Appendix

Weyl's Extension of the Riemannian Concept of Space and the Geometrical Interpretation of Electricity.

§46. Formulation of the Problem.

In the following we shall give an exposition of Weyl's extension of Riemannian space, since this area of investigation has attained considerable significance for further elaboration of the theory of relativity. The presentation will differ from the previous part of this book inasmuch as it will contain a more detailed mathematical ~~formalism~~ development than was necessary in the other chapters. This is required because of the novelty of the material, which, ~~up to now~~ so far, has hardly been treated in any of the comprehensive presentations of the theory of relativity. At the same time it is necessary, for an epistemological analysis, to present the mathematical construction in such a way that its logical form becomes evident. If the reader is not sufficiently familiar with the formalism of the general theory of relativity to follow our line of thought, he may omit this appendix which presents nothing new, philosophically speaking, but merely constitutes an application of the epistemological principles developed earlier in this book. This negative philosophical result, however, is so important that we deem it necessary to present what follows in a precise form.

The basis of Riemannian space is the <u>definition of congruence</u>; its applicability in physics rests on the fact that there are physical objects, i.e., clocks and measuring rods, which make it possible to carry through such a definition. Why are these measuring instruments adequate for this purpose? The fundamental property which makes them suitable for a definition of congruence has already been formulated as an axiom. Two measuring rods which are of equal length when lying next to each other are always found to be equally long after having been transported along different paths to a distant place. The same holds for the time units ~~for~~ of clocks. We have stated ~~before~~ above that this property does not compel us to assert that the measuring rods are of equal length when they are located at different places, and we have emphasized that such a statement can be made only in the sense of a definition. We have also pointed out that such a definition is possible only because the measuring ~~rods~~ instruments have this peculiar property. If the measuring rods were always of different length when brought back together again, it would not be possible to carry through the usual definition of congruence. It is true that this property does not constitute the empirical evidence that the measuring rods are of equal length when separated - for no such evidence could possibly exist - yet it is a necessary condition for a suitable definition of congruence.(see also section 4).

What could we do if the measuring instruments did not have this special property? To answer this question let us limit ourselves to three-dimensional space, since we can always extend our considerations to four dimensions, ~~even if~~ regardless whether or not the fourth dimension is to be interpreted as time. Even in the absence of the above mentioned special property it would still be possible to formulate a definition of congruence. We could select any measuring instrument, move it around in space, and define congruence in terms of it. It does not matter that we would obtain different congruences depending upon the path along which the measuring instrument is moved,since definitions are arbitrary. We could introduce certain rules concerning the manner of transport; for example, we could prescribe transportation from a chosen center, along certain definite paths, at definite speeds. This definition of congruence would be coordinated to physical objects in a more complicated manner and would contain a greater number of arbitrary prescriptions. It is due to these additional rules that the definition would lose its preferential status, since we could just as well have chosen different velocities and different paths. The metric of this space would, of course, be definable, but it would not express a law of nature and it would therefore lose ~~all of~~ its physical significance.

This consideration is very important. We prefer the metric of rigid rods and clocks because this metric expresses a law of nature; namely, the law which concerns the transportations of measuring instruments. This is its practical import. All metrical statements are therefore statements concerning the transportation of measuring instruments and consequently have useful physical applications. If we know the length of a room, we also know how many chairs and tables, of a given size, we can place along one of its walls, and it is therefore of practical value to know the length of the room. In the previous example, we would not know this. For instance, whether we could place six chairs next to one another along one wall would depend upon the path by which the chairs were actually brought into the room, and we might have to take our chairs around the world in order to fit them in. Similarly it is questionable whether a guest would fit a chair, since this also would depend on his previous path. Such conditions may seem very strange, but they are certainly possible; and if they were real, we would surely have adapted ourselves to them.

Obviously, the arbitrarily defined metric would be of little use ~~mf~~ for the people in such a world. Instead, they would look for a geometrical method which would characterize the law of change in length during transport -- the <u>law of displacement</u>. Thus there arises the geometrical problem of formulating a law of displacement applicable to such a case and independent of any metric.

This problem was solved by Weyl[1] and his solution certainly constitutes a mathematical achievement of extraordinary significance regardless of its physical applicability. Mathematics concerns itself mainly with the generalization of our conceptual knowledge; Weyl has discovered a type of space more general than the Riemannian space, in which the displacement rather than

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1) H. Weyl, "Gravitation and Elektrizität, <u>Berliner Akad. Ber.</u> 1918, p. 465; also <u>Raum-Zeit-Materie</u>, 3rd ed. 1919, paragraph 34.

the metric represents the most fundamental operation. Weyl's extension of geometry has carried the mathematical treatment of the problem of space considerably beyond the developments of Riemann and Gauss. We shall now outline Weyl's approach, giving it in its complete generality as developed by Eddington[1)] and Schouten[2)].

Again we shall begin the presentation from a purely mathematical point of view, without asking about its physical realization. This means that we shall first have to define the displacement operation. The only empirical question - a question of physics - is whether or not there are objects which behave according to the operation which we have defined. In view of later applications, we shall keep the mathematical exposition as general as possible, so that it will be adaptable to physical requirements. In the following we shall not only formulate definitions, but we shall also investigate the problem of the consistency of the defined space, i.e., the presentation will be entirely mathematical in the usual sense.

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1) A. S. Eddington, "A Generalization of Weyl's Theory of Electromagnetism and Gravitational Fields", Proc. Royal Soc. London, A, Vol. 99, 1921, p. 104.

2) I. A Schouten, "Ueber die verschiedenen Arten der Uebertragung in einer n-dimensionalen Mannigfaltigkeit, die einer Differentialgeometrie zugrunde gelegt werden können." Math. Zeitschr. 13, 1922, p. 56.

§47. Displacement Space and Metrical Space.

Let us suppose that a coordinate system is given for an entire space and that a vector which is given by its components

tau

A^{τ}, is placed at a point P of this space. We may think of it as an arrow or we may conceive it as the sum total of its components. Let us now move to a neighboring point P', whose distance from P

nu

is given by the coordinate differential dx_{ν} and place a vector there which we call A'^{τ}. Now we want to say that the vector A^{τ}, displaced to the point P', is given exactly by this vector A'^{τ}; i.e., we want to specify a method of coordination by means of which the vector A'^{τ} at P' is coordinated to the vector A^{τ} at P. This coordination is the definition of our process of displacement.

Every component A^{τ} changes in the displacement, by the amount

$$dA^{\tau} = A'^{\tau} - A^{\tau}$$

Let us now look for a simple relation governing the dA^{τ}. In order to do this, let us assume that dA^{τ} is a homogeneous linear function of the components A^{μ} as well as of the displacements dx_{ν}. The relation is therefore

$$dA^{\tau} = \Gamma^{\tau}_{\mu\nu} A^{\mu} dx_{\nu} \qquad (1)$$

This is a sum of several elements, each of which consists of a product of a vector component and a displacement for any arbitrary combination of indices, and contains a numerical coefficient Γ. This numerical coefficient is different for each combination of the indices μ and ν, and is therefore written with the two lower indices μ and ν. Its upper index is τ, since every set of these coefficients differs depending ~~of~~ on the ~~amount~~ increase dA^{τ} to which it is related by (1). This gives us therefore a quantity $\Gamma^{\tau}_{\mu\nu}$ which has three indices.

Why did we choose the particular relation (1)? Being a definition, it is of course arbitrary, but mathematically speaking, it has some very definite advantages. It is very general, since it makes the change of the one component dA^{τ} dependent on all of the components A^{μ}, and all of the displacements dx_{ν}. Yet it is very simple because it assumes this dependence to be linear. The linearity of this relation agrees with a basic principle which has proved useful; namely, the principle of introducing only those generalizations of concepts which behave "normally" in the infinitesimal, i.e., which satisfy the conditions of differentiability. An arbitrarily curved line, for example, which has a tangent at every point, provides us with this kind of generalization of the straight line, since such a curved line behaves in the infinitesimal exactly like a straight line.

In order to ~~make (1) as~~ generalize (1) as much as possible, we shall ~~make a~~ have to modify ~~ication.~~ it. With expression (1), we have defined the operation of displacement merely for a single point in space. The coefficients $\Gamma^{\tau}_{\mu\nu}$ need not necessarily be the same at different points. We express this by considering the $\Gamma^{\tau}_{\mu\nu}$ as functions of the coordinates. Since we add the condition that these functions must be continuous, we have preserved the necessary continuity of the operation of displacement.

Having generalized the $\Gamma^{\tau}_{\mu\nu}$ to functions, we can now define the displacement along a definite path by

$$A'^{\tau} = A^{\tau} + \int \Gamma^{\tau}_{\mu\nu} A^{\mu} dx_{\nu} \qquad (2)$$

The vector A'^{τ} which is located at P', at a definite distance from P, is coordinated to the vector A^{τ} at P. The integral in (2) will depend on whatever path we choose between P and P' and we will therefore in general get a different vector A'^{τ} if we choose a different path.

It is exactly this dependence on the path, which justifies the name "displacement" for the defined operation. Transportation is, of course, a physical process; in mathematics its place is taken by coordination. The vector A'^{τ} at P' is coordinated to the vector A^{τ} at P. Mathematically speaking it is meaningless to think of this coordination as a transportation of the vector A^{τ}. Instead, we find a certain peculiarity in the coordination: given the vector A^{τ}, the point P and the point P', no vector is yet determined at P' unless we specify, in addition, a definite path between the points P and P'. A vector A'^{τ} is coordinated uniquely only to the combination of the four elements: the vector A^{τ}, the point P, the point P', and the path PP'. This peculiarity of the coordination relation makes it possible to apply this operation to the physical process of displacement, since the logical structure of displacement is of the same type as that of coordination. It has become customary to use this physical process in order to interpret the mathematical formulation and it is common to speak, not of a coordination, but of a displacement or a transfer. It is evident, however, that we are merely using a physical picture for the characterization of a mathematical relation.

Accordingly, we may conceive the mathematical introduction of the process of displacement in the following way. Given a system of coordinates, we may give arbitrary functions $\Gamma_{\mu\nu}^{\tau}$ and use them to define, according to (1), a certain operation of displacement. This definition does not make use of a metric. The comparison of lengths by means of a metric, is now replaced by a comparison of lengths through displacement.

The defined displacement actually accomplishes <u>more</u> than a mere comparison of lengths. It also defines a comparison of <u>direction</u>, since it coordinates to a vector at P only a single vector at P'. The relation of direction which is defined in (1) is commonly referred to as <u>parallelism</u>. We must understand, of course, that this concept of parallelism is not identical with that of Euclidean geometry. It has a much wider meaning. In common with the parallelism of Euclidean geometry it defines a comparison of direction. When applied to the special case of Euclidean space, this general comparison of direction becomes Euclidean parallelism. This means that it is related to this narrower concept of parallelism just as the vector sum is related to the algebraic sum. We should not let the use of the name "parallelism" for the more general concept lead us to believe that it means the same as the narrower concept. This would, of course, be impossible.

The claim that the displacement defines a comparison of length requires some further elucidation. If we are given the components of a vector, we do not yet know its length. Its length would be a single number which would have to be calculated from the numerical values of the components. This is generally done by means of the rule

$$l^2 = g_{\mu\nu} A^{\mu} A^{\nu} \qquad (3)$$

which therefore requires knowledge of the $g_{\mu\nu}$. The displacement is defined without reference to the $g_{\mu\nu}$ and is consequently determined even in a space in which the metric has not been defined. Hence, when we call the displacement a "comparison of length", this must be understood as follows: the displacement defines what is meant by two vectors having equal length, but it does not define the magnitude of this length.

The same can be said regarding the comparison of direction. The displacement defines only the equality of direction, but not a measure of ~~directions~~ the direction. This has the consequence that the displacement is unable to provide an order of unequal lengths and directions. It can only assert an inequality, without informing us concerning the relations of smaller and larger. It can make only negative statements regarding vectors which do not correspond.

Even these negative statements are not exhaustive, however, since the coordination established by the displacement deals only with the combination of the equality of length and direction. If we symbolize this complex relation, which Weyl calls congruence, by "$\cong$", the displacement will then read

$$A'^{\tau} \underset{s}{\cong} A^{\tau} \qquad (4a)$$

or in words: A'^{τ} is congruent to A^{τ} for the path s. The negation of this relation, which exists between non-corresponding vectors, reads

$$B'^{\tau} \underset{s}{\not\equiv} A^{\tau} \qquad (4b)$$

which expresses merely that B'^{τ} is not <u>simultaneously</u> equal to A^{τ} with respect to both length and direction, for the path s. It remains an open question whether B'^{τ} does satisfy at least one of the two partial relations, i.e., whether it is equal to A^{τ} at least with respect to length <u>or</u> direction. In regard to this the displacement tells us nothing. The assertion of equality through the displacement is therefore not complete with respect to length <u>or</u> direction. There may be vectors B'^{τ} at P' which are equal in length to A^{τ}, yet which do not coincide with A'^{τ}.

This implies a far-reaching limitation of the comparison of length and direction given by means of the displacement. Without an additional operation it is impossible to resolve the complex relation which we have called congruence, and it is for this reason that we shall need a metric in addition to the operation of displacement.

How can we now introduce a metric in addition to the displacement and avoid the possibility that the two operations might lead to contradictory results in the case where they both yield statements of the same type? Generally speaking we cannot do this, because the comparison of length which is given by the metric is not restricted to any definite path. Here something has already been coordinated to the combination of the three elements, the vector A^{ν}, the point P, and the point P'; namely, the element "class of vectors at P' which are equal in length to the vector A^{ν}". Whereas, according to the operation of displacement, this combination would still correspond to the *entire manifold* of vectors at P'; according to the metric, this combination corresponds to a *narrower class of vectors* only. In general, we cannot avoid *this* contradiction. Two vectors which are located at different points and which are equal in the sense of the metric, may yet be unequal according to the displacement, depending on ~~what~~ the path which is chosen between the two points.

The most general method of avoiding this difficulty would be to consider the two operations, the metric and the displacement, as two mutually independent basic operations. They would then present us with two fundamental geometrical processes of comparison which would have nothing to do with each other. One of them supplies us with a comparison of length which is independent of the path and may also be used for the comparison of unequal lengths, due to the fact that it supplies us with a numerical measure. The other presents a comparison of length which depends on the path, allowing only assertions concerning the equality of length, but which at the same time gives a similar type of comparison of direction. To assert the independent existence of these two operations is far from meaningless. It is not difficult to think of cases, especially in physics, where such a duality has to be chosen; i.e., where each of the two operations has its own physical realization, and these two physical processes are independent of one another.

A certain limitation of this most general approach results from introducing a dependence between these two basic operations. We might, for example, express one of the two systems as a function of the other. Physically speaking, this would imply a dependence, according to a law of nature, between the physical processes represented by the two operations. Geometrically speaking, however, there remains the duality of using two operations which lead to certain contradictory statements. If we speak, in this case, of a comparison of length, we would always have to specify to which of the two operations ~~of~~ the comparison refers.

Although this situation is logically permissible, it is geometrically unsatisfactory. It seems reasonable to ask whether it is not possible to achieve an agreement between the two operations such that the assertions which they make in common would no longer be contradictory. Although we cannot say that such a geometry must be applicable to reality, the problem still has purely geometrical interest. In any case, such a geometry provides an even more general geometrical frame than Riemannian geometry for the description of reality. In the following it will be shown how such a unified geometry can be constructed by means of a displacement and a metric.

Two methods are available to us. Either we may limit the scope of the metric, so that it will no longer refer to statements which result from use of the displacement, or we may limit the displacement so that statements common to the displacement and the metric no longer contradict one another. We shall call the type of space which is obtained by the first method a *displacement space*, because in it, the displacement is the dominant principle to which the metric will have to be adapted. The type of space which results from the second approach will be called a *metrical space*, because here the metric dominates and the displacement is subordinate to it.

Let us start with the first approach. We shall construct a displacement space in which, for any two vectors, there is a comparison of length which is dependent on the path. This comparison will be based mainly on the operation of displacement. What task remains for the metric?

A single task will now be assigned to the metric: namely, the task of providing the *comparison of length at one point*. We have already seen that the displacement does not permit a separation of the comparison of ~~length from the comparison of direction~~ length from the complex relation of congruence, and it is for this reason that we now introduce the metric. It defines the equality of the length of differently directed vectors, if the vectors are located at the same point. Furthermore, it orders these vectors with respect to their magnitude, and provides a measure of the relative length of two vectors.

We also need a corresponding rule for the comparison of direction at one point, in order to compare vectors which have the same direction but differ in length. Such a rule will not require the introduction of a new field. We may simply lay down the definition: Vectors are equally directed at the same point if the components of one result from those of the other through multiplication by a factor κ. We may, on the other hand, consider this rule to be implicitly contained in the concept of a metric, since we can always define an angular measurement at any point in terms of measurements of length by means of trigonometric functions. If this is done, we will find that the equalities of direction are the same for any choice of the $g_{\mu\nu}$. The numerical measure of the angles, however, will only be determined for a definite set of $g_{\mu\nu}$.

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In assigning to the metric the comparison of length at a point, we added to the comparison through displacement that extension without which a geometry would be impossible. We must ask, however, whether this stipulation implies the possibility of contradictions with the comparison which has previously been defined by displacement.

This actually does happen, because the comparison of lengths by means of displacement, in spite of its incompleteness, tells us something concerning lengths at the same point. We may perform the displacement along a closed path obtaining vector B^{τ} at P which is congruent to A^{τ}, but which is not equal in length to the vector A^{τ} in the sense of the metric. In order to exclude the possibility of such a contradiction, we shall introduce the rule that the comparison by displacement may not be used for a closed path.

At the same time, another consideration suggests this procedure. The comparison by displacement is not a reflexive relation, since it generally leads to

$$A^{\tau} \underset{s}{\neq} A^{\tau} \qquad (5)$$

and a vector would therefore not be equal to itself in respect to length and direction. Logically speaking, this is, of course, not contradictory, since the comparison by displacement does not assert the <u>identity</u>, but merely the <u>coordination</u>, of vectors. If we were to permit relation (5) to stand, however, this would make our geometry so complicated that it is better to eliminate this possibility. Actually this difficulty occurs only if the displacement is used for closed paths, since it is only in this case that the relation of ~~in~~ non-congruence, as defined by the displacement in (4b), can lead back to the original vector. The negative statement (4b) is limited to vectors B'^{τ} at P', for a given vector A^{τ} and a given path s, and says nothing in regard to vectors at any other point, not even that they are ~~in~~ non-congruent. We shall therefore exclude closed paths ~~for~~ from the comparison by displacement. This is accomplished by means of the definition which states that the comparison of length and direction at a point is not to be performed by displacement.

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This rule is used de-facto in an "unbalanced" type of space, in which the metric and the displacement are left independent of each other. We say in this case that the vector has <u>changed</u> its length if it returns to the same point along a closed path. This avoids relation (5), which we would otherwise have been forced to use.

We can also formulate this requirement differently. In regard to the comparison of length at one point - and only for this - there exists a special path; namely, the path which touches no other point, i.e., the vanishing path. In this special case of comparison of length at one point a path of comparison, namely, this vanishing path, may be prescribed. Consequently every vector is coordinated only to itself by this displacement. Thus, it is impossible to make any statement by means of the displacement regarding the comparison of length of different vectors at the same point. - It should be noted that the use of this special path is not based on the concept of the shortest path, since such a concept would already presuppose a metric. The concept of the vanishing path is a purely topological one, which may be formed on the basis of a given coordinate system alone.

In that case our rule would be: the comparison of length at one point is to be obtained by rotation and not by displacement. A comparison by rotation, however, requires a metric.[1)] This

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1) Another possibility would be to formulate a special law of rotation, which would be similar to the law of displacement, and would be based on the assumption that the vector would no longer have the same length after a rotation. However, this method, too, would make the equality of length a non-reflexive relation and it will therefore not be pursued any further. Helmholtz formulated the assertion that rigid bodies are congruent to themselves after a rotation as the principle of Monodromy. See also Helmholtz, Schriften zur Erkenntnistheorie, edited by Hertz and Schlick, Berlin 1921, pages 42 and 62.

divides the task between the metric and the displacement as follows. The metric is employed for the comparison of length and direction at one point, while the displacement is used for the comparison of length and direction at separate points.

