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Compactness in Translation Invariant Banach Spaces of Distributions and Compact Multipliers

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It is shown that for a comprehensive family of translation invariant Banach spaces $(B, \| \|_B)$ of (classes of) measurable functions or distributions on a locally compact group (including most of the spaces of interest in harmonic analysis) the following *compactness criterion* generalizing the well-known results due to Kolmogorov–Riesz–Weil concerning compact sets in $L^p(G)$, $1 \leqslant p < \infty$, holds true: A closed subset $M \subseteq B$ is compact in B if and only if it satisfies the following conditions: (a) $\sup_{f \in M} \|f\|_B < \infty$; (b) $\forall \varepsilon > 0 \ \exists k \in \mathscr{K}(G): \|k * f - f\|_B < \varepsilon$ for all $f \in M$; (c) $\forall \varepsilon > 0 \ \exists h \in \mathscr{K}(G): \|hf - f\|_B < \varepsilon$ for all $f \in M$. Among various applications a characterization of the space of all compact multipliers between suitable pairs of such spaces can be derived.

INTRODUCTION

The purpose of this paper is threefold. First of all the above-mentioned compactness criterion is to be presented (in Section 2) in its perhaps most general form. It applies to left invariant Banach spaces of distributions (and even ultradistributions, [11]) on locally compact groups. The main interest will concern noncompact groups. In compensation we shall suppose that these spaces have a sufficiently rich multiplicative structure, i.e., with a regular Banach algebra A of continuous functions acting on them by pointwise multiplication. For Banach spaces of locally integrable functions it coincides essentially with the main result of [38]. The second purpose is a discussion of translation invariant spaces having such a multiplicative structure, called Banach spaces of distributions in "standard situation" (Section 1). In this section it is to be shown that most Banach spaces occurring in harmonic analysis are in "standard situation," and several basic observations concerning this family of spaces are made. The generality of the main result heavily depends on the results of this first section, which also serve as a reference for further publications. We only mention [15], where a more detailed study of the double module structure (i.e., with respect to

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convolution and to pointwise multiplication) is to be given, and [37], where it is explained among others that spaces in standard situation are well suited as local components in the definition of Banach spaces of distributions of local–global type (=Wiener-type spaces). Finally, it is shown in Section 3 that for suitable pairs of translation invariant Banach spaces on noncompact groups a characterization of the space of all compact multipliers can be given that is quite similar to the results known to hold in the case of compact groups.

The reader who is mainly interested in the compactness criterion and its applications to compact multipliers is advised to skip the major part of Section 1 at a first reading and to think of $A = C^0(G)$, $A_0 = \mathscr{K}(G)$ and $A_0' := R(G)$, the space of all Radon measures with the vague topology, in order to avoid notational complications. Some comments on existing literature are given in Section 4.

1. Notations and Description of Standard Situations

Let G be a lc. (= locally compact) group with identity e. As usual measurable functions that coincide l.a.e. (= locally almost everywhere) are identified. For a (continuous) function f on G the action of the left and right translation operators L_y and R_y, $y \in G$, are given by

$$L_y f(x) := f(y^{-1}x), \qquad R_y f(x) := f(xy^{-1}) \Delta^{-1}(y)$$

where Δ denotes the Haar modul on G.[1] $\mathscr{K}(G)$ denotes the space of continuous functions with compact support (supp), endowed with its natural inductive limit topology. The usual Lebesgue spaces are denoted by $(L^p(G), \| \ \|_p)$, $1 \leqslant p \leqslant \infty$. The closure of $\mathscr{K}(G)$ in $L^\infty(G)$ is identified with $C^0(G)$. $(L^1(G), \| \ \|_1)$ is considered as a Banach algebra with convolution as multiplication. $B^1 \hookrightarrow B^2$ will indicate a continuous embedding between topological vector spaces.

General Assumption

Throughout the paper $(A, \| \ \|_A)$ will always denote a (fixed) (left and right) invariant regular, self-adjoint Banach algebra of complex-valued, continuous functions on G which is continuously embedded in $(C^0(G), \| \ \|_\infty)$ such that $A \cap \mathscr{K}(G)$ is dense in A. We assume further that left and right translation are continuous in A, i.e., that one has for all $h \in A$:

$$\lim_{y \to e} \|L_y h - h\|_A = 0 = \lim_{y \to e} \|R_y h - h\|_A .$$

[1] The inversion of the argument is denoted by $f^\vee : f^\vee(x) := f(x^{-1})$.

If left translation is isometric in A ($\|L_y h\|_A = \|h\|_A$, $y \in G$), then A is a homogeneous Banach space (of locally integrable functions) in the sense of Shilov–Katznelson [97, 59]. First examples are $C^0(G)$ or Eymard's Fourier algebra $A(G)$ (cf. [28], it coincides with the ordinary Fourier algebra if G is abelian, see [89] for details).

We shall have to consider $A_0 := A \cap \mathscr{K}(G)$, which is a topological vector space with its natural inductive limit topology. The assumptions imply that A_0 is a dense subspace of $\mathscr{K}(G)$. It follows that A_0', the topological dual of A_0, is a topological vector space. Whenever $(A')_{loc}$ is defined in a reasonable way it may be identified A_0' as a topological vector space. It contains the space $R(G) = \mathscr{K}(G)'$ of Radon measures in a natural way. In the case of $A = A(G)$ the space $A_0' =: Q(G)$ is just the space of quasimeasures introduced by Gaudry ([45], see [18]). Furthermore, the support (=: supp) of $\sigma \in A_0'$ is well defined. For a subspace $B \subseteq A_0'$ we write B_Q for the space $\{f \mid f \in B,\ \operatorname{supp} f \subseteq Q\}$. The action of the translation operators may be extended to A_0' by transposition. $B \subseteq A_0'$ will be called *left* (*right*) *invariant* if $L_y B \subseteq B$ ($R_y B \subseteq B$) for all $y \in G$. If $(B, \|\ \|_B)$ is a Banach space, continuously embedded in A_0' the invariance implies (by means of the closed graph theorem) already the continuity of the operators $L_y(R_y)$, $y \in G$, on B for any $y \in G$. Its norm will be denoted by $\|L_y\|_B$. If B is left and right invariant we call B *translation invariant*. *Isometric invariance* is defined in the obvious way for normed left (right) invariant subspaces of A_0'. If $(B, \|\ \|_B)$ is left invariant we write $B_{\ell c} := \{f \mid y \mapsto L_y f$ is continuous from G into $B\}$. It can be shown to be a closed subspace of $(B, \|\ \|_B)$. If $B = B_{\ell c}$ (or $B = B_{rc}$, the corresponding (right version) one says that B has continuous *left* (*right*) *translation*. If $(B, \|\ \|_B)$ is isometrically left invariant and has continuous left translation it is called *homogeneous Banach space of distributions* (or quasimeasures ...). Since the usual homogeneous Banach spaces in $L^1_{loc}(G)$ are exactly the homogeneous Banach spaces of measures (i.e., for $A = C^0(G)$) this terminology is consistent with that used earlier (cf. [31, 106]). *Segal algebras* as introduced by Reiter (cf. [89, 90]) are then homogeneous Banach spaces which are dense in $L^1(G)$.

Banach spaces continuously embedded in $L^1_{loc}(G)$ have been called *BF-spaces* in earlier publications by the author (cf. [30, 32], e.g.). A *BF*-space is called *solid* if $g \in B$, $f \in L^1_{loc}(G)$ and $|f(x)| \leqslant |g(x)|$ l.a.e. implies $f \in B$ and $\|f\|_B \leqslant \|g\|_B$. Such spaces have been treated under the name of *Banach function spaces* by Luxemburg–Zaanen (cf. [113, Chapter 15], for example).

A Banach space of distributions on an abelian group is called (*strongly*) *character invariant*, if multiplication with any character $\chi \in \hat{G}$ defines a bounded operator (isometry) on B. The extended *Fourier transform* (either for tempered distributions on the Euclidean space, or for the space of translation bounded quasimeaasures $S_0'(G)$, introduced in [34]) is denoted by $\mathscr{F}$. We write $A_w(G)$ for $\mathscr{F}[L^1_w(\hat{G})]$.

A Banach space $(B, \| \|_B)$ is called a *left Banach module* over a Banach algebra $(C, \| \|_C)$ if B is a left module over C is the algebraic sense for some multiplication $(c, b) \mapsto c \cdot b$, satisfying $\|c \cdot b\|_B \leqslant \|c\|_C \|b\|_B$ for $c \in C$, $b \in B$. Right and two-sided Banach modules are defined in a similar way (left and right actions are supposed to commute). Since only Banach modules will be considered we shall speak of C-modules for short. In the applications below the module operation (written as $\cdot$ above) will be pointwise multiplication and convolution, respectively. Thus, a BF-space on G is solid if and only if it is a (pointwise) Banach module over $L^\infty(G)$. In particular, any solid BF-space is a $C^0(G)$-module. A left (right, two-sided) C-module is called *essential* if the closed linear span of $C \cdot B$ $(B \cdot C, C \cdot B \cdot C)$ coincides with B. If the Banach algebra $(C, \| \|_C)$ contains a *bounded left approximate unit*, i.e., a bounded net $(u_\alpha)_{\alpha \in I}$ such that $\lim_{\alpha \to \infty} \|u_\alpha c - c\|_C = 0$ for all $c \in C$, then a Banach module over C is an essential one if and only if $\lim_{\alpha \to \infty} \|u_\alpha b - b\|_B = 0$ for all $b \in B$ (cf. [92]; detailed information can also be found in [22], and various formulas of relevance in harmonic analysis are to be found in [52, 92, 65]).

It is obvious that the Banach algebra $C^0(G)$ has bounded approximate units of norm one, and it is well known that $A(G)$ has bounded approximate units if and only G is amenable [27].

If w is a (continuous) *weight function* on G, i.e., $w(x) \geqslant 1$, $w(xy) \leqslant w(x)\, w(y)$ for $x, y \in G$, then the space $L^1_w(G) := \{f \mid fw \in L^1(G)\}$ is a Banach algebra on G with respect to convolution, with the norm $\|f\|_{1,w} := \|fw\|_1$. It is called a *Beurling algebra* (cf. [89, Chapter III, Section 7i]; as explained in [32] it is no loss of generality to consider only continuous weights). Without loss of generality we shall assume that the weight functions w occurring below are symmetric, i.e., satisfy $w(x) = w(x^{-1})$ for $x \in G$. In that case $L^1_w(G)$ is stable with respect to the involution $f \to f^*$, $f^* := (f^\vee)^- \Delta^{-1}$. In any case $L^1_w(G)$ has bounded, two-sided approximate units $(u_\gamma)_{\gamma \in J}$, e.g., the normalized characteristic functions of a family of sets forming a basis of neighborhoods of the identity in G. We set $C_w := \sup_{\gamma \in J} \|u_\gamma\|_{1,w} < \infty$ (cf. [89, Chapter VI] for details).

In Section 3 the following notations will be needed: Given two Banach spaces B^1 and B^2 the space of all bounded linear operators from B^1 to B^2 is denoted by $H(B^1, B^2)$, and the norm of $T \in H(B^1, B^2)$ is denoted by $\|T\|_{B^1 \to B^2}$. If we write $\circ B^1$ for the unit ball $\{f \mid f \in B^1, \|f\|_{B^1} \leqslant 1\}$ of B^1, one has $\|T\|_{B^1 \to B^2} = \sup\{\|Tf\|_{B^2} \mid f \in \circ B^1\}$. If the spaces B^i, $i = 1, 2$, are left invariant Banach spaces of distributions one calls T a *right multiplier* if it commutes with left translations, i.e., if one has $TL_y = L_y T$ for all $y \in G$. The closed subspace of all right multipliers is denoted by $H_G(B^1, B^2)$. It is folklore (to be proved by vector-valued integration) that the following is true in case left translation is continuous in B^1 (it is an essential $L^1_w(G)$-module in that case, cf. Section 2): $T \in H(B^1, B^2)$ is a right multiplier if and only if one has $T(k * f) = k * Tf$ for all $f \in B$, $k \in \mathscr{K}(G)$. The subspace of all *compact right multipliers* is denoted by $C_G(B^1, B^2)$.

Throughout this paper the following situation will be referred as STANDARD SITUATION ("the space B is in standard situation with respect to A"):

$(B, \| \|_B)$ is a Banach space, such that for some Banach algebra $(A, \| \|_A)$ satisfying the general assumptions above one has $A_0 \hookrightarrow B \hookrightarrow A_0'$ (for the $\sigma(A_0', A_0)$-topology). It is assumed that $(B, \| \|_B)$ is a left Banach module over some Beurling algebra $L^1_w(G)$ with respect to convolution, and that $(B, \| \|_B)$ is a Banach module over $(A, \| \|_A)$ with respect to pointwise multiplication, and that there exists a net $(\tau_\alpha)_{\alpha \in I}$ in A_0 of trapezoid functions of bounded action on B (i.e., satisfying the following two conditions:

$$\sup_{\alpha \in I} \sup_{h \in \circ B} \|\tau_\alpha h\|_B =: C_B < \infty,$$

and for any compact subset $K \subseteq G$ there exists $\alpha_0 = \alpha(K)$ such that $\tau_\alpha(x) = 1$ for all $x \in K$, and $\alpha \geqslant \alpha_0$).

