

How to Guard an Art Gallery
and Other
Discrete Mathematical Adventures

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Preface

The adventures in this book are launched by easily understood questions from the realm of *discrete mathematics*, a wide-ranging subject that studies fundamental properties of the counting numbers $1, 2, 3, \dots$ and arrangements of finite sets

The book grew from talks for mathematically inclined secondary school students and college students interested in problem solving. The aim is high, but the prerequisites are modest—mostly elementary algebra and geometry. Occasionally, a perspective gained from more advanced subjects is mentioned. A sampling of questions conveys the spirit and scope of the topics.

- **The art gallery problem.** What is the minimum number of stationary guards (or security cameras) needed to protect a given art gallery?
- **The pizza-cutter's problem.** What is the maximum number of pizza pieces we can make with four straight cuts through a circular pizza? What about n cuts?
- **The computer line drawing problem.** Which pixels should a computer select to represent a given straight line on a monitor?
- **A quadratic residue question.** Is there an integer whose square is 257 more than a multiple of 641? In the jargon of number theory, is 257 a quadratic residue modulo 641?

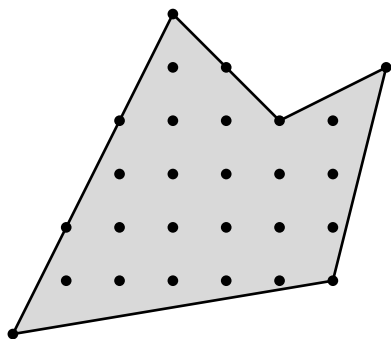
Our interest extends beyond answers to individual questions, no matter how accessible and enticing. The questions are gateways to deeper mathematical material that can be discussed without a lot of background. For instance, the following puzzle (taken from a memorable scene in the movie

Die Hard: With a Vengeance) leads to a discussion of a famous result of Fermat in number theory.

- **The Bruce Willis problem.** We are at a fountain with two unmarked jugs with capacities 3 and 5 gallons. How can we measure exactly 4 gallons of water?

Our goal is to impart a genuine feel for discovery and mature mathematical thinking by attacking problems from several points of view and in various degrees of generality. We also reveal hidden connections between seemingly unrelated topics. For instance, we will discover a relationship between computer line drawing and quadratic residues. You will also likely be surprised to learn that the following two questions are related.

- **An area question.** What is the area of the oddly shaped orchard shown in the figure if the rows and columns of trees are 1 unit apart?



- **A dollar-changing question.** How many ways are there to make change for a dollar with quarters, dimes, and nickels?

Readers inspired to chart their own mathematical adventures can explore the problems at the end of each chapter. The more challenging problems include hints or are broken into smaller steps. The lightly annotated references are a starting point for further reading.

Among the many people who helped and encouraged me as I wrote this book, several deserve special thanks. Amy Myers, Courtney Moen, and Sommer Gentry gave me valuable feedback on all aspects of early drafts and pointed out ways to improve each chapter. I also greatly appreciate the patience and guidance of my editor, Trevor Lipscombe.

Finally, I dedicate this book to Tom Apostol, who set me on the path to mathematical maturity 30 years ago.

How to Guard an Art Gallery
and Other
Discrete Mathematical Adventures

How to Guard an Art Gallery

*I found I could say things with color and shapes
that I couldn't say any other way—
things I had no words for.*

GEORGIA O'KEEFFE

3.1 The Sunflower Art Gallery

Figure 3.1 shows the unusual floor plan of the Sunflower Art Gallery and the locations of four guards. Each guard is stationary but can rotate in place to scan the surroundings in all directions. Guards cannot see through walls or around corners. Every point in the gallery is visible to at least one guard, and theft of the artwork is prevented. Of course, it

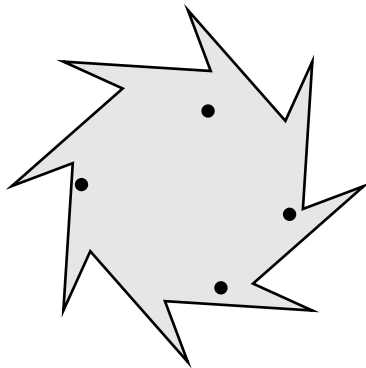


Figure 3.1: The Sunflower Art Gallery

would be more economical to protect the gallery with fewer guards, if possible.

Question. What is the smallest number of guards required to protect the Sunflower Art Gallery?

Removing any one of the four guards in Figure 3.1 leaves part of the gallery unprotected. Nonetheless, it *is* possible to protect the gallery with three suitably positioned guards. We can dismiss the lowest guard if we move the leftmost guard slightly downward. However, we cannot get by with two guards. To see why, consider the eight outer corners of the gallery. It is not possible for one guard anywhere in the gallery to keep an eye on more than three of these corners. So two guards could protect at most six of the eight outer corners.

The Sunflower Art Gallery has 16 walls and needs three guards. This raises a broader question.

Question. What is the smallest number of guards needed to protect any 16-walled gallery, regardless of its shape?

Our art gallery questions involve issues in *computational geometry*, a large and active field that blends geometry with ideas from discrete mathematics and optimization. Applications of computational geometry include:

- Barcode scanners that read prices at grocery stores
- Digital special effects common in today's movies and video games¹
- Calculations performed by global positioning satellite (GPS) receivers to determine location, speed, and direction

¹The special case of representing straight lines on a computer monitor is the topic of Chapter 4.

- Algorithms executed by machines and robotic arms on assembly lines to carry out complex tasks in a specific order
- Computerized fingerprint recognition schemes used in security systems and forensics

At the core of most problems in computational geometry is a connection between theory and algorithms. The theory describes or defines desired geometric configurations, which the algorithms construct using known mathematical procedures.

Theoretical and algorithmic issues are tightly linked in art gallery problems. For instance, we will discover a theorem asserting that every w -walled art gallery can be protected by at most $w/3$ guards. Our demonstration of this result leads to an algorithm telling us exactly where to post the guards. We also look at several variations, including an unsolved three-dimensional guarding problem.

3.2 Art Gallery Problems

Let us define our terms carefully. For our purposes, an *art gallery* is a polygon in the plane.² The polygon need not serve as the floor plan of any real-world art gallery. An art gallery includes the interior region as well as the boundary segments—the *walls*. We let G denote an arbitrary art gallery and write G_w for an art gallery with w walls.

Let p be any point in an art gallery. The point q is *visible* to p provided the line segment joining p and q does not exit the gallery. (We also assume that every point is visible to itself.) The segment represents the sight line of a guard. A set of guards protects an art gallery provided every point in

²More precisely, an art gallery is a *simple polygon*. We exclude polygons with holes, boundaries that cross, and other oddities.

the gallery is visible to at least one guard. Note that a guard at a corner protects the two adjacent walls.

Example 1. (a) The four guards in Figure 3.1 protect the Sunflower Art Gallery.

(b) The Sunflower Art Gallery is not protected by guards at the eight outer corners (Figure 3.2). Even though all of the walls are protected, a region in the center of the gallery remains invisible to all the guards.

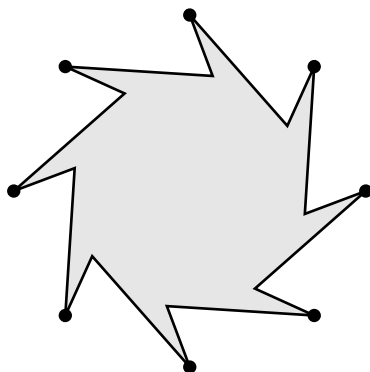


Figure 3.2: The eight guards protect the walls, but not the interior

(c) Each gallery in Figure 3.3 is protected by one or two guards, as shown.

An art gallery is *convex* provided every point in it is visible to every other point. A convex gallery is easy to guard; a guard can be posted anywhere in the gallery. Every triangle is convex, as are the first two galleries in the top row of Figure 3.3. The other galleries in the figure are nonconvex.

