use such programs. On the other hand, Mathematics and Physics have been closely intertwined since ancient times (the Russian mathematician V. I. Arnold, said that "*Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap.*") We consider that it is a serious mistake to teach Mathematics with no physical intuitions. Physics do help in the understanding of mathematics. Unfortunately, the elimination of the natural sciences from the teaching of Maths is usual. To remedy this situation we propose an integration of the teaching of mathematics and the other matters of science with a historical perspective. As M. Kline [1] said

"*The use of real and especially physical problems serves not only to motivate mathematics but to give meaning to it… Mathematical concepts arose from such physical situations or phenomena and their meanings were physical for those who created mathematics in the first place. To rob the concepts of their meaning is to keep the rind and to throw away the fruit*".

In this paper we present four *Matlab* files (these files, with minor differences, can be used with *GNU Octave*, which is a high-level language, mostly compatible with *Matlab*, and free software). The main purpose of these files is to show graphically and kinematically the three Kepler's laws of planetary motion. Meantime, we provide the

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mathematical and physical background for the sake of completeness and aiming to show an example of how Mathematics and Physics are inextricably linked.

Throughout this paper, vectors of $\mathbb{R}^3$ are written in bold font, the position vector is denoted by $\mathbf{r}(t)$ or simply $\mathbf{r}$, where t is reserved for time. Furthermore, the derivative of any function g with respect to t is denoted by $\dot{g}$. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^3$, then $|\mathbf{u}|$, $\mathbf{u} \cdot \mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ denote, respectively, the norm (or modulus) of $\mathbf{u}$, the scalar (dot) product of $\mathbf{u}$, $\mathbf{v}$, and the vector (cross) product of $\mathbf{u}$, $\mathbf{v}$. Finally, given a 3D nonzero vector $\mathbf{u}$, the symbol $\hat{\mathbf{u}}$ denotes the unitary vector $\mathbf{u} / |\mathbf{u}|$.

2. Historical Background

The movements of the Sun and the Moon, seen from the Earth, can be described as approximate circular orbits. However, there are deviations already observed by the ancient Greeks. Furthermore, the movements of the planets are even more complex. The first mathematical scheme was proposed by Eudoxus (408 BC-355 BC), where a rather complicated set of 27 imaginary spheres describes the movements of the Sun, the Moon, and the planets. Aristarchus of Samos (about 310 BC-about 230 BC) proposed a heliocentric model of the solar system; but this model had no influence. The first great attempt to describe celestial mechanics was made by Ptolemy (about 85-about 165). In its "Almagest", he propounded the geocentric theory in a form that prevailed for 1400 years until Copernicus (1473-1543) presented his heliocentric theory in "De revolutionibus orbium coelestium".

One of the most important advances of the physical sciences after the medieval ages was noting that we can mathematically describe the nature and not asking why does the nature behave in a concrete way. Kepler (1571 - 1630) did so in describing the solar system in his very famous three (empirical) laws:

Law I: The path of a planet is an ellipse being the Sun one focus.

Law II: An imaginary line drawn from the Sun to the planet sweeps out equal areas in equal intervals of time.

Law III: The ratio of the squares of the periods of any two planets is equal to the ratio of the cubes of their average distances from the sun (being the orbit closed, the planet will return to its starting point in a time T called the period of the orbit).

From Kepler's laws, Newton (1643-1727) was able to deduce the law of universal gravitation or the inverse-square law, i.e., all matter attracts all other matter with a force proportional to the product of their masses and inversely proportional to the square of the distance between them. As a suggested lecture, we recommend [2].

Why did Kepler state his laws in this way? This question is not dealt in textbooks and we want to answer it with more detail. The first law is a geometrical law: it tells us about the form of the orbit, but not about how the planet moves. The second law is a

kinematical law: it tells us about the velocity of the planet or how the planet moves; at a first sight, this law is strange because the scalar velocity is not mentioned, unlike the areal velocity. Why? As we will see, in Kepler's time, the scalar velocity could not be exactly calculated, and this impossibility forced Kepler to state his second law in that manner. The third law gives us a scale of the solar system: let a be the average distance of the Earth to the Sun; since the period of the Earth is, obviously, 1 year, we have $T_p^{2/3} = a_p/a$ when T_p is the period of another planet (a number easy to find) and a_p is the average distance of this same planet to the Sun.