If we now want to base a consistent displacement space upon this division of labor we must deprive the metric of one of its functions because, in general, the metric also defines a comparison of length at different points. Thus the possibility of a contradiction with the comparison by displacement arises. This idea may be expressed as follows. We lay down a metric $g_{\mu\nu}$, but make only statements which do not change when the $g_{\mu\nu}$ are multiplied

lambda by a scalar factor $\lambda(x_1 \ldots x_n)$. This implies that we may not use the $g_{\mu\nu}$ for the comparison of length at different points, because two vectors which are located at different points and which are equal in length according to the metric $g_{\mu\nu}$ would become unequal if we were to multiply the $g_{\mu\nu}$ by the factor $\lambda(x_1 \ldots x_n)$.

We can also express this idea mathematically as follows. We no longer specify all of the functions $g_{\mu\nu}$; instead, we give only their ratios $q_{\mu\nu}$, which are one less in number than the functions. These may be obtained by choosing at random one of the functions $g_{\mu\nu}$ and dividing all others by it. The $q_{\mu\nu}$ are therefore defined, for example, as

$$q_{\mu\nu} = \frac{g_{\mu\nu}}{g_{44}} \qquad (6)$$

The functions $g_{\mu\nu}$ are now determined but for a scalar factor $\lambda(x_1 \ldots x_n)$, which depends on the position. This factor does not matter in the comparison of vectors at a single point, since all statements regarding the relation of length of vectors at one point are independent of the choice of λ. The comparison of length at different points is impossible, however, by means of the $g_{\mu\nu}$, since this would require the determination of the factor λ. This procedure is, of course, exactly the same as the first. The permissible statements of the first method are, at the same time, the only possible statements of the second. In the following we shall use the first of these two methods.

So far we have obtained two kinds of statements in the displacement space; namely, the comparison of length at one point and the comparison of corresponding vectors at different points. Let us now inquire into the comparison of non-corresponding vectors at different points. As we shall see, this will involve a peculiar kind of indeterminacy.

Given a vector A^{ν} at P, a vector B'^{ν} at P' and a connecting path s, what is the ratio of the lengths of these two vectors? In this case it is permissible to specify the connecting path, since it is a characteristic of the displacement space that a comparison of vectors at different points depends upon the path. Yet another indeterminacy arises because the desired comparison ~~of length~~ may be perfomred in two ways. First of all, we can look for the vector A'^{ν} at P' which is congruent to the vector A^{ν} at P according to the law of displacement; then we would have to compare A'^{ν}

with B'^{τ} with the aid of the $g'_{\mu\nu}$ (or the $q'_{\mu\nu}$ as the case may be) at the point P'. This method would give us an answer to our question. On the other hand, we might look for the vector B^{τ} at P which according to the operation of displacement will be congruent to the vector B'^{τ} at P', and we would then have to compare B^{τ} with A^{τ} using the $g_{\mu\nu}$ (or the $q_{\mu\nu}$ as the case may be) at the point P. This will again give us a solution to our problem, although these two solutions will, in general, not be the same.

In order to be able to define a general comparison of length an additional rule is required. We may prescribe, for example, that a vector is to be displaced before it is rotated. This leads to the result, however, that comparison of length becomes a non-symmetrical relation. The ratio

$$\ell(A^{\tau}) : \ell(B'^{\tau}) = k$$

yields a different numerical value from the ratio

$$\ell(B'^{\tau}) : \ell(A^{\tau}) = k'$$

i.e.,

$$k \neq \frac{1}{k'} .$$

(A symmetrical comparison would yield $k = \frac{1}{k'}$.)

In particular this lack of symmetry ~~would~~ applies also to the case of equality. If for example

$$\ell(A^{\tau}) \underset{s}{=} \ell(B'^{\tau})$$

we would find that in general

$$\ell(B'^{\tau}) \underset{s}{\neq} \ell(A^{\tau}).$$

Here, even the equality of length is a non-symmetrical relation. Only the equality of corresponding vectors is a symmetrical relation, since both types of comparison become identical in this case.

Let us finally present a fourth possible determination in the displacement space. Until now we have applied the operation of displacement to vectors of arbitrary orientation. We can, however, displace a vector along its own direction. In this case we choose the dx_τ in such a manner that they are proportional to the components of the vector A^τ. If we specifically choose a line element δx_τ as our vector, the displacement which is applied to this case will then define what is meant by "displacement of a line element along its own direction". Such a displacement will therefore define a <u>straightest line</u>. However there is no <u>shortest line</u> in the displacement space, because a distance does not have a defined length but only an "extension" in the topological sense (page 288).

We shall call the space, developed above, the <u>general displacement space</u>, since in it the operation of displacement has not been restricted apart from the elimination of the comparison by displacement of vectors at the same point. This space is therefore determined if the two fields $\Gamma^\tau_{\mu\nu}$ and $q_{\mu\nu}$ are given independently of each other. In this space the following four types of assertions are permissible or possible:

1. The comparison of the length of vectors at the same point, given by the $g_{\mu\nu}$ (or the $q_{\mu\nu}$ ~~are~~ as the case may be),
2. The comparison of corresponding vectors at different points for a given connecting path, given by the $\Gamma^\tau_{\mu\nu}$,
3. The comparison of non-corresponding vectors at different points for a given connecting path and a given direction of comparison, given by the $g_{\mu\nu}$ (or the $q_{\mu\nu}$ as the case may be) and the $\Gamma^\tau_{\mu\nu}$,
4. The identification of straightest lines.

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Hans Reichenbach Papers, 1884-1972, ASP.1973.01, Archives of Scientific Philosophy, Special Collections Department, University of Pittsburgh

Let us now turn to restrictions of this general displacement space. First of all, let us ask which restricting condition must be satisfied in order that the general comparison of length be symmetrical. This means, obviously, that the ratio $A^{\tau}:B^{\tau}$ must be the same as the ratio $A'^{\tau}:B'^{\tau}$; or in other words, the ratio of the lengths of differently directed vectors must not change if they are displaced along the same path. This condition leads us to a restriction regarding the $\Gamma^{\tau}_{\mu\nu}$.

We shall carry out this simple calculation[1], because we shall make use of it later on. By differentiating (3) we obtain

$$d(\ell^2) = \left[\frac{\partial g_{\mu\nu}}{\partial x_\sigma} A^\mu A^\nu + g_{\mu\nu} A^\nu \frac{\partial A^\mu}{\partial x_\sigma} + g_{\mu\nu} A^\mu \frac{\partial A^\nu}{\partial x_\sigma} \right] dx_\sigma \qquad (7)$$

for the transition to a neighboring point. If we assume, in addition, that this transition consists of a parallel displacement of the vector A^μ, then we may substitute for the partial derivatives of the A^μ with respect to the coordinates the following expressions which are directly obtained from (1).

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1) Although a comprehensive mathematical presentation has not yet been given from the viewpoint we have developed above, the necessary calculations have already been carried out in the mathematical literature. The proofs given in this section follow the clear exposition given by Eddington in ~~Relativitätstheorie in Mathematischer Behandlung, Springer 1925, page 324,~~ The mathematical Theory of Relativity, Cambridge University Press, 1924, except for a few changes in notation.

$$\frac{\delta A^\mu}{\delta x_\sigma} = \Gamma^\mu_{\alpha\sigma} A^\alpha \qquad (8)$$

If we also exchange some of the indices of summation, (7) becomes

$$d(l^2) = \left[\frac{\delta g_{\mu\nu}}{\delta x_\sigma} + g_{\alpha\nu}\Gamma^\alpha_{\mu\sigma} + g_{\mu\alpha}\Gamma^\alpha_{\nu\sigma}\right] A^\mu A^\nu dx_\sigma \qquad (9)$$

Let us now define the quantities $\Gamma_{\mu\sigma,\alpha}$ by means of the customary lowering of the indices according to the rule 1)

$$\Gamma_{\mu\sigma,\nu} = g_{\alpha\nu}\Gamma^\alpha_{\mu\sigma} \qquad (10)$$

Finally, we get

$$d(l^2) = \left[\frac{\delta g_{\mu\nu}}{\delta x_\sigma} + \Gamma_{\mu\sigma,\nu} + \Gamma_{\nu\sigma,\mu}\right] A^\mu A^\nu dx_\sigma \qquad (11)$$

In order to abbreviate this expression, let us write

$$K_{\mu\nu,\sigma} = \left[\frac{\delta g_{\mu\nu}}{\partial x_\sigma} + \Gamma_{\mu\sigma,\nu} + \Gamma_{\nu\sigma,\mu}\right] \qquad (12)$$

The desired restricting condition for the $\Gamma^\tau_{\mu\nu}$ is therefore expressed by the fact that we must put

$$K_{\mu\nu,\sigma} = g_{\mu\nu}\cdot\kappa_\sigma \qquad (13)$$

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1) The quantities which have a lower index are commonly called the covariant components while those which have an upper index are called the contravariant components of the same vector or tensor. Between them we have the relation

$$A_\mu = g_{\mu\nu}A^\nu.$$

With respect to this and to the corresponding law of transformation see Einstein, <u>Vier Vorlesungen über Relativitätstheorie</u>, Vieweg, 1922, pages 42-44. Quantities which indices of both kinds are called mixed components and transform according to (10). Although the $\Gamma^\tau_{\mu\nu}$ have these properties of tensors, they are not general tensors, but merely linear tensors. See Weyl, <u>Raum-Zeit-Materie</u>, 3rd ed. Berlin 1920, page 102.

This gives us a functional relation between the $\Gamma_{\mu\nu}^{\tau}$ and the $g_{\mu\nu}$. Instead of the $\Gamma_{\mu\nu}^{\tau}$ only the vector (or tensor of rank one) κ_σ is to be arbitrarily given and (11) reduces to

$$d(\ell^2) = \ell^2 \kappa_\sigma \, dx_\sigma \qquad (14)$$

The magnitude of the change in length is now dependent only upon ℓ^2, not upon the direction of the vector A^τ, and it is proportional to ℓ^2. Equally long vectors at P will then also be equally long at P' and the ~~ratios~~ ratio of the lengths will be preserved in the displacement.

small print

The relation given by (12) and (13) is an In-equation (in Eddington's terminology), i.e., it applies in the same form to a metric $g'_{\mu\nu}$ which is introduced by $g_{\mu\nu} = \lambda(x_1 \ldots x_n)\, g'_{\mu\nu}$. This requirement is necessary because we want to admit only those statements which hold for all of the $g_{\mu\nu}$-systems having the same $q_{\mu\nu}$. For the proof we introduce the substitution and obtain

$$\lambda g'_{\mu\nu} \kappa_\sigma = g'_{\mu\nu} \frac{\partial \lambda}{\partial x_\sigma} + \lambda \frac{\partial g'_{\mu\nu}}{\partial x_\sigma} + \lambda g'_{\nu\tau} \Gamma_{\mu\sigma}^{\tau} + \lambda g'_{\mu\sigma} \Gamma_{\mu\sigma}^{\tau}$$

$$\lambda g'_{\mu\nu} \left[\kappa_\sigma - \frac{1}{\lambda} \frac{\partial \lambda}{\partial x_\sigma} \right] = \lambda \frac{\partial g'_{\mu\nu}}{\partial x_\sigma} + \lambda g'_{\nu\tau} \Gamma_{\mu\sigma}^{\tau} + \lambda g'_{\mu\sigma} \Gamma_{\nu\sigma}^{\tau}$$

If we write κ'_σ for the bracket on the left side and divide by λ, the assertion is proved. The unchanged functions $\Gamma_{\mu\sigma}^{\tau}$ are therefore in the same relation to the $g'_{\mu\nu}$ as to the $g_{\mu\nu}$. It should be noted, however, that the functions $\Gamma_{\mu\sigma,\nu}$ have changed, since $\Gamma_{\mu\sigma,\nu}$ in the old metric equals $g_{\nu\tau}\Gamma_{\mu\sigma}^{\tau}$, while it is equal to $g'_{\nu\tau}\Gamma_{\mu\sigma}^{\tau}$ in the new metric. Similarly ~~the~~ κ'_σ differs from κ_σ, but since κ_σ is arbitrary this does not affect our assertion.

Weyl makes assumption (13), and his space is therefore a specialization of the general displacement space. The specialization consists in the fact that, for a given connecting path, not only the comparison of length of corresponding vectors but also the comparison of length of all other vectors becomes symmetrical. Weyl's space involves, however, an additional specialization, which we shall indicate in the following.

The second restriction of the general displacement space is introduced independently of the first. It consists in the requirement that

$$\Gamma^{\tau}_{\mu\nu} = \Gamma^{\tau}_{\nu\mu} \tag{15}$$

The significance of this condition can readily be seen if we consider a small displacement δx_{τ} (Fig. 50)[1] as the vector A^{τ} which is to be displaced. If we displace the vector δx_{τ} by the distance dx_{τ}, its endpoint will be at a point P_1, whose distance from the starting point P is given by the coordinate differences

$$dx_{\tau} + \delta x_{\tau} + d(\delta x_{\tau}).$$

The only law of summation for vectors contained in this equation is that which may be considered valid for neighboring points separated by infinitesimal distances. (Rigorously it applies only to vectors which are located at the same point and thus our assertions are correct in the limit.) If we displace, on the other hand, the vector dx_{τ} by the displacement δx_{τ}, its endpoint will reach a point P_2, which is determined relative to P by the coordinate differences

Fig. 50. The non-existence of infinitesimal parallelograms.

$$\delta x_{\tau} + dx_{\tau} + d(dx_{\tau}).$$

1) At this point it becomes obvious that the writing of coordinate differentials with a lower index is a mistake. Coordinate differentials correspond to contravariant vectors and should therefore be written with an upper index. We shall, however, follow the notation as used in the literature.

Under what conditions will P_1 coincide with P_2? Since the first two terms of the two expressions are identical, this condition is satisfied if

$$d(dx_\tau) = d(\delta x_\tau).$$

Together with (1) this yields

$$\Gamma^{\tau}_{\mu\nu}\,\delta x_\mu dx_\nu = \Gamma^{\tau}_{\mu\nu}\,\delta x_\mu dx_\nu \qquad (16)$$

where μ and ν are interchanged on the two sides of the equation. This is permissible since the summation runs over both indices. Equation (16) will always be satisified if (15) is true, and (15) is therefore the condition that P_1 and P_2 coincide.

We may denote this property therefore as the existence of infinitesimal parallelograms. If four neighboring infinitesimal vectors are parallel in pairs and equally long in the sense of the displacement, they will form a quadrilateral. In the general displacement space there are no infinitesimal parallelograms. If there are infinitesimal parallelograms, this would constitute a geometrical singularity for the field $\Gamma^{\tau}_{\mu\nu}$ to be given and therefore a restriction which is given by (15). There is, of course, no logical need for such a restriction.

The restrictions (13) and (15) may also be imposed simultaneously upon the general displacement space. Since this is done by Weyl, we shall call the resulting space, namely, the third specialized displacement space, a Weylian space. It is the displacement space in which there are infinitesimal parallelograms and in which the comparison of the length of arbitrary vectors is symmetrical for any given path. If we use the Christoffel abbreviation as given in (19), the dependence between the $\Gamma^{\tau}_{\mu\nu}$ and the $g_{\mu\nu}$ holding for the Weylian space is given by

$$\Gamma^{\tau}_{\mu\nu} = -\left\{ {\mu\nu \atop \tau} \right\} + \tfrac{1}{2} g^{\tau}_{\mu} K_{\nu} + \tfrac{1}{2} g^{\tau}_{\nu} K_{\mu} - \tfrac{1}{2} g_{\mu\nu} K^{\tau} \qquad (17)$$

small print

Proof (due to Eddington): Corresponding to (12) we may form the three equations

$$K_{\mu\nu,\sigma} = \frac{\partial g_{\mu\nu}}{\partial x_\sigma} + \Gamma_{\mu\sigma,\nu} + \Gamma_{\nu\sigma,\mu}$$
$$K_{\mu\sigma,\nu} = \frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \Gamma_{\mu\nu,\sigma} + \Gamma_{\sigma\nu,\mu}$$
$$K_{\nu\sigma,\mu} = \frac{\partial g_{\nu\sigma}}{\partial x_\mu} + \Gamma_{\nu\mu,\sigma} + \Gamma_{\sigma\mu,\nu}$$

Adding the last two equations and subtracting the first, we obtain, because of (15)

$$1/2 \left[K_{\mu\sigma,\nu} + K_{\nu\sigma,\mu} - K_{\mu\nu,\sigma} \right] = 1/2 \left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \frac{\partial g_{\nu\sigma}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\sigma} \right) + \Gamma_{\nu\mu,\sigma} . \qquad (18)$$

If we raise the index σ and use condition (13), we will obtain condition (17), if the Christoffel abbreviation

$$\left\{ {\mu\nu \atop \tau} \right\} = 1/2 \, g^{\tau\sigma} \left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \frac{\partial g_{\nu\sigma}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\sigma} \right) \qquad (19)$$

is introduced. It can easily be shown that (17) is again an In-equation.

The specializations of the general displacement space developed above are still called special displacement spaces, because they still accomplish the comparison of length by means of displacement; i.e., the comparison depends upon the path, yet it is restricted. Let us now turn to the second method of bringing the metric and displacement into conformity, in which the metric supplies the basis of the comparison of length, while the displacement is specialized so that it fits the metrical comparison of length. These spaces are therefore called metrical spaces. The most general type is contrasted with narrower specializations.

We are now going to use the $g_{\mu\nu}$, and not only their ~~relations~~ ratios; that is, we include among the admissible statements those which presuppose the actual values $g_{\mu\nu}$. Among these we shall find, first of all, the comparison of length of vectors which are located at different points, independent of the path. If this comparison is not to contradict the displacement, $d(\ell^2)$ must equal zero for an infinitesimal displacement. Together with (11) and (12) this yields

$$K_{\mu\nu,\sigma} = 0 \tag{20}$$

<u>Equation (20) is the characterization of the general metrical space.</u> In such a space the transfer of length by the displacement can be integrated; i.e., it is independent of the path, and the length thus transferred is identical with that of the metrical comparison. The function of the displacement in this space is limited to the transfer of direction; the transfer of length is left entirely to the metric.

To make this clear, let us return to our previous characterization of the difference between metric and displacement (see page 472). By means of the metric a certain class of vectors at P' (the class of vectors equal in length to the vector A^{ν}), is coordinated to the combination of the three elements, a vector A^{ν}, a point P, and a point P', such that to every direction at P' there belongs only one vector of this class.

The displacement, on the other hand, coordinates to such a combination the total class of vectors at P' and makes a choice among these vectors only if a connecting path s is added to this combination as a fourth element. Condition (20) specializes the operation of displacement in such a manner that the combination of the three elements is no longer coordinated to the entire class of vectors at P', but to a narrower class which coincides with the class of vectors equal in length to A^J. For any direction specified at P', only <u>one</u> vector remains which can be made congruent to a given vector at P by choosing a path s. Therefore only the transfer of direction of the displacement but not its transfer of length now depends upon the path.

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Relation (20) is not an In-equation, since it can be satisfied only for a definite metric $g_{\mu\nu}$. This is because we are now making use of the $g_{\mu\nu}$ and are no longer restricting ourselves to statements which assume only the $q_{\mu\nu}$.

Due to the existence of a metric, there will be <u>shortest lines</u> in the general metrical space, but in general, they are not identical with the straightest lines which are defined by the displacement. It would require an additional restriction to make these two special lines coincide and the Riemannian space provides this specialization, although in this space the transfer of direction of the displacement can not yet be integrated. It is specialized in such a way that the displacement of a line element along its own direction leads to the same line as the shortest direction metrically speaking. In Riemannian space there are therefore <u>geodesics</u>; i.e., lines which are straightest and shortest at the same time.

<u>The general metrical space is therefore different from the Riemannian space and the latter is a special metrical space.</u>

The specialization which leads to Riemannian space is therefore given by the condition

$$\Gamma^{\tau}_{\mu\nu} = -\left\{ {\mu\nu \atop \tau} \right\} \tag{21}$$

(for the Christoffel abbreviation,see (19)).