Galleries in Particular

Our desire to post as few guards as possible raises two general problems about art galleries. The first problem deals

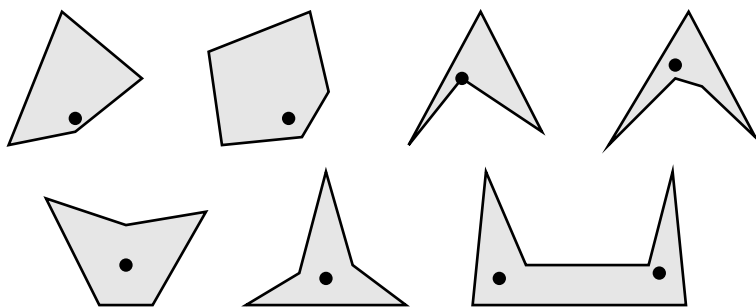


Figure 3.3: The first two galleries in the top row are convex

with specific galleries, and the second deals with all galleries with a fixed number of walls. These are generalizations of the two questions we posed earlier. Let

$\text{guard}(G)$ = the minimum number of guards
needed to protect the art gallery G .

Gallery problem 1. Find the value of $\text{guard}(G)$ for every art gallery G . In other words, find the minimum number of guards needed to protect every art gallery.

Example 2. (a) A convex gallery G satisfies $\text{guard}(G) = 1$.

(b) We have seen that the Sunflower Art Gallery G_{16} satisfies $\text{guard}(G_{16}) = 3$.

To show that $\text{guard}(G) = g$, we must demonstrate two facts:

- The gallery G can be protected by g guards.
- The gallery G cannot be protected by fewer than g guards.

The first fact implies that $\text{guard}(G) \leq g$, while the second gives $\text{guard}(G) \geq g$. The second fact becomes increasingly

difficult to demonstrate as the number of walls increases and the shape of the gallery becomes more complicated.

Ideally, we would have an efficient algorithm that takes an arbitrary gallery G as its input and produces the value of $\text{guard}(G)$ as its output. Such an algorithm could be carried out by a computer (or a patient, careful person) to determine the minimum number of guards needed to protect any given gallery. Researchers in *computational complexity*, an advanced area of discrete mathematics, have strong evidence that we will never find an efficient algorithm of the desired type. The crux of the matter is that the number of essentially different guard configurations to examine increases exponentially as a function of the number of walls. Any proposed general algorithm becomes effectively worthless, even with the fastest computers available. In this sense, gallery problem 1 remains unsolved.

Galleries in General

Now suppose we know an art gallery has w walls, but we do not know its exact shape. Let

$$g(w) = \begin{array}{l} \text{the maximum number of guards required} \\ \text{among all art galleries with } w \text{ walls.} \end{array}$$

In other words, $g(w)$ is the maximum value of $\text{guard}(G_w)$ among all w -walled galleries G_w .

Example 3. (a) Any triangular art gallery can be protected with one guard. Therefore, $g(3) = 1$.

(b) The Sunflower Art Gallery has 16 walls and requires three guards. Therefore, $g(16) \geq 3$. We cannot conclude that $g(16) = 3$ since there could be a 16-walled gallery that requires more than three guards. In fact, we will soon see a 16-walled gallery requiring five guards.

Gallery problem 2. Find the value of the function $g(w)$ for $w = 3, 4, 5, \dots$. In other words, find the largest number of guards required among all w -walled art galleries.

To show that $g(w) = g$, we must demonstrate two facts:

- Every w -walled gallery can be protected by g guards.
- There is a w -walled gallery that cannot be protected by fewer than g guards.

The first fact shows that $g(w) \leq g$, while the second shows that $g(w) \geq g$. We will solve gallery problem 2 by establishing both facts. Naturally, we must first find the correct relationship between g and w .

Crown Galleries

To establish a lower bound for $g(w)$, we construct “hard to guard” galleries—those that require at least as many guards as any other gallery with the same number of walls.

We have already noted that $g(3) = 1$. Also, $g(4) = 1$ since a convex quadrilateral clearly requires just one guard, and a nonconvex quadrilateral can be protected by posting one guard at the corner with the largest interior angle (see Figure 3.3). Moreover, it is not difficult to convince oneself that $g(5) = 1$.

The situation is more complicated for galleries with at least six walls, but we can take a hint from the nonconvex, “horned” hexagonal art gallery in Figure 3.3. Because no lone guard can possibly cover both of the two upper corners, we know that $g(6) \geq 2$. The crown-shaped galleries in Figure 3.4 extend this idea. The Crown Gallery G_{3t} has t tines³ and $3t$ walls and requires at least t guards since

³The crown with one tine is more suitable for a dunce than a prince.

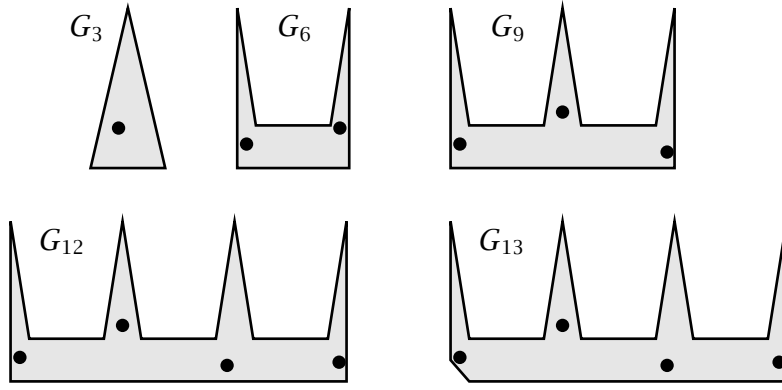


Figure 3.4: The Crown Gallery G_w requires $\lfloor w/3 \rfloor$ guards

no guard can see more than one of the uppermost corners. Therefore, $\text{guard}(G_{3t}) \geq t$ and

$$g(3t) \geq t.$$

If w is one more than a multiple of 3, say, $w = 3t + 1$, then we put a small dent in the crown G_{3t} to produce the gallery G_{3t+1} . Figure 3.4 includes the dented gallery G_{13} , for instance. Because G_{3t+1} requires t guards, we have

$$g(3t + 1) \geq t.$$

Similarly, if $w = 3t + 2$, a twice-dented crown shows that

$$g(3t + 2) \geq t.$$

Some notation helps us state our findings concisely. The *floor* of the real number x is

$\lfloor x \rfloor$ = the largest integer less than or equal to x .

So the floor function “rounds down.” For example, $\lfloor 16/3 \rfloor = \lfloor 17/3 \rfloor = 5$ and

$$\left\lfloor \frac{3t}{3} \right\rfloor = \left\lfloor \frac{3t+1}{3} \right\rfloor = \left\lfloor \frac{3t+2}{3} \right\rfloor = t$$

for each positive integer t . Our crown-shaped galleries thus give the lower bound

$$g(w) \geq \left\lfloor \frac{w}{3} \right\rfloor.$$

3.3 The Art Gallery Theorem

We are now ready to solve the second gallery problem. The answer confirms that the crown-shaped galleries are indeed the hardest to guard.

Art gallery theorem. We have

$$g(w) = \left\lfloor \frac{w}{3} \right\rfloor \quad \text{for } w = 3, 4, 5, \dots$$

In other words, $\lfloor w/3 \rfloor$ guards are sufficient and sometimes necessary to protect an art gallery with w walls.

The art gallery theorem was first stated and proved by Vasek Chvátal in 1975 in response to a query from Victor Klee (1925–2007), an expert in combinatorial problems with a geometric flavor. We have already discovered that $g(w) \geq \lfloor w/3 \rfloor$. Chvátal's crucial contribution was to establish the reverse inequality by showing that every w -walled gallery can be protected by at most $\lfloor w/3 \rfloor$ guards. His proof uses mathematical induction on the number of walls (the validity of the inequality for galleries with w walls is deduced from its validity for galleries with fewer walls) and requires some care in its execution. Problem 24 at the end of this chapter outlines his argument.