3. Mathematical and Physical Background

This is not the place to explain the main mathematical and physical developments of Astronomy (there are hundred of textbooks on vector calculus or physics explaining this branch of Science). As we pointed out in the introduction, physical sciences can not be developed without mathematics, and mathematics can be better explained by means of physical models. Thus, we will try to link these scientific disciplines. At the same time, in our opinion, it is important to stress the kinematical models in contraposition the statical models. For a comprehensive text we can cite [3].

Let us fix a coordinate system with the sun at the origin. Newton's laws lead to study the vector equation

$$m \, \mathbf{r}'' = -\frac{k}{|\mathbf{r}|^2} \, \hat{\mathbf{r}} \,. \tag{1}$$

As it is well known, constant k depends on the strength of the force. For Newton's law of gravitation, one has $k = GMm$, where G is a universal constant, M is the mass of the Sun, and m is the mass of the planet.

Let us make a parenthesis by studying (1) assuming that the orbit is a circumference (such situation is not academic, it happens when some artificial satellites orbit around the Earth). This study is so simple that it is worth to present it to any student of vector calculus. Let

$$\mathbf{r}(t) = R \, (\cos \theta(t), \sin \theta(t))$$

be the orbit, being R its radius. By differentiating twice with respect to t, we get

$$\mathbf{r}'' = R \, \theta'' \hat{\boldsymbol{\theta}} - R(\theta')^2 \, \hat{\mathbf{r}} \,, \tag{2}$$

where $\hat{\boldsymbol{\theta}} = (-\sin \theta, \cos \theta)$. From (1), (2), and having in mind that $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are linearly independent vectors, one has

$$\theta'' = 0 \quad \text{and} \quad mR(\theta')^2 = k/R^2. \tag{3}$$

From the first relation (3), we obtain that θ' is a constant, say ω; i.e., the satellite has a uniform circular movement with angular velocity ω. This is a simplified version of the Kepler's second law. From the second relation of (3), one can deduce $R^3\omega^2 = GM$, which, as we will see, is a particular case of Kepler's third law.

For the sake of completeness, we will deduce Kepler's laws by using vector methods (see, for example, [2, Ch. 6] or [4, Ch. 9] for another deduction). Let us mention that Kepler's first law is not exactly correct: ellipses are the true shape of an orbit only for two isolated masses; two masses orbit around their center of mass, which is at a focus of their elliptical orbits; and unbounded orbits are described by other conic sections (parabolas and hyperbolas).

3.1 Kepler's second law

From (1) and having in mind that $\mathbf{r} \times \hat{\mathbf{r}} = \mathbf{0}$, we get $(\mathbf{r} \times \mathbf{r}')' = \mathbf{r} \times \mathbf{r}'' = \mathbf{0}$, and thus, $\mathbf{r} \times \mathbf{r}'$ is a constant vector. The vector $\mathbf{L} = \mathbf{r} \times (m\,\mathbf{r}')$ is called the angular momentum of the planet, and therefore, the directed area $\mathbf{A} = \mathbf{A}(t)$ swept out by $\mathbf{r}$ is given by

$$\mathbf{A}(t) = \frac{1}{2}\int_0^t \mathbf{r}(\tau)\times\mathbf{r}'(\tau)\,d\tau = \frac{t-t_0}{2m}\,\mathbf{L}. \tag{4}$$

This is the second Kepler's law (parenthetically, we can deduce that $\mathbf{r}\cdot\mathbf{L} = 0$, which means that the orbit is planar).

