FLEXURE OF BEAMS

Symmetrical Flexure of Beams

In the following theory of flexure of beams it is assumed that the cross-section of the beam has an axis of symmetry which coincides with the y-axis of the Cartesian frame of reference. The x-axis coincides with the axis of the beam. The axis of the beam represents the locus of the cross-sectional centroids of the beam in the form of a straight line as shown in Figure 1.

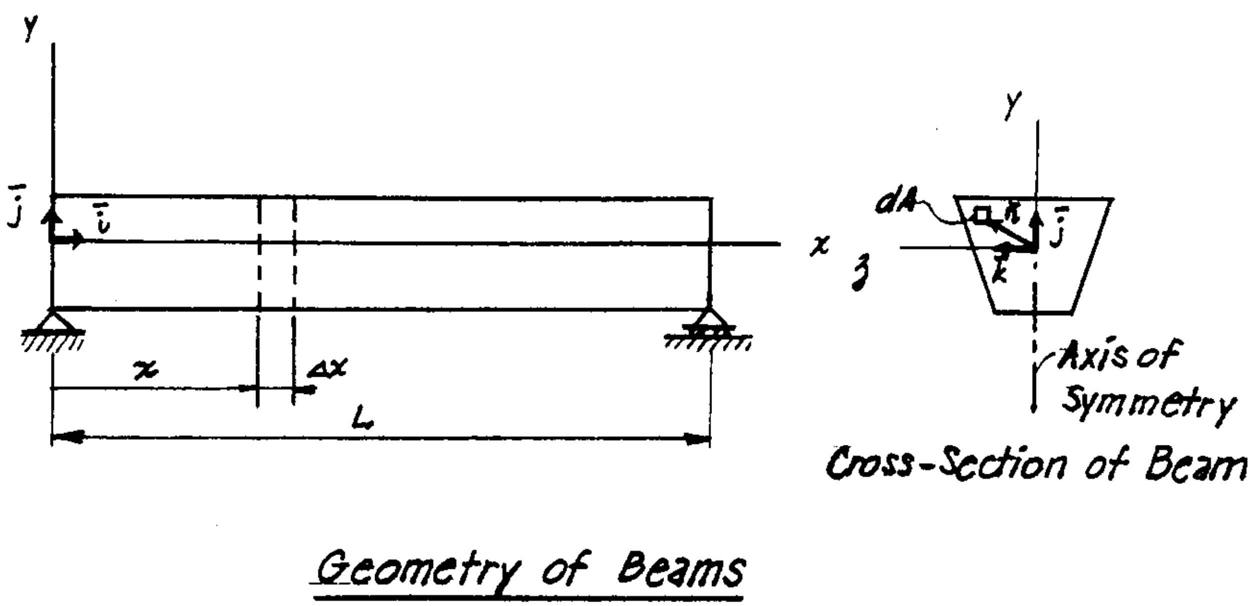


Figure 1

If a beam is slightly bent as shown in Figure 2, then the line elements of initial length Δx located by y relative to the z axis suffer a deformation which can be given from the observed geometry of deformation. The great Swiss mathematician Jacob Bernoulli (1655-1705), the first man to study mathematically deformable body mechanics in 1691-1694, observed from his simple experiments on large deformation of beams that plane cross-sections perpendicular to the axis of the beam before bending remain plane and perpendicular to the bent axis of the beam after bending. He used this observation as the central hypothesis in his pioneering theory of large bending of beams. Bernoulli was led to this problem when he acted as a consulting engineer to a well-known carriage manufacturer in Zurich, Switzerland, who was puzzled about the reason why his very well-made carriage wheels sometimes developed a crack in the rim despite the high quality material and expert craftsmanship that went into the production of the wheels. The reasonableness of Bernoulli's hypothesis can easily be checked by bending an eraser on the surface of which are drawn parallel lines perpendicular to the longitudinal axis of the eraser. See Figure 2 for illustration.

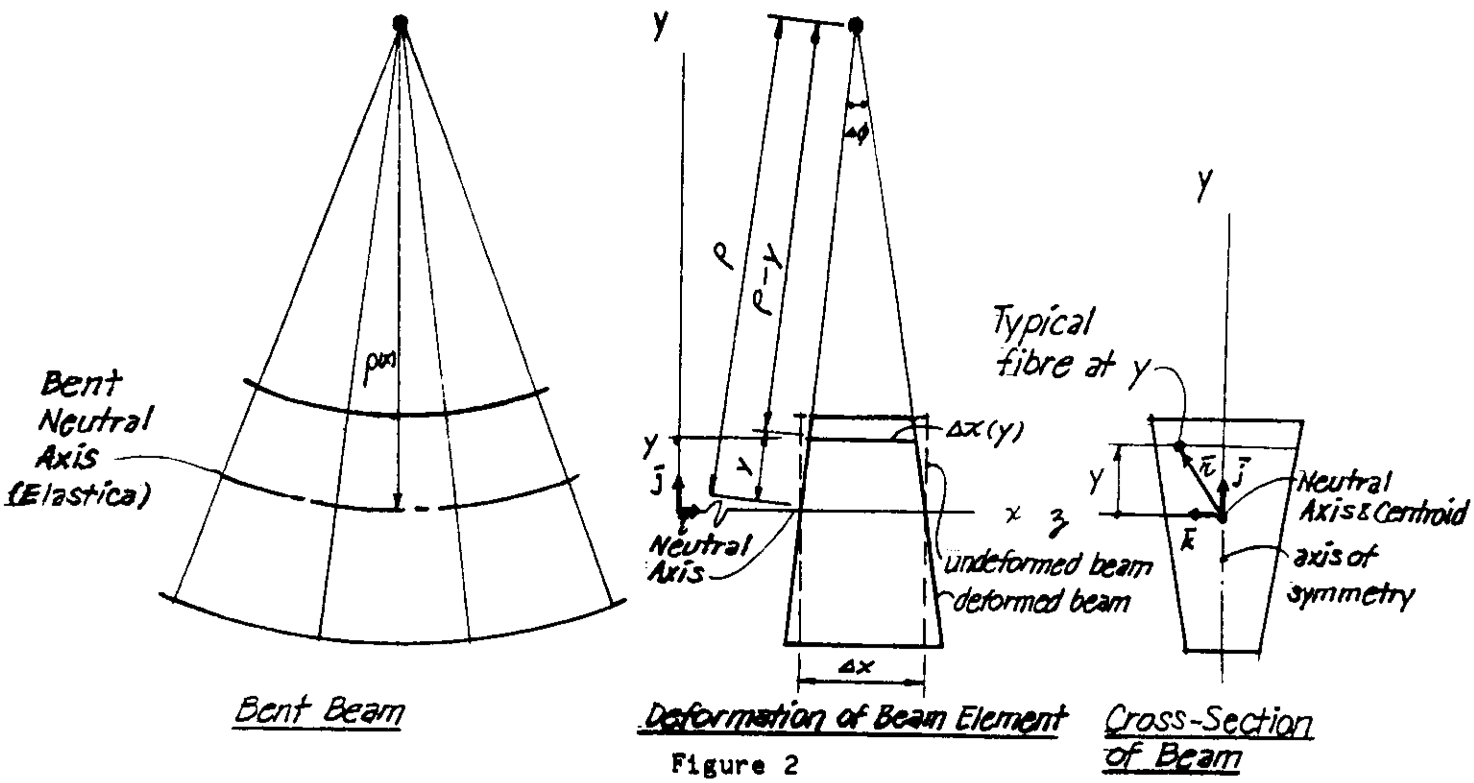


Figure 2

Bending of the beam will deform the initial length of the fibres of the small element of the beam, all of which are initially parallel to the x-axis and have the same length Δx.

In contrast with the difficult large bending theory of Bernoulli, attention in engineering theory is restricted to very small bending deformations of beams for which the curved longitudinal fibres can be approximated by straight lines as depicted in Figure 2.

From the geometry of deformation of the small element of the bent beam which is based on the Bernoulli Hypothesis, the length of the deformed fibre $\Delta x(y)$ at y is given by the proportion:

$$\Delta x/\rho = \Delta x(y)/(\rho-y)$$

where $\rho=\rho(x)$ denotes the radius of curvature of the bent axis of the beam. It is important to observe that in bending the centre line of the beam which passes through the centroid of the cross-section suffers no stretching in the bending of the beam. This line was called curva elastica by Bernoulli; later, it was simply called elastica. In contemporary literature elastica is called the neutral axis of the beam. Already in 1620, Beeckman had observed that in the bending of bars, the outer fibres of the bar extend and the inner fibres contract, which implies the existence of an unextended neutral fibre between the outer and inner fibres of the beam. Therefore, the bent line segment Δx at the centroid $y=0$, still has the initial undeformed length Δx. Solving for the deformed line segment at y:

$$\Delta x(y) = [(\rho-y)/\rho]\Delta x = [1-(y/\rho)]\Delta x$$

The Beeckman idea of linear strain measure gives the average engineering strain of the line segment $\Delta x(y)$:

$$\underset{\sim}{\varepsilon}_{xx} = [\Delta x(y)- \Delta x]/\Delta x = [\Delta x(y)/\Delta x]-1 = [1-(y/\rho)]-1$$

At the point, the linear strain is given by the limit

$$\varepsilon_{xx} = \lim_{\Delta x\to 0} \underset{\sim}{\varepsilon}_{xx} = \lim_{\Delta x\to 0} \{[1-(y/p)] - 1\} = -(y/\rho)$$

where $\rho=\rho(x)$. The constitutive equation in the form of Hooke's Law gives

$$\sigma_{xx} = E\varepsilon_{xx} = E[-(y/\rho] = -E(y/\rho) = -Ey\kappa$$

where the curvature of the bent axis of the beam is

$$\kappa(x) = 1/\rho(x)$$

The great Swiss mathematician, physicist and engineer, Leonhard Euler (1708-1783), demonstrated in his first scientific paper written in 1727, that Jacob Bernoulli's equation of bending,

$$M_z(\sigma) \propto \kappa(x)$$

which infers that the bending moment is proportional to the curvature of the bent elastica of the beam, can be obtained by finding the moment of the stresses (stress couple) relative to a particular point of the cross-section at x, and then by using Hooke's Law expressed the normal stresses σ_{xx} in terms of curvature κ of the bend elastica, and finally, like Jacob Bernoulli, Euler expressed the curvature in terms of the vertical displacement u_y of the beam. In order to accomplish this, Euler had to invent the concept of the Modulus of Elasticity E as a material constant, and separate the equilibrium problem from the constitutive property of the material. It was a brilliant contribution by a 19-year old student, and in honour of this great achievement the Equation of Flexure of Beams established in this way is called the Bernoulli-Euler Equation of Flexure. Euler's only mistake was to locate the neutral axis in the bottom of the beam, which is what Jacob Bernoulli had done before him.

