This is an extension of De Villiers, M. (2005). Feedback: Feynman's Triangle. *The Mathematical Gazette*, 89 (514), March 2005, p. 107.

Feynman's Triangle: Some Feedback and More

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According to Cook & Wood (2004), the following problem was posed to the famous physicist Richard Feynman at dinner one evening after a colloquium at Cornell University, and he spent most of the evening first trying to disprove it, but finally proved it. Cook & Wood (2004) proceed further to provide four different instructive proofs.

Problem 1

For a triangle in the plane, if each vertex is joined to the point one-third along the opposite side (measured say anti-clockwise), prove that the area of the inner triangle formed by these lines is exactly one-seventh of the area of the initial triangle.

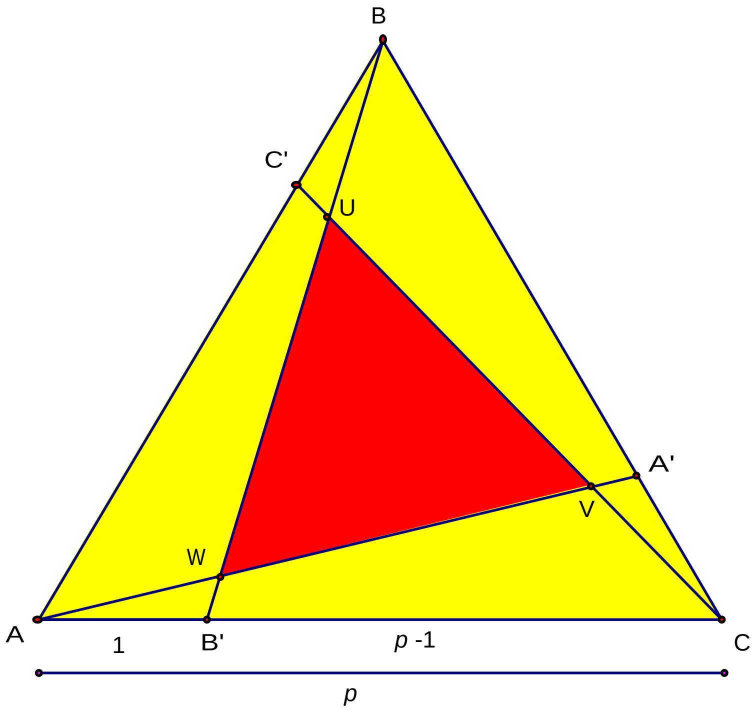

Figure 1

As pointed out by De Villiers (2005) and others in feedback in a later issue of the *Mathematical Gazette*, this delightful result can be generalised as follows.

Theorem 1

For a triangle in the plane, if each vertex is joined to the point $\frac{1}{p}$ ($p > 2$) along the opposite side (measured say anti-clockwise), then the area of the inner triangle formed by these lines is exactly $\frac{(p-2)^2}{p^2-p+1}$ of the area of the initial triangle.

So for $p = 4$, the fraction is $\frac{4}{13}$, for $p = 5$, it is $\frac{3}{7}$, etc. Probably the easiest way to derive this formula is to generalise the affine proof given in Cook & Wood (1994) as follows.

Proof

Consider an equilateral triangle *ABC* as shown in Figure 1. Let $AB = BC = CA = p$, thus $C'B = 1$. Then by the cosine rule $CC'^2 = p^2 + 1 - 2p\cos 60° = p^2 - p - 1$. Therefore, $CC' = \sqrt{p^2 - p + 1}$. The triangles CBC' and BUC' are similar since $\angle UC'B = \angle BC'C$ and $\angle UBC' = \angle BCC'$. Thus: $C'U = \frac{1}{\sqrt{p^2-p+1}}$, BU (= CV) = $\frac{p}{\sqrt{p^2-p+1}}$, and so VU = $\sqrt{p^2-p+1} - \frac{1}{\sqrt{p^2-p+1}} - \frac{p}{\sqrt{p^2-p+1}} = \frac{p(p-2)}{\sqrt{p^2-p+1}}$. By symmetry, *UVW* is also an equilateral triangle with sides $\frac{p(p-2)}{\sqrt{p^2-p+1}} \times \frac{1}{p} = \frac{p-2}{\sqrt{p^2-p+1}}$ of the sides of *ABC*. Therefore the area *UVW* is $\frac{(p-2)^2}{p^2-p+1}$ of the area *ABC*.

