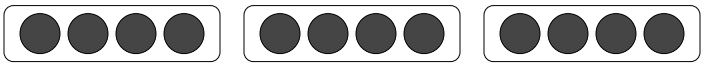
Is Multiplication Just Repeated Addition?
Insights from Japanese Mathematics Textbooks for Expanding the Multiplication Concept

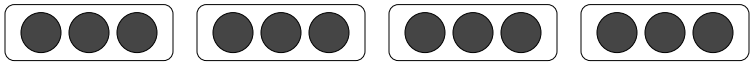
Makoto Yoshida, Ph.D.
William Paterson University
YoshidaM@wpunj.edu

2009 NCTM Annual Conference
April 23, 2009

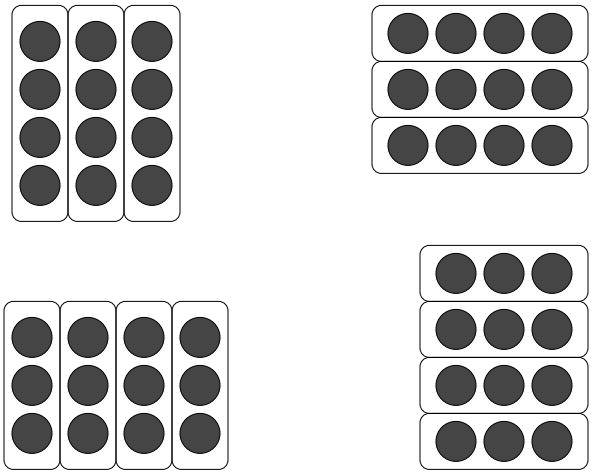
Can you draw a picture that represents 3x4?



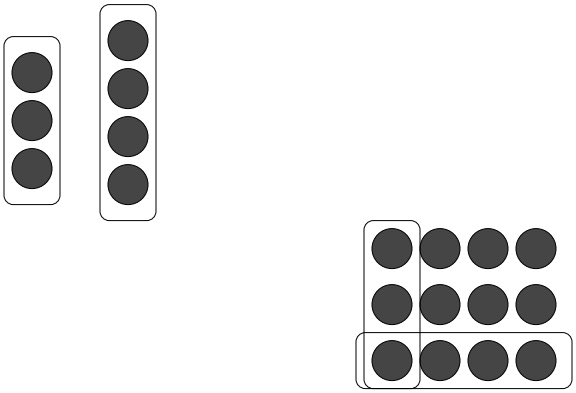
or



or



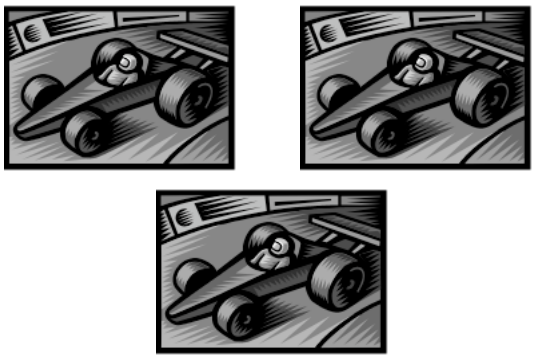
Or it does not matter?



How many wheels?

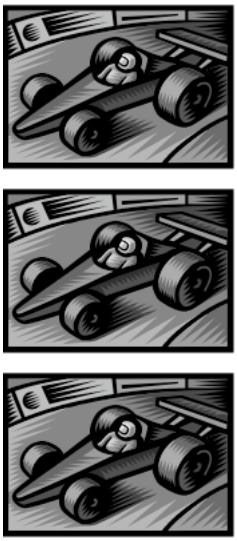
3x4, or 4x3, or it doesn't matter?

How many wheels?



3x4, or 4x3, or it doesn't matter?

Are these both 3x4 and 4x3?



If you say
multiplication is repeated addition,
are the both
3+3+3+3 and 4+4+4?

What is the total length?

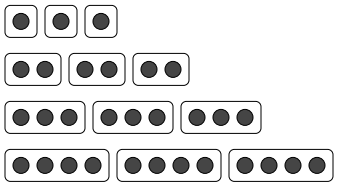
Is this 3+3+3+3, or 4+4+4, or it doesn't matter?
How should we write this usnig multiplication 3x4 or 4x3?

How did you learn to remember multiplication table of 3?

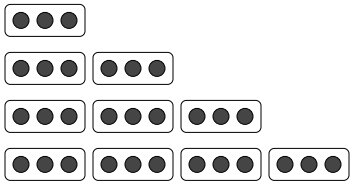
3 x 1 = 3		1 x 3 = 3
3 x 2 = 6		2 x 3 = 6
3 x 3 = 9		3 x 3 = 9
3 x 4 = 12		4 x 3 = 12
3 x 5 = 15	or	5 x 3 = 15
3 x 6 = 18		6 x 3 = 18
3 x 7 = 21		7 x 3 = 21
3 x 8 = 24		8 x 3 = 24
3 x 9 = 27		9 x 3 = 27

If the first number represents number of groups…
(3 groups of …)

3 x 1 = 3
3 x 2 = 6
3 x 3 = 9
3 x 4 = 12

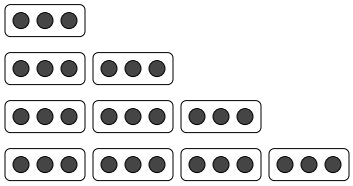


1 x 3 = 3
2 x 3 = 6
3 x 3 = 9
4 x 3 = 12

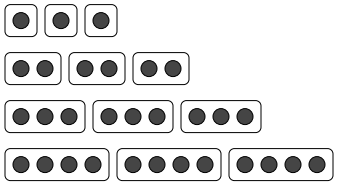


If the second number represents number of groups…

3 x 1 = 3
3 x 2 = 6
3 x 3 = 9
3 x 4 = 12



1 x 3 = 3
2 x 3 = 6
3 x 3 = 9
4 x 3 = 12



NCTM: *Focal Points*

Grade 3: Developing understandings of multiplication and division and strategies for basic multiplication facts and related division facts.

Grade 4: Developing quick recall of multiplication facts and related division facts and fluency with whole number multiplication.

Grade 5: Developing an understanding of and fluency with division of whole numbers.

NCTM: *Focal Points*

Grade 6: Developing an understanding of and fluency with multiplication and division of fractions and decimals.

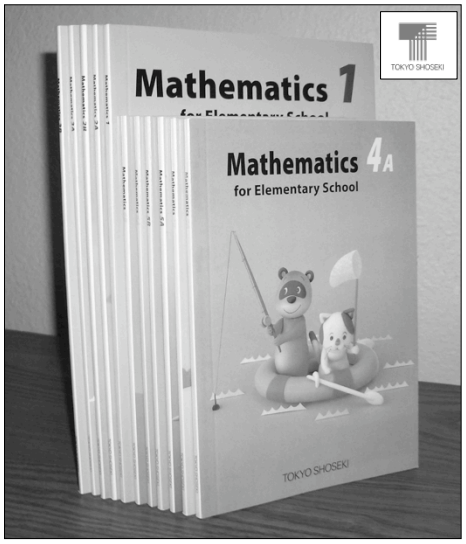
Grade 6: Connecting ratio and rate to multiplication and division.

Grade 6: Writing, interpreting, and using mathematical expressions and equations

Devlin on Multiplication

Multiplication simply is not repeated addition, and telling young pupils it is inevitably leads to problems when they subsequently learn that it is not. Multiplication of natural numbers certainly gives the same result as repeated addition, but that does not make it the same... Telling students falsehoods on the assumption that they can be corrected later is rarely a good idea. And telling them that multiplication is repeated addition definitely requires undoing later.