This condition is derivable in a different way. If the condition of symmetry (15) is added to condition (20), (21) results. The Riemannian space can also, therefore, be characterized as a metrical space in which there are infinitesimal parallelograms. This specialization is identical with that which is given by the concept of geodesics.

small print

Proof: Because of (20), if the left-hand side of (18) is equated to zero, (21) results.Here, (15) is already presupposed by (18). The proof that the displacement defined by (21) provides a straightest line which is at the same time the shortest line, has been given in the literature.[1)]

Let us finally make the only other possible specialization. Thus far, the transfer of length, and not the transfer of direction, has been made independent of the path. If we now make this additional demand we arrive at Euclidean space, in which

$$\Gamma^{\tau}_{\mu\nu} = -\left\{ {\mu\nu \atop \tau} \right\} = 0 \tag{22}$$

for rectilinear coordinates. The characterization of this condition, which is invariant, i.e., independent of the coordinate system, is formulated in the familiar way (see page german book 283) as the vanishing of the Riemannian tensor

$$R_{\mu\nu\sigma\tau} = 0 \tag{23}$$

This no longer signifies a specialization of the $\Gamma^{\tau}_{\mu\nu}$ alone, but also a specialization of the $g_{\mu\nu}$.

1) H. Weyl, Raum-Zeit-Materie, 3rd ed., Berlin 1920, p. 121.

§ 48. The Geometrical Interpretation of Electricity.

Weyl did not develop the space which we presented in the previous section purely for its mathematical interest; it was his purpose to make it suitable for a physical application. The great success which Einstein had attained with his geometrical interpretation of gravitation, led Weyl to believe that similar success might be obtained from a geometrical interpretation of electricity. Since the tensor of Riemannian space was already appropriated by gravitation, he constructed a wider geometrical frame which contained some unassigned geometrical elements which he could ascribe to electricity.

Before we ~~criticize~~ examine this idea, let us first review the actual achievements of the geometrical interpretation of gravitation. The field of force of gravitation affects the behavior of measuring instruments. Besides serving in their customary capacity of determining the geometry of space and time, they serve, therefore, also as indicators of the gravitational field. The geometrical interpretation of gravitation is consequently an expression of a real situation; namely, of the actual effect of gravitation on measuring rods and clocks. This constitutes the physical value of this interpretation. It has been confirmed to a very high degree by physical experience and is formulated by the principle of equivalence.

Anything beyond is added by the imagination and constitutes picture-thinking. If we think of a Riemannian space with its special relations of congruence, instead of thinking of a field of force, this is a permissible representation of the gravitational field. But one cannot say that it is a necessary one. There is no need to consider the behavior of measuring instruments as "voluntary"; we might just as well conceive that they are influenced by a force, i.e., gravitation, whose properties are no different from a temperature field, for instance, which also imposes certain relations of congruence upon the measuring instruments. Indeed, there is no need at all to use measuring instruments for the representation of the gravitational field. We might, for example, recognize the gravitational field by the motion of mass points. Such a motion would, of course, suggest a relationship to the geometrical interpretation, since it can be visualized as a motion along the geodesics. Yet this assertion goes beyond what is given by the motion of the mass points, alone, since it puts this motion into a relation with the geometrical behavior of the measuring instruments (measurement of length by ds^2). We are not compelled to think of this relation at all times, but we might also stop at the conception of force which, before Einstein, was associated with the falling mass points. It makes no sense to call this pictorial conception of an attractive force <u>false</u>. The new insight of Einstein consists merely in recognizing the fact that the well-known complex of relations concerning the motions of mass points is supplemented by their relations to the behavior of measuring instruments. It is not necessary, however, to consider this relation as the primary basis of the pictorial conception, to which all other gravitational facts must be referred. If we take more complicated gravitational effects, e.g. the state of tension in a beam, we could also interpret this as an attempt of matter to adapt itself to the Riemannian space. But this is just a kind of visualization, not a necessary representation of this physical state.

The geometrical representation of gravitation would therefore be nothing but a form of visualization, if it did not actually assert this relation between measuring instruments and the gravitational field. This empirical assertion is the basis of the epistemological value of Einstein's theory of gravitation. The geometrical interpretation of gravitation is merely the visual cloak in which the factual assertion is dressed. It would be a mistake to confuse the cloak with the body which it covers; rather, we may infer the shape of the body from the shape of the cloak which it wears. After all, only the body is the object of interest in physics.

If we want to do the same for electricity, we must search for a similar physical fact which relates the electrical field to the behavior of measuring instruments, thus permitting a geometrical expression of the electrical field. However, the fundamental fact which would correspond to the principle of equivalence is lacking. Although Weyl has constructed the operation of displacement, the displacement space is not the type of space which characterizes their geometrical behavior, since their behavior of measuring instruments can be integrated. We are referring to the conceivable behavior of rigid bodies as explained in section 46; however, it is precisely this behavior which does <u>not</u> occur in reality. This means that we have found a cloak in which we can dress the new theory, but we do not have the body which this new cloak would fit.

An extension of Einstein's theory to the Weylian space would look somewhat as follows. Measuring rods and clocks are indicators of the gravitational field and, at the same time, indicators of the electrical field. As long as a gravitational field exists alone, the behavior of the measuring rods can be integrated, i.e., they define a comparison of length independent of the path. For this purpose the Riemannian space would suffice. As soon as an electrical field is added, however, the integrability ceases, the behavior of the measuring instruments is describable only in terms of the operation of displacement, and the Weylian space now constitutes the natural cloak for the field which is composed of electricity and gravitation.

Unfortunately, however, this picture does not agree with the physical facts. Even if electrical fields are added, the behavior of the measuring instruments can still be integrated. This should be understood as follows. As long as there is an effect of the electrical field, the behavior of the measuring instruments differs from that in a gravitational field alone; but two measuring instruments transported along different paths will be equally long when they meet again, even in the electrical field. This is true even in the case where two measuring instruments meet outside of the electrical field after one of them has traveled through the electrical field while the other one has traveled entirely outside of it. All of our previous experience seems to confirm this fact, especially (according to Einstein) the existence of sharp spectral lines. If atomic clocks changed their periods as a function of their space-time paths, a large number of atoms with completely different pasts could hardly radiate light of the same frequency. The displacement space is therefore not suited to describe the behavior of measuring instruments in a combined electrical and gravitational field.

Weyl, therefore, tried a different approach. He conceived of two kinds of physical entities, one of which indicates the metrical field while the other represents the operation of displacement. Measuring rods and clocks are of the first kind and therefore only indicators of the gravitational field, as in Einstein's theory. Only in the field $\Gamma^{\tau}_{\mu\nu}$, is electricity expressed in addition to gravitation, and only indicators of the second kind can therefore react to electricity.

What are these indicators of the second kind? Weyl introduced the distinction between <u>adaptation</u> and <u>perseverance</u> for their characterization. He called the behavior of the ~~entities of the~~ first kind of entities <u>adaptation</u> and that of the second type <u>perseverance</u>. This second term requires a justification. We can just as well call the behavior of the second kind of entities <u>adaptation</u>; namely, an adaptation to the field of the $\Gamma^{\tau}_{\mu\nu}$. Since the field $\Gamma^{\tau}_{\mu\nu}$ is integrated, however, due to the definition of the process of displacement, we may call this special kind of adaptation perseverance. Perseverance is therefore an adaptation accompanying the simultaneous integration of the field along a described path. With the above definition of these two concepts, we may now accept Weyl's distinction.

From the geometrical viewpoint as well there is no doubt that we are justified in rejecting the measuring instruments as indicators of the operation of displacement. Measuring rods define a comparison of length, but no comparison of direction; they are not indicators of an operation of parallel displacement. If a measuring rod is to be transported along a given path in the sense of the displacement, a special rule is required, telling us how it is to be directed, since it cannot define such a rule by itself. The axis of a gyroscope, for example, which adjusts itself in a definite fashion in transport, is an indicator of this type, or, for that matter, the velocity vectors of freely moving mass points. Weyl decided not to use the above mentioned objects for reasons which we shall discuss later on.

Fundamentally, Weyl's assertion is certainly correct: it is not necessary to consider measuring rods and clocks,in particular, as the objects which realize the operation of displacement. If there are special indicators which react to electricity and behave in the sense of the displacement, the geometrical interpretation accomplishes the same thing as for gravitation. The fact that clocks and measuring rods,which realize the process of adaptation, are not indicators of this type is irrelevant, as long as there are other indicators which realize the process of perseverance. The question of realization may therefore be left open, for the time being. Let us assume that indicators of the second kind have been found. What would be the ~~the~~ geometrical interpretation of electricity? ~~be like?~~

We would then be confronted with the case described on page 471, in which we had two essentially different basic processes. The processes would not be entirely independent of each other, and we would develop the type of space described in the second para- graph of page 473, for instance, in which the two basic operations are made dependent but contradict one another in some respects. This contradiction cannot be avoided as long as there exist indicators for a metric $g_{\mu\nu}$ which determine these functions themselves and not only their ratios. The dependence of these processes will have to be adjusted in conformity with the laws of nature and their choice is therefore subject to experience. This approach, except for a restriction to be mentioned later, is used by Eddington and Einstein.

Weyl, himself, pointed out that measuring rods and clocks are none too accurate and should, therefore, never be used as a foundation of geometry. (Physically speaking this would be correct only if there were no atomic clocks.) A better foundation for geometry is supplied by the motion of light. Since this motion satisfies the equation $ds^2 = 0$, however, it defines only the ratios of the $g_{\mu\nu}$ and not the $g_{\mu\nu}$ themselves. Thus, it is possible to construct a "balanced" displacement space. Weyl decided upon the type of space we have called the Weylian space, identifying the vector x_σ, which is still unassigned in this space, with the electrical potential.

A third and last method is given by the following considerations. If we recognize measuring rods and clocks as indicators of the metric, the only balanced type of space which we would still have at our disposal, would be the general metrical space. As we have shown before (page 489), this space is not necessarily identical with the Riemannian space. It is, therefore, possible to define an operation of displacement which contains the effect of the electrical field, but which, on the other hand, does not contradict the metric. The geometrical interpretation of electricity would then be expressed by the special kind of displacement of direction, but no longer by an effect upon the comparison of length.

Which of these three methods is the correct one? No general answer is possible, since it would depend upon the behavior of the indicators. The correct method is the one which expresses in its law of displacement the behavior of the indicator in a combined electrical and gravitational field. The choice of the indicator is, of course, arbitrary, since no rule can tell us what entities we should use for the realization of the process of displacement: the geometrical interpretation of electricity, therefore, presupposes a coordinative definition of the objects of the displacement. Only after a coordinative definition has been chosen, can we apply the judgements "true" or "false", since such judgements would concern only the question whether the objects which we have chosen for the displacement will satisfy the law which we have formulated for the $\Gamma_{\mu\nu}^{\tau}$. Since the choice of the coordinative definition is arbitrary, there are different methods of obtaining a geometrical interpretation of electricity. Between them there is no difference in truth value.

We shall illustrate this idea by actually carrying through one of these methods; in fact, it is the which the physicists have avoided so far.

§ 49. An Example of a Geometrical Interpretation of Electricity. [1)]

When we wish to choose an indicator for the process of displacement, we must select a physical phenomenon in which the gravitational and electric forces together produce a geometrical effect. Such a phenomenon occurs in the mechanical effect of an electrical field on charged mass points, since the mechanical force of electricity is superimposed upon the mechanical force of gravitation and the two forces together produce a motion, i.e., a geometrical effect. Consequently we shall choose the electrically charged mass point as the indicator of the field $\Gamma^{\tau}_{\mu\nu}$.

The well-known law concerning the mechanical force of the electrical field states that the force F which acts on a mass point of charge ρ is given by the equation

use German K — delete dot, write ρE

$$K = \rho \cdot E + \rho[v, H]. \qquad (1)$$

The first term symbolizes the effect of the electrical field E, while the second term symbolizes the effect of the magnetic field H, which vanishes in the case of a static point charge, but which occurs for a charge which moves with a velocity v, and is given by the vector product of v and H, multiplied by the charge ρ. In four-dimensional notation

use Latin K — close up, no dot

$$K^{\tau} = - f^{\tau}_{\nu}\, i^{\nu}$$

$$i^{\nu} = \rho \cdot u^{\nu} \qquad u^{\nu} = \frac{dx^{\nu}}{ds} \qquad (2)$$

where f^{τ}_{ν} or $f_{\mu\nu}$ as the case may be is the electrical field

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1) The content of this section, together with an abstract of the entire appendix was presented by the author to a regional meeting of the German Physical Soc. in Stuttgart, May 16, 1926. See Verhandl. d.d. Phys. Ges. 7, 1926, page 25.

composed in the usual way of the three-dimensional vectors E and H, i^{ν} is the electrical current, also called four-vectors and u^{ν} is the velocity vector. From (2) we can now obtain the law of motion of a charged mass point of mass 1, which according to the general theory of relativity is

$$\frac{du^{\tau}}{ds} = -\left\{ {\mu\nu \atop \tau} \right\} u^{\mu} u^{\nu} - f^{\tau}_{\nu}\, i^{\nu} \qquad (3)$$

On the left-hand side we have the product of mass and acceleration and on the right-hand side the sum of the forces.

We shall now have to define a process of displacement which represents this law (3) in the form of a vector displacement. This can be done, if we choose the <u>velocity vector</u> of a charged mass point as the object of the displacement.

With this choice we have decided to use the last of the above mentioned approaches, since the length of the vector is, by definition,

$$l^2 = g_{\mu\nu}\, u^{\mu} u^{\nu} = 1. \qquad (4)$$

The displacement of this vector, therefore, cannot not change its length. We must now define a displacement $\Gamma^{\tau}_{\mu\nu}$ for which $d(l^2) = 0$, but which is not symmetrical in the indices μ and ν. This means that we are imbedding electricity in a general metrical space, which, in its transfer of direction, is different from the general Riemannian space. Such a space may be obtained as follows.

Assume that a coordinate system is given in this space, together with a fundamental tensor $G_{\mu\nu}$ which combines the electrical and gravitational fields, in a manner suggested by Eddington, so that its symmetrical component $g_{\mu\nu}$ and its anti-symmetrical component $f_{\mu\nu}$ characterize the gravitational and electrical field respectively. We therefore put

$$G_{\mu\nu} = g_{\mu\nu} + f_{\mu\nu} \tag{5}$$

where the $g_{\mu\nu}$ and $f_{\mu\nu}$ are defined by the two equations

$$g_{\mu\nu} = 1/2(G_{\mu\nu} + G_{\nu\mu}) \qquad f_{\mu\nu} = 1/2(G_{\mu\nu} - G_{\nu\mu}). \tag{5a}$$

This combination of electrical field and gravitation is based on the mathematical fact that every tensor can be written as the sum of a symmetrical and an anti-symmetrical tensor, according to (5a). Expression (5) is therefore a mathematical abbreviation of two factual assertions:

1. There are two fundamental tensors.

2. One of them is symmetrical, the other is anti-symmetrical.

We cannot say, therefore, that the combination of electricity and gravitation, which is given by schema (5) is purely formal, since it could not have been carried out unless the second statement were also true. It is really nothing but a different mathematical formulation of these two assertions, which adds nothing to their content; it is merely equivalent to the two assertions.

Let the metric be defined by

$$ds^2 = G_{\mu\nu}\, dx_\mu\, dx_\nu = g_{\mu\nu}\, dx_\mu\, dx_\nu . \tag{6}$$

The physical objects which realize the invariant ds^2 are clocks and measuring rods of unit length. Such objects will therefore follow the process of adaptation and we may consequently use them as <u>indicators</u> of the field $g_{\mu\nu}$. Their behavior does not express the field $f_{\mu\nu}$ since the skew-symmetrical part of the fundamental tensor $G_{\mu\nu}$ vanishes in (6).

We may regard the $f_{\mu\nu}$ as being determined by the customary experimental methods of electro-magnetism; however, we shall describe a different method presently. In addition we assume Einstein's equations of gravitation for the $g_{\mu\nu}$, for instance in the form (5, section 40), but now we shall write "r" instead of "R" in order to indicate that r is a function of the $g_{\mu\nu}$ alone. We can therefore write

$$r_{\mu\nu} - 1/2\, g_{\mu\nu}\, r = -\varkappa\, T_{\mu\nu} \,. \tag{7a}$$

For the $f_{\mu\nu}$ we assume Maxwell's equations

$$\frac{\partial f_{\mu\nu}}{\partial x_\rho} + \frac{\partial f_{\nu\rho}}{\partial x_\mu} + \frac{\partial f_{\rho\mu}}{\partial x_\nu} = 0 \tag{7b}$$

$$\frac{\partial f^{\sigma\rho}}{\partial x_\rho} = i^\sigma \,. \tag{7c}$$

In the following we shall make use of (7c) only. The "raising" of the indices is performed in the usual way by means of the (inner) multiplication by the $g^{\mu\nu}$.

Let us now define the coefficients ~~mf~~ of the displacement by

$$\Gamma^\tau_{\mu\nu} = \gamma^\tau_{\mu\nu} + \varphi^\tau_{\mu\nu}$$

$$\gamma^\tau_{\mu\nu} = -\left\{ {\mu\nu \atop \tau} \right\} \qquad \varphi^\tau_{\mu\nu} = g_{\mu\sigma}\, f^\tau_\nu \frac{\partial f^{\sigma\rho}}{\partial x_\rho} \,. \tag{8}$$

The $\Gamma^\tau_{\mu\nu}$ are expressed in terms of the $g_{\mu\nu}$ and $f_{\mu\nu}$ alone, i.e., in terms of the $G_{\mu\nu}$ according to (5a). The process of displacement expresses, therefore, the field $g_{\mu\nu}$ as well as the field $f_{\mu\nu}$. The physical objects which satisfy this process of displacement are the velocity vectors u^τ of mass points of unit mass and arbitrary electrical charge. Such mass points are therefore <u>indicators</u> of the field $\Gamma^\tau_{\mu\nu}$ (and consequently also of the field $f_{\mu\nu}$).

We must now prove that the given expression represents the motion of the charged mass point in a combined gravitational and electrical field as formulated in (3). We therefore maintain that this motion can be expressed by the equation

$$du^\tau = \Gamma^\tau_{\mu\nu}\, u^\mu\, dx_\nu \tag{9}$$

where we define

$$u^\tau = \frac{dx_\tau}{ds} \tag{10}$$

For the proof we divide (9) by ds and write it as

$$\frac{du^\tau}{ds} = \Gamma^\tau_{\mu\nu}\, u^\mu u^\nu = \gamma^\tau_{\mu\nu}\, u^\mu u^\nu + \varphi^\tau_{\mu\nu}\, u^\mu u^\nu. \tag{11}$$

Due to the definition of $\gamma^\tau_{\mu\nu}$ given in (8), this agrees with (3) except for the last term. Therefore, we need to investigate only the last term, ~~but first of all~~ which we shall ~~re~~write ~~it~~ in a different form. Using the definition of $\varphi^\tau_{\mu\nu}$ as given in (8) we may write

$$\varphi^\tau_{\mu\nu}\, u^\mu u^\nu = -\, g_{\mu\sigma}\, f^\tau_\nu \frac{\partial f^{\sigma\rho}}{\partial x_\rho}\, u^\mu u^\nu .$$

From (7c) we have

$$\frac{\partial f^{\sigma\rho}}{\partial x_\rho} = i^\sigma = \rho . u^\sigma \tag{12}$$

where ρ is again the charge of the mass point. If we make this substitution and a suitable simplification by means of expression (4), we will now ~~get~~ obtain

$$\varphi^\tau_{\mu\nu}\, u^\mu u^\nu = -\, f^\tau_\nu\, i^\nu . \tag{13}$$

Thus (11) becomes identical with (3). ~~completing our proof.~~

What have we achieved by means of this development? We have expressed the combined electrical and gravitational effects as the geometry of a space-time manifold which is no longer Riemannian, but of a more general type. Gravitation and electricity determine a unified field $G_{\mu\nu}$. The metric of this space-time manifold is given by the ds^2 according to (6) and is realized by means of measuring rods, light-rays and clocks. These will therefore represent only the gravitational component of the field. The displacement is ~~given~~ represented by the $\Gamma^{\tau}_{\mu\nu}$, which are given by (8) as a function of the field $G_{\mu\nu}$; ~~and which~~ it is realized by the velocity vectors of unit mass points of arbitrary charge. It is an expression of the gravitational as well as the electrical field.

The type of space we have chosen will still belong to the class of metrical spaces, because we have used the actual [redacted] values of the $g_{\mu\nu}$. Consequently we have recognized measuring rods and clocks as metrical indicators. The displacement defines only a comparison of direction. Using velocity vectors as its realization, we define, in particular, the displacement of a vector along its own direction, i.e., we define the straightest line. Our law of displacement formulates therefore the basic law of motion: <u>the electrically charged mass point of unit mass moves along the straightest line</u>. Only if its charge is zero, or if there exists no external electrical field, will this straightest line also be the shortest.