Remark 1.1. The particular choice of the Banach algebra $(A, \| \|_A)$ used to fulfill the conditions of the "standard situation" for a given Banach space $(B, \| \|_B)$ is quite irrelevant; only the existence of such a Banach algebra is of relevance for the proofs of the main results (because there will be a need for sufficiently smooth cutoff functions). In particular, it is possible to replace a Banach algebra satisfying the general assumptions by a smaller one (e.g., $C^0(G)$ by $A(G)$) whenever convenient.

Remark 1.2. If $(A, \| \|_A)$ has bounded approximate units it follows from the general assumptions that there is a bounded family (in A) of trapezoid functions in A_0 (cf. Lemma 1.5a) below). These can be obtained using the argument given in [3]. Such a family is of course of bounded action on any A-module $(B, \| \|_B)$. Conversely, any family of trapezoid functions in A_0 that is bounded in A defines a bounded approximate unit for A if A satisfies the general assumptions.

Remark 1.3. If $(A, \| \|_A)$ has bounded approximate units, and if B^1 and B^2 are in standard situation with respect to A, then their intersection, $(B^1 \cap B^2, \| \|_{B^1} + \| \|_{B^2})$ is in standard situation as well.

Remark 1.4. The embeddings $A_0 \hookrightarrow B \hookrightarrow A_0'$ are continuous if and only if for any compact set $K \subseteq G$ there exists $C_K > 0$ and $C'_K > 0$ such that one has for all $h \in A_0$ with $\operatorname{supp} h \subseteq K$, and for all $f \in B$:[2]

$$\|h\|_B \leqslant C_K \|h\|_A, \qquad \text{and} \qquad |\langle h, f \rangle| \leqslant C'_K \|f\|_B \|h\|_A. \tag{1.1}$$

[2] The second assertion follows from the closed graph theorem.

Some comments concerning the "standard situation" are in order: First of all it must be said that it occurs quite naturally, as will be seen below. Secondly, the presence of a double module structure (i.e., pointwise multiplication and convolution) on B is of basic importance as it allows one to prove many results that hold for L^p-spaces for much more general translation invariant spaces of distributions on locally compact groups, for which the compactness criterion below is just one illustration. In the proofs the operations "multiplication" and convolution" are therby to the understood as refined methods replacing the "cut down to a compact set" (for L^p-functions) and as generalized smoothing process, respectively.

The following proposition contains some folklore statements about the connection of strongly continuous Banach representations of G and corresponding representations of Beurling algebras:

PROPOSITION 1.1. *Let* $(B, \| \|_B)$ *be a Banach space, continuously embedded in* A_0'. *Then* $(B, \| \|_B)$ *is an essential left (right) Banach module over a suitable Beurling algebra* $L^1_w(G)$ *if and only if* B *is translation invariant and has continuous left (right) translation. In that case for any* $f \in B$ *and any bounded (two-sided) approximate unit* $(u_\gamma)_{\gamma \in J}$ *in* $L^1_w(G)$ *the relation*

$$\lim_{\gamma \to \infty} \|u_\gamma * f - f\|_B = 0 = \lim_{\gamma \to \infty} \|f * u_\gamma - f\|_B \tag{1.2}$$

holds true. Any homogeneous Banach space is an essential left Banach module over $L^1(G)$.

Proof. Using standard methods involving vector-valued integration one can show that for a Banach space with continuous translations left (right) representations of some Beurling algebra $L^1_w(G)$ on B can be defined. Since $\mathscr{K}(G)$ is contained in $L^1_w(G)$ and since translations are continuous in B these actions are essential ones. (cf. [59, Chapter VI, 1.14 and Exercises 11–14] for a special case, see also [23]). The converse as well as formula (1.2) can be derived from the Cohen–Hewitt factorization theorem ([52, 32.22]; see [23] for a typical special case), stating that for $\varepsilon > 0$, $f \in B$ there exist $f^1, f^2 \in B$, with $\|f - f^i\|_B < \varepsilon$, and $h^i \in L^1_w(G)$, $\|h^i\|_{1,w} \leqslant C_w$, $i = 1, 2$, such that $f = h^1 * f^1 = f^2 * h^2$.

We shall give now a couple of statements revealing somewhat the connection between various conditions constituting the standard situation, and we present simple methods of checking that all these conditions are satisfied in concrete cases. As application of these results a list of spaces is given that are in standard situation, i.e., to which the main result applies.

LEMMA 1.2. *Let* $(B, \| \|_B) \hookrightarrow A_0'$ *be a Banach module over a Banach algebra* $(A, \| \|_A)$ *satisfying the general assumptions and having bounded* *approximate units. If B is translation invariant and if $A_0 = A \cap \mathcal{K}(G)$ is densely and continuously embedded in B, then B is in standard situation. In particular, A itself is in standard situation.*

Proof. In view of Proposition 1.1 it will be sufficient to show that translation is continuous in B. Since the general assumption implies continuity of translation in A_0, translation is continuous in B for the elements of a dense subset. It follows therefrom that $y \mapsto \|L_y\|_B$ ($\|R_y\|_B$) is a semicontinuous, submultiplicative function on G, hence locally bounded. This allows to show continuity of $y \mapsto L_y f (R_y f)$ for all $f \in B$ by an approximation argument (cf. [32] for similar arguments).

Remark 1.5. It follows from Lemma 1.2 that the closure of A_0 in B' satisfies the standard assumptions, whenever B satisfies the conditions stated in the lemma. If $(B, \| \|_B)$ is a BF-space, then one has automatically $B \hookrightarrow L^1_{loc}(G) \subseteq R(G) \hookrightarrow A_0'$. Thus one has, among others, the following Corollary.

COROLLARY 1.3. *Let $(B, \| \|_B)$ be a solid, translation invariant BF-space containing $\mathcal{K}(G)$ as a dense subspace. Then B is a homogeneous Banach space (in $L^1_{loc}(G)$) satisfying the standard assumption (for $A = C^0(G)$).*

That any solid BF-space of a certain kind contains $\mathcal{K}(G)$ (e.g., for any rearrangement invariant space B) is usually easy to verify. The density of $\mathcal{K}(G)$ may be derived from the absolute continuity of the norm $\| \|_B$ (cf. [113, Chapter 15] for the definition and equivalent characterizations).

PROPOSITION 1.4. *Let $(B, \| \|_B)$ be a solid BF-space on G containing $\mathcal{K}(G)$. If B has absolutely continuous norm (e.g., if B is reflexive), $\mathcal{K}(G)$ is dense in B.*

Proof. It is no loss of generality to suppose σ-compactness of G. Let $f \in B$, $f \geqslant 0$ be given. By truncation we may obtain a sequence of bounded functions $(f_n)_{n \geqslant 1}$ in B which are concentrated on compact sets and satisfy $0 \leqslant f_n \uparrow f$. Since B has absolutely continuous norm this implies $\|f_n - f\|_B \to 0$ as $n \to \infty$. Let now $n \geqslant 1$ be fixed. Since f_n is in $L^1(G)$ there is a sequence $\{k_j^n\}_{j \geqslant 1}$ in $\mathcal{K}(G)$, $k_j^n \geqslant 0$ with $f_n(x) = \lim k_j^n(x)$ a.e. Moreover we may suppose without loss of generality that the sequence $\{k_j^n\}_{j \geqslant 1}$ is uniformly bounded in j for each n and that all functions k_j^n, $j \geqslant 1$, vanish outside some compact set. The analogon of Lebesgue's theorem on dominated convergence implies $\|k_j^n - f_n\|_B \to 0$ as $j \to \infty$ (cf. [113, Chapter 15, Section 72, Theorem 2]). The supplementary assertion is true since reflexivity of B implies the absolute continuity of the norm of B [113, Section 75].

The class of solid BF-spaces on a locally compact group is quite comprehensive. The basic examples satisfying the standard conditions are of course the $L^p(G)$-spaces for $1 \leqslant p < \infty$, and many of the other spaces to be mentioned here may be seen as generalizations in one or the other direction. One important subclass of all solid BF-spaces is the family of all rearrangement invariant Banach spaces (see [71] for details) containing $\mathscr{K}(G)$ as a dense subspace. Typical members of this class are the so-called (Birnbaum–) Orlicz spaces, the Lorentz and the Lorentz–Zygmund spaces (see [16, 7]).

Weighted L^p-spaces $L^p_w(G) := \{f \mid fw \in L^p(G)\}$, $1 \leqslant p < \infty$, with the norm $\|f\|_{p,w} := \|fw\|_p$, are other examples of solid BF-spaces in standard situation. Further examples are the mixed norm spaces considered in [6] or the "amalgam spaces" considered in [53, 17, 37], for example. Finally, the so-called Morrey spaces should be mentioned (cf. [68]).

Before we treat more general spaces on l.c.a. groups we present some information concerning the interdependence of the various conditions:

LEMMA 1.5. *Let $(A, \| \|_A)$ be a Banach algebra satisfying the general assumptions and $(B, \| \|_B)$ be a non-trivial Banach space continuously embedded in A_0'. Then one has:*

(a) *If B is a pointwise A-module as well as a $L^1_w(G)$-module for convolution, then $A_0 \hookrightarrow B$. In particular, $B \cap \mathscr{K}(G)$ is dense in $\mathscr{K}(G)$.*

(b) *A_0 is dense in B if and only if B is an essential $L^1_w(G)$-module as well as an essential A-module.*

Proof. (a) Let $0 \neq f \in B$, and $k \in A_0$ be given. Since $A_0 \subseteq \mathscr{K}(G) \subseteq L^1_w(G)$ and $B \subseteq A_0'$ there exists $u \in \check{A}_0$ such that $0 \neq u * f \in B$ is a continuous function (cf. Lemma 1.10 below for details). Choosing $h \in A_0$, positive, and with suitable small support one obtains $f^1 := h(u * f) \in AB \subseteq B$ satisfying $\beta := \int_G f^1(y^{-1})\, dy \neq 0$. Choose now $h^1 \in \mathscr{K}^+(G) \subseteq L^1_w(G)$ so that $h_1(x) = \beta^{-1}$ for all $x \in (\operatorname{supp} k) \cdot (\operatorname{supp} h)^{-1}$. It is then clear that $h_1 * f_1(x) \equiv 1$ on $\operatorname{supp} k$. Consequently $k = k(h^1 * f^1) \subseteq A(\mathscr{K}(G) * B) \subseteq B$. It also follows that the embedding $A_0 \hookrightarrow B$ is continuous.

(b) It is obvious that the density of A_0 in B implies that B is an essential $L^1_w(G)$-module, beause $\mathscr{K}(G) * A_0$ is dense in A_0, and an essential A-module, since $A_0 A_0 = A_0$ by the existence of trapezoid functions in A_0(this follows among others from the proof of (a), if one takes $B = A$). Let now B be an essential module in both senses. Then for $f \in B$ and $\varepsilon > 0$ there exists $k_1, k_2 \in A_0$ such that $\|k_2(k_1 * f) - f\|_B < \varepsilon$, and $f^2 := k_2(k_1 * f) \in \mathscr{K}(G)$. Choose now $k_3 \in A_0$ (dense in $L^1_w(G)$!) such that $\|k_3 * f_2 - f_2\|_B < \varepsilon$. Since $k_3 * f^2 \in A_0 * \mathscr{K}(G) \subseteq A_0$ by the continuity of right translations in A, and since $\|k_3 * f^2 - f\|_B < 2\varepsilon$, (b) is proved.

A number of examples having all properties of the standard situation are to be given at the end of this section.

Remark 1.6. It can also be shown that for B as above, having continuous translation, the density of $B_0 := \{f \mid f \in B, \operatorname{supp} f \text{ compact}\}$ implies already density of A_0.

Remark 1.7. If $(A, \| \|_A)$ has bounded approximate units it follows from Banach module theoretic arguments that reflexivity of $(B, \| \|_B)$, as a Banach space, implies that it is essential as A- and as L^1_w-module (cf. [92]). Hence, A_0 is dense in B if B is a reflexive Banach space with double module structure (cf. Proposition 1.4 above; for a related result see [15]).

In order to prove that various Banach spaces of quasimeasures on a locally compact abelian group are in standard situation the following result turns out to be quite effective:

PROPOSITION 1.6. *Let G be a l.c.a. group, and let $(B, \| \|_B)$ be a homogeneous Banach space of quasimeasures (i.e., $B \hookrightarrow Q(G)$). If $B \cap L^1(G)$ is dense in B and if B is strongly character invariant, then B is in standard situation with respect to the Fourier algebra $A(G)$, and $A_0 = A \cap \mathscr{K}(G)$ is a dense subspace of B.*

Proof. Since we may assume that $B \neq \{0\}$ it follows that $B \cap L^1(G) \neq \{0\}$ is a homogeneous Banach space in $L^1(G)$, with its natural norm $\| \| := \| \|_B + \| \|_1$, hence a left ideal in $L^1(G)$ (cf. Proposition 1.1). The strong character invariance of $B \cap L^1(G)$ and the minimality of a certain Segal algebra $S_0(G)$ in the family of all strongly character invariant homogeneous Banach spaces in $L^1(G)$ (the density in $L^1(G)$ follows from Wiener's Tauberian theorem, [89, VI.1.3]) implies $A_0 \hookrightarrow S_0(G) \hookrightarrow B \cap L^1(G) \hookrightarrow B$, the embeddings being continuous and dense. The mapping $\chi \mapsto \chi f$ being continuous from $\hat{G}$ into $S_0(G)$ (cf. [36]) for $f \in S_0(G)$ once more vector-valued integration may be applied in order to obtain a representation of $A(G)$ $(=\mathscr{F}L^1(\hat{G}))$ via pointwise multiplication on B. The density of A_0 in B then implies that the A-module structure has all features of a "standard situation."