The stress resultant system at the centroid of a generic cross-section x is:

$$\bar{F}(\sigma) = \int_A \bar{\sigma}_x dA = \int_A [\sigma_{xx}\bar{i} + \sigma_{xy}\bar{j}]dA = [\int_A -E\kappa(x)ydA]\bar{i} + [\int_A \sigma_{xy}dA]\bar{j}$$

$$= -E\kappa(x)\ [\int_A ydA]\bar{i} + [\int_A \sigma_{xy}dA]\bar{j} = 0\bar{i} + [\int_A \sigma_{xy}]\bar{j} = F_{xy}(\sigma)\bar{j}$$

where $\kappa(x)$ is constant for the entire cross-section and, therefore, independent of the cross-sectional integral, the bending stress

$$\sigma_{xx} = -Ey\kappa(x)$$

and

$$\int_A ydA = 0$$

when y is measured from the centroid of the cross-section. The corresponding couple-moment of stresses, representing the so-called stress-couple, is usually called the bending moment by engineers:

$$\bar{C}_x = \int_A \bar{r} x \bar{\sigma}_x dA = \int_A (y\bar{j}+z\bar{k}) \times (\sigma_{xx}\bar{i}+\sigma_{xy}\bar{j}) dA = \int_A y\sigma_{xx} dA(\bar{j}x\bar{i}) + \int_A y\sigma_{xy} dA(\bar{j}x\bar{j})$$

$$+ \int_A z\sigma_{xx} dA(\bar{k}x\bar{i}) + \int_A z\sigma_{xy} dA(\bar{k}x\bar{j}) = (-\int_A y\sigma_{xx} dA)\bar{j} + (\int_A z\sigma_{xx} dA)\bar{j}$$

$$+ (-\int_A z\sigma_{xy} dA)\bar{i} = (-\int_A y\sigma_{xx} dA)\bar{k} = M_z(\sigma)\bar{k}$$

for owing to the symmetry of σ_{xx} with respect to y-axis,

$$\int_A z\sigma_{xx} dA = \int_A z[-E\kappa(x)y]dA = -E\kappa(x) \int_A zydA = 0$$

because for any cross-sectional area with at least one axis of symmetry passing through the origin,

$$\int_A yzdA = 0$$

and, for $\sigma_{xy}(z) = \sigma_{xy}(-z)$, which implies planar stress,

$$\int_A z\sigma_{xy} dA = 0$$

Therefore,

$$M_z(\sigma) = -\int_A y\sigma_{xx} dA$$

Substituting for the bending stress σ_{xx} from the Hooke's Law

$$\sigma_{xx} = -E\kappa(x)y$$

yields after arbitrarily grouping it into three physically and geometrically distinct factors,

$$M_z(\sigma) = \int_A y[E\kappa(x)y]dA = E\kappa(x)\int_A y^2 dA = E\kappa(x)I_{zz}$$

where the Second Moment of the Cross-Sectional Area about the z-axis is

$$I_{zz} = \int_A y^2 dA > 0$$

which is always a positive definite scalar quantity. The Bernoulli-Euler Equation of Flexure,

$$M_z(\sigma) = EI_{zz}\kappa(x)$$

expresses the stress-couple by a product consisting of three factors, E, I_{zz} and $\kappa(x)$, each factor signifying a distinct conceptual feature of the bending of the beam so that the influence of each physically distinct factor on the bending of the beam can be separately studied. The Modulus of Elasticity E refers to the material property of the beam, $I_{zz}(x)$ refers to the cross-sectional geometry of the beam at x, and $\kappa(x) = 1/\rho(x)$ refers to the geometry of the bent axis of the beam at the cross section x. It should be observed that I_{zz} has no physical meaning, or explanation, beyond the fact that as an arbitrarily constructed factor it is entirely a geometrical function of the cross-section of the beam.

From elementary differential geometry the curvature of a curve corresponding to the bent axis of the beam can be expressed in terms of the ordinate of the curve, which in this case corresponds to the displacement $u_y(x)$ of the bent axis of the beam:

$$\kappa(x) = 1/\rho(x) = (d^2u_y/dx^2)/[1+(du_y/dx)^2]^{3/2}$$

This curvature expression can be derived as follows:

The curvature of a curve, such as the bent axis of a beam, can be defined as the arc-rate of change of the unit tangent vector $\bar{t}(s)$ of the curve:

$$d\bar{t}/ds = \bar{\kappa}(s) = [1/\rho(s)]\bar{n}$$

where $\bar{\kappa}$ denotes the curvature vector of the curve, and the unit normal vector

$$\bar{n}(s) = (d\bar{t}/ds)/|d\bar{t}/ds|$$

The rotation of the tangent vector $\bar{t}$ over a unit arc-length of the curve,

$$|d\bar{t}/ds| = |\bar{\kappa}(s)| = \kappa(s)$$

measures the rate of bending of the curve relative to its arc-length, as shown in Figure 3, where

$$|\bar{t}|^2 = \bar{t} \cdot \bar{t} = 1$$

This can be proven from the geometry in Figure 3 when it is observed that

$$|\Delta\bar{t}| = |\bar{t}| \sin\Delta\phi$$

and for small Δs, and thus for small $\Delta\phi$, the finite ratio

$$|\Delta\bar{t}|/\Delta s = |\Delta\bar{t}/\Delta s| = (|\bar{t}|\sin\Delta\phi/\Delta s) = \sin\Delta\phi/\Delta s \approx \Delta\phi/\Delta s$$

because for small $\Delta\phi$, $\sin\Delta\phi \approx \Delta\phi$.

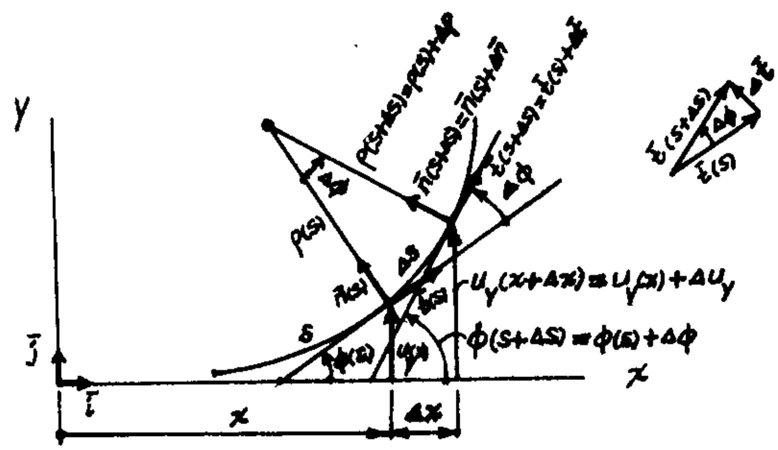


Figure 3

In the limit $\Delta s \to 0$:

$$|d\bar{t}/ds| = \lim_{\Delta s \to 0} (\Delta\phi/\Delta s) = d\phi/ds$$

From geometry of the bent axis in Figure 3,

$$\tan\phi = \lim_{\Delta x \to 0} (\Delta u_y(x)/\Delta x) = du_y(x)/dx$$

and thus,

$$\phi = \tan^{-1}(du_y/dx)$$

However,

$$\Delta s \approx \rho\ \Delta\phi$$

where ρ denotes the radius of curvature, and

$$\Delta\phi/\Delta s \simeq 1/\rho$$

Thus,

$$d\phi/ds = \lim_{\Delta s \to 0} (\Delta\phi/\Delta s) = 1/\rho = \kappa$$

Therefore,

$$\kappa = 1/\rho = d\phi/ds = (dx/ds)(d\phi/dx)$$

and

$$d\phi/dx = d[\tan^{-1} (du_y/dx)]/dx = (du_y^2/dx^2)/[1 + (du_y/dx)^2]$$

The arc-length can be found from the position vector

$$\bar{r} = x\bar{i} + u_y(x)\ \bar{j}$$

of the bent axis of the beam as a function of x by the arc-length square

$$ds^2 = d\bar{r} \cdot d\bar{r} = [dx(d\bar{r}/dx)] \cdot [dx(d\bar{r}/dx)]$$

$$= dx^2\ [(d\bar{r}/dx) \cdot (d\bar{r}/dx)] = dx^2\ [1 + (du_y/dx)^2]$$

as

$$d\bar{r}/dx = \bar{i} + (du_y/dx)\ \bar{j}$$

Since x can be expressed as a function of arc-length s, then

$$x = x(s)$$

and

$$dx = ds\ (dx/ds)$$

Therefore,

$$ds^2\ \{1 - (dx/ds)^2\ [1 + (du_y/dx)^2]\} = 0$$

which for $ds^2 \neq 0$ gives

$$(dx/ds) = 1/[1 + (du_y/dx)^2]^{1/2}$$

Finally, the curvature of the bent axis of the beam can be expressed by means of its transverse displacement $u_y(x)$:

