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A new approach to the inversion of the Riesz potential operator

Stefan G. Samko

I. Introduction

It is known that the operator (left) inverse to the Riesz potential operator

$$I^\alpha \varphi = \frac{1}{\gamma_n(\alpha)} \int_{R^n} \frac{\varphi(y)\, dy}{|x-y|^{n-\alpha}} \ , \quad x \in R^n \ , \tag{1.1}$$

where $\gamma_n(\alpha) = \frac{2^\alpha \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}$ is the well known normalizing constant, has the form of a hypersingular integral, see Samko [1], [4] or Samko, Kilbas and Marichev [1], Section 26. Namely,

$$(I^\alpha)^{-1} f = \mathbb{D}^\alpha f \ := \frac{1}{d_{n,\ell}(\alpha)} \int_{R^n} \frac{\left(\Delta_y^\ell f\right)(x)}{|y|^{n+\alpha}} dy \ , \quad \alpha > 0 \ , \tag{1.2}$$

where $\left(\Delta_y^\ell f\right)(x)$ is a centered or non-centered finite difference and $d_{n,\ell}(\alpha)$ is some normalizing constantes, see details in Samko [4] or Samko, Kilbas and Marichev [1], Section 26. This integral, known also as the *Riesz fractional derivative,* is treated as the limit

$$\mathbb{D}^\alpha f \ := \lim_{\varepsilon \to 0} \mathbb{D}^\alpha f \ . \tag{1.3}$$

where

$$\mathbb{D}^\alpha_\varepsilon f = \frac{1}{d_{n,\ell}(\alpha)} \int_{|y|>\varepsilon} \frac{\left(\Delta_y^\ell f\right)(x)}{|y|^{n+\alpha}} dy \ , \quad \alpha > 0 \ . \tag{1.4}$$

More generally, a potential type operator of the form

$$K^\alpha_\omega \varphi = \int_{R^n} \omega\left(\frac{y}{|y|}\right) \frac{\varphi(x-y)}{|y|^{n-\alpha}} \ dy \ \ , \alpha > 0, \tag{1.5}$$

with a sufficiently smooth homogeneous *characteristic* $\omega\left(\frac{y}{|y|}\right)$ is known to be inverted in the so called elliptic case by means of the hypersingular construction

$$(I^\alpha_\omega)^{-1} f = \mathbb{D}^\alpha_\Omega f \ := \int_{R^n} \Omega\left(\frac{y}{|y|}\right) \frac{\left(\Delta_y f\right)(x)}{|y|^{n+\alpha}} dy \ , \quad \alpha > 0 \ , \tag{1.6}$$

where the *characteristic* Ω of the hypersingular integral (1.6) can be effectively computed via the characteristic ω of the initial potential operator K^α_ω, see Samko [3]. See also the book Samko, Kilbas and Marichev [1], Sections 27-28, and the surveying paper Samko [5] for more other types of potential operators, when hypersingular constructions may be applied for inverting these operators.

Recently some other approach was also developed for inverting potential type, based on the idea of approximative inverse operators (called sometimes the method of AIO), see, for example, the papers Zavolzhenskii and Nogin [1]-[4], Nogin and Sukhinin [1], Sukhinin [1] and the surveying papers Samko [5] and Nogin and Samko [3]. This method gives an inverse operator as a limit of "nice" operators, one of advantages of this approach being in the fact that it allows to avoid usage of finite differences of the right-hand side $f(x)$ in the construction of the inverse operator.

The main idea of the method of AIO is the following. The problem to invert this or that convolution operator $A\varphi = a * \varphi$ reduces to multiplication of the Fourier transform of a function φ by the reciprocal $\frac{1}{\hat{a}(\xi)}$ of the Fourier transform of the kernel $a(x)$. This reciprocal, in case of potential operators, increases at infinity. We may introduce some "nice" factor $m_\epsilon(\xi)$ depending on ε, so that $\frac{m_\epsilon(\xi)}{\hat{a}(\xi)}$ vanishes at infinity (and at some other set, if necessary) and then return to Fourier pre-images and calculate the corresponding convolution $(A^{-1})_\varepsilon = F^{-1}\frac{m_\epsilon(\xi)}{\hat{a}(\xi)}F\varphi$ as an operator depending on ϵ. This is the initial idea of the method of AIO, which should be accomplished by the justification that this will really generate the inverse operator as $\epsilon \to 0$ in the space under consideration. The usual approach was based on a "nice" choice of such a factor $m_\varepsilon(\xi)$ in Fourier transforms, which provides an inverting ε-dependent kernel in Fourier pre-images as a result of calculation. In concrete cases this kernel is usually expressed in terms of these or those special functions.

In this paper we give a further development of some ideas of this approach and present a new glance at its application. The main difference in comparison with what was done before, is the opposite approach in the sense that we wish to choose a "nice" kernel directly on functions $f(x)$ themselves, not in Fourier transforms, therefore, not worrying about simplicity of the Fourier transform of this kernel. It goes without saying, that the simplicity of the kernel itself should be a final goal.

We demonstrate such a possibility of a construction of a "nice" kernel by our will in the case of the Riesz potential operator (1.1).

II. Preliminaries

We remind some formulas for the Riesz transforms. Let

$$(Ff)(\xi) = \widehat{f}(\xi) \ = \ \int_{R^n} e^{ix\xi} f(x)dx$$

be the Fourier transform of a function $f(x)$. It is known that

$$F^{-1}(|\xi|^{-\alpha}) = \frac{|x|^{\alpha-n}}{\gamma_n(\alpha)} \tag{2.1}$$

for all $\alpha \in \mathbf{C}$ except for $\alpha = n + 2k$ and $\alpha = -2k, k \in \mathbb{N}$, where the Fourier transform is understood in the sense if distributions. Hence

$$F(I^\alpha\varphi) = \frac{1}{|\xi|^\alpha}\widehat{\varphi}(\xi) \quad \text{and} \quad F(\mathbb{D}^\alpha\varphi) = |\xi|^\alpha\widehat{\varphi}(\xi) \ . \tag{2.2}$$

For a radial function $f = \varphi(|x|)$ the following Bochner formula is valid

$$\int_{R^n} e^{ix\xi}\varphi(|x|)dx = \frac{(2\pi)^{\frac{n}{2}}}{|\xi|^{\frac{n}{2}-1}} \int_0^\infty \varphi(\rho)\rho^{\frac{n}{2}} J_{\frac{n}{2}-1}(\rho|\xi|)d\rho \ , \tag{2.3}$$

under the assumption that the integral in the right-hand side converges, $J_{\frac{n}{2}-1}(z)$ being the Bessel function.

For the Wiener ring

$$\mathcal{W}_0 = \mathcal{W}_0(R^n) = \{f : f(\xi) = \widehat{\varphi}(\xi),\ \varphi(x) \in L_1(R^n)\} \tag{2.4}$$

of Fourier transforms of L_1-functions the following statement is valid (see, e.g. Samko [4]).

Lemma 2.1. *Let $f(x) \in L_1(R^n)$. If $f(x)$ has the mixed derivatives $D^j f \in L_p(R^n)$ for all $j \in \{0,1\}^n, j \neq 0$, , where $1 < p \leq 2$, then $f(x) \in \mathcal{W}_0(R^n)$ and*

$$\|f\|_{W_0} \ \leq \ c\left(\|f\|_1 \ + \sum_{j\ \in\{0,1\}^n} \|D^j f\|_p \right)$$

with $c > 0$ not depending on f .

