STOCHASTIC ANALYSIS OF TIDAL DYNAMICS EQUATION

U. MANNA*, J.L. MENALDI, AND S.S. SRITHARAN*

Abstract. In this paper the deterministic and stochastic tidal equation in 2-D have been studied. The main results of this work are the existence and uniqueness theorem for strong solutions. These results are obtained by utilizing a global monotonicity property of the sum of the linear and nonlinear operators and exploiting the generalized Minty-Browder technique.

1. Introduction

The ocean tides have long been of interest to humans. A full account of the general theory and history of the tidal waves can be found Lamb [8] and summarized as follows. First, Newton [16] established the foundations for the mathematical explanation of tides, which motivated Maclaurin [11] to investigate the effect on tidal dynamics due to Earth's rotation. Although Euler realized that the horizontal component of the tidal force has more effect than the vertical component to drive the tide, the first major theoretical formulation for water tides on a rotating globe was made by Laplace [9], who formulated a system of partial differential equations relating the horizontal flow to the surface height of the ocean. The Laplace theory was developed further by Thomson and Tait [21] and Poincaré [19].

In the last few decades many rapid progress in theoretical and experimental studies of ocean tides can be observed. These days both the experimental and the theoretical information on ocean tides are being used to study important problems not only in oceanography but also in atmospheric sciences, geophysics as well as in electronics and telecommunications. For extensive study on the recent progress in this field we refer the readers [13, 14, 18].

Marchuk and Kagan [13] considered the tidal dynamics model which can be obtained from taking the shallow water model on a rotating sphere and is a slight generalization of the Laplace model. The existence and uniqueness of the deterministic tide equation by using the classical compactness method have been proved in [5, 13]. In this paper we have considered the model described in [5, 13] and proved

2000 *Mathematics Subject Classification.* Primary 76D05; Secondary 35Q30, 60H15, 76D03, 76D06.

Key words and phrases. Tidal dynamics equations, Maximal monotone operator.

* This research is supported by Army Research Office, Probability and Statistics Program, grant number DODARMY1736.

the existence and uniqueness of strong solutions for the stochastic tide equation in bounded domains. A brief description of the model has been given in Section 2. In Section 3 we have defined few standard well known function spaces and then proved the global monotonicity property of the nonlinear operator of the tidal equation. Then we establish certain new a priori estimates which play a fundamental role in the proof of existence and uniqueness of weak and strong solution proved in the second half of Section 3 and in the Section 4 for the deterministic and stochastic cases respectively. The monotonicity argument used here is the generalization of the classical Minty-Browder method for dealing with global monotonicity. Here the use of global monotonicity avoids the classical method based on compactness and thus the results apply to unbounded domains and hence the existence and the uniqueness results are new even in the deterministic case. The Minty-Browder technique for dealing with local monotonicity was first used by Menaldi and Sritharan [15] for the stochastic Navier-Stokes equation and by Manna, Menaldi and Sritharan [12] for the stochatic Navier-Stokes equation with artificial density. Similar ideas were used by Sritharan and Sundar [20] for the stochastic Navier-Stokes equation with multiplicative noise and also in Barbu and Sritharan [2, 3].

2. Tidal Dynamics: The Model

Under the following assumptions: (1) the Earth is a perfectly solid body, (2) tides in the ocean do not change the Earth's gravitational field, and (3) there is no energy exchange between the mid-ocean and the shelf zone, Marchuk and Kagan [13] obtain the following mathematical model

$$\partial_t \mathbf{w} + A_1 \mathbf{w} - \kappa_h \triangle \mathbf{w} + \frac{r}{h}|\mathbf{w}|\mathbf{w} + g\nabla\xi = \mathbf{f}, \tag{2.1}$$

$$\partial_t \xi + \mathrm{Div}(h\mathbf{w}) = 0, \tag{2.2}$$

in $\mathcal{O} \times [0,T]$, where $\mathcal{O}$ is a bounded 2-D domain (horizontal ocean basin) with coordinates $x = (x_1, x_2)$ and t represents the time. Here ∂_t denotes the time-derivative, $\triangle$, ∇ and Div are the Laplacian, gradient and the divergence operators respectively.

The unknown (dependent) variables $(\mathbf{w}, \xi)$, functions of (x,t), represent the total transport 2-D vector (i.e., the vertical integral of the velocity)and the deviation of the free surface with respect to the ocean bottom.

The coefficients $A_1 = [a_{ij}]$ is a 2-dimensional antisymmetric square matrix with constant coefficients $a_{11} = a_{22} = 0$ and $-a_{12} = a_{21} = 2\omega_z$, the Coriolis parameter (i.e., $\omega_z = \omega\cos(\varphi)$, ω is the angular velocity of the Earth rotation and φ the latitude), $\kappa_h > 0$ the constant horizontal macro turbulent viscosity coefficient, $r > 0$ the constant bottom friction coefficient equal to a numerical constant, g the Earth gravitational constant, $h = h(x)$ is the (vertical) depth at x in the region $\mathcal{O}$ and $\mathbf{f} = \gamma_L g \nabla\xi^+$ is the known tide-generating force with γ_L the Love factor.

The domain contour $\partial\mathcal{O}$ consists of two parts, a solid part Γ_1 coinciding with the shelf edge and the open boundary Γ_2. The vector $\mathbf{w^0}$ of full flow is considered a known function of the horizontal coordinates and time. Thus a non-homogeneous Dirichlet boundary condition are added to the above PDE, namely

$$\mathbf{w} = \mathbf{w^0} \quad \text{on} \quad \partial\mathcal{O} \times [0,T]. \tag{2.3}$$

The full flow satisfies

$$\mathbf{w^0} = 0 \quad \text{on} \quad \Gamma_1 \tag{2.4}$$

and

$$\oint \mathrm{d}t \int_{\Gamma_2} \mathbf{w^0} \cdot \mathbf{n} \mathrm{d}x = 0, \tag{2.5}$$

where the time integral is extended to one time-period and $\mathbf{n}$ is a normal to the contour Γ_2.

Here (2.4) represents the fact that the no-slip boundary condition is fulfilled on the contour Γ_1 while (2.5) means that the water transfer via the open boundary Γ_2, integrated during the tidal cycle turns to zero.

To take into consideration the redistribution of water masses, one should add an integro-differential term of the form

$$g\nabla \int_{\mathcal{O}} K(x,y)\xi(y)\mathrm{d}y, \tag{2.6}$$

where $K(x,y) = G(\lambda', \varphi', \lambda, \varphi)$ with λ the longitude and φ the latitude has the form

$$\sum_n (1 + k_n - h_n)\alpha_n \sum_{m\le n} g_{nm} P_{nm}(\sin\varphi) P_{nm}(\sin\varphi') \cos[m(\lambda + \lambda')], \tag{2.7}$$

where k_n, h_n are the Love factors of order n, $\alpha_n := (0.18)(3/2n+1)$, $g_{n0} := (2n+1)/(4\pi)$, $g_{nm} := [(2n+1)(n!)^2]/[2\pi(n-m)!(n+m)!]$, and P_{nm} are the associated Legendre functions. Note that the n-term correspond to the expansion in spherical harmonics.

Denote by $\mathbf{A}$ the following matrix operator

$$\mathbf{A} := \begin{pmatrix} -\alpha\triangle & -\beta \\ \beta & -\alpha\triangle \end{pmatrix} \tag{2.8}$$

and the nonlinear vector operator

$$\mathbf{v} \mapsto \gamma(x)|\mathbf{v}|\mathbf{v} := \begin{pmatrix} \gamma(x)v_1\sqrt{(v_1)^2 + (v_2)^2} \\ \gamma(x)v_2\sqrt{(v_1)^2 + (v_2)^2} \end{pmatrix}, \tag{2.9}$$

where $\alpha := \kappa_h$ and $\beta := 2\omega_z$ are positive constants, $\gamma(x) := r/h(x)$ is strictly positive smooth function. In this model we assume the depth $h(x)$ to be continuously differentiable function of x, nowhere becoming zero, so that

$$\min_{x\in\mathcal{O}} h(x) = \varepsilon > 0, \quad \max_{x\in\mathcal{O}} h(x) = \mu, \quad \text{and} \quad \max_{x\in\mathcal{O}} |\nabla h(x)| \leq M, \tag{2.10}$$

where M is a some positive constant which equals to zero at a constant ocean depth.

To reduce to homogeneous Dirichlet boundary conditions the natural change of unknown functions

$$\mathbf{u}(x,t) := \mathbf{w}(x,t) - \mathbf{w^0}(x,t), \tag{2.11}$$

and

$$z(x,t) := \xi(x,t) + \int_0^t \mathrm{Div}(h\mathbf{w^0}(x,s))\mathrm{d}s, \tag{2.12}$$

which are referred to as the *tidal flow* and the *elevation*. The full flow $\mathbf{w^0}$, which is given a priori on the boundary $\partial\mathcal{O}$, has been extended to the whole domain $\mathcal{O}\times]0,T]$ as a smooth function and still denoted by $\mathbf{w^0}$.

Then the tidal dynamic equation can be written as

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{A}\mathbf{u} + \gamma|\mathbf{u}+\mathbf{w^0}|(\mathbf{u}+\mathbf{w^0}) + g\nabla z = \mathbf{f}' \quad \text{in} \quad \mathcal{O}\times[0,T], \\ \partial_t z + \mathrm{Div}(h\mathbf{u}) = 0 \quad \text{in} \quad \mathcal{O}\times[0,T], \\ \mathbf{u} = \mathbf{0} \quad \text{on} \quad \partial\mathcal{O}\times[0,T], \\ \mathbf{u} = \mathbf{u_0}, \quad z = z_0 \quad \text{in} \quad \mathcal{O}\times\{t=0\}, \end{cases} \tag{2.13}$$

where

$$\mathbf{f}' = \mathbf{f} - \frac{\partial \mathbf{w^0}}{\partial t} + g\nabla \int_0^t \mathrm{Div}(h\mathbf{w^0})\mathrm{d}t - \mathbf{A}\mathbf{w^0}, \tag{2.14}$$

$$\mathbf{u_0}(x) = \mathbf{w_0}(x) - \mathbf{w^0}(x,0), \tag{2.15}$$

$$z_0(x) = \xi_0(x). \tag{2.16}$$

So the nonlinear and linear partial differential equations in (2.13) are coupled with an integro-differential equation (2.14).

An integro-differential term of the form

$$Kz(x) := g\nabla \int_{\mathcal{O}} K(x,y)z(y)\mathrm{d}y \tag{2.17}$$

may be added, where the kernel is a symmetric function $K(x,y) = K(y,x)$ for any x, y in $\mathcal{O}$. Then the first equation in (2.13) will be replaced by

$$\partial_t \mathbf{u} + \mathbf{A}\mathbf{u} + \gamma|\mathbf{u}+\mathbf{w^0}|[\mathbf{u}+\mathbf{w^0}] + g\nabla[z + Kz] = \mathbf{f} \quad \text{in} \quad \mathcal{O}\times[0,T], \tag{2.18}$$

where K denotes the integral operator (2.17).

