MULTIPLICATIVE DEPENDENCE AND ISOLATION I

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1. Introduction. Two elements x, y of a field are called multiplicatively dependent if $xy \neq 0$ and there exists $r, s \in \mathbb{Z}$ not both zero such that $x^r y^s = 1$. In [CZ00] Cohen and Zannier prove that if $x \in \overline{\mathbb{Q}}$, the field of algebraic numbers, such that x and $1-x$ are multiplicatively dependent, then $\max\{H(x), H(1-x)\} \leq 2$, where $H(\cdot)$ is the absolute Weil height. This bound is sharp because of the exceptional values $x = -1, 1/2, 2$. Cohen and Zannier then use Bilu's Equidistribution Theorem [Bil97] to show that there exists $\epsilon > 0$ such that if x is as before but not one of the three exceptional values, then $\max\{H(x), H(1-x)\} \leq 2 - \epsilon$. In [Hab05] the author showed that if α is any non-zero rational integer and x, $\alpha - x$ are multiplicatively dependent, then $\max\{H(x), H(\alpha - x)\} \leq 2H(\alpha)$. In the same article it is shown that this bound is sharp if and only if α is a power of 2. If $\alpha = 2^n \geq 2$, a uniform isolation result is proved, namely either $x \in \{2\alpha, -\alpha\}$ or already $\max\{H(x), H(\alpha - x)\} \leq 1.98H(\alpha)$.

The purpose of this note is to give a concise proof of a slight strengthening of the height bound in the case $\alpha = 1$ based on the method used in [Hab05]. We also work out an explicit ϵ. Finally we show that there exists a sequence x_n with x_n and $1 - x_n$ multiplicatively dependent and such that the height of x_n converges to the Mahler measure of the polynomial $X + Y - 1$.

We define $\mathcal{M}$ to be the set of x such that x and $1 - x$ are multiplicatively dependent. Clearly the elements of $\mathcal{M}$ are algebraic. If $\zeta \neq 1$ is a root of unity, then ζ and $1 - \zeta$ are multiplicatively dependent and so $\zeta \in \mathcal{M}$. Thus $\mathcal{M}$ is infinite, a result which has been made quantitative by Masser in Theorem 2 of [Mas05]. Let $H(x, y)$ denote the affine absolute non-logarithmic Weil height, which will be defined further down. This height function corresponds to the compactification of the algebraic torus $\mathbb{G}_m^2 \to \mathbb{P}^2$. We have:

Theorem 1. *Let $x \in \mathcal{M}$, then $H(x, 1-x) \leq 2$ with equality if and only if $x \in \{-1, 1/2, 2\}$.*

Theorem 1 implies Theorem 1 of [CZ00] since $\max\{H(x), H(y)\} \leq H(x, y)$ for algebraic x and y. We choose the particular height function $H(x, 1-x)$ because it is invariant under the maps $x \mapsto 1 - x$ and $x \mapsto x^{-1}$. Incidently $\mathcal{M}$ is stable under these two maps. Our method of proof for Theorem 1 exploits this fact and relies on elementary local estimates combined with the product formula.

Theorem 1 can be put in the context of a more general result by Bombieri, Masser, and Zannier (Theorem 1, [BMZ99]). In their article it is shown that if $\mathcal{C}$ is an irreducible algebraic curve defined over $\overline{\mathbb{Q}}$ embedded in the multiplicative torus and not contained in the translate of a proper algebraic subgroup, then the points of the intersection of $\mathcal{C}$ with the union of all proper algebraic subgroups have bounded height. In our special situation the curve is defined by the polynomial $X + Y - 1$.

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The proof of the isolation result in [Hab05] made use of 2-adic estimates and works only if $\alpha = 2^n \geq 2$. To handle the case $\alpha = 1$ we apply an explicit result by Mignotte [Mig89] on the angular distribution of conjugates of an algebraic number of small height and big degree to find an explicit ϵ. Large degree is guaranteed by a theorem of Smyth [Smy71] on lower bounds for heights of non-reciprocal algebraic numbers.