We also note the following obvious boundedness of the operator I^α:

$$I^\alpha \ : L_1(R^n) \ \rightarrow \ L_q\Big(R^n; (1+|x|)^{-\lambda}\Big)\ , \quad 0 < \Re\alpha < n, \tag{2.5}$$

where $1 \leq q < \frac{n}{n-\Re\alpha},\ \lambda > n-(n-\alpha)q$.

For the hypersingular integral operator (1.3) the following fractional "integration by parts" formula holds, in which $W_p^m(R^n), 1 \leq p \leq \infty, m \in \mathbb{N}$, is the Sobolev space of functions $f(x) \in L_p(R^n)$, which have all the distributional derivatives $D^j f(x) \in L_p(R^n), 0 < |j| \leq m$. In the case $p = \infty$, by $W_\infty^m(R^n) = BC^m(R^n)$ we understand the space of functions on R^n which are differentiable in the usual sense up to order m and have all the bounded derivatives $D^j f(x), 0 \leq |j| \leq m$.

Lemma 2.2. *Let $f(x) \in W_p^m(R^n),\ 1 \leq p \leq \infty, m > \alpha$, and let $k(x) \in L_{p'}(R^n)$ and admit $(\mathbb{D}^\alpha k)(x) = \lim\limits_{\substack{\epsilon\to 0\\(L_{p'})}} (\mathbb{D}^\alpha_\epsilon k)(x)$. Then the formula of "integration by parts" is valid:*

$$(\mathbb{D}^\alpha k, f) = (k, \mathbb{D}^\alpha f)\ .$$

Proof. Indeed, the equality $(\mathbb{D}^\alpha_\epsilon k, f) = (k, \mathbb{D}^\alpha_\epsilon f)$ is obvious. It remains to pass to the limit as $\epsilon \to 0$ which is possible, by the well-known propertes of finite differences, see for example, Samko, Kilbas and Marichev [1], formula (26.20), and by the assumption on $k(x)$. ■

Lemma 2.3. *Let $f(x) \in L_1(R^n)$ and $x^j f(x) \in L_1(R^n), 0 \leq |j| < \alpha$. Then for any $\epsilon > 0$*

$$M^j_\epsilon(f)\ :=\int_{R^n} x^j (\mathbb{D}^\alpha_\epsilon f)(x)dx \equiv 0\ .$$

Proof. Evidently,

$$M^j_\epsilon(f) = \frac{1}{d_{n,\ell}(\alpha)} \int_{|y|>\epsilon} \frac{dy}{|y|^{n+\alpha}} \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} \int_{R^n} f(x-ky)x^j dx\ ,$$

the interchange of the order of integration being valid by the Fubini theorem. Hence, the change of variable $x - ky = \xi$ and the formula $(\xi + ky)^j = \sum_{0\le\nu\le j} \binom{j}{\nu}\xi^{j-\nu}(ky)^\nu$ with $\binom{j}{\nu} = \binom{j_1}{\nu_1}\cdots\binom{j_n}{\nu_n}$ yield the equation

$$M_\epsilon^j(f) = \frac{1}{d_{n,\ell}(\alpha)} \int_{|y|>\epsilon} \frac{P_{|j|}(y)dy}{|y|^{n+\alpha}} ,$$

where $P_{|j|}(y) = \sum_{0\le\nu\le j} \binom{j}{\nu} f_{j-\nu} A_\ell(|\nu|) y^\nu$ with $f_j = \int_{R^n} f(\xi)\xi^j d\xi$. But $A_\ell(|\nu|) = 0$ for all $|\nu| < \ell$, as is known, see Samko, Kilbas and Marichev [1], Lemma 26.1, which proves our lemma. ■

Corollary. *Let $f(x)$ satisfy the assumptions of Lemma 2.3 and let* $\lim_{\epsilon\to 0}(\mathbb{D}_\epsilon^\alpha f)(x) \in L_1(R^n; |x|^{|j|})$. *Then*

$$\int_{R^n} x^j (\mathbb{D}_\epsilon^\alpha f)(x)dx = 0 \ , 0 \le |j| < \alpha \ .$$

The space of Riesz potentials

$$I^\alpha(L_p) \ = \ \{f : f = I^\alpha\varphi, \ \varphi \in L_p(R^n)\} \tag{2.6}$$

is well studied, see Samko [1]-[2], [4] or Samko, Kilbas and Marichev [1], Theorem 26.8, and characterized in terms of convergence of hypersingular integrals in the case $p > 1$. The following is some counterpart of Theorem 26.8 from Samko, Kilbas and Marichev [1] for $p = 1$.

Lemma 2.4. *1) Let $0 < \alpha < n$ or $0 < \Re\alpha < 2$. Then*

$$I^\alpha(L_1) = \{f(x) : \ f(x) \in L_1 + L_s, \mathbb{D}^\alpha f \in L_1\} \ , \quad s > \frac{n}{n - \Re\alpha}.$$

2) Let $\Re\alpha > 0$. The conditions

$$f(x) \in L_1(R^n) \quad \textit{and} \quad |x|^\alpha \widehat{f}(x) \in \mathcal{W}_0(R^n)$$

are sufficient for $f(x)$ to be in $I^\alpha(L_1)$. If $0 < \alpha < \infty$ or $0 < \Re\alpha < 2$, then these conditions guarantee the convergence of $\mathbb{D}_\epsilon^\alpha f$ in the norm of the space $L_1(R^n)$.

Proof. 1) The imbedding

$$I^\alpha(L_1) \subseteq \{f(x) : \ f(x) \in L_1 + L_s, \mathbb{D}^\alpha f \in L_1\} \ , \quad s > \frac{n}{n - \Re\alpha} \ ,$$

is a consequence of the "only if" part of Theorem 26.8 from Samko, Kilbas and Marichev [1] for $p = 1$ (we observe that the "only if" part is valid for $p = 1$ in that theorem) and of the simple property $I^\alpha : L_1 \to L_1 + L_s, s > \frac{n}{n-\Re\alpha}$. The latter is easily obtained by splitting the Riesz kernel $k_\alpha(x)$ to its restrictions to $|x| < 1$ and $|x| > 1$.

To show the inverse imbedding, we notice that

$$(I^\alpha\mathbb{D}^\alpha f, \varphi) = (f, \varphi) \ , \quad \varphi \in \Phi \ ,$$

if $f \in L_p$ and $\mathbb{D}_\epsilon^\alpha$ converges in $L_p, p \ge 1$, Φ being the Lizorkin test function space, see [Samko, Kilbas and Marichev [1], which can be verified directly. Then f and $I^\alpha\mathbb{D}^\alpha$ may differ only by a polynomial, as is known, see for example, Samko, Kilbas and Marichev [1], Subsection 25.1. But both f and $I^\alpha \mathbb{D}^\alpha f$ are in $L_1 + L_s$, so that they cannot "contain" a polynomial. Therefore, $f \in I^\alpha(L_1)$.