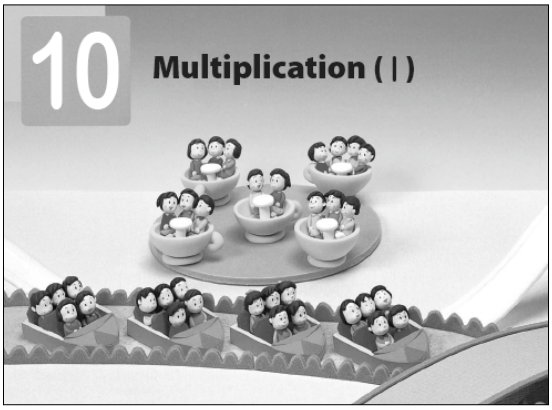
Tokyo Shoseki's Mathematics for Elementary School (Grades 1 to 6)



The textbook follows 1989 Japanese National Course of Study that was examined in the 1995 TIMSS.

It is translated into English and available at www.globaledresources.com.

2B, p. 14

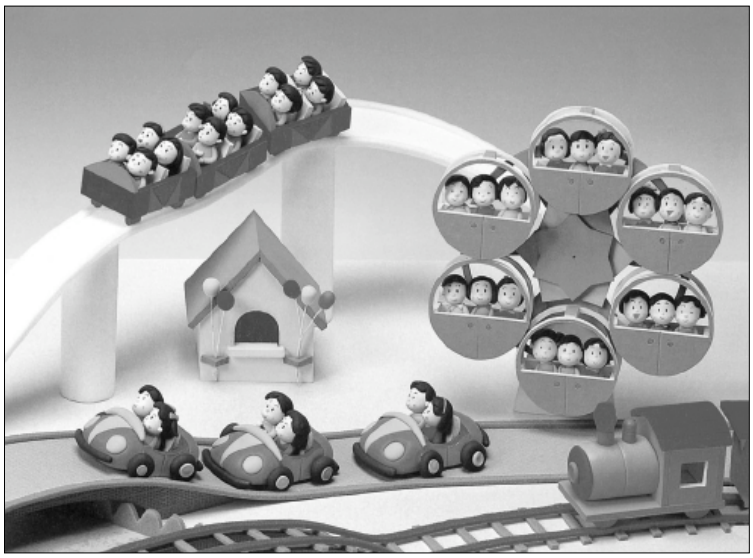


Let's find out how many children are on each ride at the amusement park!

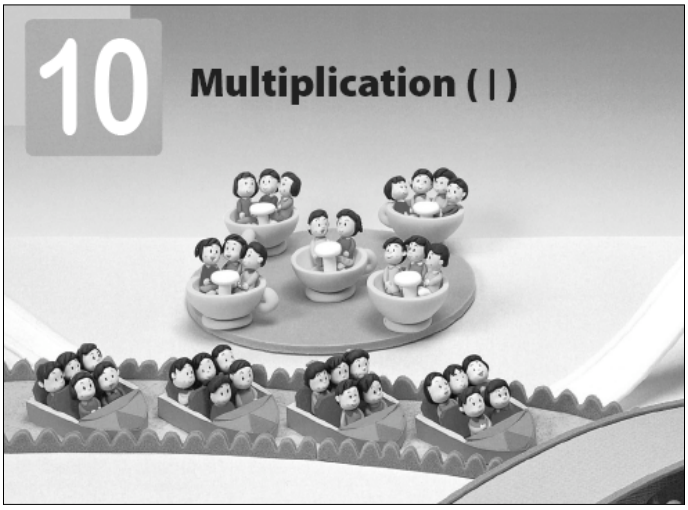
- How many children are on the tea-cup ride?
- How many children are on the boat ride?

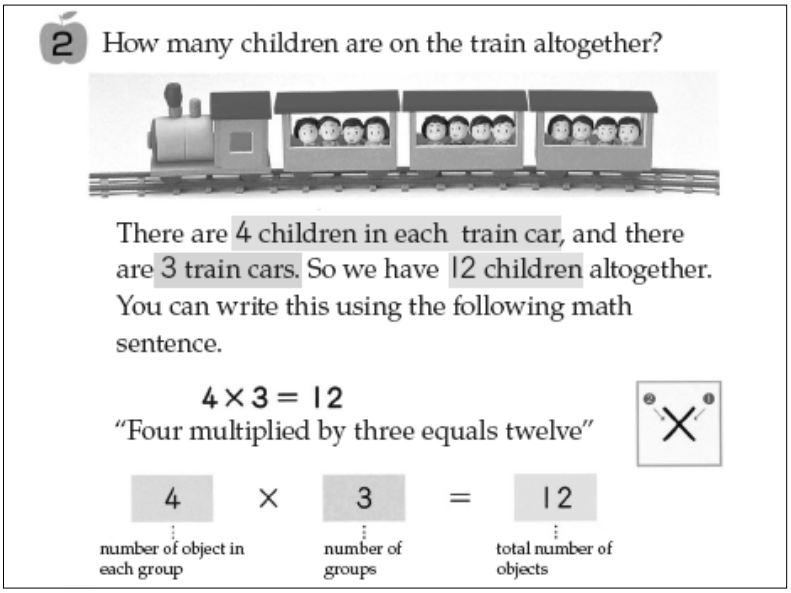
There are 5 children on each boat and there are 4 boats. There are 20 children altogether.

Let's check other rides!



Let's look at the teacup ride again!
Can we use multiplication for it?





2 How many children are on the train altogether?

There are 4 children in each train car, and there are 3 train cars. So we have 12 children altogether. You can write this using the following math sentence.

$4 \times 3 = 12$

"Four multiplied by three equals twelve"

4	×	3	=	12
number of object in each group		number of groups		total number of objects

p. 16

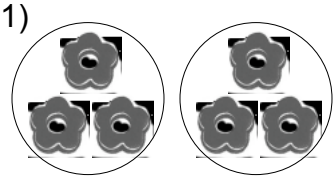
Important ideas of multiplication

- Multiplication sentences describe equal set situations.
  - Repeated addition and skip counting are ways to find the total (product).
- The numbers in a multiplication sentences mean something specific:
  - Number in a group - multiplicand
  - Number of groups - multiplier
  - Total number of objects - product

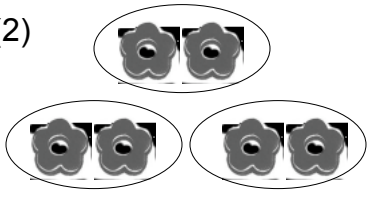
2 Please arrange your counters to show the following math sentences.

(1) 3×2 (2) 2×3 (3) 5×2

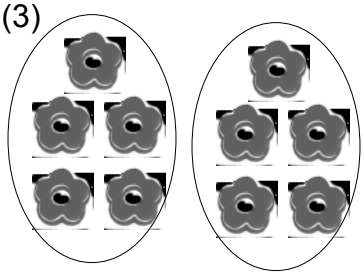
(1)



(2)



(3)

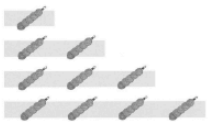


2 Multiplication Facts for 5 and 2

The multiplication facts for 5

1 Each bag has 5 oranges. There are 4 bags of oranges.
How many oranges do we have altogether?

1 How many oranges are there in 2 bags?
Let's find out how many oranges there are in 3, 4, and more bags!

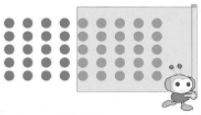


5 × 1 = ☐
5 × 2 = ☐
5 × 3 = ☐
5 × 4 = ☐

There are 20 oranges altogether.

2 How many oranges are there in 5 bags?
3 How about in 6, 7, 8, and 9 bags?

5 × 5 = ☐
5 × 6 = ☐
5 × 7 = ☐
5 × 8 = ☐
5 × 9 = ☐



If you remember the answers for 5×4 is "five, four, 20" and 5×5 is "five, five, 25", they will be useful. This way of remembering multiplication facts is called "kuku" in Japanese.

