Roger D. Maddux

The Origin of Relation Algebras in the Development and Axiomatization of the Calculus of Relations

Abstract. The calculus of relations was created and developed in the second half of the nineteenth century by Augustus De Morgan, Charles Sanders Peirce, and Ernst Schröder. In 1940 Alfred Tarski proposed an axiomatization for a large part of the calculus of relations. In the next decade Tarski's axiomatization led to the creation of the theory of relation algebras, and was shown to be incomplete by Roger Lyndon's discovery of nonrepresentable relation algebras. This paper introduces the calculus of relations and the theory of relation algebras through a review of these historical developments.

1. Introduction

One of the purposes of this paper is to present an introduction to both the calculus of relations and relation algebras in a historical context. This is done in §§2–3 through a close look at two of the seminal papers of the subject, *On the Syllogism: IV, and on the Logic of Relations*[1] by Augustus De Morgan, and *Note B: the logic of relatives*[2] by Charles Sanders Peirce. In §3 we accumulate a representative sample of results in the calculus of relations, presented in a uniform notation. In §4 we discuss Tarski's axiomatization of the calculus of relations. The definition of relation algebras evolved from Tarski's axiomatization, as is shown in §5. Tarski's axiomatization is strong enough to prove a huge number of results in the calculus of relations, so it is natural to ask whether this axiomatization is complete. In the context of relation algebras, this question turns out to be closely connected to the question of whether there are relation algebras which are not representable. Both questions were answered by Roger Lyndon[3], as we shall see in §6.

The theory of relation algebras is currently classified as part of algebraic logic. (In the American Mathematical Society's Subject Classification, 03G15 is relation, cylindric, and polyadic algebras, while 03Gxx is algebraic logic.) The reason for this is largely historical: the founding of the theories of cylindric and polyadic algebras by Tarski and Halmos, respectively, was a conscious effort to create algebra out of logic, more specifically, to create algebra out of first order predicate calculus. (This is also true, but less so, for relation algebras, as we shall see.) It is interesting to note, and

[1] [De Morgan1864b].
[2] [Peirce1883b].
[3] [Lyndon1950], §8.

it is one of the purposes of this paper to show, that the actual historical development is the other way around: first order predicate calculus has its origins in the calculus of relations. The calculus of relations is indeed the result of Peirce's efforts to create algebra out of logic, but these efforts took place decades before the emergence of first order logic in the 1920's, and are instead based on the pioneering work of Boole.[4] Peirce's efforts to get a "good general algebra of logic" (as he called it) led him not only to develop the algebra of relations, but also to find convenient ways to explicate and work with his algebra, ways which led directly to first order logic. The main contributor to the development of first order logic is Frege, but Frege and Peirce worked independently. The early notation for quantifiers, as well as the name "quantifier", originate with Peirce. Löwenheim's original version of the Löwenheim-Skolem Theorem[5] is not a theorem about first order logic, but about the calculus of relations. Indeed, it is in Löwenheim's 1915 paper that "first order expressions" are first singled out for special attention.[6] This important step toward the emergence of first logic is connected to the initial work of Peirce through the only thorough treatise on the calculus of relations, namely Volume III of Ernst Schröder's **Algebra der Logik**[7]. In §3 we will see the beginnings of Peirce's move toward first order logic and how it was motivated by his algebra of relations.

2. The calculus of relations: history and name

The most important figures in the creation of the calculus of relations in the nineteenth century were Augustus De Morgan, Charles Sanders Peirce, and F. W. K. Ernst Schröder. The rôles of these three founders are summarized in a brief historical sketch in the 1911 edition of the **Dictionary of Philosophy and Logic.**[8] The sketch occurs at the end of the article on "Relatives". This article was written by Peirce himself.[9]

> *Literature:* relatives have, since Aristotle, been a recognized topic of logic. The first germ of the modern doctrine appears in a somewhat trivial remark of *Robert Leslie Ellis*. *De Morgan* did the first systematic work in his fourth memoir on the syllogism in 1860 (*Cambridge Philosophical Transactions*, x. 331–358); he here sketched out the theory of dyadic relations. C. S. Peirce, in

[4][Boole1847].
[5][Löwenheim1915].
[6]See [Moore1987] for more details.
[7][Schröder1895].
[8]See [DPP1911]. For another brief history of the calculus of relations, including remarks on the forty years following Peirce's summary, see the first four paragraphs of [Tarski1941], pp. 73–74. See also [Tarski-Givant1987], p. xv, and [Lewis1918].
[9]See [Peirce1933], Volume III, pp. 404–409, for Peirce's article on relatives. The historical sketch is paragraph 3.643. (References to [Peirce1933] of the form "$n.m$" refer to paragraph number m in volume n.)

1870, extended Boole's algebra so as to apply to them, and after many attempts produced a good general algebra of logic, together with another algebra specially adapted to dyadic relations (*Studies in Logic*, by members of the Johns Hopkins University, 1883, Note B, 187–203). Schröder developed the last in a systematic manner (which brought out its glaring defect of involving hundreds of merely formal theorems without any significance, and some of them quite difficult) in the third volume of his *Exacte Logik* (1895). Schröder's work contains much else of great value.

De Morgan's pioneering paper, referred to by Peirce simply as De Morgan's "fourth memoir", is *On the Syllogism: IV, and on the Logic of Relations*[10]. It was completed by November 12, 1859, and was read before the Cambridge Philosophical Society on April 23, 1860. It appeared in print in 1864, in the Transactions of the Cambridge Philosophical Society.[11] De Morgan opens with the following statement of purpose.

In my second and third papers[12] on logic I insisted on the ordinary syllogism being one case, and one case only, of the *composition of relations*. In this fourth paper I enter further on the subject of *relation*, as a branch of logic.[13]

In one section of his paper, De Morgan considers the complement and converse relations as well as the composition of relations, and states some laws governing combinations of these operations.

De Morgan's fourth paper greatly influenced Peirce's paper of 1870, *Description of a notation for the logic of relatives, resulting from an amplification of the conceptions of Boole's calculus of logic.*[14] This paper is sometimes referred to as DNLR in Peirce circles, and it has been the object of some study.[15] It has also had contemporary mathematical influence[16]. This large and complex paper opens as follows:

Relative terms usually receive some slight treatment in works upon logic, but the only considerable investigation in the formal laws which govern them is contained in a valuable paper by Mr. De Morgan in the tenth volume of the *Cambridge Philosophical Transactions*. He there uses a convenient algebraic notation, which is formed by adding to the well-known *spiculæ* of that writer the signs used in the following examples.

X..LY signifies that X is some one the objects of thought which stand to Y in the relation L, or is one of the L's of Y.

X.LMY signifies that X is not an L of an M of Y.

X.(L,M)Y signifies that X is either an L or an M of Y.

[10] [De Morgan1864b].

[11] Volume **10**, pp. 331–358.

[12] De Morgan's second and third papers are [De Morgan1856] and [De Morgan1864a].

[13] [De Morgan1966], p. 208.

[14] [Peirce1870].

[15] See, for example, [Brink1978], [Brink1979], [Brunning1980], [Martin1976], [Martin1978], and [Merrill1984].

[16] See [Brink1981] and [Brink1988].

> LM′ an L of every M. L,M an L of none but M's.
> $L^{-1}Y$ something to which Y is L. l (small L) non-L.
> This system still leaves something to be desired. Moreover, Boole's logical algebra has such singular beauty, so far as it goes, that it is interesting to inquire whether it cannot be extended over the whole realm of formal logic, instead of being restricted to that simplest and least useful part of the subject, the logic of absolute terms, which, when he wrote, was the only formal logic known. The object of this paper is to show that an affirmative answer can be given to this question. I think there can be no doubt that a *calculus*, or art of drawing inferences, based upon the notation I am to describe, would be perfectly possible and even practically useful in some difficult cases, and particularly in the *investigation* of logic. I regret that I am not in a situation to perform this labor, but the account here given of the notation itself will afford the ground of a judgment concerning its probable utility.[17]

In the years following Peirce's initial paper in 1870, he "produced a good general algebra of logic". The evolution of Peirce's algebra of logic can be traced through his major published papers on this subject.[18] On their basis one could defend the thesis that first order logic, and, to a lesser extent, second order logic evolved from Peirce's algebraic logic.

De Morgan speaks of the "logic of relations". But Peirce refers to the logic of "relatives" instead of "relations" throughout most of his papers on the subject. One might therefore expect that the calculus of relations should actually be called the "calculus of relatives". Eventually Peirce regretted this change in terminology.

> I must, with pain and shame, confess that in my early days I showed myself so little alive to the decencies of science that I presumed to change the name of this branch of logic, a name established by its author and my master, Augustus De Morgan, to "the logic of relatives." I consider it my duty to say that this thoughtless act is a bitter reflection to me now, so that young writers may be warned not to prepare for themselves similar sources of unhappiness. I am the more sorry, because my designation has come into general use.[19]

In his historical sketch, Peirce made special reference to his small paper called *Note B*, which appears at the end of the book **Studies in Logic**. This book, which was edited by Peirce, contains papers by students in Peirce's courses on logic at the Johns Hopkins University, 1879–82. These papers were first presented in meetings of the Metaphysical Club, which Peirce started in 1879 and over which he presided for its first three years. **Studies in Logic** ends with a paper by Peirce himself, called *A theory of probable inference*. The running head for the second note (*Note B*) appended to this paper is *The logic of relatives*. This note outlines the final form of Peirce's

[17] [Peirce1933], 3.45.
[18] The most important papers are [Peirce1870], [Peirce1880], [Peirce1882], [Peirce1883b], [Peirce1885], [Peirce1892] and [Peirce1897].
[19] This is the second footnote in [Peirce1903]; see [Peirce1933], 3.574.

"algebra specially adapted to dyadic relations". It is the system sketched in this paper which Schröder develops into 649 pages, perhaps thereby incurring Peirce's assessment that Schröder "brought out its glaring defect of involving hundreds of merely formal theorems without any significance".

3. The calculus of relations: basic concepts and formulæ

We present the calculus of relations by following the presentation in Peirce's *Note B*, with comparisons to De Morgan's *On the Syllogism: IV.* The notation of this paper is primarily that of Schröder. At the end of this section is a list of nearly all the formulæ of the calculus of relations which can be gleaned from Peirce's *Note B.* In this section, any reference of the form "(Bn)", with $1 \leq n \leq 50$, refers to that list.