Remark 1.8. After suitable modifications the above result may be extended to Banach spaces $(B, \| \|_B)$ that are only invariant under multiplication by characters and satisfying certain (non-quasianalyticity) conditions. $A(G)$ has to be replaced by $A_w(G) := \mathscr{F}[L^1_w(\hat{G})]$, for a suitable weight function w on $\hat{G}$.

Remark 1.9. It follows from the proof that the density of $B \cap L^1(G)$ in the above proposition may be replaced by the density of $A(G) \cap \mathscr{K}(G)$, or better by the density of $S_0(G)$ in B. The invariance of $S_0(G)$ under Fourier transforms (see [36, Theorem 7]) then implies that the family of spaces considered in Proposition 1.6 is mapped on the corresponding family of spaces on the dual group by means of the (extended) Fourier transform. A closely related result is given in the following Lemma.

PROPOSITION 1.7. *Let* $(B, \| \|_B)$ *be a character invariant, homogeneous Banach space of quasimeasures on a l.c.a. group G such that* $B \cap L^1(G)$ *is dense in B. Then the space* $\mathscr{F}B$, *with its natural norm* $\|\hat{f}\|_{\mathscr{F}B} := \|f\|_B$, *is a Banach of quasimeasures on* $\hat{G}$, *in standard situation with respect to* $A(\hat{G})$, *and* A_0 *is dense in* $\mathscr{F}B$.

Proof. As in the proof of 1.6 it follows that $B \cap L^1(G)$ is a character invariant Segal algebra on G with its natural norm. It follows that $A^K := \{f \mid f \in L^1(G), \hat{f} \in \mathscr{K}(\hat{G})\}$ is a dense subspace of $B \cap L^1(G)$, hence of B. Since it can be shown that any homogeneous Banach space of quasimeasures is contained in $S_0'(G)$—the Banach dual of the Segal algebra $S_0(G)$—the extended Fourier transform, defined for $S_0'(G)$ by transposition of the ordinary Fourier transform $\mathscr{F}: S_0(G) \to S_0(\hat{G})$ (cf. [34]), maps B isometrically onto $\mathscr{F}B \subseteq S_0'(\hat{G}) \subseteq Q(G)$, and $A_0 = \mathscr{F}(A^K)$ is dense in $\mathscr{F}B$. The character invariance of B implies translation invariance of $\mathscr{F}B$. Finally, the (essential) $L^1(G)$-module structure of B corresponds to an $A(\hat{G})$-module structure (pointwise) of B. An application of Lemma 1.2 then yields the desired result.

It follows from Proposition 1.7 that spaces such as $\mathscr{F}L^q$, $1 \leqslant q < \infty$, are in standard situation for $A(G)$, even if it does not follow from some type of Hausdorff–Young inequality (cf. [9]) that it is a BF-space. Applying Remark 1.3 one sees that spaces such as $B := \{f \mid f \in L^p(G), \mathscr{F}f \in L^r(G)\}$ are in standard situation for $A(G)$, if $1 \leqslant p, r < \infty$.

Let us return to the Banach spaces on l.c.a. groups as considered in Proposition 1.6. As we have seen in Proposition 1.7 this class, which is larger than the (at first sight perhaps more natural) class of solid *BF*-space considered in 1.3 and 1.4, has the advantage of being in a sense invariant under Fourier transforms. We mention at this occasion that the necessity of introducing quasimeasures arises already if one wants to treat $\mathscr{F}L^p$ for $p > 2$. Another advantage will be discussed now: Whenever the *convolution tensor product* $B^1 \circledast B^2$ (cf. [34]) of two spaces in this class is well defined as a Banach space of quasimeasures (e.g., if $B^1 \subseteq L^1_w(G)$, or $B^1 \subseteq L^p(G)$ and $B^2 \subseteq L^{p'}(G)$, for $1/p + 1/p' = 1$) it also satisfies the conditions stated in Proposition 1.6.

Proof. The density of $(B^1 \circledast B^2) \cap L^1(G)$ is dense in $B^1 \circledast B^2$ as it contains the linear span of $(B^1 \cap L^1(G)) * (B^2 \cap L^1(G))$. The strong character invariance follows from the formula $\chi(f * g) = \chi f * \chi g$ for $\chi \in \hat{G}$, $f, g \in L^1(G)$.

Remark 1.10. It should be observed that the convolution tensor product of two solid *BF*-spaces is usually no longer a solid *BF*-space. If one takes

$B^1 = L^p(G)$, and $B^2 = L^{p'}(G)$, $1/p + 1/p' = 1$ for $1 < p < \infty$, then $B^1 \circledast B^2$ is just the space $A_p(G)$ introduced in [42] (where $A_2(G) = A(G)$!). For that particular example different methods (cf. [28]) may be used to show that $A_p(G)$ (suitable defined for general G) is always a Banach algebra—the so-called Herz algebra—satisfying the general assumptions, having bounded approximate units if G is amenable (cf. [28]). The above arguments show, however, that at least in the abelian setting the two spaces B^1 and B^2 need not be in duality in order to have the standard situation. For results in that direction for general l.c. groups cf. [18].

Since it is possible to identify in most concrete cases $(B^1 \circledast B^2)'$ with the set (of kernels) of multipliers from B^1 into $(B^2)'$, i.e., $(B^1 \circledast B^2)' \cong H_G(B^1, (B^2)')$, the standard situation also applies to the Banach space of those quasimeasures that define multipliers from B^1 into B^2 and can be approximated by "elementary" operators if the form T_f: $g \mapsto f * g$, for some $f \in S_0(G)$ (cf. [91, 14, 34]). Details will be given elsewhere.

As will be shown in Section 3 this space coincides with the space of distributions inducing (via convolution) compact multipliers, if G is compact.

We conclude this section with a short discussion of the standard situation for spaces of smooth functions or distributions on $\mathbb{R}^n$, such as Sobolev spaces $L^p_k(\mathbb{R}^m)$, $1 \leqslant p < \infty$, $k \in \mathbb{N}$, the (Bessel-) potential spaces $\mathscr{L}^p_\alpha(\mathbb{R}^m)$, $1 \leqslant p < \infty$, $\alpha \in \mathbb{R}$, or the Besov–Lipschitz spaces $\Lambda^{p,q}_\alpha(\mathbb{R}^m) = B^\alpha_{p,q}(\mathbb{R}^m)$, $1 \leqslant p$, $q \leqslant \infty$, $\alpha \in \mathbb{R}$. For details concerning these spaces cf. [101, Chapter V; 75, 10; or 68]. Since all these spaces are members of two scales of Banach spaces of tempered distributions considered in detail by H. Tribel (cf. [104, 106]) we prefer to state the corresponding result in this more general frame.

PROPOSITION 1.8. *The Banach spaces $B^s_{p,q}(\mathbb{R}^m)$ and $F^s_{p,q}(\mathbb{R}^m)$, $s \in \mathbb{R}$, $1 \leqslant p$, $q < \infty$, are in standard situation.*

Proof. According to Theorem 2.3.3 of [106] the Schwartz space $\mathscr{S}(\mathbb{R}^m)$ is continuously and densely embedded in the Banach spaces under consideration. Since $y \to L_y f$ is continuous from $\mathbb{R}^m$ to $\mathscr{S}(\mathbb{R}^m)$ for $f \in \mathscr{S}(\mathbb{R}^m)$ these spaces are homogeneous Banach spaces of distributions, hence $L^1(\mathbb{R}^m)$-convolution modules. Furthermore, they are Banach modules (with respect to pointwise multiplication) over the Banach algebra $\mathscr{C}^\rho(\mathbb{R}^m)$ (Zygmund-space) for $\rho = \rho(s, p, q)$ sufficiently large (see [106, Theorem 2.6.1]). If one defines A to be the closure of $\mathscr{K}(\mathbb{R}^m) \cap \mathscr{C}^\rho(\mathbb{R}^m)$ in $\mathscr{C}^\rho(\mathbb{R}^m)$ for some (perhaps larger) ρ one has also $B \hookrightarrow A_0'$, i.e., the standard assumptions are satisfied since A has always bounded approximate units (sufficiently flat and smooth trapezoid functions).

The following lemma gives sufficient conditions for similar, perhaps more general Banach spaces of distributions on $\mathbb{R}^m$. By means of suitable modification it can be extended to more general situations.

LEMMA 1.9. *Let* $(B, \| \|_B)$ *be an isometrically translation invariant Banach space of distributions on* $\mathbb{R}^m$, *containing* $\mathscr{D}(\mathbb{R}^m)$ *as a dense subspace. Assume further that* $\mathscr{D} \cdot B \subseteq B$ *(i.e., that* $\mathscr{D}$ *operates on* B *by pointwise multiplication), and that there exists a family* $(\tau_y)_{y \in J}$ *of trapezoid functions in* $\mathscr{D}$ *of bounded action on* B. *Then* B *is in standard situation (for a suitable Banach algebra* $(A, \| \|_A)$*), and* A_0 *is a dense subspace in* B.

Proof. Set

$$M_0(B) := \{h \mid h \in C^0(G), \|h\|_M := \sup_{f \in \circ B} \|hf\|_B + \sup_{f \in \circ B} \|\bar{h}f\|_B < \infty\}.$$

This space is of course a Banach algebra with respect to pointwise multiplication, under the norm $\|h\|_M + \|h\|_\infty$. We let $(A, \| \|_A)$ be the closure of $\mathscr{D}(\mathbb{R}^m)$ in $M_0(B) \cap B'$ (since B contains $\mathscr{D}$ as a dense subspace B' is a space of distributions!) with its natural norm. Then one can check that $(A, \| \|_A)$ is a Banach algebra satisfying the general assumptions, the continuity of translation following from the isometric translation invariance of $(A, \| \|_A)$ and the continuity of the translation in $\mathscr{D}$. That $(B, \| \|_B)$ is in standard situation follws from the construction of $(A, \| \|_A)$ and the assumptions made.

As a last example let us mention $C_{\text{abs}}(\mathbb{R}) =: C^0(\mathbb{R}) \cap \text{Abs}(\mathbb{R})$, the space of all C^0-functions that are absolutely continuous. It is another Banach algebra with bounded approximate units satisfying the general assumptions, with the norm $\| \| := \| \|_\infty + \| \|_{BV}$ (where $\| \|_{BV}$ denotes the variation norm on $BV(\mathbb{R})$, cf. [83, 82]).

Remark 1.11. It is easy to check that any Banach algebra $(A, \| \|_A)$ satisfying the general assumptions and having approximate units that are bounded in the operator norm is in "standard situation" (over itself).

We conclude this section with a Lemma that has been used already above (Lemma 1.5(a)) and that will be of importance in the proof of the main result (Theorem 2.1) for general l.c. G.

LEMMA 1.10. (a) *The mapping* $h \mapsto \Delta^{-1} h$ *defines a bounded linear operator on* A_0.

(b) *There is a mapping* $z \mapsto N_z$ *from* G *into* $H(A_0, A_0)$ *which is strongly continuous, and such that* $N_x k(y) = R_y k(x)$, *and*

$$k * \sigma(x) = \langle N_x k^{\vee}, \sigma \rangle \text{ for all } x \in G,\ k_0 \in A_0^{\vee}, \sigma \in A_0'.$$

In particular, $k^{\vee} * \sigma \in C(G)$ *in this case.*

Proof. (a) (cf. [81]). For $h \in A_0$ and $g \in A_0$ with $\int_G g(y)\,dy = 1$ the mapping $y \mapsto (L_{y^{-1}} g)h$ is continuous from G into $(A, \|\ \|_A)$, with compact support. Hence $w := \int_G (L_{y^{-1}} g)\, h\, dy \in A$. But $w(x) = \int_G g(yx)\, h(x)\, dy$ $=$ $\int_G g(y)\, \Delta^{-1}(x)\, h(x)\, dy = \Delta^{-1}(x)\, h(x)$, hence $\Delta^{-1} h \in A \cap \mathscr{K}(G) = A_0$.

(b) Set $N_z(k) := \Delta(z)\, \Delta^{-1} R_z k$ for $k \in A_0$, then $z \mapsto N_z(k)$ is continuous from G into A_0, and N_z is norm bounded over compact sets of z, as operator on each A_K (K compact). Direct computation shows that one has

$$\langle \check{k}_1 * g, h \rangle = \int_G \langle N_z k_1, g \rangle\, a_0(z)\, dz \qquad \text{for} \quad k_1, g, a_0 \in A_0,$$

which implies the above formula for $\sigma \in A_0'$ by an approximation argument.

The formula can also be shown to hold true whenever $(B. \|\ \|_B)$ is a Banach space with continuous left translation continuously embedded in A_0', and if the convolution is interpreted via vector-valued integrals. More detailed explanations are to be given in [15].

Convention. Throughout this paper expressions of the form $g * hf$ or $hg * f$ should be read as $g * (hf)$ or $(hg) * f$, i.e., pointwise multiplication being carried out before convolution!

2. Compactness Conditions

This section contains the main result of this paper. We start with the statement of sufficient conditions for a closed subset of a Banach space B in standard situation to be compact.