3.2 Kepler's first law

From $\mathbf{r}' = (|\mathbf{r}|\,\hat{\mathbf{r}})' = |\mathbf{r}|'\,\hat{\mathbf{r}} + |\mathbf{r}|\,\hat{\mathbf{r}}'$ we get $\mathbf{L} = m\,\mathbf{r}\times\mathbf{r}' = m\,|\mathbf{r}|(\mathbf{r}\times\hat{\mathbf{r}}')$. Observe that $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}' = 0$ because $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}$ is constant. Now (1) leads to

$$\mathbf{r}''\times\mathbf{L} = m\,|\mathbf{r}|\,[\,\mathbf{r}''\times(\mathbf{r}\times\hat{\mathbf{r}}')] = -\frac{k}{|\mathbf{r}|}[\hat{\mathbf{r}}\times(\mathbf{r}\times\hat{\mathbf{r}}')]$$

$$= -\frac{k}{|\mathbf{r}|}[(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')\mathbf{r} - (\hat{\mathbf{r}}\cdot\mathbf{r})\,\hat{\mathbf{r}}'] = k\hat{\mathbf{r}}'.$$

Since $\mathbf{L}$ is constant, we get $(\mathbf{r}'\times\mathbf{L} - k\,\hat{\mathbf{r}})' = \mathbf{0}$. Therefore, there exists a constant vector $\mathbf{w}\in\mathbb{R}^3$ such that $\mathbf{r}'\times\mathbf{L} = k\,\hat{\mathbf{r}} + \mathbf{w}$. Hence

$$(\mathbf{r}'\times\mathbf{L})\cdot\mathbf{r} = (k\hat{\mathbf{r}} + \mathbf{w})\cdot\mathbf{r}. \tag{5}$$

The left side of (5) reduces to $(\mathbf{r}'\times\mathbf{L})\cdot\mathbf{r} = (\mathbf{r}\times\mathbf{r}')\cdot\mathbf{L} = \mathbf{L}\cdot\mathbf{L}/m = |\mathbf{L}|^2/m$. The right side of (5) becomes

$$(k\hat{\mathbf{r}} + \mathbf{w})\cdot\mathbf{r} = k\,|\mathbf{r}| + |\mathbf{w}||\mathbf{r}|\cos\theta = |\mathbf{r}|(k + |\mathbf{w}|\cos\theta),$$

being θ the angle between **r** and **w**. Hence (5) leads to

$$|\mathbf{r}| = \frac{\dfrac{|\mathbf{L}|^2}{km}}{1+\dfrac{|\mathbf{w}|}{k}\cos\theta}.$$

(6)

This is the equation of a conic with eccentricity e = |**w**|/k. Recall that if e > 1, the orbit is an hyperbola, if e = 1, the orbit is a parabola, and if 0 ≤ e < 1, the orbit is an ellipse.

In the sequel, we will suppose that the orbit of the planet under consideration is an ellipse (see Fig. 1). Let **î** and **ĵ** be the two unitary vectors depicted in Fig. 1, and $\hat{\mathbf{k}}$ = **î** × **ĵ**. Let a be the semimajor axis and b the semiminor axis (with b ≤ a). It can be proved that $c = (a^2 - b^2)^{1/2}$ is the distance between the center of the ellipse and any of the two focus. Furthermore, the eccentricity e is defined as e = c/a. The area of such ellipse is πab.

*********** Here comes figure 1 ************

Figure 1: The geometry of the orbit of a planet.

Since the orbit of the planet is closed, the planet will return to its starting point in a time T called the period of the orbit. From (4) it follows

$$\pi ab\,\hat{\mathbf{k}} = \frac{T}{2m}\mathbf{L}.$$

(7)

It is necessary to be able to locate the planet at any given time t and the second Kepler's law will be the main tool to this end. Let us remark that the problem of the localization of the planet is not developed in the standard textbooks; thus we will study it.