This last statement requires some further explanation. In free space we have $\frac{\partial f^{\sigma\rho}}{\partial x_\rho} = 0$, according to (7c), and therefore also $\phi^{\tau}_{\mu\nu} = 0$ according to (8), and the displacement will have become identical with the corresponding operation in the Riemannian space. The same is true for an uncharged mass point, which would therefore describe the geodesic. Only a charged mass point produces such an additional field, due to its own charge, ~~such~~ so that the divergence $\frac{\partial f^{\sigma\rho}}{\partial x_\rho}$ for the entire electrical field $f_{\mu\nu}$ no longer vanishes, and the $\varphi^{\tau}_{\mu\nu}$ appear. A charged mass point will therefore engender its own displacement geometry, depending on the strength of its own charge.

One might object that ~~thus~~ the geometrical meaning of this interpretation becomes questionable, because a field $\Gamma^{\tau}_{\mu\nu}$ does not exist independently, since the value of $\Gamma^{\tau}_{\mu\nu}$ depends on the nature of the indicator. Conversely, however, we could also interpret this argument in support of Weyl's conception of perseverance, which would then obtain an even deeper significance. The independently existing field is the field $G_{\mu\nu}$ with its two components $g_{\mu\nu}$ and $f_{\mu\nu}$, whereas the coefficients of the displacement $\Gamma^{\tau}_{\mu\nu}$ represent a resultant of the field and the indicator. Straightest lines are therefore <u>perseverance lines</u>, which are not determined by the field alone. Only in the absence of electrical forces, i.e., when there are no $f_{\mu\nu}$ or the body is not charged is there a motion of pure adaptation.

There is a way, however, to avoid this peculiarity of our formulation, and to construct a diplacement field which is independent of the indicator. We can require, that in addition to its unit <u>mass</u>, the indicator must also have a certain unit <u>charge</u>. This would make the system of straightest lines dependent on the electrical field alone and the $\Gamma^{\tau}_{\mu\nu}$ become a field with independent existence. In this case, expression (9) would no longer comprehend the motion of the charged mass point as well as that of the uncharged one, since the displacement can now be applied only to charged mass points, whereas uncharged ones describe the shortest lines. Such a division is, of course, necessary, if we wish to construct a space which is independent of the indicator and ~~whose~~ the different geometrical elements of which must be realized by different indicators.

Instead of (8) we must now put

$$\Gamma^{\tau}_{\mu\nu} = \gamma^{\tau}_{\mu\nu} + \varphi^{\tau}_{\mu\nu}$$

$$\gamma^{\tau}_{\mu\nu} = -\left\{ {\mu\nu \atop \tau} \right\} \qquad \varphi^{\tau}_{\mu\nu} = -f^{\tau}_{\nu}\, u_{\mu} \tag{14}$$

where only the $\varphi^{\tau}_{\mu\nu}$ have been changed. It can easily be seen that equation (3) is now derivable from (9), if we remember that we have set $\rho = 1$.

Expression (14) will have to be explained in more detail. So far (also in section 47), we have considered the $\Gamma^{\tau}_{\mu\nu}$ as functions of the coordinates alone, i.e., the set of components $\Gamma^{\tau}_{\mu\nu}$ are determined for every point or point event as the case may be. Whereas the f^{τ}_{ν} are also a field given in such a way, this is not the case for ~~the~~ u_{μ} which appears as a variable in expression (14). u is the direction vector, because three-dimensional velocity becomes a direction in four-dimensional space; namely, the direction of the element of the world-line. Expression (14) will therefore make $\Gamma^{\tau}_{\mu\nu}$ a function of the displacement.

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The resulting mathematical complication disappears, if we substitute (14) into (9). Since $u_{\mu}\, u^{\mu} = 1$ this yields

$$du^\tau = -\left\{ {\mu\nu \atop \tau} \right\} u^\mu \, dx_\nu - f^\tau_\nu \, dx_\nu . \qquad (15)$$

Next to the Riemannian term which includes $\left\{ {\mu\nu \atop \tau} \right\}$ there appears a term which depends only on the position and not on the displacement, and whose coefficient has only two indices. The fact that expression (14) satisfies the condition $d\,(\ell^2) = 0$ can be verified through a substitution in (11, section 47), noting that $f_{\mu\nu} = -f_{\nu\mu}$ and putting $A^\nu = u^\nu$.

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If (14) is applied to some other vector A^τ and not to the vector u^τ, ~~then~~ the displacement is non-linear, since ~~Then the~~ dA^τ according to (1, section 47) is no longer a linear function of the dx_ν because u_ν contains the dx_ν. If, on the other hand, we assume (14) to be derived from an expression

$$\varphi^\tau_{\mu\nu} = -f^\tau_\nu A_\mu \qquad (16)$$

the displacement again is no longer a linear function of the A^μ, because, if we put $\gamma^\tau_{\mu\nu} = 0$, this yields

$$dA^\tau = -f^\tau_\nu A_\mu A^\mu dx_\nu = -f^\tau_\nu \ell^2 dx_\nu \qquad (17)$$

according to (1, section 47). The increment dA^τ is therefore proportional to the second power of the length of the vector. For vectors other than u^τ, our displacement procedure will therefore constitute a generalization of the basic assumptions developed on page .

Expression (14), together with (9), therefore represents the realization of a space in which the shortest lines and straightest lines are not identical, both being determined by the field of the $G_{\mu\nu}$ alone. Uncharged mass points realize the shortest lines; charged mass points realize the straightest. We have made the demand that in (14) the mass point must have a unit charge besides its unit mass, but it is evident that its motion will depend only on the ratio of its mass to its charge. The indicator of the straightest line, therefore, is merely required to have a definite unit ratio between mass and charge. The fact that we must specify a unit value at all is no more surprising than the fact that we had to make similar specifications of unit clocks and unit rods for the metric.

Let us remember that there is a natural unit for this ratio; namely, the ratio e/m between the charge and mass of the electron. However, this natural choice of the geometry is not unique because the corresponding ratio for the atom of positive electricity, the nucleus of the hydrogen atom, has a different value. Therefore, there are two natural geometries, depending upon whether we let positive or negative electricity describe the straightest line.

§ 50. The Epistemological Value of a Geometrical Interpretation of Electricity.

In the preceding section we have carried through a complete geometrical interpretation of electricity. What is the physical significance of this theory?

Actually, it adds no content to the physical assertions of Einstein's theory. Expression (8), developed for the $\Gamma^{\tau}_{\mu\nu}$, says nothing new, but leads to the well-known law of the mechanical force of an electric field (3), as the calculations have shown. The operation of displacement therefore embodies nothing but a geometrical presentation of this law, i.e., a visualization, not a new physical idea.

Consequently we might call the geometrical interpretation of electricity a graphical representation and distinguish it in this manner from the geometrical interpretation of gravitation.[1] Yet this distinction is due to a misconception of the nature of a geometrical interpretation.

In section 15 we ~~made a~~ examined in detail~~ed analysis~~ the significance of graphical representations, according to ~~which~~ this analysis they stand for the coordination of a system b of physical objects to the system a of rigid bodies. Such a coordination is possible because the two systems a and b satisfy the same relations A. Although the system a of rigid bodies is the normal system for the graphical representation, it

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1) Eddington, op. cit, ~~p. 312~~, makes this distinction, but applies it only to Weyl's theory.

is not the normal system for geometry in general. Instead, we would have to choose the four-dimensional system α of clocks and rigid bodies as the normal system for geometry. This system α can also be represented graphically: in which case it is coordinated to the system a (with the separation of one spatial ~~diemenion~~ dimension) just as the graphical representation of the indefinite metric was given in Fig. 32 (page 294). This is not necessary, however, since we have direct visual experience of the four-dimensional geometry α, in which we perceive space and time in terms of their distinct sensible qualities, without resorting to graphical representations. We may therefore accept α itself as a normal system which can be visualized directly.

Why, then, is the geometrical interpretation of gravitation not a graphical representation? The answer is that it contains assertions about the system α itself, not about the coordination of some other system to α. Of course, if we want to identify the concept "geometrical interpretation" with "reduction to the system a of rigid bodies", then only a graphical representation is possible even for gravitation. Since Einstein's theory of gravitation necessarily contains assertions about clocks, it refers to α and not to a; therefore, the geometrical interpretation could be achieved only in that kind of a graphical representation which is given by the indefinite metric of Minkowski (section 29). But this restriction is not necessary. Let us understand by "geometrical interpretation" the same as "reduction to the system α of space-time measuring instruments".

This reduction can be a "four-dimensional graphical representation" if a system is coordinated to the normal system α; it can also be a "genuine geometrical interpretation" if it contains direct assertions about α itself. Einstein's theory of gravitation is therefore not a graphical representation, since it contains assertions about α itself and not about a system β of other objects which is equivalent to α.

What about the geometrical interpretation of electricity? We have to include the moving mass points within the system α; this must be done because measuring rods define only a congruence and not a displacement of direction and therefore by themselves are not sufficient for the physical representation of the displacement space. Thus our geometrical interpretation of electricity is not a graphical representation, but a genuine geometrical interpretation, ~~just like~~ as is the theory of gravitation.

Our geometrical interpretation is therefore no "coarse" than the geometrical interpretation of gravitation. Above we objected to the geometrical interpretation of electricity (as has frequently been done in the literature) on the grounds that it lacks a basic physical fact analogous to the principle of equivalence. This objection is valid only if we limit the measuring instruments to measuring rods and clocks - it does not, therefore, apply to our theory. While the principle of equivalence asserts a relation between gravitation and the behavior of measuring rods and clocks, we now use a relation between electricity and other measuring instruments, namely, charged mass points. Furthermore, these measuring instruments are chosen so that they will react simultaneously to both the electrical and gravitational field. They are, therefore, precisely the measuring instruments which we need in order to find a geometrical expression of this combined field. What is new in this geometrical interpretation of electricity is merely the fact that the elec~~el~~trical fields are included within the metrical forces. Our interpretation accomplishes, therefore, something similar to the representation of a temperature field by means of the geometry of the rods which are placed in this field. Whereas different substances would supply us in this case with different geometries, our world geometry was chosen wide enough to express, within a single geometry, the corresponding difference in the behavior of ~~both~~ charged and uncharged unit mass points. By means of their motion, the first represent the straightest line, the second the shortest. The geometrical interpretation of electricity is therefore made possible by a suitable extension of the fundamental geometry, such that the differences in the measuring instruments due to their physical properties can be interpreted again as differences in the basic geometrical forms.

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It should be noted that according to (8, section 49) only unit mass points realize the displacement geometry. Expression (14, section 49) which prescribes only the unit <u>ratio</u> of charge and mass is therefore more advantageous than (9, section 49) even though we obtain different natural geometries for the positive and the negative charge. It is, indeed, of extraordinary significance that this procdeure yields only <u>two</u> natural geometries. The fundamental idea here may, incidentally, be taken as an analogy to Einstein's principle of equivalence, since the equality of gravitational and inertial mass originally represents only a proportionality (i.e., if the two masses are proportional, their equality can always be established by definition through a suitable choice of the units of measurement). The corresponding proportionality between the charge and mass of the electron or proton lead, in (14, section 49), to one or the other of the two natural geometries, as the case may be. The fact that the difference between positive and negative electricity produces a bifurcation expresses geometrically the asymmetry, lacking with respect to mass, between the two kinds of electricity. It would be too much, to expect that there should be only one choice of a natural geometry. Let us remember that there are similar divisions of geometry in Einstein's theory of gravitation, depending on whether light, four-dimensional measuring instruments, or three-dimensional rods are regarded as indicators of the geometry (see pages 412-418).

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Although the presentation developed above has provided a complete geometrical interpretation of the two fields, we must recognize that the geometrical interpretation of electricity constitutes no more advance in physical knowledge than the geometrical interpretation of gravitation. This is because we could have rewritten Einstein's original theory of ~~gravitation~~ relativity, without changing its physical content, so that electricity as well would have acquired a geometrical interpretation. Rewriting the theory in this fashion would tell us nothing about reality that we did not know before.

This fact, as trivial as it may appear, is nevertheless of great significance. Physical theory so far, especially under the influence of Weyl, has always defended the idea that the geometrical interpretation of electricity constitutes something which is physically essential. We find that this is not the case.

Is it not true, though, that the geometrical interpretation of gravitation has brought about an advance of physics? It has *brought* it about, yes, but it is not *identical* with this progress. It has led, in its effects, to a physical discovery, but in itself it *is* not this discovery. Two things which are asserted by Einstein's theory of gravitation are physically new.

In the first place, the relation between measuring instruments and gravitation was not previously known; this relations was therefore asserted by Einstein as a new fact. Due to the chosen realization of the process of displacement, the corresponding relation in our geometrical interpretation of electricity is a well-known fact. The geometrical interpretation has therefore acted as a heuristic principle in the theory of gravitation.

In the second place, the geometrical interpretation of gravitation has led to changes in the physical theory even in domains where its relation to clocks and measuring rods is of no significance, e.g., in planetary motion. Einstein describes the gravitational field by different and more precise equations than Newton. The geometrical interpretation has acted again as a heuristic principle. Unfortunately this is not the case with our geometrical interpretation of electricity, since the new theory has remained identical with the old.

We must therefore recognize that the geometrical interpretation of gravitation has attained its important position in the historical development of science, because it has led to new physical insights. The geometrical interpretation itself is merely a formulation, a visualization of these new insights. What we have attained with our geometrical interpretation of electricity is an analogous formulation of physical insights regarding electricity, but these insights are not ~~physically~~ new.

As long as the geometrical interpretation of electricity does not act as a heuristic principle, its sole value will lie in the visualization it provides. Since such a heuristic principle is what physicists who have worked in this field have hoped to attain, we shall now investigate this question in more detail. For this reason, Einstein, in particular, has devised several new formulations in which the geometrical interpretation is reduced to the role of a mathematical tool. There is a definite problem for which a solution is sought through this approach; namely, the problem of the electron.

Until recently the electron has been viewed as a particle charged with electricity, which is no different from a pith sphere, for example, which has been brought into contact with the pole of an induction machine. The field theory, on the other hand, conceives the elementary structure of matter in an essentially different way. It is considered a region of the field, wherein the laws of the field apply just as they do in free space. The only difference is that the field magnitudes are to be distributed so that a negative electrical charge can maintain itself in a concentrated form. One might imagine, for example, that gravitational forces acting cohesively press the negative charge together.

But how can such a structure be derived from the field theory? Maxwell's equations may be valid in a part of the space in which there is a charge, but they cannot explain why such a charge does not fly apart in all directions, in accordance with Coulomb's force of repulsion. According to Maxwell's theory, the cohesion of the charge has always been attributed to a "foreign" force; namely, the force of cohesion in the material body of the electron. The equations of gravitation, on the other hand, in their present form have no effect on the charge and cannot, therefore, yield the cohesive force. They describe only the state of equilibrium between matter and the gravitational field, without consideration of the electrical state. If this state is to be included in the equilibrium, the field equations would have to state a connection between the $g_{\mu\nu}$ and the $f_{\mu\nu}$. The resulting differential equation would have to have a solution corresponding to the electron, and would have to show the discrete nature of the electron as a mathematical necessity.

It is not difficult to combine the three equations of (7, section 49) into one by means of the action-principle, but the resulting equation will not yet give us the desired solution. Rather the amalgamation of the three equations (7, section 49) would have to incorporate <u>slight changes</u> in them. Only slight, however, because we know that (they) do apply to a high degree of approximation; yet changed, because otherwise they would never give us the electron as a solution. We have to "guess" therefore what kind of change has to be performed.

In this "guessing" the geometrical interpretation of electricity is supposed to be the guide. The point of departure in this approach is the (unwritten) assumption that whatever looks <u>simple</u> and <u>natural</u> from the viewpoint of the geometrical interpretation will lead to the desired changes in the equations of the field. The physicist needs a kind of instinct in his research, a feeling for the hidden paths of nature, and he believes that the adaptation of his concepts to the geometrical interpretation of electricity will direct him toward the desired field equations. Epistemologically speaking, there is no objection to such an approach, which is a <u>working hypothesis</u> logically speaking. Only its <u>success</u> can decide its correctness, for it is purely a <u>matter of experience</u>, whether the way to a simple and natural geometry also leads to an approximation to reality.[1)] It is noteworthy that contemporary discussions of these problems are filled with concepts like "most natural assumption", "simplest invariant", etc.; this tells us that we are dealing with virgin territory in the field of physics which cannot yet be developed systematically

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1) Weyl, in particular, through his concept of "Gauge-invariance", has developed a method for narrowing the choice among the available equations. This is, indeed, a rigorously formulated principle which is more than a mere guide to geometrical feeling. Whether this principle is correct is, of course, a purely empirical question.

and which is not yet ripe for an epistemological evaluation. It is only the method employed that arouses epistemological interest.

Another important consideration must be added at this point. So far, we have always ~~made the point~~ maintained that a simple physical realization must be given for the process of displacement. In our example we ourselves gave such a realization and we obtained, in this way, an actual geometrical interpretation of electricity. Attempts which were made by Weyl, Eddington and Einstein, on the other hand, renounced such a realization of the process of displacement. It is generally believed that such "tangible" realizations do not lead to the desired field equations. Consequently the problem of realization is left open for the time being.

This point of view can be justified logically as follows: Expression (8, section 49) defines a process of displacement as a function of $g_{\mu\nu}$ and $f_{\mu\nu}$. This definition is such that the velocity vectors of charged mass points provide us with the objects of the displacement. Where we to choose a different equation, the objects of the displacement would be vectors other than the velocity vectors. For arbitrarily defined $\Gamma^{\tau}_{\mu\nu}$, we can ask, what vector is moved, due to the motion of the charged mass point, so that it will realize the displacement? Perhaps this will give us a complicated vector which is a function of the velocity vector, the electrical, and the gravitational field; but some answer will be found. Nor is it necessary to limit ourselves to mass points in motion. It might be possible, for example, to define an entity like the magnetic flow through a closed electrical circuit, which would realize the displacement. It is afterall always possible to find some such entity if we do not require that such a combination of known entities has previously been given a particular name.

Logically speaking, there are no objections to such a method. It is based on the fundamental principle that the geometrical interpretation of electricity always rests on an arbitrary coordination; namely, the coordinative definition of an object of displacement; and there is no logical necessity to prefer one particular coordination. There simply are <u>several</u> geometrical interpretations of electricity. From the viewpoint of simplicity and naturalness, a lot can be said <u>against</u> this approach, and it is quite noticeable that this argument, in particular, is employed in the physical discussions objecting to the attempt to formulate a unified field theory. It would be pointless to enter this discussion from a philosophical point of view. An epistemological analysis can tell us only whether a chosen method is <u>permissible</u> or not. Only the physical instinct, whose content lies completely outside the realm of epistemological criticism, can judge, for the time being, whether it will lead us to the physical goal we have described.

If the goal of a field-theoretical interpretation of the electron were attained, then conversely, the realization defined by the expression used, in this case for the $\Gamma^{\tau}_{\mu\nu}$, would be singled out as the "most natural" geometrical interpretation of electricity. As long as this aim has not been reached, less remote realizations will be preferred, among which, the one given in section 49 might very well be the "most natural". Its existence proves, in any case, that a geometrical interpretation of electricity can be carried out, although our formulation will hardly lead to the desired explanation of the electron, even if it is changed so that it is no longer identical with Maxwell's theory.

The final decision regarding this new physical territory must be left to the physicist, whose physical instinct provides the sole illumination. Our epistemological discussion was given merely to show that the geometrical interpretation of electricity, as such, has no physical meaning as yet; i.e., it is nothing but a form for descri~~bing~~ption of physical results. It is, therefore, meaningless to ask for the "correct" geometrical interpretation of electricity. Every consistent interpretation is equally admissible, since the coordination between electricity and geometry is not an assertion ~~having~~ which has the character of physical knowledge. This epistemological insight might very well be helpful to the physicist, by showing him the limitations of his method and making it easier for him to free himself from the enchantment of a unified field theory. The many ruins along this road urgently suggest that solutions should be sought in an entirely different direction. It is not the geometrical interpretation of electricity, but an assumption of an entirely different character, which is fundamental to all these attempts; namely, the assumption that the road to a simple conception, in the sense of a geometrical interpretation, is also the road to true relationships in nature. It is this assumption which constitutes the <u>physical hypothesis</u> contained in these attempts. The geometrical interpretation of electrcity can be carried through in any case, but it by no means follows that this added hypothesis must also be correct.

