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Spectral Methods of Automorphic Forms

Problem Set 3 (due Nov 9)

3.1. a) Let T_n be the n-th Hecke operator. Show $T_nT_m = \sum_{d|(n,m)} T_{nm/d^2}$ and deduce by Möbius inversion $T_{nm} = \sum_{d|(n,m)} \mu(d)T_{m/d}T_{n/d}$.

b) As formal series, show

$$\sum_{k=0}^{\infty} \frac{T_{p^k}}{p^{ks}} = \left(1 - \frac{T_p}{p^s} + \frac{1}{p^{2s}}\right)^{-1}$$

and conclude

$$\sum_{n=1}^{\infty} \frac{T_n}{n^s} = \prod_p \left(1 - \frac{T_p}{p^s} + \frac{1}{p^{2s}}\right)^{-1}.$$

3.2. For $n \in \mathbb{N}$, $k \geq 4$, $z, w \in \mathbb{H}$ define

$$h_n(z,w) := \sum_{ad-bc=n} \frac{(\Im z \Im w)^{k/2}}{(czw + dw + az + b)^k} = \sum_{ad-bc=n} (\Im z \Im w)^k (cz+d)^{-k} \left(w + \frac{az+b}{cz+d}\right)^{-k}.$$

a) Show that $(\Im z \Im w)^{-k/2} h_n(z,w)$ is holomorphic in both z and w, vanishes at $(i\infty, w)$ and $(z, i\infty)$, and that $h_n(z,w)$ is in both variables invariant with respect to all $R_\gamma^{(k)}$, $\gamma \in SL_2(\mathbb{Z})$. Conclude that h_n is in both variables a cusp form for $SL_2(\mathbb{Z})$ of weight k and eigenvalue $k/2(1-k/2)$, in other words, h_n is in both variables a holomorphic cusp form of weight k for the full modular group.

b) Let F be any a holomorphic cusp form of weight k for the full modular group, equivalently $f = y^{k/2}F$ is a Maaß cusp form of weight k. Show that $\langle f, h_1(., \overline{-w})\rangle = c_k f(w)$ for some constant c_k. *Hint:* Unfold the fundamental domain getting

$$\langle f, h_1(., \overline{-w})\rangle = 2(\Im w)^{k/2} \int_{\mathbb{H}} (\overline{z} - w)^{-k} F(z) (\Im z)^{k-2} dz.$$

Calculate this integral by Cauchy's integral formula.

c) Conclude $T_n f = c_{k,n}\langle f, h_n(., \overline{-w})\rangle$ for some constant $c_{k,n}$. In other words, in this special case, we found a kernel for T_n.

3.3. Let X be the reflection conjugation operator: $(Xf)(z) := f(-\overline{z})$. X is $\mathbb{R}$-linear, but not $\mathbb{C}$-linear.

a) Show that X maps forms of weight $-k$ to forms of weight k forms, $X^2 = 1$, X is self-adjoint, $\Delta_k X = X\Delta_{-k}$, and X commutes with the Hecke operators. So for a weight 0 newform one can assume that it is an eigenform of X. Thus we call a (weight 0) newform f *even* if $Xf = f$, and *odd* if $Xf = -f$. What can you say about the positive and negative Fourier coefficients of odd and even newforms?

b) Have a look at Invent. Math. 149 (2002), p.509-511 to see what one does for general weight k.

3.4. Denote by $[a, b, c]$ the binary quadratic form $ax_1^2 + bx_1x_2 + cx_2^2 = \mathbf{x}^t A\mathbf{x}$ with $A = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$. Two forms with matrices A, A' are equivalent if $A' = \gamma^t A\gamma$ for some $\gamma \in SL_2(\mathbb{Z})$. The discriminant of $[a, b, c]$ is $D = b^2 - 4ac$. We call $\gamma \in SL_2(\mathbb{Z})$ and automorph of $[a, b, c]$ if $A = \gamma^t A\gamma$.

a) Show that for given $D \in \mathbb{Z}$ there are only finitely many classes of integral quadratic forms. (Show that each form is equivalent to a form with $|b| \leq |a| \leq |c|$)

b) Assume that D (if it is positive) is not a perfect square, and assume that $(a, b, c) = 1$ (i.e. the quadratic form is primitive). Let (t, u) be a solution of $t^2 - Du^2 = 4$. Show that $\begin{pmatrix} (t-bu)/2 & -cu \\ au & (t+bu)/2 \end{pmatrix}$ is an automorph of $[a, b, c]$; conversely, every automorph is of this form.

c) Let $\theta_{1,2} = (-b \pm \sqrt{D})/(2a)$. Show that the group of automorphs of the primitive form $[a, b, c]$ is the stabilizer of $\theta_{1,2}$.

(to be continued...)

3.5. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ be the Lie algebra of $SL(2, \mathbb{R})$ generated as a vector space by

$$R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These matrices generate the (associative) envelopping algebra $U(\mathfrak{g})$ (with mutliplication denoted by $\cdot$) as an $\mathbb{R}$-algebra. Let $-4\Delta = H^2 + 2R \cdot L + 2L \cdot R \in U(\mathfrak{g})$ be the Casimir element. Verify that Δ is in the centre of $U(\mathfrak{g})$. *Hint:* It is enough to show that Δ commutes with R, L, and H. Use the commutator relations $H \cdot R - R \cdot H = 2R, H \cdot L - L \cdot H = -2L, R \cdot L - L \cdot R = H$.