The orbit of the planet can be described by **x** = a cos φ **î** + b sin φ **ĵ**, where φ = φ(t). It is a common mistake among the students saying that φ is the angle formed by **x** and **î**. Now, we have

r = -c **î** + **x** = -a e **î** + a cos φ **î** + b sin φ **ĵ** = a(-e + cos φ) **î** + b sin φ **ĵ**,

(8)

hence,

r' = -a φ' sin φ **î** + b φ' cos φ **ĵ**,

(9)

and thus, after simplifying,

$$\mathbf{L} = m(\mathbf{r} \times \mathbf{r'}) = m\,a\,b\,\varphi\,'\,(1 - e\cos\varphi)\hat{\mathbf{k}}. \tag{10}$$

Kepler's second law explains why the nearer each planet approaches the Sun, the faster it moves. Observe that (10) states it in a quantitative way since $|\mathbf{L}|$ is constant. Let us remark that using (9) one gets that the scalar velocity satisfies $v = |\mathbf{r'}| = |\varphi\,'|\,(a^2\sin^2\varphi + b^2\cos^2\varphi)^{1/2}$, and an expression of the form $\int v(t)\,dt$ leads to an elliptical integral, which cannot be exactly solved. Now, we can admire the genius of Kepler, many times underestimate. In fact, the following reasoning is just from Kepler with a modern parlance.

Having in mind that $\mathbf{L}$ is constant, integrating (10) with respect t, and assuming $\varphi(0) = 0$ (i.e., associating the initial time with the position at the point nearest to the center of attraction), we get $t\mathbf{L} = mab(\varphi - e\sin\varphi)\,\hat{\mathbf{k}}$, and using (7) yields

$$\frac{2\pi t}{T} = \varphi - e\,\text{sen}\,\varphi \tag{11}$$

Since $0 \le e < 1$, for any t, there exists a unique $\varphi = \varphi(t)$ such that (11) holds. This angle φ enables us to locate the planet at any time t. Unfortunately, this equation has not an analytic solution, and therefore, we have to solve it by using non exact methods. Matlab and Octave use the function `fzero` (Dekker's algorithm is used in both programs; see [5, Ch. 4] for a deep study of this algorithm). But we have another choice: we can use that the values of e in Table 1 are small. To find the angle φ satisfying (11), let us fix t in (11), consider φ as a function of e, and use a Taylor approximation:

$$\varphi \approx \frac{2\pi t}{T} + e\sin\left(\frac{2\pi t}{T}\right) + \frac{e^2}{2}\sin\left(\frac{4\pi t}{T}\right). \tag{12}$$

However, the angle θ depicted in Figure 1 has a more geometric meaning than φ. Thus, we will express θ in terms of φ. From (8) and $b^2 = a^2(1 - e^2)$, we get $|\mathbf{r}| = a(1 - e\cos\varphi)$. Now, we can compute $\mathbf{r}\cdot\hat{\mathbf{i}}$ in two different ways (one of them uses again (8)):

$$\mathbf{r}\cdot\hat{\mathbf{i}} = \begin{cases} |\mathbf{r}||\hat{\mathbf{i}}|\cos\theta = a(1 - e\cos\varphi)\cos\theta, \\ a(-e + \cos\varphi), \end{cases} \implies \cos\theta = \frac{-e + \cos\varphi}{1 - e\cos\varphi}.$$

Finally, by making some algebra and using the half-angle formula for the tangent we get

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{\varphi}{2}\right). \tag{13}$$

Since φ/2 and θ/2 are always located in the same quadrant, the above relation allows us to find θ for a given φ.

*********** *Here comes table 1* *************

Table 1. This table was obtained from [2] and [6]. As a curious remark, let us say that in [2] Pluto was still considered a planet. The value of a is measured in astronomical units and T in years.

3.3 Kepler's third law

Now, we will prove Kepler's third law. If we assume that the origin of coordinates is located at the center of the ellipse, then the equation of the orbit is $x^2/a^2 + y^2/b^2 = 1$. Let us define $l > 0$ by means of $\mathbf{r}(\pi/2) = (c, l)$. Hence $c^2/a^2 + l^2/b^2 = 1$, which yields $b^2 = al$. Now (6) leads to

$$l = |\mathbf{r}(\pi/2)| = |\mathbf{L}|^2/(mk).$$

Using (7) yields

$$T^2 = 4\pi^2 a^2 b^2 m^2 / |\mathbf{L}|^2 = 4\pi^2 a^3 m/k,$$

or using the value of k we have $T^2/a^3 = 4\pi^2/GM$.