2 Let's memorize the multiplication facts for 5!

1 How are the 5's facts increasing?

The Multiplication Table of 5

5 × 1 = 5	five, one,	5
5 × 2 = 10	five, two,	10
5 × 3 = 15	five, three,	15
5 × 4 = 20	five, four,	20
5 × 5 = 25	five, five,	25
5 × 6 = 30	five, six,	30
5 × 7 = 35	five, seven,	35
5 × 8 = 40	five, eight,	40
5 × 9 = 45	five, nine,	45

1 There are 5 strawberries on each plate and there are 6 plates.
How many strawberries do you have altogether?

2 Recite the 5's table starting with the largest answer.

3 Multiplication Facts for 3 and 4

The multiplication facts for 3

1 Let's make a multiplication table for 3!

3 × 1 = ☐
3 × 2 = ☐
3 × 3 = ☐
3 × 4 = ☐

1 To get an answer for 3×5, what do you need to add to the answer for 3×4?

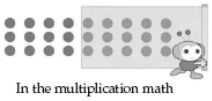
3 × 5 = ☐

2 Use the answer for 3×5 to find 3×6.

3 × 6 = ☐

3 Find the answers for 3×7, 3×8, and 3×9.

3 × 7 = ☐
3 × 8 = ☐
3 × 9 = ☐

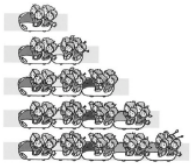


In the multiplication math sentence 3×9:
3 is called the **multiplicand** and 9 is called the **multiplier**.

3 × 9 = 27
Multiplicand Multiplier

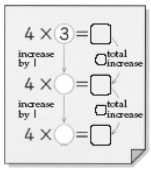
The multiplication facts for 4

3 Let's make a multiplication table for 4!



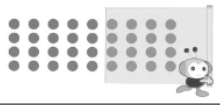
4 × 1 = ☐
4 × 2 = ☐
4 × 3 = ☐
4 × 4 = ☐
4 × 5 = ☐

1 When the multiplier of 4×3 increases by 1, how many does the answer increase?
Also, what happens when the multiplier of 4×4 increases by 1?



2 Please find the answers for 4×6, 4×7, 4×8, and 4×9.

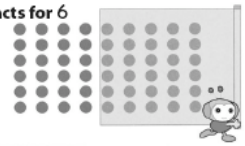
4 × 6 = ☐
4 × 7 = ☐
4 × 8 = ☐
4 × 9 = ☐



1 Multiplication Facts for 6 and 7

The multiplication facts for 6

1 Let's make a multiplication table for 6 using the ideas you learned before.



In the 6's table, each answer increases by 6 so . . .

$6 \times 1 = 6$
$6 \times 2 = 12$ ($6+6=12$)
$6 \times 3 = 18$ ($6+6+6=18$)

Sayuri

In the 6s table, when the multiplier increases by 1, the answer increases by 6 so . . .

$6 \times 1 = 6$
↓ increase 1 → $6+6$
$6 \times 2 = 12$
↓ increase 1 → $12+6$
$6 \times 3 = 18$

Minoru

$6 \times 1 =$ ☐
$6 \times 2 =$ ☐
$6 \times 3 =$ ☐
$6 \times 4 =$ ☐
$6 \times 5 =$ ☐
$6 \times 6 =$ ☐
$6 \times 7 =$ ☐
$6 \times 8 =$ ☐
$6 \times 9 =$ ☐

1 Please write the answers for the multiplication facts for 6 in the multiplication table on page 31.

2 Let's memorize the multiplication facts for 6!

Let's do the practice on page 29.

1 Let's look at the multiplication table of 6 carefully.

The answers for 6×4 and 4×6 are the same.

$6 \times 4 = 4 \times 6$

Yuji

The Multiplication Table of 6

$6 \times 1 = 6$	……six, one,	6
$6 \times 2 = 12$	……six, two,	12
$6 \times 3 = 18$	……six, three,	18
$6 \times 4 = 24$	……six, four,	24
$6 \times 5 = 30$	……six, five,	30
$6 \times 6 = 36$	……six, six,	36
$6 \times 7 = 42$	……six, seven,	42
$6 \times 8 = 48$	……six, eight,	48
$6 \times 9 = 54$	……six, nine,	54

The answers to the multiplication facts for 6 are the same as the multiplication facts for 3 added to themselves.

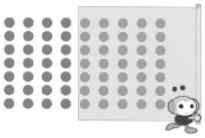
Multiplication facts for 6 = Multiplication facts for 3 + Multiplication facts for 3

Naoko

1 Let's find the same answer to 6×5 in the 5's table.

The multiplication facts for 7

3 Let's make a multiplication table for 7 using the ideas you learned before.



In the 7's table, each answer increases by 7 so . . .

Ritsuko

In the multiplication table of 7, when the multiplier increases by 1, the answer increases by 7 so . . .

Makoto

The answers for 7×2 and 2×7 are the same so . . .

Kazuya

$7 \times 1 =$ ☐
$7 \times 2 =$ ☐
$7 \times 3 =$ ☐
$7 \times 4 =$ ☐
$7 \times 5 =$ ☐
$7 \times 6 =$ ☐
$7 \times 7 =$ ☐
$7 \times 8 =$ ☐
$7 \times 9 =$ ☐

1 Please write the answers for the multiplication facts for 7 in the multiplication table on page 31.

4 Let's memorize the multiplication facts for 7!

1 Let's look at the multiplication table of 7 carefully.

$7 \times 3 = 21$, $3 \times 7 = 21$, $7 \times 4 = 28$, $4 \times 7 = 28$. Even if the multiplier and the multiplicand are switched, the answers are still the same.

Mami

The Multiplication Table of 7

$7 \times 1 = 7$	……seven, one,	7
$7 \times 2 = 14$	……seven, two,	14
$7 \times 3 = 21$	……seven, three,	21
$7 \times 4 = 28$	……seven, four,	28
$7 \times 5 = 35$	……seven, five,	35
$7 \times 6 = 42$	……seven, six,	42
$7 \times 7 = 49$	……seven, seven,	49
$7 \times 8 = 56$	……seven, eight,	56
$7 \times 9 = 63$	……seven, nine,	63

The multiplication facts for 7 are the same as adding the multiplication facts for 5 with the multiplication facts for 2.

Multiplication facts for 7 = Multiplication facts for 5 + Multiplication facts for 2

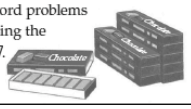
Naoko

The multiplication table of 7 is the same answers as adding the multiplication tables of 3 and 4.

Yuji

1 7 pieces of origami paper are to be given to each of 6 children. How many papers do you need altogether?

2 Please make up some word problems that can be solved by using the multiplication facts for 7.



2 Multiplication Facts for 8, 9 and 1

The multiplication facts for 8

1 Let's make a multiplication table for 8.

$8 \times \square$

Please write the answers for the multiplication facts for 8 in the multiplication table on page 31.

2 Let's memorize the multiplication facts for 8!

Let's look at the multiplication table of 8 carefully.

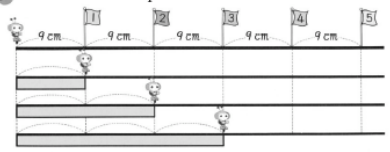
The Multiplication Table of 8

8 × 1 =	8 ····· eight, one,	8
8 × 2 =	16 ····· eight, two,	16
8 × 3 =	24 ····· eight, three,	24
8 × 4 =	32 ····· eight, four,	32
8 × 5 =	40 ····· eight, five,	40
8 × 6 =	48 ····· eight, six,	48
8 × 7 =	56 ····· eight, seven,	56
8 × 8 =	64 ····· eight, eight,	64
8 × 9 =	72 ····· eight, nine,	72

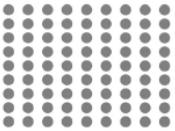
1 There are 6 benches. 8 children can sit on each bench. How many total children can sit on the benches?

The multiplication facts for 9

3 Let's make a multiplication table for 9.



$9 \times \square$



Please write the answers for the multiplication facts for 9 in the multiplication table on page 31.

4 Let's memorize the multiplication facts for 9!

Let's look at the multiplication table of 9 carefully.