Appendix

Weyl's Extension of the Riemannian Concept of Space and the Geometrical Interpretation of Electricity.

§ 46. Formulation of the Problem.

In the following we shall give an exposition of Weyl's extension of Riemannian space, since this area of investigation has attained considerable significance for further elaboration of the theory of relativity. The presentation will differ from the previous part of this book inasmuch as it will contain a more detailed mathematical ~~formalism~~ development than was necessary in the other chapters. This is required because of the novelty of the material, which, ~~up to now~~ so far, has hardly been treated in any of the comprehensive presentations of the theory of relativity. At the same time it is necessary, for an epistemological analysis, to present the mathematical construction in such a way that its logical form becomes evident. If the reader is not sufficiently familiar with the formalism of the general theory of relativity to follow our line of thought, he may omit this appendix which presents nothing new, philosophically speaking, but merely constitutes an application of the epistemological principles developed earlier in this book. This negative philosophical result, however, is so important that we deem it necessary to present what follows in a precise form.

The basis of Riemannian space is the definition of congruence; its applicability in physics rests on the fact that there are physical objects, i.e., clocks and measuring rods, which make it possible to carry through such a definition. Why are these measuring instruments adequate for this purpose? The fundamental property which makes them suitable for a definition of congruence has already been formulated as an axiom. Two measuring rods which are of equal length when lying next to each other are always found to be equally long after having been transported along different paths to a distant place. The same holds for the time units ~~for~~ of clocks. We have stated ~~before~~ above that this property does not compel us to assert that the measuring rods are of equal length when they are located at different places, and we have emphasized that such a statement can be made only in the sense of a definition. We have also pointed out that such a definition is possible only because the measuring ~~rods~~ instruments have this peculiar property. If the measuring rods were always of different length when brought back together again, it would not be possible to carry through the usual definition of congruence. It is true that this property does not constitute the empirical evidence that the measuring rods are of equal length when separated - for no such evidence could possibly exist - yet it is a necessary condition for a suitable definition of congruence.(see also section 4).

What could we do if the measuring instruments did not have this special property? To answer this question let us limit ourselves to three-dimensional space, since we can always extend our considerations to four dimensions, ~~even if~~ regardless whether or not the fourth dimension is to be interpreted as time. Even in the absence of the above mentioned special property it would still be possible to formulate a definition of congruence. We could select any measuring instrument, move it around in space, and define congruence in terms of it. It does not matter that we would obtain different congruences depending upon the path along which the measuring instrument is moved,since definitions are arbitrary. We could introduce certain rules concerning the manner of transport; for example, we could prescribe transportation from a chosen center, along certain definite paths, at definite speeds. This definition of congruence would be coordinated to physical objects in a more complicated manner and would contain a greater number of arbitrary prescriptions. It is due to these additional rules that the definition would lose its preferential status, since we could just as well have chosen different velocities and different paths. The metric of this space would, of course, be definable, but it would not express a law of nature and it would therefore lose ~~all of~~ its physical significance.

This consideration is very important. We prefer the metric of rigid rods and clocks because this metric expresses a law of nature; namely, the law which concerns the transportations of measuring instruments. This is its practical import. All metrical statements are therefore statements concerning the transportation of measuring instruments and consequently have useful physical applications. If we know the length of a room, we also know how many chairs and tables, of a given size, we can place along one of its walls, and it is therefore of practical value to know the length of the room. In the previous example, we would not know this. For instance, whether we could place six chairs next to one another along one wall would depend upon the path by which the chairs were actually brought into the room, and we might have to take our chairs around the world in order to fit them in. Similarly it is questionable whether a guest would fit a chair, since this also would depend on his previous path. Such conditions may seem very strange, but they are certainly possible; and if they were real, we would surely have adapted ourselves to them.

Obviously, the arbitrarily defined metric would be of little use for the people in such a world. Instead, they would look for a geometrical method which would characterize the law of change in length during transport -- the <u>law of displacement</u>. Thus there arises the geometrical problem of formulating a law of displacement applicable to such a case and independent of any metric.

This problem was solved by Weyl[1)] and his solution certainly constitutes a mathematical achievement of extraordinary significance regardless of its physical applicability. Mathematics concerns itself mainly with the generalization of our conceptual knowledge; Weyl has discovered a type of space more general than the Riemannian space, in which the displacement rather than

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1) H. Weyl, "Gravitation and Elektrizität, <u>Berliner Akad. Ber.</u> 1918, p. 465; also <u>Raum-Zeit-Materie</u>, 3rd ed. 1919, paragraph 34.

the metric represents the most fundamental operation. Weyl's extension of geometry has carried the mathematical treatment of the problem of space considerably beyond the developments of Riemann and Gauss. We shall now outline Weyl's approach, giving it in its complete generality as developed by Eddington[1] and Schouten[2].

Again we shall begin the presentation from a purely mathematical point of view, without asking about its physical realization. This means that we shall first have to define the displacement operation. The only empirical question - a question of physics - is whether or not there are objects which behave according to the operation which we have defined. In view of later applications, we shall keep the mathematical exposition as general as possible, so that it will be adaptable to physical requirements. In the following we shall not only formulate definitions, but we shall also investigate the problem of the consistency of the defined space, i.e., the presentation will be entirely mathematical in the usual sense.

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1) A. S. Eddington, "A Generalization of Weyl's Theory of Electromagnetism and Gravitational Fields", Proc. Royal Soc. London, A, Vol. 99, 1921, p. 104.

2) I. A Schouten, "Ueber die verschiedenen Arten der Uebertragung in einer n-dimensionalen Mannigfaltigkeit, die einer Differentialgeometrie zugrunde gelegt werden können." Math. Zeitschr. 13, 1922, p. 56.

§ 47. Displacement Space and Metrical Space.

Let us suppose that a coordinate system is given for an entire space and that a vector which is given by its components

greek tau

A^τ, is placed at a point P of this space. We may think of it as an arrow or we may conceive it as the sum total of its components. Let us now move to a neighboring point P', whose distance from P is given by the coordinate differential dx_ν and place a vector there which we call A'^τ. Now we want to say that the vector A^τ, displaced to the point P', is given exactly by this vector A'^τ; i.e., ~~We~~ we want to specify a method of coordination by means of which the vector A'^τ at P' is coordinated to the vector A^τ at P. <u>This coordination is the definition of our process of displacement.</u>

greek mu

Every component A^τ changes in the displacement, by the amount

$$dA^\tau = A'^\tau - A^\tau$$

Let us now look for a simple relation governing the dA^τ. In order to do this, let us assume that dA^τ is a homogeneous linear function of the components A^μ as well as of the displacements dx_ν. The relation is therefore

greek mu

capital greek gamma

$$dA^\tau = \Gamma^\tau_{\mu\nu} A^\mu dx_\nu \qquad (1)$$

This is a sum of several elements, each of which consists of a product of a vector component and a displacement for any arbitrary combination of indices, and contains a numerical coefficient Γ. This numerical coefficient is different for each combination of the indices μ and ν, and is therefore written with the two lower indices μ and ν. Its upper index is τ, since every set of these coefficients differs depending ~~of~~ on the ~~amount~~ increase dA^τ to which it is related by (1). This gives us therefore a quantity $\Gamma^\tau_{\mu\nu}$ which has three indices.

Why did we choose the particular relation (1)? Being a definition, it is of course arbitrary, but mathematically speaking, it has some very definite advantages. It is very general, since it makes the change of the one component dA^{τ} dependent on all of the components A^{μ}, and all of the displacements dx_{ν}. Yet it is very simple because it assumes this dependence to be linear. The linearity of this relation agrees with a basic principle which has proved useful; namely, the principle of introducing only those generalizations of concepts which behave "normally" in the infinitesimal, i.e., which satisfy the conditions of differentiability. An arbitrarily curved line, for example, which has a tangent at every point, provides us with this kind of generalization of the straight line, since such a curved line behaves in the infinitesimal exactly like a straight line.

In order to ~~make (1) as~~ generalize (1) as much as possible, we shall have to ~~make a~~ modif~~ication.~~y it. With expression (1), we have defined the operation of displacement merely for a single point in space. The coefficients $\Gamma^{\tau}_{\mu\nu}$ need not necessarily be the same at different points. We express this by considering the $\Gamma^{\tau}_{\mu\nu}$ as functions of the coordinates. Since we add the condition that these functions must be continuous, we have preserved the necessary continuity of the operation of displacement.

Having generalized the $\Gamma^{\tau}_{\mu\nu}$ to functions, we can now define the displacement along a definite path by

$$A'^{\tau} = A^{\tau} + \int \Gamma^{\tau}_{\alpha\nu} A^{\alpha} dx_{\nu} \qquad (2)$$

The vector A'^{τ} which is located at P', at a definite distance from P, is coordinated to the vector A^{τ} at P. The integral in (2) will depend on whatever path we choose between P and P' and we will therefore in general get a different vector A'^{τ} if we choose a different path.

It is exactly this dependence on the path, which justifies the name "displacement" for the defined operation. Transportation is, of course, a physical process; in mathematics its place is taken by coordination. The vector A'^{τ} at P' is coordinated to the vector A^{τ} at P. Mathematically speaking it is meaningless to think of this coordination as a transportation of the vector A^{τ}. Instead, we find a certain peculiarity in the coordination: given the vector A^{τ}, the point P and the point P', no vector is yet determined at P' unless we specify, in addition, a definite path between the points P and P'. A vector A'^{τ} is coordinated uniquely only to the combination of the four elements: the vector A^{τ}, the point P, the point P', and the path PP'. This peculiarity of the coordination relation makes it possible to apply this operation to the physical process of displacement, since the logical structure of displacement is of the same type as that of coordination. It has become customary to use this physical process in order to interpret the mathematical formulation and it is common to speak, not of a coordination, but of a displacement or a transfer. It is evident, however, that we are merely using a physical picture for the characterization of a mathematical relation.

Accordingly, we may conceive the mathematical introduction of the process of displacement in the following way. Given a system of coordinates, we may give arbitrary functions $\Gamma^{\tau}_{\mu\nu}$ and use them to define, according to (1), a certain operation of displacement. This definition does not make use of a metric. The comparison of lengths by means of a metric is now replaced by a comparison of lengths through displacement.

The defined displacement actually accomplishes more than a mere comparison of lengths. It also defines a comparison of direction, since it coordinates to a vector at P only a single vector at P'. The relation of direction which is defined in (1) is commonly referred to as parallelism. We must understand, of course, that this concept of parallelism is not identical with that of Euclidean geometry. It has a much wider meaning. In common with the parallelism of Euclidean geometry it defines a comparison of direction. When applied to the special case of Euclidean space, this general comparison of direction becomes Euclidean parallelism. This means that it is related to this narrower concept of parallelism just as the vector sum is related to the algebraic sum. We should not let the use of the name "parallelism" for the more general concept lead us to believe that it means the same as the narrower concept. This would, of course, be impossible.

The claim that the displacement defines a comparison of length requires some further elucidation. If we are given the components of a vector, we do not yet know its length. Its length would be a single number which would have to be calculated from the numerical values of the components. This is generally done by means of the rule

$$l^2 = g_{\mu\nu} A^{\mu} A^{\nu} \tag{3}$$

which therefore requires knowledge of the $g_{\mu\nu}$. The displacement is defined without reference to the $g_{\mu\nu}$ and is consequently determined even in a space in which the metric has not been defined. Hence, when we call the displacement a "comparison of length", this must be understood as follows: the displacement defines what is meant by two vectors having equal length, but it does not define the magnitude of this length.

The same can be said regarding the comparison of direction. The displacement defines only the equality of direction, but not a measure of ~~direction~~ the direction. This has the consequence that the displacement is unable to provide an order of unequal lengths and directions. It can only assert an inequality, without informing us concerning the relations of smaller and larger. It can make only negative statements regarding vectors which do not correspond.

Even these negative statements are not exhaustive, however, since the coordination established by the displacement deals only with the combination of the equality of length and direction. If we symbolize this complex relation, which Weyl calls congruence, by "$\cong$", the displacement will then read

$$A'^{\tau} \underset{s}{\cong} A^{\tau} \tag{4a}$$

or in words: A'^{τ} is congruent to A^{τ} for the path s. The negation of this relation, which exists between non-corresponding vectors, reads

$$B'^{\tau} \underset{s}{\overset{\downarrow}{\not\equiv}} A^{\tau} \qquad (4b)$$

which expresses merely that B'^{τ} is not simultaneously equal to A^{τ} with respect to both length and direction, for the path s. It remains an open question whether B'^{τ} does satisfy at least one of the two partial relations, i.e., whether it is equal to A^{τ} at least with respect to length or direction. In regard to this the displacement tells us nothing. The assertion of equality through the displacement is therefore not complete with respect to length or direction. There may be vectors B'^{τ} at P' which are equal in length to A^{τ}, yet which do not coincide with A'^{τ}.

This implies a far-reaching limitation of the comparison of length and direction given by means of the displacement. Without an additional operation it is impossible to resolve the complex relation which we have called congruence, and it is for this reason that we shall need a metric in addition to the operation of displacement.

How can we now introduce a metric in addition to the displacement and avoid the possibility that the two operations might lead to contradictory results in the case where they both yield statements of the same type? Generally speaking we cannot do this, because the comparison of length which is given by the metric is not restricted to any definite path. Here something has already been coordinated to the combination of the three elements, the vector A^{τ}, the point P, and the point P'; namely, the element "class of vectors at P' which are equal in length to the vector A^{τ}". Whereas, according to the operation of displacement, this combination would still correspond to the entire manifold of vectors at P'; according to the metric, this combination corresponds to a narrower class of vectors only. In general, we cannot avoid this contradiction. Two vectors which are located at different points and which are equal in the sense of the metric, may yet be unequal according to the displacement, depending on ~~what~~ the path which is chosen between the two points.

The most general method of avoiding this difficulty would be to consider the two operations, the metric and the displacement, as two mutually independent basic operations. They would then present us with two fundamental geometrical processes of comparison which would have nothing to do with each other. One of them supplies us with a comparison of length which is independent of the path and may also be used for the comparison of unequal lengths, due to the fact that it supplies us with a numerical measure. The other presents a comparison of length which depends on the path, allowing only assertions concerning the equality of length, but which at the same time gives a similar type of comparison of direction. To assert the independent existence of these two operations is far from meaningless. It is not difficult to think of cases, especially in physics, where such a duality has to be chosen; i.e., where each of the two operations has its own physical realization, and these two physical processes are independent of one another.

A certain limitation of this most general approach results from introducing a dependence between these two basic operations. We might, for example, express one of the two systems as a function of the other. Physically speaking, this would imply a dependence, according to a law of nature, between the physical processes represented by the two operations. Geometrically speaking, however, there remains the duality of using two operations which lead to certain contradictory statements. If we speak, in this case, of a comparison of length, we would always have to specify to which of the two operations the comparison refers.

Although this situation is logically permissible, it is geometrically unsatisfactory. It seems reasonable to ask whether it is not possible to achieve an agreement between the two operations such that the assertions which they make in common would no longer be contradictory. Although we cannot say that such a geometry must be applicable to reality, the problem still has purely geometrical interest. In any case, such a geometry provides an even more general geometrical frame than Riemannian geometry for the description of reality. In the following it will be shown how such a unified geometry can be constructed by means of a displacement and a metric.

Two methods are available to us. Either we may limit the scope of the metric, so that it will no longer refer to statements which result from use of the displacement, or we may limit the displacement so that statements common to the displacement and the metric no longer contradict one another. We shall call the type of space which is obtained by the first method a *displacement space*, because in it, the displacement is the dominant principle to which the metric will have to be adapted. The type of space which results from the second approach will be called a *metrical space*, because here the metric dominates and the displacement is subordinate to it.

Let us start with the first approach. We shall construct a displacement space in which, for any two vectors, there is a comparison of length which is dependent on the path. This comparison will be based mainly on the operation of displacement. What task remains for the metric?

A single task will now be assigned to the metric: namely, the task of providing the *comparison of length at one point*. We have already seen that the displacement does not permit a separation of the comparison of ~~length from the comparison of direction~~ length from the complex relation of congruence, and it is for this reason that we now introduce the metric. It defines the equality of the length of differently directed vectors, if the vectors are located at the same point. Furthermore, it orders these vectors with respect to their magnitude, and provides a measure of the relative length of two vectors.

We also need a corresponding rule for the comparison of direction at one point, in order to compare vectors which have the same direction but differ in length. Such a rule will not require the introduction of a new field. We may simply lay down the definition: Vectors are equally directed at the same point

greek Kappa

if the components of one result from those of the other through multiplication by a factor κ. We may, on the other hand, consider this rule to be implicitly contained in the concept of a metric, since we can always define an angular measurement at any point in terms of measurements of length by means of trigonometric functions. If this is done, we will find that the equalities of direction are the same for any choice of the $g_{\mu\nu}$. The numerical measure of the angles, however, will only be determined for a definite set of $g_{\mu\nu}$.

In assigning to the metric the comparison of length at a point, we added to the comparison through displacement that extension without which a geometry would be impossible. We must ask, however, whether this stipulation implies the possibility of contradictions with the comparison which has previously been defined by displacement.

This actually does happen, because the comparison of lengths by means of displacement, in spite of its incompleteness, tells us something concerning lengths at the same point. We may perform the displacement along a closed path obtaining vector B^τ at P which is congruent to A^τ, but which is not equal in length to the vector A^τ in the sense of the metric. In order to exclude the possibility of such a contradiction, we shall introduce the rule that the comparison by displacement may not be used for a closed path.

At the same time, another consideration suggests this procedure. The comparison by displacement is not a reflexive relation, since it generally leads to

$$A^{\tau} \underset{s}{\not\equiv} A^{\tau} \qquad (5)$$

and a vector would therefore not be equal to itself in respect to length and direction. Logically speaking, this is, of course, not contradictory, since the comparison by displacement does not assert the identity, but merely the coordination, of vectors. If we were to permit relation (5) to stand, however, this would make our geometry so complicated that it is better to eliminate this possibility. Actually this difficulty occurs only if the displacement is used for closed paths, since it is only in this case that the relation of ~~in~~non-congruence, as defined by the displacement in (4b), can lead back to the original vector. The negative statement (4b) is limited to vectors B'^{τ} at P' for a given vector A^{τ} and a given path s, and says nothing in regard to vectors at any other point, not even that they are ~~in~~non-congruent. We shall therefore exclude closed paths ~~for~~ from the comparison by displacement. This is accomplished by means of the definition which states that the comparison of length and direction at a point is not to be performed by displacement.

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This rule is used de-facto in an "unbalanced" type of space, in which the metric and the displacement are left independent of each other, We say in this case that the vector has changed its length if it returns to the same point along a closed path. This avoids relation (5), which we would otherwise have been forced to use.

We can also formulate this requirement differently. In regard to the comparison of length at one point - and only for this - there exists a special path; namely, the path which touches no other point, i.e., the vanishing path. In this special case of comparison of length at one point a path of comparison, namely, this vanishing path, may be prescribed. Consequently every vector is coordinated only to itself by this displacement. Thus, it is impossible to make any statement by means of the displacement regarding the comparison of length of different vectors at the same point. - It should be noted that the use of this special path is not based on the concept of the shortest path, since such a concept would already presuppose a metric. The concept of the vanishing path is a purely topological one, which may be formed on the basis of a given coordinate system alone.

In that case our rule would be: the comparison of length at one point is to be obtained by rotation and not by displacement. A comparison by rotation, however, requires a metric.[1] This

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1) Another possibility would be to formulate a special law of rotation, which would be similar to the law of displacement, and would be based on the assumption that the vector would no longer have the same length after a rotation. However, this method, too, would make the equality of length a non-reflexive relation and it will therefore not be pursued any further. Helmholtz formulated the assertion that rigid bodies are congruent to themselves after a rotation as the principle of Monodromy. See also Helmholtz, Schriften zur Erkenntnistheorie, edited by Hertz and Schlick, Berlin 1921, pages 42 and 62.

divides the task between the metric and the displacement as follows. The metric is employed for the comparison of length and direction at one point, while the displacement is used for the comparison of length and direction at separate points.