A Colorful Idea

Steve Fisk produced a new and colorful proof of the art gallery theorem in 1978. His ingenious argument is less sophisticated than Chvátal's and has a visual appeal. He assigns

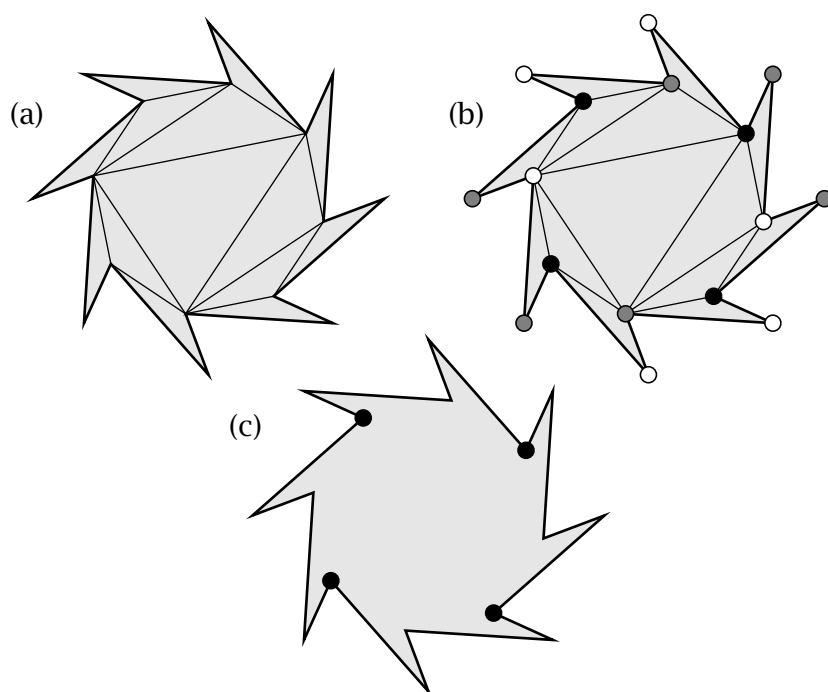


Figure 3.5: Fisk's colorful proof of the art gallery theorem

colors to the corners of the art gallery in a special way and then posts guards based on the arrangement of colors. Figure 3.5 illustrates the steps for the Sunflower Art Gallery.

First, partition the gallery into triangles by inserting suitable noncrossing diagonals, as in (a). The diagonals *triangulate* the gallery.⁴ Then assign one of three colors—black, gray, or white, say—to each of the w corners so that every triangle has one corner of each color. The resulting configuration is called a *polychromatic 3-coloring* of the triangulation. In (b) we have $w = 16$, and there are four black, six

⁴The vertices of the triangles must be corners of the gallery; interior vertices are forbidden. More general triangulations appear in Chapter 2 and in Problem 18 of Chapter 1.

gray, and six white corners. Finally, if we post guards at the four black corners, then every triangle is certainly protected (since every triangle has a black corner), and hence the entire gallery is protected by the guards in (c). The six white corners or the six gray corners also protect the gallery, but the black corners give us fewer guards in this case.

The same argument applies to any w -walled gallery. In a polychromatic 3-coloring of a triangulation, the least frequently used color occurs at most $\lfloor w/3 \rfloor$ times. Guards at those corners protect every triangle and hence the entire gallery.

3.4 Colorful Consequences

Fisk's colorful proof of the art gallery theorem has several consequences.

How to Guard an Art Gallery: An Algorithm

The colorful proof not only guarantees that $\lfloor w/3 \rfloor$ guards suffice to protect any w -walled gallery but also tells us exactly where to post at most $\lfloor w/3 \rfloor$ guards. Briefly, triangulate, color, and post. The *art gallery algorithm* (Algorithm 3.1) formalizes the process.

The interactive site

cut-the-knot.org/Curriculum/Combinatorics/Chvatal.shtml

lets you build your own art galleries and triangulations; the applet then produces a 3-coloring and posts the guards.

What Is an Algorithm?

We have seen the first of several algorithms in this book, and it is appropriate to make a few comments here. An algorithm is a recipe—a list of precise instructions—that be-

Algorithm 3.1. Art gallery algorithm**Input:** art gallery G_w with w walls**Output:** positions for at most $w/3$ guards that protect G_w

1. Triangulate G_w by inserting suitable diagonals.
 2. Find a polychromatic 3-coloring of the corners of the triangulation.
 3. Post guards at the corners with the least frequently used color.
-

gins with given ingredients (the input) and ends at a specified goal (the output). Algorithms occur throughout mathematics but are especially prevalent in discrete mathematics. Reading and writing a well-constructed algorithm hones our problem-solving skills and focuses our attention on the essential aspects of a mathematical problem.

An algorithm to be used in a real-world application must be written with great formality in a suitable programming language to avoid the glitches for which computers have become infamous. The algorithms we present in this book are intended for human edification, not actual computer implementation. They are therefore less formal and written in ordinary English.

Nice Try, But ...

Now that we know that $\lfloor w/3 \rfloor$ guards suffice to protect any w -walled gallery, it is natural to seek a simpler and direct process to post the guards. For instance, one attempt to avoid the fuss of triangulation and coloring in Algorithm 3.1 merely posts guards at every third corner of the gallery.

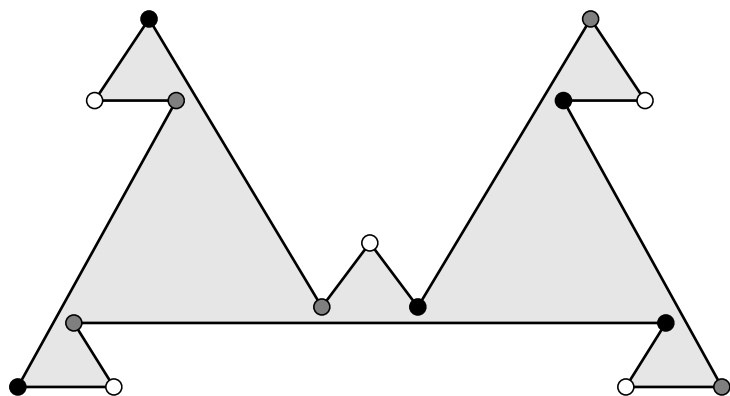


Figure 3.6: Guards at every third corner fail to protect the gallery

This naive strategy works for many galleries but fails for others. Consider the 15-walled gallery in Figure 3.6 with successive corners colored in a repeating black-white-gray pattern. If we post guards at all the black corners, then part of the gallery is unprotected. Guards at the white or gray corners also fail to protect the entire gallery.

Cornered Guards

The art gallery algorithm does not necessarily post the minimum number of guards needed to protect a given gallery. For instance, the algorithm posts four guards in the Sunflower Art Gallery in Figure 3.5, and we know that three guards suffice. In other words, Algorithm 3.1 solves the second gallery problem but not the first.

The algorithm also shows that we can protect the gallery G_w by placing at most $\lfloor w/3 \rfloor$ guards at *corners*. There is no need to place guards in the interior of the gallery—although such placements might be helpful in trying to find the true minimum number of guards required to protect G_w . For instance, one suitably placed guard along the “horizon” of

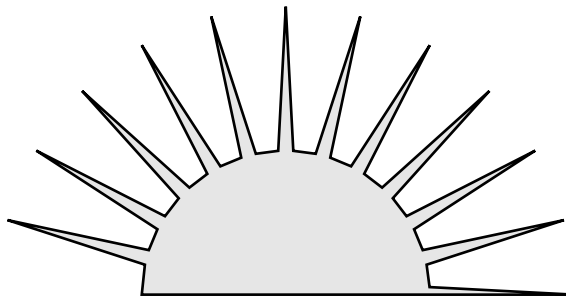


Figure 3.7: The Sunrise Gallery

the Sunrise Gallery in Figure 3.7 protects the entire gallery. But if we must place the guards at corners, then many more guards are needed to cover each ray of the sun.

3.5 Triangular and Chromatic Assumptions

A careful reader might object that our colorful proof of the art gallery theorem is incomplete because we relied on two assumptions without justifying them. Both assumptions are so plausible, you likely did not identify them as potential causes for concern.

Assumption 1. Every art gallery has a triangulation.

Assumption 2. Every triangulation has a polychromatic 3-coloring.

It is wise to question the assumptions we make in mathematics. Plausible assertions sometimes turn out to be false on closer inspection (as we will see later in this chapter), invalidating an entire line of reasoning. The correctness of the colorful proof art gallery theorem is not in doubt, however. Triangulations and polychromatic 3-colorings occur in several contexts in discrete mathematics and have been studied in detail. Rigorous justifications of both assumptions have

been known for a long time. See Problems 22 and 23 for a verification of assumption 1.