The Multiplication Table of 9

9 × 1 =	9 ····· nine, one,	9
9 × 2 =	18 ····· nine, two,	18
9 × 3 =	27 ····· nine, three,	27
9 × 4 =	36 ····· nine, four,	36
9 × 5 =	45 ····· nine, five,	45
9 × 6 =	54 ····· nine, six,	54
9 × 7 =	63 ····· nine, seven,	63
9 × 8 =	72 ····· nine, eight,	72
9 × 9 =	81 ····· nine, nine,	81

Multiplication Table

Multipliers	1	2	3	4	5	6	7	8	9
1's	1								
2's	2								
3's	3								
4's	4								
5's	5								
6's	6								
7's	7								
8's	8								
9's	9								

Multiplicands

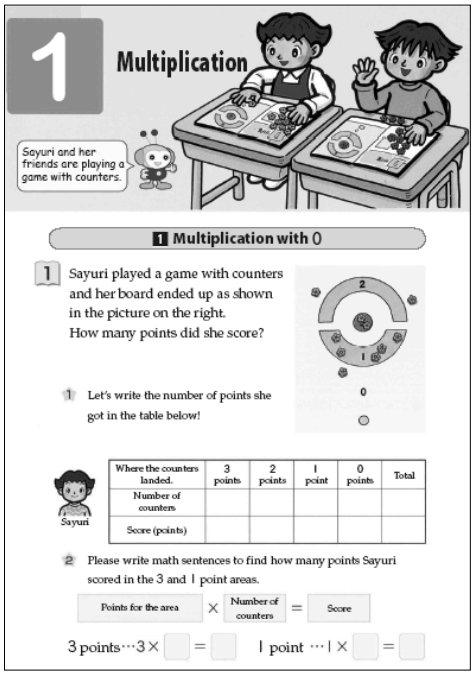
31

Developing Multiplication Facts

- Organized according to the multiplicand.
- Emphasis on students developing the multiplication table.
- Focusing on one specific property: when the multiplier increases by 1, the product increases by the multiplicand.
- Array diagrams are used to promote the understanding of the basic facts.
- 1 as the multiplicand is treated last. 0 as the multiplicand is discussed in Grade 3.

0 as a factor

3A p. 2



1 Multiplication

Sayuri and her friends are playing a game with counters.

1 Multiplication with 0

1 Sayuri played a game with counters and her board ended up as shown in the picture on the right. How many points did she score?

2 / 1 / 0

1 Let's write the number of points she got in the table below!

Sayuri

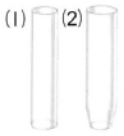
Where the counters landed.	3 points	2 points	1 point	0 points	Total
Number of counters					
Score (points)					

2 Please write math sentences to find how many points Sayuri scored in the 3 and 1 point areas.

Points for the area × Number of counters = Score

3 points…3 × ☐ = ☐ 1 point …1 × ☐ = ☐

2 Properties of Proportional Relationships



1 The depth of water is measured in the 2 containers shown above.
Let's investigate how the depth of water changes as the amount of water increases!

(1)

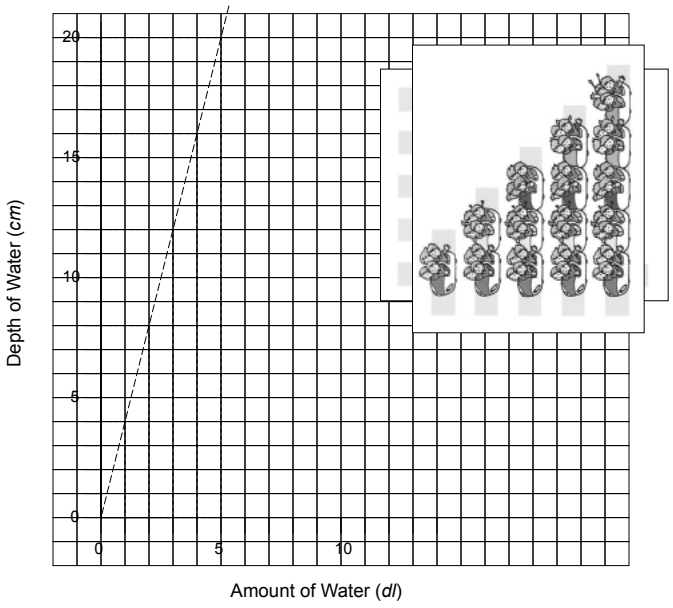
Amount of Water ($d\ell$)	1	2	3	4	5	6
Depth of Water (cm)	4	8	12	16	20	24

(2)

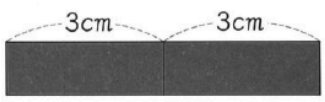
Amount of Water ($d\ell$)	1	2	3	4	5	6
Depth of Water (cm)	4	7	10	13	16	19

1 In Table (1), how does the depth of water change as the amount of water doubles, triples, and so on?

6A, p.8



4 What is the length of two 3cm tapes put together?



Math Sentence ☐

Answer ☐ cm

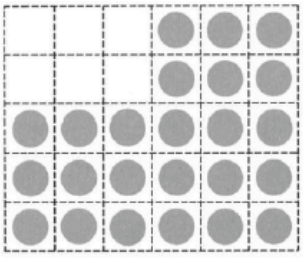
If a tape is as long as two 3cm tapes put together, we can say the tape is "**2 times**" **as long as** a 3cm tape.

You can use the multiplication math sentence 3×2 to find the length that is two times as long as 3cm.

p. 18

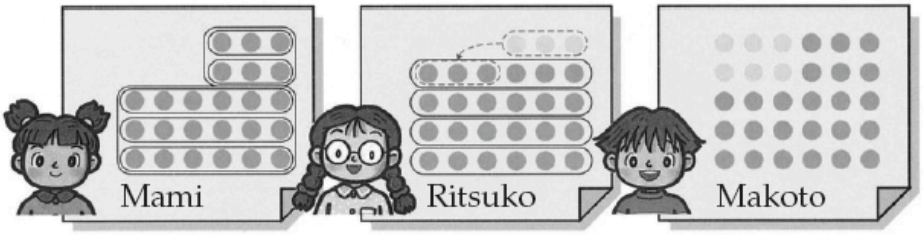
Book 2B, p. 44

2 How many chocolates are in the box?
Think about many different ways to find the answer.

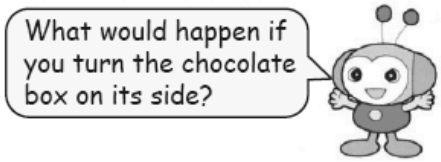


Book 2B, p. 44

1 Please describe how these friends found the answer.



2 Let's think about different ideas from them!



Properties of Multiplication

Kazuya

I don't remember the answer for 4×6. I forgot.

Let's all think together with Kazuya about how to find the answer to 4×6.

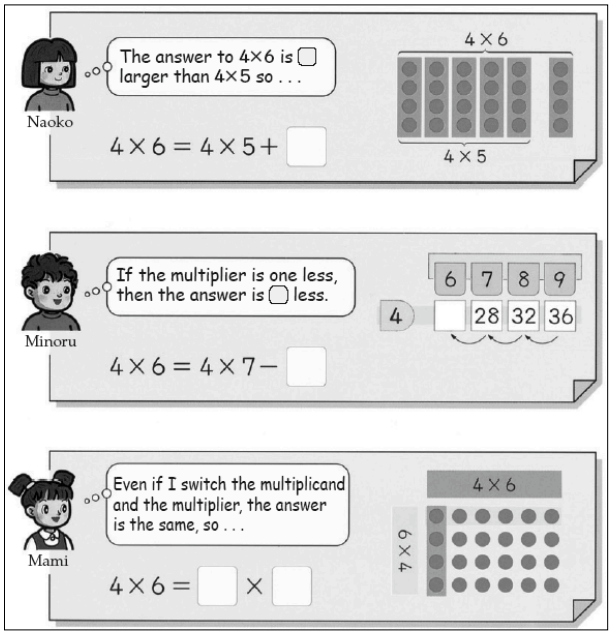
Teacher

2 Properties of Multiplication

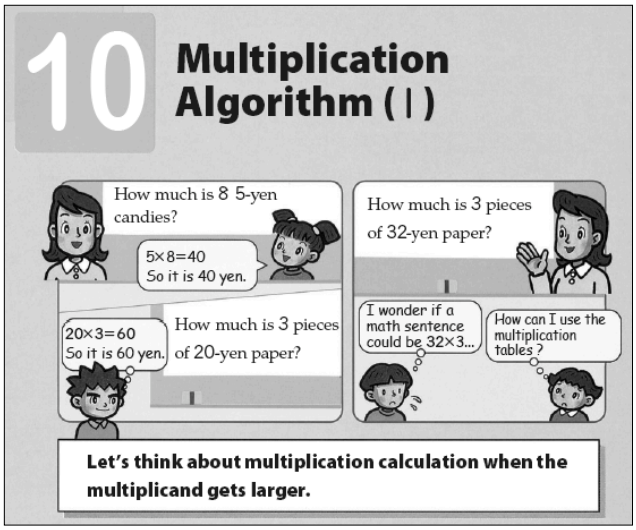
1 Let's think about the way to find the answer to 4×6.