Let U be an arbitrary nonempty set, called the "universe of discourse". In the laws stated below, x, y and z are arbitrary binary relations between elements of U. In symbols, $x, y, z \leq U \times U$. (Here we use "$\leq$" to symbolize set inclusion.) Hence, for example,

(A1) $$x = \{\langle a, b\rangle : \langle a, b\rangle \in x\}.$$

As Peirce puts it,

> A dual relative term, such as "lover," "benefactor," "servant," is a common name signifying a pair of objects. Of the two members of the pair, a determinate one is generally the first, and the other the second; so that if the order is reversed, the pair is not considered as remaining the same.
>
> Let A, B, C, D, etc., be all the individual objects in the universe; then all the individual pairs may be arrayed in a block, thus :—

A : A	A : B	A : C	A : D	etc.
B : A	B : B	A : C	B : D	etc.
C : A	C : B	C : C	C : D	etc.
D : A	D : B	D : C	D : D	etc.
etc.	etc.	etc.	etc.	etc.

> A general relative may be conceived as a logical aggregate of a number of such individual relatives. Let l denote "lover"; then we may write
>
> $$l = \Sigma_i \Sigma_j (l)_{ij} (I : J)$$
>
> where $(l)_{ij}$ is a numerical coefficient, whose value is 1 in case I is a lover of J, and 0 in the opposite case, and where the sums are to be taken for all individuals in the universe.[20]

De Morgan mixes symbols for the individual objects with symbols for the relations:

[20] [Peirce1983], p. 107, and [Peirce1933], 3.328.

> Just as in ordinary logic *existence* is implicitly predicated for all the terms, so in this subject every relation employed will be considered as actually connecting the terms of which it is predicated. Let $X..L\,Y$ signify that X is some one of the objects of thought which stand to Y in the relation L, or is one of the Ls of Y. Let $X.L\,Y$ signify that X is not any one of the Ls of Y.[21]

De Morgan and Peirce consider two unary operations on relations: complementation and conversion. The **complement** of x is $\overline{x}$. Its definition depends on the universe of discourse:

$$\overline{x} = \{\, \langle a,b\rangle : a,b \in U \text{ and } \langle a,b\rangle \notin x \,\}. \tag{A2}$$

The **converse** of x is $\breve{x}$, obtained by "turning around" all the pairs in x:

$$\breve{x} = \{\, \langle b,a\rangle : \langle a,b\rangle \in x \,\}. \tag{A3}$$

Here is how De Morgan explained these operations:

> The *converse* relation of L, L^{-1}, is defined as usual: if $X..L\,Y$, $Y..L^{-1}X$: if X be one of the Ls of Y, Y is one of the L^{-1}s of X. And $L^{-1}X$ may be read '*L-verse* of X.' Those who dislike the mathematical symbol in L^{-1} might write L^{v}. This language would be very convenient in mathematics: $\phi^{-1}x$ might be the 'ϕ-verse of x,' read as 'ϕ-verse x.'
>
> Relations are assumed to exists between any two terms whatsoever. If X be not any L of Y, X is to Y in some not-L relation: let this *contrary* relation be signified by l; thus $X.L\,Y$ gives and is given by $X..l\,Y$. Contrary relations may be compounded, though contrary terms cannot: Xx, both X and not-X, is impossible; but LlX, the L of a not-L of X, is conceivable. Thus a man may be the partisan of a non-partisan of X.[22]

Peirce's introduction of these operations runs as follows:[23]

> Every relative has a negative (like any other term) which may be represented by drawing a straight line over the sign for the relative itself. The negative of a relative includes every pair that the latter excludes, and *vice versa*. Every relative has a *converse*, produced by reversing the order of the members of the pair. Thus, the converse of "lover" is "loved". The converse may be represented by drawing a curved line over the sign for the relative, thus: $\breve{l}$. It is defined by the equation
>
> $$(\breve{l})_{ij} = (l)_{ji}.$$

After this explanation, Peirce notes that (B1)–(B5) hold. De Morgan explains (B1)–(B5) more verbally. His first three paragraphs below explain (B1)–(B3) and their consequences, while his fourth paragraph expresses (B4)–(B5) and their converses.

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[21] [De Morgan1966], p. 220.

[22] [De Morgan1966], pp. 222–223.

[23] [Peirce1983], p. 188, and [Peirce1933], 3.330.

> Contraries of converses are converses: thus not-L and not-L^{-1} are converses. For $X..LY$ and $Y..L^{-1}X$ are identical; whence $X..$not-LY and $Y..($not-$L^{-1})X$, their simple denials, are identical; whence not-L and not-L^{-1} are converses.
>
> Converses of contraries are contraries: thus L^{-1} and $($not-$L)^{-1}$ are contraries. For since $X..LY$ and $X..$not-LY are simple denials of each other, so are their converses $Y..L^{-1}X$ and $Y..($not-$L)^{-1}X$; whence L^{-1} and $($not-$L)^{-1}$ are contraries.
>
> The contrary of a converse is the converse of a contrary: not-L^{-1} is $($not-$L)^{-1}$. For $X..LY$ is identical with $Y.$not-$L^{-1}X$ and with $X.($not-$L)Y$, which is also identical with $Y.$not-$L^{-1}X$. Hence the term not-L-verse is unambiguous in meaning, although ambiguous in form.
>
> If a first relation be contained in a second, then the converse of the first is contained in the converse of the second: but the contrary of the *second* in the contrary of the *first*.[24]

If we use "$-x$" as notation for complementation instead of "$\overline{x}$", we can echo De Morgan by saying that "$-x^{\smile}$" is unambiguous in meaning, although ambiguous in form.

We let $x+y$ denote the **union** of x and y, and $x \cdot y$ the **intersection** of x and y. These are defined as usual:

$$\text{(A4)} \qquad x \cdot y = \{\langle a,b\rangle : \langle a,b\rangle \in x \text{ and } \langle a,b\rangle \in y\},$$

$$\text{(A5)} \qquad x + y = \{\langle a,b\rangle : \langle a,b\rangle \in x \text{ or } \langle a,b\rangle \in y\}.$$

Here are Peirce's remarks on these operations:[25]

> Relative terms can be aggregated and compounded like others. Using + for the sign of logical aggregation, and the comma for the sign of logical composition (Boole's multiplication, here to be called non-relative or internal multiplication), we have the definitions
>
> $$(l+b)_{ij} = (l)_{ij} + (b)_{ij}$$
> $$(l,b)_{ij} = (l)_{ij} \times (b)_{ij}.$$
>
> The first of these equations, however, is to be understood in a peculiar way: namely, the + in the second member is not strictly addition, but an operation by which
>
> $$0+0=0 \qquad 0+1=1+0=1+1=1.$$

De Morgan, on the other hand, had no immediate application for these operations. After pointing out that he does not need to make a certain symbolic distinction, he says,

> Neither do I at present find it necessary to use relations which are aggregates of other relations: as in $X..(L,M)Y$, X is either one of the Ls of Y, or one of the Ms, or both.[26]

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[24] [De Morgan1966], p. 223.
[25] [Peirce1983], p. 188, and [Peirce1933], 3.331.
[26] [De Morgan1966], p. 221.

Peirce notes a few familiar laws governing union and intersection, and then declines to give additional ones, "being the same as in non-relative logic."

Peirce then proceeds to introduce relative multiplication and relative addition. The **relative product** of x and y is $x;y$. It is defined as follows:

$$x;y = \{\langle a,c\rangle : \text{for some } b,\ \langle a,b\rangle \in x \text{ and } \langle b,c\rangle \in y\}. \tag{A6}$$

The operation of forming the relative product is called **relative multiplication**. Relative multiplication is heavily used throughout mathematics. For example, if x is a function mapping X to Y, and y is a function mapping Y to Z, then $x;y$ is the *composition* of x and y. It is a function mapping X to Z. Relative multiplication also occurs frequently in everyday life, and numerous examples of its use are easily found. As De Morgan puts it, "The most apposite instances are taken from the relations between human beings: among which the *relations* which have almost monopolized the name, those of consanguinity and affinity, are conspicuously convenient, as being in daily use."[27] One of De Morgan's examples is that "*brother of parent* is identical with *uncle*, by mere definition."[28] This same example used by Whitehead and Russell, and also by Lewis and Langford.[29] As another example, Whitehead and Russell say "the relative product of *father* and *father* is *paternal grandfather*".

Intersection and union are dual to each other, in the sense that

$$x + y = \overline{\overline{x} \cdot \overline{y}}, \tag{A7}$$

$$x \cdot y = \overline{\overline{x} + \overline{y}}. \tag{A8}$$

The operation which is dual to relative multiplication is **relative addition**. For any two relations x and y, the **relative sum** of two relations x and y is $x \dagger y$. It is defined by

$$x \dagger y = \{\langle a,c\rangle : \text{for every } b \in U,\ \text{either } \langle a,b\rangle \in x \text{ or } \langle b,c\rangle \in y\}. \tag{A9}$$

That $\dagger$ and ; are duals is expressed by the relations

$$x \dagger y = \overline{\overline{x};\overline{y}}, \tag{A10}$$

$$x;y = \overline{\overline{x} \dagger \overline{y}}. \tag{A11}$$

While relative multiplication is relatively familiar, relative addition is not. For example, it is easy to see that the relative product of two functions is

[27] [De Morgan1966], p. 222.
[28] [De Morgan1966], p. 225.
[29] See [Whitehead-Russell1910], *34, and [Lewis-Langford1959], p. 112.

also a function, even if that product is empty.[30] On the other hand, it is not so obvious that the relative sum of two functions is also a function. In fact, at first glance it may appear to be false.[31]

Peirce explains relative multiplication and addition as follows.