Polychromatic 3-Colorings

There is a convincing, constructive way to verify assumption 2. To show that the particular triangulation of the gallery G_9 in Figure 3.8 has a polychromatic 3-coloring, we first assign three different colors to the corners of an arbitrary triangle, say, triangle acd , as shown. Since each triangle is to contain one corner of each color, corner b must be the same color as corner d . Also, corner f must be the same color as corner c , and then i must be the same color as d . We continue in this manner and eventually produce the desired polychromatic 3-coloring for the entire triangulation.

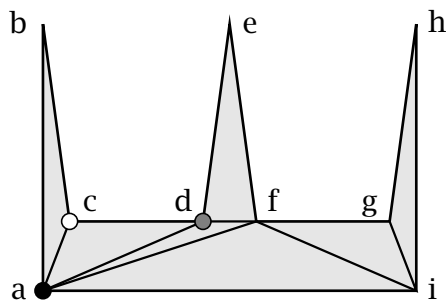


Figure 3.8: The start of a polychromatic 3-coloring

The same process works in general. Once three colors are assigned to the corners of any triangle, the colors for the remaining corners of the gallery are forced.

Degenerate Quadrilaterals

The triangulation of the gallery G_9 in Figure 3.8 illustrates a technical issue that sometimes arises in triangulating a gallery. We maintain that deleting diagonal ad destroys the tri-

angulation even though each remaining region would indeed be triangular. The problem is that the four corners a , c , d , and f of G_9 occur on the boundary of one triangular region. We regard such regions as *degenerate quadrilaterals*, not triangles, and exclude them from our triangulations. This exclusion is necessary for our colorful argument to work.

3.6 Modern Art Galleries

The art gallery theorem has inspired work on related problems in which the rules are changed in some manner to make the model more realistic or more interesting. The changes are of two types. First, we can restrict or relax the allowed shapes for the galleries. Second, we can bestow new powers on the guards or alter their responsibilities. All such variants are referred to as *art gallery problems*. The goals are the same as before. We want to find the minimum number of guards needed and write an efficient algorithm that posts a relatively small number of guards.

The remainder of this chapter is devoted to some of these art gallery problems. A few have been solved, usually by adapting Chvátal's inductive approach or Fisk's colorful argument, but many remain unsolved. The more realistic an art gallery problem, the more difficult it is to discover, to state, and to prove a counterpart to the basic guard formula $g(w) = \lfloor w/3 \rfloor$.

Fortresses, Prisons, and Zoos

Here are some examples of art gallery problems.

In the *fortress problem*, we view a polygon not as an art gallery to be protected against theft from the inside but as a fortress to be alerted to attack from the outside. The goal is to post the minimum number of guards along the fortress

walls so that every point outside the fortress is visible to at least one guard.

The *prison yard problem* asks us to post guards on the boundary of a polygon so that every point in the plane—both inside and outside the polygon—is visible to at least one guard. From a mathematical perspective, a prison yard is an art gallery on the inside but a fortress from the outside, a viewpoint presumably not shared by the prison yard's occupants.

In some realistic art gallery problems, the guards are mobile. We can ask for a path of minimum length inside a polygon such that every point in the polygon is visible to some point on the path. Such a path is an efficient route for a lone guard patrolling a large art gallery.

In the *zookeeper problem* we have a collection of disjoint polygons (the animals' cages) inside a large polygon (the zoo). We seek a path of minimum length inside the zoo that meets the boundary of each cage, while avoiding the interior. Such a path traces an efficient and safe route for a zookeeper at feeding time.

The whimsical names bestowed on art gallery problems do not limit the scope of possible applications. For example, the scientists directing the actions of a rover on Mars confront a type of zookeeper problem. The goal is to maneuver the rover to various locations, gather images and measurements of interesting features in the vicinity of the landing spot, and send the data to Earth. There are constraints on time and energy, and steep terrain must be avoided.

3.7 Art Gallery Sketches

We now state some art gallery theorems with proofs omitted.

Galleries with Holes

Most art galleries in the real world contain obstacles that block the sight lines of the guards. We model this situation by allowing *holes* in the interior of the galleries. We assume

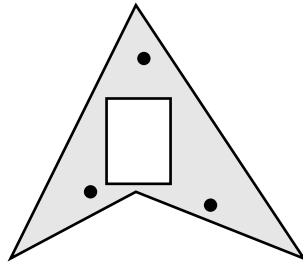


Figure 3.9: A gallery with eight walls and one hole

that each hole is a simple polygon. Guards must not be posted in the interiors of the holes, of course. Figure 3.9 shows a gallery with eight walls and one hole. The gallery is protected by three guards. It is not difficult to verify that the gallery cannot be protected by two guards.

The general problem asks for the minimum number of guards sufficient to protect any gallery with w walls and h holes. Note that the walls surrounding the holes contribute to w . Here is the main theorem in the area.

Theorem. Any art gallery with w walls and h holes can be protected by $\lfloor (w + h) / 3 \rfloor$ guards.

Half-Guards: Restricted Field of Vision

Suppose we want to protect a w -walled gallery with stationary cameras, each of which has a fixed 180° field of vision. We call this type of camera a *half-guard*. It seems likely that more than $w/3$ half-guards might be needed to compensate

for the restricted field of vision of the guards. In any case, $2 \lfloor w/3 \rfloor$ half-guards suffice since we can use $\lfloor w/3 \rfloor$ pairs of back-to-back half-guards. Surprisingly, we can always get away with just $\lfloor w/3 \rfloor$ half-guards.

Half-guard theorem. Any art gallery with w walls can be protected by $\lfloor w/3 \rfloor$ half-guards.

The half-guard theorem was established by the Hungarian mathematician Csaba Tóth in 2000. The new twist is that suitable corners for the half-guards can no longer be found by a colorful argument. In fact, it is sometimes necessary to place half-guards in the interior or along the walls of the gallery. Figure 3.10 shows a gallery protected by one half-guard along a wall. No corner placement of a lone half-guard does the job for this gallery.

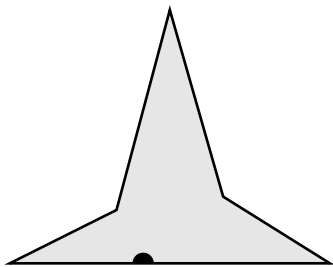


Figure 3.10: A gallery guarded by a half-guard

Rectangulated Galleries: Guarding the Met

Figure 3.11 shows a slightly modified floor plan of one section of the Metropolitan Museum of Art in New York City. (Some walls were adjusted, and a few new doorways were included.) Interior walls partition the large rectangle into rectangular rooms, and each pair of adjacent rooms is joined by

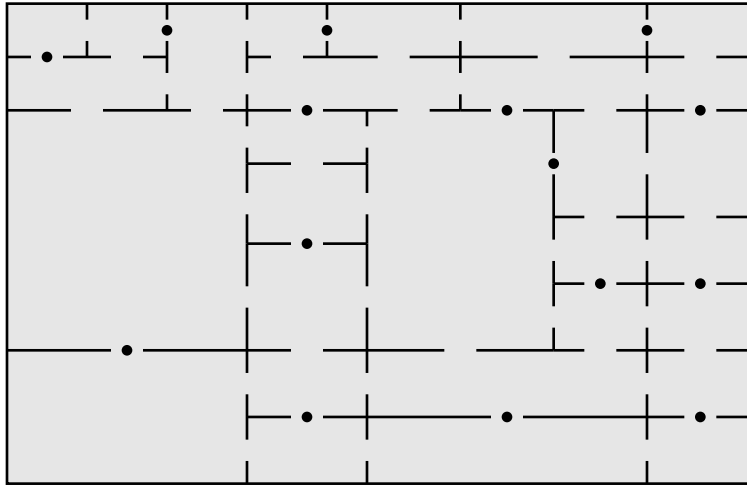


Figure 3.11: A rectangulated gallery with 29 rooms and 15 guards

a narrow doorway. We call this type of configuration a *rectangulated gallery*. As usual, we want to protect the gallery with as few guards as possible. Guards in doorways protect two rooms simultaneously. The gallery in Figure 3.11 has 29 rooms and is protected by 15 guards, which is the best we can hope for since 14 guards can protect at most 28 rooms. In general, if there are r rooms, then at least $\lceil r/2 \rceil$ guards are required. We have used the *ceiling* function, defined by

$\lceil x \rceil$ = the smallest integer greater than or equal to x .