3. Deterministic Setting: Global Monotonicity and Solvability

In this paper the standard spaces used are as follows:
$\mathbb{H}_0^1(\mathcal{O})$ with the norm

$$\|\mathbf{v}\|_{\mathbb{H}_0^1} := \Big(\int_{\mathcal{O}} |\nabla \mathbf{v}|^2 dx\Big)^{1/2}, \tag{3.1}$$

and $\mathbb{L}^2(\mathcal{O})$ with the norm

$$\|\mathbf{v}\|_{\mathbb{L}^2} := \Big(\int_{\mathcal{O}} |\mathbf{v}|^2 dx\Big)^{1/2}. \tag{3.2}$$

Using the Gelfand triple $\mathbb{H}_0^1(\mathcal{O}) \subset \mathbb{L}^2(\mathcal{O}) \subset \mathbb{H}^{-1}(\mathcal{O})$ we may consider $\triangle$ or ∇ as a linear map from $\mathbb{H}_0^1(\mathcal{O})$ or $\mathrm{L}^2(\mathcal{O})$ into the dual of $\mathbb{H}_0^1(\mathcal{O})$ respectively. The inner product in the $\mathbb{L}^2$ or L^2 is denoted by $(\cdot,\cdot)$ and the induced duality by $\langle\cdot,\cdot\rangle$. So clearly

$$(\mathbf{u},\mathbf{v})_{\mathbb{L}^2} = \int_{\mathcal{O}} \mathbf{u}(x)\cdot\mathbf{v}(x)\mathrm{d}x$$

for any $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{L}^2(\mathcal{O})$.

Notice that by the divergence theorem,

$$(g\nabla z, h\mathbf{v})_{\mathbb{L}^2} = -(gz, \mathrm{Div}(h\mathbf{v}))_{\mathrm{L}^2}, \quad \forall z \in \mathrm{L}^2(\mathcal{O}),\ \mathbf{v} \in \mathbb{L}^2(\mathcal{O}). \tag{3.3}$$

Lemma 3.1. *For any real-valued smooth functions φ and ψ with compact support in $\mathbb{R}^2$, the following hold:*

$$\|\varphi\,\psi\|^2_{\mathrm{L}^2} \le \|\varphi\,\partial_1\varphi\|_{\mathrm{L}^1}\|\psi\,\partial_2\psi\|_{\mathrm{L}^1}, \tag{3.4}$$

$$\|\varphi\|^4_{L^4} \le 2\|\varphi\|^2_{\mathrm{L}^2}\|\nabla\varphi\|^2_{\mathbb{L}^2}. \tag{3.5}$$

Proof. The results stated above are classical and well known [7]. □

Notice that by means of the Gelfand triple we may consider $\mathbf{A}$, given by (2.8), as mapping $\mathbb{H}_0^1(\mathcal{O})$ into its dual $\mathbb{H}^{-1}(\mathcal{O})$.

Define the non-symmetric bilinear form

$$a(\mathbf{u},\mathbf{v}) := \alpha[(\partial_1 u_1, \partial_1 v_1) + (\partial_2 u_2, \partial_2 v_2)] + \beta[(u_1, v_2) - (u_2, v_1)] \tag{3.6}$$

on the (vector-valued) Sobolev space $\mathbb{H}_0^1(\mathcal{O}) := \mathrm{H}_0^1(\mathcal{O},\mathbb{R}^2)$, where $(\cdot,\cdot)$ denotes the inner product in the (vector-valued) Lebesgue space $\mathbb{L}^2(\mathcal{O}) := \mathrm{L}^2(\mathcal{O},\mathbb{R}^2)$, e.g., see Adams [1] for detail on Sobolev spaces. Thus if $\mathbf{u}$ has a smooth second derivative then

$$a(\mathbf{u},\mathbf{v}) = (\mathbf{A}\mathbf{u},\mathbf{v})$$

for every $\mathbf{v}$ in $\mathbb{H}_0^1(\mathcal{O})$. Moreover, the bilinear form $a(\cdot,\cdot)$ is continuous and coercive in $\mathbb{H}_0^1(\mathcal{O})$, i.e.,

$$|a(\mathbf{u},\mathbf{v})| \le C_1 \|\mathbf{u}\|_{\mathbb{H}_0^1} \|\mathbf{v}\|_{\mathbb{H}_0^1}, \quad \forall \mathbf{u},\mathbf{v} \in \mathbb{H}_0^1(\mathcal{O}), \tag{3.7}$$

$$(\mathbf{A}\mathbf{u},\mathbf{u}) = a(\mathbf{u},\mathbf{u}) = \alpha \|\mathbf{u}\|^2_{\mathbb{H}_0^1}, \quad \forall \mathbf{u} \in \mathbb{H}_0^1(\mathcal{O}), \tag{3.8}$$

for some positive constants $C_1 = \alpha + \beta$.

Now we state an useful result as a lemma from [6].

Lemma 3.2. *Let X be a normed linear space and let $\Omega \subset X$ be open. Let $J : \Omega \to \mathbb{R}$ be twice differentiable in Ω. Let $K \subset \Omega$ be convex. Then J is convex if and only if, for all $\mathbf{u},\mathbf{v} \in K$,*

$$J''(\mathbf{v};\mathbf{u},\mathbf{u}) = \frac{\mathrm{d}^2}{\mathrm{d}\theta\ \mathrm{d}\alpha} J(\mathbf{v} + \theta\mathbf{u} + \alpha\mathbf{u})\Big|_{\theta,\alpha=0} \ \ge 0.$$

Let us denote the nonlinear operator $\mathbf{B}(\cdot)$ by

$$\mathbf{v} \mapsto \mathbf{B}(\mathbf{v}) := \gamma|\mathbf{v} + \mathbf{w^0}|[\mathbf{v} + \mathbf{w^0}]. \tag{3.9}$$

Then we have the following lemma:

Lemma 3.3. *Let $\mathbf{u}$ and $\mathbf{v}$ be in $\mathbb{L}^4(\mathcal{O},\mathbb{R}^2)$. Then the following estimate holds:*

$$\langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{u} - \mathbf{v} \rangle \ \ge \ 0. \tag{3.10}$$

Proof. Suppose $J(\mathbf{v})$ denotes the functional

$$\mathbf{v} \mapsto \frac{1}{3} \int_{\mathcal{O}} |\mathbf{v}(x)|^3 \gamma(x) \mathrm{d}x.$$

Then simple calculations give,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} J(\mathbf{v} + \theta\mathbf{u} + \alpha\mathbf{w}) = \int_{\mathcal{O}} \gamma \mathbf{w}(\mathbf{v} + \theta\mathbf{u} + \alpha\mathbf{w})|\mathbf{v} + \theta\mathbf{u} + \alpha\mathbf{w}| \mathrm{d}x,$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta\ \mathrm{d}\alpha} J(\mathbf{v} + \theta\mathbf{u} + \alpha\mathbf{w}) = 2 \int_{\mathcal{O}} \gamma \mathbf{u}\mathbf{w} |\mathbf{v} + \theta\mathbf{u} + \alpha\mathbf{w}| \mathrm{d}x.$$

Hence

$$J''(\mathbf{v};\mathbf{u},\mathbf{u}) = \frac{\mathrm{d}^2}{\mathrm{d}\theta\ \mathrm{d}\alpha} J(\mathbf{v} + \theta\mathbf{u} + \alpha\mathbf{u})\Big|_{\theta,\alpha=0} = 2 \int_{\mathcal{O}} \gamma |\mathbf{v}| \mathbf{u}^2 \mathrm{d}x \ge 0,$$

for any $\mathbf{u},\mathbf{v} \in \mathbb{L}^4(\mathcal{O},\mathbb{R}^2)$, since $\gamma(x)$ is a positive function.

Thus in view of Lemma 3.2, $J(\mathbf{v})$ is a convex functional and so

$$J(\theta\mathbf{u} + (1-\theta)\mathbf{v}) \le \theta J(\mathbf{u}) + (1-\theta) J(\mathbf{v})$$

which yields

$$J(\mathbf{v} + \theta(\mathbf{u} - \mathbf{v}) - J(\mathbf{v}) \le \theta[J(\mathbf{u}) - J(\mathbf{v})].$$

Dividing both sides by θ and letting $\theta \to 0$ we deduce

$$\langle J'(\mathbf{v}), \mathbf{u} - \mathbf{v} \rangle \leq J(\mathbf{u}) - J(\mathbf{v})$$

and then

$$\begin{aligned} \langle J'(\mathbf{u}) - J'(\mathbf{v}), \mathbf{u} - \mathbf{v} \rangle &\geq \langle J'(\mathbf{u}), \mathbf{u} - \mathbf{v} \rangle - [J(\mathbf{u}) - J(\mathbf{v})] \\ &\geq -J(\mathbf{v}) + J(\mathbf{u}) - [J(\mathbf{u}) - J(\mathbf{v})] \\ &= 0, \end{aligned} \tag{3.11}$$

for every $\mathbf{u}$ and $\mathbf{v}$ where the above integrations in $\mathcal{O}$ make sense.

Then with the help of (3.11) we can conclude that

$$\begin{aligned} \langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{u} - \mathbf{v} \rangle &= \langle J'(\mathbf{u} + \mathbf{w^0}) - J'(\mathbf{v} + \mathbf{w^0}), [\mathbf{u} + \mathbf{w^0}] - [\mathbf{v} - \mathbf{w^0}] \rangle \\ &\geq 0, \end{aligned}$$

i.e., inequality (3.10). $\square$

Notice that the nonlinear operator $B(\cdot)$ is a continuous operator from $\mathbb{L}^4(\mathcal{O})$ into $\mathbb{L}^2(\mathcal{O})$. We can check

$$\|\mathbf{B}(\mathbf{v})\|_{\mathbb{L}^2} \leq C_2 \|\mathbf{v}\|_{\mathbb{L}^4}, \quad \forall \mathbf{v}, \mathbf{w}^0 \in \mathbb{L}^4(\mathcal{O}) \tag{3.12}$$

$$\|\mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v})\|_{\mathbb{L}^2} \leq C_2 \big[\|\mathbf{u}\|_{\mathbb{L}^4} + \|\mathbf{v}\|_{\mathbb{L}^4}\big] \|\mathbf{u} - \mathbf{v}\|_{\mathbb{L}^4}, \quad \forall \mathbf{v}, \mathbf{w}^0 \in \mathbb{L}^4(\mathcal{O}), \tag{3.13}$$

where the constant C_2 is the sup-norm of the function γ.

With the above notation, the tidal dynamics equation can be written in a weak sense as

$$(\dot{\mathbf{u}}, \mathbf{v})_{\mathbb{L}^2} + a(\mathbf{u}, \mathbf{v}) + (\mathbf{B}(\mathbf{u}), \mathbf{v})_{\mathbb{L}^2} + (g\nabla z, \mathbf{v})_{\mathbb{L}^2} = (\mathbf{f}, \mathbf{v})_{\mathbb{L}^2}, \quad \forall \mathbf{v} \in \mathbb{H}_0^1(\mathcal{O}) \tag{3.14}$$

$$(\dot{z} + \mathrm{Div}(h\mathbf{u}), \zeta)_{\mathrm{L}^2} = 0, \quad \forall \zeta \in \mathrm{L}^2(\mathcal{O}), \tag{3.15}$$

$$\mathbf{u}(0) = \mathbf{u_0}, \quad z(0) = z_0, \tag{3.16}$$

where $\mathbf{f}$ is given by the integro-differential equation (2.14).