Theorem 2. *If* $x \in \mathcal{M}\backslash\{-1, 1/2, 2\}$ *then* $H(x, 1-x) < 1.915$.

The element of $\mathcal{M}\backslash\{-1, 1/2, 2\}$ of largest height known to the author is $1-\zeta_3$ where ζ_3 is a primitive 3rd root of unity. In fact $H(1-\zeta_3) = \sqrt{3}$. It would be interesting to know if $\sqrt{3}$ is already the second to largest height value obtained on $\mathcal{M}$.

If $\zeta \neq 1$ is a root of unity, then $1-\zeta \in \mathcal{M}$. As the degree of ζ goes to infinity we can use Bilu's Equidistribution Theorem (Theorem 1.1, [Bil97]) to show that $H(1-\zeta, \zeta) = H(1-\zeta)$ converges to

$$\exp \int_{-1/3}^{1/3} \log|1+\exp(2\pi i t)| dt = 1.381356..., \tag{1}$$

Let f be a polynomial in n variables with complex coefficients, the Mahler measure $M(f)$ of f is defined as

$$M(f) = \exp\left(\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} \log|f(\exp(it_1), \ldots, \exp(it_n))| dt_1 \cdots dt_n\right).$$

Smyth ([Smy81]) calculated the Mahler measure of the polynomial $X+Y-1$ defining our curve as

$$M(X+Y-1) = \exp(\frac{3\sqrt{3}}{4\pi}\sum_{k\geq 1}\left(\frac{k}{3}\right)\frac{1}{k^2}), \tag{2}$$

here $\left(\frac{\cdot}{\cdot}\right)$ is the Legendre symbol. By Jensen's Formula the Mahler measure in (2) is equal to the integral (1). We immediately get

Proposition. *There exists a sequence* $x_n \in \mathcal{M}$ *with* $\lim_{n\to\infty}[\mathbb{Q}(x_n):\mathbb{Q}] = \infty$ *such that* $\lim_{n\to\infty} H(x_n, 1-x_n) = M(X+Y-1)$.

In a future paper we will show that $H(x_n, 1-x_n)$ has limit $M(X+Y-1)$ for any sequence $x_n \in \mathcal{M}$ with $\lim_{n\to\infty}[\mathbb{Q}(x_n):\mathbb{Q}] = \infty$.

2. Proof of Theorem 1. First we recall some basics about places of number fields and heights. Let K be a number field. A place of K is a non-trivial absolute value normalized such that its restriction to $\mathbb{Q}$ is either a p-adic absolute value for a prime p or the standard complex absolute value. The places extending the complex absolute value will be called infinite, the others finite. It is well-known that if v is an infinite place of K, then there exists an embedding σ of K into $\mathbb{C}$, such that $|x|_v = |\sigma x|$ for all $x \in K$ and $|\cdot|$ the standard complex absolute value. For any place v of K we denote K_v the completion of K with respect to v. If $|\cdot|_v$ is a finite place of K, then $|\cdot|_v$ is ultrametric, i.e. for all $x, y \in K$ one has $|x+y|_v \leq \max\{|x|_v, |y|_v\}$. If x is in K^*, the non-zero elements of K, then $|x|_v = 1$ for all but finitely many places v of K. Furthermore for such x we have the product formula

$$\prod_v |x|_v^{[K_v:\mathbb{Q}_v]} = 1, \tag{3}$$

where the product runs over all places of K. Let $(x_1,\ldots,x_n)\in K^n$, we set

$$H(x_1,\ldots,x_n)=\prod_v \max\{1,|x_1|_v,\ldots,|x_n|_v\}^{[K_v:\mathbb{Q}_v]/[K:\mathbb{Q}]} \tag{4}$$

where the product runs over all places of K. For notation purposes we set $h(\cdot)=\log H(\cdot)$. It is well-known that (4) is independent of the number field K containing the x_i. If $P\in\mathbb{Z}[X]$ is the minimal polynomial of x (with content 1), then $H(x)^{[\mathbb{Q}(x):\mathbb{Q}]}$ is equal to the Mahler measure of P. The height function satisfies some important functional (in)equalities which we apply freely throughout this note. For example $h(x^n)=|n|h(x)$ if $x\in\overline{\mathbb{Q}}^*$ and $n\in\mathbb{Z}$ or $h(x+y)\leq\log 2+h(x)+h(y)$ for any $x,y\in\overline{\mathbb{Q}}$. For references see [Lan83].