$$\kappa(x) = 1/\rho(x) = (d^2u_y/dx^2)/[1 + (du_y/dx)^2]^{3/2}$$

If the beam undergoes very small bending, the type of bending usually permitted in engineering design, then following the important idea of Daniel Bernoulli (1700-1782), a nephew of Jacob and a friend of Euler, which he used in 1734, the beam equation can be linearised. In small bending, the slope of the beam at every x is very small, such that

$$(du_y/dx)^2 \ll 1$$

Then the <u>square of the slope</u> can be neglected in comparison with 1, and consequently,

$$[1+(du_y/dx)^2]^{3/2} \approx 1$$

This important simplification of Daniel Bernoulli <u>linearises</u> the <u>Bernoulli-Euler Bending of Beam Equation</u>:

$$EI_{zz}(d^2u_y/dx^2) = M_z(\sigma) \qquad (I)$$

For this very reason, the name Bernoulli in the practically important <u>linear Bernoulli-Euler Beam Equation</u> refers to both Jacob and Daniel Bernoulli. It is important to realise that $M_z(\sigma)$ denotes the stress-couple, i.e., the moment of the stresses acting in the cross-section about the z-axis which passes through the centroid. $M_z(\sigma)$ <u>is not</u> the couple-moment $M_z(x)$ of the external applied loads (which is measured by the moments of applied forces relative to x) acting at the same section x, even though $M_z(\sigma)$ is evaluated in terms of the applied couple-moment $M_z(x)$ from the equilibrium of the sectional free-body of the beam, which is isolated as a free-body from the entire beam by an imaginary section at x.

Euler Equation of Beam Bending:

Some years later, after Euler had established his celebrated Field Equations of Beams (which also included the inertia forces):

$$dF_y(\sigma)/dx = -\,p_y(x)$$

$$dM_z(\sigma)/dx = -\,F_y(\sigma)$$

he derived another form of the beam equation in bending. By taking a derivative of the Bernoulli-Euler Beam Equation with respect to x and substituting from the second field equation gives:

$$d[EI_{zz}(d^2u_y/dx^2)]/dx = dM_z(\sigma)/dx = -F_y(\sigma)$$

Taking another derivative with respect to x and again substituting from the first field equation gives

$$d^2[EI_{zz}(d^2u_y/dx^2)]/dx^2 = -[dF_y(\sigma)/dx] = p_y(x)$$

The Euler Equation of Flexure,

$$d^2[EI_{zz}(d^2u_y/dx^2)]/dx^2 = p_y(x) \quad \text{(II)}$$

connects the applied load intensity $p_y(x)$ to the bending properties of the beam at point x along the axis of the beam.

If the beam is homogeneous, i.e. if,

$$E \neq E(x)$$

and if, moreover, the beam is also prismatic, i.e.

$$I_{zz} \neq I_{zz}(x)$$

then the Euler Equation of Flexure becomes

$$EI_{zz}(d^4u_y/dx^4) = p_y(x) \quad \text{(II)}$$

It should be noted that for EI_{zz} = constant,

$$-EI_{zz}d^3u_y/dx^3 = F_y(\sigma)$$

which permits the calculation of $F_y(\sigma)$ as a function of displacement $u_y(x)$.

Bending Stresses $\sigma^M_{xx}(y)$:

From Hooke's Law, as demonstrated,

$$\sigma^M_{xx}(y) = E\varepsilon^M_{xx}(y) = -E\,y\,\kappa(x)$$

due to the bending caused by the stress-couple $M_z(\sigma)$. Substituting from the Bernoulli-Euler Equation of Flexure

$$\kappa(x) = M_z(\sigma)/EI_{zz}$$

gives the bending stress

$$\sigma^M_{xx}(y) = - M_z(\sigma)y/I_{zz}$$

Thus bending stresses vary linearly with y, and vanish at the centroid, where y=0:

$$\sigma^M_{xx}(y=0) = 0$$

Combined State of Stress:

A fundamental assumption in mechanics of deformable solids is that small strains are superposable as shown in Figure 4. Therefore, a small axial strain

$$\epsilon^F_{xx} = F_x(\sigma)/AE$$

which is constant over the cross-section and created by the normal stress resultant $F_x(\sigma)$, can be superimposed on the bending strain

$$\epsilon^M_{xx}(y) = - M_z(\sigma)y/I_{zz}$$

created by the bending moment $M_z(\sigma)$ such that the total strain at point y in the cross-section at x is

$$\epsilon_{xx}(y) = \epsilon^F_{xx} + \epsilon^M_{xx}(y) = [F_x(\sigma)/AE] + [- M_z(\sigma)y/I_{zz}E]$$

From Hooke's Law, the corresponding total, or compound normal stress at y is

$$\sigma_{xx}(y) = E\epsilon_{xx}(y) = [F_x(\sigma)/A] - [M_z(\sigma)y/I_{zz}]$$

Trace of the Neutral Plane in the Cross-Section
Under Compound State of Stress:

The trace of the neutral plane in a cross-section is defined as the locus of points y_N at which the normal stress vanishes:

$$\sigma_{xx}(y_N) = [F_x(\sigma)/A] - [M_z(\sigma)y_N/I_{zz}] = 0$$

from which

$$y_N = F_x(\sigma)I_{zz}/M_z(\sigma)A$$

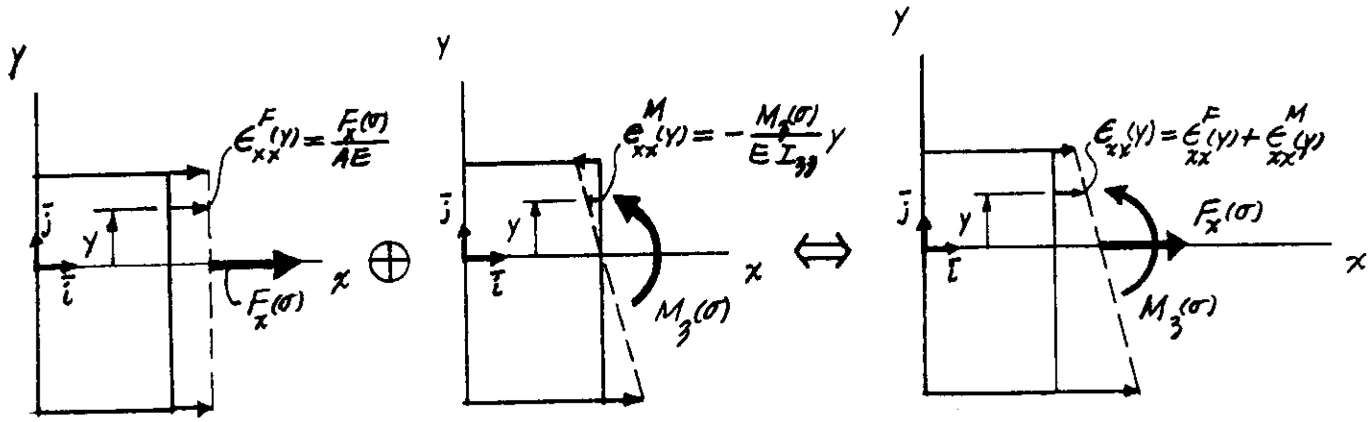
It is easy to see, that the neutral plane passes through the centroid:

$$y_N = 0$$

if, and only if,

$$F_x(\sigma) = 0$$

that is, for the case of pure bending when there is no axial stress resultant.



Superposition of Small Strains

Figure 4

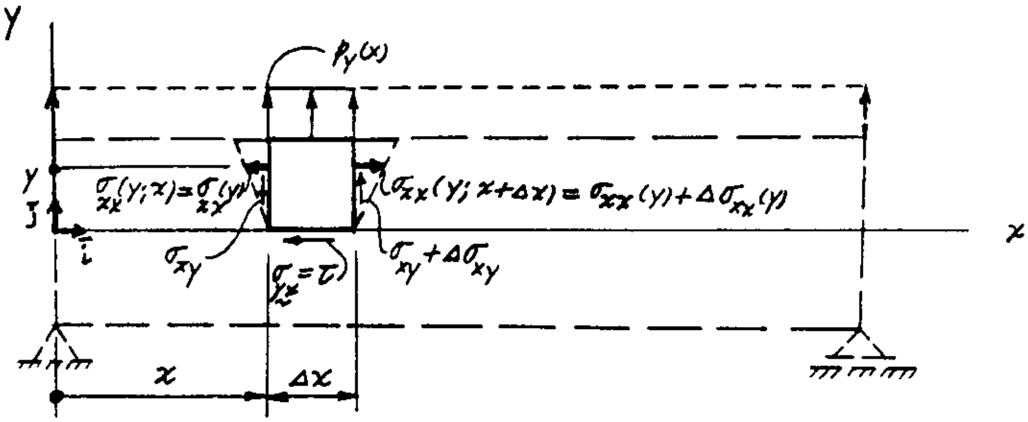
Shear Stresses in Beams Brought About by Bending

The French engineers used the theory of flexure given above, which was entirely correctly established by Henri L.C. Navier (1785-1836) in his lectures at the École des Ponts et Chaussées published in 1826. Everything in his book on mechanics of materials looks familiar to us. In it Euler's theory of beams and columns are perfected and the first statically indeterminate elastic problems are solved. The design of beams was essentially based on allowable bending stresses σ_{xx}, and professional experience of the engineer. Navier's book made structural theory into a science of mechanics. However, Navier, who also founded three dimensional theory of elasticity and mechanics of perfect fluids, seems not to have realised that another important stress had been neglected in the beam theory, a stress which has an important function in bending of beams and often governs the design of such structural members.