Proposition 3.4 (energy estimate)**.** *Under the above mathematical setting let*

$$\mathbf{w^0} \in \mathrm{L}^2(0, T; \mathbb{H}_0^1(\mathcal{O})), \mathbf{f} \in \mathrm{L}^2(0, T; \mathbb{L}^2(\mathcal{O})), \ \mathbf{u}_0 \in \mathbb{L}^2(\mathcal{O}), \ z_0 \in \mathrm{L}^2(\mathcal{O}). \tag{3.17}$$

Let $\mathbf{u}$ *in* $\mathrm{L}^2(0, T; \mathbb{H}_0^1(\mathcal{O}))$ *and* z *in* $\mathrm{L}^2(0, T; \mathrm{L}^2(\mathcal{O}))$ *be the solution of deterministic parabolic variational equality* (3.14)-(3.16) *such that* $\dot{\mathbf{u}}$ *belongs to* $\mathrm{L}^2(0, T; \mathbb{H}^{-1}(\mathcal{O}))$ *and* $\dot{z}$ *belongs to* $\mathrm{L}^2(0, T; \mathrm{L}^2(\mathcal{O}))$, *with* $\mathbb{H}^{-1}(\mathcal{O})$ *the dual space of* $\mathbb{H}_0^1(\mathcal{O})$. *Then we have the energy equality*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}(t)\|_{\mathbb{L}^2}^2+\alpha\|\mathbf{u}(t)\|_{\mathbb{H}_0^1}^2+(\mathbf{B}(\mathbf{u}(t)),\mathbf{u}(t))_{\mathbb{L}^2}+(g\nabla z(t),\mathbf{u}(t))_{\mathbb{L}^2}$$
$$=(\mathbf{f}(t),\mathbf{u}(t))_{\mathbb{L}^2}, \tag{3.18}$$

which yields the following a priori estimates

$$\sup_{0\le t\le T}\left[\|\mathbf{u}(t)\|_{\mathbb{L}^2}^2+\|z(t)\|_{\mathrm{L}^2}^2\right]\le C, \tag{3.19}$$

$$\int_0^T\|\mathbf{u}(t)\|_{\mathbb{H}_0^1}^2\mathrm{d}t\le C, \tag{3.20}$$

where the constant C depends on the coefficients and the norms $\|\mathbf{f}\|_{\mathrm{L}^2(0,T;\mathbb{H}^{-1})}$, $\|\mathbf{w}^0\|_{\mathrm{L}^2(0,T;\mathbb{H}_0^1)}$, $\|\mathbf{u}(0)\|_{\mathbb{L}^2}$ and $\|z(0)\|_{\mathrm{L}^2}$.

Proof. From (3.14) we notice that,

$$(\dot{\mathbf{u}}(t),\mathbf{u}(\mathbf{t}))_{\mathbb{L}^2}+a(\mathbf{u}(\mathbf{t}),\mathbf{u}(\mathbf{t}))+(\mathbf{B}(\mathbf{u}(\mathbf{t})),\mathbf{u}(\mathbf{t}))_{\mathbb{L}^2}+(g\nabla z(t),\mathbf{u}(\mathbf{t}))_{\mathbb{L}^2}$$
$$=(\mathbf{f}(\mathbf{t}),\mathbf{u}(\mathbf{t}))_{\mathbb{L}^2},\quad \forall\mathbf{u}\in\mathbb{H}_0^1(\mathcal{O}) \tag{3.21}$$

$$(\dot{z}(t)+\mathrm{Div}(h\mathbf{u}(\mathbf{t})),z(t))_{\mathrm{L}^2}=0,\quad \forall z\in\mathrm{L}^2(\mathcal{O}), \tag{3.22}$$

which with the help of (3.8) give the desired estimate (3.18).

From the definition of the nonlinear operator $\mathbf{B}(\cdot)$ and Lemma 3.3, we notice that,

$$\begin{aligned}(\ \mathbf{B}(\mathbf{u}(t)),\ \mathbf{u}(t)\ )_{\mathbb{L}^2}&\ge(\ \gamma\ |\mathbf{w^0}(t)|^2,\ \mathbf{u}(t)\ )_{\mathbb{L}^2}\\&\ge-\frac{r}{\varepsilon}\ \|\mathbf{w^0}(t)\|_{\mathbb{L}^4}^2\ \|\mathbf{u}(t)\|_{\mathbb{L}^2}\\&\ge-\frac{r}{2\varepsilon}\ \left[\|\mathbf{w^0}(t)\|_{\mathbb{L}^4}^4+\|\mathbf{u}(t)\|_{\mathbb{L}^2}^2\right].\end{aligned} \tag{3.23}$$

Then the energy equality (3.18) yields,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}(t)\|_{\mathbb{L}^2}^2+2\alpha\|\mathbf{u}(t)\|_{\mathbb{H}_0^1}^2\le2(\mathbf{f}(t),\mathbf{u}(t))_{\mathbb{L}^2}+\frac{r}{\varepsilon}\ \left[\|\mathbf{w^0}(t)\|_{\mathbb{L}^4}^4+\|\mathbf{u}(t)\|_{\mathbb{L}^2}^2\right]$$
$$-2(g\nabla z(t),\mathbf{u}(\mathbf{t}))_{\mathbb{L}^2}. \tag{3.24}$$

Using the divergence theorem and the inequality

$$2ab\le\delta a^2+\frac{1}{\delta}b^2,$$

we obtain,

$$\begin{aligned}|g(\nabla z(t),\mathbf{u}(\mathbf{t}))_{\mathbb{L}^2}|&=|-g(z(t),\mathrm{Div}\,\mathbf{u}(t))_{\mathbb{L}^2}|\\&\le\frac{g}{2}\Big[\frac{2g}{\alpha}\|z(t)\|_{\mathrm{L}^2}^2+\frac{\alpha}{2g}\|\,\mathrm{Div}\,\mathbf{u}(t)\|_{\mathrm{L}^2}^2\Big].\end{aligned}$$

Since in L^2-norm divergence is bounded by gradient,

$$|g(\nabla z(t),\mathbf{u}(\mathbf{t}))_{\mathbb{L}^2}| \leq \frac{g}{2}\big[\frac{2g}{\alpha}\|z(t)\|^2_{\mathrm{L}^2} + \frac{\alpha}{2g}\|\mathbf{u}(t)\|^2_{\mathbb{H}^1_0}\big] \tag{3.25}$$

Also

$$|(\mathbf{f}(t),\mathbf{u}(t))_{\mathbb{L}^2}| \leq \frac{1}{2}\big[\|\mathbf{f}(t)\|^2_{\mathbb{L}^2} + \|\mathbf{u}(t)\|^2_{\mathbb{L}^2}\big]. \tag{3.26}$$

Hence from (3.24) we have,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}(t)\|^2_{\mathbb{L}^2} + 2\alpha\|\mathbf{u}(t)\|^2_{\mathbb{H}^1_0}\\
&\quad \leq \|\mathbf{f}(t)\|^2_{\mathbb{L}^2} + \|\mathbf{u}(t)\|^2_{\mathbb{L}^2} + \frac{r}{\varepsilon}\ \big[\|\mathbf{w^0}(t)\|^4_{\mathbb{L}^4} + \|\mathbf{u}(t)\|^2_{\mathbb{L}^2}\big]\\
&\qquad + g\big[\frac{2g}{\alpha}\|z(t)\|^2_{\mathrm{L}^2} + \frac{\alpha}{2g}\|\mathbf{u}(t)\|^2_{\mathbb{H}^1_0}\big]\\
&\quad = (1+\frac{r}{\varepsilon})\|\mathbf{u}(t)\|^2_{\mathbb{L}^2} + \frac{2g^2}{\alpha}\|z(t)\|^2_{\mathrm{L}^2} + \frac{\alpha}{2}\|\mathbf{u}(t)\|^2_{\mathbb{H}^1_0} + \frac{r}{\varepsilon}\|\mathbf{w^0}(t)\|^4_{\mathbb{L}^4} + \|\mathbf{f}(t)\|^2_{\mathbb{L}^2}.
\end{aligned}$$

Integrating the above equation in t we have,

$$\begin{aligned}
&\|\mathbf{u}(t)\|^2_{\mathbb{L}^2} + \frac{3\alpha}{2}\int_0^t \|\mathbf{u}(s)\|^2_{\mathbb{H}^1_0}\mathrm{d}s\\
&\quad \leq (1+\frac{r}{\varepsilon})\int_0^t \|\mathbf{u}(s)\|^2_{\mathbb{L}^2}\mathrm{d}s + \frac{2g^2}{\alpha}\int_0^t \|z(s)\|^2_{\mathrm{L}^2}\mathrm{d}s + \frac{r}{\varepsilon}\int_0^t \|\mathbf{w^0}(s)\|^4_{\mathbb{L}^4}\mathrm{d}s\\
&\qquad + \int_0^t \|\mathbf{f}(s)\|^2_{\mathbb{L}^2}\mathrm{d}s + \|\mathbf{u}(0)\|^2_{\mathbb{L}^2}.
\end{aligned} \tag{3.27}$$

Now equation (3.22) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|^2_{\mathrm{L}^2} = -\operatorname{Div}(h\mathbf{u}(t)), z(t))_{\mathrm{L}^2}. \tag{3.28}$$

Notice that

$$\begin{aligned}
|\operatorname{Div}(h\mathbf{u}(t)), z(t))_{\mathrm{L}^2}| &= \big|(h\operatorname{Div}\mathbf{u}(t), z(t))_{\mathrm{L}^2} + (\mathbf{u}(t)\cdot\nabla h, z(t))_{\mathrm{L}^2}\big|\\
&\leq |(h\operatorname{Div}\mathbf{u}(t), z(t))_{\mathrm{L}^2}| + |(\mathbf{u}(t)\cdot\nabla h, z(t))_{\mathrm{L}^2}|\\
&\leq \|h\|_{\mathrm{L}^\infty}\ \|\operatorname{Div}\mathbf{u}(t)\|_{\mathrm{L}^2}\ \|z(t)\|_{\mathrm{L}^2}\\
&\quad + \|\mathbf{u}(t)\|_{\mathbb{L}^2}\ \|\nabla h\|_{\mathrm{L}^\infty}\ \|z(t)\|_{\mathrm{L}^2}.
\end{aligned}$$

Now using the assumption on h from (2.10), we have

$$
\begin{aligned}
|\operatorname{Div}(h\mathbf{u}(t)), z(t))_{\mathrm{L}^2}| &\leq \mu\|\mathbf{u}(t)\|_{\mathbb{H}_0^1}\ \|z(t)\|_{\mathrm{L}^2} + M\|\mathbf{u}(t)\|_{\mathbb{L}^2}\ \|z(t)\|_{\mathrm{L}^2} \\
&\leq \frac{\mu}{2}\Big[\frac{\alpha}{2\mu}\|\mathbf{u}(t)\|^2_{\mathbb{H}_0^1} + \frac{2\mu}{\alpha}\|z(t)\|^2_{\mathrm{L}^2}\Big] \\
&\quad + \frac{M}{2}\big[\|\mathbf{u}(t)\|^2_{\mathbb{L}^2} + \|z(t)\|^2_{\mathrm{L}^2}\big].
\end{aligned}
$$