> We now come to the combination of relatives. Of these, we denote two by special symbols; namely, we write
>
> lb for lover of a benefactor,
>
> and
>
> $l \dagger b$ for lover of everything but benefactors.
>
> The former is called a particular combination, because it implies the *existence* of something *loved by* its relate and a *benefactor of* its correlate. The second combination is said to be *universal*, because it implies the *non-existence* of anything except what is either loved by its relate or a benefactor of its correlate. The combination lb is called a relative product, $l \dagger b$ a relative sum.[32]

Peirce notes that (B6)–(B9) hold, and continues:

> The two combinations are defined by the equations
>
> $$\begin{aligned}(lb)_{ij} &= \Sigma_x (l)_{ix}(b)_{xj} \\ (l \dagger b)_{ij} &= \Pi_x \{(l)_{ix} + (b)_{xj}\}\end{aligned}$$
>
> The sign of addition in the last formula has the same signification as in the equation defining non-relative multiplication.[33]

The last formula also shows that $l \dagger b$ is perhaps better expressed by "lover of all non-benefactors". Peirce then presents (B10)–(B25), along with some commentary. For example, (B12) and (B13) are "[t]wo formulæ so constantly used that hardly anything can be done without them," while (B16)–(B19) are "curious development formulæ", in which "[t]he summations and multiplications denoted by Σ and Π are to be taken non-relatively, and all relative terms are to be successively substituted for p."

De Morgan discusses three binary operations on relations. One of them, which he calls composition, coincides with relative multiplication. The other two are different from relative addition.

> When the predicate is itself the subject of a relation, there may be a composition: thus if $X..L(MY)$, if X be one of the Ls of one of the Ms of Y, we may think of X as an 'L of M' of Y, expressed by $X..(LM)Y$, or simply by

[30] The empty relation $\emptyset$ is a function since it contains no two pairs which begin with the same element and end with different elements.

[31] It is stated in [Chin-Tarski1951] that the relative sum of two functions may not be a function. This mistake was noted in [Monk1961].

[32] [Peirce1983], p. 189, and [Peirce1933], 3.332.

[33] [Peirce1983], p. 189, and [Peirce1933], 3.333.

> $X..LM\,Y$. A wider treatment of the subject would make it necessary to effect a symbolic distinction between 'X is not any L of any M of Y' and 'X is not any L of some of the Ms of Y'. For my present purpose this is not necessary: so that $X.LM\,Y$ may denote the first of the two. ...
>
> We cannot proceed further without attention to forms in which *universal* quantity is an inherent part of the compound relation, as belonging to the notion of the relation itself, intelligible in the compound, unintelligible in the separated component.
>
> First, let LM' signify an L of *every* M, $LM'X$ being an individual in the same relation to many. Here the accent is a sign of universal quantity which forms part of the description of the relation: LM' is not an aggregate of cases of LM. Next let $L_{\prime}M$ signify an L of an M in every way in which it is an L at all: an L of none but Ms. Here the accent is also a sign of universal quantity: and logic seems to dictate to grammar that this should be read 'an every-L of M.'[34]
>
> ...
>
> We have thus three symbols of compound relation: LM, an L of an M; LM', an L of every M, $L_{\prime}M$, an L of none but Ms. No other compounds will be needed in the syllogism, until the premises themselves contain compound relations.[35]

De Morgan's $L_{\prime}M$ and LM' are definable from previous operations.

$$x_{\prime}y = \overline{x} \dagger y = \overline{x;\overline{y}} \tag{A12}$$

$$xy' = x \dagger \overline{y} = \overline{\overline{x};y} \tag{A13}$$

These operations are just two out of a system of sixty-four operations considered by Peirce.[36] Define four unary operations on relations by

$$\begin{array}{ll} f_1(x) = x & f_2(x) = \overline{x} \\ f_3(x) = \breve{x} & f_4(x) = \overline{\breve{x}} \end{array}$$

For each triple $\langle i,j,k\rangle$ with $i,j,k = 1,\ldots,4$, there is an operation defined by

$$g_{ijk}(x,y) = f_i(f_j(x);f_k(y))$$

There are sixty-four such operations. Peirce first rejects all those obtained by using f_3 or f_4, and then rejects all those which use f_2 an odd number of times. This leaves four operations, namely, relative multiplication and the following three operations (with Peirce's names and notation for them):

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[34] [De Morgan1966], pp. 221.

[35] [De Morgan1966], pp. 222.

[36] See Part III, §5, "The Composition of Relatives", in [Peirce1880], or [Peirce1933], 3.236–242.

(A14)	${}^{x}y = \overline{x} \dagger y = \overline{x;\overline{y}}$	regressive involution
(A15)	$x^{y} = x \dagger \overline{y} = \overline{\overline{x};y}$	progressive involution
(A16)	$x \circ y = \overline{x} \dagger \overline{y} = \overline{x;y}$	transaddition

The three operations discussed by De Morgan are relative multiplication, regressive involution, and progressive involution.[37] Peirce used the exponential notation for the involutions because of some remarkable formulæ, for example,[38]

$$(x \cdot y)^{z} = x^{z} \cdot y^{z} \tag{A17}$$

$$x^{y+z} = x^{y} \cdot x^{z} \tag{A18}$$

$$(x^{y})^{z} = x^{y \cdot z} \tag{A19}$$

Nevertheless, in his later papers Peirce abandoned transaddition and the involutions. Instead he adopted relative addition in *Note B*.

The formulæ (B26)–(B50) involve some distinguished relations which we now define. Let 1 be the **Cartesian square**, or **direct square**, of U. Thus 1 consists of all ordered pairs $\langle a, b\rangle$, where a and b are elements of U:

$$1 = U \times U = \{\langle a, b\rangle : a, b \in U\} \tag{A20}$$

The relation 1 is the **universal relation**. It is that relation which always holds between any two members of the universe of discourse. Let 0 be the relation which never holds between two members of the universe of discourse; 0 is the **empty relation**. We distinguish two other special binary relations:

$$1' = \{\, \langle a, a\rangle : a \in U \,\} \tag{A21}$$

$$0' = \{\, \langle a, b\rangle : a, b \in U \text{ and } a \neq b \,\} = \overline{1'} \tag{A22}$$

The relation 1' is the **identity relation on** U and 0' is the **diversity relation on** U. Peirce's explanation of these distinguished relations goes as follows:

> There is but one relative which pairs every object with itself and with every other. It is the aggregate of all pairs, and is denoted by ∞. It is translated

[37] Except for transaddition, these operations already occur in [Peirce1870]. In that paper, the two involutions are called involution and backward involution.

[38] See [Peirce1933], 3.249–250, for these and dozens of other formulæ. These operations are also mentioned in [Schröder1895], §29(5).

> into ordinary language by "is coexistent with." Its negative is 0. There is but one relative which pairs every object with itself and none with any other. It is
>
> $$(A:A)+(B:B)+(C:C)+\text{etc.};$$
>
> is denoted by 1, and in ordinary language is "identical with –." Its negative, denoted by n, is "other than –," or "not."[39]

Immediately after defining these special relations, Pierce lists various laws involving them, including (B26)–(B31). We follow Schröder here in replacing ∞, 0, 1, and n, by 1, 0, 1', and 0', respectively.[40] De Morgan did not deal with these distinguished relations.

Peirce goes on to make some interesting observations regarding the algebraic system he has invented:

> The logic of relatives is highly multiform; it is characterized by innumerable immediate conclusions from the same sets of premises. ... The effect of these peculiarities is that this algebra cannot be subjected to hard and fast rules like those of the Boolian calculus; and all that can be done in this place is to give a general idea of the way of working with it.[41]

One could perhaps regard this as a prophetic statement, for Tarski proved that there is no algorithm for determining whether a given formula is a conclusion of a given set of premises.[42] Peirce goes on to illustrate how to eliminate a variable from a given set of equations, pointing out the utility of (B12), (B13), (B30)–(B32). Near the end of *Note B* he states (B33)–(B50), and refers to (B33) as a "remarkable property". Peirce then embarks on the road to first order logic:[43]

> When the relative and non-relative operations occur together, the rules of the calculus become pretty complicated. In these cases, as well as in such as involve *plural* relations (subsisting between three or more objects), it is often advantageous to recur to the numerical coefficients on page 187. Any proposition whatever is equivalent to saying that some complexus of aggregates and products of such numerical coefficients is greater than zero. Thus,
>
> $$\Sigma_i \Sigma_j l_{ij} > 0$$
>
> means that something is a lover of something; and
>
> $$\Pi_i \Sigma_j l_{ij} > 0$$

[39] [Peirce1983], p. 191, and [Peirce1933], 3.339.
[40] See the remarks after 38) in §25 of [Schröder1895], and [Peirce1933], 3.510. Peirce was not happy with Schröder's changes.
[41] [Peirce1983], pp. 192–193, and [Peirce1933], 3.342.
[42] This result, which was first announced in [Tarski1941], p. 88, is equivalent to the undecidability of the equational theory of representable relation algebras. See also [Tarski-Givant1987], §8.7, p. 268.
[43] [Peirce1983], p. 200, and [Peirce1933], 3.351.

> means that everything is a lover of something. We shall, however, naturally omit, in writing the inequalities, the > 0 which terminates them all; and the above two propositions will appear as
>
> $$\Sigma_i \Sigma_i l_{ij} \quad \text{and} \quad \Pi_i \Sigma_j l_{ij}.$$

Here is another one of Peirce's examples:

> Let α denote the triple relative "accuser to — of —," and ε the triple relative "excuser to — of —." Then
>
> $$\Sigma_i \Pi_j \Sigma_k (\alpha)_{ijk} (\varepsilon)_{jki}$$
>
> means that an individual, i, can be found, such, that taking any individual whatever, j, it will always be possible to select a third individual, k, that i is an accuser to j of k, and j an excuser of k of i.[44]

Peirce gives some rules for the deduction of conclusions from sets of premises. One such rule is

$$\Sigma_i \Pi_j \leq \Pi_j \Sigma_i$$

by which he means that a universal quantifier can be moved in front of an existential quantifier. For example, $\Sigma_i \Pi_j l_{ij} \leq \Pi_j \Sigma_i l_{ij}$, which asserts "if someone loves everyone then everyone is loved by someone". In the notation of *Note B*, this becomes $1;(l \dagger 0) \leq 1;l \dagger 0$, a special case of (B12), one of the "[t]wo formulæ so constantly used that hardly anything can be done without them".