2) Let $\varphi = F^{-1}|x|^\alpha \widehat{f}(x)$, which is in $L_1(R^n)$, by the assumption. For any $\omega(x) \in \Phi$ we have

$$(I^\alpha \varphi, \omega) = (\varphi, I^\alpha \omega) = \left(F^{-1}\varphi, \frac{1}{|x|^\alpha}\widehat{\omega}(x)\right) = \left(F^{-2}|x|^\alpha F f, \frac{1}{|x|^\alpha}\widehat{\omega}(x)\right)$$

$$= \frac{1}{(2\pi)^n}\left(|x|^\alpha \widehat{f}(-x), \frac{1}{|x|^\alpha}\widehat{\omega}(x)\right) = \left(F^{-1}f, F\omega\right) = (f, \omega).$$

Hence $I^\alpha \varphi$ coincides with f as an element of the space Φ' . When $\alpha \geq n$, this already means that $f \in I^\alpha(L_1)$ in accordance with the definition of the space $I^\alpha(L_p)$ when $\alpha \geq \frac{n}{p}$. If $0 < \alpha < n$, to show that f and $I^\alpha \varphi$ coincide as functions, it remains to see that they both "do not contain" a polynomial. For f this is clear, since $f \in L_1(R^n)$, while for $I^\alpha \varphi$ with $\varphi \in L_1$ we should refer to the weak-type estimate

$$|x : |(I^\alpha \varphi)(x)| > t| \leq c\left(\frac{c}{t}\|\varphi\|_1\right)^{\frac{n}{n-\Re\alpha}} \quad ,$$

known for the Riesz potentials, see Zygmund [1].

The inclusion $f \in I^\alpha(L_1)$ having been obtained, the convergence of $\mathbb{D}^\alpha_\epsilon f$ in L_1 is then a consequence of the "only if" part of Theorem 26.8 from Samko, Kilbas and Marichev [1]. ■

Corollary. *The intersection*

$$\{f(x) : \ f(x) \in L_1 + L_s,\} \bigcap \{f(x) : \ \mathbb{D}^\alpha f \quad converges \quad in \quad L_1\}$$

does not depend on $s > \frac{n}{n-\Re\alpha}(0 < \alpha < n \ or \ 0 < \Re\alpha < 2)$.

III. Ideas leading from hypersingular constructions to the method of AIO

Keeping in mind that the hypersingular integral of the form (1.6) is interpreted as the limit of the truncated integrals as in (1.3)-(1.4), we rewrite (1.6) in the form

$$\mathbb{D}^\alpha_\Omega f = \lim_{\epsilon\to 0}\int_{R^n} \chi_\epsilon(y)\Omega(y)\frac{(\Delta^\ell_y f)(x)}{|y|^{n+\alpha}}dy \ , \tag{3.1}$$

where $\chi_\epsilon(y) = \chi_{R^n\setminus B(0,\epsilon)}(y)$ is the characteristic function of the exterior of the ball $B(0,\epsilon) = \{y \in R^n : |y| < \epsilon\}$. The first question arising is whether it is possible to use any other truncation different from the spherical one. That is, may one take

$$\chi_\epsilon(y) = \chi_{R^n\setminus G_\epsilon}(y) \ ,$$

where G_ϵ is an arbitrary small neighbourhood of the origin, which tends to it when $\epsilon \to 0$? The question of equivalence of this approach to the case of the spherical truncation is not trivial since we deal with a non-absolute convergence of the integrals. In Emgusheva and Nogin [1]-[2],[5] it was shown that in the case $\Omega(y) \equiv 1$ the convergence in (3.1) does not depend on the choice of the sets $G_\epsilon \ni 0$ under the only assumption that

$$\lim_{\epsilon \to 0} |G_\epsilon \bigcap K| = 0$$

for any compact set $K \subset R^n$ (within the framework of the spaces $L^\alpha_{p,r}(R^n)$, see Samko [2],[4] or Samko, Kilbas and Marichev [1] for these spaces).

We may go further and, instead of the truncation by means of the characteristic function $\chi_\epsilon(y)$, take, for example, a smooth truncation" $\mu\left(\frac{|y|}{\epsilon}\right)$, where $\mu(r) \in C^\infty(R^1_+), \mu(r) \equiv 0$ near the origin and $\mu(r) \equiv 1$ for $r \geq 2$ - see Alisultanova and Nogin [1]-[3], where not only smooth such variable truncations were dealt with in the framework of the spaces $L^\alpha_{p,r}(R^n)$.

On the other hand, the hypersingular integral (1.6), in case of homogeneous characteristics $\Omega(y) = \Omega\left(\frac{y}{|y|}\right)$ may be represented in terms not involving finite differences:

$$\mathbb{D}^\alpha_\Omega f = \lim_{\epsilon \to 0} \int_{R^n} \chi_\epsilon(y) \frac{\Omega(y) f(x-y)}{|y|^{n+\alpha}} dy = \lim_{\epsilon \to 0} \frac{1}{\epsilon^{n+\alpha}} \int_{R^n} m_\alpha\left(\frac{y}{\epsilon}\right) f(x-y) dy \ , \tag{3.2}$$

where $m_\alpha(y) = \Omega(y)|y|^{-n-\alpha}$ when $|y| > 1$ and $m_\alpha(y) = 0$ when $|y| < 1$, if $\Omega(y)$ satisfies the conditions

$$\int_{S^{n-1}} \sigma^j \Omega(\sigma) d\sigma = 0 \ , \quad |j| \leq [\alpha]$$

(compare this with the equalities (3.4) and (4.4) below). See a study of hypersingular integrals in the form (3.2) under the latter condition on $\Omega(\sigma)$ in Nogin and Samko [1] and [2].

The next natural step is to consider modifications of the hypersingular integral in the form

$$\lim_{\epsilon \to 0} \int_{R^n} q_\alpha(y, \epsilon) f(x-y) dy \ , \tag{3.3}$$

where the kernel $q_\alpha(y,\epsilon)$ has a singularity of the type $\frac{c}{|y|^{n+\alpha}}$ typical for hypersingular integrals when $\epsilon = 0$, but is "nice" when $\epsilon > 0$. Thus, we avoid usage of finite differences, but already do not try to represent the limit (3.3) as an integral which converges even if non-absolutely. Naturally, there may be a large choice for the kernels $q_\alpha(y,\epsilon)$ and under different choices of these kernels we may get these or those hypersingular integrals. We shall consider the constructions (3.3) in this Section, the main strategy being the realization of the operators inverse to potential type operators, in the form (3.3).

Since the Riesz potential operator is a convolution with a homogeneous kernel, it is natural to look for an approximative inverse operator not in the general form (3.3), but in the form

$$(I^\alpha)^{-1} f = \lim_{\epsilon \to 0} \frac{1}{\epsilon^{n+\alpha}} \int_{R^n} q_\alpha\left(\frac{y}{\epsilon}\right) f(x-y) dy = \lim_{\epsilon \to 0} \frac{1}{\epsilon^\alpha} \int_{R^n} q_\alpha(y) f(x - \epsilon y) dy \ . \tag{3.4}$$

as in (3.2).

IV. General requirements to the kernel $q_\alpha(y)$

While choosing the kernel $q_\alpha(y)$, we wish to have it sufficiently nice, at least integrable over R^n:

$$q_\alpha(x) \in L_1(R^n) \tag{4.1}$$

and, at the same time, such that the limit in (3.4) gives the real inverse of I^α. We can rewrite formally the relation (3.4) in Fourier transforms as

$$F\left[(I^\alpha)^{-1}f\right] = \lim_{\epsilon\to 0}\frac{1}{\epsilon^\alpha}\widehat{q}_\alpha(\epsilon\xi)\widehat{f}(\xi) \ .$$

Therefore we may try to look for $q_\alpha(x)$ via the relation

$$\frac{1}{\epsilon^\alpha}\widehat{q}_\alpha(\epsilon\xi) \ \to \ |\xi|^\alpha$$

as $\epsilon \to 0$, by (2.2). This is equivalent, in a sense, to writing

$$\widehat{q}_\alpha(\xi) = |\xi|^\alpha \mathcal{K}(\xi) \ , \tag{4.2}$$

where $\lim_{\xi\to 0}\mathcal{K}(\xi) = 1$. In other words, we can take an arbitrary "nice" function $\mathcal{K}(\xi)$ which vanishes at infinity rapidly enough and has $\mathcal{K}(0) = 1$, and get the kernel $q_\alpha(x) \in L_1(R^n)$. But we should keep in mind that we are interested in obtaining $q_\alpha(x)$ in an explicit and constructive form, preferably as an elementary or well-known special function.