I wonder if we can use the rules for multiplication that we learned in the 2nd grade.

3A, p. 6



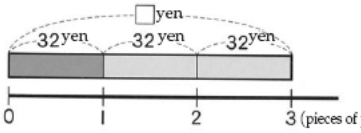
Multiplication Algorithm (Gr. 3)



3A, p. 71

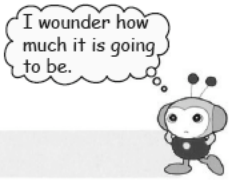
1 Multiplication of 2-digit Number and 1-digit Number

1 We bought 3 pieces of 32-yen construction paper. How much did they cost altogether?



Math Sentence

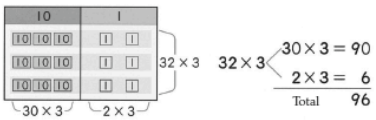
Price for 1 piece × Numbers you bought = Total Price



Let's think about a way to calculate!

3A, p. 71

1 Tell how to calculate 32×3 using the picture below.



$32 \times 3 = 30 \times 3 = 90$, $2 \times 3 = 6$, Total 96

$32 \times 3 = 96$ Answer: 96 yen

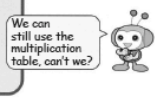
You can do multiplication 32×3 using an algorithm shown below.

How to calculate 32×3

Line up the numbers vertically in each place value.

Write 6 from 3×2=6 in the ones place.

Write 9 from 3×3=9 in the tens place.



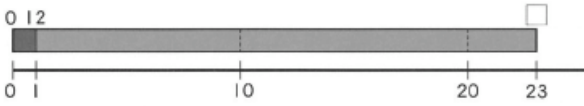
32 × 3 = 6 ··· 2×3, 90 ··· 30×3, 96

① 42×2 13×3 34×2 20×4

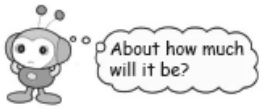
3A, p. 72

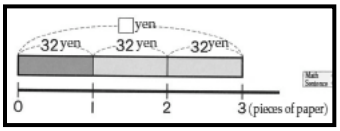
2 Multiplication by a Two-digit Number

1 You are buying 23 sheets of paper. If each sheet costs 12 yen, how much will it cost altogether?



Math Sentence





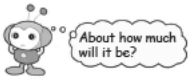
3B, p. 49

2 Multiplication by a Two-digit Number

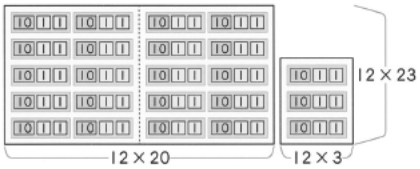
1 You are buying 23 sheets of paper. If each sheet costs 12 yen, how much will it cost altogether?

0 12
0 1 10 20 23

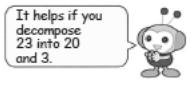
Math Sentence



1 Let's think about how to calculate!



12 × 23 < 12 × 20 = 240
12 × 3 = 36
Total 276

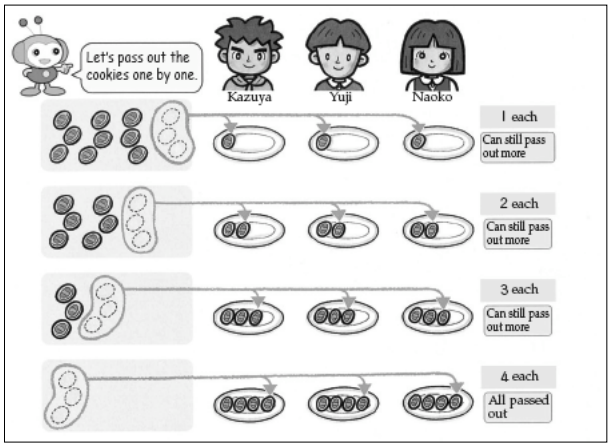


12 × 23 = 276 Answer: 276 yen

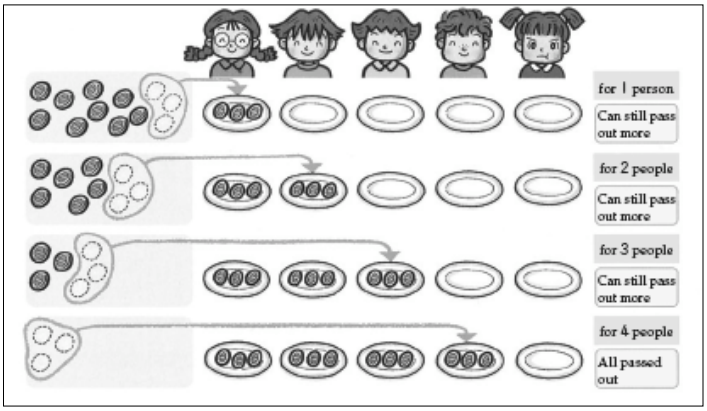
Division

- Partitive Division
  - Partitioning, fair share
- Quotitive Division
  - Measurement division, repeated subtraction

There are 12 cookies. If 3 people divide them evenly, how many cookies will one person get?



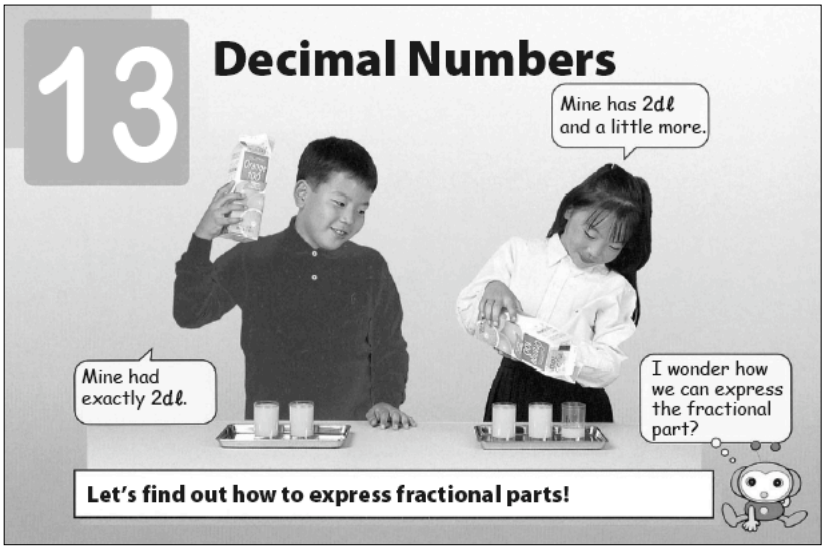
There are 12 cookies. If one child gets 3 cookies, how many children can get cookies?