In 1844, a young Russian engineer, Dimitrii Ivanovich Zhuravskii (1821-1891), whilst designing bridges for the first Russian railroad, noted that many of the simply-supported timber baulks in the bridge decks which he had designed split longitudinally in the centre of the cross-section, a place where according to Navier the bending stress σ_{xx} is zero. Zhuravskii concluded that another important stress component in bending of beams had been neglected by Navier, a stress which is important in equilibrating the beam. The basic problem he faced was how this stress could be found. Zhuravskii hit on a brilliant idea: he sliced a secondary free-body, the so-called Zhuravskii Free-Body, out of the transversely cut free-body of length Δx by means of a horizontal section. The first such free-body he sliced along the split in the beam at the centre of the cross-section. By investigating the equilibrium state of this free-body he readily discovered that it was not in axial force equilibrium under the bending stresses. The only stress component that could provide axial force equilibrium of his free-body had to be a shear stress acting along the surface of the horizontal cut. See Figure 5 for illustration. Zhuravskii evaluated these longitudinal shear stresses for sufficiently narrow beams by using an average shear stress across the width of the beam,

$$\underset{\sim}{\sigma}_{yx} = \int_{-b/2}^{b/2} \sigma_{yx} dz/b(y) = \tau$$

where b(y) denotes the width of the beam at y.



Zhuravskii Free-Body

Figure 5

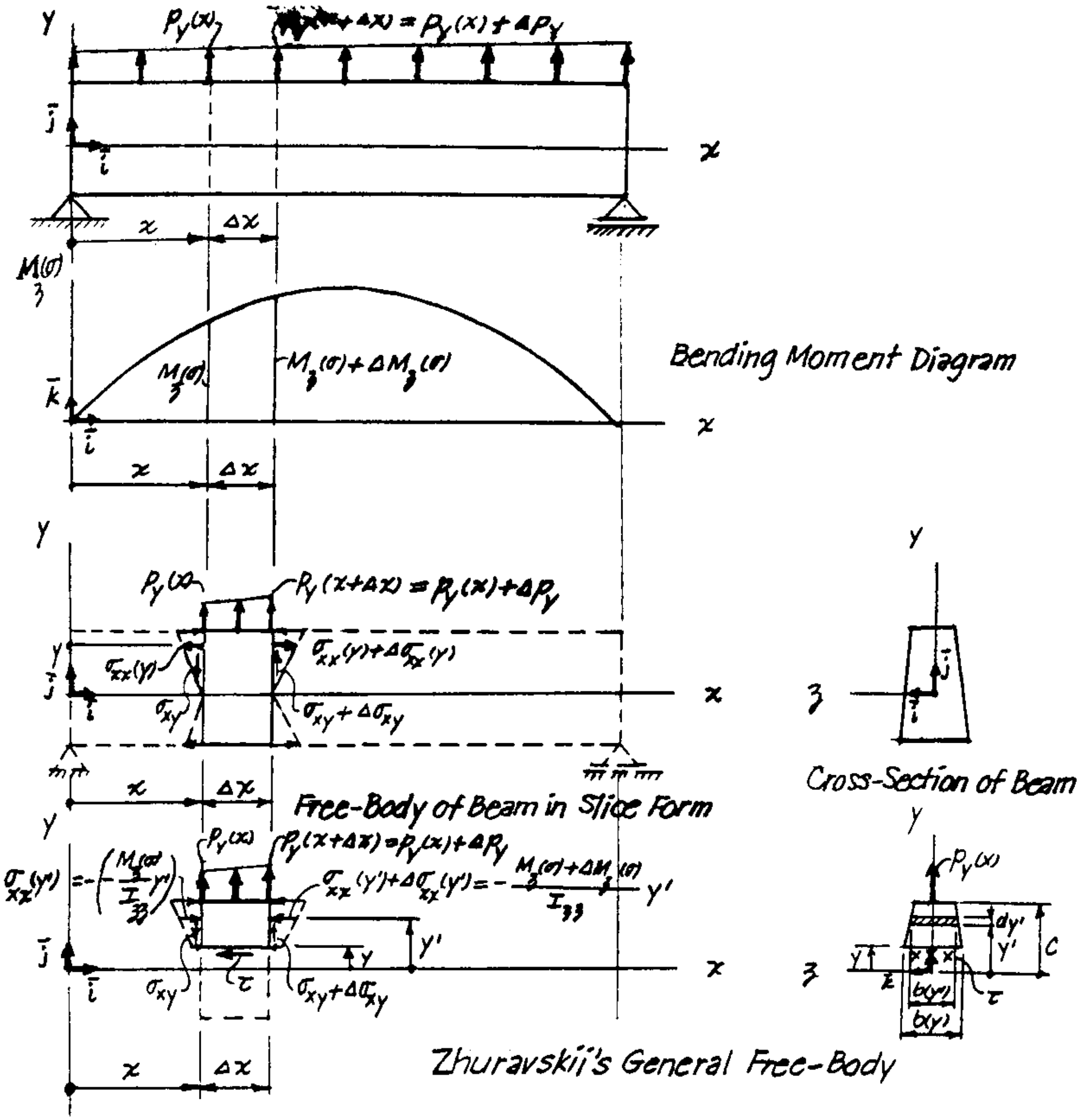
Zhuravskii established a simple shear stress formula, the so-called Zhuravskii Shear Stress Formula, for shear stress distribution over the depth of the beam by cutting the Zhuravskii free-body with a horizontal section at arbitrary y as shown in Figure 6. The stress resultant of bending stresses $\sigma_{xx}(y') + \Delta\sigma_{xx}(y')$ acting on the transverse cross-section of the Zhuravskii free-body at $x+\Delta x$ is given by

$$F_x(\sigma_{xx}+\Delta\sigma_{xx}) = \int_y^c (\sigma_{xx}+\Delta\sigma_{xx})\ b(y')dy' = -\int_y^c [M_z(\sigma)+\Delta M_z(\sigma)/I_{zz}]y'b(y')dy'$$

$$= -[M_z(\sigma)/I_{zz}] \int_y^c y'b(y')dy' - [\Delta M_z(\sigma)/I_{zz}] \int_y^c y'b(y')dy'$$

The stress resultant of bending stresses $-\sigma_{xx}(y')$ acting on the transverse cross-section of the Zhuravskii free-body at x is

$$F_x(-\sigma_{xx}) = -\int_y^c -[M_z(\sigma)/I_{zz}]\ y'b(y')dy' = [M_z(\sigma)/I_{zz}] \int_y^c y'b(y')dy'$$



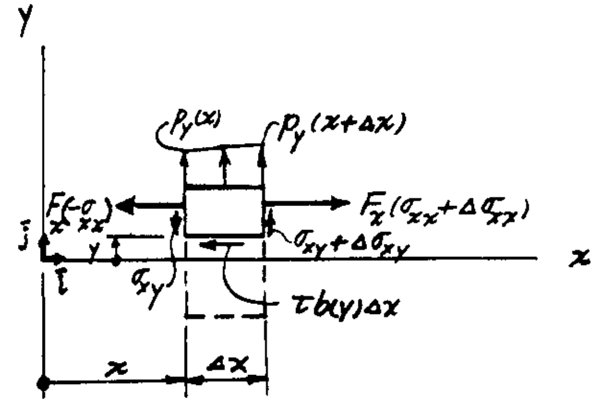
Zhuravskii's Analysis of Shear Stresses in Bending of Beams

Figure 6

The horizontal <u>average</u> shear stress acting on the lower horizontal surface at y contributes a stress resultant in x-direction

$$-\tau b(y)\,\Delta x$$

It is important to realise that the horizontal face has a normal vector $\bar{n} = -\bar{j}$, therefore the average stress τ acts in the $-\bar{i}$ direction as shown in Figure 7.



Axial Stress Resultants of Zhuravskii's General Free-Body

Figure 7

Since Zhuravskii's free-body must be in equilibrium, the Equivalent Force System constructed at x must vanish:

$$F_x = \bar{0} : F_{x,x} = -\tau b(y)\Delta x + F_x(\sigma_{xx}+\Delta\sigma_{xx}) + F_x(-\sigma_{xx})$$

$$= -\tau b(y)\Delta x - [\Delta M_z(\sigma)/I_{zz}] \int_y^c y'b(y')dy' = 0$$

Solving for the unknown average shear stress at x:

$$\tau = -\lim_{\Delta x \to 0} [\Delta M_z(\sigma)/\Delta x] \int_y^c y'b(y')dy'/I_{zz}b(y)$$

$$= -[dM_z(\sigma)/dx][\int_y^c y'b(y')dy']/I_{zz}b(y)$$

However, by substituting from Euler's Field Equation

$$dM_z(\sigma)/dx = - F_y(\sigma)$$

Zhuravskii Formula for the average shear stress due to bending of beams becomes

$$\tau = F_y(\sigma)Q(y)/I_{zz}b(y)$$

where

$$Q_y = \int_y^c y'b(y')dy'$$

represents the First Moment of the Cross-Section of the Zhuravskii free-body about the z-axis located at the centroid of the entire cross-section. It is important to realise that Zhuravskii's shear stress τ represents the average shear stress $\underset{\sim}{\sigma}_{yx}$ acting across the width of the beam. The average shear stress τ is only then a good approximation to the actual shear stress σ_{xy} when the width of the beam b(y) is small relative to the height of the beam, i.e. when the cross-section is sufficiently narrow. Otherwise the maximum shear stress σ_{xy} may be considerably larger than the calculated average shear stress τ.