Lemma 4.1. *Let*

$$y^j q_\alpha(y) \in L_1(R^n), 0 \le |j| < \Re\alpha \ . \tag{4.3}$$

Then the condition

$$\int_{R^n} y^j q_\alpha(y)dy = 0 \ , \qquad 0 \le |j| < \Re\alpha \ , \tag{4.4}$$

is necessary and sufficient for existence of the limit (3.4) on nice functions f ($\in \mathcal{S}$, say).

Proof. It suffices just to use the Cauchy-L'Hôspital rule in the second equation in (3.4). ■

As we know, $F(\mathbb{D}^\alpha f) = |x|^\alpha \widehat{f}(x)$,

Passing to Fourier pre-images in (4.2), again formally, in view of (2.2) we can write

$$q_\alpha(x) = \mathbb{D}^\alpha k(x) \ , \tag{4.5}$$

where $k(x) = (F^{-1}\mathcal{K})(x)$, and

$$\int_{R^n} k(x)dx = 1 \ . \tag{4.6}$$

Hence we arrive at the following conclusion.

Conclusion 4.2. *The limit (3.4) coincides with $\mathbb{D}^\alpha f$, which is the real inverse to $I^\alpha\varphi$, only if the kernel $q_\alpha(x)$ has the form (4.5) with $k(x)$ satisfying the condition (4.6). In other words, the kernel $q_\alpha(x)$ which we may take in (3.4), should be the Riesz fractional derivative of an identity approximation kernel.*

We expect also that under a concrete choice of $q_\alpha(x)$ we shall encounter the property

$$q_\alpha(x) \sim \frac{c}{|x|^{n+\alpha}} \ , \qquad c = \frac{1}{\gamma_n(-\alpha)} = -\frac{\sin\frac{\alpha\pi}{2}}{\beta_n(\alpha)} \tag{4.7}$$

as $|x| \to \infty$. Indeed, by (4.2),

$$q_\alpha(x) = \frac{1}{(2\pi)^n |x|^{n+\alpha}} \int_{R^n} e^{-i\frac{x}{|x|}y} |y|^\alpha \mathcal{K}\left(\frac{y}{|x|}\right) dy \tag{4.8}$$

and, since $\mathcal{K}(0) = 1$, this formally yields (4.7) by (2.1).

Let us note that the conditions (4.3) used above, are in evident agreement with (4.7).

The following lemma shows that the kernels of the form (4.5), in general, automatically posess the property (4.4).

Lemma 4.3. *Let $k(x) \in L_1(R^n)$ and $x^j k(x) \in L_1(R^n)$ and let $k(x)$ have the Riesz derivative $\mathbb{D}^\alpha k(x) = \lim\limits_{\substack{\epsilon\to 0\\ (L_1)}} \mathbb{D}^\alpha_\epsilon k(x)$. If, besides this, $q_\alpha(x) = \mathbb{D}^\alpha k(x)$ itself satisfies the condition (4.3), then the equation (4.4) is satisfied.*

Proof. It suffices to refer to Corollary of Lemma 2.3 $\blacksquare$.

For further goals we find it convenient to introduce the following

Definition 4.4. *The identity approximation kernel $k(x)$ is called* **admissible** *for the inversion of the Riesz potential operator I^α, if*

$$k(x) \in L_1(R^n) \bigcap I^\alpha(L_1) \ . \tag{4.9}$$

V. The construction (3.4) as the inverse operator to I^α on $I^\alpha(L_p)$

First of all we show that the construction (3.4) converges on nice functions.

Theorem 5.1. *Let $k(x) \in L_1(R^n)$ be an arbitrary identity approximation kernel admissible in the sense of Definition 4.4. Then*

$$\lim_{\substack{\epsilon\to 0\\ (BC(R^n))}} \frac{1}{\epsilon^{n+\alpha}} \int_{R^n} q_\alpha\left(\frac{y}{\epsilon}\right) f(x-y)dy = \mathbb{D}^\alpha f \tag{5.1}$$

for any $f \in W^m_\infty(R^n), m > \alpha$.

Proof. Firstly, we observe that $\mathbb{D}^\alpha f$ exists for $f \in W^m_\infty(R^n), m > \alpha$, and is in $BC(R^n)$, by the properties of hypersingular integrals, see Samko [4] or Samko, Kilbas and Marichev, Subsection 26.2. We have

$$\frac{1}{\epsilon^{n+\alpha}} \int_{R^n} q_\alpha\left(\frac{y}{\epsilon}\right) f(x-y)dy = \int_{R^n} k(y)(\mathbb{D}^\alpha f)(x-\epsilon y)dy \ \underset{(BC(R^n))}{\longrightarrow} \ (\mathbb{D}^\alpha f)(x) \ , \tag{5.2}$$

where we have used Lemma 2.2 and the fact that $k(x)$ is the identity approximation kernel. ■

Theorem 5.2. *Let $k(x)$ be any admissible identity approximation kernel and let $f(x) = I^\alpha\varphi,\ \varphi \in L_p(R^n), 1 < p < \frac{n}{\Re\alpha}$. Then*

$$\lim_{\substack{\epsilon\to 0\\(L_p)}} \frac{1}{\epsilon^{n+\alpha}} \int_{R^n} q_\alpha\left(\frac{y}{\epsilon}\right) f(x-y)dy = \varphi(x), \tag{5.3}$$

where $q_\alpha(x)$ is the function (4.5). If, also $q_\alpha(x)(1+|x|)^\delta \in L_1(R^n)$, for some $\delta > 0$, then (5.3) holds in the case $p = 1$ as well. If $k(x)$ has a radial decreasing majorant in $L_1(R^n)$, then the almost everywhere limit may be also taken in (5.3).

Proof. Let $\varphi \in \mathcal{S}$ first. Then we have the equality written in (5.2), that is ,

$$\frac{1}{\epsilon^{n+\alpha}} \int_{R^n} q_\alpha\left(\frac{y}{\epsilon}\right) f(x-y)dy = \int_{R^n} k(y)\varphi(x-\epsilon y)dy\ , \tag{5.4}$$

which can be also easily verified via Fourier transforms since $\varphi \in \mathcal{S}$. We wish to extend this relation to the case of functions $\varphi \in L_p$, seeing that $\mathcal{S}$ is dense in $L_p(R^n)$. We may take $\epsilon = 1$ and observe that (5.4) is nothing else but

$$q_\alpha * I^\alpha\varphi = I^\alpha q_\alpha * \varphi\ . \tag{5.5}$$

It remains to show that both the left- and right-hand side of (5.5) are operators bounded, with respect to φ, from L_p into $L_q, \frac{1}{q} = \frac{1}{p} - \frac{\Re\alpha}{n}, p > 1$. The case $p = 1$ is to be considered separately. The left-hand side of (5.5) is obviously $(L_p \to L_q)$-bounded since $q_\alpha \in L_1$. For the right-hand side of (5.5) we have

$$I^\alpha q_\alpha * \varphi = I^\alpha(q_\alpha * \varphi)\ , \tag{5.6}$$

which is obvious on nice functions and is extended to $\varphi \in L_p$ since both q_α and $I^\alpha q_\alpha$ are in L_1. Therefore, by (5.6), the right-hand side of (5.5) is $(L_p \to L_q)$-bounded as well.