4. Files

In this section we provide the `m` files that show in a interactive way the three Kepler's laws.

The first file (called `k1.m`) draws the orbit with eccentricity $e \in [0,1[$, locates the planet at any time $t \in [0, T]$ (it is assumed that the planet starts at the perihelion), and shows the area swept out by the position vector. If $t \leq T/2$, the file makes the same, but starting from the perihelion and the aphelion. This shows clearly Kepler's second law as is seen in Fig. 2. The file computes the angle(s) swept out by the position vector. The file also prints the area swept starting from the aphelion.

Now, we give some explanations about this file: Throughout the file it is assumed that $a = 1$, $T = 1$, and the variable `Tempus` is reserved for the final time. In lines 12-14 the file draws the ellipse. In line 15, it is found the angle φ satisfying φ − e sin φ = 2πTempus. Note that the function `fzero` and the "inner anonymous function" `fkepler` are used (alternatively, under the assumption that e is small, we can use (12) in order to perform fewer operations). Lines 16-19 paint the area swept out by the position vector. Lines 20-22 compute the angle (in radians) swept out by using

(13). Lines 23-35 are analogous to the former ones. Observe that if the origin of coordinates is the center of the ellipse, the foci are located at ($\pm c$, 0). We will assume that the Sun is located at (c, 0); but recall that we have assumed $a = 1$; hence $c = e$, which explains line 36.

********** *Here comes figure 2* ************

Figure 2: This figure has been obtained by making `k1(0.7,0.1)`. The output is [122.32 9.07].

```
1     function angle = k1(e,Tempus)

2     % a = k1(e,Tempus) draws the orbit a planet
3     % and the area swept out by the position vector.
4     % Tempus is the elapsed time
5     % (if Tempus = 0, the planet is at the perihelion)
6     % The period of the orbit is 1 (0 <= Tempus <=1).
7     % e is the eccentricity of the ellipse (0 <= e < 1)
8     % If Tempus < 0.5, the movement from the aphelion is also drawn.
9     % angle is (are) the angle(s) swept by the position vectors.

10    fig=figure; set(fig,'color',[1 1 1])

11    b=sqrt(1-e^2);

12    draw_ellipse = linspace(0,2*pi);
13    x = cos(draw_ellipse); y = b*sin(draw_ellipse);
14    plot(x,y); axis off; axis equal; hold on;

15    phi = fzero(@(x) fkepler(x,e,Tempus),0.01);

16    phi_color = linspace(0,phi);
17    xx = cos(phi_color); yy = b*sin(phi_color);
18    xx = [xx e]; yy = [yy 0];
19    fill(xx,yy,'r');

20    factor1 = sqrt((1+e)/(1-e)); factor2 = tan(phi/2);
21    product = factor1*factor2;
22    theta = 2*atan(product);

23    if Tempus <= 0.5
24       phi1 = fzero(@(x) fkepler(x,e,Tempus+0.5),0.01);
25       phi1_color = linspace(pi,phi1);
26       xxx = cos(phi1_color); yyy = b*sin(phi1_color);
27       xxx = [xxx e]; yyy = [yyy 0];
28       fill(xxx,yyy,'y')
29       factor2 = tan(phi1/2);
30       product = factor1*factor2;
31       theta1 = 2*atan(product);
32       angle = [(180*theta)/pi (180*theta1)/pi+180];
33    else
34       angle = (180*theta)/pi+360;
35    end

36    sun=plot(e,0,'ko'); set(sun,'MarkerSize',12)

37    function f = fkepler(x,e,t)
38    f = x-e*sin(x)-2*pi*t;
```