THEOREM 2.1 (Sufficiency). *Let M be a closed subset of a Banach space $(B, \|\ \|_B)$ in standard situation. Suppose that M is bounded, equicontinuous and tight (uniformly concentrated) in the following sense:*

(a) $\sup_{f \in M} \|f\|_B := C < \infty$;

(b) $\forall \varepsilon > 0\ \exists k \in \mathscr{K}(G)$ *such that* $\|k * f - f\|_B < \varepsilon$ *for all* $f \in M$;

(c) $\forall \varepsilon > 0\ \exists h \in \mathscr{K}(G)$ *such that* $\|hf - f\|_B < \varepsilon$ *for all* $f \in M$.

Then M is compact in B.

Proof. (i) First of all, we claim that M is relatively compact in A_0' with respect to the $\sigma(A_0', A_0)$-topology. In view of the Alaoglu–Bourbaki theorem (cf. [94, III.4.3]) it will be sufficient to show that M is equicontinuous in the following sense: Given $k_0 \in A_0$ and $\varepsilon > 0$ a neighborhood U of k_0 in A_0 can be found such that $|\langle k - k_0, f \rangle| < \varepsilon$ for all $k \in U$ and all $f \in M$. Actually, a suitable set U can be obtained by first choosing any compact set $K \subseteq G$ containing supp k_0, and defining U by

$$U := \{k \mid k \in A, \operatorname{supp} k \subseteq K, \|k - k_0\|_A < \varepsilon (C_K' \cdot C)^{-1}\}$$

(cf. Remark 1.4). The required estimate now follows from (1.1).

Consequently, any net in M contains a weakly convergent subnet $(f_\alpha)_{\alpha \in I}$ in A_0', i.e., there exists $\sigma \in A_0'$ such that

$$\lim_{\alpha \to \infty} \langle f_\alpha, k \rangle = \langle \sigma, k \rangle \qquad \text{for all} \quad k \in A_0.$$

It will therefore be sufficient to show that $\sigma(A_0', A_0)$-convergence of such a net in M implies that it is a Cauchy net with respect to the norm $\| \; \|_B$.

(ii) Before we show the above assertion we note that one may suppose without loss of generality that the functions k and h used in (b) and (c) satisfy $k^\vee \in A_0$ and $h \in A_0$, respectively. In fact, the density of A_0 (hence $A_0^\vee$) in $\mathscr{K}(G)$ implies that there exists for $k \in \mathscr{K}(G)$ some $k_1 \in A_0^\vee$ such that $\|k_1 - k\|_{1,w} < \varepsilon / C$. Then: $\|k_1 * f - f\|_B \leqslant \|k_1 - k\|_{1,w} \|f\|_B + \|k * f - f\|_B < 2\varepsilon$ for all $f \in M$. In order to replace $h \in \mathscr{K}(G)$ by $h_1 \in A_0$ we recall that the conditions made concerning the standard situation imply the existence of $C_B > 0$, such that for each $h \in \mathscr{K}(G)$ some $h_1 \in A_0$ can be found, satisfying $\|h_1 f\|_B \leqslant C_B \|f\|_B$ for $f \in B$ and $hh_1 = h$. If $h \in \mathscr{K}(G)$ is chosen such that $\|hf - f\|_B < \varepsilon C_B^{-1}$ for $f \in M$ the above choice of h_1 implies

$$\begin{aligned}\|h_1 f - f\|_B &\leqslant \|h_1 f - h_1 hf\|_B + \|hf - f\|_B \\ &\leqslant C_B \|f - hf\|_B + \varepsilon \leqslant 2\varepsilon \qquad \text{for} \quad f \in M.\end{aligned}$$

(iii) Let now $\varepsilon > 0$ be given, and without loss of generality suppose $\varepsilon \leqslant C$. Combining (b) and (c) (with the modifications of (ii)) we can choose $k \in A_0^\vee$ such that

$$\|k * f - f\|_B < \varepsilon \qquad \text{for} \quad f \in M,$$

and $h \in A_0$, such that

$$\|hf - f\|_B < \varepsilon \min(1, \|k\|_{1,w}^{-1}) \qquad \text{for} \quad f \in M.$$

Combining these two inequalities one obtains

$$\|f - k * hf\|_B < 2\varepsilon \qquad \text{for all} \quad f \in M.$$

Writing now $K_1 := \operatorname{supp} k$ and $K_2 := K_1(\operatorname{supp} h)$ we observe that the family $\{k * hf \mid f \in M\}$ has common compact contained in K_2. Moreover, it is equicontinuous in $\mathscr{K}(G) \subseteq C^0(G)$. Since for $k \in A_0^\vee$ $z \mapsto N_z k^\vee$ is equicontinuous from G into A over the compact set K_2 (cf. Lemma 1.10), which implies for $x_1 y \in K_2$ and $f \in M$:

$$|k * hf(x) - k * hf(y)| = |\langle N_x k^{\check{}} - N_y k^{\check{}}, hf \rangle|$$
$$\leqslant C_{K_2 K_1} \|N_x k^{\check{}} - N_y k^{\check{}}\|_A \sup_{f \in M} \|hf\|_B .$$

Next, using the existence of left approximate units for $L^1_w(G)$ in the (dense!) subspace A_0 one can find $k' \in A_0 \subseteq B$ such that

$$\|k' * k - k\|_{1,w} < \varepsilon (2C)^{-1}.$$

It follows

$$\|f - k' * k * hf\|_B < \|f - k * hf\|_B + \|k - k' * k\|_{1,w} \sup_{f \in M} \|hf\|_B$$
$$\leqslant 2\varepsilon + \varepsilon (2C)^{-1} (C + \varepsilon) \leqslant 3\varepsilon \qquad \text{for all} \quad f \in M.$$

(iv) Returning to (i) let a $\sigma - (A_0', A_0)$-convergent net $(f_\alpha)_{\alpha \in I}$ in M be given. The identity

$$k * hf_\alpha(x) = \langle (N_x k^{\check{}})\, h, f_\alpha \rangle$$

(together with the assumptions $k^{\check{}} \in A_0$, $h \in A_0$!) then implies pointwise convergence of $k * hf_\alpha$, hence uniform convergence: (equicontinuity + compact support!). Hence for some α_0,

$$\|k * hf_\alpha - k * hf_\beta\|_\infty$$
$$\leqslant \varepsilon (C_{K_3} \|c_{K_2}\|_{1,w} \|k'\|_A)^{-1} \qquad \text{for} \quad \alpha, \beta \geqslant \alpha_0, f \in M$$

where $K_3 := (\operatorname{supp} k')(\operatorname{supp} k)(\operatorname{supp} h)$ (hence $\operatorname{supp}(k' * k * hf) \subseteq K_3$ for all $f \in M$). Since $k * hf \in C^b_{K^3} \subseteq L^1_w$ the right L^1_w-convolution structure on A and the continuous embedding of A_0 in B imply

$$\|k' * k * hf_\alpha - k' * k * hf_\beta\|_B \leqslant C_{K_3} \|\cdots\|_A$$
$$\leqslant C_{K_3} \|k'\|_A \|k * (hf_\alpha - hf_\beta)\|_{1,w} < \varepsilon \qquad \text{for } \alpha, \beta \geqslant \alpha_0 \text{ and all } f \in M.$$

Together with the last inequality in (iii) this implies

$$\|f_\alpha - f_\beta\|_B \leqslant 7\varepsilon \qquad \text{for} \quad \alpha, \beta \geqslant \alpha_0,$$

and the proof is complete.

Remark 2.1. The above proof (steps (iii) and (iv)) shows that any $\sigma(A_0', A_0)$-convergent net $(f_\alpha)_{\alpha \in I}$ in B that is bounded, equicontinuous and tight in B (i.e., satisfies (a), (b) and (c)) is already convergent with respect to the norm of B.

Remark 2.2. In the above proof the pointwise A-module structure of B and the existence of trapezoid functions of bounded action on B in A_0 has only be used in step (ii). If one assumes instead of (c) to have for any $\varepsilon > 0$ some $h \in A_0(!)$ such that $\|hf - f\|_B < \varepsilon$ for all $f \in M$ the assertion of the Theorem still remains true. In particular, Theorem 2.1 applies to Banach algebras satisfying the general assumptions, even if they do not have (perhaps not even operator) norm bounded approximate units, such as $A_p(G)$, $1 < p < \infty$, for a non-amenable group G.

Remark 2.3. The right $L^1_w(G)$-module structure of A_0 has only been used in step (iv). If one checks the proof it is clear that it is sufficient to find a dense subset D of $\mathscr{K}(G)$ such that $T_d: f \to d * f$ defines a continuous linear oeprator from $\mathscr{K}(G)$ into B for any $d \in D$ (k' above has to be chosen simply in D in that case). It is trivial that such a choice is possible if—for example—B instead of A has a right $L^1_w(G)$-module structure and if $\mathscr{K}(G) \cap B$ is dense in $\mathscr{K}(G)$ (set $D := \mathscr{K}(G) \cap B!$).

It turns out that for most concrete situations the sufficient conditions are necessary as well. The precise conditions are given in the following theorem that may be considered a general version of the Kolmogorov–Riesz–Weil compactness criterion for $L^p(G)$-spaces. We have preferred to use the more symmetric assumptions in the theorem, because the additional conditions are satisfied in most concrete cases of interest.

THEOREM 2.2 (Compactness Criterion). *Let $(B, \|\ \|_B)$ be a Banach space of distributions on G in standard situation. Suppose in addition that A_0 is a dense subspace of B. Then a closed subset $M \subseteq B$ is compact in B if and only if conditions* (a), (b) *and* (c) *of Theorem* 2.1 *are satisfied.*

Proof. That the assumptions are sufficient has been proved above. In order to prove necessity of (b) and (c) ((a) is obvious) let a compact set $M \subseteq B$, and $\varepsilon > 0$ be given. Since A_0 is dense in B it is possible to choose a finite sequence $(f_i)_{i=1}^n$ in A_0 such that for any $f \in M$ there exists i, $1 \leqslant i \leqslant n$, satisfying

$$\|f - f_i\|_B < \varepsilon/2 \cdot \max(1 + C_w, 1 + C_B).$$

Then it is possible to find k and h in A_0, satisfying $\|k\|_{1,w} \leqslant C_w$, $\|hf\|_B \leqslant C_B \|f\|_B$ for all $f \in B$, $\|h * f_i - f_i\| < \varepsilon/2$ and $hf_i = f_i$ for $1 \leqslant i \leqslant n$ (cf. the proof of Lemma 1.5(b)) (h has to be chosen out of a family of trapezoid functions in A_0 of bounded action on B). Combining these estimates implies (b) and (c): e.g.,

$$\|k*f-f\|_B \leqslant \|k*f-k*f_i\|_B + \|k*f_i-f_i\|_B + \|f_i-f\|_B$$
$$\leqslant (\|k\|_{1,w}+1)\,\|f-f_i\|_B + \varepsilon/2 < \varepsilon \qquad \text{for all} \quad f \in M.$$

In view of the relevance of conditions (b) and (c) for the above criterion let us discuss now various equivalent characterizations of these conditions.

PROPOSITION 2.3. *Let $(B, \|\ \|_B)$ be a left invariant Banach space of distributions, and let M be a bounded subset in B. Then the following conditions are equivalent:*

(b) $\forall\, \varepsilon > 0\ \exists\, k \in \mathscr{K}(G)$ *such that* $\|k*f-f\|_B < \varepsilon$ *for all* $f \in M$;

(b_1) $\forall\, \varepsilon > 0\ \exists\, k_1 \in L^1_w(G)$ *such that* $\|k_1*f-f\|_B < \varepsilon$ *for all* $f \in M$;

(b_2) *For every bounded approximate unit* $(u_\gamma)_{\gamma\in I}$ *in* $L^1_w(G)$ *one has* $\lim_{\gamma\to\infty} \|u_\gamma * f - f\|_B = 0$, *uniformly for* $f \in M$;

(b_3) *There exists a bounded subset* $M' \subseteq B$ *and* $k_0 \in L^1_w(G)$ $(\|k_0\|_{1,w} \leqslant C_w)$ *such that* $M \subseteq k_0 * M'$;

(b_4) $\forall\, \varepsilon > 0\ \ \exists\, U$, *neighborhood of the identity, such that* $\|L_y f - f\|_B < \varepsilon$ *for all* $y \in U$ *and for all* $f \in M$.

Proof. One considers B as a Banach module over $L^1_w(G)$. The equivalence $(b_2) \Leftrightarrow (b_1) \Leftrightarrow (b)$ is well known (cf. Proposition 1.1, or [22, Proposition 1.2]). That (b_3) implies (b) is shown as usual by vector-value integration (cf. Proposition 1.1). That (b_2) implies (b_3) can be shown using a variant of the Cohen–Hewitt factorization theorem (see [96, Theoem 2.1]; cf. [22, Theorem 17.1]). That (b_3) implies (b_4) follows from the estimate

$$\|L_y(g*f) - g*f\|_B \leqslant \|L_y\, g - g\|_{1,w}\, \|f\|_B$$

and continuity of translation in $L^1_w(G)$ (of course $(b_1) \Rightarrow (b_4)$ can easily be shown directly).

The following Proposition describes conditions equivalent to (c). For simplicity we assume that $(A, \|\ \|_A)$ has bounded approximate units.