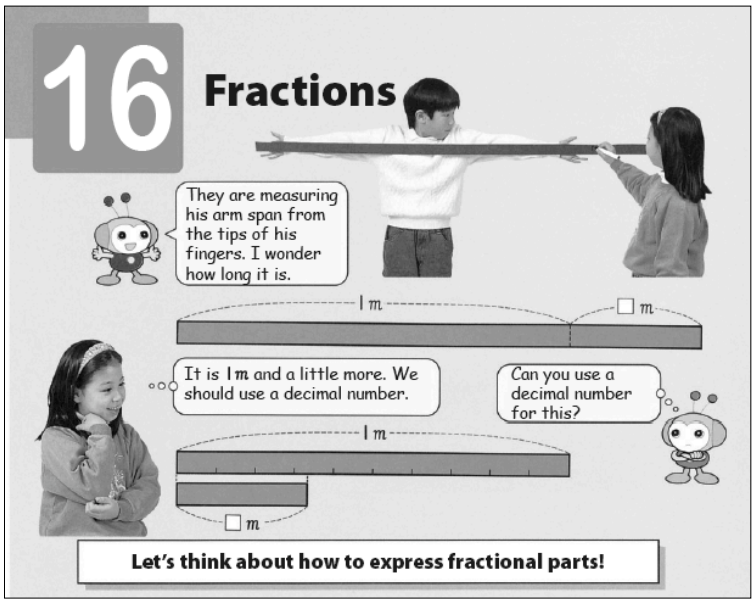
3 cookies x 4 people = 12 cookies

12 cookies ÷ 4 people = 3 cookies

12 cookies ÷ 3 cookies = 4 people


13
Decimal Numbers
Mine has 2dℓ and a little more.
Mine had exactly 2dℓ.
I wonder how we can express the fractional part?
Let's find out how to express fractional parts!

3B p. 29

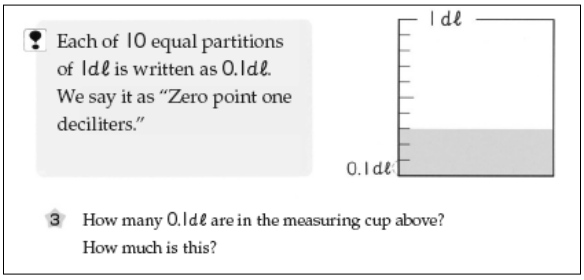

16
Fractions
They are measuring his arm span from the tips of his fingers. I wonder how long it is.
1 m
□ m
It is 1 m and a little more. We should use a decimal number.
Can you use a decimal number for this?
1 m
□ m
Let's think about how to express fractional parts!

3B p. 57

Similarities/Differences

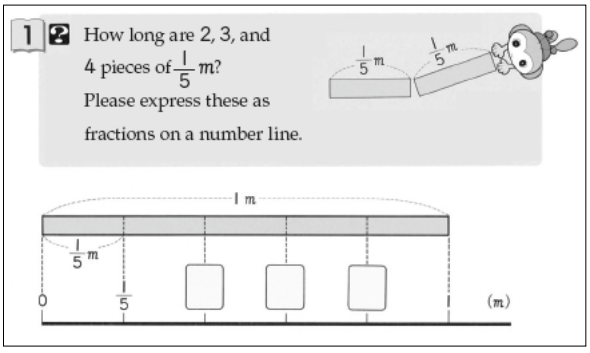
- Both problems involve measurements.
  - One involves liquid measures while the other is about linear measurements.
- Both problems involve quantities greater than 1 unit
  - With decimals, the whole quantity is represented using the notation “2.3” but with fractions, mixed numbers are not used to express the whole quantity.

Decimals



3B, p. 30

Fractions

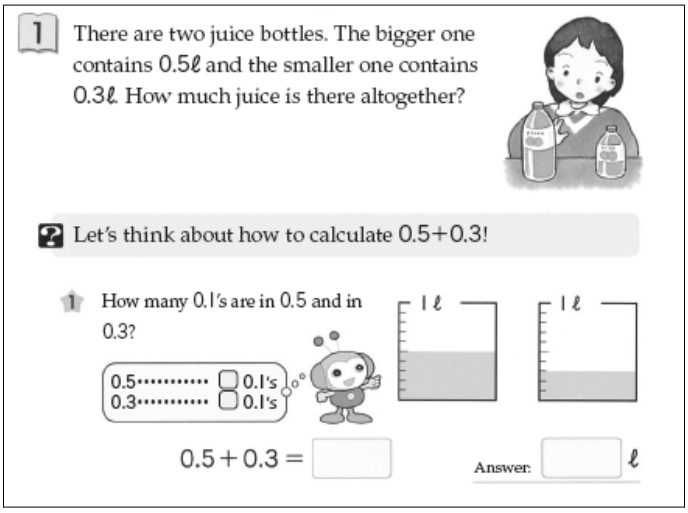


3B, p. 60

Similarities/Differences

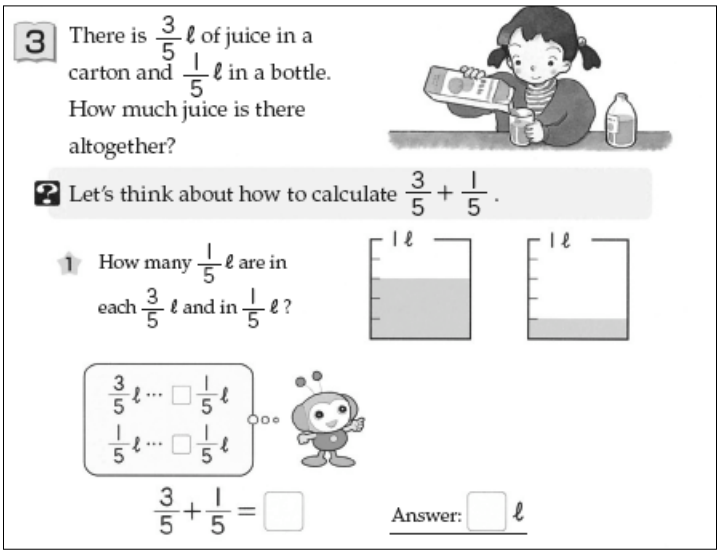
- Decimals and fractions as collections of decimal/fraction units, that is, “0.3” is *three* “0.1” units, and 3/4 is *three* “1/4” units.

Addition of Decimals

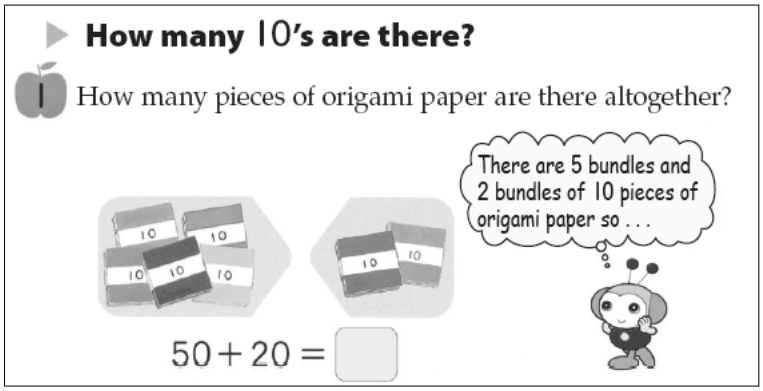


3B, p. 34

Addition of Fractions



3B, p. 62



1, p. 93

Teaching Guide

p. 60 (Grade 3):

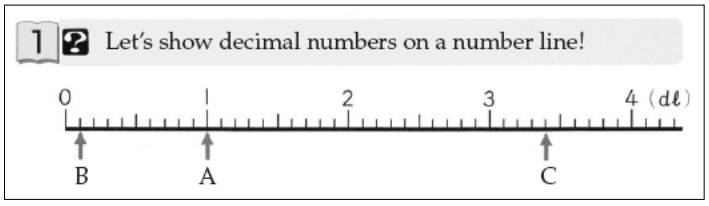
Calculations such as 1/5 + 2/5 can be thought of as the same as that of whole numbers, taking 1/5 as a unit.

p. 69 (Grade 4):

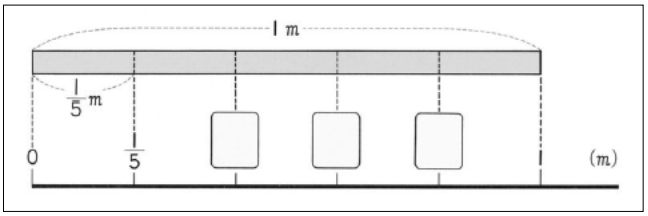
… it is important to help children think of it as counting numbers of a fraction whose numerator is one so that they can see the similarity between addition and subtraction of whole numbers and fractions.