Equality of the Magnitude of Shear Stresses $\sigma_{ij} = \sigma_{ji}$:

The average shear stress σ_{yx} acting on the horizontal plane at a point (x,y) in the beam can be found from the Zhuravskii Shear Stress Formula:

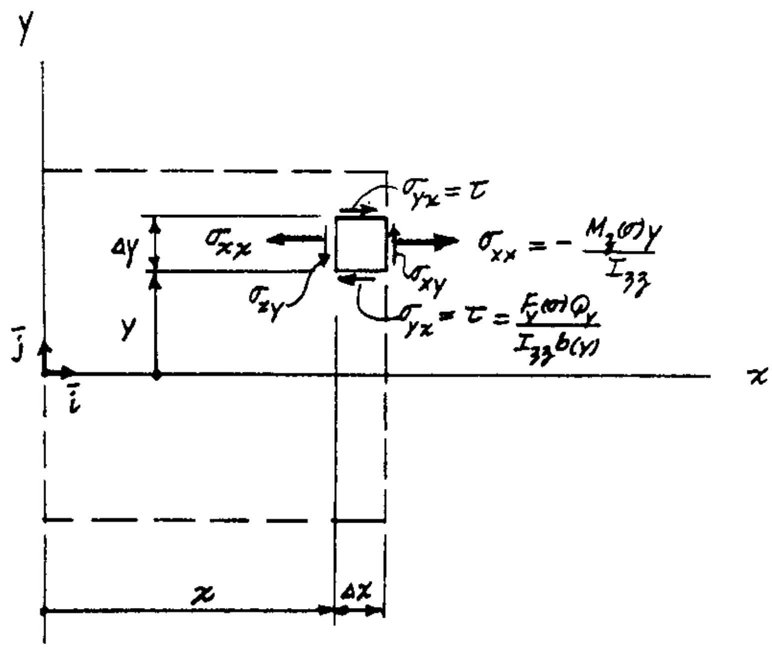
$$\underset{\sim}{\sigma}_{yx} = \tau = F_y(\sigma)Q_y/I_{zz}b(y)$$

The bending stress acting on the cross-section at the point (x,y) can be found from the Bending Stress Formula:

$$\sigma_{xx} = - M_z(\sigma)y/I_{zz}$$

If a rectangular element of the beam is isolated as shown in Figure 8 the following stresses are acting on it:

The stresses σ_{xx} and σ_{yx} are given by the two formulae, but the shear stress σ_{xy} acting in the cross-section is not given. It can be found from the moment equilibrium of the finite rectangular element Δx by Δy in the Figure 9. The Equivalent Force System constructed at the centroid c of the rectangular element must vanish for equilibrium.



Stresses Acting on Beam Element $\Delta x \Delta y \to 0$

Figure 8

Using only the Equivalent Couple-Moment for equilibrium gives:

$$\bar{C}_c = \bar{0} : M_{c,z} = (\Delta x/2)(\sigma_{xy} + \Delta\sigma_{xy})\Delta y + (\Delta x/2)\sigma_{xy}\Delta y$$

$$- (\Delta y/2)(\sigma_{yx}+\Delta\sigma_{yx})\Delta x - (\Delta y/2)(\sigma_{yx})$$

$$= \Delta x \ \Delta y \ \sigma_{xy} + (\Delta x \ \Delta y \ \Delta\sigma_{xy}/2)$$

$$- \Delta y \ \Delta x \ \sigma_{yx} - (\Delta y \ \Delta x \ \Delta\sigma_{yx}/2) = 0$$

Dividing this expression by $\Delta x \Delta y$ and taking the limit $\Delta x \to 0$, $\Delta y \to 0$, gives

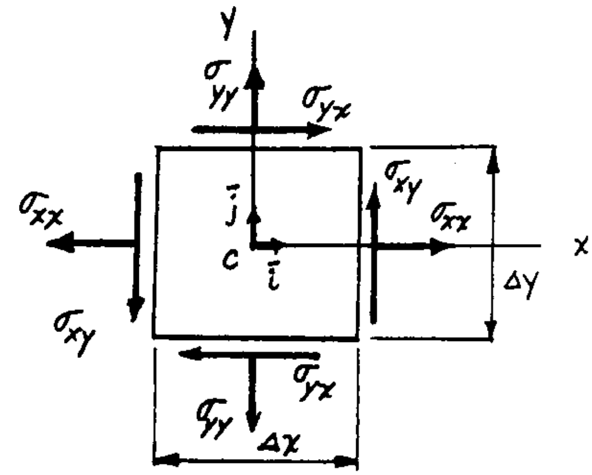
$$\sigma_{xy} - \sigma_{yx} = 0$$

which implies that the magnitudes, <u>but not the directions</u>, of the shear stresses σ_{xy} and σ_{yx} are equal:

$$\sigma_{xy} = \sigma_{yx}$$

The shear stresses must always act in the way shown in Figure 9:

the two sets of parallel shear stresses must always constitute two opposing force couples if moment equilibrium of the element is to be maintained.



Stresses Acting on Beam Element $\Delta x \Delta y \to 0$

Figure 9

EXAMPLE I-1

Second moment of Cross-Section:

The Bernoulli-Euler Equation of Flexure can be factored into a product of three factors

$$E I_{33} \frac{d^2 u_y}{dx^2} = M_3(x)$$

in which one of the factors

$$I_{33} = \int_A y^2 dA$$

is an integral of the geometry of the cross-section, which can be interpreted as the Second moment of the cross-section about the centroidal z-axis.

This integral is erroneously called "moment of Inertia" of the cross-section about the centroidal z-axis. This incompetent terminology was coined by a French engineer and teacher, Nicolas PERSY, in early 1840s. Persy encountered this integral in the book of structural mechanics of Henri C.L.M. NAVIER (1785-1836), the founder of structural mechanics as a science of mechanics. Since this integral looks similar to the mass integral

$$I_{33} = \int_m y^2 dm$$

in Euler's theory of rotating solid bodies, which Euler properly named moment of Inertia, PERSY improperly named the cross-sectional integral $I_{33} = \int_A y^2 dA$ also a moment of Inertia of the cross-section.

The remarkable fact about this ignorantly conceived neologism of Persy is that it has lasted till this very day. Engineers, and other professionals are notorious for their propagation of errors because they tend to be hide-bound by bad traditional practices.

Frequently the concept of radius of gyration ρ_3 of a cross-section $\rho_3 = \sqrt{\frac{I_{33}}{A}}$ from $I_{33} = \int_A y^2 dA = \rho_3^2 A$ is introduced in mechanics of materials in analogy to radius of gyration in dynamics: $\rho_3 = \sqrt{\frac{I_{33}}{m}}$

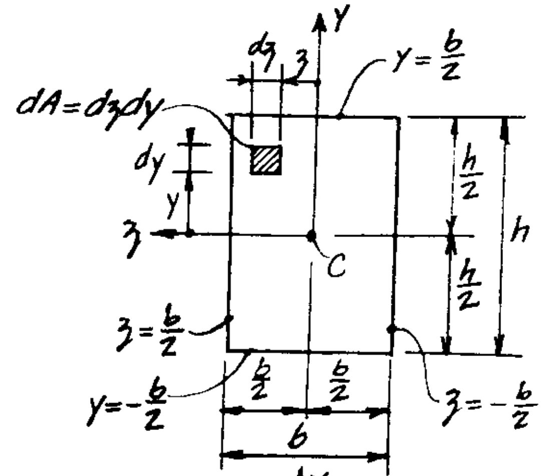
(Example I-2)

Second moment of a Rectangular Cross-Section about the ζ-axis:

$$I_{\zeta\zeta} = \int_A y^2 dA = \int_{y=-\frac{h}{2}}^{y=\frac{h}{2}} \int_{\zeta=-\frac{b}{2}}^{\zeta=\frac{b}{2}} y^2 d\zeta dy$$

$$= \int_{y=-\frac{h}{2}}^{y=\frac{h}{2}} y^2 \left[\int_{\zeta=-\frac{b}{2}}^{\zeta=\frac{b}{2}} d\zeta \right] dy = \int_{y=-\frac{h}{2}}^{y=\frac{h}{2}} y^2 dy \left[\zeta \right]_{-\frac{b}{2}}^{\frac{b}{2}}$$

$$= \int_{y=-\frac{h}{2}}^{y=\frac{h}{2}} y^2 b dy$$

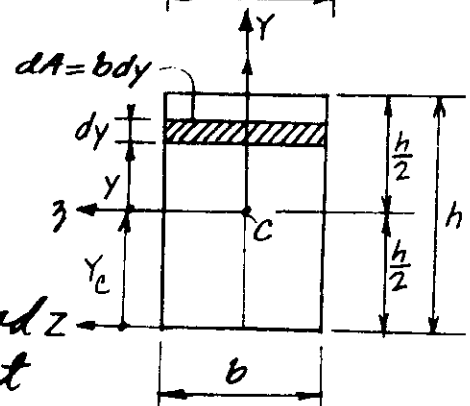


Now,

$$I_{\zeta\zeta} = \int_{y=-\frac{h}{2}}^{y=\frac{h}{2}} y^2 dA$$

where $dA = b dy$

can be interpreted as the strip method of calculating the Second Moment of Cross-Section. Very often engineers use the strip method and the single integral calculation of $I_{\zeta\zeta}$ as shown in the lower figure.