The following file (`k2.m`) is "more kinematical" because it shows the movement of the planet. Observe lines 12-13 and 20-21, where the planet and its position vector move without using the function `delete`. This file does not require further explanation because is similar to `k1.m`. Observe that if `Time` = 0, the planet is at the perihelion.

```
1    function k2(e,Tempus)

2    % k2(e,Tempus) draws the orbit a planet and the position vector.
3    % Time is the elapsed time
4    % The period of the orbit is 1 (0 <= Time ).
5    % e is the eccentricity of the ellipse (0 <= e < 1)

6    fig=figure; set(fig,'color',[1 1 1])

7    b=sqrt(1-e^2);

8    draw_ellipse = linspace(0,2*pi);
9    x = cos(draw_ellipse); y = b*sin(draw_ellipse);
10   plot(x,y); axis off; axis equal;
11   hold on; plot(e,0,'ko'); axis([-1 1 -1 1]);

12    position_vector = line([e 1],[0,0]);
13    planet = plot(1,0,'ko');
14    Press = title('Press any key to continue');
15    pause
16    set(Press,'Visible','off')

17    for t=0:0.001:Tempus
18      phi = fzero(@(x) fkepler(x,e,t),0.01);
19      x_planet = cos(phi); y_planet=b*sin(phi);
20      set(position_vector,'xdata',[e x_planet],'ydata',[0,y_planet]);
21      set(planet,'xdata',x_planet,'ydata',y_planet);
22      pause(0.01)
23    end

24    function f = fkepler(x,e,t)
25    f = x-e*sin(x)-2*pi*t;
```

The following two files draw the orbits of two planets in the same solar system. To this end, let us recall that the periods of the orbits must satisfy Kepler's third law. The first file draws the orbits with a fixed sun, whereas the second file draws the trajectories of the Sun and one planet from the point of view of one inhabitant of the other planet.

Let us briefly explain the foregoing file `k3`. Kepler's third law is codified in line 15 (it is assumed that the period of the first planet is 1). We shall use the polar expression (6) to draw the orbit of the two planets with the sun being plotted at the origin (line 14). This expression can be written as $|\mathbf{r}| = l/(1 + e\cos\theta)$, with the positive number l satisfying $l = b^2/a$ (this number l also appears in subsection 3.3). Line 13 draws the two orbits. Finally, the variable `planet_1` stores in lines 16 and 33 the coordinates of the position of the first planet (the variable `planet_2` is analogous). Also the file draws the line joining the sun and the planets (lines 17, 19 and 34, 36). If `Tempus` = 0, the planets are at the perihelion.

Figure 3: This figure has been obtained by issuing `k3([0.1 0.5],[1 2], 3)`.

```
1    function k3(e,a,Tempus)
2    % k3(e,a,Tempus) draws the orbits of two planets and
3    % their position vectors with fixed sun.
4    % The sun and the two perihelion are collinear
5    % e is a 2D vector containing the eccentricities
6    % a is a 2D vector containing the semimajor axis
7    % Tempus is the elapsed time
8    fig=figure; set(fig,'color',[1 1 1])

9    theta = linspace(0,2*pi); ct=cos(theta)'; st=sin(theta)';
10   b = a.*sqrt(1-e.^2); l=b.^2./a;
11   r1=l(1)./(1+e(1)*ct); r2=l(2)./(1+e(2)*ct);
12   x1=r1.*ct; y1=r1.*st; x2=r2.*ct; y2=r2.*st;
13   plot(x1,y1,'r',x2,y2,'b'); axis off; axis equal; hold on
14   sun=plot(0,0,'ok'); set(sun,'Markersize',12)

15   T=1; T2=sqrt((a(2)/a(1))^3);

16   planet1 = plot(l(1)/(1+e(1)),0,'or');
17   vector1 = plot([0, l(1)/(1+e(1))],[0 0],'r');
18   planet2 = plot(l(2)/(1+e(2)),0,'o');
19   vector2 = plot([0, l(2)/(1+e(2))],[0 0],'b');
20   Press = text(-0.35,-(1.2)*b,'Press any key to continue');
21   pause; set(Press,'Visible','off')

22   ffactor1 = sqrt((1+e(1))/(1-e(1)));
23   ffactor2 = sqrt((1+e(2))/(1-e(2)));

24   for t=0:0.01:Tempus
25     phi1 = fzero(@(x) fkepler(x,e(1),t,T),0.01);
26     sfactor1 = tan(phi1/2); factor1 = ffactor1*sfactor1;
27     theta1=2*atan(factor1); r1=l(1)./(1+e(1)*cos(theta1));
28     x1=r1.*cos(theta1); y1=r1.*sin(theta1);

29     phi2 = fzero(@(x) fkepler(x,e(2),t,T2),0.01);
30     sfactor2 = tan(phi2/2); factor2 = ffactor2*sfactor2;
31     theta2=2*atan(factor2); r2=l(2)./(1+e(2)*cos(theta2));
32     x2=r2.*cos(theta2); y2=r2.*sin(theta2);
33     set(planet1,'xdata',x1,'ydata',y1);
34     set(vector1,'xdata',[0 x1],'ydata',[0 y1])
35     set(planet2,'xdata',x2,'ydata',y2);
36     set(vector2,'xdata',[0 x2],'ydata',[0 y2])
37     pause(0.001)
38   end

39   function f = fkepler(x,e,t,T)
40   f = x-e*sin(x)-2*pi*t/T;
```