Since the theorem only involves affine properties such as ratios of lines and areas, and every triangle is affinely equivalent to an equilateral triangle the general result follows.

Feynman's triangle can also be generalized to a parallelogram as pointed out be De Villiers (2005) by observing as follows that the side opposite a vertex in a triangle is the same as the alternate side from that vertex:

Theorem 2

For a parallelogram in the plane, if each vertex is joined to the point $\frac{1}{p}$ ($p \geq 2$) along the alternate side (measured say anti-clockwise), then the area of the inner parallelogram

EFGH formed by these lines is exactly $\frac{p^2 - 2p + 1}{p^2 + 1}$ of the area of the initial parallelogram *ABCD* (see Figure 2).

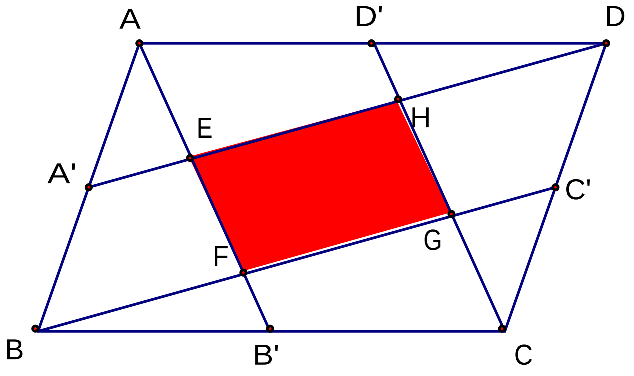

Figure 2

So for p = 2, the fraction is $\frac{1}{5}$, for p = 3, it is $\frac{2}{5}$, for p = 4, it is $\frac{9}{17}$, etc. Since a square is affinely equivalent to a parallelogram, the easiest way to derive and prove this formula is to consider the special case of a square.

Exercise

1. For a triangle *ABC* in the plane, if each side is divided by a point $\frac{1}{p}$ ($p \geq 2$) (measured say anti-clockwise), then find (and prove) a formula relating the area of *ABC* with that of *A'B'C'* (triangle formed by these points). See Figure 3.

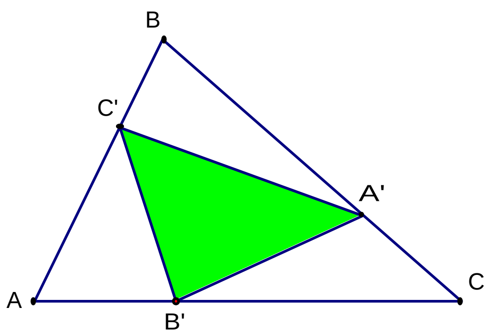

Figure 3

2. For a parallelogram *ABCD* in the plane, if each side is divided by a point $\frac{1}{p}$ $(p \geq 2)$ (measured say anti-clockwise), then find (and prove) a formula relating the area of *ABCD* with that of *A'B'C'D'* (parallelogram formed by these points). See Figure 4.

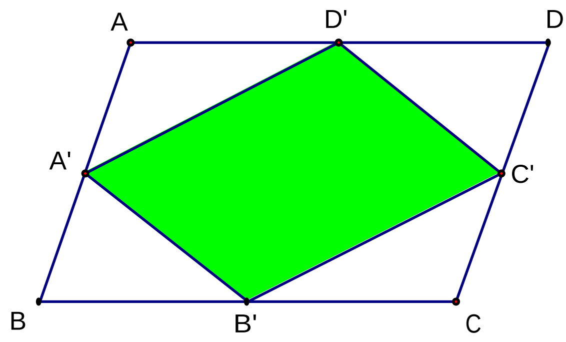

Figure 4

Note

Sketchpad sketches for exploring and demonstrating all four results and problems can be downloaded in zipped format from:

http://mysite.mweb.co.za/residents/profmd/feynman.zip

These dynamic geometry figures can also be viewed and manipulated with the Free Demo of *Sketchpad* which can be downloaded from:

http://www.keypress.com/sketchpad/sketchdemo.html

References

Cook, R.J. & Wood, G.V. (2004). Note 88.46: Feynman's Triangle. *The Mathematical Gazette*, July, Vol. 88, No. 512, pp. 299-302.

De Villiers, M. (2005). Feedback: Feynman's Triangle. *The Mathematical Gazette*, 89 (514), March 2005, p. 107.