$$I_{\zeta\zeta} = \int_{y=-\frac{h}{2}}^{y=\frac{h}{2}} y^2 b dy = b \int_{y=-\frac{h}{2}}^{y=\frac{h}{2}} y^2 dy = b \left. \frac{y^3}{3} \right|_{-\frac{h}{2}}^{\frac{h}{2}} = b\left[\frac{h^3}{24} - \left(-\frac{h^3}{24}\right)\right] = \frac{bh^3}{12}$$

The Second Moment of Area about the non-central z-axis can be obtained from the Parallel Axes Theorem:

$$I_{zz} = I_{\zeta\zeta} + Y_c^2 A = \frac{bh^3}{12} + \left(\frac{h}{2}\right)^2 (bh) = \frac{bh^3}{3}$$

where

$$Y_c = \frac{h}{2}$$

and

$$A = bh$$

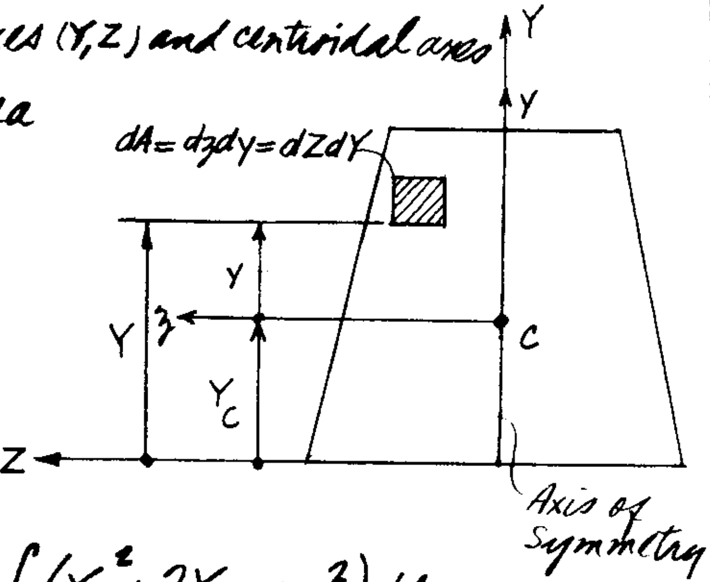
Parallel Axes Theorem:

Consider two sets of parallel axes (Y,Z) and centroidal axes (y,z). The Second Moment of Area relative to (Y,Z) axes is

$$I_{ZZ} = \int_A Y^2 \, dA$$

The transformation between the two coordinates is

$$Y = Y_c + y$$



Then

$$I_{ZZ} = \int_A (Y_c + y)^2 dA = \int_A (Y_c^2 + 2Y_c y + y^2) dA$$

$$= Y_c^2 \int_A dA + 2Y_c \int_A y\,dA + \int_A y^2 dA = Y_c^2 A + I_{zz}$$

where

$$I_{zz} = \int_A y^2 dA$$

is the Second moment of the cross-section about the centroidal axis z, and

$$\int_A y\,dA = 0$$

for centroidal axes.

Then

$$I_{zz} = I_{ZZ} - Y_c^2 A$$

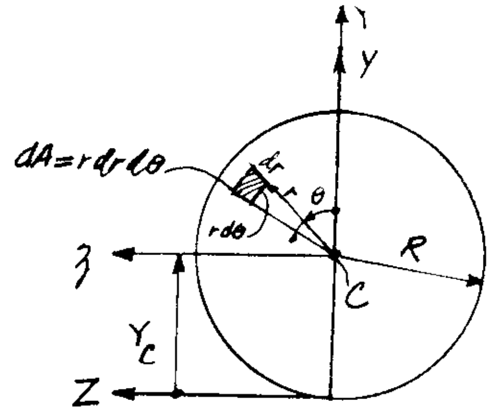
is called the Parallel Axes Theorem, which was first established for moments of inertia by Christiaan HUYGENS (1629-1695) in 1654. It is usually easier to calculate the Second moment of the cross-section relative to non-central axes such as (Y,Z)-axes. Then the Second moment of the cross-section relative to the centroidal axes, I_{zz}, can be obtained from I_{ZZ} by the Parallel Axes Theorem. Since

$$Y_c^2 A > 0$$

it is obvious that $I_{zz} > 0$ is the smallest possible Second moment of the Cross-section that can be calculated.

(Example I-4)

Second moment of Area of a Circular Cross-Section:



In order to find the Second moment of Area of the circular cross-section about the Z-axis, it is more convenient to find first the Second moment of Area about the centroidal axis z.

For a circular area it is more convenient to use polar coordinates:

$$y = r\cos\theta \quad \text{and} \quad dA = r\,dr\,d\theta$$

Then

$$I_{zz} = \int_A y^2\,dA = \int_0^{2\pi}\int_0^R (r\cos\theta)^2 r\,dr\,d\theta = \int_0^{2\pi}\cos^2\theta\,d\theta \int_0^R r^3\,dr$$

$$= \int_0^{2\pi}\frac{1}{2}(1+\cos 2\theta)\,d\theta \; \frac{r^4}{4}\Big|_0^R = \frac{R^4}{8}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = \frac{\pi R^4}{4}$$

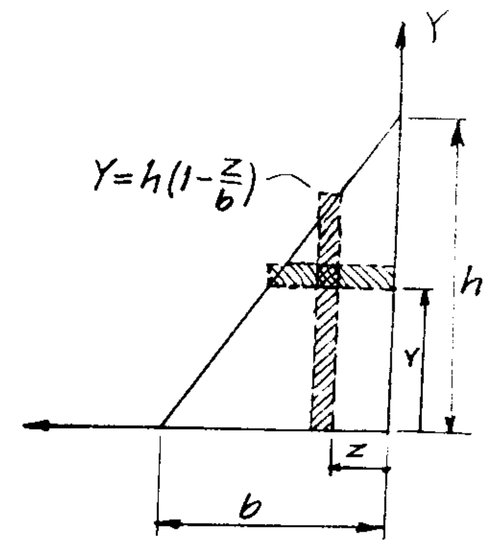
By Parallel Axes Theorem,

$$I_{ZZ} = I_{zz} + Y_C^2 A = \frac{\pi R^4}{4} + R^2(\pi R^2) = \frac{5}{4}\pi R^4$$

(Example I-5)

Second Moment of Area of a Triangular Cross-Section:

In order to find the centroidal Second Moment of Area I_{zz} for a triangular cross-section, it is useful to use the Parallel Axes Theorem.



Solution:

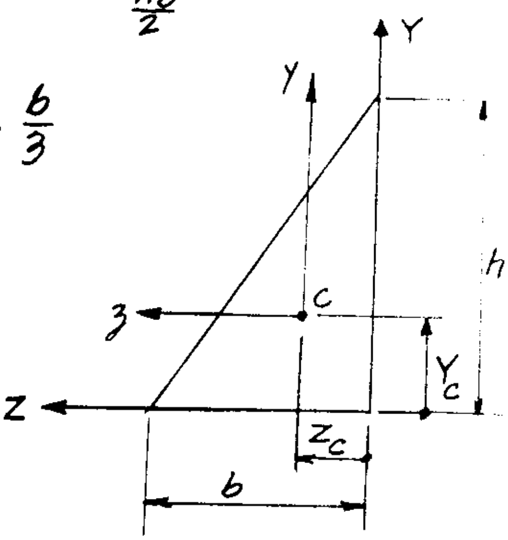
Centroid:

$$Y_c = \frac{\int_A Y\,dA}{\int_A dA} = \frac{\int_0^b \int_0^{Y=h(1-\frac{Z}{b})} Y\,dY\,dZ}{\int_0^b \int_0^Y dZ\,dY} = \frac{\int_0^b dZ\left[\frac{Y^2}{2}\right]_0^{h(1-\frac{Z}{b})}}{\int_0^b dZ\left[Y\right]_0^{h(1-\frac{Z}{b})}}$$

$$= \frac{\frac{h^2}{2}\int_0^b\left[1-2\frac{Z}{b}+\frac{Z^2}{b^2}\right]dZ}{h\int_0^b\left[1-\frac{Z}{b}\right]dZ} = \frac{\frac{h^2}{2}\left[Z-\frac{Z^2}{b}+\frac{Z^3}{3b^2}\right]_0^b}{h\left[Z-\frac{Z^2}{2b}\right]_0^b} = \frac{\frac{h^2 b}{6}}{\frac{hb}{2}} = \frac{h}{3}$$

$$Z_c = \frac{\int_A Z\,dA}{\int_A dA} = \frac{\int_0^b \int_0^Y Z\,dZ\,dY}{\int_0^b \int_0^Y dZ\,dY} = \frac{\int_0^b Z\,dZ\int_0^{Y=h(1-\frac{Z}{b})} dY}{\frac{hb}{2}} = \frac{\int_0^b Z\,dZ\left[Y\right]_0^{h(1-\frac{Z}{b})}}{\frac{hb}{2}}$$

$$= \frac{h\int_0^b Z\left[1-\frac{Z}{b}\right]dZ}{\frac{hb}{2}} = \frac{h\left[\frac{Z^2}{2}-\frac{Z^3}{3b}\right]_0^b}{\frac{hb}{2}} = \frac{\frac{hb^2}{6}}{\frac{hb}{2}} = \frac{b}{3}$$



$$I_{ZZ} = \int_A Y^2 dA = \int_0^b \int_0^{Y=h(1-\frac{Z}{b})} Y^2\,dY\,dZ = \int_0^b dZ\left[\frac{Y^3}{3}\right]_0^{h(1-\frac{Z}{b})}$$

$$= \int_0^b \frac{h^3}{3}\left(1-\frac{Z}{b}\right)^3 dZ = -\frac{h^3 b}{3}\int_1^0 \xi^3 d\xi = -\frac{h^3}{3}\left[\frac{\xi^4}{4}\right]_1^0$$

$$= \frac{bh^3}{12}$$

where $\xi = 1-\frac{Z}{b}$ and $d\xi = -\frac{1}{b}dZ$

with the boundary conditions: when $Z=b$, then $\xi=0$, when $Z=0$, then $\xi=1$.