Decimals/Fractions as Numbers

- Representing decimals/fractions on a number line

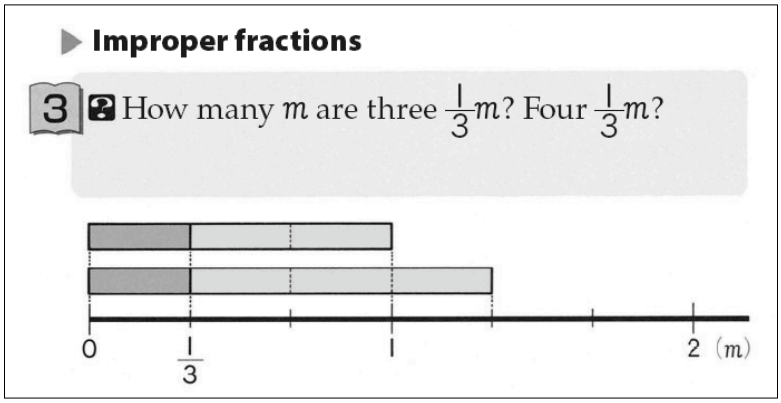


3B, p. 32



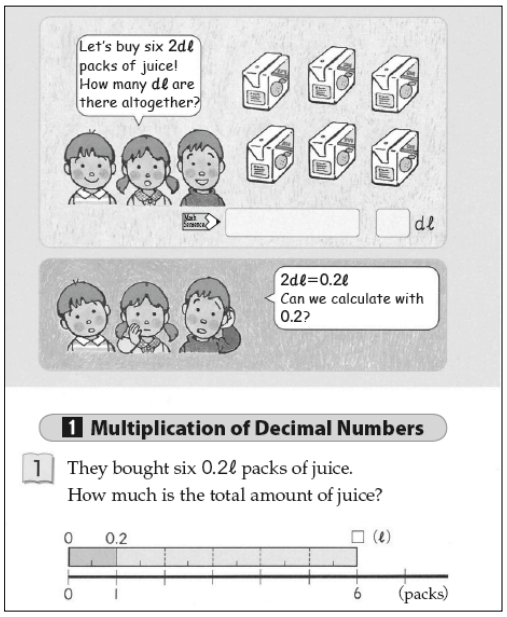
3B, p. 60

Improper fractions



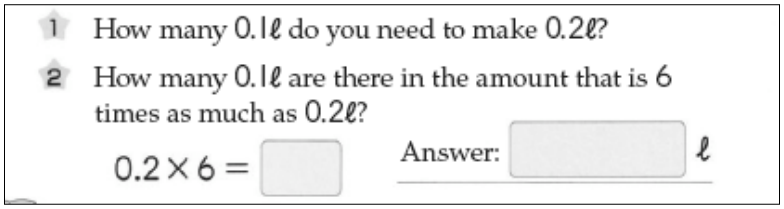
4B, p. 41

Multiplication of Decimals: Decimal x Whole Number



4B, p. 60

Using Number Structure



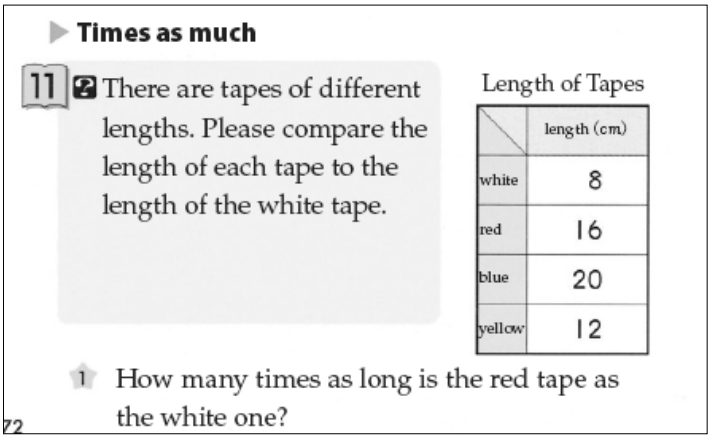
1 How many 0.1ℓ do you need to make 0.2ℓ?

2 How many 0.1ℓ are there in the amount that is 6 times as much as 0.2ℓ?

$0.2 \times 6 =$ ☐ Answer: ☐ ℓ

4B, p. 62

Laying Foundation



Times as much

11 2 There are tapes of different lengths. Please compare the length of each tape to the length of the white tape.

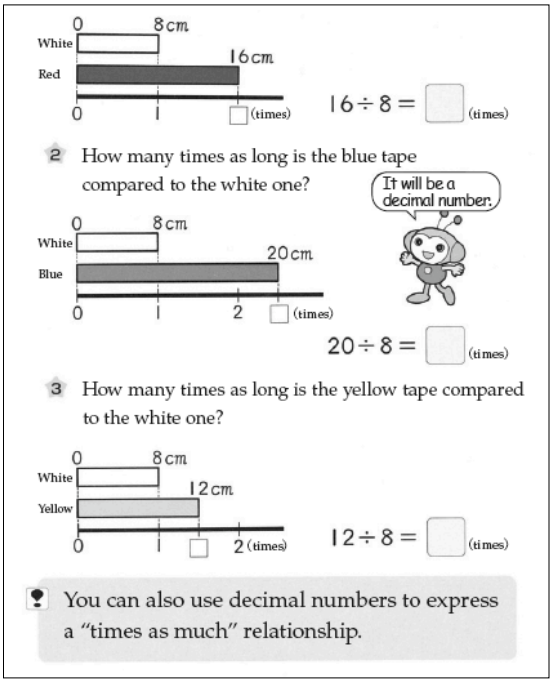
Length of Tapes

	length (cm)
white	8
red	16
blue	20
yellow	12

1 How many times as long is the red tape as the white one?

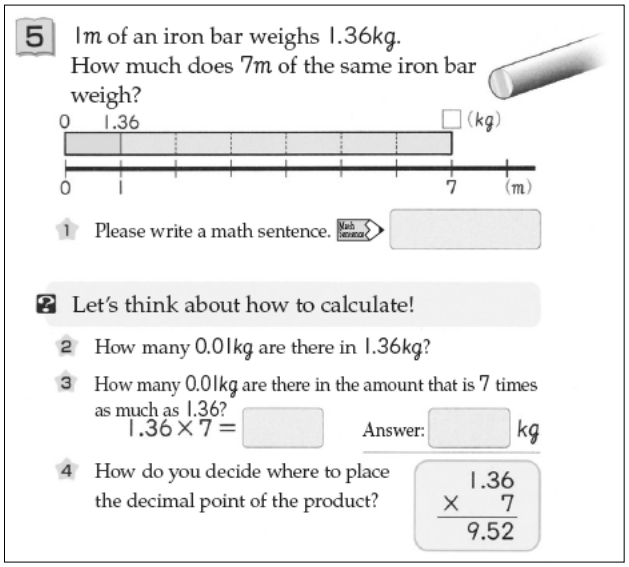
72

4B, p. 72



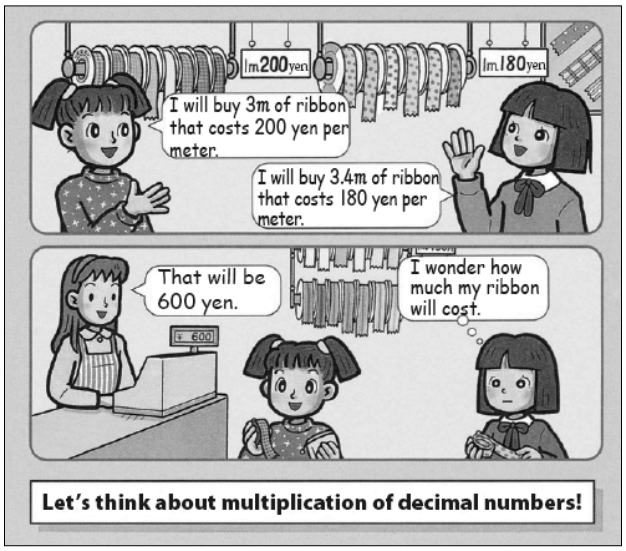
4B, p. 73

Use of Representations



4B, p. 64

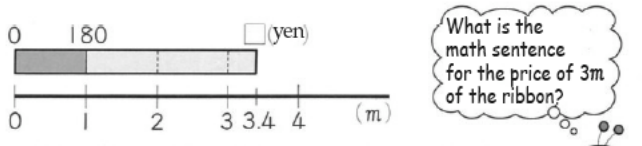
Decimal Multipliers



5A, p.27

1 Multiplying by a Decimal Number

1 We bought 3.4 meters of ribbon that cost 180 yen per meter. What was the price of the ribbon?



What is the math sentence for the price of 3m of the ribbon?