PROPOSITION 2.4. *Let M be a bounded subset of a Banach space $(B, \|\ \|_B)$ in standard situation. Then the following conditions are equivalent, whenever A has bounded approximate units:*

(c) $\forall\, \varepsilon > 0\ \exists\, h \in \mathscr{K}(G)$ *such that* $\|hf-f\|_B < \varepsilon$ *for all* $f \in M$;

(c_1) $\forall\, \varepsilon > 0\ \exists\, h^1 \in A_0$ *such that* $\|h^1 f - f\|_B \leqslant \varepsilon$ *for all* $f \in M$;

(c_2) *For every bounded approximate unit* $(\tau_\alpha)_{\alpha\in I}$ *in* A *one has* $\lim_{\alpha\to\infty} \|u_\alpha f - f\|_B = 0$, *uniformly for* $f \in M$;

(c_3) *There exists a bounded subset* $M'' \subseteq B$ *and some* $h_0 \in A$ *such that* $M \subseteq h_0 M''$;

(c_4) $\forall \varepsilon > 0\ \exists K \subseteq G$, *K compact, such that for any* $f \in M$ *there exists* $f_1 \in B$, $\operatorname{supp} f \subseteq K$ *and* $\|f - f_1\|_B < \varepsilon$.

Proof. The equivalence of (c)–(c_3) is the same as the equivalence of (b)–(b_3) from a Banach module theoretic point of view. The implication (c_4) $\Rightarrow$ (c) is trivial ($K := \operatorname{supp} k$), and the converse is shown using any bounded family of trapezoid functions in A_0.

For abelian groups the equicontinuity of M can be described via "almost" compactness of the spectrum (cf. [76]). We have the following:

LEMMA 2.5. *Let G be a l.c.a. group, and let B be a strongly translation invariant Banach space of quasimeasures (i.e.,* $B \hookrightarrow Q(G)$*) that is a* $L^1_w(G)$*-module for a weight function w satisfying the Beurling–Domar condition (BD; see* [21]; *cf.* [89, VI, *Section* 3.1]). *Then a bounded subset* $M \subseteq B$ *is equicontinuous, i.e., satisfies* (b)–(b_4) *if and only* (b_5) *is satisfied*:

(b_5) $\forall \varepsilon > 0\ \exists K^1 \subseteq \hat{G}$, K^1 *compact, such that for any* $f \in M$ *there exists* $f_1 \in B$ *such that* $\operatorname{supp} \hat{f}_1 \subseteq K^1$ *and* $\|f - f_1\|_B < \varepsilon$.

Proof. (Cf. [31] for a special case.) The assumptions imply $B \hookrightarrow S_0'(G)$ (cf. [34] for the definition of $S_0'(G)$), and therefore $\mathscr{F}B$ is well defined as a Banach space of quasimeasures on $\hat{G}$. To the $L^1_w(G)$-convolution structure on B there corresponds a pointwise A_w-structure on $\mathscr{F}B$. Condition (BD) on w implies the regularity of $A_w(\hat{G}) = \mathscr{F}(L^1_w)$; the symmetry of w and the existance of bounded approximate units in $A_w(\hat{G})$ imply that $A_w(\hat{G})$ satisfies the standard assumptions. A combination of Propositions 2.4 and 2.5 then gives the result.

Remark 2.4. The assumptions are of course satisfied if B is a homogeneous Banach space of quasimeasures, because then one may choose $w(x) \equiv 1$.

We conclude this section with a result concerning products of sets. It contains a result of Georgakis as a special case (cf. [47]).

PROPOSITION 2.6. *Let* $(B^i, \|\ \|_{B^i})$, $i = 1, 2, 3$, *be a triple of Banach spaces in standard situation and suppose that they form a multiplication triple, i.e., that*

$$\|f^1 f^2\|_{B^3} \leqslant C\, \|f^1\|_{B^1} \|f^2\|_{B^2} \qquad \textit{for all} \quad f^i \in B^i, i = 1, 2.$$

Assume that A_0 *is dense in* B^2 *and that translation is isometric in* $(B^2, \|\ \|_{B^2})$. *Then the following holds true: Given a relatively compact set* $M^2 \subseteq B^2$, *and a bounded, equicontinuous set* $M^1 \subseteq B^1$, *the complex product* $M^1 M^2$ *is a relatively compact set in* B^3.

Proof. Suppose that one has $\|f^i\|_{B^i} \leqslant C^i$ for all $f^i \in M^i$, $i = 1, 2$. It is then clear that M^1M^2 is bounded in B^3 by CC^1C^2. Since M^2 is equicontinuity of M^1, M^2 follows from

$$\begin{aligned}
&\|L_y(f^1f^2) - f^1f^2\|_{B^3} \\
&\quad \leqslant C\,\|L_yf^1 - f^1\|_{B^1}\,\|L_yf^2\|_{B^2} \\
&\qquad + C\,\|f^1\|_{B^1}\,\|L_yf^2 - f^2\|_{B^2} \qquad \text{for all} \quad f^i \in M^i, i = 1, 2.
\end{aligned}$$

Since M^2 has to be tight in B^2 as well, there exists for $\varepsilon > 0$ $h \in A_0$ such that $\|f^2 - hf^2\|_{B^2} < \varepsilon(CC^1)^{-1}$ for $f^2 \in M^2$, hence

$$\|f^1f^2 - h(f^1f^2)\|_{B^3} < \varepsilon \qquad \text{for all} \quad f^i \in M^i,\ i = 1, 2.$$

COROLLARY 2.7. *Let $(B, \|\ \|_B)$ be a solid, isometrically translation invariant BF-space on a locally compact abelian group, containing $\mathscr{K}(G)$ as a dense subspace. Let $f \in B^\alpha$ (the Köthe dual) and $g \in B$ be given. Then the Fourier transforms of $\{(L_yf)\,g\}_{y \in G}$ tend to zero at infinity, uniformly in $y \in G$, i.e., there exists $h \in C^0(\hat{G})$ such that $|\mathscr{F}[(L_yf)\,g](t)| \leqslant h(t)$ for all $t \in \hat{G}$.*

Proof. By the definition of B^α one has $\|fg\|_{L^1} \leqslant \|f\|_{B^\alpha}\,\|g\|_B$ for $f \in B^\alpha$, $g \in B$. The above theorem is therefore applicable, with $B = B^2$, $M^1 = \{L_yf \mid y \in G\}$, $M^2 = \{g\}$, and shows that $\{\mathscr{F}[(L_yf)g]\}_{y \in G}$ is compact in the Fourier algebra on $\hat{G}$, which implies the assertion.

A corresponding result can be proved for convolution triples of Banach spaces in standard situation (as considered in [106, 30, 37]). The basic example of such a triple is $(L^p(G), L^q(G), L^r(G))$, for $1/p + 1/q - 1 =: 1/r \geqslant 0$ (Young's inequality).

PROPOSITION 2.8. *Let $(B^i, \|\ \|_{B^i})$, $i = 1, 2, 3$, be a triple of Banach spaces in standard situation. Assume further that A_0 is dense in B^2. Let M^1 be a bounded tight subset of B^1, and let M^2 be a relatively compact subset of B^2. Then $M^1 * M^2$ is a relatively compact subset of $(B^3, \|\ \|_{B^3})$.*

Proof. It is clear that $M^1 * M^2$ is an equicontinuous, bounded subset of B^3. The tightness of $M^1 * M^2$ (cf. [31, Lemma 2.1]) follows from the fact that one has for $f^i \in B^i$, $k^i \in A_0^i$, some $\tau \in A_0$ satisfying $\tau(x) \equiv 1$ on $(\operatorname{supp} k^1) \cdot (\operatorname{supp} k^2)$, such that the following estimate holds:

$$\begin{aligned}
&\|(1 - \tau)(f^1 * f^2)\|_{B^3} \\
&\quad = \|(1 - \tau)(f^1 * f^2 - f^1k^1 * f^2k^2)\|_{B^3} \\
&\quad \leqslant (1 + C_{B^3})[\|f^1 * (1 - k^2)f^2\|_{B^3} + \|f(1 - k^1)f^1 * k^2f^2\|_{B^3}] \\
&\quad \leqslant (1 + C_{B^3})[\|f^1\|_{B^1}\,\|(1 - k^2)f^2\|_{B^2} + \|(1 - k^1)f^1\|_{B^1}\,C_A\,\|f^2\|_{B^2}].
\end{aligned}$$

As a typical example of a more general consequence of the above result let us give a Corollary. We shall call a multiplier between two homogeneous Banach spaces of distributions (over a Banach algebra A having bounded approximate units) on a l.c.a. group G an *elementary* one, if there exists a sequence

$$(k_n)_{n\geqslant 1} \text{ in } \mathscr{K}(G) \text{ (or } A_0) \text{ such that } T = \lim_{n\to\infty} T_{k_n}$$

in the operator norm $(T_{k_n}(f) := k_n * f)$.

COROLLARY 2.9. *Any elementary multiplier maps bounded, tight subsets of B^1 into compact subsets of B^2.*

Proof. Using approximation arguments one shows that T does not only preserve boundedness, but also tightness. Since T_{k_n} apparently maps bounded sets into equicontinuous sets the same holds true for T, and the proof is complete.

Another consequence is the following characterization of compact sets via factorization:

COROLLARY 2.10. *Let $(B, \| \|_B)$ be a Banach space in standard situation, such that A_0 is dense in B, and such that $(A, \| \|_A)$ has bounded approximate units, then one has:*

A closed, bounded subset $M \subseteq B$ is compact if and only if there exists a bounded subset $M^1 \subseteq B$, and $k \in L^1_w(G)$ and $h \in A$ such that $M \subseteq h(k * M^1)$. Equivalently, one may choose M^1, h, k such that one has $M \subseteq k * (hM^1)$. M^1 may even be chosen to be compact in B as well.

Proof. That sets of the form $h(k * M^1)$ or $k * (hM^1)$ are relatively compact follows from a combination of Propositions 2.6 and 2.8 and Theorem 2.1. The converse follows from Theorem 2.2, and the factorizations that can be derived from tightness and equicontinuity as stated in Propositions 2.3 and 2.4., using the well known fact that it is possible to factorize compact sets through other compact sets (cf. [22, Section 17], [96]).

Remark 2.5. A more detailed inspection of the proof of Corollary 2.9, in particular of the applications of the Cohen–Hewitt factorization theorem, shows that any compact set in B can be approximated as close as one wants (in the norm of B) by subsets that are compact in $\mathscr{K}(G)$ or even in A_0 as well.

We conclude this section with a summary of a characterization of compact sets in Segal algebras, because the corresponding theorem is slightly more general in the particular situation (only tightness in the bigger space $L^1(G)$ is required). We present only one typical version. This is a special case of the following result:

THEOREM 2.11 [31, Theorems 2.3 and 2.4]. *Let $(S, \|\ \|_S)$ be a symmetric or pseudosymmetric Segal algebra on G. Then a bounded closed subset is compact in $(S, \|\ \|_S)$ if and only if it satisfies:*

(b_r) $\forall \varepsilon > 0\ \exists v$, *v neighborhood of identity, such that*

$$\|R_y f - f\|_S < \varepsilon \qquad \textit{for all} \quad f \in M;$$

(c′) $\forall \varepsilon > 0\ \exists h \in \mathscr{K}(G)$ *such that* $\|hf - f\|_1 < \varepsilon$ *for all* $f \in M$.

THEOREM 2.12. *Let B, B^1 be two Banach spaces in standard situation with respect to A and A^1, respectively, such that $B \subseteq B^1$. Assume in addition that B is right invariant and that there exists a two-sided approximate unit $(u_\alpha)_{\alpha \in I}$ for $L^1_w(G)$ in $\mathscr{K}(G)$, such that $B^1 * u_\alpha \subseteq B$ for all $\alpha \in I$ (e.g., that there is a dense subspace D of $\mathscr{K}(G)$ such that $B^1 * D \subseteq B$). Then a bounded subset M of B is relatively compact in B whenever it is left equicontinuous in B and right equicontinuous and tight in B^1 (only).*

Proof. The right equicontinuity of $M \subseteq B^1$ and the right L^1_w-module structure on B^1 imply that for $\varepsilon > 0$ there exists $k \in \mathscr{K}(G)$ such that $\|f * k - f\|_{B^1} < \varepsilon$ for all $f \in M$. Choosing now α_0 appropriately one has $\|k * u_\alpha - k\|_{1,w} < \varepsilon / \sup_{f \in M} \|f\|_B$ for $\alpha \geqslant \alpha_0$. It will therefore be sufficient to show that any set $M' = M * k * u_\alpha$ is relatively compact in B^1. Since, by the closed graph theorem, each u_α defines a bounded convolution operator from B^1 into B boundedness of M' is clear. Left equicontinuity in B follows from the fact that right convolution commutes with left translation. Tightness is shown as in the proof of Proposition 2.8, with $B^3 = B$, $M^1 = M * k$, $M^2 = \{u_\alpha\}$, $u_\alpha = f^2$(α fix!). Choose $k^2 \in A^2_0$ such that $k^2 f^2 = k^2 u_\alpha = u_\alpha$, and $k^1 \in A^1_0$ such that $\|(1 - k^1)\ (f * k)\|_{B^1} \leqslant (1 + C_B)\, \|u_\alpha\|_{B^1 \to B} \cdot \varepsilon$ for all $f \in M$. This is possible by the tightness of $M * k \subseteq B^1 * L^1_w \subseteq B^1$ (cf. Proposition 2.8). Choosing $\tau_\alpha \in A_0$ such that $\tau_\alpha(x) \equiv 1$ on $(\operatorname{supp} k^1)(\operatorname{supp} k^2)$ and satisfying $\|\tau_\alpha\, g\|_B \leqslant C_B\, \|g\|_B$ for all $g \in B$ one obtains (cf. proof of 2.8):

$$\begin{aligned}\|(1 - \tau_\alpha)(f * k) * u_\alpha\| &\leqslant (1 + C_B)\, \|(1 - k^1)(f * k) * k^2 u_\alpha\|_B \\ &\leqslant (1 + C_B)\, \|(1 - k^1)(f * k)\|_{B^1}\, \|u_\alpha\|_{B^1 \to B} < \varepsilon.\end{aligned}$$

A typical example for the situation arises if B is some kind of generalized Lipschitz space or portential space derived from B^1, e.g., $B = g_0 * B^1$ for a l.c.a. group, with $g_0 \in L^1(G)$ satisfying $\hat{g}_0(t) \neq 0$ for all $t \in \hat{G}$, such as the classical Bessel potentials $\mathscr{L}_s^p = G_s * L^p$ (cf. [101]). By the above argument only tightness in L^p is required (tightness in $\mathscr{L}_s^p$, for example, is indeed a consequence; cf. the above proof!).