Thus from (3.28) and from the above equation we get,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|^2_{\mathrm{L}^2} \leq M\|\mathbf{u}(t)\|^2_{\mathbb{L}^2} + (\frac{2\mu^2}{\alpha} + M)\|z(t)\|^2_{\mathrm{L}^2} + \frac{\alpha}{2}\|\mathbf{u}(t)\|^2_{\mathbb{H}_0^1}. \tag{3.29}
$$

Integrating in t we have,

$$
\begin{aligned}
\|z(t)\|^2_{\mathrm{L}^2} &\leq M\int_0^t \|\mathbf{u}(s)\|^2_{\mathbb{L}^2}\mathrm{d}s + (\frac{2\mu^2}{\alpha} + M)\int_0^t \|z(s)\|^2_{\mathrm{L}^2}\mathrm{d}s + \\
&\quad + \frac{\alpha}{2}\int_0^t \|\mathbf{u}(s)\|^2_{\mathbb{H}_0^1}\mathrm{d}s + \|z(0)\|^2_{\mathrm{L}^2}.
\end{aligned} \tag{3.30}
$$

Next we add (3.27) and (3.30) to get

$$
\begin{aligned}
&\|\mathbf{u}(t)\|^2_{\mathbb{L}^2} + \|z(t)\|^2_{\mathrm{L}^2} + \alpha\int_0^t \|\mathbf{u}(s)\|^2_{\mathbb{H}_0^1}\mathrm{d}s \\
&\quad\leq (1 + M + \frac{r}{\varepsilon})\int_0^t \|\mathbf{u}(s)\|^2_{\mathbb{L}^2}\mathrm{d}s + (\frac{2g^2}{\alpha} + \frac{2\mu^2}{\alpha} + M)\int_0^t \|z(s)\|^2_{\mathrm{L}^2}\mathrm{d}s \\
&\qquad + \frac{r}{\varepsilon}\int_0^t \|\mathbf{w}^{\mathbf{0}}(s)\|^4_{\mathbb{L}^4}\mathrm{d}s + \int_0^t \|\mathbf{f}(s)\|^2_{\mathbb{L}^2}\mathrm{d}s + \|\mathbf{u}(0)\|^2_{\mathbb{L}^2} + \|z(0)\|^2_{\mathrm{L}^2}.
\end{aligned}
$$

Let

$$
K = \max\{1 + M + \frac{r}{\varepsilon}, \frac{2g^2}{\alpha} + \frac{2\mu^2}{\alpha} + M\},
$$

then the above equation becomes

$$
\begin{aligned}
&\big[\|\mathbf{u}(t)\|^2_{\mathbb{L}^2} + \|z(t)\|^2_{\mathrm{L}^2}\big] + \alpha\int_0^t \|\mathbf{u}(s)\|^2_{\mathbb{H}_0^1}\mathrm{d}s \\
&\quad\leq K\int_0^t \big[\|\mathbf{u}(s)\|^2_{\mathbb{L}^2} + \|z(s)\|^2_{\mathrm{L}^2}\big]\mathrm{d}s + \frac{r}{\varepsilon}\int_0^t \|\mathbf{w}^{\mathbf{0}}(s)\|^4_{\mathbb{L}^4}\mathrm{d}s \\
&\qquad + \int_0^t \|\mathbf{f}(s)\|^2_{\mathbb{L}^2}\mathrm{d}s + \|\mathbf{u}(0)\|^2_{\mathbb{L}^2} + \|z(0)\|^2_{\mathrm{L}^2}.
\end{aligned}
$$

Now by virtue of equation (3.5) in Lemma 3.1 and by the assumption on $\mathbf{w}^{\mathbf{0}}$ in the proposition we have,

$$
\|\mathbf{w}^{\mathbf{0}}(t)\|_{\mathrm{L}^2(0,T;\mathbb{L}^4)} \leq \|\mathbf{w}^{\mathbf{0}}(t)\|_{\mathrm{L}^2(0,T;\mathbb{H}_0^1)} \leq K'.
$$

Hence using the Gronwall's inequality we have the desired a priori estimates (3.19) and (3.20). □

Corollary 3.5. *Note that since h=$h(x)$ is a bounded function, we can get similar energy estimate by taking the inner product of the tidal dynamics equation with $h\mathbf{u}$. If we denote*

$$F(\mathbf{u}) := \mathbf{A}\mathbf{u} + \mathbf{B}(\mathbf{u}) - \mathbf{f},$$

where the operators $\mathbf{A}$ *and* $\mathbf{B}$ *are defined by* (2.8) *and* (3.9) *respectively, then the tidal dynamics equation can be written in a week sense as*

$$\begin{cases} (\dot{\mathbf{u}}(t), h\mathbf{u}(\mathbf{t}))_{\mathbb{L}^2} + (F(\mathbf{u}(t)), h\mathbf{u}(\mathbf{t}))_{\mathbb{L}^2} + (g\nabla z(t), h\mathbf{u}(\mathbf{t}))_{\mathbb{L}^2} = 0, \\ \hfill \forall \mathbf{u} \in \mathbb{H}_0^1(\mathcal{O}) \\ (\dot{z}(t) + \mathrm{Div}(h\mathbf{u}(t)), z(t))_{\mathrm{L}^2} = 0, \quad \forall z \in \mathrm{L}^2(\mathcal{O}). \end{cases} \tag{3.31}$$

Now by divergence theorem,

$$(g\nabla z(t), h\mathbf{u}(\mathbf{t}))_{\mathbb{L}^2} = -(gz(t), \mathrm{Div}(h\mathbf{u}(t)))_{\mathrm{L}^2} = (gz(t), \dot{z}(t))_{\mathrm{L}^2} = \frac{g}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|_{\mathrm{L}^2}^2.$$

Thus from (3.31) *we have the energy equality,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\sqrt{h}\mathbf{u}(t)\|_{\mathbb{L}^2}^2 + g\|z(t)\|_{\mathrm{L}^2}^2\right] + 2(F(\mathbf{u}(t)), h\mathbf{u}(\mathbf{t}))_{\mathbb{L}^2} = 0. \tag{3.32}$$

Note that the energy equality (3.32) *also yields a priori estimates similar to* (3.19)-(3.20)*.*

Proposition 3.6 (Uniqueness)**.** *Let* $(\mathbf{u}, z)$ *be a solution of the deterministic tide equation* (3.14)-(3.16) *with regularity*

$$\mathbf{u} \in C^0(0,T;\mathbb{L}^2(\mathcal{O})) \cap \mathrm{L}^2(0,T;\mathbb{H}_0^1(\mathcal{O})), \quad z \in \mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{O})),$$

and let the data $\mathbf{f}$, $\mathbf{u}_0$ *and* z_0 *satisfy the condition*

$$\mathbf{f} \in \mathrm{L}^2(0,T;\mathbb{H}^{-1}(\mathcal{O})), \quad \mathbf{u}_0 \in \mathbb{L}^2(\mathcal{O}), \quad z_0 \in \mathrm{L}^2(\mathcal{O}).$$

If $(\mathbf{v}, \tilde{z})$ *is another solution of the deterministic tide equation* (3.14)-(3.16)*, such that* $\mathbf{v} \in C^0(0,T;\mathbb{L}^2(\mathcal{O})) \cap \mathrm{L}^2(0,T;\mathbb{H}_0^1(\mathcal{O}))$ *and* $\tilde{z} \in \mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{O}))$*, then*

$$\begin{aligned} \big[\|\mathbf{u}(t) - \mathbf{v}(t)\|_{\mathbb{L}^2}^2 + \|z(t) - \tilde{z}(t)\|_{\mathrm{L}^2}^2\big]e^{-Kt} \\ \le \|\mathbf{u}(0) - \mathbf{v}(0)\|_{\mathbb{L}^2}^2 + \|z(0) - \tilde{z}(0)\|_{\mathrm{L}^2}^2, \end{aligned} \tag{3.33}$$

for any $0 \le t \le T$*, where* K *is a positive constant.*

Proof. Notice that if $\mathbf{u}$ and $\mathbf{v}$ are two solutions then $\mathbf{w} = \mathbf{u} - \mathbf{v}$ solves the deterministic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{w}(t) + \mathbf{A}\mathbf{w}(t) + g\nabla\big(z(t) - \tilde{z}(t)\big) = \mathbf{B}(\mathbf{v}(t)) - \mathbf{B}(\mathbf{u}(t)) \text{ in } \mathbb{L}^2(0,T;\mathbb{H}^{-1}(\mathcal{O})).$$

Multiplying the above equation by $\mathbf{w}(t)$, taking the inner product and applying the result from Lemma 3.3, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}(t)\|^2_{\mathbb{L}^2} + \alpha\|\mathbf{w}(t)\|^2_{\mathbb{H}^1_0} + \Big(g\nabla\big(z(t)-\tilde{z}(t)\big), \mathbf{w}(t)\Big)_{\mathbb{L}^2} \leq 0.$$

Using the result from (3.25) in the above equation we have,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}(t)\|^2_{\mathbb{L}^2} + 2\alpha\|\mathbf{w}(t)\|^2_{\mathbb{H}^1_0} &\leq -\Big(g\nabla\big(z(t)-\tilde{z}(t)\big), \mathbf{w}(t)\Big)_{\mathbb{L}^2}\\ &\leq \frac{2g^2}{\alpha}\|z(t)-\tilde{z}(t)\|^2_{\mathrm{L}^2} + \frac{\alpha}{2}\|\mathbf{w}(t)\|^2_{\mathbb{H}^1_0}.\end{aligned}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}(t)\|^2_{\mathbb{L}^2} + \frac{3\alpha}{2}\|\mathbf{w}(t)\|^2_{\mathbb{H}^1_0} \leq \frac{2g^2}{\alpha}\|z(t)-\tilde{z}(t)\|^2_{\mathrm{L}^2}. \tag{3.34}$$

Again notice that

$$\frac{\mathrm{d}}{\mathrm{d}t}(z(t)-\tilde{z}(t)) + \mathrm{Div}(h\mathbf{w}(t)) = 0.$$

Taking inner product with $z(t)-\tilde{z}(t)$ we have as before (3.29)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)-\tilde{z}(t)\|^2_{\mathrm{L}^2} \leq M\|\mathbf{w}(t)\|^2_{\mathbb{L}^2} + (\frac{2\mu^2}{\alpha} + M)\|z(t)-\tilde{z}(t)\|^2_{\mathrm{L}^2}\\ +\frac{\alpha}{2}\|\mathbf{w}(t)\|^2_{\mathbb{H}^1_0}.\end{aligned} \tag{3.35}$$

Let us denote

$$K = \frac{2g^2}{\alpha} + \frac{2\mu^2}{\alpha} + M.$$

Then adding (3.34) and (3.35) and rearranging we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\|\mathbf{w}(t)\|^2_{\mathbb{L}^2} + \|z(t)-\tilde{z}(t)\|^2_{\mathrm{L}^2}\big] \leq K\big[\|\mathbf{w}(t)\|^2_{\mathbb{L}^2} + \|z(t)-\tilde{z}(t)\|^2_{\mathrm{L}^2}\big]. \tag{3.36}$$

Hence using Gronwall's inequality we have the desired estimate (3.33) for any $0 \leq t \leq T$. □

A finite-dimensional Galerkin approximation of the deterministic tide equation can be defined as follows. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots\}$ be a complete orthonormal system (i.e., a basis) in the Hilbert space $\mathbb{L}^2(\mathcal{O})$ belonging to the space $\mathbb{H}^1_0(\mathcal{O})$ (and $\mathbb{L}^4(\mathcal{O})$). Denote by $\mathbb{L}^2_n(\mathcal{O})$ the n-dimensional subspace of $\mathbb{L}^2(\mathcal{O})$ and $\mathbb{H}^1_0(\mathcal{O})$ of all linear combinations of the first n elements $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$.