Similarly,

$$I_{YY} = \int_A Z^2 dA = \int_0^b \int_0^Y Z^2\,dZ\,dY = \int_0^b Z^2\left[\int_0^{Y=h(1-\frac{Z}{b})} dY\right]dZ = \int_0^b Z^2\left[Y\right]_0^{h(1-\frac{Z}{b})} dZ$$

$$= \int_0^b hZ\left[1-\frac{Z}{b}\right]dZ = h\left[\frac{Z^3}{3}-\frac{Z^4}{4b}\right]_0^b = \frac{b^3 h}{3}$$

as was to be expected.

(Example I-6)

Even a Second Product of Area can be calculated:

$$I_{ZY} = \int_A ZY\,dA = \int_0^b \int_0^{Y=h(1-\frac{Z}{b})} ZY\,dY\,dZ = \int_0^b Z\left[\int_0^{Y=h(1-\frac{Z}{b})} Y\,dY\right]dZ = \int_0^b Z\left[\frac{Y^2}{2}\right]_0^{h(1-\frac{Z}{b})} dZ$$

$$= \frac{h^2}{2}\int_0^b Z\left[1-\frac{Z}{b}\right]^2 dZ = \frac{h^2}{2}\left[\frac{Z^2}{2} - \frac{2}{3}\frac{Z^3}{b} + \frac{Z^4}{4b^2}\right]_0^b = \frac{b^2h^2}{24}$$

Obviously

$$I_{YZ} = \int_A YZ\,dA = \int_A ZY\,dA = I_{ZY}$$

$I_{YZ} = I_{ZY}$ can be positive, negative, or zero scalar: $I_{YZ} = I_{ZY} \gtreqless 0$!

In unsymmetrical, or the so-called skew, bending all four quantities

$$I_{YY},\ I_{ZZ} \text{ and } I_{YZ} = I_{ZY}$$

appear in the flexure equation.

I_{zz} can now be found from the Parallel Axes Theorem:

$$I_{zz} = I_{ZZ} - Y_C^2 A = \frac{bh^3}{12} - \left(\frac{h}{3}\right)^2\left(\frac{bh}{2}\right) = \frac{bh^3}{36}$$

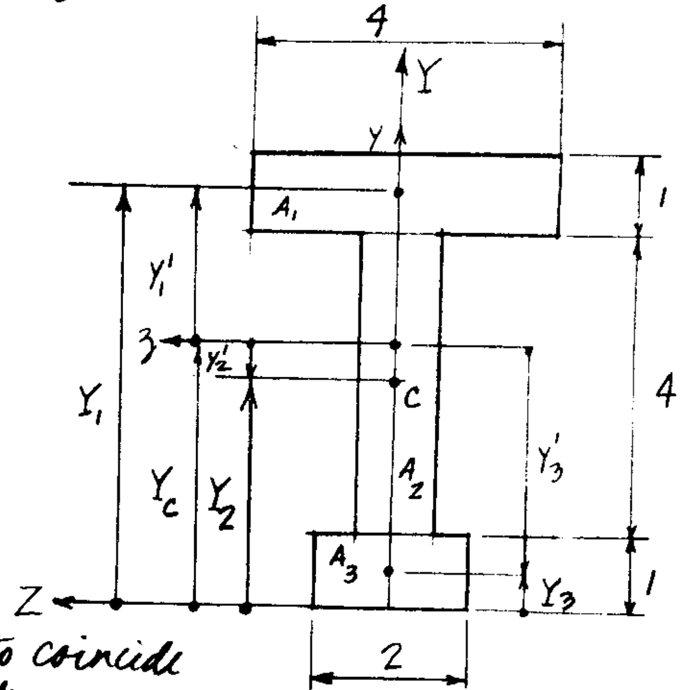
Similarly,

$$I_{yy} = I_{YY} - Z_C^2 A = \frac{b^3h}{12} - \left(\frac{b}{3}\right)^2\left(\frac{bh}{2}\right) = \frac{b^3h}{36}$$

$$I_{zy} = I_{ZY} - Y_C Z_C A = \frac{b^2h^2}{24} - \left(\frac{b}{3}\right)\left(\frac{h}{3}\right)\left(\frac{bh}{2}\right) = -\frac{b^2h^2}{72}$$

It should be noted that both Y_c and Z_c could be positive or negative depending upon the relative location of the two parallel axes. Y_c and Z_c always measure the centroid of the cross-section from the origin of the (Y, Z) axes, and their signs depend upon the location of the centroid of the cross-section relative to the (Y, Z)-axes.

EXAMPLE II



Find the centroid, and I_{33} and I_{YY}.

Solution:

Since this is a symmetrical cross-section, its centroid lies on the axis of symmetry. The Y and y axes of the frames (Y,Z) and (y,z) are made to coincide with the axis of symmetry.

Total Area $A = A_1 + A_2 + A_3$

Solution:

Centroid Y_c is given by

$$Y_c(A_1 + A_2 + A_3) = Y_1A_1 + Y_2A_2 + Y_3A_3$$

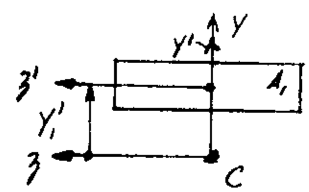
Thus

$$Y_c = \frac{Y_1A_1 + Y_2A_2 + Y_3A_3}{A_1 + A_2 + A_3} = \frac{(5.5)(4)+(3)(4)+(0.5)(2)}{(4+4+2)} = 3.5$$

The Second moment of Area about the central z-axis can be obtained by the Parallel Axes Theorem and superposition of Second moment of Rectangular Areas:

$$I_{33} = \left[I^1_{3'3'} + A_1Y_1'^2\right] + \left[I^2_{3'3'} + A_2Y_2'^2\right] + \left[I^3_{3'3'} + A_3Y_3'^2\right] = I^1_{33} + I^2_{33} + I^3_{33}$$

$$I^1_{3'3'} = \frac{4\times 1^3}{12} \quad \text{and} \quad I^1_{33} = I^1_{3'3'} + A_1Y_1'^2 = \frac{(4)(1)^3}{12} + (4)(2)^2$$



$$I_{33} = \left[\frac{(4)(1)^3}{12} + (4)(2)^2\right] + \left[\frac{(1)(4)^3}{12} + (-0.5)^2(4)\right] + \left[\frac{(2)(1)^3}{12} + (-3)^2(2)\right]$$

$$= 40.83$$

$$I_{YY} = \frac{(1)(4)^3}{12} + \frac{(4)(1)^3}{12} + \frac{(1)(2)^3}{12} = \frac{1}{12}(64+4+8) = \frac{76}{12} = \frac{19}{3}$$

NB! I_{33} of A_1:

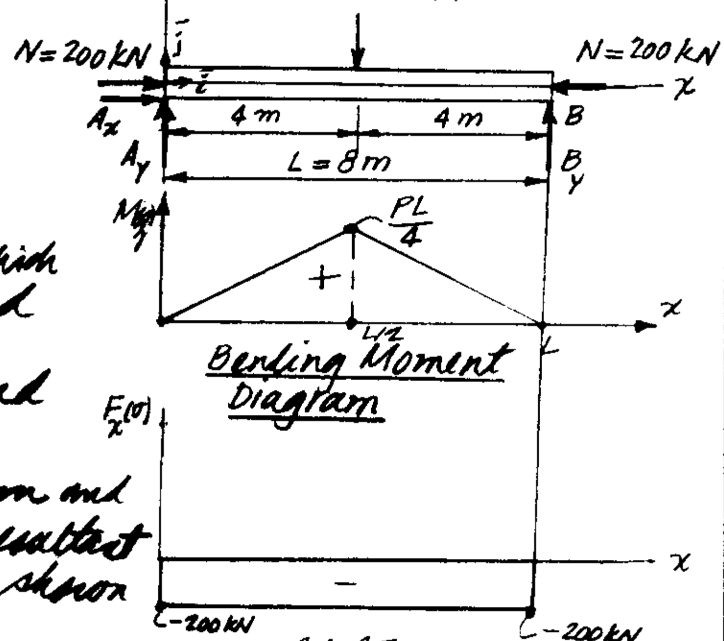
$$I_{33} = I_{3'3'} + (Y_1')^2A_1$$

EXAMPLE III-1

Find the maximum resultant stress acting over parts of the rectangular cross-section of a simply supported beam shown which is subjected to a vertical load $P = 8\,kN$ in the middle of the 8 m span, and an axial load $N = 200\,kN$.

The cross-section of the beam and cross-sectional areas over which the resultant stress is to be evaluated are shown in Figure 1.