3. Compact Multipliers

As an application of the results in Section 2 we are able to present a characterization of compact (right) multipliers between certain pairs of translation invariant Banach spaces as considered above. In spite of various results in that direction that hold for compact groups we have not been able to find such results for non-compact groups in the literature. This probably stems from the fact that in the case of a non-compact group there do not exist compact multipliers between two isometrically translation invariant spaces as considered usually (e.g., L^p-spaces, Segal algebras, homogeneous Banach spaces; cf., e.g., [66, Proposition 2.2]), simply because the set of translates of a single element is always bounded in such a space, but never compact, because it cannot be tight (cf. Theorem 2.2). Exceptions lead to spaces of strongly almost periodic functions (cf. [39]). The situation changes drastically if the domain is a Banach space that is small enough, e.g., a Beurling algebra with a weight tending to infinity. Although the space of multipliers from that Beurling algebra into a homogeneous Banach space coincides with that from $L^1(G)$ into that space (cf. Theorem 3.5 below) the subspace of all compact multipliers is far from being trivial, and can in fact be characterized as the closure of the space of "elementary" multipliers in the norm topology. As will be explained in Section 4 the situation is completely similar to that on compact groups.

Since there are different situations where it is possible to give (the same) characterizations of compact multipliers we have tried to emphasize their common aspects. This has made the presentation somewhat technical (and for concrete examples it is likely that parts of the proofs could be shortened), but we hope that the reader will not be confused by that fact. Before we state the main result of this section we prove two of the main steps in its proof separately.

Proposition 3.1. *Let G be a $[SIN]$-group and let $(B^i, \| \|_{B^i})$ be three Banach spaces in standard situation such that A_0 is dense in B^1. Suppose the following condition* (A) *is satisfied:*

$$\text{(A)} \qquad \begin{array}{l} {}_\circ B^1 = \{f \mid f \in B^1, \|f\|_{B^1} \leqslant 1\} \textit{ is a tight subset of } B^3 \\ \textit{and for any } k \in A_0,\ T_k : f \to f * k \textit{ maps } B^3 \textit{ into } B^2. \end{array}$$

Then T_k defines a compact multiplier from B^1 to B^2 for any $k \in A_0$.

Proof. Before we come to the relevant estimates we have to fix several constants. Here we assume $k_0 \in A_0$ to be fixed.

First of all the closed graph theorem implies the existence of a constant $C_1 > 0$ such that

$$\|f\|_{B^3} \leqslant C_1 \|f\|_{B^1} \qquad \text{for all} \quad f \in B^1, \tag{3.1}$$

and the continuity of the operators $T_k: B^3 \to B^2$, for any $k \in A_0$. Therefore $k \mapsto T_k$ is well defined as a linear mapping from $A_K = \{k \mid k \in A, \operatorname{supp} k \subseteq A\}$ to $H(B^1, B^2)$ for any compact subset $K \subseteq G$. This, in turn, implies the existence of constants $\alpha_K > 0$ such that

$$\|f * k\|_{B^2} \leqslant \alpha_K \|f\|_{B^3} \|k\|_A \qquad \text{for all} \quad k \in A_K, f \in B^3. \tag{3.2}$$

For later use we fix besides $k_0 \in A_0$ some relatively compact neighborhood U_0 of the identity, and write α_0 for α_{K_p}, where K_0 is an arbitrary compact set satisfying $K_0 \supseteq U_0(\operatorname{supp} k_0)$.

We observe further that the mapping $y \mapsto \|L_y\|_{B^2}$ is semicontinuous and submultiplicative, hence locally bounded on G. We write

$$C_2 := \sup_{y \in U_0} \|L_y\|_{B^2}. \tag{3.3}$$

We are now in the position to give the relevant estimates. In fact, since $T_{k_0}: B^1 \to B^2$ is a continuous operator and since $A_0 \cap \circ B^1$ is dense in $\circ B^1$ it will by sufficient to show that $T_k(A_0 \cap \circ B^1)$ is tight and equicontinuous in B^2 (by Theorem 2.1). Given $\varepsilon > 0$ and $f \in A_0 \cap \circ B^1$ we proceed as follows:

The continuity of $y \mapsto L_y k_0$ from G to A implies that there exists $U \subseteq U_0$ such that

$$\|L_y k_0 - k_0\|_A < \varepsilon[2C_1 C_2(\alpha_0 + 1)]^{-1}. \tag{3.4}$$

We then write $g := |U_1|^{-1} c_{U_1}$ for the normalized characteristic function of a suitable invariant neighborhood $U_1 \subseteq U_0$. A simple estimate (derived from the representation of $g * k_0$ as vector-valued integral) implies

$$\|g * k_0 - k_0\|_A < \varepsilon[2C_1 C_2(\alpha_0 + 1)]^{-1}. \tag{3.5}$$

The invariance of U_1 implies centrality of g in $L^1_w(G)$ $[g \in ZL^1_w(G)]$. In particular

$$f * g = g * f \qquad \text{for all} \quad f \in A_0. \tag{3.6}$$

Using the continuity of $y \mapsto L_y g$ from G to $L^1_w(G)$ one finds a neighborhood $U_2 \subseteq U_1$ such that

$$\|L_y g - g\|_{1,w} < \varepsilon[2C_1 C_2 \|k_0\|_A]^{-1} \qquad \text{for} \quad y \in U_2. \tag{3.7}$$

Combining (3.4), (3.6) and (3.7) one obtains for $f \in A_0 \cap \circ B^1$:

$$\begin{aligned}\|L_y(f * k_0) - f * k_0\|_{B^2} &\leqslant \|L_y(f * k_0) - L_y(f * g * k_0)\|_{B^2} \\ &\quad + \|(L_y g - g) * f * k_0\|_{B^2} + \|f * g * k_0 - f * k_0\|_{B^2} \\ &\leqslant (\alpha_0 + 1)\|f * (g * k_0 - k_0)\|_{B^2} + \|L_y g - g\|_{1,w}\|f * k_0\|_{B^2} \\ &\leqslant (\alpha_0 + 1) C_1 C_2 \|g * k_0 - k_0\|_A + \|L_y g - g\|_{1,w} C_1 C_2 \|k_0\|_A < \varepsilon \\ &\qquad \text{for all} \quad y \in U_2.\end{aligned}$$

In order to show tightness of $T_k(\circ B^1)$ in B^2 we choose $h \in A_0$ such that

$$\|hf - f\|_{B^3} < \varepsilon[\alpha_0(1 + C_{B^2})\|k_0\|_A]^{-1} \qquad \text{for} \quad f \in \circ B^{-1}. \tag{3.8}$$

According to the assumptions it is possible to find $h_1 \in A_0$ such that $h_1(x) = 1$ on $(\operatorname{supp} h)(\operatorname{supp} k_0)$ and such that $\|h_1 f\|_{B^2} \leqslant C_{B^2}\|f\|_{B^2}$ for all $f \in B^2$. This implies $h_1(hf * k_0) = hf * k_0$ and further

$$\begin{aligned}\|f * k_0 - h_1(f * k_0)\|_{B^2} &\leqslant \|f * k_0 - hf * k_0\|_{B^2} + \|h_1(hf * k_0) - h_1(f * k_0)\|_{B^2} \\ &\leqslant (1 + C_{B^2})\|hf - f\|_{B^3} \cdot \|k_0\|_A < \varepsilon \qquad \text{for all} \quad f \in M.\end{aligned} \tag{3.9}$$

It follows then from Theorem 2.1 that $T_{k_0}(A_0 \cap \circ B^1)$, hence $T_{k_0}(\circ B^1)$ is relatively compact in B^2. Q.E.D.

We shall now give a short list of typical examples for which assumption (A) is satisfied.

(A1) $\circ B^1$ is tight in B^2 and B^2 is also a right L^1_w-convolution module. [Choose $B^3 := B^2$, then $B^3 * A_0 \subseteq B^2 * L^1_w \subseteq B^2$.]

(A2) $\circ B^1$ is tight in $L^1_{w_2}(G)$. [Again $B^3 := B^2$, then $B^3 * A_0 \subseteq L^1_{w_2} * B^2 \subseteq B^2$.]

Remark 3.1. For the case that B^2 is a homogeneous Banach space tightness of $\circ B^1$ in $L^1(G)$ is sufficient for (A2) to be applicable. For example, $\circ L^1_w(G)$ is tight in $L^1(G)$ whenever $w^{-1} \in C^0(G)$, or $\circ L^p_w(G)$ is tight in $L^1(G)$ if $w^{-1} \in L^{p'}(G)$, $1 < p < \infty$, $1/p + 1/p' = 1$ (by Hölder's inequality).

Remark 3.2. If B^2 is two-sided L^1_w-convolution module then (A1) and (A2) may be combined to assume tightness of $\circ B^1$ in $B^2 + L^1_{w_2} =: B^3$, the perhaps largest simple choice of B^3 in Proposition 3.1. Another example explaining the role of the auxiliary space B^3 is the following:

(A3) $B^1 = L^1 \cap L^p_w$, $B^2 = L^q$, for w such that $w^{-1} \in C^0(G)$. [For $1 < q \leqslant p$ one can show that $\circ B^1$ is a tight subset of $B^2 = L^q$, i.e., (A1) applies. For $1 \leqslant p < q$ neither (A1) nor (A2) applies, however, the choice $B^3 := W(L^1, L^q) = 1^q(L^1)$, as considered in [53, 17, or 37], is possible. Since $A_0 \subseteq \mathscr{K}(G) \subseteq W(C^0, L^1) = W(G)$ one has, taking into account that G is an $[IN]$-group,

$$B^3 * A_0 \subseteq W(L^1, L^q) * W(C^0, L^1) \subseteq W(C^0, L^1) \subseteq L^q(G). \tag{3.10}$$

Tightness of $\circ(L^1 \cap L^p_w) \subseteq L^p \subseteq W(L^1, L^p) \subseteq W(L^1, L^q)$ follows from the assumption $w^{-1} \in C^0(G)$.]

PROPOSITION 3.2. *Let G be a $[SIN]$-group, and let B^1, B^2 be two Banach spaces on G in standard situation over the same Banach algebra A, both containing A_0 as a dense subspace. Assume that B^2 is right invariant and that the following condition* (B) *is satisfied*:

$$\text{(B)} \qquad \begin{cases} B^2 \subseteq H_G(B^1, B^2), \text{ i.e., there exists } C_4 \text{ such that} \\ \|f * g\|_{B^2} \leqslant C_4 \|f\|_{B^1} \|g\|_{B^2} \text{ for } f \in A_0,\ g \in B_2 \\ (\text{i.e., } g \mapsto T_g \text{ defines a continuous injection}). \end{cases}$$

Then one has: If $\lim_{y \to e} \|L_y T - T\|_{B^1 \to B^2} = 0$ for some $T \in H_G(B^1, B^2)$, then there exists a sequence of "elementary" multipliers T_n (i.e., $T_n f = f * k_n$ for some $k_n \in A_0$, $n \geqslant 1$) such that $\lim_{n \to \infty} \|T_n - T\|_{B^1 \to B^2} = 0$.

Proof. Let $\varepsilon > 0$ be given. Since the assumption $L_y T \to T$ for $y \to e$ can be rewritten as equicontinuity of $T(\circ B^1)$ in B^2 there exists $g \in ZL^1_w(G)$ (center of $L^1_w(G)$) such that $\|g * Tf - Tf\|_{B^2} < \varepsilon$ for $f \in A_0 \cap \circ B^1$ (cf. the proof of 3.1). Observe that the assumptions imply that right translation is continuous in B^2. Therefore B^2 is a right convolution module over a Beurling algebra $L^1_{w_2}(G)$. Using the identity $g * Tf = T(g * f) = T(f * g)$ and the existence of $k \in A_0$ with $\|k - g\|_{1,w_2} < \varepsilon \|T\|^{-1} B^1 \to B^2$ one obtains (using now $T(f * k) = f * Tk$):

$$\begin{aligned} \|Tf - f * Tk\|_{B^2} &\leqslant \|Tf - g * Tf\|_{B^2} + \|T(f * (g - k))\|_{B^2} \\ &< \varepsilon + \|T\|_{B^1 \to B^2} \|f\|_{B^1} \|g - k\|_{1,w_2} \\ &< 2\varepsilon \qquad \text{for all } f \in A_0 \cap \circ B^1. \end{aligned} \tag{3.11}$$

The density of A_0 in B^2 allows one to choose now $k_0 \in A_0$ such that $\|Tk - k_0\|_{B^2} < \varepsilon / C_4$, i.e., such that one obtains by applying condition (B):

$$\|f * Tk - f * k_0\|_{B^2} \leqslant C_4 \|f\|_{B^1} (\varepsilon / C_4) \leqslant \varepsilon \qquad \text{for } f \in \circ B^1. \tag{3.12}$$

Combining (3.11) and (3.12) one obtains

$$\|Tf - f * k_0\|_{B^2} < 3\varepsilon \qquad \text{for all} \quad f \in A_0 \cap \circ B^1,$$

i.e., $\|T - T_{k_0}\|_{B^1 \to B^2} \leqslant 3\varepsilon$. Q.E.D.