We prove Theorem 1 via an elementary estimate which holds for any field K with any absolute value $|\cdot|:K\to\mathbb{R}$.

Lemma 1. *Let $x\in K\backslash\{0,1\}$, $r,t\in\mathbb{Z}$ with $0\neq t\geq r\geq 0$ and $x^r=(1-x)^t$. We have*

$$|1-x|^{-1}\max\{1,|x|\}\leq\delta \tag{5}$$

where $\delta=1$ if $|\cdot|$ is ultrametric and $\delta=2$ otherwise. Furthermore, equality in (5) implies $\delta=1$ or $r=0$ or $r=t$.

Proof. Let q denote the left-hand side of (5).

First let us assume

$$|x|<\delta^{-1} \text{ or } |x|>\delta. \tag{6}$$

If $\delta=1$, then $|x|\neq 1$, so $|1-x|=\max\{1,|x|\}$, hence $q=1$. If $\delta=2$ we use the triangle inequality to bound

$$|1-x|\geq\begin{cases}|x|-1>|x|\delta^{-1} & : \text{ if } |x|>\delta,\\ 1-|x|>\delta^{-1} & : \text{ if } |x|<\delta^{-1}\end{cases} \tag{7}$$

which implies $q<\delta$. So in the case (6) we have $q\leq\delta$ and furthermore $q=\delta$ can only hold if $\delta=1$.

Now let us assume $\delta^{-1}\leq|x|\leq\delta$. If $|x|<1$, then $q=|1-x|^{-1}=|x|^{-r/t}\leq\delta^{r/t}\leq\delta$, and if $|x|\geq 1$, then $q=|x|/|1-x|=|x|^{1-r/t}\leq\delta^{1-r/t}\leq\delta$. It is clear that if we have the equalities $q=\delta=2$, then $r=0$ or $r=t$. $\square$

We recall Lemma 7 of [Hab05]:

Lemma 2. *Let $\zeta\neq 1$ be a root of unity, then $h(1+\zeta)\leq\frac{1}{2}\log(2\sqrt{3})$.*

We note that $\frac{1}{2}\log(2\sqrt{3})$ is an improvement of the trivial bound $h(1+\zeta)\leq\log 2$ which holds for all roots of unity ζ. In the proof of Theorem 1 we need only a weak form of Lemma 2. For the sake of completeness we include its proof:

Lemma 3. *Let $\zeta\neq 1$ be a root of unity, then $h(1+\zeta)<\log 2$.*

Proof. Since $|1+\zeta|_v\leq 1$ for all finite places v of $\mathbb{Q}(\zeta)$ we have

$$[\mathbb{Q}(\zeta):\mathbb{Q}]h(1+\zeta)=\sum_\sigma \log\max\{1,|1+\sigma\zeta|\} \tag{8}$$

where the sum runs over all embeddings σ of $\mathbb{Q}(\zeta)$ into $\mathbb{C}$. Let σ be such an embedding. Since $\sigma\zeta\neq 1$ the parallelogram equality $|1+\sigma\zeta|^2+|1-\sigma\zeta|^2=2|\sigma\zeta|^2+2=4$ implies

$|1+\sigma\zeta| < 2$. Therefore $h(1+\zeta) < \log 2$ by (8) and since $\mathbb{Q}(\zeta)$ has $[\mathbb{Q}(\zeta):\mathbb{Q}]$ embeddings σ. □