Let us consider the following ODE in $\mathbb{R}^n$

$$\begin{cases} \mathrm{d}(\mathbf{u}^n(t),\mathbf{v}(t))_{\mathbb{L}^2}+a(\mathbf{u}^n(t),\mathbf{v}(t))\mathrm{d}t+(B(\mathbf{u}^n(t)),\mathbf{v}(t))_{\mathbb{L}^2}\mathrm{d}t \\ \qquad\qquad +(g\nabla z^n(t),\mathbf{v}(t))_{\mathbb{L}^2}\mathrm{d}t=(\mathbf{f}(t),\mathbf{v}(t))_{\mathbb{L}^2}\mathrm{d}t, \\ \big(\dot{z}^n(t)+\mathrm{Div}(h\mathbf{u}^n(t)),\zeta(t)\big)_{\mathrm{L}^2}=0, \end{cases} \tag{3.37}$$

in $(0,T)$, with the initial conditions

$$(\mathbf{u}(0),\mathbf{v})_{\mathbb{L}^2}=(\mathbf{u}_0,\mathbf{v})_{\mathbb{L}^2}, \quad \text{and} \quad (z(0),\zeta)_{\mathrm{L}^2}=(z_0,\zeta)_{\mathrm{L}^2}, \tag{3.38}$$

for any $\mathbf{v}$ in the space $\mathbb{L}^2_n(\mathcal{O})$ and ζ in $\mathrm{L}^2_n(\mathcal{O})$. The coefficients involved are locally Lipschitz and we use the *a priori* estimates (3.19) and (3.20) to show global existence of a solution $\mathbf{u}^n(t)$ in the space $C^0(0,T,\mathbb{L}^2_n(\mathcal{O}))$.

Proposition 3.7 (2-D existence)**.** *Let* $\mathbf{f}$, $\mathbf{u}_0$ *and* z_0 *be such that*

$$\begin{cases} \mathbf{f}\in \mathrm{L}^2(0,T;\mathbb{H}^{-1}(\mathcal{O})), \\ \mathbf{u}_0\in\mathbb{L}^2(\mathcal{O}), \quad z_0\in\mathrm{L}^2(\mathcal{O}). \end{cases} \tag{3.39}$$

Then there exists a solution $(\mathbf{u}(t,x),z(t,x))$ *with the regularity*

$$\begin{cases} \mathbf{u}\in C^0(0,T;\mathbb{L}^2(\mathcal{O}))\cap \mathrm{L}^2(0,T;\mathbb{H}^1_0(\mathcal{O})), \\ z,\ \dot{z}\in \mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{O})) \end{cases} \tag{3.40}$$

satisfying the deterministic tide equation (3.14)-(3.16) *and the a priori bounds* (3.19)-(3.20).

Proof. Let us denote

$$F(\mathbf{u}):=\mathbf{A}\mathbf{u}+\mathbf{B}(\mathbf{u})-\mathbf{f},$$

where the operators $\mathbf{A}$ and $\mathbf{B}$ are defined by (2.8) and (3.9) respectively.
Then

$$\mathrm{d}\mathbf{u}^n(t)+F(\mathbf{u}^n(t))\mathrm{d}t+g\nabla z^n(t)\mathrm{d}t=0.$$

Using the a priori estimates (3.19)-(3.20), it follows from the Banach-Alaoglu theorem that along a subsequence, the Galerkin approximations $\{\mathbf{u}^n\}$ have the following limits:

$$\begin{aligned} &\mathbf{u}^n\longrightarrow\mathbf{u} \quad \text{weakly star in } \mathrm{L}^\infty(0,T;\mathbb{L}^2(\mathcal{O}))\cap\mathrm{L}^2(0,T;\mathbb{H}^1_0(\mathcal{O})), \\ &z^n\longrightarrow z \text{ weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2(\mathcal{O})), \\ &F(\mathbf{u}^n)\longrightarrow F_0 \quad \text{weakly in } \mathrm{L}^2(0,T;\mathbb{H}^{-1}(\mathcal{O})), \end{aligned}$$

where $\mathbf{u}$ has the differential form

$$\mathrm{d}\mathbf{u}(t)+F_0(t)\mathrm{d}t+g\nabla z(t)\mathrm{d}t=0, \qquad \text{in } \mathrm{L}^2(0,T;\mathbb{H}^{-1}(\mathcal{O}))$$

and the energy equality similar to (3.32) holds, i.e.,

$$\mathrm{d}\big[\|\sqrt{h}\mathbf{u}(t)\|^2_{\mathbb{L}^2}+g\|z(t)\|^2_{\mathrm{L}^2}\big]+2\big(F_0(t),h\mathbf{u}(t)\big)_{\mathbb{L}^2}\mathrm{d}t=0.$$

Since the Galerkin approximations $\{\mathbf{u}^n\}$ satisfy the energy equality

$$\mathrm{d}\big[\|\sqrt{h}\mathbf{u}^n(t)\|^2_{\mathbb{L}^2}+g\|z^n(t)\|^2_{\mathrm{L}^2}\big]+2\big(F(\mathbf{u}^n(t)),h\mathbf{u}^n(t)\big)_{\mathbb{L}^2}\mathrm{d}t=0.$$

Integrating in $0\le t\le T$ we have,

$$\|\sqrt{h}\mathbf{u}^n(T)\|^2_{\mathbb{L}^2}+g\|z^n(T)\|^2_{\mathrm{L}^2}+2\int_0^T\big(F(\mathbf{u}^n(t)),h\mathbf{u}^n(t)\big)_{\mathbb{L}^2}\mathrm{d}t$$
$$=\|\sqrt{h}\mathbf{u}^n(0)\|^2_{\mathbb{L}^2}+g\|z^n(0)\|^2_{\mathrm{L}^2}.$$

Hence

$$-2\int_0^T\big(F(\mathbf{u}^n(t)),h\mathbf{u}^n(t)\big)_{\mathbb{L}^2}\mathrm{d}t=\|\sqrt{h}\mathbf{u}^n(T)\|^2_{\mathbb{L}^2}+g\|z^n(T)\|^2_{\mathrm{L}^2}$$
$$-\|\sqrt{h}\mathbf{u}^n(0)\|^2_{\mathbb{L}^2}-g\|z^n(0)\|^2_{\mathrm{L}^2}. \tag{3.41}$$

Considering the fact that the initial conditions $\mathbf{u}^n(0)$ and $z^n(0)$ converge to $\mathbf{u}(0)=\mathbf{u}_0$ and $z(0)=z_0$ respectively in $\mathbb{L}^2$, and the lower-semi-continuity of the $\mathbb{L}^2$-norm, we deduce

$$\liminf_n\Big[-2\int_0^T\big(F(\mathbf{u}^n(t)),h\mathbf{u}^n(t)\big)_{\mathbb{L}^2}\mathrm{d}t\Big]$$
$$\ge\|\sqrt{h}\mathbf{u}(T)\|^2_{\mathbb{L}^2}+g\|z(T)\|^2_{\mathrm{L}^2}-\|\sqrt{h}\mathbf{u}(0)\|^2_{\mathbb{L}^2}-g\|z(0)\|^2_{\mathrm{L}^2}$$
$$=-2\int_0^T\big(F_0(t),h\mathbf{u}(t)\big)_{\mathbb{L}^2}\mathrm{d}t. \tag{3.42}$$

Here notice that from the equation (3.8) and the monotonicity property of the nonlinear operator $\mathbf{B}(\cdot)$, i.e. from Lemma 3.3, we have

$$\big(F(\mathbf{u}^n(t))-F(\mathbf{v}(t)),h\mathbf{u}^n(t)-h\mathbf{v}(t)\big)_{\mathbb{L}^2}\ge 0. \tag{3.43}$$

Multiplying both sides of (3.43) by 2, integrating in $0\le t\le T$ and rearranging the terms we have

$$\int_0^T\big(2F(\mathbf{v}(t)),h\mathbf{v}(t)-h\mathbf{u}^n(t)\big)_{\mathbb{L}^2}\mathrm{d}t$$
$$\ge\int_0^T\big(2F(\mathbf{u}^n(t)),h\mathbf{v}(t)-h\mathbf{u}^n(t)\big)_{\mathbb{L}^2}\mathrm{d}t$$
$$=-2\int_0^T\big(F(\mathbf{u}^n(t)),h\mathbf{u}^n(t)\big)_{\mathbb{L}^2}\mathrm{d}t+2\int_0^T\big(F(\mathbf{u}^n(t)),h\mathbf{v}(t)\big)_{\mathbb{L}^2}\mathrm{d}t.$$

Taking limit in n and using (3.42) we get

$$
\begin{aligned}
&\int_0^T \big(2F(\mathbf{v}(t)), h\mathbf{v}(t) - h\mathbf{u}(t)\big)_{\mathbb{L}^2}\mathrm{d}t \\
&\quad \geq\ -2\int_0^T \big(F_0(t), h\mathbf{u}(t)\big)_{\mathbb{L}^2}\mathrm{d}t + 2\int_0^T \big(F_0(t), h\mathbf{v}(t)\big)_{\mathbb{L}^2}\mathrm{d}t \\
&\quad = \int_0^T \big(2F_0(t), h\mathbf{v}(t) - h\mathbf{u}(t)\big)_{\mathbb{L}^2}\mathrm{d}t.
\end{aligned}
$$

Let us consider $\mathbf{v} := \mathbf{u} + \lambda\mathbf{w}$ with $\lambda > 0$ and $\mathbf{w} \in \mathrm{L}^\infty(0,T;\mathbb{L}^2(\mathcal{O})) \cap \mathrm{L}^2(0,T;\mathbb{H}_0^1(\mathcal{O}))$. Then we have

$$
\lambda\int_0^T \big(2F(\mathbf{u}(t) + \lambda\mathbf{w}(t)), h\mathbf{w}(t)\big)_{\mathbb{L}^2}\mathrm{d}t\ \geq\ \lambda\int_0^T \big(2F_0(t), h\mathbf{w}(t)\big)_{\mathbb{L}^2}\mathrm{d}t. \tag{3.44}
$$