If we now want to base a consistent displacement space upon this division of labor we must deprive the metric of one of its functions because, in general, the metric also defines a comparison of length at different points. Thus the possibility of a contradiction with the comparison by displacement arises. This idea may be expressed as follows. We lay down a metric $g_{\mu\nu}$, but make only statements which do not change when the $g_{\mu\nu}$ are multiplied by a scalar factor $\lambda(x_1 \ldots x_n)$. This implies that we may not use the $g_{\mu\nu}$ for the comparison of length at different points, because two vectors which are located at different points and which are equal in length according to the metric $g_{\mu\nu}$ would become unequal if we were to multiply the $g_{\mu\nu}$ by the factor $\lambda(x_1 \ldots x_n)$.

greek lambda

We can also express this idea mathematically as follows. We no longer specify all of the functions $g_{\mu\nu}$; instead, we give only their ratios $q_{\mu\nu}$, which are one less in number than the functions. These may be obtained by choosing at random one of the functions $g_{\mu\nu}$ and dividing all others by it. The $q_{\mu\nu}$ are therefore defined, for example, as

$$q_{\mu\nu} = \frac{g_{\mu\nu}}{g_{44}} \qquad (6)$$

The functions $g_{\mu\nu}$ are now determined but for a scalar factor $\lambda(x_1 \ldots x_n)$, which depends on the position. This factor does not matter in the comparison of vectors at a single point, since all statements regarding the relation of length of vectors at one point are independent of the choice of λ. The comparison of length at different points is impossible, however, by means of the $q_{\mu\nu}$, since this would require the determination of the factor λ. This procedure is, of course, exactly the same as the first. The permissible statements of the first method are, at the same time, the only possible statements of the second. In the following we shall use the first of these two methods.

So far we have obtained two kinds of statements in the displacement space; namely, the comparison of length at one point and the comparison of corresponding vectors at different points. Let us now inquire into the comparison of non-corresponding vectors at different points. As we shall see, this will involve a peculiar kind of indeterminacy.

Given a vector A^{ν} at P, a vector B'^{ν} at P' and a connecting path s, what is the ratio of the lengths of these two vectors? In this case it is permissible to specify the connecting path, since it is a characteristic of the displacement space that a comparison of vectors at different points depends upon the path. Yet another indeterminacy arises because the desired comparison ~~of length~~ may be performed in two ways. First of all, we can look for the vector A'^{ν} at P' which is congruent to the vector A^{ν} at P according to the law of displacement; then we would have to compare A'^{ν}

with B'^{τ} with the aid of the $g'_{\mu\nu}$ (or the $q'_{\mu\nu}$ as the case may be) at the point P'. This method would give us an answer to our question. On the other hand, we might look for the vector B^{τ} at P which according to the operation of displacement will be congruent to the vector B'^{τ} at P', and we would then have to compare B^{τ} with A^{τ} using the $g_{\mu\nu}$ (or the $q_{\mu\nu}$ as the case may be) at the point P. This will again give us a solution to our problem, although these two solutions will, in general, not be the same.

In order to be able to define a general comparison of length an additional rule is required. We may prescribe, for example, that a vector is to be displaced before it is rotated. This leads to the result, however, that comparison of length becomes a non-symmetrical relation. The ratio

$$\ell(A^{\tau}) : \ell(B'^{\tau}) = k$$

yields a different numerical value from the ratio

$$\ell(B'^{\tau}) : \ell(A^{\tau}) = k'$$

i.e.,

$$k \neq \frac{1}{k'} .$$

(A symmetrical comparison would yield $k = \frac{1}{k'}$.)

In particular this lack of symmetry ~~would~~ applies also to the case of equality. If for example

$$\ell(A^{\tau}) \underset{s}{=} \ell(B'^{\tau})$$

we would find that in general

$$\ell(B'^{\tau}) \underset{s}{\neq} \ell(A^{\tau}).$$

Here, even the equality of length is a non-symmetrical relation. Only the equality of corresponding vectors is a symmetrical relation, since both types of comparison become identical in this case.

Let us finally present a fourth possible determination in the displacement space. Until now we have applied the operation of displacement to vectors of arbitrary orientation. We can, however, displace a vector along its own direction. In this case we choose the dx_τ in such a manner that they are proportional to the components of the vector A^τ. If we specifically choose a line element δx_τ as our vector, the displacement which is applied to this case will then define what is meant by "displacement of a line element along its own direction". Such a displacement will therefore define a straightest line. However there is no shortest line in the displacement space, because a distance does not have a defined length but only an "extension" in the topological sense (page 288).

We shall call the space, developed above, the general displacement space, since in it the operation of displacement has not been restricted apart from the elimination of the comparison by displacement of vectors at the same point. This space is therefore determined if the two fields $\Gamma^\tau_{\mu\nu}$ and $q_{\mu\nu}$ are given independently of each other. In this space the following four types of assertions are permissible or possible:

1. The comparison of the length of vectors at the same point, given by the $g_{\mu\nu}$ (or the $q_{\mu\nu}$ ~~are~~ as the case may be),
2. The comparison of corresponding vectors at different points for a given connecting path, given by the $\Gamma^\tau_{\mu\nu}$,
3. The comparison of non-corresponding vectors at different points for a given connecting path and a given direction of comparison, given by the $g_{\mu\nu}$ (or the $q_{\mu\nu}$ as the case may be) and the $\Gamma^\tau_{\mu\nu}$,
4. The identification of straightest lines.

Let us now turn to restrictions of this general displacement space. First of all, let us ask which restricting condition must be satisfied in order that the general comparison of length be symmetrical. This means, obviously, that the ratio $A^\tau : B^\tau$ must be the same as the ratio $A'^\tau : B'^\tau$; or in other words, the ratio of the lengths of differently directed vectors must not change if they are displaced along the same path. This condition leads us to a restriction regarding the $\Gamma^\tau_{\mu\nu}$.

We shall carry out this simple calculation[1], because we shall make use of it later on. By differentiating (3) we obtain

$$d(\ell^2) = \left[\frac{\partial g_{\mu\nu}}{\partial x_\sigma} A^\mu A^\nu + g_{\mu\nu} A^\nu \frac{\delta A^\mu}{\partial x_\sigma} + g_{\mu\nu} A^\mu \frac{\delta A^\nu}{\partial x_\sigma} \right] dx_\sigma \qquad (7)$$

for the transition to a neighboring point. If we assume, in addition, that this transition consists of a parallel displacement of the vector A^μ, then we may substitute for the partial derivatives of the A^μ with respect to the coordinates the following expressions which are directly obtained from (1).

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1) Although a comprehensive mathematical presentation has not yet been given from the viewpoint we have developed above, the necessary calculations have already been carried out in the mathematical literature. The proofs given in this section follow the clear exposition given by Eddington in ~~Relativitätstheorie in Mathematischer Behandlung, Springer 1925, page 324,~~ The mathematical Theory of Relativity, Cambridge University Press, 1924 except for a few changes in notation.

$$\frac{\delta A^\mu}{\delta x_\sigma} = \Gamma^\mu_{\alpha\sigma} A^\alpha \qquad (8)$$

If we also exchange some of the indices of summation, (7) becomes

$$d(l^2) = \left[\frac{\delta g_{\mu\nu}}{\delta x_\sigma} + g_{\alpha\nu}\Gamma^\alpha_{\mu\sigma} + g_{\mu\alpha}\Gamma^\alpha_{\nu\sigma} \right] A^\mu A^\nu dx_\sigma \qquad (9)$$

Let us now define the quantities $\Gamma_{\mu\sigma,\nu}$ by means of the customary lowering of the indices according to the rule [1]

$$\Gamma_{\mu\sigma,\nu} = g_{\alpha\nu}\Gamma^\alpha_{\mu\sigma} \qquad (10)$$

Finally, we get

$$d(l^2) = \left[\frac{\delta g_{\mu\nu}}{\delta x_\sigma} + \Gamma_{\mu\sigma,\nu} + \Gamma_{\nu\sigma,\mu} \right] A^\mu A^\nu dx_\sigma \qquad (11)$$

In order to abbreviate this expression, let us write

$$K_{\mu\nu,\sigma} = \left[\frac{\delta g_{\mu\nu}}{\delta x_\sigma} + \Gamma_{\mu\sigma,\nu} + \Gamma_{\nu\sigma,\mu} \right] \qquad (12)$$

The desired restricting condition for the $\Gamma^\tau_{\mu\nu}$ is therefore expressed by the fact that we must put

$$K_{\mu\nu,\sigma} = g_{\mu\nu}\cdot\kappa_\sigma \qquad (13)$$

small Kappa

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1) The quantities which have a lower index are commonly called the covariant components while those which have an upper index are called the contravariant components of the same vector or tensor. Between them we have the relation

$$A_\mu = g_{\mu\nu}A^\nu .$$

With respect to this and to the corresponding law of transformation see Einstein, <u>Vier Vorlesungen über Relativitätstheorie</u>, Vieweg, 1922, pages 42-44. Quantities with indices of both kinds are called mixed components and transform according to (10). Although the $\Gamma^\tau_{\mu\nu}$ have these properties of tensors, they are not ~~pure~~ general tensors, but merely linear tensors. See Weyl, <u>Raum-Zeit-Materie</u>, 3rd ed. Berlin 1920, page 102.

This gives us a functional relation between the $\Gamma^{\tau}_{\mu\nu}$ and the $g_{\mu\nu}$. Instead of the $\Gamma^{\tau}_{\mu\nu}$ only the vector (or tensor of rank one) $\varkappa_\sigma$ is to be arbitrarily given and (11) reduces to

$$d(l^2) = l^2 \varkappa_\sigma \, dx_\sigma \, . \tag{14}$$

The magnitude of the change in length is now dependent only upon l^2, not upon the direction of the vector A^{τ}, and it is proportional to l^2. Equally long vectors at P will then also be equally long at P' and the ratio~~s~~ of the lengths will be preserved in the displacement.

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The relation given by (12) and (13) is an In-equation (in Eddington's terminology), i.e., it applies in the same form to a metric $g'_{\mu\nu}$ which is introduced by $g_{\mu\nu} = \lambda(x_1 \ldots x_n)\, g'_{\mu\nu}$. This requirement is necessary because we want to admit only those statements which hold for all of the $g_{\mu\nu}$-systems having the same $q_{\mu\nu}$. For the proof we introduce the substitution and obtain

$$\lambda g'_{\mu\nu} \varkappa_\sigma = g'_{\mu\nu} \frac{\partial \lambda}{\partial x_\sigma} + \lambda \frac{\partial g'_{\mu\nu}}{\partial x_\sigma} + \lambda g'_{\nu\tau} \Gamma^{\tau}_{\mu\sigma} + \lambda g'_{\mu\tau} \Gamma^{\tau}_{\nu\sigma}$$

$$\lambda g'_{\mu\nu} \left[\varkappa_\sigma - \frac{1}{\lambda} \frac{\partial \lambda}{\partial x_\sigma} \right] = \lambda \frac{\partial g'_{\mu\nu}}{\partial x_\sigma} + \lambda g'_{\nu\tau} \Gamma^{\tau}_{\mu\sigma} + \lambda g'_{\mu\tau} \Gamma^{\tau}_{\nu\sigma}$$

If we write $\varkappa'_\sigma$ for the bracket on the left side and divide by λ, the assertion is proved. The unchanged functions $\Gamma^{\tau}_{\mu\sigma}$ are therefore in the same relation to the $g'_{\mu\nu}$ as to the $g_{\mu\nu}$. It should be noted, however, that the functions $\Gamma_{\mu\sigma,\nu}$ have changed, since $\Gamma_{\mu\sigma,\nu}$ in the old metric equals $g_{\nu\tau}\Gamma^{\tau}_{\mu\sigma}$, while it is equal to $g'_{\nu\tau}\Gamma^{\tau}_{\mu\sigma}$ in the new metric. Similarly ~~the~~ $\varkappa'_\sigma$ differs from $\varkappa_\sigma$, but since $\varkappa_\sigma$ is arbitrary this does not affect our assertion.

Weyl makes assumption (13), and his space is therefore a specialization of the general displacement space. The specialization consists in the fact that, for a given connecting path, not only the comparison of length of corresponding vectors but also the comparison of length of all other vectors becomes symmetrical. Weyl's space involves, however, an additional specialization, which we shall indicate in the following.

The second restriction of the general displacement space is introduced independently of the first. It consists in the requirement that

$$\Gamma^{\tau}_{\mu\nu} = \Gamma^{\tau}_{\nu\mu} \qquad (15)$$

The significance of this condition can readily be seen if we consider a small displacement δx_τ (Fig. 50)[1] as the vector A^τ which is to be displaced. If we displace the vector δx_τ by the distance dx_τ, its endpoint will be at a point P_1, whose distance from the starting point P is given by the coordinate differences

$$dx_\tau + \delta x_\tau + d(\delta x_\tau).$$

The only law of summation for vectors contained in this equation is that which may be considered valid for neighboring points separated by infinitesimal distances. (Rigorously it applies only to vectors which are located at the same point and thus our assertions are correct in the limit.) If we displace, on the other hand, the vector dx_τ by the displacement δx_τ, its endpoint will reach a point P_2, which is determined relative to P by the coordinate differences

$$\delta x_\tau + dx_\tau + d(dx_\tau).$$

Fig. 50. The nonexistence of infinitesimal parallelograms.

1) At this point it becomes obvious that the writing of coordinate differentials with a lower index is a mistake. Coordinate differentials correspond to contravariant vectors and should therefore be written with an upper index. We shall, however, follow the notation as used in the literature.

Under what conditions will P_1 coincide with P_2? Since the first two terms of the two expressions are identical, this condition is satisfied if

$$d(dx_\tau) = d(\delta x_\tau).$$

Together with (1) this yields

$$\Gamma^\tau_{\mu\nu}\, \delta x_\mu dx_\nu = \Gamma^\tau_{\nu\mu}\, \delta x_\mu dx_\nu \tag{16}$$

where μ and ν are interchanged on the two sides of the equation. This is permissible since the summation runs over both indices. Equation (16) will always be satisified if (15) is true, and (15) is therefore the condition that P_1 and P_2 coincide.

We may denote this property therefore as the <u>existence of infinitesimal parallelograms</u>. If four neighboring infinitesimal vectors are parallel in pairs and equally long in the sense of the displacement, they will form a quadrilateral. In the general displacement space there are no infinitesimal parallelograms. If there are infinitesimal parallelograms, this would constitute a geometrical singularity for the field $\Gamma^\tau_{\mu\nu}$ to be given and therefore a restriction which is given by (15). There is, of course, no logical need for such a restriction.

The restrictions (13) and (15) may also be imposed simultaneously upon the general displacement space. Since this is done by Weyl, we shall call the resulting space, namely, the third specialized displacement space, a Weylian space. It is the displacement space in which there are infinitesimal parallelograms and in which the comparison of the length of arbitrary vectors is symmetrical for any given path. If we use the Christoffel abbreviation as given in (19), the dependence between the $\Gamma^\tau_{\mu\nu}$ and the $g_{\mu\nu}$ holding for the Weylian space is given by

$$\Gamma^{\tau}_{\mu\nu} = -\left\{ {\mu\nu \atop \tau} \right\} + \frac{1}{2} g^{\tau}_{\mu} \chi_{\nu} + \frac{1}{2} g^{\tau}_{\nu} \chi_{\mu} - \frac{1}{2} g_{\mu\nu} \chi^{\tau} \tag{17}$$

Proof (due to Eddington): Corresponding to (12) we may form the three equations

$$K_{\mu\nu,\sigma} = \frac{\partial g_{\mu\nu}}{\partial x_{\sigma}} + \Gamma_{\mu\sigma,\nu} + \Gamma_{\nu\sigma,\mu}$$
$$K_{\mu\sigma,\nu} = \frac{\partial g_{\mu\sigma}}{\partial x_{\nu}} + \Gamma_{\mu\nu,\sigma} + \Gamma_{\sigma\nu,\mu}$$
$$K_{\nu\sigma,\mu} = \frac{\partial g_{\nu\sigma}}{\partial x_{\mu}} + \Gamma_{\nu\mu,\sigma} + \Gamma_{\sigma\mu,\nu}$$

Adding the last two equations and subtracting the first, we obtain, because of (15)

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$$1/2 \left[K_{\mu\sigma,\nu} + K_{\nu\sigma,\mu} - K_{\mu\nu,\sigma} \right] = 1/2\left(\frac{\partial g_{\mu\sigma}}{\partial x_{\nu}} + \frac{\partial g_{\nu\sigma}}{\partial x_{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x_{\sigma}}\right) + \Gamma_{\nu\mu,\sigma}. \tag{18}$$

If we raise the index σ and use condition (13), we will obtain condition (17), if the Christoffel abbreviation

$$\left\{ {\mu\nu \atop \tau} \right\} = 1/2\ g^{\tau\sigma} \left(\frac{\partial g_{\mu\sigma}}{\partial x_{\nu}} + \frac{\partial g_{\nu\sigma}}{\partial x_{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x_{\sigma}} \right) \tag{19}$$

is introduced.) It can easily be shown that (17) is again an In-equation.

The specializations of the general displacement space developed above are still called special <u>displacement</u> spaces, because they still accomplish the comparison of length by means of displacement; i.e., the comparison depends upon the path, yet it is restricted. Let us now turn to the second method of bringing the metric and displacement into conformity, in which the metric supplies the basis of the comparison of length, while the displacement is specialized so that it fits the metrical comparison of length. These spaces are therefore called <u>metrical</u> spaces. The most general type is contrasted with narrower specializations.

We are now going to use the $g_{\mu\nu}$, and not only their ~~relations~~ ratios; that is, we include among the admissible statements those which presuppose the actual values $g_{\mu\nu}$. Among these we shall find, first of all, the comparison of length of vectors which are located at different points, independent of the path. If this comparison is not to contradict the displacement, $d(\ell^2)$ must equal zero for an infinitesimal displacement. Together with (11) and (12) this yields

$$K_{\mu\nu,\sigma} = 0 \qquad (20)$$

<u>Equation (20) is the characterization of the general metrical space.</u> In such a space the transfer of length by the displacement can be integrated; i.e., it is independent of the path, and the length thus transferred is identical with that of the metrical comparison. The function of the displacement in this space is limited to the transfer of direction; the transfer of length is left entirely to the metric.

To make this clear, let us return to our previous characterization of the difference between metric and displacement (see page 472). By means of the metric a certain class of vectors at P' (the class of vectors equal in length to the vector A^τ), is coordinated to the combination of the three elements, a vector A^τ, a point P, and a point P', such that to every direction at P' there belongs only one vector of this class.

The displacement, on the other hand, coordinates to such a combination the total class of vectors at P' and makes a choice among these vectors only if a connecting path s is added to this combination as a fourth element. Condition (20) specializes the operation of displacement in such a manner that the combination of the three elements is no longer coordinated to the entire class of vectors at P', but to a narrower class which coincides with the class of vectors equal in length to A^{τ}. For any direction specified at P', only one vector remains which can be made congruent to a given vector at P by choosing a path s. Therefore only the transfer of direction of the displacement but not its transfer of length now depends upon the path.

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Relation (20) is not an In-equation, since it can be satisfied only for a definite metric $g_{\mu\nu}$. This is because we are now making use of the $g_{\mu\nu}$ and are no longer restricting ourselves to statements which assume only the $q_{\mu\nu}$.

Due to the existence of a metric, there will be shortest lines in the general metrical space, but in general, they are not identical with the straightest lines which are defined by the displacement. It would require an additional restriction to make these two special lines coincide and the Riemannian space provides this specialization, although in this space the transfer of direction of the displacement can not yet be integrated. It is specialized in such a way that the displacement of a line element along its own direction leads to the same line as the shortest direction metrically speaking. In Riemannian space there are therefore geodesics; i.e., lines which are straightest and shortest at the same time.

The general metrical space is therefore different from the Riemannian space and the latter is a special metrical space.

The specialization which leads to Riemannian space is therefore given by the condition

$$\Gamma^{\tau}_{\mu\nu} = -\left\{ \begin{matrix} \mu\nu \\ \tau \end{matrix} \right\} \qquad (21)$$

(for the Christoffel abbreviation,see (19)).

This condition is derivable in a different way. If the condition of symmetry (15) is added to condition (20), (21) results. The Riemannian space can also, therefore, be characterized as a metrical space in which there are infinitesimal parallelograms. This specialization is identical with that which is given by the concept of geodesics.