Peirce gives a couple similar rules, and then, as an example, shows how to deduce

$$\Sigma_x \Sigma_u \Sigma_y \Sigma_v (\varepsilon_{uyx} \alpha_{xuv} + \varepsilon_{uyx} l_{yu} + \alpha_{xuv} b_{vx})$$

from the premises

$$\Sigma_h \Pi_i \Sigma_j \Pi_k (\alpha_{hik} + s_{jk} l_{ji}),$$
$$\Sigma_u \Sigma_v \Pi_x \Pi_y (\varepsilon_{uyx} + \bar{s}_{yv} b_{vx}).$$

Peirce's next paper[45] on the algebra of logic is truly remarkable for its farsightedness. Parts of this paper deal with topics from contemporary textbooks of logic, such as truth values,[46] an axiomatization of propositional logic based on the connectives $\rightarrow$ and $\neg$,[47] a decision procedure for propositional logic,[48] quantifiers and first order formulæ, prenex normal form,[49] second order logic,[50] and axioms for set theory.[51]

---

[44] [Peirce1983], p. 201, and [Peirce1933], 3.353.
[45] [Peirce1885].
[46] [Peirce1933], 3.366–370. The first explicit use of truth values appears [Peirce1885]; see [Church1956], footnote 67, p. 25.
[47] [Peirce1933] 3.376–384.
[48] [Peirce1933], 3.387.
[49] [Peirce1933], 3.396.
[50] [Peirce1933], 3.398–400.
[51] [Peirce1933], 3.398–400.

In this paper, Peirce abandons the convention that formulæ of the kind just illustrated assert facts about sums and products of numerical coefficients and that each such formula is therefore equivalent to a certain corresponding proposition. Instead, he allows each formula to assert its corresponding proposition directly. In this interpretation, Σ_x is not really a sum, but a quantifier. Peirce credits this device to O. H. Mitchell, but his notation is carried over from *Note B*.

> Mr. Mitchell has a very interesting and instructive extension of his notation for *some* and *all*, to a two-dimensional universe, that is, to the logic of relatives. Here, in order to render the notation as iconical as possible we may use Σ for *some*, suggesting a sum, and Π for *all*, suggesting a product. Thus $\Sigma_i x_i$ means that x is true of some one of the individuals denoted by i or
>
> $$\Sigma_i x_i = x_i + x_j + x_k + \text{etc.}$$
>
> In the same way, $\Pi_i x_i$ means that x is true of all these individuals, or
>
> $$\Pi_i x_i = x_i x_j x_k, \text{etc.}$$
>
> ... It is to be remarked that Σ_i and $\Pi_i x_i$ are only *similar* to a sum and a product; they are not strictly of that nature, because the individuals of the universe may be innumerable.[52]

On the basis of this paper, Peirce and Mitchell share credit with Frege for the introduction of quantifiers.[53]

Now we turn to a theorem which is among De Morgan's most important contributions to the calculus of relations. In what follows, De Morgan uses "))" in the way that "$\leq$" is used in this paper.

> If a compound relation be contained in another relation, by the nature of the relations and not by casualty of the predicate, the same may be said when either component is converted, and the contrary of the other component and of the compound change places. That is if, be Z whatever it may, every L of M of Z be an N of Z, say $LM))N$, then $L^{-1}n))m$, and $nM^{-1}))l$. If $LM))N$, then $n))lM'$ and $nM^{-1}))lM'M^{-1}$. But an l of *every* M of an M^{-1} of Z must be an l of Z: hence $nM^{-1}))l$. Again, if $LM))N$, then $n))L,m$, whence $L^{-1}n))L^{-1}L,m$. But an L^{-1} of an L of none but ms of Z must be an m of Z; whence $L^{-1}n))m$.
>
> I shall call this result theorem K, in remembrance of the office of that letter in Baroko and Bokardo; it is the theorem on which the formation of what I called opponent syllogisms is founded.[54] ...

Here is De Morgan's Theorem K in the notation of this paper:

$$\text{(A23)} \qquad x;y \leq z \ \rightarrow \ \breve{x};\overline{z} \leq \overline{y} \ \wedge \ \overline{z};\breve{y} \leq \overline{x}$$

[52] [Peirce1933], 3.393.
[53] [Church1956], footnote 103, p. 45, and footnote 453, p. 288. For more details, see [Moore1987].
[54] [De Morgan1966], p. 224.

De Morgan was well aware that all three formulæ in (A23) are equivalent.[55]

Given De Morgan's emphasis on Theorem K, and the fact that Peirce read De Morgan's paper, it is puzzling that Theorem K is given no prominent treatment in any of Peirce's papers. Schröder, on the other hand, presents the following elaborate version of Theorem K, but with no mention of De Morgan. In (A24)–(A27) below there are four groups of twelve formulæ each. Any two formulæ in the same group are equivalent.[56] The three formulæ of De Morgan's Theorem K occur in (A26).

$$
\begin{array}{lllll}
\text{(A24)} & y;z \leq \breve{\overline{x}} & z \leq \breve{\overline{y}} \dagger \breve{\overline{x}} & \breve{z};\breve{y} \leq \overline{x} & \breve{z} \leq \overline{x} \dagger \overline{y} \\
 & z;x \leq \breve{\overline{y}} & y \leq \breve{\overline{x}} \dagger \breve{\overline{z}} & \breve{x};\breve{z} \leq \overline{y} & \breve{y} \leq \overline{z} \dagger \overline{x} \\
 & x;y \leq \breve{\overline{z}} & x \leq \breve{\overline{z}} \dagger \breve{\overline{y}} & \breve{y};\breve{x} \leq \overline{z} & \breve{x} \leq \overline{y} \dagger \overline{z} \\
\text{(A25)} & \breve{\overline{x}} \leq y \dagger z & \breve{\overline{y}};\breve{\overline{x}} \leq z & \overline{x} \leq \breve{z} \dagger \breve{y} & \overline{x};\overline{y} \leq \breve{z} \\
 & \breve{\overline{y}} \leq z \dagger x & \breve{\overline{x}};\breve{\overline{z}} \leq y & \overline{y} \leq \breve{x} \dagger \breve{z} & \overline{z};\overline{x} \leq \breve{y} \\
 & \breve{\overline{z}} \leq x \dagger y & \breve{\overline{z}};\breve{\overline{y}} \leq x & \overline{z} \leq \breve{y} \dagger \breve{x} & \overline{y};\overline{z} \leq \breve{x} \\
\text{(A26)} & x;y \leq z & x \leq z \dagger \breve{\overline{y}} & \breve{y};\breve{x} \leq \breve{z} & \breve{x} \leq \overline{y} \dagger \breve{z} \\
 & \breve{\overline{z}};x \leq \breve{\overline{y}} & y \leq \breve{\overline{x}} \dagger z & \breve{x};\overline{z} \leq \overline{y} & \breve{y} \leq \breve{z} \dagger \overline{x} \\
 & y;\breve{\overline{z}} \leq \breve{\overline{x}} & \breve{\overline{z}} \leq y \dagger \breve{\overline{x}} & \overline{z};\breve{y} \leq \overline{x} & \overline{z} \leq \overline{x} \dagger \overline{y} \\
\text{(A27)} & x \leq y \dagger z & \breve{\overline{y}};x \leq z & \breve{x} \leq \breve{z} \dagger \breve{y} & \breve{x};\overline{y} \leq \breve{z} \\
 & \breve{\overline{y}} \leq z \dagger \breve{\overline{x}} & x;\breve{\overline{z}} \leq y & \overline{y} \leq \overline{x} \dagger \breve{z} & \overline{z};\breve{x} \leq \breve{y} \\
 & \breve{\overline{z}} \leq \breve{\overline{x}} \dagger y & \breve{\overline{z}};\breve{\overline{y}} \leq \breve{\overline{x}} & \overline{z} \leq \breve{y} \dagger \overline{x} & \overline{y};\overline{z} \leq \overline{x}
\end{array}
$$

Furthermore, as Schröder notes, each group of twelve can be expanded to a group of 60 equivalent formulæ by using the following law:[57]

(A28)

$$x;y \leq z \;\leftrightarrow\; 1 \leq \overline{x} \dagger \overline{y} + z \;\leftrightarrow\; x;y \cdot \overline{z} \leq 0 \;\leftrightarrow\; x;y \cdot z = x;y \;\leftrightarrow\; x;y + z = z$$

Schröder proved that every Boolean combination of equations is equivalent to a single equation, and, in fact, an equation of the form "$x = 1$".[58] This

[55] See [De Morgan1966], pp. 186–187.
[56] [Schröder1895], §17(2)(3).
[57] [Schröder1895], §17, p. 244.
[58] [Schröder1895], §11, pp. 153ff.

follows from the following formulæ:

$$
\begin{array}{lrcl}
\text{(A29)} & x = y & \leftrightarrow & x \cdot y + \overline{x} \cdot \overline{y} = 1 \\
\text{(A30)} & x \neq 1 & \leftrightarrow & 1;\overline{x};1 = 1 \\
\text{(A31)} & x = 1 \ \wedge \ y = 1 & \leftrightarrow & x \cdot y = 1 \\
\text{(A32)} & x = 1 \ \vee \ y = 1 & \leftrightarrow & 0 \dagger x \dagger 0 + y = 1 \\
 & & \leftrightarrow & x + 0 \dagger y \dagger 0 = 1 \\
 & & \leftrightarrow & 0 \dagger x \dagger 0 \dagger y \dagger 0 = 1 \\
\text{(A33)} & (x = 1 \ \rightarrow \ y = 1) & \leftrightarrow & 1;\overline{x};1 + y = 1
\end{array}
$$

We close this section with some laws about transitive relations, due to Peirce, 1893:

(A34) $(x = y \dagger \breve{\overline{y}} \ \vee \ x = \breve{\overline{y}} \dagger y) \ \rightarrow \ (x;x \leq x \ \wedge \ 1' \leq x)$

(A35) $(x = x \dagger \breve{\overline{x}}) \ \leftrightarrow \ (x = \breve{\overline{x}} \dagger x) \ \leftrightarrow \ (x;x \leq x \ \wedge \ 1' \leq x)$

Here is Peirce's manner of expressing these results:

> Yet really, the form $l \dagger \breve{\bar{l}}$ is all-important, inasmuch as it is the basis of all quantitative thought. For the relation expressed by it is transitive. ... This is not only a transitive relation, but it is one which contains the identity under it. ... But it is further demonstrable that *every* transitive relation which includes identity under it is of the form $l \dagger \breve{\bar{l}}$.[59]

Formulas from Peirce's *Note B*[60]

Any missing parentheses in the following formulæ should be restored according to the convention that unary operations are performed first, followed by binary operations in the following order: ;, ·, †, and finally +. The computation should proceed from left to right in case of repeated operations. Thus, for example, $0 \dagger x + 1;z \cdot x = (0 \dagger x) + ((1;z) \cdot x)$ and $x;y;z = (x;y);z$.