Ancient astronomers observed that Mars generally travelled eastward, except every couple of years, when this planet reversed course, going westward for a few months. Let us remember the system proposed by Ptomely: planets orbit Earth in quasi circular paths; the variations were explained by adding complex sets of circles upon circles. In the 16th century, Copernicus suggested that the movement of Mars could be explained by assuming that all planets (including Earth) orbit around the Sun. Later, Kepler proposed that the orbit were ellipses. How is the movement of Mars seen from the

Earth? Seeing this movement helps to admire more the talent of Copernicus and Kepler, because this movement is very different from an ellipse (as is easily seen in Fig. 4, in where the true eccentricities and the semimajor axis of the Earth and Mars were introduced as inputs).

The mathematics inherent of file `k4.m` are very similar to those of file `k3.m`. There are only two main differences: (a) In order to change the origin coordinates, we make a translation (lines 15-16 and 38-39); and (b) The file does not draw directly the orbit of the non fixed planet, but this orbit is pictured step by step in lines 44−45 (this is the reason for storing the values of `old_sun` and `old_planet2` in line 27). Again, if `Tempus` = 0, the planets are at the perihelion.

********** Here comes figure 4 ************

Figure 4: We obtain this figure by doing `k4([0.017 0.093],[1 1.52], 3)`.

```
1    function k4(e,a,Tempus)
2    % k4(e,a,Tempus); Orbits of two planets and their position vectors
3    % with one fixed planet.
4    % The sun and the two perihelion are collinear
5    % e is a 2D vector containing the eccentricities
6    % a is a 2D vector containing the semimajor axis
7    % Tempus is the elapsed time.
8    fig=figure; set(fig,'color',[1 1 1])

9    b = a.*sqrt(1-e.^2); l=b.^2./a;
10   T=1; T2=sqrt((a(2)/a(1))^3);

11   plot(0,0,'or'); hold on
12   axis off; axis equal; axis([-2*a(2) 2*a(2) -2*a(2) 2*a(2)]);
13   position_planet1 = [l(1)/(1+e(1)) 0];
14   position_planet2 = [l(2)/(1+e(2)) 0];
15   planet2 = position_planet2 - position_planet1;
16   sun = -position_planet1;
17   plot_sun = plot(sun(1),sun(2),'ok');
set(plot_sun,'Markersize',12);
18   plot_planet2 = plot(planet2(1),planet2(2),'o');
19   v1 = plot([sun(1) 0],[sun(2) 0],'r');
20   v2 = plot([sun(1) planet2(1)], [sun(2) planet2(2)], 'b');
21   legend('non fixed planet','sun')
22   Press = title('Press any key to continue');
23   pause; set(Press,'Visible','off')

24   firstfactor1 = sqrt((1+e(1))/(1-e(1)));
25   firstfactor2 = sqrt((1+e(2))/(1-e(2)));

26   for t=0:0.01:Tempus
27     old_sun = sun; old_planet2 = planet2;
28     phi1 = fzero(@(x) fkepler(x,e(1),t,T),0.01);
29     secondfactor1 = tan(phi1/2); factor1 = firstfactor1*secondfactor1;
30     theta1=2*atan(factor1);
31     r1=l(1)./(1+e(1)*cos(theta1));
32     position_planet1 = [r1.*cos(theta1) r1.*sin(theta1)];
33     phi2 = fzero(@(x) fkepler(x,e(2),t,T2),0.01);
```

```
34    secondfactor2 = tan(phi2/2); factor2 = firstfactor2*secondfactor2;
35    theta2=2*atan(factor2);
36    r2=l(2)./(1+e(2)*cos(theta2));
37    position_planet2 = [r2.*cos(theta2) r2.*sin(theta2)];
38    planet2 = position_planet2 - position_planet1;
39    sun = -position_planet1;
40    set(plot_sun,'xdata',sun(1),'ydata',sun(2));
41    set(plot_planet2,'xdata',planet2(1),'ydata',planet2(2));
42    set(v1,'xdata',[sun(1) 0],'ydata',[sun(2) 0]);
43    set(v2,'xdata',[sun(1) planet2(1)],'ydata',[sun(2) planet2(2)]);
44    plot([old_sun(1) sun(1)],[old_sun(2) sun(2)],'k')
45    plot([old_planet2(1) planet2(1)],[old_planet2(2) planet2(2)])
46    pause(0.001)
47  end

48  function f = fkepler(x,e,t,T)
49  f = x-e*sin(x)-2*pi*t/T;
```