Entire Beam as Free-Body



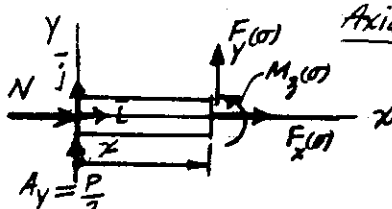
Axial Force Diagram

Solution:

Equilibrium of the Entire Beam as Free-Body:

Equivalent Force System at A vanishes for Equilibrium:

$$\begin{cases} \bar{F}_A = \bar{0}: F_{A,x} = 200 + A_x - 200 = 0 \therefore A_x = 0 \\ \quad F_{A,y} = A_y - 8 + B_y = 0 \therefore A_y = 8 - B_y = 4kN \\ \bar{C}_A = \bar{M}_A = \bar{0}: M_{A,z} = -(4)(8) + (8)B_y = 0 \therefore B_y = 4kN \end{cases}$$



Sectional Free-Body

Sectional Free-Body $0 \le x < \frac{L}{2}$:

Equivalent Force System at x vanishes for equilibrium:

$$\begin{cases} \bar{F}_x = \bar{0}: F_{x,x} = N + F_x(\sigma) = 0 \therefore F_x(\sigma) = -N = -200kN \\ \quad F_{x,y} = A_y + F_y(\sigma) = 0 \therefore F_y(\sigma) = -A_y = -4kN \\ \bar{C}_x = \bar{M}_x = \bar{0}: M_{x,z} = -xA_y + M_z(\sigma) = 0 \therefore M_z(\sigma) = xA_y = x(4) = 4x\ kNm \end{cases}$$

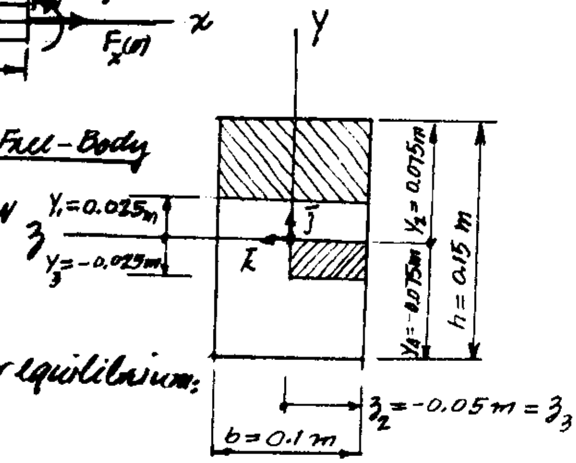


Figure 1

maximum Bending moment is at $x = 4$
(i.e. at the right boundary point of the region $0 \le x \le \frac{L}{2}$!)

$$\max M_z(\sigma) = (4)(4) = 16\,kN\cdot m$$

The Second moment of the Cross-Sectional Area

$$I_{zz} = \frac{bh^3}{12} = \frac{(0.1)(0.15)^3}{12} = (2.81)(10^{-5})\,m^4$$

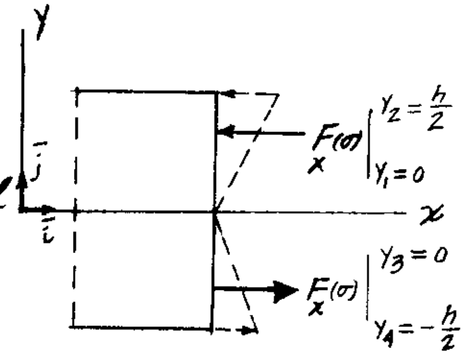


Figure 2

(Example III-2)

(a) Assume there is no axial applied force, i.e. $N=0$. Then the beam is only subject to bending owing to the transverse load $P = 8kN$ applied at the center of the beam. Find the maximum resultant of stresses caused by bending between y_1, z_1 and y_2, z_2 in the cross-section.

The normal stress due to bending is given by the formula

$$\sigma_{xx}(y) = -\frac{M_z(\sigma)\,y}{I_{zz}}$$

The magnitude of the stress σ_{xx} depends upon the magnitude of $M_z(\sigma)$, and the magnitude of y. To maximize σ_{xx}, the bending moment $M_z(\sigma)$ ought to be the largest possible, which occurs at the center of the span $x = L/2 = 4\,m$. It should be noted that I_{zz} is a fixed value for prismatic beams!

The resultant stress acting over any arbitrary part, such as over a rectangular area between y_1, z_1 and y_2, z_2 in the cross-section, which is created only by bending of the beam is

$$F_x(\sigma)\Big|_{y_1,z_1}^{y_2,z_2} = \int_{z_1}^{z_2}\int_{y_1}^{y_2} \sigma_{xx}(y)\,dz\,dy = \int_{z_1}^{z_2}\int_{y_1}^{y_2} -\frac{M_z(\sigma)}{I_{zz}}\,y\,dz\,dy = -\frac{M_z(\sigma)}{I_{zz}}\int_{z_1}^{z_2} dz \int_{y_1}^{y_2} y\,dy$$

$$= -\frac{M_z(\sigma)}{I_{zz}}\; z\Big|_{z_1}^{z_2} \frac{y^2}{2}\Big|_{y_1}^{y_2} = -\frac{M_z(\sigma)}{2I_{zz}}(z_2 - z_1)(y_2^2 - y_1^2)$$

(i) Consider the cross-sectional area between $y_1 = 0.025m$, $z_1 = -0.05m$ and $y_2 = 0.075m$, $z_2 = 0.05m$:

$$F_x(\sigma)\Big|_{y_1,z_1}^{y_2,z_2} = -\frac{(16)(10^3)}{2(2.81)(10^{-5})}\left[0.05 - (-0.05)\right]\left[0.075^2 - 0.025^2\right] = -0.142\ MN$$

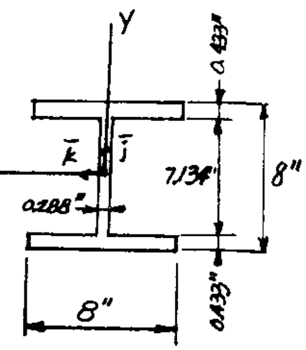
where max. $M_z(\sigma) = \frac{PL}{4} = 16\,kN\cdot m$

EXAMPLE XXII

Find the distribution of the shear stress $\tau(y)$ over y in a wide-flange section shown. The Second Moment of Area of the cross-section about the centroidal z axis is

$$I_{zz} = 108.0 \text{ in}^4$$

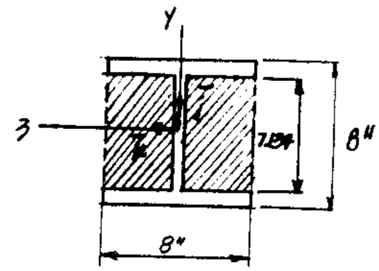
All cross-sectional dimensions are in inches. The shear force in the cross-section is $F_y(\sigma)$.



Solution:

I_{zz} can easily be calculated by composition based on the result that for a rectangular cross-section $b \times h$:

$$I_{zz} = \frac{bh^3}{12}$$



Thus

$$I_{zz} = \frac{1}{12}(8)(8)^3 - \frac{1}{12}(7.712)(7.134)^3 = \frac{1}{12}\left[8^4 - (7.712)(7.134)^3\right] = \frac{1295.45}{12}$$

$$= 108.0 \text{ in}^4$$

The width of the cross-hatched cross-sectional area:

$8 - 0.288 = 7.712$ in

In Web: $-3.567 < y < 3.567$:

then

$$\frac{Q_y}{b(y)} = \frac{8(0.433)(3.784) + (0.288)(3.567-y)(3.567+y)/2}{(0.288)} = 51.86 - (0.5)y^2$$

$$\tau(y) = \frac{F_y(\sigma)}{I_{zz}}\frac{Q_y}{b(y)} = F_y(\sigma)\frac{[51.86 - (0.5)y^2]}{(108.0)}$$

At $y = 0$: $\tau_{max} = (0.480)F_y(\sigma)$

because

$$\frac{d\tau(y)}{dy} = \frac{F_y(\sigma)}{108.0}(-y) = 0 \quad \therefore y = 0!$$

In Flanges: $3.567 < y < 4.0$:

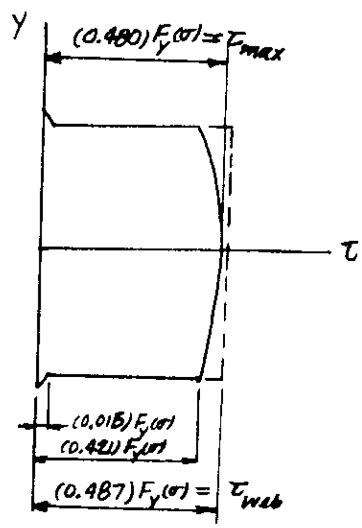
then

$$\frac{Q_y}{b(y)} = \frac{(8)[4-y](4+y)/2}{(8)} = 8 - (0.5)y^2$$

$$\tau(y) = \frac{F_y(\sigma)}{I_{zz}}\frac{Q_y}{b(y)} = F_y(\sigma)\frac{[8-(0.5)y^2]}{(108)} = F_y(\sigma)[0.074 - 0.00046y^2]$$

At $y = 3.567$ in:

$$\tau = (0.015)F_y(\sigma)$$



The total shear force in the web:

$$F_y^w(\sigma) = \int_{-3.567}^{3.567} \tau(y)[(0.288)dy] = \frac{(0.288)}{(108.0)}F_y(\sigma)\int_{-3.567}^{3.567}[51.86 - (0.5)y^2]dy = (0.947)F_y(\sigma)$$

$\therefore$ Therefore, in the I beams the web provides 94.7% of the vertical shear force $F_y(\sigma)$ in the cross-section! Only 5.3% of the shear force $F_y(\sigma)$ is provided by the flanges. Therefore, in design the shear force is assumed to be resisted entirely by the web and is approximated by the average shear stress in the web:

$$\tau_{web} \simeq \frac{F_y(\sigma)}{A_{web}} = \frac{F_y(\sigma)}{(0.288)(7.134)} = (0.487)F_y(\sigma)$$