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Proof: Because of (20), if the left-hand side of (18) is equated to zero, (21) results. Here, (15) is already presupposed by (18). The proof that the displacement defined by (21) provides a straightest line which is at the same time the shortest line, has been given in the literature.[1)]

Let us finally make the only other possible specialization. Thus far, the transfer of length, and not the transfer of direction, has been made independent of the path. If we now make this additional demand we arrive at Euclidean space, in which

$$\Gamma^{\tau}_{\mu\nu} = -\left\{ \begin{matrix} \mu\nu \\ \tau \end{matrix} \right\} = 0 \qquad (22)$$

for rectilinear coordinates. The characterization of this condition, which is invariant, i.e., independent of the coordinate system, is formulated in the familiar way (see page german book 283) as the vanishing of the Riemannian tensor

$$R_{\mu\nu\sigma\tau} = 0 \qquad (23)$$

This no longer signifies a specialization of the $\Gamma^{\tau}_{\mu\nu}$ alone, but also a specialization of the $g_{\mu\nu}$.

1) H. Weyl, Raum-Zeit-Materie, 3rd ed., Berlin 1920, p. 121.

§48. The Geometrical Interpretation of Electricity.

Weyl did not develop the space which we presented in the previous section purely for its mathematical interest; it was his purpose to make it suitable for a physical application. The great success which Einstein had attained with his geometrical interpretation of gravitation, led Weyl to believe that similar suceess might be obtained from a geometrical interpretation of electricity. Since the tensor of Riemannian space was already appropriated by gravitation, he constructed a wider geometrical frame which contained some unassigned geometrical elements which he could ascribe to electricity.

Before we ~~criticize~~ examine this idea, let us first review the actual achievements of the geometrical interpretation of gravitation. The field of force of gravitation affects the behavior of measuring instruments. Besides serving in their customary capacity of determining the geometry of space and time, they serve, therefore, also as indicators of the gravitational field. The geometrical interpretation of gravitation is consequently an expression of a real situation; namely, of the actual effect of gravitation on measuring rods and clocks. This constitutes the physical value of this interpretation. It has been confirmed to a very high degree by physical experience and is formulated by the principle of equivalence.

Anything beyond is added by the imagination and constitutes picture-thinking. If we think of a Riemannian space with its special relations of congruence, instead of thinking of a field of force, this is a permissible representation of the gravitational field. But one cannot say that it is a necessary one. There is no need to consider the behavior of measuring instruments as "voluntary"; we might just as well conceive that they are influenced by a force, i.e., gravitation, whose properties are no different from a temperature field,for instance, which also imposes certain relations of congruence upon the measuring instruments. Indeed, there is no need at all to use measuring instruments for the representation of the gravitational field. We might, for example, recognize the gravitational field by the motion of mass points. Such a motion would, of course, suggest a relationship to the geometrical interpretation, since it can be visualized as a motion along the geodesics. Yet this assertion goes beyond what is given by the motion of the mass points alone, since it puts this motion into a relation with the geometrical behavior of the measuring instruments (measurement of length by ds^2). We are not compelled to think of this relation at all times, but we might also stop at the conception of force which, before Einstein, was associated with the falling mass points. It makes no sense to call this pictorial conception of an attractive force false. The new insight of Einstein consists merely in recognizing the fact that the well-known complex of relations concerning the motions of mass points is supplemented by their relations to the behavior of measuring instruments. It is not necessary, however, to consider this relation as the primary basis of the pictorial conception, to which all other gravitational facts must be referred. If we take more complicated gravitational effects, e.g. the state of tension in a beam, we could also interpret this as an attempt of matter to adapt itself to the Riemannian space. But this is just a kind of visualization, not a necessary representation of this physical state.

The geometrical representation of gravitation would therefore be nothing but a form of visualization, if it did not actually assert this relation between measuring instruments and the gravitational field. This empirical assertion is the basis of the epistemological value of Einstein's theory of gravitation. The geometrical interpretation of gravitation is merely the visual cloak in which the factual assertion is dressed. It would be a mistake to confuse the cloak with the body which it covers; rather, we may infer the shape of the body from the shape of the cloak which it wears. After all, only the body is the object of interest in physics.

If we want to do the same for electricity, we must search for a similar physical fact which relates the electrical field to the behavior of measuring instruments, thus permitting a geometrical expression of the electrical field. However, the fundamental fact which would correspond to the principle of equivalence is lacking. Although Weyl has constructed the operation of displacement, the displacement space is not the type of space which characterizes the~~ir~~ geometrical behavior, since their behavior of measuring instruments can be integrated. We are referring to the conceivable behavior of rigid bodies as explained in section 46; however, it is precisely this behavior which does not occur in reality. This means that we have found a cloak in which we can dress the new theory, but we do not have the body which this new cloak would fit.

An extension of Einstein's theory to the Weylian space would look somewhat as follows. Measuring rods and clocks are indicators of the gravitational field and, at the same time, indicators of the electrical field. As long as a gravitational field exists alone, the behavior of the measuring rods can be integrated, i.e., they define a comparison of length independent of the path. For this purpose the Riemannian space would suffice. As soon as an electrical field is added, however, the integrability ceases, the behavior of the measuring instruments is describable only in terms of the operation of displacement, and the Weylian space now constitutes the natural cloak for the field which is composed of electricity and gravitation.

Unfortunately, however, this picture does not agree with the physical facts. Even if electrical fields are added, the behavior of the measuring instruments can still be integrated. This should be understood as follows. As long as there is an effect of the electrical field, the behavior of the measuring instruments differs from that in a gravitational field alone; but two measuring instruments transported along different paths will be equally long when they meet again, even in the electrical field. This is true even in the case where two measuring instruments meet outside of the electrical field after one of them has traveled through the electrical field while the other one has traveled entirely outside of it. All of our previous experience seems to confirm this fact, especially (according to Einstein) the existence of sharp spectral lines. If atomic clocks changed their periods as a function of their space-time paths, a large number of atoms with completely different pasts could hardly radiate light of the same frequency. The displacement space is therefore not suited to describe the behavior of measuring instruments in a combined electrical and gravitational field.

Weyl, therefore, tried a different approach. He conceived of two kinds of physical entities, one of which indicates the metrical field while the other represents the operation of displacement. Measuring rods and clocks are of the first kind and therefore only indicators of the gravitational field, as in Einstein's theory. Only in the field $\Gamma_{\mu\nu}^{\tau}$, is electricity expressed in addition to gravitation, and only indicators of the second kind can therefore react to electricity.

What are these indicators of the second kind? Weyl introduced the distinction between adaptation and perseverance for their characterization. He called the behavior of the ~~entities of the~~ first kind of entities adaptation and that of the second type perseverance. This second term requires a justification. We can just as well call the behavior of the second kind of entities adaptation; namely, an adaptation to the field of the $\Gamma_{\mu\nu}^{\tau}$. Since the field $\Gamma_{\mu\nu}^{\tau}$ is integrated, however, due to the definition of the process of displacement, we may call this special kind of adaptation perseverance. Perseverance is therefore an adaptation accompanying the simultaneous integration of the field along a described path. With the above definition of these two concepts, we may now accept Weyl's distinction.

From the geometrical viewpoint as well there is no doubt that we are justified in rejecting the measuring instruments as indicators of the operation of displacement. Measuring rods define a comparison of length, but no comparison of direction; they are not indicators of an operation of parallel displacement. If a measuring rod is to be transported along a given path in the sense of the displacement, a special rule is required, telling us how it is to be directed, since it cannot define such a rule by itself. The axis of a gyroscope, for example, which adjusts itself in a definite fashion in transport, is an indicator of this type, or, for that matter, the velocity vectors of freely moving mass points. Weyl decided not to use the above mentioned objects for reasons which we shall discuss later on.

Fundamentally, Weyl's assertion is certainly correct: it is not necessary to consider measuring rods and clocks, in particular, as the objects which realize the operation of displacement. If there are special indicators which react to electricity and behave in the sense of the displacement, the geometrical interpretation accomplishes the same thing as for gravitation. The fact that clocks and measuring rods, which realize the process of adaptation, are not indicators of this type is irrelevant, as long as there are other indicators which realize the process of perseverance. The question of realization may therefore be left open, for the time being. Let us assume that indicators of the second kind have been found. What would be the ~~the~~ geometrical interpretation of electricity? ~~be like?~~

We would then be confronted with the case described on page 471, in which we had two essentially different basic processes. The processes would not be entirely independent of each other, and we would develop the type of space described in the second para-graph of page 473, for instance, in which the two basic operations are made dependent but contradict one another in some respects. This contradiction cannot be avoided as long as there exist indicators for a metric $g_{\mu\nu}$ which determine these functions themselves and not only their ratios. The dependence of these processes will have to be adjusted in conformity with the laws of nature and their choice is therefore subject to experience. This approach, except for a restriction to be mentioned later, is used by Eddington and Einstein.

Weyl, himself, pointed out that measuring rods and clocks are none too accurate and should, therefore, never be used as a foundation of geometry. (Physically speaking this would be correct only if there were no atomic clocks.)A better foundation for geometry is supplied by the motion of light. Since this motion satisfies the equation $ds^2 = 0$, however, it defines only the ratios of the $g_{\mu\nu}$ and not the $g_{\mu\nu}$ themselves. Thus, it is possible to construct a "balanced" displacement space. Weyl decided upon the type of space we have called the Weylian space, identifying the vector χ_σ, which is still unassigned in this space, with the electrical potential.

A third and last method is given by the following considerations. If we recognize measuring rods and clocks as indicators of the metric, the only balanced type of space which we would still have at our disposal, would be the general metrical space. As we have shown before (page 489), this space is not necessarily identical with the Riemannian space. It is, therefore, possible to define an operation of displacement which contains the effect of the electrical field, but which, on the other hand, does not contradict the metric. The geometrical interpretation of electricity would then be expressed by the special kind of displacement of direction, but no longer by an effect upon the comparison of length.

Which of these three methods is the correct one? No general answer is possible, since it would depend upon the behavior of the indicators. The correct method is the one which expresses in its law of displacement the behavior of the indicator in a combined electrical and gravitational field. The choice of the indicator is, of course, arbitrary, since no rule can tell us what entities we should use for the realization of the process of displacement: the geometrical interpretation of electricity, therefore, presupposes a coordinative definition of the objects of the displacement. Only after a coordinative definition has been chosen, can we apply the judgements "true" or "false", since such judgements would concern only the question whether the objects which we have chosen for the displacement will satisfy the law which we have formulated for the $\Gamma^{\tau}_{\mu\nu}$. Since the choice of the coordinative definition is arbitrary, there are different methods of obtaining a geometrical interpretation of electricity. Between them there is no difference in truth value.

We shall illustrate this idea by actually carrying through one of these methods; in fact, it is the which the physicists have avoided so far.

§ 49. An Example of a Geometrical Interpretation of Electricity.[1]

When we wish to choose an indicator for the process of displacement, we must select a physical phenomenon in which the gravitational and electric forces together produce a geometrical effect. Such a phenomenon occurs in the mechanical effect of an electrical field on charged mass points, since the mechanical force of electricity is superimposed upon the mechanical force of gravitation and the two forces together produce a motion, i.e., a geometrical effect. Consequently we shall choose the electrically charged mass point as the indicator of the field $\Gamma^{\tau}_{\mu\nu}$.

The well-known law concerning the mechanical force of the electrical field states that the force F which acts on a mass point of charge ρ is given by the equation

use German K ρE

$$K \cancel{F} = \rho \cancel{F} E + \rho[v,H]. \tag{1}$$

The first term symbolizes the effect of the electrical field E, while the second term symbolizes the effect of the magnetic field H, which vanishes in the case of a static point charge, but which occurs for a charge which moves with a velocity v, and is given by the vector product of v and H, multiplied by the charge ρ.

In four-dimensional notation

use Latin K

$$K^{\tau} = - f^{\tau}_{\nu}\, i^{\nu}$$

$$i^{\nu} = \rho \mu u^{\nu} \qquad u^{\nu} = \frac{dx^{\nu}}{ds} \tag{2}$$

where f^{τ}_{ν} or $f_{\mu\nu}$ as the case may be is the electrical field

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1) The content of this section, together with an abstract of the entire appendix was presented by the author to a regional meeting of the German Physical Soc. in Stuttgart, May 16, 1926. See <u>Verhandl. d.d. Phys. Ges.</u> 7, 1926, page 25.

composed in the usual way of the three-dimensional vectors E and H, i^{ν} is the electrical current, also called four-vector, and u^{ν} is the velocity vector. From (2) we can now obtain the law of motion of a charged mass point of mass 1, which according to the general theory of relativity is

$$\frac{du^{\tau}}{ds} = -\left\{ {\mu\nu \atop \tau} \right\} u^{\mu} u^{\nu} - f^{\tau}_{\nu}\, i^{\nu} \qquad (3)$$

On the left-hand side we have the product of mass and acceleration and on the right-hand side the sum of the forces.

We shall now have to define a process of displacement which represents this law (3) in the form of a vector displacement. This can be done, if we choose the <u>velocity vector</u> of a charged mass point as the object of the displacement.

With this choice we have decided to use the last of the above mentioned approaches, since the length of the vector is, by definition,

$$\ell^2 = g_{\mu\nu}\, u^{\mu}\, u^{\nu} = 1. \qquad (4)$$

The displacement of this vector, therefore, cannot not change its length. We must now define a displacement $\Gamma^{\tau}_{\mu\nu}$ for which $d(\ell^2) = 0$, but which is not symmetrical in the indices μ and ν. This means that we are imbedding electricity in a general metrical space, which, in its transfer of direction, is different from the general Riemannian space. Such a space may be obtained as follows.

Assume that a coordinate system is given in this space, together with a fundamental tensor $G_{\mu\nu}$ which combines the electrical and gravitational fields, in a manner suggested by Eddington, so that its symmetrical component $g_{\mu\nu}$ and its anti-symmetrical component $f_{\mu\nu}$ characterize the gravitational and electrical field respectively. We therefore put

$$G_{\mu\nu} = g_{\mu\nu} + f_{\mu\nu} \tag{5}$$

where the $g_{\mu\nu}$ and $f_{\mu\nu}$ are defined by the two equations

$$g_{\mu\nu} = 1/2(G_{\mu\nu} + G_{\nu\mu}) \qquad f_{\mu\nu} = 1/2(G_{\mu\nu} - G_{\nu\mu}). \tag{5a}$$

This combination of electrical field and gravitation is based on the mathematical fact that every tensor can be written as the sum of a symmetrical and an anti-symmetrical tensor, according to (5a). Expression (5) is therefore a mathematical abbreviation of two factual assertions:

1. There are two fundamental tensors.

2. One of them is symmetrical, the other is anti-symmetrical.

We cannot say, therefore, that the combination of electricity and gravitation, which is given by schema (5) is purely formal, since it could not have been carried out unless the second statement were also true. It is really nothing but a different mathematical formulation of these two assertions, which adds nothing to their content; it is merely equivalent to the two assertions.

Let the metric be defined by

$$ds^2 = G_{\mu\nu}\, dx_\mu\, dx_\nu = g_{\mu\nu}\, dx_\mu\, dx_\nu\,. \tag{6}$$

The physical objects which realize the invariant ds^2 are clocks and measuring rods of unit length. Such objects will therefore follow the process of adaptation and we may consequently use them as <u>indicators</u> of the field $g_{\mu\nu}$. Their behavior does not express the field $f_{\mu\nu}$ since the skew-symmetrical part of the fundamental tensor $G_{\mu\nu}$ vanishes in (6).

We may regard the $f_{\mu\nu}$ as being determined by the customary experimental methods of electro-magnetism; however, we shall describe a different method presently. In addition we assume Einstein's equations of gravitation for the $g_{\mu\nu}$, for instance in the form (5, section 40), but now we shall write "r" instead of "R" in order to indicate that r is a function of the $g_{\mu\nu}$ alone. We can therefore write

$$r_{\mu\nu} = 1/2\, g_{\mu\nu}\, r = - \varkappa\, T_{\mu\nu}\ . \tag{7a}$$

For the $f_{\mu\nu}$ we assume Maxwell's equations

$$\frac{\partial f_{\mu\nu}}{\partial x_\rho} + \frac{\partial f_{\nu\rho}}{\partial x_\mu} + \frac{\partial f_{\rho\mu}}{\partial x_\nu} = 0 \tag{7b}$$

$$\frac{\partial f^{\sigma\rho}}{\partial x_\rho} = i^\sigma\ . \tag{7c}$$

In the following we shall make use of (7c) only. The "raising" of the indices is performed in the usual way by means of the ~~reduced~~ inner multiplication by the $g^{\mu\nu}$. mu nu

Let us now define the coefficients ~~of~~ $\Gamma^\tau_{\mu\nu}$ of the displacement by

$$\Gamma^\tau_{\mu\nu} = \gamma^\tau_{\mu\nu} + \varphi^\tau_{\mu\nu}$$

$$\gamma^\tau_{\mu\nu} = -\left\{ {\mu\nu \atop \tau} \right\} \qquad \varphi^\tau_{\mu\nu} = - g_{\mu\sigma}\, f^\tau_\nu \frac{\partial f^{\sigma\rho}}{\partial x_\rho}\ . \tag{8}$$

The $\Gamma^\tau_{\mu\nu}$ are expressed in terms of the $g_{\mu\nu}$ and $f_{\mu\nu}$ alone, i.e., in terms of the $G_{\mu\nu}$ according to (5a). The process of displacement expresses, therefore, the field $g_{\mu\nu}$ as well as the field $f_{\mu\nu}$. The physical objects which satisfy this process of displacement are the velocity vectors u^τ of mass points of unit mass and arbitrary electrical charge. Such mass points are therefore indicators of the field $\Gamma^\tau_{\mu\nu}$ (and consequently also of the field $f_{\mu\nu}$).

We must now prove that the given expression represents the motion of the charged mass point in a combined gravitational and electrical field as formulated in (3). We therefore maintain that this motion can be expressed by the equation

$$du^\tau = \Gamma^\tau_{\mu\nu} u^\mu dx_\nu \tag{9}$$

where we define

$$u^\tau = \frac{dx_\tau}{ds} \tag{10}$$

For the proof we divide (9) by ds and write it as

$$\frac{du^\tau}{ds} = \Gamma^\tau_{\mu\nu} u^\mu u^\nu = \gamma^\tau_{\mu\nu} u^\mu u^\nu + \varphi^\tau_{\mu\nu} u^\mu u^\nu . \tag{11}$$

Due to the definition of $\gamma^\tau_{\mu\nu}$ given in (8), this agrees with (3) except for the last term. Therefore, we need to investigate only the last term, ~~but first of all~~ which we shall ~~re~~write ~~it~~ in a different form. Using the definition of $\varphi^\tau_{\mu\nu}$ as given in (8) we may write

$$\varphi^\tau_{\mu\nu} u^\mu u^\nu = - g_{\mu\sigma} f^\tau_\nu \frac{\partial f^{\sigma\rho}}{\partial x_\rho} u^\mu u^\nu .$$

From (7c) we have

$$\frac{\partial f^{\sigma\rho}}{\partial x_\rho} = i^\sigma = \rho . u^\sigma \tag{12}$$

where ρ is again the charge of the mass point. If we make this substitution and a suitable simplification by means of expression (4), we will now ~~get~~ obtain

$$\varphi^\tau_{\mu\nu} u^\mu u^\nu = - f^\tau_\nu i^\nu . \tag{13}$$

Thus (11) becomes identical with (3), ~~completing our proof.~~

What have we achieved by means of this development? We have expressed the combined electrical and gravitational effects as the geometry of a space-time manifold which is no longer Riemannian, but of a more general type. Gravitation and electricity determine a unified field $G_{\mu\nu}$. The metric of this space-time manifold is given by the ds^2 according to (6) and is realized by means of measuring rods, light-rays and clocks. These will therefore represent only the gravitational component of the field. The displacement is ~~given~~ represented by the $\Gamma_{\mu\nu}^{\tau}$ which are given by (8) as a function of the field $G_{\mu\nu}$; ~~and which~~ it is realized by the velocity vectors of unit mass points of arbitrary charge. It is an expression of the gravitational as well as the electrical field.

The type of space we have chosen will still belong to the class of metrical spaces, because we have used the ~~absolute~~ actual values of the $g_{\mu\nu}$. Consequently we have recognized measuring rods and clocks as metrical indicators. The displacement defines only a comparison of direction. Using velocity vectors as its realization, we define, in particular, the displacement of a vector along its own direction, i.e., we define the straightest line. Our law of displacement formulates therefore the basic law of motion: <u>the electrically charged mass point of unit mass moves along the straightest line</u>. Only if its charge is zero, or if there exists no external electrical field, will this straightest line also be the shortest.