(B1) $\overline{\overline{x}} = x$

(B2) $\breve{\breve{x}} = x$

(B3) $\breve{\overline{x}} = \overline{\breve{x}}$

(B4) if $x \leq y$ then $\overline{y} \leq \overline{x}$

[59] [Peirce1933], 4.94.

[60] Peirce did not actually include the second equation of (B14) in [Peirce1883b], but both forms occur in [Peirce1880]; see [Peirce1933], 3.249. With two exceptions, formulæ belonging strictly to the calculus of classes have not been included.

(B5) if $x \leq y$ then $\breve{x} \leq \breve{y}$

(B6) if $x \leq z$ then $x;y \leq z;y$

(B7) if $x \leq z$ then $x \dagger y \leq z \dagger y$

(B8) if $y \leq z$ then $x;y \leq x;z$

(B9) if $y \leq z$ then $x \dagger y \leq x \dagger z$

(B10) $x \dagger (y \dagger z) = (x \dagger y) \dagger z$

(B11) $x;(y;z) = (x;y);z$

(B12) $x;(y \dagger z) \leq x;y \dagger z$

(B13) $(x \dagger y);z \leq x \dagger y;z$

(B14) $(x + y);z = x;z + y;z$ and $x;(y + z) = x;y + x;z$

(B15) $(x \cdot y) \dagger z = (x \dagger z) \cdot (y \dagger z)$

(B16) $(x \cdot y);z = \Pi_p(x;(z \cdot p) + y;(z \cdot \overline{p}))$

(B17) $x;(y \cdot z) = \Pi_p((x \cdot p);y + (x \cdot \overline{p});z)$

(B18) $(x + y) \dagger z = \Sigma_p\{[x \dagger (z + p)] \cdot [y \dagger (z + \overline{p})]\}$

(B19) $x \dagger (y + z) = \Sigma_p\{[(x + p) \dagger y] \cdot [(x + \overline{p}) \dagger z]\}$

(B20) $\overline{x \dagger y} = \overline{x};\overline{y}$

(B21) $\overline{x;y} = \overline{x} \dagger \overline{y}$

(B22) $(x + y)\breve{} = \breve{x} + \breve{y}$

(B23) $(x \cdot y)\breve{} = \breve{x} \cdot \breve{y}$

(B24) $(x \dagger y)\breve{} = \breve{y} \dagger \breve{x}$

(B25) $(x;y)\breve{} = \breve{y};\breve{x}$

(B26) $x \dagger 1 = 1 = 1 \dagger x$

(B27) $x;0 = 0 = 0;x$

(B28) $x \dagger 0' = x = 0' \dagger x$

(B29) $x;1' = x = 1';x$

(B30) $1' \leq x \dagger \breve{\overline{x}}$

(B31) $x;\breve{\overline{x}} \leq 0'$

(B32) $1' \leq y \dagger x$ iff $1' \leq x \dagger y$ iff $1' \leq \breve{y} \dagger \breve{x}$ iff $1' \leq \breve{x} \dagger \breve{y}$

(B33) each of $0 \dagger x \dagger 0$, $(0 \dagger x);1$, $0 \dagger x;1$, and $1;x;1$ is either 0 or 1

(B34) $x \dagger 0 \leq x$ and $x \leq x;1$

(B35) $x;z \cdot (y \dagger \overline{z}) \leq (x \cdot y);z$

(B36) $z;x \cdot (\overline{z} \dagger y) \leq z;(x \cdot y)$

(B37) $(0 \dagger x \dagger 0) \cdot (0 \dagger y \dagger 0) = 0 \dagger (x \cdot y) \dagger 0$

(B38) $(0 \dagger x);1;(0;x;\breve{y} \dagger 0) \leq (0 \dagger x);(\breve{y} \dagger 0)$

(B39) $(0 \dagger x);1;(1;(\breve{y} \dagger 0)) = (0 \dagger x);(\breve{y} \dagger 0) + (0 \dagger x);0';(\breve{y} \dagger 0)$

(B40) $(0 \dagger x);1;((0 \dagger \breve{y});1) \leq (0 \dagger x);\breve{y};1$

(B42) $(0 \dagger x \dagger 0);(0 \dagger \breve{y};1) = 0 \dagger (\breve{y};x \cdot x) \dagger 0$

(B43) $(0 \dagger x);1;(0 \dagger \breve{y};1) = (0 \dagger \breve{y};x \cdot x);1$

(B44) $(0 \dagger x);1 \cdot (0 \dagger y;1) = (0 \dagger x \cdot y;1);1$

(B45) $(0 \dagger x);1;(0 \dagger y;1) = 0 \dagger (x;\breve{y} \cdot \breve{y};x);1$

(B46) $(0 \dagger x;1) \cdot (0 \dagger y;1) = 0 \dagger x;1 \cdot y;1$

(B47) $(0 \dagger x \dagger 0);(1;\breve{y};1) = 0 \dagger (x;\breve{y};x \cdot x) \dagger 0$

(B48) $(0 \dagger x);1;(1;\breve{y};1) = (0 \dagger x);\breve{y};1 + (0 \dagger x);0';\breve{y};1$

(B49) $(0 \dagger x;1);(1;\breve{y};1) = (0 \dagger x;\breve{y};1) + (0 \dagger x;0';\breve{y};1)$

(B50) $1;x;1;(1;\breve{y};1) = 1;x;\breve{y};1 + 1;x;0';\breve{y};1$

4. The calculus of relations: axiomatization

De Morgan, Peirce, and Schröder were certainly interested in deducing complicated formulæ from simpler ones, but they were not particularly interested in the axiomatic approach to the calculus of relations. After listing several laws as "the axiomatic principles of this branch of logic, not deducible from others", Peirce says, "But these axioms are mere substitutes for definitions of the universal logical relations, and so far as these can be given, all axioms may be dispensed with."[61]

Tarski took a different view, for in 1940 he proposed[62] an axiomatization for the fragment of the calculus of relations consisting of all Boolean combinations of equations. All the formulæ from Peirce's *Note B* belong to this fragment, except for (B16)–(B19). Tarski noted that this fragment may be developed within the framework of first order predicate calculus, and outlined how this may be done. The beginnings of this method were reviewed above and are due to Peirce.[63] But Tarski thought that

> this method has certain defects from the point of view of simplicity and elegance. We obtain the calculus of relations in a very roundabout way, and in proving theorems of this calculus we are forced to make use of concepts and

[61] See the "Conclusion" of [Peirce1870], or [Peirce1933], 3.148.

[62] See [Tarski1941], the text of an invited address delivered at a meeting in Philadelphia, December 28, 1940.

[63] Later works following in this tradition are [Whitehead-Russell1910], [Lewis1918], and [Lewis-Langford1959], pp. 104ff. The approach outlined in [Tarski1941] is worked out in more detail in [Tarski-Givant1987].

> statements which are outside the calculus. It is for this reason that I am going outline another method of developing this calculus.[64]

Tarski therefore proposed axioms which are specific to the calculus of relations. This second method led to the theory of relation algebras. Tarski's axiomatization consists of an axiomatization of the sentential calculus, together with the following axioms:

$$
\begin{array}{rl}
\text{(I)} & (x = y \ \wedge \ x = z) \ \rightarrow \ y = z \\
\text{(II)} & x = y \ \rightarrow \ (x + z = y + z \ \wedge \ x \cdot z = y \cdot z) \\
\text{(III)} & x + y = y + x \ \wedge \ x \cdot y = y \cdot x \\
\text{(IV)} & (x + y) \cdot z = x \cdot z + y \cdot z \ \wedge \ (x \cdot y) + z = (x + z) \cdot (y + z) \\
\text{(V)} & x + 0 = x \ \wedge \ x \cdot 1 = x \\
\text{(VI)} & x + \overline{x} = 1 \ \wedge \ x \cdot \overline{x} = 0 \\
\text{(VII)} & \neg(1 = 0) \\
\text{(VIII)} & \breve{\breve{x}} = x \\
\text{(IX)} & (x\,;y)^{\smile} = \breve{y}\,;\breve{x} \\
\text{(X)} & x\,;(y\,;z) = (x\,;y)\,;z \\
\text{(XI)} & x\,;1' = x \\
\text{(XII)} & x\,;1 = 1 \ \vee \ 1\,;\overline{x} = 1 \\
\text{(XIII)} & (x\,;y) \cdot \breve{z} = 0 \ \rightarrow \ (y\,;z) \cdot \breve{x} = 0 \\
\text{(XIV)} & 0' = \overline{1'} \\
\text{(XV)} & x \dagger y = \overline{\overline{x}\,;\overline{y}}
\end{array}
$$

The only two rules of inference to be used with these axioms are the rule of substitution and the rule of detachment.

Axioms (I)–(VII) form an axiomatization of the calculus of classes, due essentially to E. V. Huntington.[65] The remaining axioms are selected from laws already noted by De Morgan, Peirce, and Schröder. In particular, (VIII) is (B2), (IX) is (B25), (X) is (B11), (XI) is part of (B29), (XII) corresponds to (B33), (XIII) corresponds to (A23), (XIV) defines 0', and (XV) defines †. Therefore, the collection of all binary relations on an arbitrary nonempty set U, together with the appropriate operations and distinguished relations, is a model of (I)–(XV).

Tarski proves sixteen theorems in this system, many of which are among the formulæ already listed. Tarski then proves the result of Schröder mentioned above, that every sentence (Boolean combination of equations) is

---

[64] [Tarski1941], p. 77.

[65] See [Huntington1904] and [Tarski1941], p. 78, footnote 3.

equivalent to an equation of the form "$x = 1$". But Tarski's proof is based just on axioms (I)–(XV), so he actually proves a generalization of Schröder's theorem, namely, that if A is a model of (I)–(XV), then a sentence holds in A if and only if its corresponding equation holds in A.