Let $p = 1$. By (5.6) and (2.5), the right-hand side of (5.5) is bounded from $L_1(R^n) \to L_{q,\lambda} := L_q(R^n;(1+|x|)^{-\lambda}), 1 \le q < \frac{n}{n-\Re\alpha}, \lambda > n-(n-\Re\alpha)q$. As for the left-hand side, it is not hard to check, using the Minkowsky inequality, that the assumption $q_\alpha(x)(1+|x|)^\delta \in L_1(R^n)$ guarantees existence of $q \in \left[1, \frac{n}{n-\Re\alpha}\right)$ and $\lambda > n-(n-\Re\alpha)q$ such that $q_\alpha * I^\alpha\varphi$ is $(L_1 \to L_{q\lambda})$-bounded $\big($taking $\delta \le \Re\alpha$, one may choose $q = \frac{n}{n-\alpha+\delta}\big)$.

The above boundednesses enable us to have the equality (5.4) for all $\varphi \in L_p(R^n),\ 1 \le p < \frac{n}{\Re\alpha}$. Having (5.4) for $\varphi \in L_p$, it suffices to use the fact that $k(x)$ is an identity approximation kernel with the reference to the known properties of such kernels, see Stein [1], Chapter III, Subsection 2.2, Theorem 2. ■

VI. Approximative inverse operator under the choice $k(x) = P(x,1)$

We begin with the choice $q_\alpha(x) = \mathbb{D}^\alpha k$ with $k(x) = P(x,1)$, where $P(x,t)$ is the Poisson kernel

$$P(x,t) = \frac{c_n t}{\left(|x|^2 + t^2\right)^{\frac{n+1}{2}}} \ , \quad c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \ , \tag{6.1}$$

so that $\mathcal{K}(\xi) = e^{-|\xi|}$ in (4.2). The main reason for this choice is just the fact that $P(x,1)$ is a famous identity approximation kernel. This choice for the application of the method of AIO was made in Zavolzhenskii and Nogin [2], but for reader's convenience we present here the main result from that paper with proofs, which are somewhat simplified. Under this choice the kernel $q_\alpha(x)$ proves to be a special function, the Gauss hypergeometric function (which can be expressed in terms of elementary functions in case of odd n). But below in Section 7 we show that it is possible to make a choice which is opposite in a sense: we may choose $\mathcal{K}(x)$ not an elementary function, but such that $q_\alpha(x)$ proves to be a very nice elementary function. In other words, as was already noted in Introduction, we may achieve simplicity of the inversion not in the Fourier transforms, but directly, not caring about the picture we can have in Fourier transforms.

We start from the consideration of properties of the function $q_\alpha(x) = \mathbb{D}^\alpha P(x,1)$.

Lemma 6.1. *The formula is valid*

$$\mathbb{D}^\alpha\Big(P(\cdot,1)\Big) = F^{-1}\left(|\xi|^\alpha e^{-|\xi|}\right) = \frac{\Gamma(n+\alpha)}{2^{n-1}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} F\left(\frac{n+\alpha}{2}, \frac{n+\alpha+1}{2}; \frac{n}{2}; -|x|^2\right) \ , \tag{6.2}$$

where $\Re\alpha \geq 0$ and $F(a,b;c;z)$ is the Gauss hypergeometric function.

Proof. The first equality in (6.2) is a consequence of (2.2) and the fact that $\widehat{P}(\cdot,t) = e^{-t|\xi|^2}$. To get the second one, we use the Bochner formula (2.3) and obtain

$$F^{-1}\left(|\xi|^\alpha e^{-|\xi|}\right) = \frac{1}{(2\pi)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \rho^{\frac{n}{2}+\alpha} e^{-\rho} J_{\frac{n}{2}-1}(\rho|x|) d\rho \ .$$

It remains to apply the formula N 6.621 in Gradshtein and Ryzhik [1] to the integral in the right-hand side. ■

Lemma 6.2. *The function*

$$q_\alpha(x) = \frac{\Gamma(n+\alpha)}{2^{n-1}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} F\left(\frac{n+\alpha}{2}, \frac{n+\alpha+1}{2}; \frac{n}{2}; -|x|^2\right) \ , \quad \Re\alpha > 0 \ , \tag{6.3}$$

is in $L_1(R^n)\bigcap C_0(R^n)$ and

$$|q_\alpha(x)| \leq A(1+|x|)^{-n-\Re\alpha} \ . \tag{6.4}$$

Proof. We have

$$|\xi|^\alpha e^{-|\xi|} \in \mathcal{W}_0(R^n) \tag{6.5}$$

by Lemma 2.1, if $\Re\alpha > 0$. Hence $q_\alpha(x) \in L_1(R^n)$ as the inverse Fourier transform of this function. Since also $|\xi|^\alpha e^{-|\xi|} \in L_1(R^n)$, we have $q_\alpha(x) \in \mathcal{W}_0(R^n) \subset C_0(R^n)$.

From the transformation formula

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-1)^a z^{-a} F\left(a, a+1-c; a+1-b; \frac{1}{z}\right)$$
$$+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-1)^b z^{-b} F\left(b, b+1-c; b+1-a; \frac{1}{z}\right)$$

for the Gauss function (Gradshtein and Ryzhik [1], N 9.132), the estimate (6.4) is easily derived. ■

Below in this Section we write $q_\alpha(|x|)$ instead of $q_\alpha(x)$ without danger of confusion.

Remark 6.3. *In case of the space of odd dimension, $n = 2k+1, k \in \mathbb{N}_0$, the function $q_\alpha(r)$ is an elementary function:*

$$q_\alpha(r) = \frac{\Gamma(1+\alpha)}{2\pi^{\frac{n+1}{2}}} \frac{d^k}{dz^k} \left[(1+\sqrt{z})^{-\alpha-1} + (1-\sqrt{z})^{-\alpha-1}\right]\Big|_{\sqrt{z}=ir} \tag{6.6}$$

In particular, in the cases $n = 1$ and $n = 3$, we have

$$q_\alpha(r) = \frac{\Gamma(1+\alpha)}{2\pi}\left[(1+ir)^{-\alpha-1} + (1-ir)^{-\alpha-1}\right] = \frac{\Gamma(1+\alpha)\cos\left[(1+\alpha)arctg\ r\right]}{\pi(1+r^2)^{\frac{1+\alpha}{2}}}$$

and

$$q_\alpha(r) = -i\frac{\Gamma(2+\alpha)}{8\pi^2 r}\left[(1+ir)^{-\alpha-2} - (1-ir)^{-\alpha-2}\right] = -\frac{\Gamma(2+\alpha)\sin\left[(2+\alpha)arctg\ r\right]}{4\pi^2 r(1+r^2)^{\frac{2+\alpha}{2}}},$$

respectively.

Indeed, (6.6) follows immediately from the formula (7), page 71 (Russian edition) in Erdelyi *et al* [1] and the equality 15.19 in Abramowitz and Stegun [1].