This last statement requires some further explanation. In free space we have $\frac{\partial f^{\sigma\rho}}{\partial x_\rho} = 0$, according to (7c), and therefore also $\varphi_{\mu\nu}^{\tau} = 0$ according to (8), and the displacement will have become identical with the corresponding operation in the Riemannian space. The same is true for an uncharged mass point, which would therefore describe the geodesic. Only a charged mass point produces such an additional field, due to its own charge, ~~such~~ so that the divergence $\frac{\partial f^{\sigma\rho}}{\partial x_\rho}$ for the entire electrical field $f_{\mu\nu}$ no longer vanishes, and the $\varphi^{\tau}_{\mu\nu}$ appear. A charged mass point will therefore engender its own displacement geometry, depending on the strength of its own charge.

One might object that ~~thus~~ the geometrical meaning of this interpretation becomes questionable, because a field $\Gamma^{\tau}_{\mu\nu}$ does not exist independently, since the value of $\Gamma^{\tau}_{\mu\nu}$ depends on the nature of the indicator. Conversely, however, we could also interpret this argument in support of Weyl's conception of perseverance, which would then obtain an even deeper significance. The independently existing field is the field $G_{\mu\nu}$ with its two components $g_{\mu\nu}$ and $f_{\mu\nu}$, whereas the coefficients of the displacement $\Gamma^{\tau}_{\mu\nu}$ represent a resultant of the field and the indicator. Straightest lines are therefore perseverance lines, which are not determined by the field alone. Only in the absence of electrical forces, i.e., when there are no $f_{\mu\nu}$ or the body is not charged is there a motion of pure adaptation.

There is a way, however, to avoid this peculiarity of our formulation, and to construct a diplacement field which is independent of the indicator. We can require, that in addition to its unit mass, the indicator must also have a certain unit charge. This would make the system of straightest lines dependent on the electrical field alone and the $\Gamma^{\tau}_{\mu\nu}$ become a field with independent existence. In this case, expression (9) would no longer comprehend the motion of the charged mass point <u>as well as</u> that of the uncharged one, since the displacement can now be applied only to <u>charged</u> mass points, whereas uncharged ones describe the shortest lines. Such a division is, of course, necessary, if we wish to construct a space which is independent of the indicator and ~~whose~~ the different geometrical elements of which must be realized by different indicators.

Instead of (8) we must now put

$$\Gamma^{\tau}_{\mu\nu} = \gamma^{\tau}_{\mu\nu} + \varphi^{\tau}_{\mu\nu}$$

$$\gamma^{\tau}_{\mu\nu} = -\left\{ {\mu\nu \atop \tau} \right\} \qquad \varphi^{\tau}_{\mu\nu} = -f^{\tau}_{\mu}\, u_{\mu} \tag{14}$$

where only the $\varphi^{\tau}_{\mu\nu}$ have been changed. It can easily be seen that equation (3) is now derivable from (9), if we remember that we have set $\rho = 1$.

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Expression (14) will have to be explained in more detail. So far (also in section 47), we have considered the $\Gamma^{\tau}_{\mu\nu}$ as functions of the coordinates alone, i.e., the set of components $\Gamma^{\tau}_{\mu\nu}$ are determined for every point or point event as the case may be. Whereas the f^{τ}_{ν} are also a field given in such a way, this is not the case for ~~the~~ u_{ν} which appears as a variable in expression (14). u_{μ} is the <u>direction vector</u>, because three-dimensional velocity becomes a direction in four-dimensional space; namely, the direction of the element of the world-line. Expression (14) will therefore make $\Gamma^{\tau}_{\mu\nu}$ a function of the displacement.

The resulting mathematical complication disappears, if we substitute (14) into (9). Since $u_{\mu}\, u^{\mu} = 1$ this yields

$$du^{\tau} = - \left\{ {\mu\nu \atop \tau} \right\} u^{\mu}\, dx_{\nu} - f_{\nu}^{\tau}\, dx_{\nu} \, . \qquad (15)$$

Next to the Riemannian term which includes $\left\{ {\mu\nu \atop \tau} \right\}$ there appears a term which depends only on the position and not on the displacement, and whose coefficient has only two indices. The fact that expression (14) satisfies the condition $d\,(l^2) = 0$ can be verified through a substitution in (11, section 47), noting that $f_{\mu\nu} = - f_{\nu\mu}$ and putting $A^{\nu} = u^{\nu}$.

If (14) is applied to some other vector A^{τ} and not to the vector u^{τ}, ~~then~~ the displacement is non-linear, since ~~Then the~~ dA^{τ} according to (1, section 47) is no longer a linear function of the dx_{ν} because u_{μ} contains the dx_{ν}. If, on the other hand, we assume (14) to be derived from an expression

$$\varphi^{\tau}_{\mu\nu} = - f^{\tau}_{\nu} A_{\mu} \qquad (16)$$

the displacement again is no longer a linear function of the A^{μ}, because, if we put $\gamma^{\tau}_{\mu\nu} = 0$, this yields

$$dA^{\tau} = - f^{\tau}_{\nu} A_{\mu} A^{\mu}\, dx_{\nu} = - f^{\tau}_{\nu}\, l^2\, dx_{\nu} \qquad (17)$$

according to (1, section 47). The increment dA^{τ} is therefore proportional to the second power of the length of the vector. For vectors other than u^{τ}, our displacement procedure will therefore constitute a generalization of the basic assumptions developed on page 467.

Expression (14), together with (9), therefore represents the realization of a space in which the shortest lines and straightest lines are not identical, both being determined by the field of the $G_{\mu\nu}$ alone. Uncharged mass points realize the shortest lines; charged mass points realize the straightest. We have made the demand that in (14) the mass point must have a unit charge besides its unit mass, but it is evident that its motion will depend only on the ratio of its mass to its charge. The indicator of the straightest line, therefore, is merely required to have a definite unit ratio between mass and charge. The fact that we must specify a unit value at all is no more surprising than the fact that we had to make similar specifications of unit clocks and unit rods for the metric.

Let us remember that there is a natural unit for this ratio; namely, the ratio e/m between the charge and mass of the electron. However, this natural choice of the geometry is not unique because the corresponding ratio for the atom of positive electricity, the nucleus of the hydrogen atom, has a different value. Therefore, there are two natural geometries, depending upon whether we let positive or negative electricity describe the straightest line.

§ 50. The Epistemological Value of a Geometrical Interpretation of Electricity.

In the preceding section we have carried through a complete geometrical interpretation of electricity. What is the physical significance of this theory?

Actually, it adds no content to the physical assertions of Einstein's theory. Expression (8), developed for the $\Gamma^{\tau}_{\mu\nu}$, says nothing new, but leads to the well-known law of the mechanical force of an electric field (3), as the calculations have shown. The operation of displacement therefore embodies nothing but a geometrical presentation of this law, i.e., a visualization, not a new physical idea.

Consequently we might call the geometrical interpretation of electricity a graphical representation and distinguish it in this manner from the geometrical interpretation of gravitation.[1] Yet this distinction is due to a misconception of the nature of a geometrical interpretation.

In section 15 we ~~made a~~ examined in detail~~ed analysis~~ the significance of graphical representations; according to ~~which~~ this analysis they stand for the coordination of a system b of physical objects to the system a of rigid bodies. Such a coordination is possible because the two systems a and b satisfy the same relations A. Although the system a of rigid bodies is the normal system for the graphical representation, it

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1) Eddington, op. cit, ~~p. 312~~, makes this distinction, but applies it only to Weyl's theory.

is not the normal system for geometry in general. Instead, we would have to choose the four-dimensional system α of clocks and rigid bodies as the normal system for geometry. This system α can also be represented graphically: in which case it is coordinated to the system a (with the separation of one spatial ~~diemsnion~~ dimension) just as the graphical representation of the indefinite metric was given in Fig. 32 (page 294). This is not necessary, however, since we have direct visual experience of the four-dimensional geometry α, in which we perceive space and time in terms of their distinct sensible qualities, without resorting to graphical representations. We may therefore accept α itself as a normal system which can be visualized directly.

Why, then, is the geometrical interpretation of gravitation not a graphical representation? The answer is that it contains assertions about the system α itself, not about the coordination of some other system to α. Of course, if we want to identify the concept "geometrical interpretation" with "reduction to the system a of rigid bodies", then only a graphical representation is possible even for gravitation. Since Einstein's theory of gravitation necessarily contains assertions about clocks, it refers to α and not to a; therefore, the geometrical interpretation could be achieved only in that kind of a graphical representation which is given by the indefinite metric of Minkowski (section 29). But this restriction is not necessary. Let us understand by "geometrical interpretation" the same as "reduction to the system α of space-time measuring instruments".

This reduction can be a "four-dimensional graphical representation" if a system is coordinated to the normal system α; it can also be a "genuine geometrical interpretation" if it contains direct assertions about α itself. Einstein's theory of gravitation is therefore not a graphical representation, since it contains assertions about α itself and not about a system β of other objects which is equivalent to α.

What about the geometrical interpretation of electricity? We have to include the moving mass points within the system α; this must be done because measuring rods define only a congruence and not a displacement of direction and therefore by themselves are not sufficient for the physical representation of the displacement space. Thus our geometrical interpretation of electricity is not a graphical representation, but a genuine geometrical interpretation, ~~just~~ as is ~~like~~ the theory of gravitation.

Our geometrical interpretation is therefore no "worse" than the geometrical interpretation of gravitation. Above we objected to the geometrical interpretation of electricity (as has frequently been done in the literature) on the grounds that it lacks a basic physical fact analogous to the principle of equivalence. This objection is valid only if we limit the measuring instruments to measuring rods and clocks - it does not, therefore, apply to our theory. While the principle of equivalence asserts a ~~x~~ relation between gravitation and the behavior of measuring rods and clocks, we now use a relation between electricity and other measuring instruments, namely, charged mass points. Furthermore, these measuring instruments are chosen so that they will react simultaneously to both the electrical and gravitational field. They are, therefore, precisely the measuring instruments which we need in order to find a geometrical expression of this combined field. What is new in this geometrical interpretation of electricity is merely the fact that the elec~~el~~trical fields are included within the metrical forces. Our interpretation accomplishes, therefore, something similar to the representation of a temperature field by means of the geometry of the rods which are placed in this field. Whereas different substances would supply us in this case with different geometries, our world geometry was chosen wide enough to express, within a single geometry, the corresponding difference in the behavior of ~~both~~ charged and uncharged unit mass points. By means of their motion, the first represent the straightest line, the second the shortest. The geometrical interpretation of electricity is therefore made possible by a suitable extension of the fundamental geometry, such that the differences in the measuring instruments due to their physical properties can be interpreted again as differences in the basic geometrical forms.

Small print

It should be noted that according to (8, section 49) only unit mass points realize the displacement geometry. Expression (14, section 49) which prescribes only the unit <u>ratio</u> of charge and mass is therefore more advantageous than (9, section 49) even though we obtain different natural geometries for the positive and the negative charge. It is, indeed, of extraordinary significance that this procdeure yields only <u>two</u> natural geometries. The fundamental idea here may, incidentally, be taken as an analogy small print

to Einstein's principle of equivalence, since the equality of gravitational and inertial mass originally represents only a proportionality (i.e., if the two masses are proportional, their equality can always be established by definition through a suitable choice of the units of measurement). The corresponding proportionality between the charge and mass of the electron or proton lead, in (14, section 49), to one or the other of the two natural geometries, as the case may be. The fact that the difference between positive and negative electricity produces a bifurcation expresses geometrically the asymmetry, lacking with respect to mass, between the two kinds of electricity. It would be too much, to expect that there should be only <u>one</u> choice of a natural geometry. Let us remember that there are similar divisions of geometry in Einstein's theory of gravitation, depending on whether light, four-dimensional measuring instruments, or three-dimensional rods are regarded as indicators of the geometry (see pages 412-418).

Although the presentation developed above has provided a complete geometrical interpretation of the two fields, we must recognize that the geometrical interpretation of electricity constitutes no more advance in physical knowledge than the geometrical interpretation of gravitation. This is because we could have rewritten Einstein's original theory of ~~gravitation~~ relativity, without changing its physical content, so that electricity as well would have acquired a geometrical interpretation. Rewriting the theory in this fashion would tell us nothing about reality that we did not know before.

This fact, as trivial as it may appear, is nevertheless of great significance. Physical theory so far, especially under the influence of Weyl, has always defended the idea that the geometrical interpretation of electricity constitutes something which is physically essential. We find that this is not the case.

Is it not true, though, that the geometrical interpretation of gravitation has brought about an advance of physics? It has brought it about, yes, but it is not identical with this progress. It has led, in its effects, to a physical discovery, but in itself it is not this discovery. Two things which are asserted by Einstein's theory of gravitation are physically new.

In the first place, the relation between measuring instruments and gravitation was not previously known; this relations was therefore asserted by Einstein as a new fact. Due to the chosen realization of the process of displacement, the corresponding relation in our geometrical interpretation of electricity is a well-known fact. The geometrical interpretation has therefore acted as a heuristic principle in the theory of gravitation.

In the second place, the geometrical interpretation of gravitation has led to changes in the physical theory even in domains where its relation to clocks and measuring rods is of no significance, e.g., in planetary motion. Einstein describes the gravitational field by different and more precise equations than Newton. The geometrical interpretation has acted again as a heuristic principle. Unfortunately this is not the case with our geometrical interpretation of electricity, since the new theory has remained identical with the old.

We must therefore recognize that the geometrical interpretation of gravitation has attained its important position in the historical development of science, because it has led to new physical insights. The geometrical interpretation itself is merely a formulation, a visualization of these new insights. What we have attained with our geometrical interpretation of electricity is an analogous formulation of physical insights regarding electricity, but these insights are not ~~physically~~ new.

As long as the geometrical interpretation of electricity does not act as a heuristic principle, its sole value will lie in the visualization it provides. Since such a heuristic principle is what physicists who have worked in this field have hoped to attain, we shall now investigate this question in more detail. For this reason, Einstein, in particular, has devised several new formulations in which the geometrical interpretation is reduced to the role of a mathematical tool. There is a definite problem for which a solution is sought through this approach; namely, the problem of the electron.

Until recently the electron has been viewed as a particle charged with electricity, which is no different from a pith sphere, for example, which has been brought into contact with the pole of an induction machine. The field theory, on the other hand, conceives the elementary structure of matter in an essentially different way. It is considered a region of the field, wherein the laws of the field apply just as they do in free space. The only difference is that the field magnitudes are to be distributed so that a negative electrical charge can maintain itself in a concentrated form. One might imagine, for example, that gravitational forces acting cohesively press the negative charge together.

But how can such a structure be derived from the field theory? Maxwell's equations may be valid in a part of the space in which there is a charge, but they cannot explain why such a charge does not fly apart in all directions, in accordance with Coulomb's force of repulsion. According to Maxwell's theory, the cohesion of the charge has always been attributed to a "foreign" force; namely, the force of cohesion in the material body of the electron. The equations of gravitation, on the other hand, in their present form have no effect on the charge and cannot, therefore, yield the cohesive force. They describe only the state of equilibrium between matter and the gravitational field, without consideration of the electrical state. If this state is to be included in the equilibrium, the field equations would have to state a connection between the $g_{\mu\nu}$ and the $f_{\mu\nu}$. The resulting differential equation would have to have a solution corresponding to the electron, and would have to show the discreet nature of the electron as a mathematical necessity.

It is not difficult to combine the three equations of (7, section 49) into one by means of the action-principle, but the resulting equation will not yet give us the desired solution. Rather the amalgamation of the three equations (7, section 49) would have to incorporate slight changes in them. Only slight, however, because we know that they do apply to a high degree of approximation; yet changed, because otherwise they would never give us the electron as a solution. We have to "guess" therefore what kind of change has to be performed.

In this "guessing" the geometrical interpretation of electricity is supposed to be the guide. The point of departure in this approach is the (unwritten) assumption that whatever looks simple and natural from the viewpoint of the geometrical interpretation will lead to the desired changes in the equations of the field. The physicist needs a kind of instinct in his research, a feeling for the hidden paths of nature, and he believes that the adaptation of his concepts to the geometrical interpretation of electricity will direct him toward the desired field equations. Epistemologically speaking, there is no objection to such an approach, which is a working hypothesis logically speaking. Only its success can decide its correctness, for it is purely a matter of experience, whether the way to a simple and natural geometry also leads to an approximation to reality.[1)] It is noteworthy that contemporary discussions of these problems are filled with concepts like "most natural assumption", "simplest invariant", etc.; this tells us that we are dealing with virgin territory in the field of physics which cannot yet be developed systematically

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1) Weyl, in particular, through his concept of "Gauge-invariance", has developed a method for narrowing the choice among the available equations. This is, indeed, a rigorously formulated principle which is more than a mere guide to geometrical feeling. Whether this principle is correct is, of course, a purely empirical question.

and which is not yet ripe for an epistemological evaluation. It is only the method employed that arouses epistemological interest.

Another important consideration must be added at this point. So far, we have always ~~made the point~~ maintained that a simple physical realization must be given for the process of displacement. In our example we ourselves gave such a realization and we obtained, in this way, an actual geometrical interpretation of electricity. Attempts which were made by Weyl, Eddington and Einstein, on the other hand, renounced such a realization of the process of displacement. It is generally believed that such "tangible" realizations do not lead to the desired field equations. Consequently the problem of realization is left open for the time being.

This point of view can be justified logically as follows: Expression (8, section 49) defines a process of displacement as a function of $g_{\mu\nu}$ and $f_{\mu\nu}$. This definition is such that the velocity vectors of charged mass points provide us with the objects of the displacement. Where we to choose a different equation, the objects of the displacement would be vectors other than the velocity vectors. For arbitrarily defined $\Gamma^{\tau}_{\mu\nu}$, we can ask, what vector is moved, due to the motion of the charged mass point, so that it will realize the displacement? Perhaps this will give us a complicated vector which is a function of the velocity vector, the electrical, and the gravitational field; but some answer will be found. Nor is it necessary to limit ourselves to mass points in motion. It might be possible, for example, to define an entity like the magnetic flow through a closed electrical circuit, which would realize the displacement. It is afterall always possible to find some such entity if we do not require that such a combination of known entities has previously been given a particular name.

Logically speaking, there are no objections to such a method. It is based on the fundamental principle that the geometrical interpretation of electricity always rests on an arbitrary coordination; namely, the coordinative definition of an object of displacement; and there is no logical necessity to prefer one particular coordination. There simply are several geometrical interpretations of electricity. From the viewpoint of simplicity and naturalness, a lot can be said against this approach, and it is quite noticeable that this argument, in particular, is employed in the physical discussions objecting to the attempt to formulate a unified field theory. It would be pointless to enter this discussion from a philosophical point of view. An epistemological analysis can tell us only whether a chosen method is permissible or not. Only the physical instinct, whose content lies completely outside the realm of epistemological criticism, can judge, for the time being, whether it will lead us to the physical goal we have described.

If the goal of a field-theoretical interpretation of the electron were attained, then conversely, the realization defined by the expression used, in this case for the $\Gamma^{\tau}_{\mu\nu}$, would be singled out as the "most natural" geometrical interpretation of electricity. As long as this aim has not been reached, less remote realizations will be preferred, among which the one given in section 49 might very well be the "most natural". Its existence proves, in any case, that a geometrical interpretation of electricity can be carried out, although our formulation will hardly lead to the desired explanation of the electron, even if it is changed so that it is no longer identical with Maxwell's theory.

The final decision regarding this new physical territory must be left to the physicist, whose physical instinct provides the sole illumination. Our epistemological discussion was given merely to show that the geometrical interpretation of electricity, as such, has no physical meaning as yet; i.e., it is nothing but a form for descri~~bing~~ption of physical results. It is, therefore, meaningless to ask for the "correct" geometrical interpretation of electricity. Every consistent interpretation is equally admissible, since the coordination between electricity and geometry is not an assertion ~~having~~ which has the character of physical knowledge. This epistemological insight might very well be helpful to the physicist, by showing him the limitations of his method and making it easier for him to free himself from the enchantment of a unified field theory. The many ruins along this road urgently suggest that solutions should be sought in an entirely different direction. It is not the geometrical interpretation of electricity, but an assumption of an entirely different character, which is fundamental to all these attempts; namely, the assumption that the road to a simple conception, in the sense of a geometrical interpretation, is also the road to true relationships in nature. It is this assumption which constitutes the <u>physical hypothesis</u> contained in these attempts. The geometrical interpretation of electrcity can be carried through in any case, but it by no means follows that this added hypothesis must also be correct.