Tarski's axiomatization also suffices to prove (B1)–(B15), (B20)–(B50), Schröder's version of De Morgan's Theorem K, and Peirce's theorem on transitive relations containing the identity relation. It is natural to ask whether Tarski's axiomatization is complete. Tarski put the problem this way:

> Is it the case that every sentence of the calculus of relations which is true in every domain of individuals is derivable from the axioms adopted under the second method? This problem presents some difficulties and remains open. I can only say that I am practically sure that I can prove with the help of the second method all of the hundreds of theorems to be found in Schröder's *Algebra und Logik der Relative*.[66]

It would be interesting to know whether there is a theorem in Schröder's book which is part of the fragment axiomatized by Tarski, but which is not derivable from his axioms. However, Tarski was probably right. It seems likely that there is no such sentence.

Altogether, Tarski considers five questions:

1. Is the axiomatization complete?
2. Is every model of the axioms representable?
3. Is there a decision method for valid equations?
4. Is every first order sentence expressible as an equation?
5. Is there a decision method for expressible first order sentences?

The answer is "no" in every case. Roger Lyndon answered the first two questions with a single construction.[67] Tarski already had the solution to his third question.[68] The answer to the fourth question is due to Korselt and had been known for many years.[69] The fifth question was finally answered by M. Kwatinetz, a student of Tarski's.[70]

The first three questions and their answers are best explained in a framework which emerged a few years later, that of Tarski's relation algebras.

5. Relation algebras: definitions

We will look closely at four definitions of relation algebras, in chronological order.[71]

[66] [Tarski1941], pp. 87–88.
[67] [Lyndon1950], §8.
[68] See [Tarski1941], p. 88.
[69] Korselt's result is reported in [Löwenheim1915], Theorem 1. See [vanHeijenoort], p. 233.
[70] [Kwatinetz1981].
[71] Relation algebras have been defined in many different ways by various authors; see, for

The first definition of relation algebras appears in an abstract, *Representation problems for relation algebras*, by Jónsson and Tarski.[72] They define a relation algebra as a Boolean algebra together with a binary operation ; and a unary operation $\breve{}$, such that (B2), (B11), (B14), and (B22) hold, (B29) holds for some element 1', and

$$\breve{x};(x;y)^{-} \leq \overline{y}. \tag{C1}$$

This is almost an equational definition: all the postulates are equations except for the one which asserts the existence of an element 1' satisfying (B29). In the presence of the axioms listed before it, (C1) is equivalent to De Morgan's Theorem K. (C1) does not occur in Peirce's Note B, but it can be replaced by either (B12) or (B13), if the occurrences of † in those formulæ are eliminated by (XV).

Now we describe the connection between relation algebras and models of Tarski's axioms. The language which Tarski uses for his axiomatization of the calculus of relations includes variables denoting arbitrary relations, symbols for the distinguished relations 0, 1, 1', and 0', symbols for the six operations of *Note B*, namely $+$, $\cdot$, $;$, $\dagger$, $\breve{}$, and $^{-}$, a symbol for equality, and propositional connectives. Thus a structure which is appropriate for this language must be an algebra of the form

$$A = \langle A, +, \cdot, {}^{-}, 0, 1, ;, \dagger, \breve{}, 1', 0' \rangle .$$

Suppose A is a model of Tarski's axioms (I)–(XV). Then the reduct $\langle A, +, \cdot, {}^{-}, 0, 1\rangle$ is a nontrivial Boolean algebra by (I)–(VII). Using (I)–(XI) and (XIII) it can be shown[73] that the reduct $\langle A, +, \cdot, {}^{-}, 0, 1, ;, \breve{}, 1'\rangle$ is a relation algebra as defined by Jónsson and Tarski.

Conversely, if $\langle A, +, \cdot, {}^{-}, 0, 1, ;, \breve{}, 1'\rangle$ is a relation algebra, and we take (XIV) and (XV) as definitions of 0' and †, respectively, then $A = \langle A, +, \cdot, {}^{-}, 0, 1, ;, \dagger, \breve{}, 1', 0'\rangle$ is a model of (I)–(VI), (VIII)–(XI), and (XIII)–(XV). A is also a model of (VII) just in case A has at least two elements, and A is also a model of (XII) just in case A is simple, that is, A has no nontrivial congruence relations. Thus every nontrivial simple relation algebra yields a model of Tarski's axioms (I)–(XV). Relation algebras could have been defined as models of (I)–(XV), but the resulting class of algebras would not be a variety, for it would not be closed under the formation of homomorphic images (due to axiom (VII)) or direct products (due to axiom (XII)). If

example, [Birkhoff1948], Ch. XIII, §5, pp. 209–211. Such definitions are not pursued here, since the relation algebras of this paper are just the ones due to Tarski.

[72] [Jónsson-Tarski1948], received October 21, 1947, presented November 29, 1947.

[73] See [Chin-Tarski1951], p. 352, and [Jónsson-Tarski1952], p. 128, footnote 15.

axioms (VII) and (XII) are deleted, then the remaining ones form an alternative axiomatization of relation algebras. This alternative axiomatization is almost equational. The identity element 1' is treated as a distinguished element, but axiom (XIII) is an implication. Axiom (XIII) is an alternative form of De Morgan's Theorem K.

The second definition of relation algebras is by Lyndon,[74] who defines a relation algebra as a Boolean algebra together with a unary operation, a binary operation, and a distinguished element, satisfying (VIII)–(XI) and (XIII), with (XII) replaced by

$$x \neq 0 \;\rightarrow\; 1;(x;1) = 1, \tag{C2}$$

a variant of (A29). He does not include (VII), so Lyndon's relation algebras are actually simple (possibly trivial) relation algebras. Thus Lyndon's definition is not only not equational, but is not equivalent to any equational definition.

The third definition of relation algebras, due to Chin and Tarski, is the first in which the similarity type of relation algebras is explicitly specified.[75] Chin and Tarski require a relation algebra to be of the form $\mathfrak{A} = \langle A, +, ^-, ;, ^\smile \rangle$, where $\langle A, +, ^- \rangle$ is a Boolean algebra (note that $\cdot$, 0, and 1 can be obtained from $+$ and $^-$ by composition), satisfying (B2), (B11), the first equation in (B14), (B22), (B25), and (C1). They do not have 1' as a distinguished element, and hence require an existential axiom, which asserts that there is some $u \in A$ such that $x;u = x$ for all $x \in A$.

The fourth definition of relation algebras is by Jónsson and Tarski.[76] Some of their results require that complementation cannot be obtained from the fundamental operations by composition, and hence they do not include complementation in the similarity type of relation algebras.[77] They define relation algebras as algebras of the form $\mathfrak{A} = \langle A, +, 0, \cdot, 1, ;, ^\smile, 1' \rangle$, where $\langle A, +, 0, \cdot, 1 \rangle$ is a Boolean algebra, satisfying (B11), (B29), and the following version of De Morgan's Theorem K:

$$x;y \cdot z = 0 \;\leftrightarrow\; \breve{x};z \cdot y = 0 \;\leftrightarrow\; z;\breve{y} \cdot x = 0 \tag{C3}$$

This definition of relation algebras is equivalent to the others given above (except for Lyndon's inclusion of an axiom guaranteeing simplicity), and several other variations on the definition are possible.[78]

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[74] [Lyndon1950], p. 708.

[75] See [Chin-Tarski1951], Definition 1.1. The first two definitions do not specify the similarity type of the underlying Boolean algebra.

[76] [Jónsson-Tarski1952], Definition 4.1.

[77] See [Jónsson-Tarski1951], p. 897, for further comments on this situation.

[78] See [Chin-Tarski1951], p. 352, footnote 10, p. 354, footnote 12, and [Jónsson-Tarski1952], p. 128, footnote 15.

Three of the definitions mentioned so far are in papers co-authored by Tarski, and perhaps therefore express some evolution in his thoughts on the way relation algebras should be presented. But Tarski soon adopted the attitude that relation algebras should be defined with purely equational postulates.[79] Such a definition makes it obvious that relation algebras form a variety. Indeed, his final version[80] is quite explicitly equational: a relation algebra is an algebra of the form $A = \langle A, +, {}^{-}, ;, {}^{\smile}, 1' \rangle$ which satisfies (B2), (B11), the first equation in (B14), (B22), (B25), (XI), (C1), and the following three equations:

$$x + y = y + x, \tag{C4}$$

$$x + (y + z) = (x + y) + z, \tag{C5}$$

$$\overline{\overline{x} + y} + \overline{\overline{x} + \overline{y}} = x. \tag{C6}$$

These three equations assert that $\langle A, +, {}^{-} \rangle$ is a Boolean algebra, and are due essentially to E. V. Huntington.[81]

This axiomatic algebraic approach to the calculus of relations, marked by the advent of relation algebras, has two important features which distinguish it from the viewpoint of Peirce and Schröder. First, Peirce and Schröder work with *all* binary relations on the universe of discourse. But Tarski's axioms hold in certain models formed by taking only *some* of those relations. Indeed, a model of (I)–(XV) can be built from any set of relations which contains 1' and is closed under union, complementation, relative multiplication, and conversion. In such models, (B16)–(B19) make sense, but may not hold. For example, if U has three or more elements, then 0, 1, 1', and 0' are distinct relations forming a 4-element model of (I)–(XV) in which (B16) fails with $x = 0'$, $y = 1'$, and $z = 1$, since $(x \cdot y); z = (0' \cdot 1'); 1 = 0; 1 = 0$, but

$$\begin{aligned}
\Pi_p(x;(z \cdot p) + y;(z \cdot \overline{p})) &= \Pi_p(0';(1 \cdot p) + 1';(1 \cdot \overline{p})) \\
&= \Pi_p(0';p + \overline{p}) \\
&= (0';0 + \overline{0}) \cdot (0';1 + \overline{1}) \cdot (0';1' + \overline{1'}) \cdot (0';0' + \overline{0'}) \\
&= (0 + 1) \cdot (1 + 0) \cdot (0' + 0') \cdot (0';0' + 1') \\
&= 0'
\end{aligned}$$

However, Peirce made no mistake, for (B16)–(B19) do hold in $Re\,U$, which is the relation algebra of all binary relations on U.[82]

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[79] See [Tarski1955], p. 60, and [Tarski1956], §3.
[80] [Tarski-Givant1987], Definition 8.1.
[81] [Huntington1933].
[82] For a proof, see [Schröder1895], §29, pp. 491–494.