The main result of this section is now easily available.

THEOREM 3.3. *Let B^1, B^2 be two Banach spaces in standard situation (over the same Banach algebra A) on a $[SIN]$-group G, such that B^2 is right invariant and both spaces contain A_0 as a dense subspace. If conditions* (A) *and* (B) *are satisfied the following conditions are equivalent for $T \in H_G(B^1, B^2)$*:

(i) *T is a compact multiplier*;

(ii) $\|L_y T - T\|_{B^1 \to B^2} \to 0$ *for* $y \to e$;

(iii) $\lim_{n \to \infty} \|T - T_{k_n}\|_{B^1 \to B^2} = 0$ *for a sequence* $(k_n)_{n \geqslant 1}$ *in* A_0;

(iv) $T = T_g \circ T^1$ *for some* $g \in L^1_w(G)$ and $T^1 \in H_G(B^1, B^2)$.

Proof. Since left translation is continuous in B^2 it is clear that (i) $\Rightarrow$ (ii). Proposition 3.2 shows that under the hypotheses made (ii) $\Rightarrow$ (iii). $H_G(B^1, B^2)$ is now considered as a Banach module over the commutative Banach algebra $ZL^1_w(G)$, via $g \cdot T = T_g \circ T = T \circ T_g$, for $T \in H_G(B^1, B^2)$, $g \in ZL^1_w(G)$. G being a $[SIN]$-group $ZL^1_w(G)$ is a Banach convolution algebra having bounded approximate units. Applying the Cohen–Hewitt factorization theorem one obtains the implication (ii) $\Rightarrow$ (iii), as it is evident that (ii) implies that T belongs to the essential part of $H_G(B^1, B^2)$ with respect to that algebra (cf. Proposition 1.1). Conversely, $T = T_g \circ T^1$ implies

$$\begin{aligned}\|L_y T - T\|_{B^1 \to B^2} &\leqslant \|L_y T_g - T_g\|_{B^2 \to B^2} \|T^1\|_{B^1 \to B^2} \\ &\leqslant \|L_y g - g\|_{1,w} \|T^1\|_{B^1 \to B^2} \to 0 \qquad \text{for} \quad y \to e.\end{aligned}$$

We give now some comments concerning the conditions (A) and (B), and sufficient conditions implying that both ot them are satisfied.

PROPOSITION 3.4. *Let B^1, B^2 be two left and right invariant Banach spaces in standard situation on a $[SIN]$-group G, both containing A_0 as a dense subspace. Then Theorem* 3.3 *is applicable whenever one of the following conditions is satisfied by the pair (B^1, B^2)*:

(C1) *$\circ B^1$ is tight in B^2 and B^2 is a Banach algebra with convolution as multiplication (e.g., $\|f * g\|_{B^2} \leqslant \|f\|_{B^2} \|g\|_{B^2}$ for $f, g \in A_0$*, cf. [106]).

(C2) = (A2) $\circ B^1$ *is tight in* $L^1_{w_2}(G)$; *e.g.*, $\circ B$ *is tight in* $L^1(G)$ *and* B^2 *is a homogeneous Banach space.*

(C3) = (A3) $B^1 \subseteq L^1 \cap L^p_w$, $B^2 = L^q$, *for some* w: $w^{-1} \in C^0(G)$.

Proof. (i) Since (C1) is stronger than (A1) it is sufficient to verify that it also implies (B): $B^1 \subseteq B^2$ implies $B^1 * B^2 \subseteq B^2 * B^2 \subseteq B^2$, together with the corresponding norm inequalities. That (B) is a consequence of (C2) follows essentially from $B^1 * B^2 \subseteq L^1_{w2} * B^2 \subseteq B^2$, and in a similar way it follows from (A3): $(L^1 \cap L^p_w) * L^q \subseteq L^1 * L^q \subseteq L^q$.

It should be mentioned that for a fixed space B^2 conditions (A) and (B) as conditions on B^1 are inherited by any subspace of B^1 satisfying the standard assumptions. Furthermore, both properties are stable with respect to any (uniform) interpolation functor F: Given two such spaces B^1_i, $i = 1, 2$, satisfying (A) and (B), $F^1 := F(B^1_1, B^1_2)$ satisfies these properties as well. The easy proof is left to the interested reader. We only want to explain a (perhaps more useful) variant of this result, showing that only tightness of $\circ B^1_1$ in B^3_1 is required in order to prove (A) (without tightness of $\circ B^1_2$ in B^3_2):

Suppose that B^1_i, B^3_i are translation invariant Banach spaces containing A_0 as a dense subspace, and such that $B^3_i * A_0 \subseteq B^2$ and $B^1_i \subseteq B^3_i$ for $i = 1, 2$. Assume that $\circ B^1_1$ is a tight subset of B^3, and that F is an interpolation functor of exponent $\theta \in (0, 1)$ (cf. [8, Section 2.4]). Then $F^1 := F(B^1_1, B^1_2)$ satisfies (A).

Proof. Let us set $B^3 := B^3_1 + B^3_2$ with its natural norm. It is then clear that $B^3 * A_0 \subseteq B^2$ holds. It only has to be shown that $\circ F^1$ is tight in B^3. Let $\varepsilon > 0$ be given. Since F is of exponent θ there exists $C_1 > 0$ such that for any operator $T: B^1_1 + B^1_2 \to B^3_1 + B^3_2$ $FT := T|F^1$ (restriction) satisfies

$$\|FT\|_{F^1 \to B^3} < C \, \|T\|^{\theta}_{B^1_1 \to B^3_1} \|T\|^{1-\theta}_{B^1_2 \to B^3_2}.$$

Let now $(h_\alpha)_{\alpha \in I}$ be a family of trapezoid functions in A_0 of bounded action on B^3_i, $i = 1, 2$. The assumptions imply that the family (N_α) of operators defined by $N_\alpha : f \mapsto f - h_\alpha f$ is bounded by some constant C_1 in $H(B^1_2, B^3_2)$. On the other hand it follows from the tightness of $\circ B^1_1$ in B^3_1 that there exists $\alpha_0 \in I$ such that one has, for $\alpha \geqslant \alpha_0$,

$$\|h_\alpha f - \|_{B^3_1} < (\varepsilon C_1^{-1} C_2^{\theta - 1})^{1/\theta} \qquad \text{for all} \quad f \in \circ B^1_1.$$

Hence $\|N_\alpha\|_{F^1 \to B^3} \leqslant C_1(\varepsilon C_1^{-1} C_2^{\theta-1}) \, C_2^{1-\theta} = \varepsilon$ for $\alpha \geqslant \alpha_0$, i.e., $\|N_\alpha\|_{F^1 \to B^3} \to 0$ for $\alpha \to \infty$. This in turn is equivalent to tightness of $\circ F^1$ in B^3. Q.E.D.

THEOREM 3.5. *Let* w_1, w_2 *be two weight functions on* G, *such that* $L^1_{w_1}(G) \hookrightarrow L^1_{w_2}(G)$, *and let* B *be an essential Banach module over* $L^1_{w_2}(G)$ *with respect to convolution. Then one has*

$$H_G(L^1_{w_1}, B) = H_G(L^1_{w_2}, B).$$

Proof. Since one inclusion is trivial we only have to verify that $T \in H_G(L^1_{w_1}, B)$ may be extended to a multiplier from $L^1_{w_2}$ to B. Since $\mathscr{K}(G)$ is dense in any Beurling algebra only continuity of T (on $\mathscr{K}(G)$) with respect to $\| \|_{1,w_2}$ has to be shown. To this end observe that a slight modification of the arguments of [23] shows that the Cohen–Hewitt factorization theorem [52, Section 32.22], applied to $(B, \| \|_B)$ as an essential $L^1_{w_2}(G)$-module, implies that $\|L_y\|_B \leqslant C_{w_2} w_2(y)$, where C_{w_2} indicates the bound for some approximate unit in $L^1_{w_2}(G)$. Let now $f \in \mathscr{K}(G)$ be given. Using a bounded uniform partition of unity $(\psi_i)_{i \in I}$ in $C^0(G)$ with $\|\psi_i\|_\infty \subseteq 1$ for $i \in I$ (see [35]) one may write $f = \sum f\psi_i$, a finite sum, with supp $(f\psi_i) \subseteq y_i Q$ for some fixed compact set $Q \subseteq G$ and $y_i \in G$ suitable chosen for any $i \in I$. Since w_1 is bounded over compact sets the norms $\| \|_{1,w_1}$ and $\| \|_1$ are equivalent on the space $L^1_Q(G)$, and there exists $C > 0$ such that $\|f\|_{1,w_1} \leqslant C \|f\|_1$ for $f \in L^1(G)$, $\operatorname{supp} f \subseteq Q$. Writing $f_i := L_{y_i}^{-1}(f\psi_i)$ we have $f = \sum L_{y_i} f_i$, and $f_i \in L^1_Q(G)$ for all $i \in I$. This allows one to obtain for any $f \in \mathscr{K}(G)$ the following estimate:

$$\begin{aligned} \|Tf\|_B &\leqslant \sum \|TL_{y_i} f_i\|_B \leqslant \sum \|L_{y_i}\|_B \, \|Tf_i\|_B \\ &\leqslant C_{w_2} \|T\|_{L^1_{w_1} \to B} \sum w_2(y_i) \, \|f_i\|_{1w_1} \\ &\leqslant C \cdot C_{w_2} \|T\|_{L^1_{w_1} \to B} \sum w_2(y_i) \, \|f_i\|_1 . \end{aligned}$$

Taking into account that (due to the regularity of w) $f \mapsto \sum w_2(y_i) \|f\psi_i\|_1$ defines an equivalent norm on $L^1_{w_2}(G)$ the assertion follows. (In the terminology of Wiener-type spaces the norm equivalence follows from the identity $L^1_{w_2}(G) = W(L^1, L^1_{w_2})$, cf. [37].) In conclusion we mention that $L^1_{w_1} \hookrightarrow L^1_{w_2}$ if and only if there exists $C > 0$ such that $w_2(x) \leqslant C w_1(x)$ for all $x \in G$ (cf. [89, 6.3.6], or [32]).

COROLLARY 3.6. *Let B be a homogeneous Banach space on G in standard situation, containing A_0 as a dense subspace, and let w be a weight function such that $w^{-1} \in C^0(G)$. Then*

$$C_G(L^1_w(G), B) \cong B.$$

Consequently, any multiplier from $L^1_w(G)$ into B is compact whenever B is reflexive as a Banach space.

Proof. In view of the above theorem $H_G(L^1_w, G)$ can be identified with $H_G(L^1, B)$. The existence of 1-bounded approximate units in $L^1(G)$ implies that $(B, \| \|_B)$ may be considered (via convolution from the right) as a closed subspace of $H_G(L^1, B)$ (cf. [31]). Since condition (A2) applies to the present situation Theorem 3.3 yields that $C_G(L^1_w, B)$ coincides with the closure of A_0 in $B \subseteq H_G(L^1, B)$. According to the assumptions this is just B, and the proof of the formula is complete. Since $H_G(L^1, B) \cong$ for any reflexive homogeneous Banach space (for Banach module-theoretic reasons, cf. [92]) the additional assertion follows therefrom.

Remark 3.3. It is clear that one may consider, more generally, essential Banach modules B over $L^1_{w_2}(G)$, if one replaces w by w_1, satisfying $w_2/w_1 \in C^0(G)$ above.

4. Some Comments on the Relations to Existing Literature

In this final section we shall point out the connection to results in the literature, usually concerning particular spaces of families of spaces, such as solid BF-spaces or Besov-spaces. Also some comments concerning the characterization of compact multipliers are given. Finally, we indicate possible extensions of the results presented in this paper.

(A) First of all the classical results due to Kolmogorov and Riesz, concerning L^p-spaces on the real line, have to be mentioned [61, 93]. Kolmogorov makes use of a spacial kind of approximate unit in $L^1(\mathbb{R})$, the so-called Stecklov-means, which are nothing else but normalized characteristic functions of small intervals (and which are good enough to approximate integrable functions by continuous ones). M. Riesz describes eqicontinuity essentially by means of the modulus of continuity (cf. Proposition 2.3 for the equivalence in the general context). Tightness in L^p-spaces (or more general, in solid BF-spaces) can of course be described as follows (cf. Proposition 2.4):

(C_5) For $\varepsilon > 0$ there exists a compact set $K \subseteq G$ such that $\| f - c_K f \|_B < \varepsilon$ for all $f \in M$.

The extension of this result to L^p-spaces on arbitrary locally compact groups has been given by Weil ([112, p. 53], or [26]; cf. also [109]). The corresponding results for Orlicz-spaces can be found in the book of Krasnoselskij–Rutickij ([64, Chapter II, Section 11], or [68, Section–3.14]). For Orlicz-spaces on groups cf. [16]. The most general results concerning compactness in solid BF-spaces (including different compactness criteria) as well as further information in that direction are given in the paper by Goes–Welland [48].