Theorem 6.4. *Let $1 \le p < \frac{n}{\Re\alpha}$. The inversion (5.3) of the Riesz potential operator holds under the choice (6.3) of $q_\alpha(x)$.*

Proof. We have $q_\alpha(x) = \mathbb{D}^\alpha k, k(x) = P(x,1)$ and by Theorem 5.2, it suffices to show that the kernel $k(x)$ is admissible, that is, satisfies the condition (4.9). Evidently, $P(x,1) \in L_1(R^n)$. Also, $P(x,1) \in I^\alpha(L_1)$, by Theorem 2.4, the condition $|x|^\alpha \widehat{k}(x) \in \mathcal{W}_0$ of Theorem 2.4 being fullfilled by (6.5). It remains to note that the condition $P(x,1)(1+|x|)^\delta \in L_1(R^n)$ of Theorem 5.2 is also satisfied

Corollary. *The Laplace operator may be approximated by integral operators in the form*

$$-\Delta f = \frac{(n+1)!}{2^n \pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\int_{R^n} v(y) f(x-\epsilon y)dy\ , \tag{6.7}$$

where $v(y) = \frac{1}{(1+|x|^2)^{\frac{n+3}{2}}}\left(3 - \frac{n+3}{1+|x|^2}\right)$.

Indeed, for $\alpha = 2$ from (6.3) we get $q_2(x) = \frac{(n+1)!}{2^n\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} v(x)$ by means of the Gauss recursion formula 9.137.17 from Gradshtein and Ryzhik [1] for hypergeometric functions and the relation $F(a,b;b;z) = (1-z)^{-a}$.

Remark 6.5. *Taking $k(x) = W(x,1)$, where $W(x,t)$ is the Gauss-Weierstrass kernel, instead of the Poisson kernel, we obtain the corresponding kernel $q_\alpha(x)$ in the form*

$$q_\alpha(x) = \frac{\Gamma\left(\frac{n+\alpha}{2}\right)}{2^n \pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} \ {}_1F_1\left(\frac{n+\alpha}{2}; \frac{n}{2}; -\frac{|x|^2}{4}\right) \ , \tag{6.8}$$

where ${}_1F_1(a;b;z) = \sum_{k=0} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}$ is the confluent hypergeometric function.

Indeed, (6.8) may be obtained by means of the formula N 6.631.1 in Gradshtein and Ryzhik [1].

VII. Approximative inverse operator under the direct choice of $q_\alpha(x)$

Now we pass to the idea mentioned in Introduction and at the beginning of Section 6. We look for the answer to the question: can we choose by ourselves some simple elementary function $q_\alpha(x)$ which fits the inversion (5.3), not caring about how complicated might be its Fourier transform. (Before, on the contrary, we wrote a prescribed simple Fourier transform of $q_\alpha(x)$ and then calculated $q_\alpha(x)$ itself). The direct search of $q_\alpha(x)$ is restricted by the conditions (4.4) and (4.7), which are necessary in a sense. An immediate idea to satisfy both these conditions in a very simple manner, at least, in the case $0 < \Re\alpha < 2$, is to take

$$q_\alpha(x) = \frac{1}{\gamma_n(-\alpha)} \left[\frac{1}{(1+|x|^2)^{\frac{n+\alpha}{2}}} - \frac{\lambda}{(1+|x|^2)^{\frac{n+\alpha}{2}+1}}\right] \ , \tag{7.1}$$

which surely satisfies the requirement (4.7) and we certainly can determine λ in such a fashion that (4.4) is satisfied for $j = 0$, the latter being sufficient in the case $0 < \Re\alpha < 2$, since $q_\alpha(x)$ is radial. Direct easy calculations provide

$$\lambda = \frac{n+\alpha}{\alpha} \ . \tag{7.2}$$

And we shall show that this choice (7.1)-(7.2) does work !

In the case $\Re\alpha \geq 2$ we may proceed in a similar way and consider the linear combination

$$q_\alpha(x) = \frac{1}{\gamma_n(-\alpha)} \sum_{k=0}^{m} \frac{\lambda_k}{(1+|x|^2)^{\frac{n+\alpha}{2}+k}} \tag{7.3}$$

instead of (7.1). The choice of the λ_k will be dictated by the conditions (4.4) and (4.7).

Naturally, just to find coefficients is not enough. In accordance with Theorem 5.2, we should prove that under our choice of $q_\alpha(x)$ the condition $k(x) = I^\alpha q_\alpha \in L_1(R^n)$ is satisfied, which will be the main point.

It is known that

$$F\left(\frac{1}{(1+|x|^2)^{\frac{\beta}{2}}}\right) = (2\pi)^n G_\alpha(\xi) \ , \tag{7.4}$$

where $G_\alpha(x)$ is the Bessel kernel,

$$G_\alpha(x) \ = \ \frac{2^{1-\frac{\alpha+n}{2}}}{\pi^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})}\frac{K_{\frac{n-\alpha}{2}}(|x|)}{|x|^{\frac{n-\alpha}{2}}} \ . \tag{7.5}$$

So, the study of the function (7.3) will be closely connected with properties of this kernel.

The following lemma is crucial for our goals. It gives some remarkable recursion relation for the Bessel kernels $G_\alpha(x)$.

Lemma 7.1. *Let $-n < \Re\alpha < n+2m$. Then*

$$\sum_{k=0}^{m}(-1)^k a_{m,k}(\alpha)G_{n+\alpha+2k}(\xi) = (-1)^m a_m(\alpha)|\xi|^\alpha G_{n+2m-\alpha(\xi)} \ , \tag{7.6}$$

where

$$a_{m,k}(\alpha) = \alpha(\alpha-2)\cdots(\alpha-2m+2k+2)(n+\alpha(n+\alpha+2)\cdots(n+\alpha+2k-2)\binom{m}{k} =$$

$$= 2^m\binom{m}{k}\frac{\Gamma\left(\frac{\alpha}{2}+1\right)\Gamma\left(\frac{n+\alpha}{2}+k\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}-m+k+1\right)} \tag{7.7}$$

and

$$a_m(\alpha) = 2^{m-\alpha}\frac{\Gamma\left(\frac{n-\alpha}{2}+m\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)} \ .$$

Its particular cases for $m=1$ and $m=2$ are, respectively,

$$\alpha G_{n+\alpha}(\xi) - (n+\alpha)G_{n+\alpha+2}(\xi) = -\frac{2^{1-\alpha}\Gamma\left(\frac{n-\alpha}{2}+1\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)}|\xi|^\alpha G_{n+2-\alpha}(\xi) \tag{7.8}$$

and

$$\alpha(\alpha-2)G_{n+\alpha}(\xi) - 2\alpha(n+\alpha)G_{n+\alpha+2}(\xi) + (n+\alpha)(n+\alpha+2)G_{n+\alpha+4}(\xi)$$
$$= \frac{2^{2-\alpha}\Gamma\left(\frac{n-\alpha}{2}+2\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)}|\xi|^\alpha G_{n+4-\alpha}(\xi) \tag{7.9}$$

Proof. Let us agree to write $G_{n+\alpha}(|\xi|) = G_{n+\alpha}(r)$ instead of $G_{n+\alpha}(\xi)$ By (7.5) we have

$$G_{n+\alpha}(r) = \frac{2^{1-n-\frac{\alpha}{2}}}{\pi^{\frac{n}{2}}\Gamma\left(\frac{n+\alpha}{2}\right)}r^{\frac{\alpha}{2}}K_{\frac{\alpha}{2}}(r) \ . \tag{7.10}$$

For the McDonald function $K_\nu(r)$ the recurrence relation

$$r\Big[K_{\nu+1}(r) - K_{\nu-1}(r)\Big] = 2\nu K_\nu(r) \tag{7.11}$$

is well known, see Gradshtein and Ryzhik [1], N 8.486.10. We wish to extend this to

$$r^m\Big[K_{\nu+m}(r)-K_{\nu-m}(r)\Big]=\sum_{j=0}^{m-1}(-1)^{m-j-1}b_m(j)r^jK_{\nu+j}(r) \tag{7.12}$$

with explicitely calculated coefficients $b_m(j)$. It proves to be that

$$b_m(j)=2^{m-j}\binom{m}{j}\nu(\nu-1)...(\nu+j-m+1)=\Gamma(\nu+1)\frac{2^{m-j}\binom{m}{j}}{\Gamma(\nu+j-m+1)}\ . \tag{7.13}$$