The second distinguishing feature of the relation algebraic approach to the calculus of relations is that there are relation algebras of binary relations in which the universal relation need not relate every element of the universe of discourse to every other. Such algebras are called *proper relation algebras.* To define them, the definition of complement must be slightly modified from (A2) to make it suitable for the case in which $1 \neq U \times U$:

$$\text{(C7)} \qquad \overline{x} = \{ \langle a,b \rangle : \langle a,b \rangle \in 1 \text{ and } \langle a,b \rangle \notin x \} .$$

Then a proper relation algebra $\mathfrak{A} = \langle A, +, ^{-}, ;, ^{\smile}, 1' \rangle$ is a relation algebra whose universe A is a family of binary relations (all contained in some largest binary relation $1 \in A$), such that A is closed under union, complementation with respect to 1, relative multiplication, and conversion, A contains the identity relation on the field of the largest relation, $+$ is union, $^{-}$ is complementation with respect to 1, ; is relative multiplication, $^{\smile}$ is conversion, and 1' is the identity relation on the field of 1.[83]

6. Relation algebras: representability and incompleteness

Jónsson and Tarski posed representation problems in their abstract, but they do not define the notion of a representable relation algebra. Instead, they define the notion of proper relation algebra, and ask whether every relation algebra is isomorphic to one which is proper. A relation algebra is **representable** if it is isomorphic to a proper relation algebra.[84] The representation problem for relation algebras then reads, "Is every relation algebra representable?"

Roger Lyndon showed that the answer is "no", by constructing a finite nontrivial simple relation algebra which is not representable.[85] Because his algebra is simple and nontrivial, it is also a model of Tarski's axioms (I)–(XV), and hence yields a negative answer to Tarski's second question at the end of §4.

[83] This last requirement, that a proper relation algebra must contain the identity relation on its underlying set, is dropped in [Jónsson-Tarski1952]; see Definition 4.23. Since a proper relation algebra is a relation algebra, there must be a relation which acts as an identity element for relative multiplication, but this relation need not be an identity relation, although it does have to be an equivalence relation. It is shown in Theorem 4.27 of [Jónsson-Tarski1952] that a relation algebra is isomorphic to a proper relation algebra in the sense of [Jónsson-Tarski1952] if and only if it is isomorphic to a proper relation algebra in the sense of [Jónsson-Tarski1948], so for the representation problem this difference in the definition of proper relation algebra makes no difference. However, in [Tarski-Givant1987], §8.3, the definition of proper relation algebra in [Jónsson-Tarski1952] has been dropped in favor of the original one in [Jónsson-Tarski1948].
[84] This meaning is implicit in [Lyndon1950].
[85] See [Lyndon1950], §8.

Lyndon found an infinite set of conditions which are necessary and sufficient for a finite relation algebra to be representable.[86] His finite relation algebra fails to satisfy one of these conditions, and is therefore not representable. Each condition has an integer attached to it. The simplest of these conditions, for which that integer is 4 or less, can be shown to hold in all relation algebras. For $n = 5$, there are, on grounds of symmetry, only three conditions which are not obviously equivalent to each other. These three conditions are equivalent to the following valid sentences of the calculus of relations. In these equations, if "x_{ji}" occurs after "x_{ij}", then "x_{ji}" is an abbreviation of "$\breve{x}_{ij}$". This notational device is due to Lyndon.

(D1) $x_{02}\,;x_{21}\cdot x_{03}\,;x_{31}\cdot x_{04}\,;x_{41} \leq$

$\qquad x_{02}\,;[x_{20}\,;x_{03}\cdot x_{21}\,;x_{13}\cdot(x_{20}\,;x_{04}\cdot x_{21}\,;x_{14});(x_{40}\,;x_{03}\cdot x_{41}\,;x_{13})];x_{31}$

(D2) $x_{01}\cdot(x_{02}\cdot x_{03}\,;x_{32});(x_{21}\cdot x_{24}\,;x_{41}))\leq$

$\qquad x_{03}\,;[(x_{30}\,;x_{01}\cdot x_{32}\,;x_{21});x_{14}\cdot x_{32}\,;x_{24}\cdot x_{30}\,;(x_{01}\,;x_{14}\cdot x_{02}\,;x_{24})];x_{41}$

(D3) $x_{01}\leq x_{02}\,;x_{21}\cdot x_{03}\,;x_{31}\ \wedge\ x_{20}\,;x_{03}\cdot x_{21}\,;x_{13}\leq x_{24}\,;x_{43}\ \rightarrow$

$\qquad x_{01}\leq(x_{02}\,;x_{24}\cdot x_{03}\,;x_{34});(x_{42}\,;x_{20}\cdot x_{43}\,;x_{31})$

Lyndon's nonrepresentable relation algebra fails to satisfy (D1). It follows that (D1) is not derivable from Tarski's axioms (I)–(XV). Tarski's axiomatization is therefore incomplete.

Lyndon original versions of (D1)–(D3) were first order conditions on the "atom structure" of a finite relation algebra. To explain what this is we present Lyndon's analysis of the structure of finite relation algebras.

Since every relation algebra A has a Boolean algebra as a reduct we may apply all the standard terminology and theory of Boolean algebras to relation algebras. In particular, every finite relation algebra has a finite Boolean part, and the structure of every finite Boolean algebra is completely determined by one number, namely, the number of atoms. Every finite relation algebra A has cardinality 2^n, where n is the number of atoms of A. The structure of A thus depends entirely on the choice of 1' and on the way the operations ; and $\breve{}$ are defined. In fact, since ; and $\breve{}$ are distributive over $+$, and every element in a finite Boolean algebra is the join of the atoms below it, the action of ; and $\breve{}$ is completely determined by their restrictions to the atoms. For example, if $x = a+b+c$ and $y = b+c+d$, where a, b, c, and d are atoms, then $x;y = (a+b+c);(b+c+d) = a;b+a;c+a;d+b;b+b;c+b;d+c;b+c;c+c;d$

[86] See §5 of [Lyndon1950] for the conditions, Theorem II for their sufficiency, and Theorem I for their necessity, with "complete" in Theorem I replaced by "finite", in accordance with the corrections listed in [Lyndon1956].

and $\breve{x} = (a+b+c)\breve{} = \breve{a} + \breve{b} + \breve{c}$. In general, for all $x, y \in A$,

$$\begin{aligned} x;y &= \sum\{a;b : x \geq a \in AtA,\ y \geq b \in AtA\}, \\ \breve{x} &= \sum\{\breve{a} : x \geq a \in AtA\}, \end{aligned}$$

where AtA is the set of atoms of A.

In every relation algebra, if $x \leq 1'$, then $\breve{x} = x$. Furthermore, if x is an atom, then so is $\breve{x}$.[87] Thus $\breve{}$ is an involution on the atoms which leaves the atoms below 1' fixed. An atom a of a finite relation algebra A is an **identity atom** if $a \leq 1'$, and a **diversity atom** if $a \leq 0'$. The atom a is **symmetric** if $\breve{a} = a$, and **antisymmetric** if $\breve{a} \cdot a = 0$. Every identity atom is symmetric, and every antisymmetric atom is a diversity atom. Since $\breve{}$ is an involution, the antisymmetric atoms occur in pairs.

The structure of a finite relation algebra is completely determined by the following items: a list of atoms below 1', an involution $\breve{}$ on the atoms which leaves the atoms below 1' fixed, and a table listing the products $a;b$ for all pairs of atoms a, b. However, the action of ; and $\breve{}$ on AtA is in turn completely determined by the ternary relation

$$C(A) = \{\, \langle a, b, c \rangle : a, b, c \in AtA,\ a;b \geq c \,\}\,.$$

Indeed, for every $a, b \in AtA$, we have

$$a;b = \sum\{c : \langle a, b, c \rangle \in C(A)\}.$$

The triples in $C(A)$ will be called **cycles**. Lyndon suggested[88] that a triple $\langle a, b, c \rangle$ of atoms be called a *cycle* if $a;b \geq \breve{c}$. Any such triple also satisfies the conditions $b;c \geq \breve{a}$, $c;a \geq \breve{b}$, $\breve{b};\breve{a} \geq c$, $\breve{a};\breve{c} \geq b$, and $\breve{c};\breve{b} \geq a$. We have departed somewhat from Lyndon's suggestion, using the condition $a;b \geq c$, instead of $a;b \geq \breve{c}$, because $C(A)$ is the relation which occurs in the Jónsson-Tarski Representation Theorem[89] for Boolean algebras with operators. Note that a triple $\langle a, b, c \rangle$ is a cycle in Lyndon's sense iff $\langle a, b, \breve{c} \rangle$ is a cycle in the sense followed here.

In view of these observations, it is clear that a table listing the products of all the pairs of atoms of a given finite relation algebra is nothing more than a (rather redundant) list of its cycles. Furthermore, as Lyndon observed, the identity atoms can be characterized among all atoms as those which satisfy

[87] See [Lyndon1950], p. 710, line 5, or [Jónsson-Tarski1952], Theorem 4.3(xii).

[88] [Lyndon1950], p. 710.

[89] See [Jónsson-Tarski1951], §3. Theorem 3.10 is the Representation Theorem. For the correspondence between operators and relations, see Definition 3.2 and Theorem 3.3.

$u;\breve{u} = u$. Thus he notes that a finite relation algebra may be characterized by specifying the mapping $\breve{}$ from atoms to atoms, and giving a list of cycles.

It turns out, however, that the entire structure is determined by the cycles alone. First, it can be shown that for every $a \in AtA$, $a \leq 1'$ iff $b = c$ whenever $\langle a, b, c\rangle \in C(A)$. Thus $C(A)$ determines all the identity atoms. Next, $\breve{a}$ is the unique $b \in A$ such that $\langle a, b, u\rangle \in C(A)$ for some identity atom u. Thus the cycles determine $\breve{}$ as well. However, as we now show, it is more efficient to specify $\breve{}$ independently of the cycles.