It turns out that $\lceil r/2 \rceil$ guards suffice, but this is difficult to prove.

Rectangulated gallery theorem. Any rectangulated gallery with r rooms can be protected by $\lceil r/2 \rceil$ guards, but no fewer.

3.8 Right-Angled Art Galleries

We now examine several art gallery problems in detail and illustrate how art gallery results are discovered and proved.

Adjacent walls in a *right-angled art gallery* meet at right angles, just like the floor plans of most buildings. See Figure 3.12. Each interior angle is 90° or 270° . A right-angled

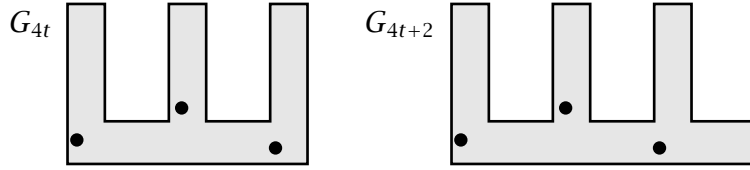


Figure 3.12: The comb-shaped galleries G_{4t} and G_{4t+2}

gallery can be drawn so that the walls run alternately north-south and east-west. It follows that the number of walls must be even.

We let

$g_{\perp}(w)$ = the maximum number of guards
required to protect a right-angled
art gallery with w walls.

The notation $g_{\perp}(w)$ is pronounced “ g perp of w .” The subscript is a visual reminder of the perpendicularity of the walls. The comb-shaped galleries in Figure 3.12 play the role of our earlier crown-shaped galleries. Each of the t teeth of the comb adds four more walls and requires one additional guard. It follows that the galleries G_{4t} and G_{4t+2} in the figure require t guards, giving us the lower bound

$$g_{\perp}(w) \geq \left\lfloor \frac{w}{4} \right\rfloor.$$

To establish the reverse inequality, we attempt to modify Fisk’s colorful argument. Figure 3.13 depicts a promising strategy. First, partition the right-angled gallery G_w into

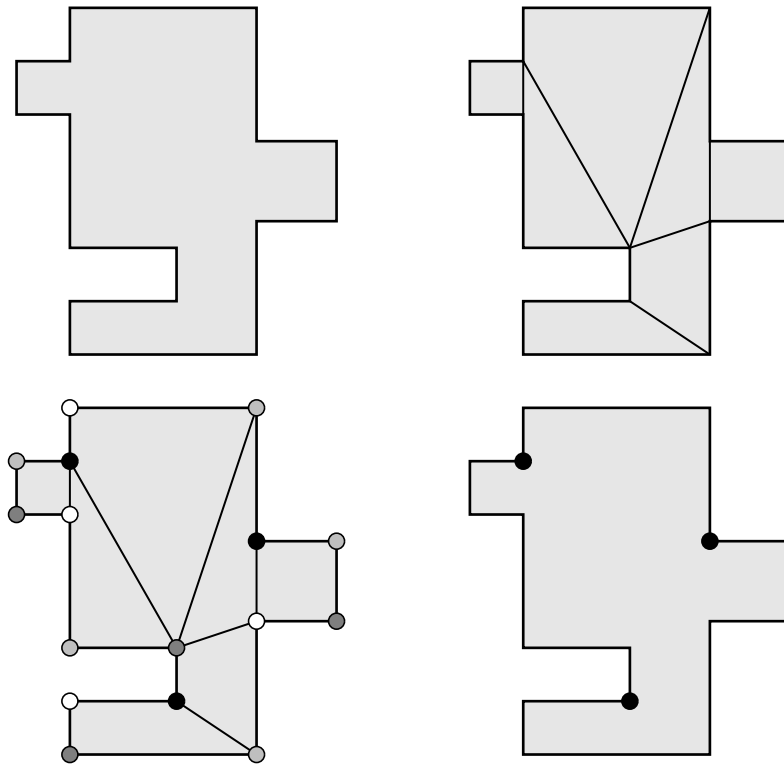


Figure 3.13: A colorful argument for right-angled galleries

quadrilaterals by inserting noncrossing diagonals. The resulting configuration is a *quadrangulation* of G_w . Each quadrilateral has exactly four corners of G_w on its boundary. If three of the corners are collinear, then the quadrilateral has a triangular shape, a degeneracy we now allow. Second, assign one of four colors (black, dark gray, light gray, white) to each of the w corners of G_w so that each quadrilateral has one corner of each color. The least frequently used color in this *polychromatic 4-coloring* occurs at most $\lfloor w/4 \rfloor$ times. Finally, post guards at these corners (the black ones in Figure 3.13); the whole gallery is protected since every quadri-

lateral has a black corner. It seems we have established the following result.

Right-angled art gallery theorem. We have

$$g_{\perp}(w) = \left\lfloor \frac{w}{4} \right\rfloor \quad \text{for } w = 4, 6, 8, \dots$$

In other words, $\lfloor w/4 \rfloor$ guards are sufficient and sometimes necessary to protect a right-angled art gallery with w walls.

Alas, our colorful argument has a flaw. It fails to post the guards correctly in some situations. Consider the polychromatic 4-coloring of the quadrangulation in Figure 3.14. The three guards at the black corners fail to protect part of the rightmost nook of the gallery. The reason is clear. A guard at a corner of nonconvex quadrilateral might not cover the entire quadrilateral.

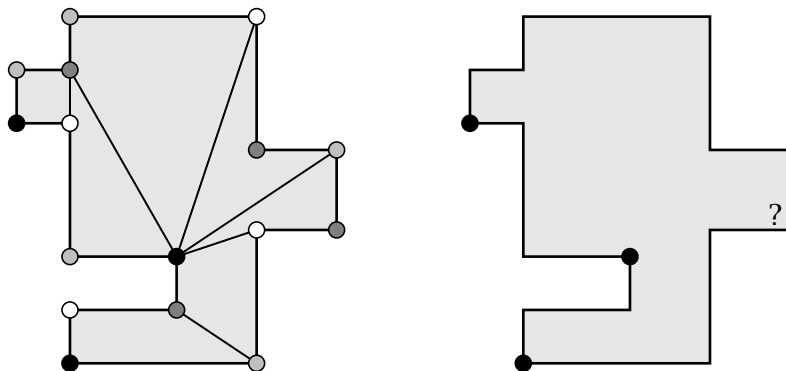


Figure 3.14: A convex quadrangulation is required

To guarantee that the colorful argument works, we must start with a *convex quadrangulation*—one whose quadrilaterals are all convex. Jeffry Kahn, Maria Klawe, and Daniel Kleitman fixed the flaw in the above argument by proving the following result in 1985.

Convex quadrangulation theorem. Every right-angled art gallery has a convex quadrangulation.

The right-angled art gallery algorithm (Algorithm 3.2) formalizes our discussion.

Algorithm 3.2. Algorithm for right-angled art galleries

Input: right-angled art gallery G_w with w walls

Output: positions for at most $w/4$ guards that protect G_w

1. Form a convex quadrangulation of G_w by inserting suitable diagonals.
 2. Find a polychromatic 4-coloring of the corners of the quadrangulation.
 3. Post guards at the corners with the least frequently used color.
-

3.9 Guarding the Guards

We now demand that our guards protect one another in addition to the art gallery. Every guard must be visible to at least one other guard. Such configurations protect against an ambush of an isolated guard. We refer to *guarded guards* in this case and study the function

$gg(w)$ = the maximum number of
 guarded guards required to protect
 an art gallery with w walls.

It is not difficult to see that $gg(w) = 2$ for $w = 3, 4, 5, 6$. We must have

$$\left\lfloor \frac{w}{3} \right\rfloor \leq gg(w) \leq 2 \left\lfloor \frac{w}{3} \right\rfloor.$$

The left inequality is clear because we need at least $g(w)$ guards just to cover the gallery. Also, by starting with a feasible configuration of $g(w)$ guards and assigning each guard a nearby partner, we see that no more than $2g(w)$ guarded guards are needed.