Lemma 4. *Let $x' \in \mathcal{M}$, then there exists $x \in \mathcal{M}$ and $r, t \in \mathbb{Z}$ with $0 \neq t \geq 2r \geq 0$ such that $x^r = (1-x)^t$ and $h(x', 1-x') = h(x)$. Furthermore if $x' \notin \{-1, 1/2, 2\}$ then we can choose x such that $x \notin \{-1, 1/2, 2\}$.*

Proof. The lemma is simple if x' is a primitive 6th root of unity, for then $1-x'$ is also a 6th root of unity and we may take $x = x'$, $t = 6$, and $r = 0$. Hence it suffices to show the lemma for $x' \in \mathcal{M}_0$ with $\mathcal{M}_0 = \mathcal{M}\backslash\{e^{\pm 2\pi i/6}\}$. For any such x' there exists a unique $\lambda(x') = [r : t] \in \mathbb{P}^1(\mathbb{Q})$ with r and t coprime integers such that $x'^r(1-x')^{-t}$ is a root of unity. The maps $\phi_1(x) = 1/x$ and $\phi_2(x) = 1-x$ are automorphisms of $\mathcal{M}_0$ and generate the symmetric group S_3. Thus we get an action of S_3 on $\mathcal{M}_0$ which also leaves $\{-1, 1/2, 2\}$ invariant. By the product formula the height $h(x, 1-x)$ is also invariant under this action. We check that if $\lambda(x) = [r : t]$, then $\lambda(\phi_1(x)) = [t-r : t]$ and $\lambda(\phi_2(x)) = [t : r]$. We get an action of S_3 on $\mathbb{P}^1(\mathbb{Q})$ with fundamental domain $\{[1 : s];\ s \geq 2\} \cup \{[0 : 1]\}$. The lemma follows immediately. □

Proof of Theorem 1. Because of Lemma 4 it suffices to show that if $x \in \overline{\mathbb{Q}}\backslash\{0, 1\}$ with $x \neq -1, 1/2, 2$ and $x^r = (1-x)^t$ for integers $0 \neq t \geq 2r \geq 0$, then $h(x) < \log 2$.

If $r = 0$, then $x = 1 + \zeta$ for some root of unity $\zeta \neq \pm 1$. In this case the theorem follows from Lemma 3.

Let us assume $r > 0$. We fix a number field K which contains x and apply the product formula (3) to $1 - x$ to deduce

$$[K : \mathbb{Q}]h(x) = \sum_v [K_v : \mathbb{Q}_v] \log \max\{1, |x|_v\} = \sum_v [K_v : \mathbb{Q}_v] \log \frac{\max\{1, |x|_v\}}{|1-x|_v}.$$

Since $0 < r < t$ we apply Lemma 1 to the local terms in the equality above to see that $[K : \mathbb{Q}]h(x) < \sum_{v \text{ infinite}} [K_v : \mathbb{Q}_v] \log 2$. This inequality completes the proof since the sum is just $[K : \mathbb{Q}] \log 2$. □

3. Proof of Theorem 2.

A non-zero algebraic number α is called reciprocal if α and α^{-1} are conjugated. We apply Mignotte's equidistribution result and Smyth's Theorem ([Smy71]) on lower bounds for heights of non-reciprocal algebraic integers to deduce the following lemma.

Lemma 5. *Let $\alpha \in \overline{\mathbb{Q}}^*$ be non-reciprocal with $h(\alpha) \leq \frac{\log 2}{3\cdot 10^5}$, then $h(1+\alpha) \leq 0.933 \cdot \log 2 + h(\alpha)$.*

Proof. Let α be as in the hypothesis and $d = [\mathbb{Q}(\alpha) : \mathbb{Q}]$, furthermore let $\theta_0 > 1$ be the unique real which satisfies $\theta_0^3 - \theta_0 - 1 = 0$. If α is an algebraic integer, then $dh(\alpha) \geq \log \theta_0$ by Smyth's Theorem ([Smy71]). The upper bound for $h(\alpha)$ implies

$$d \geq 121700. \tag{9}$$

On the other hand, if α is not an algebraic integer, then it is well-known that $dh(\alpha) \geq \log 2$. Thus (9) holds in any case.