The relation (7.12)-(7.13), valid in the case $m=1$ by (7.11), will be proved by induction. We suppose that (7.12)-(7.13) is valid for some number m and all values of ν. Taking the left-hand side of (7.12) of order $m+1$, we represent it as

$$r^{m+1}\Big[K_{\nu+m+1}(r)-K_{\nu-m-1}(r)\Big]=r^m\cdot r\Big[K_{\nu+m+1}(r)-K_{\nu+m-1}(r)\Big]+r\cdot r^m\Big[K_{\nu-1+m}(r)-K_{\nu-1-m}(r)\Big] \tag{7.14}$$

Since (7.12)-(7.13) is assumed to be valid for all ν, we may use it in the second term in the right-hand side of (7.14) with ν replaced by $\nu-1$, while the first term may be treated by the formula (7.11), with ν replaced by $\nu+m$. As a result we obtain

$$r^{m+1}\Big[K_{\nu+m+1}(r)-K_{\nu-m-1}(r)\Big]=2r^m(\nu+m)K_{\nu+m}(r)+r\sum_{j=0}^{m-1}(-1)^{m-j-1}c_jr^jK_{\nu-1+j}(r)\ ,$$

where we have denoted

$$c_j=2^{m-j}\binom{m}{j}(\nu-1)(\nu-2)\cdots(\nu+j-m)=\frac{\nu+j-m}{\nu}b_m(j)$$

for brevity. Lifting the order of the McDonald functions in every term in the sum $\sum_{j=0}^{m-1}$, after easy calculations we arrive at the right-hand side of (7.12) exactly for the order $m+1$.

The formula (7.12) being proved, to get (7.6), it remains to choose $\nu=\frac{\alpha}{2}$ in (7.12) and calculate $r^jK_{\nu+j}(r)=r^{-\frac{\alpha}{2}}r^{\frac{\alpha}{2}+j}K_{\frac{\alpha}{2}+j}(r)$ in terms of $G_{n+\alpha+2j}$ according to (7.10); in the right-hand side of (7.12), we replace similarly $r^mK_{\nu-m}(r)=r^{\frac{\alpha}{2}}r^{m-\frac{\alpha}{2}}K_{m-\frac{\alpha}{2}}(r)$. After easy calculation of constants we arrive at (7.6). ■

Corollary. *Let $0<\Re\alpha<2m, m=1,2,...$ Then*

$$\mathbb{D}^{2m-\alpha}\left(\frac{1}{(1+|x|^2)^{\frac{n+\alpha}{2}}}\right)=\sum_{k=0}^{m}(-1)^{m-k}\frac{\nu_{m,k}(\alpha)}{(1+|x|^2)^{\frac{n-\alpha}{2}+m+k}}\ , \tag{7.15}$$

where $\nu_{m,k}(\alpha)=2^{2m-\alpha}\binom{m}{k}\frac{\Gamma\left(m-1-\frac{\alpha}{2}\right)\Gamma\left(\frac{n-\alpha}{2}+m+k\right)}{\Gamma\left(k-\frac{\alpha}{2}-1\right)\Gamma\left(\frac{n+\alpha}{2}\right)}$; in particular,

$$\mathbb{D}^{2-\alpha}\left(\frac{1}{(1+|x|^2)^{\frac{n+\alpha}{2}}}\right)=-\frac{2^{1-\alpha}\Gamma\left(\frac{n-\alpha}{2}+1\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)}\left[\frac{2-\alpha}{(1+|x|^2)^{\frac{n-\alpha}{2}+1}}-\frac{n+2-\alpha}{(1+|x|^2)^{\frac{n-\alpha}{2}+2}}\right]\ ,$$

$$0<\Re\alpha<2\ .$$

Indeed, when $\Re\alpha > 0$, obviously, $G_{n+\alpha}(\xi) \in \mathcal{W}_0(R^n)$. Passing to Fourier transforms in (7.6) according to (7.4) and changing α to $2m-\alpha$, we arrive at (7.15).

It is clear that Lemma 7.1 paves the way to construction of the kernel $q_\alpha(x)$ in the simple form (7.3).

Lemma 7.2. *Let $0 < \Re\alpha < 2m, m = 1, 2, 3, \ldots, \alpha \neq 2, 4, 6, \ldots$ Then the kernel*

$$q_\alpha(x) = \frac{1}{\gamma_n(-\alpha)} \left[\frac{1}{(1+|x|^2)^{\frac{n+\alpha}{2}}} - \sum_{k=1}^{m} \frac{(-1)^{k-1} c_{m,k}}{(1+|x|^2)^{\frac{n+\alpha}{2}+k}} \right] \tag{7.16}$$

with

$$c_{m,k} = \binom{m}{k} \frac{\left(\frac{n+1}{2}\right)_k}{\left(\frac{\alpha}{2}-m+1\right)_k} \tag{7.17}$$

is equal to $\mathbb{D}^\alpha k$, where $k(x)$ is an identity approximation kernel satisfying the admissibility condition (4.9).

Proof. The relation (7.6) prompts us to look for $q_\alpha(x)$ via Fourier transforms, since the right-hand side of (7.6) includes $|\xi|^\alpha = F^{-1}\mathbb{D}^\alpha F$. So we choose $\widehat{q}_\alpha(\xi)$ as

$$\widehat{q}_\alpha(\xi) = \lambda \sum_{k=0}^{m} (-1)^k a_{m,k}(\alpha) G_{n+\alpha+2k}(\xi) \ , \tag{7.18}$$

where λ is to be determined from the condition $|\xi|^{-\alpha}\widehat{q}_\alpha(\xi)\Big|_{\xi=0} = 1$, that is, $\lambda(-1)^m a_m(\alpha)$ $G_{n+2m-\alpha}(0) = 1$. It is known that

$$G_\alpha(0) = \frac{\Gamma\left(\frac{\alpha-n}{2}\right)}{2^n \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)} \ , \quad \Re\alpha > n \ .$$

Taking the value of $a_m(\alpha)$ from Lemma 7.1, we get

$$\lambda = (-1)^m \pi^{\frac{n}{2}} 2^{n+\alpha-m} \frac{\Gamma\left(\frac{n+\alpha}{2}\right)}{\Gamma\left(m-\frac{\alpha}{2}\right)} \ .$$

Then, passing to Fourier pre-images in (7.18), we arrive at

$$q_\alpha(x) = \frac{2^\alpha \Gamma\left(1+\frac{\alpha}{2}\right)}{\pi^{\frac{n}{2}} \Gamma\left(m-\frac{\alpha}{2}\right)} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \frac{\Gamma\left(\frac{n+\alpha}{2}+k\right)}{\Gamma\left(\frac{\alpha}{2}-m+k+1\right)} \frac{1}{(1+|x|^2)^{\frac{n+\alpha}{2}+k}} \ . \tag{7.19}$$