Dividing by λ on both sides of the inequality above, and letting λ go to 0, one obtains

$$
\int_0^T \big(F(\mathbf{u}(t)) - F_0(t), h\mathbf{w}(t)\big)_{\mathbb{L}^2}\mathrm{d}t\ \geq\ 0.
$$

Since $\mathbf{w}$ is arbitrary and $h = h(x)$ is a positive, bounded and continuously differentiable function, we conclude that $F_0(t) = F(\mathbf{u}(t))$. Thus the existence of a solution of the deterministic tide equation (3.14)-(3.16) has been proved. □

4. Stochastic Tide Equation

Let us consider the tide equation subject to a random (Gaussian) term i.e., the forcing field $\mathbf{f}$ has a mean value still denoted by $\mathbf{f}$ and a noise denoted by $\dot{\mathbf{G}}$. We can write (to simplify notation we use time-invariant forces) $\mathbf{f}(t) = \mathbf{f}(x,t)$ and the noise process $\dot{\mathbf{G}}(t) = \dot{\mathbf{G}}(x,t)$ as a series $d\mathbf{G}_k = \sum_k \mathbf{g}_k(x,t)dw_k(t)$, where $\mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2, \cdots)$ and $w = (w_1, w_2, \ldots)$ are regarded as ℓ^2-valued functions in x and t respectively. The stochastic noise process represented by $\mathbf{g}(t)dw(t) = \sum_k \mathbf{g}_k(x,t)dw_k(t,\omega)$ is normal distributed in $\mathbb{H}$ with a trace-class co-variance operator denoted by $\mathbf{g}^2 = \mathbf{g}^2(t)$ and given by

$$
\begin{cases}
(\mathbf{g}^2(t)\mathbf{u}, \mathbf{v}) = \displaystyle\sum_k (\mathbf{g}_k(t), \mathbf{u})\,(\mathbf{g}_k(t), \mathbf{v}), \\
\mathrm{Tr}(\mathbf{g}^2(t)) = \displaystyle\sum_k |\mathbf{g}_k(t)|^2 < \infty.
\end{cases} \tag{4.1}
$$

We interpret the stochastic tide equations as an Itô stochastic equations in variational form

$$\begin{cases} \mathrm{d}(\mathbf{u},\mathbf{v})_{\mathbb{L}^2} + a(\mathbf{u},\mathbf{v})\mathrm{d}t + (\mathbf{B}(\mathbf{u}),\mathbf{v})_{\mathbb{L}^2}\mathrm{d}t + (g\nabla z,\mathbf{v})_{\mathbb{L}^2}\mathrm{d}t \\ \qquad\qquad = (\mathbf{f},\mathbf{v})_{\mathbb{L}^2}\mathrm{d}t + \sum_k (\mathbf{g}_k,\mathbf{v})_{\mathbb{L}^2}\,\mathrm{d}w_k(t), \\ (\dot z + \mathrm{Div}(h\mathbf{u}),\zeta)_{\mathrm{L}^2} = 0, \end{cases} \tag{4.2}$$

in $(0,T)$, with the initial condition

$$(\mathbf{u}(0),\mathbf{v})_{\mathbb{L}^2} = (\mathbf{u}_0,\mathbf{v})_{\mathbb{L}^2} \quad \text{and} \quad (z(0),\zeta)_{\mathrm{L}^2} = (z_0,\zeta)_{\mathrm{L}^2}, \tag{4.3}$$

for any $\mathbf{v}$ in the space $\mathbb{H}_0^1(\mathcal{O})$ and any ζ in $\mathrm{L}^2(\mathcal{O})$.

Proposition 4.1 (energy estimate)**.** *Let*

$$\begin{cases} \mathbf{w^0} \in \mathrm{L}^2(\Omega;\mathrm{L}^2(0,T;\mathbb{H}_0^1(\mathcal{O}))),\ \mathbf{f} \in \mathrm{L}^2(0,T;\mathbb{L}^2(\mathcal{O})), \\ \mathbf{g} \in \mathrm{L}^2(0,T;\ell_2(\mathbb{L}^2(\mathcal{O}))), \mathbf{u}_0 \in \mathbb{L}^2(\mathcal{O}),\ z_0 \in \mathrm{L}^2(\mathcal{O}). \end{cases} \tag{4.4}$$

Let $\mathbf{u}(t)$ *be an adapted process in* $C^0(0,T,\mathbb{H}_0^1)$ *which solves the stochastic ODE* (4.2). *Then we have the energy equality*

$$\begin{cases} \mathrm{d}[\|\mathbf{u}(t)\|_{\mathbb{L}^2}^2] + 2\alpha\|\mathbf{u}(t)\|_{\mathbb{H}_0^1}^2\mathrm{d}t + 2(\mathbf{B}(\mathbf{u}(t)),\mathbf{u}(t))_{\mathbb{L}^2}\mathrm{d}t + 2(g\nabla z(t),\mathbf{u}(t))_{\mathbb{L}^2}\mathrm{d}t \\ \quad = \big[2(\mathbf{f}(t),\mathbf{u}(t))_{\mathbb{L}^2} + \mathrm{Tr}(\mathbf{g}^2(t))\big]\mathrm{d}t + 2\sum_k (\mathbf{g}_k(t),\mathbf{u}(t))_{\mathbb{L}^2}\mathrm{d}w_k(t), \end{cases} \tag{4.5}$$

which yields the following a priori estimate

$$\begin{cases} E\Big\{\sup_{0\le t\le T}\big[\|\mathbf{u}(t)\|_{\mathbb{L}^2}^2 + \|z(t)\|_{\mathrm{L}^2}^2\big] + 2\alpha\int_0^T \|\mathbf{u}(t)\|_{\mathbb{H}_0^1}^2\mathrm{d}t\Big\} \\ \quad \le 2CKT + 2\big[\|\mathbf{u}(0)\|_{\mathbb{L}^2}^2 + \|z(0)\|_{\mathrm{L}^2}^2\big] + \dfrac{2r}{\varepsilon}\int_0^T E\big[\|\mathbf{w^0}(t)\|_{\mathbb{L}^4}^4\big]\mathrm{d}t \\ \qquad + 2\int_0^T \big[\|\mathbf{f}(t)\|_{\mathbb{L}^2}^2 + \mathrm{Tr}(\mathbf{g}^2(t))\big]\mathrm{d}t, \end{cases} \tag{4.6}$$

for any $0 \le t \le T$ *and* K *is a positive constant and the constant* C *depends on the coefficients and the norms* $\|\mathbf{f}\|_{\mathrm{L}^2(0,T;\mathbb{H}^{-1})}$, $\|\mathbf{u}(0)\|_{\mathbb{L}^2}$ *and* $\|z(0)\|_{\mathrm{L}^2}$.

Proof. It is straightforward to see that (4.2) and (3.8) yield the energy estimate (4.5).

Next we take the first equation of (4.2) by replacing $\mathbf{v}$ by $\mathbf{u}$ and proceed in the same way as in Proposition 3.4 to get the estimate similar to (3.27)

$$\begin{aligned}
&\|\mathbf{u}(t)\|_{\mathbb{L}^2}^2 + \frac{3\alpha}{2}\int_0^t \|\mathbf{u}(s)\|_{\mathbb{H}_0^1}^2 \mathrm{d}s \\
&\quad \leq (1+\frac{r}{\varepsilon})\int_0^t \|\mathbf{u}(s)\|_{\mathbb{L}^2}^2 \mathrm{d}s + \frac{2g^2}{\alpha}\int_0^t \|z(s)\|_{\mathrm{L}^2}^2 \mathrm{d}s + \frac{r}{\varepsilon}\int_0^t \|\mathbf{w^0}(s)\|_{\mathbb{L}^4}^4 \mathrm{d}s \\
&\qquad + \int_0^t \|\mathbf{f}(s)\|_{\mathbb{L}^2}^2 \mathrm{d}s + \int_0^t \mathrm{Tr}(\mathbf{g}^2(s))\mathrm{d}s + 2\int_0^t \sum_k (\mathbf{g}_k(s), \mathbf{u}(s))_{\mathbb{L}^2}\mathrm{d}w_k(s) \\
&\qquad + \|\mathbf{u}(0)\|_{\mathbb{L}^2}^2. \qquad (4.7)
\end{aligned}$$

Similarly we consider the second equation of (4.2) by replacing ζ by z to get the estimate like (3.30)

$$\begin{aligned}
\|z(t)\|_{\mathrm{L}^2}^2 \leq M\int_0^t \|\mathbf{u}(s)\|_{\mathbb{L}^2}^2 \mathrm{d}s + (\frac{2\mu^2}{\alpha}+M)\int_0^t \|z(s)\|_{\mathrm{L}^2}^2 \mathrm{d}s \\
+ \frac{\alpha}{2}\int_0^t \|\mathbf{u}(s)\|_{\mathbb{H}_0^1}^2 \mathrm{d}s + \|z(0)\|_{\mathrm{L}^2}^2. \qquad (4.8)
\end{aligned}$$

Now let us set

$$K = \max\{1+M+\frac{r}{\varepsilon}, \frac{2g^2}{\alpha}+\frac{2\mu^2}{\alpha}+M\}.$$

Then summing up (4.7) and (4.8) and rearranging the terms we have

$$\begin{aligned}
&\big[\|\mathbf{u}(t)\|_{\mathbb{L}^2}^2 + \|z(t)\|_{\mathrm{L}^2}^2\big] + \alpha\int_0^t \|\mathbf{u}(s)\|_{\mathbb{H}_0^1}^2 \mathrm{d}s \\
&\quad \leq K\int_0^t \big[\|\mathbf{u}(s)\|_{\mathbb{L}^2}^2 + \|z(s)\|_{\mathrm{L}^2}^2\big]\mathrm{d}s + \frac{r}{\varepsilon}\int_0^t \|\mathbf{w^0}(s)\|_{\mathbb{L}^4}^4 \mathrm{d}s \\
&\qquad + \int_0^t \|\mathbf{f}(s)\|_{\mathbb{L}^2}^2 \mathrm{d}s + \int_0^t \mathrm{Tr}(\mathbf{g}^2(s))\mathrm{d}s + 2\int_0^t \sum_k (\mathbf{g}_k(s), \mathbf{u}(s))_{\mathbb{L}^2}\mathrm{d}w_k(s) \\
&\qquad + \|\mathbf{u}(0)\|_{\mathbb{L}^2}^2 + \|z(0)\|_{\mathrm{L}^2}^2. \qquad (4.9)
\end{aligned}$$