We split $\mathbb{C}^*$ up into three sectors

$$C_k = \{r \cdot \exp(i\phi);\ r > 0 \text{ and } \frac{2\pi}{3}(k-1) \leq \phi < \frac{2\pi}{3}k\} \text{ for } 1 \leq k \leq 3$$

and define the function

$$m(z) = \frac{\max\{1, |z+1|\}}{\max\{1, |z|\}} = \frac{\max\{1, (r^2 + 2r\cos\phi + 1)^{1/2}\}}{\max\{1, r\}}$$

for $z = r \cdot \exp(i\phi)$ with $r > 0$ and $\phi \in \mathbb{R}$. Hence

$$m(z)^2 \leq \begin{cases} \frac{\max\{1, r^2+2r+1\}}{\max\{1, r^2\}} & : \text{ if } -2\pi/3 \leq \phi \leq 2\pi/3 \\ \frac{\max\{1, r^2-r+1\}}{\max\{1, r^2\}} & : \text{ if } 2\pi/3 \leq \phi \leq 4\pi/3. \end{cases}$$

Elementary calculus now leads to

$$m|_{C_1 \cup C_3} \leq 2 \text{ and } m|_{C_2} \leq 1. \tag{10}$$

We fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$. Let $\alpha_1 \ldots, \alpha_d \in \mathbb{C}^*$ be the conjugates of α. We set $N_k = \#\{i;\ \alpha_i \in C_k\}$ for $1 \leq k \leq 3$. For any finite place v of $\mathbb{Q}(\alpha)$ we have $\max\{1, |1+\alpha|_v\} = \max\{1, |\alpha|_v\}$ by the ultrametric inequality. Since the finite places of $\mathbb{Q}(\alpha)$ taken with multiplicities correspond to embeddings of $\mathbb{Q}(\alpha)$ into $\mathbb{C}$ and because of (10) we have

$$d(h(1+\alpha) - h(\alpha)) = \sum_{i=1}^{d} \log m(\alpha_i) \leq (N_1 + N_3)\log 2. \tag{11}$$

We set $\epsilon = (\frac{9}{2}c^2(\frac{\log(2d+1)}{d} + h(\alpha)))^{1/3}$ with $c = 2.62$. Since $\frac{\log(2d+1)}{d}$ is decreasing considered as function of $d \geq 1$, we use (9) and our hypothesis on $h(\alpha)$ to conclude $\epsilon < 0.1477$. We apply Mignotte's Theorem (ii) ([Mig89], p.83) to the minimal polynomial of α and to the closure of our sectors C_k to bound

$$\frac{N_k}{d} \leq \frac{1}{3} + 2.823(\frac{\log(2d+1)}{d} + h(\alpha))^{1/3}. \tag{12}$$

Our hypothesis on $h(\alpha)$ and (9) together with (12) imply $\frac{N_k}{d} < 0.4662$. This last bound applied to (11) concludes the proof. □

We note that the trivial bound $h(1+\alpha) \leq \log 2 + h(\alpha)$ holds for all algebraic α. Thus Lemma 5 gives a slight improvement for non-reciprocal α of small height. Instead of Smyth's lower bound for heights we could have used the lower bound by Dobrowolski which holds for any non-zero algebraic number not a root of unity. This approach leads to slightly worse numerical constants. Thus by taking sectors with smaller angles in the proof of Lemma 5 the constant $0.933 \cdot \log 2$ can be replaced by any real number strictly greater than the logarithm of the number (1) if the height of α is sufficiently small but positive. But the bound given in Lemma 5 is apt for our application.