Easy calculations with the properties of the Gamma function taken into account transform this to (7.16). It remains to show that that the function $k(x) = I^\alpha q_\alpha$ satisfies the condition (4.9). Due to the way in which we constructed the function $q_\alpha(x)$ via (7.18), we have from (7.6):

$$\widehat{k}(\xi) = \lambda(-1)^m a_m(\alpha) G_{n+2m-\alpha}(\xi)$$

so that

$$k(x) = \frac{const}{(1+|x|^2)^{n+2m-\alpha}} \in L_1(R^n)$$

since $\Re\alpha < 2m$. To check that $k(x) \in I^\alpha(L_1)$, we have to verify that $|\xi|^\alpha \widehat{k}(\xi) \in \mathcal{W}_0(R^n)$ according to Lemma 2.4. This is satisfied since $|\xi|^\alpha \widehat{k}(\xi) = \sum_{k=0}^m c_k G_{n+\alpha+2k}(\xi) \in \mathcal{W}_0(R^n)$. ■

Theorem 7.3. *Let $0 < \Re\alpha < 2m,\ m = 1, 2, \ldots\ , \alpha \neq 2, 4, 6, \ldots$ Then the inversion of the Riesz potential operator $f = I^\alpha\varphi, \varphi \in L_p(R^n), 1 \leq p < \frac{n}{\Re\alpha}$, can be written in the form*

$$\varphi(x) = \frac{1}{\gamma_n(-\alpha)} \lim_{\varepsilon\to 0} \int_{R^n} \left[\frac{1}{(|y|^2+\varepsilon^2)^{\frac{n+\alpha}{2}}} - \varepsilon A(y,\varepsilon) \right] f(x-y)dy \ , \qquad (7.20)$$

where

$$A(y,\epsilon) = \sum_{k=1}^m (-1)^{k-1} \frac{c_{m,k}\epsilon^{k-1}}{(|y|^2+\epsilon^2)^{\frac{n+\alpha}{2}+k}}$$

with $c_{m,k}$ given in (7.17). The limit in (7.20) exists in the usual sense, if $f(x) \in W_\infty^N(R^n), N > \Re\alpha$, and in the sense of L_p-convergence or almost everywhere, if $f \in I^\alpha(L_p)$.

Proof. In view of Lemma 7.2, Theorem 7.3 follows immediately from Theorems 5.1 and 5.2.

Corollary. *Let $0 < \Re\alpha < 2$. The inversion of the Riesz potential operator $I^\alpha\varphi$ may be taken in the form*

$$\varphi(x) = \frac{1}{\gamma_n(-\alpha)} \lim_{\varepsilon\to 0} \int_{R^n} \left[\frac{1}{(|y|^2+\varepsilon^2)^{\frac{n+\alpha}{2}}} - \frac{n+1}{\alpha} \frac{\varepsilon}{(|y|^2+\varepsilon^2)^{\frac{n+\alpha}{2}+1}} \right] f(x-y)dy \ , \qquad (7.21)$$

Bibliography

Abramowitz, M. and Stegun, I.A. [1] (1972) *Handbook of Mathematical Funcions.* Dover Publications, New York, 830 p.

Alisultanova, E.D. and Nogin, V.A. [1] (1992) *Generalized hypersingular integrals and their applications to the inversion of operators of potential type* (Russian). Deponierted in VINITI, Moscow, no 2386-92, 64 p.

[2] (1993) Generalized hypersingular integrals and their application to inversion of potential type operators. *Izv. Vysch, Uchebn. Zaved., Matematika,* no 6, 65-68

[3] (1993) *Hypersingular integrals in the principal value sense* (Russian). Deponierted in VINITI, Moscow, no 2056-93, 23 p.

Emgusheva, G.P. and Nogin, V.A. [1] (1987) *On convergence of hypersingular integrals with a non-standard truncation* (Russian). Deponierted in VINITI, Moscow, no 3714-87, 39 p.

[2] (1988) Riesz derivatives with a non-standard truncation and their application to the inversion and characterization of potentials commuting with dilatations. *Dokl. Akad. Nauk SSSR*, **300**, no 2, 277-280

[3] (1991) On convergence in $L_p(R^n)$ of hypersingular integrals with a non-standard truncation. *Izv. Vysch, Uchebn. Zaved., Matematika,* no 7, 71-74

Erdélyi, A., Magnus, W., Overhettinger, F. and Tricomi, F.G. [1] (1953) *Higher Transcindental Functions.*. In 3 vols, Vol. 1. New York: McGraw-Hill Book Co., (Reprinted Krieger), Melbourne, Florida, 1981, 302 p.

Gradshtein, I.S. and Ryzhik, I.M. [1] (1994) *Tables of Integrals, Sums, Series and Products.* Fifth Edition, Academic Press, Inc., 1204 p.

Nogin, V.A. and Samko, S.G. [1] (1981) *On $L^p(R^n)$-convergence of hypersingular integrals with a homogeneous characteristic* (Russian). Deponierted in VINITI, Moscow, no 179-81, 47 p.

[2] (1982) Convergence in $Lp(R^n)$ of hypersingular integrals with a homogeneous characteristic (Russian), *Collection of papers "Differ. i integr. uravn. i ikh prilozh."*, Elista, Kalmyskii universitet, 119-131 p.

[3] (1998) Method of approximating inverse operators and its applications to inversion of potential type integral operators. *Integr. Transform. and Special Funct*, **6**, (to appear)

Nogin, V.A. and Sukhinin, E.V. [1] (1995) Fractional powers of the Klein-Gordon-Fock operator in L_p-spaces. *Dokl. Acad. Nauk*, **341**, no 2, 166-168

Samko, S.G. [1] (1976) Spaces of Riesz potentials (Russian). *Izv. Akad. Nauk SSSR, ser. Mat.*, **40**, no 5, 1143-1172 (Transl. in Math. USSR Izvestija, **10** (1976), no 5, 1089-1117)

[2] (1977) The spaces $L^{\alpha}_{p,r}(R^n)$ and hypersingular integrals (Russian). *Studia Math. (PRL)*, **61**, no 3, 193-230

[3] (1978) Hypersingular integrals with homogeneous characteristic (Russian). *Trudy Inst. Prikl. Mat. Tbil. Univ.*, **5-6**, 235-249

[4] (1984) *Hypersingular integrals and their applications.* (Russian). Rostov-on-Don, Izdad. Rostov Univ, 208 p.

[5] (1993) Inversion theorems for potential-type integral transforms in R^n and on S^{n-1}. *Integr. Transf. and Special Funct.*, **1**, no 2, 145-163.

Samko, S.G., Kilbas, A.A. and Marichev, O.I. [1] (1993) *Fractional Integrals and Derivatives. Theory and Applications.* "Gordon & Breach.Sci.Publ.", London-New-York (Russian edition - "Fractional Integrals and Derivatives an some of their Applications", "Nauka i Tekhnika, Minsk, 1987.), 1012 p.

Stein E.M.

[1] (1973) *Singular Integrals and Differentialiability Properties of functions.* (Russian). Moscow: Mir, (English ed. in Princeton Univ. Press, 1970), 342 p.

Zavolzhenskii, M.M. and Nogin, V.A. [1] (1991) On the inversion of some generalized Riesz potentials in non-elliptic case,. *Izv. Sev. Kavk. Nauchn. Tsentra Vyssh. Shkoly, Ser. estestv.nauk*, **4**, 56-61

[2] (1991) *On a certain method of inversion of potential-type operators* (Russian). Deponierted in VINITI, Moscow, no 978-91, 81 p.

[3] (1992) Approximating approach to inversion of the generalized Riesz potentials,. *Dokl. Acad. Nauk*, **324**, no 4, 738-741.

[4] (1992) *Approximative approach to the inversion of potential type operators with a smooth characteristic* (Russian). Deponierted in VINITI, Moscow, no 2150-92, 47 p.

Zygmund, A. [1] (1956) On a theorem of Marcinkiewicz concerning interpolation of operations. *J. math. pures et appl.*, **35**, no 3, 223-248