To determine the formula for $gg(w)$, we might start with another modification to the crown galleries. The New Wave Gallery G_{5t} (shown for $t = 4$ in Figure 3.15) has t waves and $5t$ walls. Note that G_{5t} requires $2t$ guarded guards since each additional wave increases the number of walls by 5 and the number of guarded guards required by 2. We dent the gallery in suitable places if w is not divisible by 5 and conclude that

$$gg(w) \geq \left\lfloor \frac{2w}{5} \right\rfloor \quad \text{for } w = 5, 6, 7, \dots$$

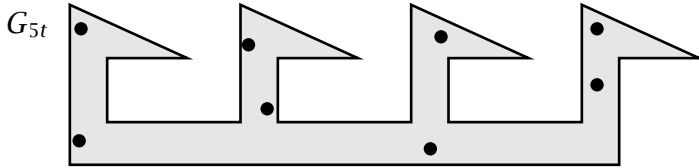


Figure 3.15: The New Wave Gallery G_{5t}

It is tempting to conjecture that $gg(w) = \lfloor 2w/5 \rfloor$, but a rude counterexample intrudes at $w = 12$. The 12-walled gallery in Figure 3.16 requires five guarded guards, contrary to the predicted maximum of $\lfloor 2w/5 \rfloor = \lfloor (2 \times 12) / 5 \rfloor = 4$. As

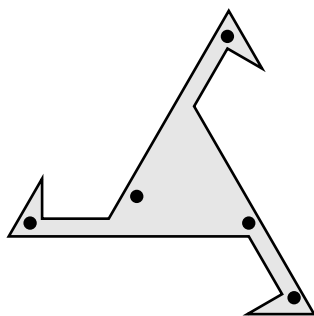


Figure 3.16: A 12-walled gallery requiring five guarded guards

this counterexample suggests, the true formula for guarded guards is more complicated than any we have encountered so far.

Guarded guards theorem. We have

$$gg(w) = \left\lfloor \frac{3w - 1}{7} \right\rfloor \quad \text{for } w = 5, 6, 7, \dots$$

In other words, $\lfloor (3w - 1) / 7 \rfloor$ guarded guards are sufficient and sometimes necessary to protect an art gallery with w walls for $w = 5, 6, 7, \dots$

Table 3.1 gives some values of the function $gg(w)$. The only known proof of the guarded guards theorem follows the same general scheme of Chvátal's inductive argument

Table 3.1: Number of guarded guards for a gallery with w walls

w	5	6	7	8	9	10	11	12	13	14	15	16	17
$gg(w)$	2	2	2	3	3	4	4	5	5	5	6	6	7

for the original art gallery theorem, but two issues complicate the argument. First, the w -walled galleries that require $\lfloor (3w - 1) / 7 \rfloor$ guarded guards have complex shapes that are difficult to discover and describe.

Challenge 1. Can you find a 17-walled gallery that requires seven guarded guards? Look at Figure 3.15 if you are stumped.

Second, the inductive step is more subtle than the one used by Chvátal. Is there a pleasant, colorful argument? Nobody has found one yet.

Research problem. Find a colorful, Fisk-like argument for the inequality $gg(w) \leq \lfloor (3w - 1) / 7 \rfloor$.

Guarded Guards in Right-Angled Galleries

The guarded guards formula for right-angled galleries turns out to be less complicated. Let

$gg_{\perp}(w)$ = the maximum number of guarded guards required to protect a right-angled art gallery with w walls.

The Square Wave Gallery G_{6t} (shown for $t = 4$ in Figure 3.17) has t waves and $6t$ walls. Note that G_{6t} requires

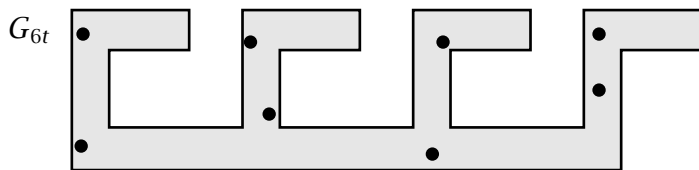


Figure 3.17: The Square Wave Gallery G_{6t}

$2t$ guarded guards since each additional wave increases the number of walls by 6 and the number of guarded guards required by 2. We truncate the gallery in suitable places if w is not divisible by 6 and conclude that

$$gg_{\perp}(w) \geq \left\lfloor \frac{w}{3} \right\rfloor \quad \text{for } w = 6, 8, 10, \dots$$

This time there are no surprises.

Guarded guards theorem for right-angled galleries. We have

$$gg_{\perp}(w) = \left\lfloor \frac{w}{3} \right\rfloor \quad \text{for } w = 6, 8, 10, \dots$$

In other words, $\lfloor w/3 \rfloor$ guarded guards are sufficient and sometimes necessary to protect a right-angled art gallery with w walls for $w = 6, 8, 10, \dots$

To show that $\lfloor w/3 \rfloor$ guarded guards are sufficient for a right-angled gallery, we again modify Fisk's colorful argument. Start with a convex quadrangulation of a right-angled gallery G_w , as in Figure 3.18(a). Then triangulate the gallery by inserting a diagonal in each quadrilateral (the thin lines in (b)). The inserted diagonals should "alternate" so that if two quadrilaterals share an edge, then their diagonals do not share a corner. After the diagonal for one quadrilateral is selected arbitrarily, the diagonals for the other quadrilaterals are all forced by this alternating condition. The resulting triangulation has a polychromatic 3-coloring, as shown in (b), and we post guards temporarily at the corners of the least frequently used color—the four black corners in (c). Of course, we have posted at most $\lfloor w/3 \rfloor$ guards, and these guards protect the entire gallery.

Alas, some guards might be invisible to all other guards. The lowest guard in (c) is invisible to the other guards, as is the rightmost guard. We remedy this situation by giving

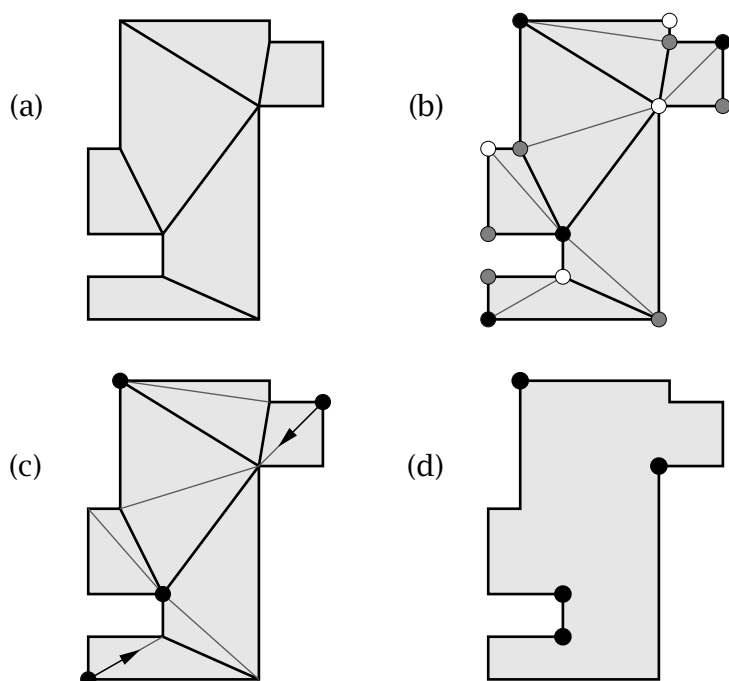


Figure 3.18: Guarded guards for right-angled galleries

marching orders to each guard standing at a corner with exactly one diagonal. Such a guard must march along the diagonal to the opposite corner, as indicated by the arrows in (c). If two or more guards end up at the same corner, then the extra ones are sent home. One can prove that the resulting configuration of guards is indeed guarded and that the entire gallery remains protected, as in (d). We do not give the details.

As with the original art gallery theorem, the resulting configuration of guarded guards need not be minimal. For instance, the process posts four guarded guards in the right-angled gallery in Figure 3.18(d), but three guarded guards suffice; simply dismiss the uppermost guard.

3.10 Three Dimensions and the Octoplex

Art galleries and other buildings in the real world are three-dimensional, a fact that has been conspicuously absent from our discussion so far. Let us model three-dimensional galleries by *polyhedra*—solid shapes bounded by polygons. Familiar polyhedra include cubes, prisms, and pyramids. As usual, we want to post cameras in the gallery so that every point is visible to at least one camera. We use security cameras instead of guards since it will sometimes be necessary to post them on the ceiling, in midair, or at other inconvenient locations. We make the somewhat unrealistic assumption that a camera can see in all directions.