In [CZ00] Cohen and Zannier introduce a function $S : (1, \infty) \to \mathbb{R}$ relevant to our problem. We briefly recall its definition. Say $\lambda > 1$ and let $\xi, \tilde{\xi} > 1$ be the unique reals such that $\xi^\lambda = \xi + 1$ and $\tilde{\xi}^{\lambda/(\lambda-1)} = \tilde{\xi} + 1$, then

$$S(\lambda) = \frac{\log(\xi+1)\log(\tilde{\xi}+1)}{\log(\xi+1) + \log(\tilde{\xi}+1)}.$$

Lemma 1 of [CZ00] implies $S < \log 2$, furthermore if $x^r = (1-x)^t$ for integers $t > r > 0$, then $h(x) \leq S(t/r)$. The proof of said lemma also shows that S increases on $[2, \infty)$.

Proof of Theorem 2. Because of Lemma 4 it suffices to show that if $x \in \mathbb{Q}\backslash\{0,1\}$, $x \neq -1, 1/2, 2$ and $x^r = (1-x)^t$ for integers $0 \neq t \geq 2r \geq 0$, then $h(x) < \log 1.915$. If $r = 0$, then $x = 1 + \zeta$ for a root of unity $\zeta \neq \pm 1$. Lemma 2 implies $h(x) \leq \frac{1}{2}\log(2\sqrt{3}) < \log 1.915$. We now assume $r > 0$ and define $\lambda = t/r \geq 2$.

If $\lambda < 3 \cdot 10^5$, Then we have $h(x) \leq S(3 \cdot 10^5)$ by the properties of $S(\cdot)$. A calculation shows that the right-hand side of the last inequality is strictly less then $\log 1.915$.

Finally we assume $\lambda \geq 3 \cdot 10^5$. Then $h(1-x) = \lambda^{-1}h(x) \leq \frac{\log 2}{3\cdot 10^5}$ by Theorem 1. Let $\alpha = x - 1$, we have

$$(-1)^t \alpha^t = (1+\alpha)^r. \tag{13}$$

Let us assume first that α and α^{-1} are not conjugated, then $h(x) \leq 0.933 \cdot \log 2 + \frac{\log 2}{3\cdot 10^5} < \log 1.915$ by Lemma 5. If α and α^{-1} are conjugated, then equality (13) must hold with α replaced by α^{-1}. Hence $1 = \alpha^{2t}(1+\alpha^{-1})^{2r}$, or $1 = x^{2r}(1-x)^{2(t-r)}$ in terms of x. Since $r \neq 0$ and $r \neq 2t$ this new dependency relation between x and $1-x$ is independent of the original relation $1 = x^r(1-x)^{-t}$. We conclude that x is a root of unity and so $h(x) = 0$. □

References

[Bil97] Y. Bilu, *Limit distribution of small points on algebraic tori*, Duke Math. J. **89** (1997), 465–476.

[BMZ99] E. Bombieri, D. Masser, and U. Zannier, *Intersecting a curve with algebraic subgroups of multiplicative groups*, Int. Math. Research Notes **20** (1999), 1119–1140.

[CZ00] P. B. Cohen and U. Zannier, *Multiplicative dependence and bounded height, an example*, Proceedings of the Algebraic Number Theory and Diophantine Approximation Conference, Graz, 1998 (Walter de Gruyter, 2000), 93–101.

[Hab05] P. Habegger, *The equation $x + y = \alpha$ in multiplicatively dependent unknowns*, Acta Arith. **119** (2005), 349–372.

[Lan83] S. Lang, *Fundamentals of Diophantine Geometry*, Springer, 1983.

[Mas05] D.W. Masser, *Counting points with multiplicatively dependent coordinates on a curve*, This volume. (23rd October 2005).

[Mig89] M. Mignotte, *Sur un théorème de M. Langevin*, Acta Arith. **54** (1989), 81–86.

[Smy71] C.J. Smyth, *On the product of the conjugates outside the unit circle of an algebraic integer*, Bull. London Math. Soc. **3** (1971), 169–175.

[Smy81] ———, *On measures of polynomials in several variables*, Bull. Austral. Math. Soc. **23** (1981), 49–63.

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