5. CONCLUSIONS

There are some other objects (many of them available in the web) that permit to study the movement of a planet around the Sun. Among them, we can find in [7] (by searching "*kepler*") a lively demonstration of the second Kepler's law. Another interesting site is [8], where a simulation created by Easy Java Simulations is given. A mathematical difference with our proposal is that in [8] Newton's laws are used, i.e., if $(x, y) = (x(t), y(t))$ is the trajectory of the planet (assuming that the Sun is located at the center of coordinates), then by solving approximately the following differential system

$$x'' = -GMx / (x^2+y^2)^{3/2}, \qquad y'' = -GMy / (x^2+y^2)^{3/2},$$

the trajectory of the orbit can be depicted. Notice that in our approach we do not deal with any differential equation.

Besides the interactivity of the files, one important feature of our approach is the openness of the code source. This fact allows the students to create more files with a better understanding of the mathematics and physics involved in the topic. We propose the following tasks for the students:

a) Include more planets
b) Include some satellites around a planet.
c) Use approximation (12) instead solving a non linear equation. Test with different values of the eccentricity.
d) Study non-closed orbits.

By supervising the work made by the students, teacher gets an impression on how much the students really understand the area of study.

These learning objects have been developed to provide a unique opportunity to understand the conceptual and practical issues of the three Kepler's laws. Simulation

of the essential aspects of dynamic testing is implemented in the learning objects. These `m.` files, using Matlab or Octave programming language, allow easy interaction for the users and providing a wide variety of the parameters for their virtual experiments. The students who use these files are expected to gain significant insight into celestial mechanics.

References

[1] M. Kline, *Why Johnny can't add: The Failure of the New Mathematics*. St. Martin's Press, 1973.
`http://www.marco-Learningsystems.com/pages/kline/kline.html`

[2] D. Acheson. *From Calculus to Chaos. An introduction to dynamics*. Oxford University Press, 1997.

[3] D.A. Vallado, W.D. McClain. *Fundamentals of astrodynamics and applications*. McGraw-Hill, 1997.

[4] C. Kittel, W.D. Knight, M. Ruderman, C. Helmholz, and B. Moyer. *Berkeley Physics Course Mechanics Vol. 1*. Tata-McGraw-Hill, 2007.

[5] R.P. Brent. *Algorithms for Minimization without Derivatives*. Englewood Cliffs, Prentice-Hall, 1973.

[6] `http://www.windows.ucar.edu/`

[7] `http://demonstrations.wolfram.com`

[8] `http://www.um.es/fem/EjsWiki/Main/ExamplesPlanetaryMotion`