Research question. What is the maximum number of security cameras required to protect a three-dimensional gallery with c corners?

We seek a simple answer to the three-dimensional art gallery problem, similar to our expressions for two-dimensional galleries. However, no one has been able to find such a formula—or even propose a plausible guess—for reasons we will explain soon.

In a two-dimensional gallery the number of corners is equal to the number of walls. But these parameters are unequal for most three-dimensional galleries. (A corner is a point where three or more walls meet.) We would be just as pleased to answer the research question in terms of the number of walls.

The Octoplex

Guards posted at each corner of a two-dimensional gallery certainly protect the whole gallery. Astonishingly, this obvious assertion is false for three-dimensional galleries. This

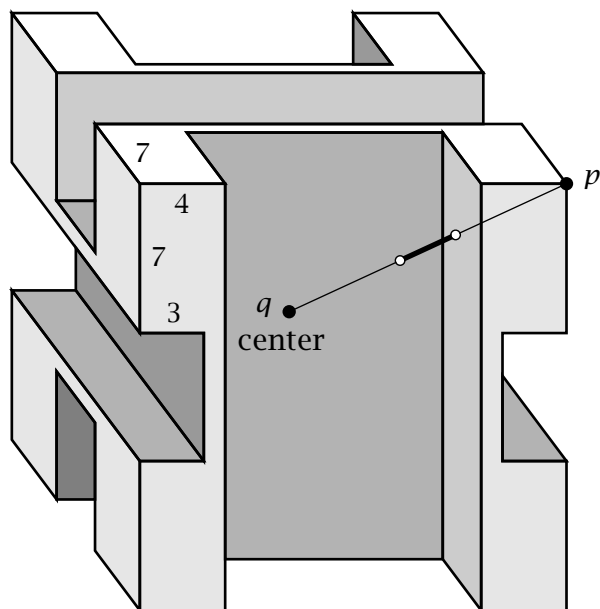


Figure 3.19: The Octoplex

fact frustrates would-be solvers of the three-dimensional problem.

The example in Figure 3.19 is constructed as follows. Start with a 20-by-20-by-20 cube. Remove a rectangular channel 12 units wide and 6 units deep from the center of the front face. There is an identical channel in the back face. The channels in the left and right faces are 6 units wide and 3 units deep, while those in the top and bottom faces are 6 units wide and 6 units deep. The figure that remains is the *Octoplex*. It consists of eight 4-by-7-by-7 theaters connected to one another and to a central lobby by passageways 1 unit wide. The Octoplex has 56 corners and 30 walls.

Claim. Even if we post a camera at every corner, part of the Octoplex is unprotected.

To see why the claim is true, observe that the center point q of the Octoplex is not visible to a camera at the corner p in Figure 3.19 since the direct line of sight from p to q exits and then reenters the Octoplex, as shown. Similar reasoning shows that point q is hidden from cameras at the other 55 corners. In fact, there is a small region in the middle of the lobby that is hidden from every corner camera.

Challenge 2. Protect the Octoplex with 25 cameras. At least one of your cameras will not be at a corner. Can you find a way to use fewer cameras?

The Megaplex

For some three-dimensional galleries, the number of cameras required greatly exceeds the number of corners. The Megaplex is formed by a cubical arrangement of m^3 abutting copies of the Octoplex, as shown for $m = 4$ in Figure 3.20. The interior walls of adjacent theaters are removed so that some theaters in the Megaplex are formed by merging two, four, or eight Octoplex theaters. For the sake of clarity, the walls separating each Octoplex from its neighbors are retained in the figure. There are m^3 lobbies in the Megaplex. Also, The many channels and shafts do not cross one another.

Claim. The Megaplex has $c = 24m^2 + 24m + 8$ corners and requires at least $m^3/8$ cameras to protect.

To see why the claim is true, observe that the Megaplex has eight outer corners, and that each of the $3m(m + 1)$ shafts and channels also contributes eight corners. The total number of corners is therefore

$$c = 8 + 3m(m + 1) \times 8 = 24m^2 + 24m + 8.$$

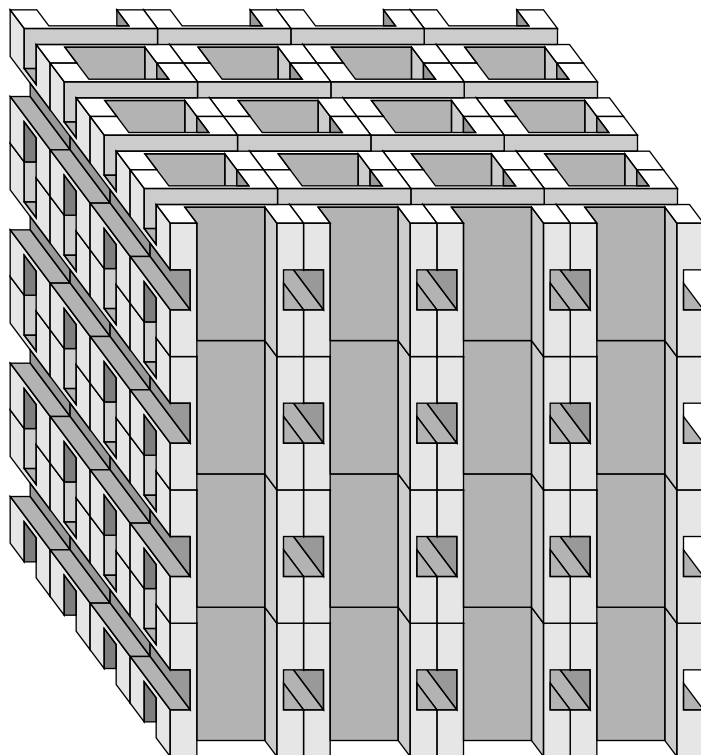


Figure 3.20: The Megaplex

A little experimentation shows that no camera could possibly cover the centers of eight lobbies, and it follows that at least $m^3/8$ cameras are required.

Notice that the ratio of cameras to corners satisfies

$$\frac{\text{cameras}}{\text{corners}} \geq \frac{m^3/8}{24m^2 + 24m + 8} \geq \frac{(m^3 - 1)/8}{24(m^2 + m + 1)} \geq \frac{m - 1}{192}.$$

The factorization $m^3 - 1 = (m - 1)(m^2 + m + 1)$ was used for the last inequality. If $m \geq 194$, then the number of cameras required exceeds the number of corners of the Megaplex. Moreover, as m increases, the ratio of cameras to corners

becomes arbitrarily large, which dashes any hope for a linear upper bound (say, $1000c$) for the number of cameras required. A similar assertion holds in terms of the number of walls. Although our Megaplex with $m \geq 194$ does not resemble any three-dimensional building in the real world, it does show that the three-dimensional guarding problem is fundamentally different from the two-dimensional problems we have seen.

The horizontal shafts passing completely through the Megaplex from front to back form undesirable “holes.” We can eliminate them by erecting a thin wall to close up the back of each shaft. The shafts in other directions can be dealt with similarly, and the essential features of the Megaplex are preserved.

3.11 Notes and References

The original proof of the art gallery theorem is by Chvátal [2]. A more leisurely treatment of Chvátal’s proof by mathematical induction appears in Honsberger’s book [5]. The book [1] includes a first-hand account of how Fisk discovered his colorful proof [4] on a bus trip in Afghanistan. The right-angled art gallery theorem was first proved by Kahn, Klawe, and Kleitman [6]. Guarded guards are examined in [7]. Żyliński [12] surveys the use of colorful arguments in proving art gallery theorems. The theorem on rectangulated galleries appears in the paper by Czyzowicz et al. [3].

In 1987, O’Rourke wrote *the* book [9] on art gallery theorems, covering both theory and algorithms. Variants examined in O’Rourke’s book include mobile guards, the fortress and prison yard problems, and three-dimensional galleries. The encyclopedic tome [11] spans all of computational geometry, and the chapter on art gallery theorems has more than 100 references. Algorithms for art gallery problems (and computational geometry in general) are treated in [10], which also covers computational complexity.

The site

<http://maven.smith.edu/~orourke/TOPP/>

lists unsolved problems in computational geometry.