Also, papers by Kaminska and Pluciennik on Orlicz spaces and modular spaces have to be mentioned here (see [56, 57]). As mentioned in Section 1 also Morrey spaces [68], Lorentz-spaces [8, 71], Lorentz–Zygmund-spaces [73], mixed norm spaces [6, 53, 17], and the partially rearrangement invariant spaces introduced by Blozinski [13] belong to the class of solid BF-spaces, and the compactness criterion applies if $\mathscr{K}(G)$ is dense in such a space. Concerning this density in Köthe-spaces Silverman has given a result that is closely related to Lemma 1.5 above (see [99]).

(B) Compactness criteria for spaces of differentiable functions on the Euclidean space were first given by Russian mathematicians, starting with Nikolskij ([74, cf. also [67]). Results for such spaces have appeared among others in the books of Nikolskij [75, Section 7.7], Besov–Ilin–Nikolskij [10], and Kufner–Fucik–John [68, Section 7.4]. Detailed information concerning the generalized Lipschitz spaces or Besov spaces $B^s_{p,q}$ or the Bessel potential (or Lebesgue) spaces $\mathscr{L}^p_s$, their anisotropic versions and their importance for fields such as partial differential equations can be found in [54, 101, 68] and in the books of Triebel [104–106].

Usually the compactness criteria for these spaces are derived from the compactness criterion for the corresponding L^p- or Orlicz-space, taking in account how the new space is derived from the given L^p-spaces (e.g., by integral conditions on the modulus of continuity). The present approach, however, makes use of a few basic properties of the constructed spaces only, such as invariance under translations and under multiplication with characters, and the density of the space of test functions (cf. Proposition 1.6 and Remark 1.8). The proof that a new constructed space has these properties usually belongs to the first information one tries to obtain about it, and does not present difficulties in most cases. As a benefit, however, this basic information is also useful for other purposes. Thus, for example, it is intended to show [41] that the compactness criterion has a particularly simple formulation (and proof) for the so-called Wiener-type spaces, as introduced in [37] (cf. [17, Proposition 3.13] for a special case), as well as for a family of much more general Banach-spaces of distributions defined by decomposition methods, including the usual Besov-spaces, their anisotropic versions as well as a number of new spaces, also to be discussed in [37]. The compactness criterion will also apply to another family of generalized Lipschitz spaces, obtained by several variants of the ordinary modulus of continuity, called smoothness indicator (cf. [40], and a special case is considered in [33]; see also the paper by Janson [55] for a related construction). It is perhaps worth mentioning that compactness criteria for these spaces can be used to prove compactness of partial differential operators (or more general, pseudodifferential operators) between suitable pairs of such spaces, which are often derived from results concerning compact embeddings (cf. [44, 75, 68, 72, 73]).

By means of suitable modifications assertions about compact embeddings between function spaces on domains can also be derived. Finally, it should be mentioned that the formulation of our results also allows one to consider Banach spaces of ultradistributions (in the description of the regularity of the corresponding test functions the Beurling–Domar non-quasianalyticity property, cf. [1], appears in a natural way). Basic information about such spaces can be obtained from papers of Björck [11], Komatsu [62] or the introductory chapter of [105].

(C) Segal algebras (see [89, 90]) as well as homogeneous Banach spaces (of locally integrable functions or quasimeasures) arise naturally in all parts of (harmonic) analysis (see [59, 31, 111], cf. [110] for a list of examples). Compactness criteria for homogeneous Banach spaces have been given by Shilov [98], Goldberg [49] and in a recent paper by the author [38]. The results of Goldberg concern homogeneous Banach spaces on the real line and are obtained by reduction to the case of periodic functions. However, the utility of a multiplicative structure on the spaces under consideration also arises in his completely different approach.

As important spacial cases of homogeneous Banach spaces we mention the Herz algebras $A_p(G)$ (cf. [28]). For the case of an amenable group G the compactness criterion has been also proved by Granirer–Leinert [51]. In our approach it can be seen as a consequence of the fact that any Banach algebra A satisfying the general assumptions and having approximate units bounded in the operator norm is in standard situation. Typical examples of Banach spaces of quasimeasures, to which the compactness criterion applies, would be the spaces $\mathscr{F}L^p$ (G l.c.a.) for $1 < p < \infty$, or certain spaces of "kernels" of compact multipliers (cf. Section 1). Another example would be the space $V_0(G_1 \times G_2)$ (Varopoulos-algebra) or the space $BM_a(G_1 \times G_2)$ of absolutely continuous bimeasures in $G_1 \times G_2$ (G_i l.c.a. groups, $i = 1, 2$), as considered by Graham–Schreiber [50]. The chosen point of view also allows one to reformulate results as stated in (B) for the corresponding local-field versions of the generalized Lipschitz spaces, without any change (see [102]).

(D) There exists an extensive literature concerning multipliers. We can only mention Larsen's book [69] as a general reference concerning the abelian case (and including a great number of examples), and Rieffel's fundamental papers [91, 92], where the concepts of Banach modules and of module tensor products is established in a rigorous way. Multipliers between Segal algebras are treated among others, in [2, 14, 15, 25, 31, 33, 34, 69, 77, 103, 110]. The question of whether there exist compact multipliers from $L^p(G)$ into $L^q(G)$ has been considered by a number of authors. Roughly speaking, one can say that compact multipliers only exist if $q = \infty$ and $p = 1$, or if G is a compact group. The first case leads to the notion of almost periodic functions (exactly the AP-function defines compact multipliers). There exists of course an extensive literature in this direction. We only mention [39], where it is shown that various concepts of strong almost periodicity, including several classical concepts, can be obtained by replacing L^∞ by a Wiener-type space $W(B, L^\infty)$ (i.e., the global L^∞-behaviour is the fact that counts for this question).

(E) The nonexistence of (weakly) compact multipliers between various pairs of homogeneous Banach spaces (starting with multipliers on $L^1(G)$) have been observed by a number of authors, which we mention in rough chronological order (without going into details): Sakai [95], Akeman [1] and Crombez–Govaerts [19, 20]. Concerning the question of compact multipliers between L^p-speces the most general results seem to be due to Lau [70]. Concerning Segal algebras this question has been considered by Krogstad [66, Proposition 2.2], Parthasarathy [77] and Dutta–Tewari [25].

At least for abelian groups these results may also be considered as special cases of a result concerning the nonexistence of compact multiplication operators on certain multiplicative (semisimple) Banach algebras, due to Friedberg [43] and Kamowitz [58].

(F) On the other hand, for compact groups G a study of compact multipliers between certain pairs of translation invariant Banach spaces of functions has been undertaken by various authors. Some of their papers contain theorems that are quite analog to Theorem 3.3 above, making use of the compactness of the group instead of a multiplicative structure on the spaces under consideration; this is no surprise as the multiplicative structure has just been used to reduce the problem on noncompact groups by multiplication with trapeziod functions to a problem over a compact set. Evidently there is no need for such a cutoff function in the compact case. Replacing all pointwise products with these trapezoid functions in the above proofs by multiplication with the constant 1 (i.e., replacing the multiplication operators by the identity operator) would give proofs for Banach spaces of distributions (i.e., subspaces of dual of some Banach algebra of test functions on compact groups) that do not make use of the multiplicative structure on B. Theorems that are very similar to Theorem 3.3 above are given by several authors. Without claiming completeness we mention Akeman ([1], G compact, $B^1 = B^2 = L^1(G)$), Kitchen ([60], G compact abelian, $B^1 = B^2 = L^1(G)$), Dunkl–Ramirez ([24], G compact, $B^1 = B^2$ a homogeneous Banach space), Bachelis–Pigno ([4], G compact, $B^1 = L^p(G)$, $B^2 = L^q(G)$), Bachelis–Gilbert ([2], certain pairs of homogeneous Banach spaces on compact groups), Racher ([86–88], L^p-spaces on compact groups) and Tewari–Parthasarathy (103], for pairs of Segal algebras on compact groups).

Combining Theorem 3.3 with various known results, concerning multipliers between Lipschitz spaces, for example, several interesting results could be obtained (for the case of a compact as well as for the noncompact situation). Cf. [12, 33, 81, 84, 85, 110] and [40, 41] for generalizations. One may expect that the existence of central approximate units in Segal algebras on [SIN]-groups (see [63]) or the characterization of the multipliers of a Beurling algebra (cf. [46]) might be useful tools in that context.

It has been shown by Bachelis–Gilbert [2] that for suitable pairs of homogeneous Banach spaces on compact groups the bidual of the Banach space of all compact multipliers between these spaces is just the space of all multipliers (cf. also [86], where some of the conditions are relaxed). In a forthcoming paper it will be shown that this relation sometimes persists to hold true in the situation of Theorem 3.3, at least if G is an abelian group (see [15]). In that paper a more detailed study of the connection between pointwise module structure and the convolution structure (arising from the relations between several multiplier spaces) will also be given.

(G) As far as we know there is until now only a small number of papers that give more quantitative information about the "degree of compactness" of compact multipliers. It must be considered a natural question to ask for sufficient and (or) necessary conditions for a given distribution (defining a compact multiplier between two given homogeneous Banach spaces) to define an operator belonging to a certain operator ideal in the sense of Pietsch [79], e.g., to define a Hilbert Schmidt multiplier, or a trace class multiplier. Usually, one will have to expect that the smoothness of the kernel will be responsible for the operator ideal to which the convolution operator will belong (cf. [80, 87]). It has to be expected that interpolation methods, cf. [44, 78, 8], will play in important role for such assertions. Some results in this direction (for L^p-spaces on compact groups) have been obtained by Bauhardt [5] and Racher [86, 87].

(H) By the symmetry of the pointwise and the convolutive module structure it is clear that it would be possible (and within the frame of locally compact abelian group, in fact, equivalent via Fourier transform) to give results on compact pointwise multipliers. Instead of assuming tightness of the domain of the operator some conditions on the equicontinuity of the unit ball of the domain of that multiplication operator have to be made. We do not go into details about such questions here (cf. [15]).

(K) Of course, the compactness criterion also might be used to derive sufficient conditions for more general operators (arising in harmonic analysis, partial differential equations,...) to be compact or to belong to a certain operator ideal. Here, for example, the so-called PC- and CP-operators (product-convolution and convolution-product operators) can be mentioned, which are of the form $T(f) = g * (hf)$ or $T(f) = g(h * f)$, for functions (distributions) g, h belonging to suitable spaces (cf. [17, 100]). Among them several pseudodifferential operators of interest can also be found. Using our results assertions concerning these operators can be easily derived from the corresponding inequalities concerning convolution or (pointwise) multiplication, i.e., using suitable Banach convolution (multiplication) triples. In the case of L^p-spaces the necessary inequalities are just Young's and Hölder's inequality, respectively. Using similar inequalities for Wiener-type spaces (cf. [53], or [41] for more general versions) it would be possible to derive the sufficiency conditions for compactness, as given in [17].

Also, for results concerning the compactness of pseudodifferential operators or even more general operators between spaces arising in harmonic analysis the compactness criterion might be useful, as Fourier analytic ideas are involved in both field. Thus, for example, it should be possible to prove that certain integral operators are compact. We add that the possibility of requiring smoothness conditions of the kernel in order to ensure that the corresponding operator is compact exists not only for operators on the Euclidean space (using of course Schwartz's kernel theorem, which allows one to think of the kernel as of a tempered distribution on the product space, cf. [107, 108]) but also for many spaces of distributions on locally compact abelian groups, using the more simple Banach space $S_0'(G)$ of translation bounded quasimeasures (cf. [34], Theorem B3).

(L) At the end of this paper we mention that the general method of the proof can also be used to obtain compactness criteria for Banach spaces $(B, \| \|_B)$ of distributions on spaces of homogeneous nature, e.g., Banach spaces on G/K, K being a compact subgroup of the locally compact group G. In the proofs the action of G on the Banach space, which had been simply translation in our proofs, has to be replaced by the natural action of G on B. The important fact to be used in the proof is the existence of suitable approximate units in $L^1(G)$ (with compact support) having the property that their action on compactly supported distributions (by generalized convolution) results in compactly supported, continuous functions (tending to the given distributions in a weak sense). Without going into details we state a typical result that can be obtained by such methods (we use the normalized dilation operators M_ρ, cf. [89, Chapter I]):

THEOREM 4.1. *Let $(B, \| \|_B)$ be a Banach space of tempered distributions on $\mathbb{R}^n$ be given, such that $\mathscr{D}(\mathbb{R}^n)$ is dense in B. Suppose that a Banach algebra $(A, \| \|_A)$ satisfying the general assumptions and having bounded approximate units acts on B by pointwise multiplication, and that A and B are dilation invariant, satisfying $\|M_\rho\|_B \leqslant C(1+|\rho|)^\alpha$ for $\rho \in \mathbb{R}^+$ $(C>0$, $\alpha>0$ chosen suitably). Then a closed, bounded subset $M \subseteq B$ is compact if and only if it is tight in B and satisfies*

(d) *For $\varepsilon>0$ there exists $\eta>0$ such that $\|M_\rho f - f\|_B < \varepsilon$ for $\rho \in (1-\eta, 1+\eta)$ and all $f \in M$.*

This theorem applies, for example, to L^p_w-spaces with weights of the form $\omega(x) = (1+|x|)^s$ or to the (inhomogeneous) Besov-spaces $B^s_{p,q}$ as well as to $F^s_{p,q}$-spaces, including the potential spaces $\mathscr{L}^p_s$ as a special case (cf. [32, 106]).

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