Suppose A is a finite relation algebra. For any $a, b, c \in AtA$, let

$$[a, b, c] = \{\langle a, b, c\rangle,\ \langle \breve{a}, c, b\rangle,\ \langle b, \breve{c}, \breve{a}\rangle,\ \langle \breve{b}, \breve{a}, \breve{c}\rangle,\ \langle \breve{c}, a, \breve{b}\rangle,\ \langle c, \breve{b}, a\rangle\}.$$

Then either $[a, b, c] \leq C(A)$ or else $[a, b, c] \cdot C(A) = 0$. In other words, $C(A)$ can be partitioned into sets of the form $[a, b, c]$, each of which may contain up to six different cycles. This follows from (C3). We will refer to the sets $[a, b, c]$ as **cyclesets**. Any finite relation algebra may therefore be succinctly specified in the following way. First list the identity atoms, then the symmetric diversity atoms, then the pairs of antisymmetric atoms, and finally the cyclesets. The action of $\breve{}$ is thus implicitly specified by the notation for the atoms. Pairs of antisymmetric atoms are denoted by a, $\breve{a}$, b, $\breve{b}$, *etc.*, while symmetric atoms are denoted by just a, b, *etc.* (The convention is that if a is listed as an atom, but $\breve{a}$ does *not* appear in the list of atoms, then $\breve{a} = a$.) We use u_1, u_2, *etc.*, for identity atoms. For example, let

$$u_0,\ u_1,\ a,\ b,\ \breve{b}$$

be a list of the five atoms in a certain finite relation algebra A with 32 elements. We can determine 1' and $\breve{}$ from this list: $1' = u_0 + u_1$, and $\breve{u}_0 = u_0$, $\breve{u}_1 = u_1$, $\breve{a} = a$, $(b)\breve{} = \breve{b}$, $(\breve{b})\breve{} = b$. What remains is to specify the cyclesets making up $C(A)$:

$$[u_0, u_0, u_0],\ [u_1, u_1, u_1],\ [u_0, a, a],\ [u_0, b, b],\ [u_1, \breve{b}, \breve{b}],\ [a, b, b],\ [a, a, a].$$

Note that $[u_1, \breve{b}, \breve{b}] = [b, u_1, b]$, so we could actually list the cyclesets using two fewer symbols. The relation algebra A thus specified happens to be isomorphic to the subalgebra of $Re\{0, 1, 2, 3\}$ which is generated by the relation $\{\langle 0, 3\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle\}$. The atoms of this subalgebra are

$$\begin{array}{l}
\{\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,2\rangle\}\\
\{\langle 3,3\rangle\}\\
\{\langle 0,1\rangle, \langle 1,0\rangle, \langle 0,2\rangle, \langle 2,0\rangle, \langle 1,2\rangle, \langle 2,1\rangle\}\\
\{\langle 0,3\rangle, \langle 1,3\rangle, \langle 2,3\rangle\}\\
\{\langle 3,0\rangle, \langle 3,1\rangle, \langle 3,2\rangle\}
\end{array}$$

which correspond respectively with the atoms u_0, u_1, a, b, and $\breve{b}$ of A.

The **atom structure**[90] of a finite relation algebra A is

$$\mathfrak{At}\, A = \langle At\, A, C(A), \breve{}\,, \{u : 1\text{'} \geq u \in At\, A\} \rangle .$$

As we have seen, the atom structure determines the algebra completely. In fact, an isomorphic copy of A can be obtained by applying the "complex algebra" construction of Jónsson and Tarski.[91] They showed that every complete and atomic (hence every finite) relation algebra is isomorphic to the complex algebra of its atom structure.[92] Lyndon worked out the first order conditions which characterize those relational structures whose complex algebras are relation algebras.[93] This characterization is part of his infinite set of first order conditions which characterize those *finite* relational structures whose complex algebras are *representable* relation algebras. The simplest conditions in this latter characterization which could possibly fail in some relation algebra are formulated above as (D1)–(D3). Indeed, as Lyndon showed, (D1) can fail in a finite relation algebra. Lyndon presented his algebra by listing its atoms, their converses, and the cycles. This will not be done here because Lyndon's algebra has fifty-six atoms, and considerably smaller nonrepresentable relation algebras were found later. Lyndon knew that all finite relation algebras with three or fewer atoms are representable,[94] so the smallest nonrepresentable relation algebra must have at least four atoms. Ralph McKenzie was the first to find such a small nonrepresentable relation algebra.[95] The atoms of McKenzie's algebra are $1\text{'}, a, b, \breve{b}$, and its cyclesets are

$$[1\text{'},1\text{'},1\text{'}],\ [1\text{'},a,a],\ [1\text{'},b,b],\ [b,1\text{'},b],\ [b,b,b],\ [a,b,a],\ [a,b,b],\ [b,a,b].$$

McKenzie's argument that his algebra is not representable does not involve (D1)–(D3). It turns out that McKenzie's algebra satisfies (D1), unlike Lyndon's algebra, but fails to satisfy (D2) and (D3). For example, (D2) fails when $x_{01} = a$, $x_{02} = b$, $x_{21} = \breve{b}$, $x_{03} = a$, $x_{32} = a$, $x_{24} = a$, and $x_{41} = a$, while (D3) fails when $x_{01} = a$, $x_{02} = b$, $x_{21} = \breve{b}$, $x_{03} = \breve{b}$, $x_{31} = b$, $x_{24} = a$,

[90] This term is not used by Lyndon. It was borrowed from the theory of cylindric algebras and applied to relation algebras in [Maddux1982].

[91] See [Jónsson-Tarski1951], Definition 3.8.

[92] [Jónsson-Tarski1951], Theorem 3.9.

[93] See [Lyndon1950], p. 710. A slightly different first order characterization is given in [Maddux1978], Theorem 3(5), and in [Maddux1982], Theorem 2.2. For symmetric relation algebras, those in which $\breve{x} = x$ holds, this characterization appears in [Jónsson1959], pp. 460–461.

[94] [Lyndon1956], p. 307, footnote 13.

[95] See [McKenzie1966], pp. 37–39, or [McKenzie1970], p. 286.

and $x_{43} = a$. Thus none of (D1)–(D3) can be proved from Tarski's axioms (I)–(XV).

7. Epilogue

This paper has examined

- the origin of the calculus of relations in the work of De Morgan and Peirce,
- Tarski's axiomatization of a portion of this calculus,
- the emergence of relation algebras from Tarski's axiomatization,
- the incompleteness of Tarski's axiomatization, as shown by Lyndon's work on finite relation algebras.

In the course of this presentation, we briefly indicated Peirce's role in the formation of first order logic. There are many other interesting topics which have not been pursued, such as

(1) Schröder's development of the calculus of relations,
(2) the development of model theory out of the Peirce-Schröder tradition, through Löwenheim and Tarski.
(3) the independent development of concepts closely related to relation algebras, and
(4) the later history of relation algebras.

Here we make just a few brief comments on these topics.

(1) Tarski had a high opinion of Schröder's book, which, he said, "contains a wealth of unsolved problems, and seems to indicate the direction for further investigations".[96] Peirce was not so enthusiastic, as shown by his brief historical summary. Schröder developed Peirce's algebra from *Note B*, but Peirce thought that "Professor Schröder attaches, it seems to me, too high a value on this algebra."[97] Nevertheless, Schröder's work deserves much more attention than it has been given.

(2) It was noted above that Löwenheim's Theorem, now conceived as the first theorem in model theory, was a theorem about the calculus of relations. Tarski is generally acknowledged as the principal creator of model theory. It is no coincidence that he was influenced by the Peirce-Schröder tradition. The ideological trappings of the work of Frege, Whitehead, and Russell precluded model theory.

(3) The atom structure of a finite relation algebra has a ternary relation (the set of cycles) which can be viewed as a set-valued binary operation. Such structures have been invented from the 1930's onward as generalizations

[96] [Tarski1941], p. 74.

[97] [Peirce1933], 3.498. For further explication of Schröder's work by Peirce, see [Peirce1897], which can be found in [Peirce1933], 3.510–525.

of groups and other mathematical objects. As a result, relation algebras have many tantalizing connections with various other areas of mathematics. Peirce himself worked on some of these connections, and many more have appeared over the last hundred years.

(4) The later history of relation algebras has many fascinating stories, a few of which we will now briefly outline. Besides constructing a nonrepresentable finite relation algebra, Lyndon proved that the class of representable relation algebras is *not* axiomatizable by any set of equations.[98] On the other hand, Tarski proved that the class of representable relation algebras *is* axiomatizable by a set of equations.[99] The apparent contradiction was resolved in Tarski's favor.[100] Lyndon responded with an equational axiomatization of the class of relation algebras.[101] Building on Lyndon's work, Jónsson axiomatized a special class of reducts of representable relation algebras.[102] Noting that the associative law for relative multiplication is equivalent to a first order condition which resembles one of the axioms of projective geometry, Jónsson constructed a nonrepresentable relation algebra from a non-DesArguesian projective plane.[103] By extending Jónsson's idea, Lyndon found a beautiful connection between projective geometries and certain relation algebras.[104] Through this connection, the nonexistence of a projective plane of order six yields the nonrepresentability of a certain relation algebra with only eight atoms, a considerable improvement over Lyndon's first example. This connection had a profound effect on Lyndon's earlier work. Lyndon felt that his infinite set of conditions (the ones characterizing finite representable relation algebras) could not be reduced to finitely many.[105] He turned out to be right. J. D. Monk used Lyndon's connection, together with the Bruck-Ryser Theorem on the nonexistence of certain projective planes, to show that there is an infinite class of finite nonrepresentable relation algebras with a representable ultraproduct.[106] This shows that the class of representable relation algebras is not finitely axiomatizable, and that Lyndon's intuition was correct. It follows from this that Tarski's axiomatization is very incomplete: no finite number of additional axioms will secure completeness. It is therefore all the more amazing that Tarski's short list of axioms seems to suffice for the derivation of all the

98 [Lyndon1950], Theorem IV.
99 [Tarski1955], Theorem 2.4.
100 See [Tarski1955], p. 61, footnote 5, and [Lyndon1956], p. 294 and the Appendix, pp. 306–307.
101 [Lyndon1956].
102 [Jónsson1959], Theorem 1.
103 [Jónsson1959], pp. 460–463.
104 [Lyndon1961].
105 [Lyndon1950], p. 713.
106 [Monk1964].

results in Schröder's book.

We close with one more remark on Lyndon's conditions. He did not say, and presumably did not know, whether (D1)–(D3) are independent, but it turns out that they are indeed independent. This has been shown by computer computations. In fact, every possible subset of (D1)–(D3) can fail in a finite relation algebra with no more than five atoms.

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Department of Mathematics
Iowa State University
Ames, Iowa 50011

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