? How should we write the math sentence?

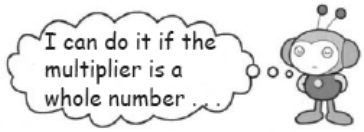
Price for 1m × Length we bought (Unit: m) = Price we have to pay

5A, p.26

Even if the length of the ribbon is shown as a decimal number, we can construct a multiplication sentence just like we do with whole numbers to find the price of the ribbon.

180×3.4

Let's think about how to calculate 180×3.4!

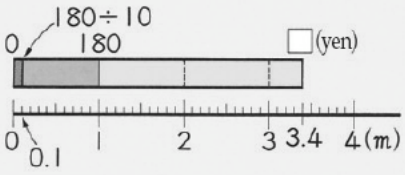


5A, p.27

1 Let's explain these two friends' methods!

Because 3.4m is made out of 34 pieces of 0.1m...

Sayuri

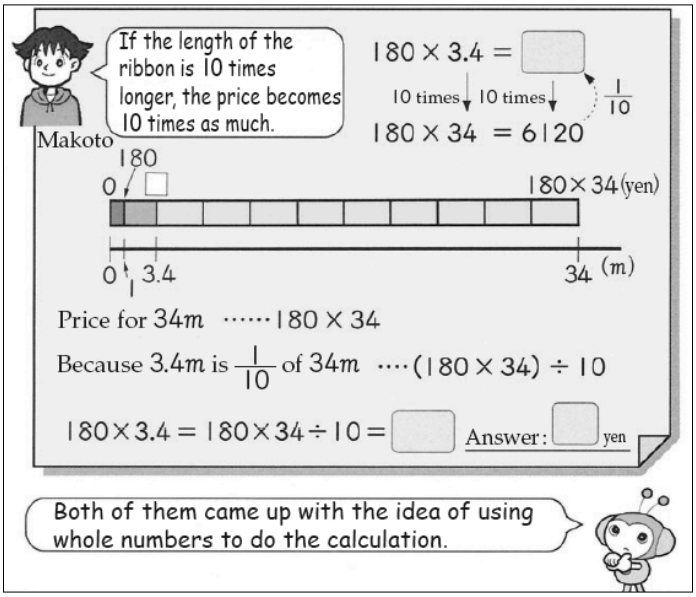


Price for 0.1m ·····180 ÷ 10

Total for 34 pieces ·····(180 ÷ 10) × 34

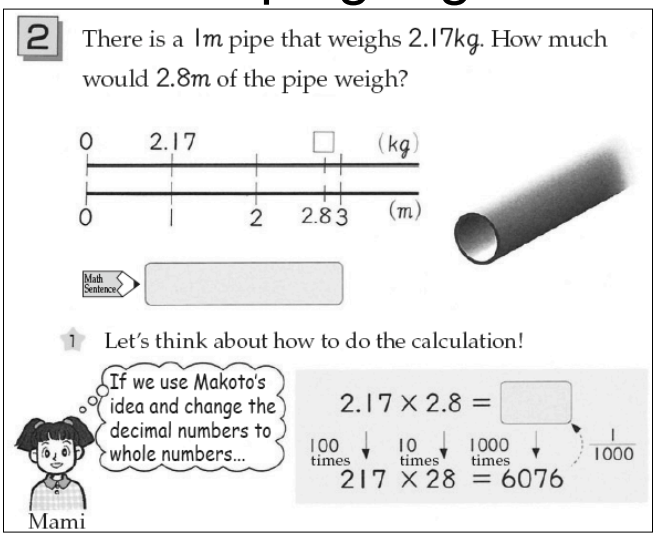
$180 \times 3.4 = 180 \div 10 \times 34 =$ ☐ Answer: ☐ yen

5A, p.28

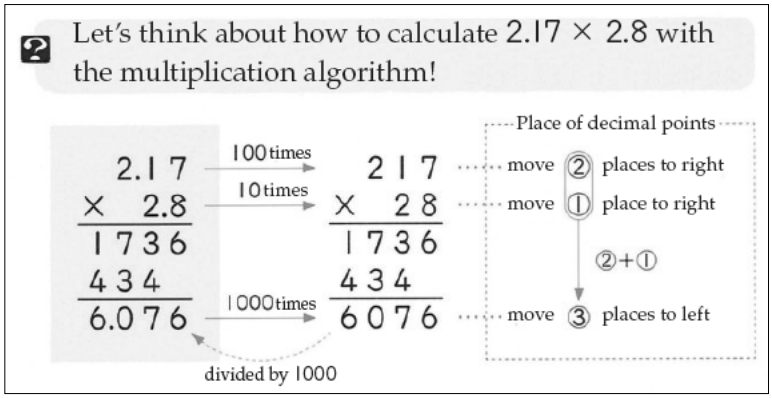


5A, p.27

Developing Algorithm

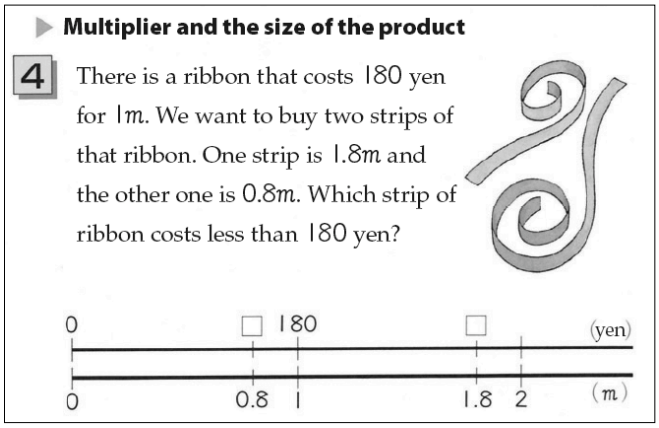


5A, p.29



5A, p.29

More than a procedure



5A, p.31

Let's think about the relationship between the multiplier and the size of the product!

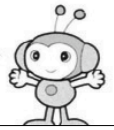
(1) $180 \times 1.8 =$ ☐

(2) $180 \times 0.8 =$ ☐

1 Which one of the products becomes smaller than 180?

In multiplication of decimal numbers, if the multiplier is less than 1, the product will be less than the multiplicand.

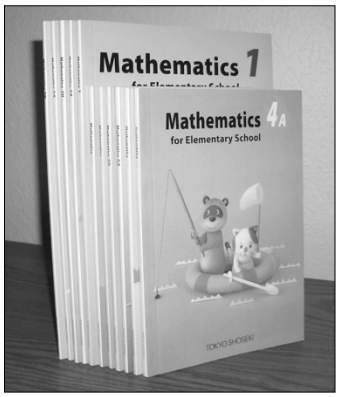
If multiplier > 1, then the product > multiplicand
If multiplier < 1, then the product < multiplicand



5A, p.31

Tokyo Shoseki's Mathematics for Elementary School (Grades 1 to 6)

TOKYO SHOSEKI



Global Education Resources
www.globaledresources.com
Booth # 1538