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3.12 Problems

The problems deal with two-dimensional art galleries.

1. Find a triangulation of the Sunflower Art Gallery and a polychromatic 3-coloring that leads to a posting of three guards.
2. True or false.
 - (a) If G is a convex gallery, then $\text{guard}(G) = 1$.
 - (b) If $\text{guard}(G) = 1$, then the gallery G is convex.

- (c) If $\text{guard}(G) \geq 2$, then G has at least six walls.
 - (d) If G has at least six walls, then $\text{guard}(G) \geq 2$.
3. (a) Exhibit an art gallery with eight walls that has a unique triangulation.
- (b) Exhibit a w -walled gallery that has a unique triangulation for each $w = 3, 4, \dots$.
4. Which corners are the same color as c when Figure 3.8 is completed to a polychromatic 3-coloring?
5. Let s be the number of 90° interior angles in a right-angled gallery with w walls. Show that $w = 2s - 4$. Hint: What is the sum of all the angles in the gallery?
6. Exhibit an art gallery with both of the following properties.
- The gallery can be protected by one guard.
 - It is possible to post guards at seven corners and not protect the entire gallery.
7. Exhibit an art gallery with both of the following properties.
- The gallery can be protected by two guards but not by one guard.
 - It is possible to post guards at 29 corners and not protect the entire gallery.
8. Let G_{15} denote the 15-walled gallery in Figure 3.6.
- (a) Protect G_{15} with five guards.
 - (b) Show that G_{15} cannot be protected by four guards.
 - (c) What is the minimum number of guarded guards needed to protect G_{15} ?
9. The galleries in Figure 3.17 show that $gg_\perp(w) \geq \lfloor w/3 \rfloor$ when w is divisible by 6. Exhibit right-angled galleries for the cases when $w - 2$ or $w - 4$ is divisible by 6.

10. Find the final configuration of guarded guards if we begin with guards at the four white corners in Figure 3.18(b).
11. Write an algorithm to post guarded guards in right-angled galleries. The input of your algorithm will be a right-angled gallery G_w with w walls ($w \geq 6$), and the output will be the positions of at most $\lfloor w/3 \rfloor$ guarded guards that protect G_w .
12. Post 10 guards in a particular 17-walled art gallery so that the entire gallery is protected, but dismissal of any guard leaves some part of the gallery unprotected.
13. Figure 3.21 shows four guards that protect a rectangulated gallery with six rooms and one rectangular hole.
 - (a) Explain why the gallery cannot be protected by three guards.
 - (b) Explain why every rectangulated gallery with r rooms and one rectangular hole can be protected by $\lceil (r + 1) / 2 \rceil$ guards.
 - (c) Explain why every rectangulated gallery with r rooms and h rectangular holes can be protected by $\lceil (r + h) / 2 \rceil$ guards.

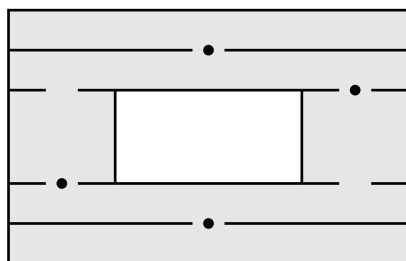


Figure 3.21: A rectangulated gallery with a hole

14. There is an art gallery with $8t$ walls and t holes that requires $3t$ guards for each $t = 1, 2, 3, \dots$. Figure 3.9 shows such a gallery for $t = 1$. Find a gallery for $t = 2, 3, \dots$
15. What is the minimum number of half-guarded needed to protect the Sunflower Art Gallery?
16. Give an example of a gallery that can be protected by one guard but not by one half-guard.
17. Explain why a triangulation of a w -walled gallery must have $w - 2$ triangles and $w - 3$ diagonals.
18. (a) Find a 10-walled gallery requiring four guarded guards.
(b) Find a 15-walled gallery requiring five guarded guards.
19. The Scorpio Gallery in Figure 3.22 has 17 walls.
 - (a) Protect the gallery with seven guarded guards.
 - (b) Show that the gallery cannot be protected by six guarded guards.

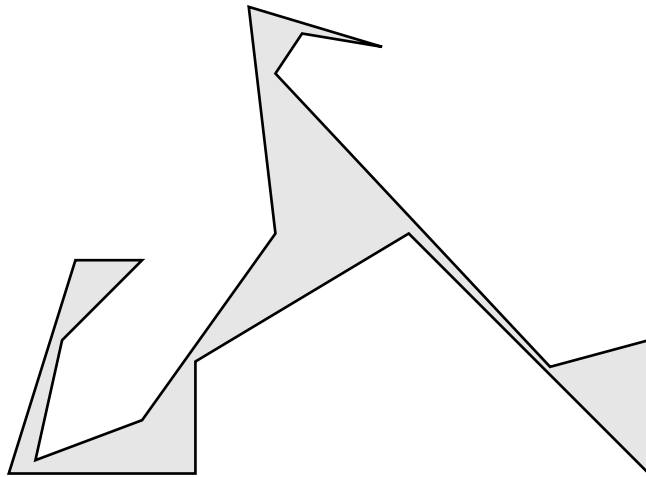


Figure 3.22: The Scorpio Gallery

20. Consider a triangulation of an art gallery with at least four walls. Identify each statement as true or false.
- (a) Every triangle in the triangulation must have at least one side in common with the boundary of the gallery.
 - (b) There is a triangle with exactly two sides in common with the boundary.
 - (c) There are at least two triangles with exactly two sides in common with the boundary.
21. Consider a triangulation of an art gallery with at least four walls. Let N_k denote the number of triangles with exactly k sides in common with the boundary of the gallery. Of course, $N_k = 0$ for $k \geq 3$.
- (a) Explain why $N_0 + N_1 + N_2 = w - 2$.
 - (b) Explain why $N_1 + 2N_2 = w$.
 - (c) Show that $3N_0 + 2N_1 + N_2 = 2w - 6$.
 - (d) Show that $N_2 = N_0 + 2$.
22. The purpose of this problem is to prove that every art gallery with at least four walls has a diagonal. Let G_w be an art gallery with w walls ($w \geq 4$), and let q be the leftmost corner of the gallery. If more than one corner is leftmost, choose the lowest of these. Let p , q , and r be consecutive corners of the gallery and consider the triangle pqr .
- (a) Suppose that p , q , and r are the only corners of G_w in pqr (including the boundary). Show that segment pr is a diagonal of G_w .
 - (b) Suppose that pqr contains at least one other corner of G_w . Show that G_w has a diagonal with one endpoint at q .
23. Show that every art gallery has a triangulation. Hint: Assume that every art gallery with at least four walls has a diagonal (Problem 22).

24. This problem outlines Chvátal's proof of the art gallery theorem. Consider a triangulated art gallery G_w with w walls ($w \geq 4$). Select any diagonal of the triangulation. The diagonal partitions G_w into two galleries with w_1 and w_2 walls.

- (a) Explain why $w_1 + w_2 = w + 2$.
- (b) Show that if $w \geq 6$, then we can always select a diagonal so that $w_1 = 5, 6$, or 7 . Hint: Among all diagonals for which $w_1 \geq 5$, choose one for which w_1 is smallest.
- (c) Assume that $w_1 = 5$. Show that

$$\left\lfloor \frac{w_1}{3} \right\rfloor + \left\lfloor \frac{w_2}{3} \right\rfloor = \left\lfloor \frac{w}{3} \right\rfloor.$$

Hint: Show that

$$\left\lfloor \frac{w_2}{3} \right\rfloor = \left\lfloor \frac{w-3}{3} \right\rfloor = \left\lfloor \frac{w}{3} \right\rfloor - 1.$$

- (d) Assume that $w_1 = 5$. Make an inductive hypothesis and apply (c) to show that $g(w) \leq \lfloor w/3 \rfloor$. Chvátal also deals with the more difficult cases $w_1 = 6$ and $w_1 = 7$.

*The picture will have charm when each color
is very unlike the one next to it.*

LEON BATTISTA ALBERTI

Who will guard the guardians?

JUVENAL

*Science is what we understand
well enough to explain to a computer.*

Art is everything else we do.

DONALD KNUTH

Mighty is geometry; joined with art, resistless.

EURIPEDES

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