Taking the sup norm in $[0,T]$ and then taking the mathematical expectation we have

$$
\begin{aligned}
&E\Big\{\sup_{0\leq t\leq T}\big[\|\mathbf{u}(t)\|_{\mathbb{L}^2}^2+\|z(t)\|_{\mathrm{L}^2}^2\big]+\alpha\int_0^T\|\mathbf{u}(t)\|_{\mathbb{H}_0^1}^2\mathrm{d}t\Big\}\\
&\quad\leq KE\Big\{\int_0^T\big[\|\mathbf{u}(t)\|_{\mathbb{L}^2}^2+\|z(t)\|_{\mathrm{L}^2}^2\big]\mathrm{d}t\Big\}+\frac{r}{\varepsilon}E\Big\{\int_0^T\|\mathbf{w^0}(t)\|_{\mathbb{L}^4}^4\mathrm{d}t\Big\}\\
&\quad+\int_0^T\big[\|\mathbf{f}(t)\|_{\mathbb{L}^2}^2+\mathrm{Tr}(\mathbf{g}^2(t))\big]\mathrm{d}t+\|\mathbf{u}(0)\|_{\mathbb{L}^2}^2+\|z(0)\|_{\mathrm{L}^2}^2\\
&\quad+2E\Big\{\sup_{0\leq t\leq T}\big|\int_0^t\sum_k(\mathbf{g}_k(s),\mathbf{u}(s))_{\mathbb{L}^2}\mathrm{d}w_k(s)\big|\Big\}. \qquad (4.10)
\end{aligned}
$$

Now by means of martingale inequality, we deduce

$$
\begin{aligned}
&E\Big\{\sup_{0\leq t\leq T}\big|\int_0^t\sum_k(\mathbf{g}_k(s),\mathbf{u}(s))_{\mathbb{L}^2}\mathrm{d}w_k(s)\big|\Big\}\\
&\quad\leq C_1E\Big\{\big(\int_0^T\sum_k(\mathbf{g}_k(t),\mathbf{u}(t))_{\mathbb{L}^2}^2\mathrm{d}t\big)^{1/2}\Big\}\\
&\quad\leq C_1E\Big\{\big(\int_0^T\mathrm{Tr}(\mathbf{g}^2(t))|\mathbf{u}(t)|^2\mathrm{d}t\big)^{1/2}\Big\}\\
&\quad\leq C_1E\Big\{\big(\sup_{0\leq t\leq T}|\mathbf{u}(t)|\big)\ \big(\int_0^T\mathrm{Tr}(\mathbf{g}^2(t))\mathrm{d}t\big)^{1/2}\Big\}\\
&\quad\leq \frac{1}{4}E\{\sup_{0\leq t\leq T}\|\mathbf{u}(t)\|_{\mathbb{L}^2}^2\}+C_1^2E\{\int_0^T\mathrm{Tr}(\mathbf{g}^2(t))\mathrm{d}t\}\\
&\quad\leq \frac{1}{4}E\{\sup_{0\leq t\leq T}\big[\|\mathbf{u}(t)\|_{\mathbb{L}^2}^2+\|z(t)\|_{\mathrm{L}^2}^2\big]\}+C_1^2E\{\int_0^T\mathrm{Tr}(\mathbf{g}^2(t))\mathrm{d}t\}. \qquad (4.11)
\end{aligned}
$$

Using (3.19) on the first term of the right hand side of (4.10), applying (4.11) and rearranging the terms we get the desired estimate (4.6). ☐

Now we deal with the existence and uniqueness of the SPDE and its finite-dimensional approximation.

Proposition 4.2 (uniqueness)**.** *Let* $\mathbf{u}$ *be a solution of the stochastic tide equation* (4.2) *with the regularity*

$$
\begin{cases}
\mathbf{u}\in\mathrm{L}^2(\Omega;C^0(0,T;\mathbb{L}^2(\mathcal{O}))\cap\mathrm{L}^2(0,T;\mathbb{H}_0^1(\mathcal{O}))),\\
z\in\mathrm{L}^2(\Omega\times\mathcal{O}\times(0,T)),
\end{cases}
\qquad (4.12)
$$

and let the data $\mathbf{f}$, $\mathbf{g}$, $\mathbf{u}_0$ *and* z_0 *satisfy the condition*

$$\begin{cases} \mathbf{f} \in \mathrm{L}^2(0,T;\mathbb{H}^{-1}(\mathcal{O})), \quad \mathbf{g} \in \mathrm{L}^2(0,T;\ell_2(\mathbb{L}^2(\mathcal{O}))), \\ \mathbf{u}_0 \in \mathbb{L}^2(\mathcal{O}), \quad z_0 \in \mathrm{L}^2(\mathcal{O}). \end{cases} \tag{4.13}$$

If $\mathbf{v}$ *in* $\mathrm{L}^2(\Omega; C^0(0,T,\mathbb{L}^2(\mathcal{O})) \cap \mathrm{L}^2(0,T,\mathbb{H}_0^1(\mathcal{O})))$ *is another solution of the stochastic tide equation* (4.2)*, then*

$$\begin{aligned} \big[\|\mathbf{u}(t)-\mathbf{v}(t)\|_{\mathbb{L}^2}^2 + &\|z(t)-\tilde{z}(t)\|_{\mathrm{L}^2}^2\big]e^{-Kt} \\ &\le \|\mathbf{u}(0)-\mathbf{v}(0)\|_{\mathbb{L}^2}^2 + \|z(0)-\tilde{z}(0)\|_{\mathrm{L}^2}^2, \end{aligned} \tag{4.14}$$

with probability 1*, for any* $0 \le t \le T$ *and* K *is a positive constant.*

Proof. Indeed if $\mathbf{u}$ and $\mathbf{v}$ are two solutions then $\mathbf{w} = \mathbf{v} - \mathbf{u}$ solves the deterministic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{w}(t) + \mathbf{A}\mathbf{w}(t) + g\nabla\big(z(t)-\tilde{z}(t)\big) = \mathbf{B}(\mathbf{v}(t)) - \mathbf{B}(\mathbf{u}(t)) \quad \text{in } \mathbb{L}^2(0,T;\mathbb{H}^{-1}(\mathcal{O})).$$

Thus the proof of uniqueness follows directly from Proposition 3.6, with probability 1. □

If a given adapted process $\mathbf{u}$ in $\mathrm{L}^2(\Omega; \mathrm{L}^\infty(0,T;\mathbb{L}^2(\mathcal{O})) \cap \mathrm{L}^2(0,T;\mathbb{H}_0^1(\mathcal{O})))$ satisfies

$$\mathrm{d}(\mathbf{u}(t),\mathbf{v})_{\mathbb{L}^2} = (\mathbf{F}(t),\mathbf{v})_{\mathbb{L}^2}\mathrm{d}t + (\mathbf{g}(t),\mathbf{v})_{\mathbb{L}^2}\mathrm{d}w(t), \tag{4.15}$$

for any function $\mathbf{v}$ in $\mathbb{H}_0^1(\mathcal{O})$ and some functions $\mathbf{F}$ in $\mathrm{L}^2(0,T;\mathbb{H}^{-1}(\mathcal{O}))$ and $\mathbf{g}$ in $\mathrm{L}^2(0,T;\ell_2(\mathbb{L}^2(\mathcal{O})))$, then we can find a version of $\mathbf{u}$ (which is still denoted by $\mathbf{u}$) in $\mathrm{L}^2(\Omega; C^0(0,T;\mathbb{L}^2(\mathcal{O})))$ satisfying the energy equality

$$\mathrm{d}\|\mathbf{u}(t)\|_{\mathbb{L}^2}^2 = \big[2(\mathbf{F}(t),\mathbf{u}(t))_{\mathbb{L}^2} + \mathrm{Tr}(\mathbf{g^2}(t))\big]\mathrm{d}t + 2(\mathbf{g}(t),\mathbf{u}(t))_{\mathbb{L}^2}\mathrm{d}w(t) \tag{4.16}$$

see e.g. Gyongy and Krylov [4], Pardoux [17].

Definition 4.3. (*Strong Solution*) A strong solution $\mathbf{u}$ is defined on a given filtered probability space $(\Omega,\ \mathcal{F},\ \mathcal{F}_t,\ P)$ as a $\mathrm{L}^2(\Omega; \mathrm{L}^\infty(0,T;\mathbb{L}^2(\mathcal{O})) \cap \mathrm{L}^2(0,T;\mathbb{H}_0^1(\mathcal{O})) \cap C^0(0,T;\mathbb{L}^2(\mathcal{O})))$ valued function which satisfies the stochastic tide equation (4.2) in the weak sense and also the energy inequality (4.6).

Proposition 4.4 (2-D existence)**.** *Let* $\mathbf{f}$, $\mathbf{g}$*,* $\mathbf{u}_0$ *and* z_0 *be such that*

$$\begin{cases} \mathbf{f} \in \mathrm{L}^2(0,T;\mathbb{H}^{-1}(\mathcal{O})), \quad \mathbf{g} \in \mathrm{L}^2(0,T;\ell_2(\mathbb{L}^2(\mathcal{O}))), \\ \mathbf{u}_0 \in \mathbb{L}^2(\mathcal{O}), \quad z_0 \in \mathrm{L}^2(\mathcal{O}). \end{cases} \tag{4.17}$$

Then there exist adapted processes $\mathbf{u}(t,x,\omega)$ *and* $z(t,x,\omega)$ *with the regularity*

$$\begin{cases} \mathbf{u} \in \mathrm{L}^2(\Omega; C^0(0,T;\mathbb{L}^2(\mathcal{O})) \cap \mathrm{L}^2(0,T;\mathbb{H}_0^1(\mathcal{O}))), \\ z,\ \dot{z} \in \mathrm{L}^2(\Omega; \mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{O}))) \end{cases} \tag{4.18}$$

satisfying the stochastic tide equation (4.2) *and the a priori bound* (4.6)*.*

Proof. Consider a finite dimensional Galerkin approximation of the stochastic tide equation.
Let us denote

$$F(\mathbf{u}) := \mathbf{A}\mathbf{u} + \mathbf{B}(\mathbf{u}) - \mathbf{f},$$

where the operators $\mathbf{A}$ and $\mathbf{B}$ are defined by (2.8) and (3.9) respectively.
Then

$$\mathrm{d}\mathbf{u}^n(t) + F(\mathbf{u}^n(t))\mathrm{d}t + g\nabla z^n(t)\mathrm{d}t = \mathbf{g}(t)\mathrm{d}w(t).$$

Then Using the a priori estimates (4.6), it follows from the Banach-Alaoglu theorem that along a subsequence, the Galerkin approximations $\{\mathbf{u}^n\}$ have the following limits:

$$\begin{aligned}
&\mathbf{u}^n \longrightarrow \mathbf{u} \quad \text{weakly star in } \mathrm{L}^2(\Omega; \mathrm{L}^\infty(0,T;\mathbb{L}^2(\mathcal{O})) \cap \mathrm{L}^2(0,T;\mathbb{H}_0^1(\mathcal{O}))),\\
&z^n \longrightarrow z \text{ weakly in } \mathrm{L}^2(\Omega; \mathrm{L}^2(0,T;\mathbb{L}^2(\mathcal{O}))),\\
&F(\mathbf{u}^n) \longrightarrow F_0 \quad \text{weakly in } \mathrm{L}^2(\Omega; \mathrm{L}^2(0,T;\mathbb{H}^{-1}(\mathcal{O}))),
\end{aligned}$$

where $\mathbf{u}$ has the Itô differential

$$\mathrm{d}\mathbf{u}(t) + F_0(t)\mathrm{d}t + g\nabla z(t)\mathrm{d}t = \mathbf{g}(t)\mathrm{d}w(t) \qquad \text{in } \mathrm{L}^2(\Omega; \mathrm{L}^2(0,T;\mathbb{H}^{-1}(\mathcal{O}))),$$

and the energy equality similar to stochastic version of (3.32) holds, i.e.,

$$\begin{aligned}
\mathrm{d}\big[\|\sqrt{h}\mathbf{u}(t)\|_{\mathbb{L}^2}^2 + g\|z(t)\|_{\mathrm{L}^2}^2\big] &+ 2\big(F_0(t), h\mathbf{u}(t)\big)_{\mathbb{L}^2}\mathrm{d}t\\
&= \mathrm{Tr}(\mathbf{g}^2(t))\mathrm{d}t + 2(\mathbf{g}(t), h\mathbf{u}(t))_{\mathbb{L}^2}\mathrm{d}w(t).
\end{aligned}$$

Since the Galerkin approximations $\{\mathbf{u}^n\}$ satisfy the energy equality

$$\begin{aligned}
\mathrm{d}\big[\|\sqrt{h}\mathbf{u}^n(t)\|_{\mathbb{L}^2}^2 + g\|z^n(t)\|_{\mathrm{L}^2}^2\big] &+ 2\big(F(\mathbf{u}^n(t)), h\mathbf{u}^n(t)\big)_{\mathbb{L}^2}\mathrm{d}t\\
&= \mathrm{Tr}(\mathbf{g}^2(t))\mathrm{d}t + 2(\mathbf{g}(t), h\mathbf{u}^n(t))_{\mathbb{L}^2}\mathrm{d}w(t).
\end{aligned}$$

Integrating between $0 \le t \le T$ and taking the mathematical expectation we have

$$\begin{aligned}
&E\Big[\|\sqrt{h}\mathbf{u}^n(T)\|_{\mathbb{L}^2}^2 + g\|z^n(T)\|_{\mathrm{L}^2}^2 - \|\sqrt{h}\mathbf{u}^n(0)\|_{\mathbb{L}^2}^2 - g\|z^n(0)\|_{\mathrm{L}^2}^2\Big]\\
&\quad + 2E\Big[\int_0^T \big(F(\mathbf{u}^n(t)), h\mathbf{u}^n(t)\big)_{\mathbb{L}^2}\mathrm{d}t\Big] = E\Big[\int_0^T \mathrm{Tr}(\mathbf{g}^2(t))\mathrm{d}t\Big].
\end{aligned}$$

Considering the fact that the initial conditions $\mathbf{u}^n(0)$ and $z^n(0)$ converge to $\mathbf{u}(0)$ and $z(0)$ respectively in $\mathbb{L}^2$, and the lower-semi-continuity of the $\mathbb{L}^2$-norm, we deduce

$$
\begin{aligned}
&\liminf_n E\Big[-\int_0^T \big(2F(\mathbf{u}^n(t)), h\mathbf{u}^n(t)\big)_{\mathbb{L}^2}\mathrm{d}t\Big]\\
&\quad= \liminf_n E\Big[\|\sqrt{h}\mathbf{u}^n(T)\|^2_{\mathbb{L}^2} + g\|z^n(T)\|^2_{\mathrm{L}^2} - \|\sqrt{h}\mathbf{u}^n(0)\|^2_{\mathbb{L}^2} - g\|z^n(0)\|^2_{\mathrm{L}^2}\\
&\qquad\qquad - \int_0^T \mathrm{Tr}(\mathbf{g}^2(t))\mathrm{d}t\Big]\\
&\quad\geq E\Big[\|\sqrt{h}\mathbf{u}(T)\|^2_{\mathbb{L}^2} + g\|z(T)\|^2_{\mathrm{L}^2} - \|\sqrt{h}\mathbf{u}(0)\|^2_{\mathbb{L}^2} - g\|z(0)\|^2_{\mathrm{L}^2}\\
&\qquad\qquad - \int_0^T \mathrm{Tr}(\mathbf{g}^2(t))\mathrm{d}t\Big]\\
&\quad= E\Big[-\int_0^T \big(2F_0(t), h\mathbf{u}(t)\big)_{\mathbb{L}^2}\mathrm{d}t\Big]
\end{aligned}
\tag{4.19}
$$

Next, equation (3.8) and Lemma 3.3 yield

$$
2E\Big[\int_0^T \big(F(\mathbf{u}^n(t)) - F(\mathbf{v}(t)), h\mathbf{u}^n(t) - h\mathbf{v}(t)\big)_{\mathbb{L}^2}\Big] \geq 0. \tag{4.20}
$$

Rearranging the terms we have

$$
\begin{aligned}
&E\Big[\int_0^T \big(2F(\mathbf{v}(t)), h\mathbf{v}(t) - h\mathbf{u}^n(t)\big)_{\mathbb{L}^2}\mathrm{d}t\Big]\\
&\quad\geq\ E\Big[\int_0^T \big(2F(\mathbf{u}^n(t)), h\mathbf{v}(t) - h\mathbf{u}^n(t)\big)_{\mathbb{L}^2}\mathrm{d}t\Big]\\
&\quad= -2E\Big[\int_0^T \big(F(\mathbf{u}^n(t)), h\mathbf{u}^n(t)\big)_{\mathbb{L}^2}\mathrm{d}t\Big] + 2E\Big[\int_0^T \big(F(\mathbf{u}^n(t)), h\mathbf{v}(t)\big)_{\mathbb{L}^2}\mathrm{d}t\Big].
\end{aligned}
$$

Taking limit in n and using (4.19) we get

$$
\begin{aligned}
&E\Big[\int_0^T \big(2F(\mathbf{v}(t)), h\mathbf{v}(t) - h\mathbf{u}(t)\big)_{\mathbb{L}^2}\mathrm{d}t\Big]\\
&\quad\geq\ -2E\Big[\int_0^T \big(F_0(t), h\mathbf{u}(t)\big)_{\mathbb{L}^2}\mathrm{d}t\Big] + 2E\Big[\int_0^T \big(F_0(t), h\mathbf{v}(t)\big)_{\mathbb{L}^2}\mathrm{d}t\Big]\\
&\quad= E\Big[\int_0^T \big(2F_0(t), h\mathbf{v}(t) - h\mathbf{u}(t)\big)_{\mathbb{L}^2}\mathrm{d}t\Big].
\end{aligned}
$$

Now we take $\mathbf{v} := \mathbf{u} + \lambda\mathbf{w}$ with $\lambda > 0$ and $\mathbf{w}$ is an adapted process in $\mathrm{L}^2(\Omega; \mathrm{L}^\infty(0,T;\mathbb{L}^2(\mathcal{O})) \cap \mathrm{L}^2(0,T;\mathbb{H}^1_0(\mathcal{O})))$.

Then we have

$$\lambda E\Big[\int_0^T \big(2F(\mathbf{u}(t)+\lambda\mathbf{w}(t)), h\mathbf{w}(t)\big)_{\mathbb{L}^2}\mathrm{d}t\Big] \geq \lambda E\Big[\int_0^T \big(2F_0(t), h\mathbf{w}(t)\big)_{\mathbb{L}^2}\mathrm{d}t\Big].$$

Dividing by λ on both sides of the inequality above, and letting λ go to 0, we obtain

$$E\Big[\int_0^T \big(F(\mathbf{u}(t)) - F_0(t), h\mathbf{w}(t)\big)_{\mathbb{L}^2}\mathrm{d}t\Big] \;\geq\; 0.$$

Since $\mathbf{w}$ is arbitrary and $h = h(x)$ is a positive, bounded and continuously differentiable function, we conclude that $F_0(t) = F(\mathbf{u}(t))$. Hence the existence of a strong solution of the stochastic tide equation (4.2) has been proved. □

References

1. Adams, R.A.: *Sobolev spaces*, Academic Press, New York, 1975.
2. Barbu, V. and Sritharan, S.S.: m-accretive quantization of the vorticity equation, *Semigroups of operators: theory and applications (Newport Beach, CA, 1998)*, 296–303, *Progr. Nonlinear Differential Equations Appl.*, **42**, Birkhuser, Basel, 2000.
3. Barbu, V. and Sritharan, S.S.: Flow invariance preserving feedback controllers for the Navier-Stokes equation, *J. Math. Anal. Appl.*, **255**(2001), no. 1, 281–307.
4. Gyongy, I. and Krylov, N. V.: On stochastic equations with respect to semimartingales Itô formula in Banach spaces, *Stochastics*, **6** (1982), 153–173.
5. Ipatova, V.M.: Solvability of a tide dynamics model in adjacent seas, *Russ. J. Numer. Anal. Math. Modelling*, **20**, No. 1 (2005), 67–79.
6. Kesavan, S.: *Nonlinear Functional Analysis. A First Course*, Hindustan Book Agency, New Delhi, 2004.
7. Ladyzhenskaya, O.A.: *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1969.
8. Lamb, H.: *Hydrodynamics*, Dover Publications, New York, 1932.
9. Laplace, P.S.: Recherches sur quelques points du système du monde, *Mem. Acad. Roy. Sci, Paris*, **88** (1775), 75–182.
10. Lions, J.L.: *Quelque méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.
11. Maclaurin, C.: *De Causâ Physicâ Fluxus et Refluxus Maris*, 1740.
12. Manna, U., Menaldi, J.L. and Sritharan, S.S.: Stochastic 2-D Navier-Stokes Equation with Artificial Compressibility, *Communications on Stochastic Analysis*, In press(2007).
13. Marchuk, G.I. and Kagan, B.A.: *Ocean tides. Mathematical models and numerical experiments*, Pergamon Press, Elmsford, NY, 1984.
14. Marchuk, G.I. and Kagan, B.A.: *Dynamics of Ocean Tides*, Kluwer Academic Publishers, Dordrecht / Boston / London, 1989.
15. Menaldi, J.L. and Sritharan, S.S.: Stochastic 2-D Navier-Stokes Equation, *Appl. Math. Optim.*, **46**(2002), 31-53.
16. Newton, I.: *Philosophiae Naturalis Principia Mathematica*, 1687.
17. Pardoux, E.: Stochastic partial differential equations and filtering of diffusion processes, *Stochastics*, **6** (1979), 127–167.
18. Pedlosky, J.: *Geophysical Fluid Dyanmics I, II*, Springer, Heidelberg, 1981.

19. Poincaré, H.: *Lecons de Mécanique Celeste. 3. Théroe des marées*, Cauthier Villars, Paris, 1910.
20. Sritharan, S.S. and Sundar, P.: Large deviations for two dimensional Navier-Stokes equations with multiplicative noise, *Stochastic Processes & Their Applications*, **116**, No. 11(2006), 1636–1659.
21. Thomson, W. and Tait, P.G.: *Treatise of Natural Philosophy*, Vol. I, 1883.

U. Manna: Department of Mathematics, University of Wyoming, Laramie, WY 82071, USA
E-mail address: utpal@uwyo.edu
URL: http://uwyo.edu/utpal/

J.L. Menaldi: Department of Mathematics, Wayne State University, Detroit, MI 48202, USA
E-mail address: jlm@math.wayne.edu
URL: http://www.math.wayne.edu/~menaldi/

S.S. Sritharan: Department of Mathematics, University of Wyoming, Laramie, WY 82071, USA
E-mail address: sri@uwyo.edu
URL: http://math.uwyo.edu/Prof_home/Sri.html