CGI 2010 , Singapore

"Shape and Image Cognition, Construction and Compression via Tools from Differential Geometry"

F.- E. Wolter

Welfen Laboratory

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Institute of Man-Machine-Communication
Leibniz University of Hannover

Three Fundamental Problems in Geometry and Image Processing

Shape and image…

- Cognition
- Construction
- Compression

Overview

- Laplace-Beltrami spectra for shape and image recognition
- Euclidean Medial Axis (Inverse) Transform for Shape Modeling
- Generalization of Medial Axis and Voronoi Diagrams to Curved Surfaces and Riemannian Spaces
- Application of Medial Modeling to Shape Optimization

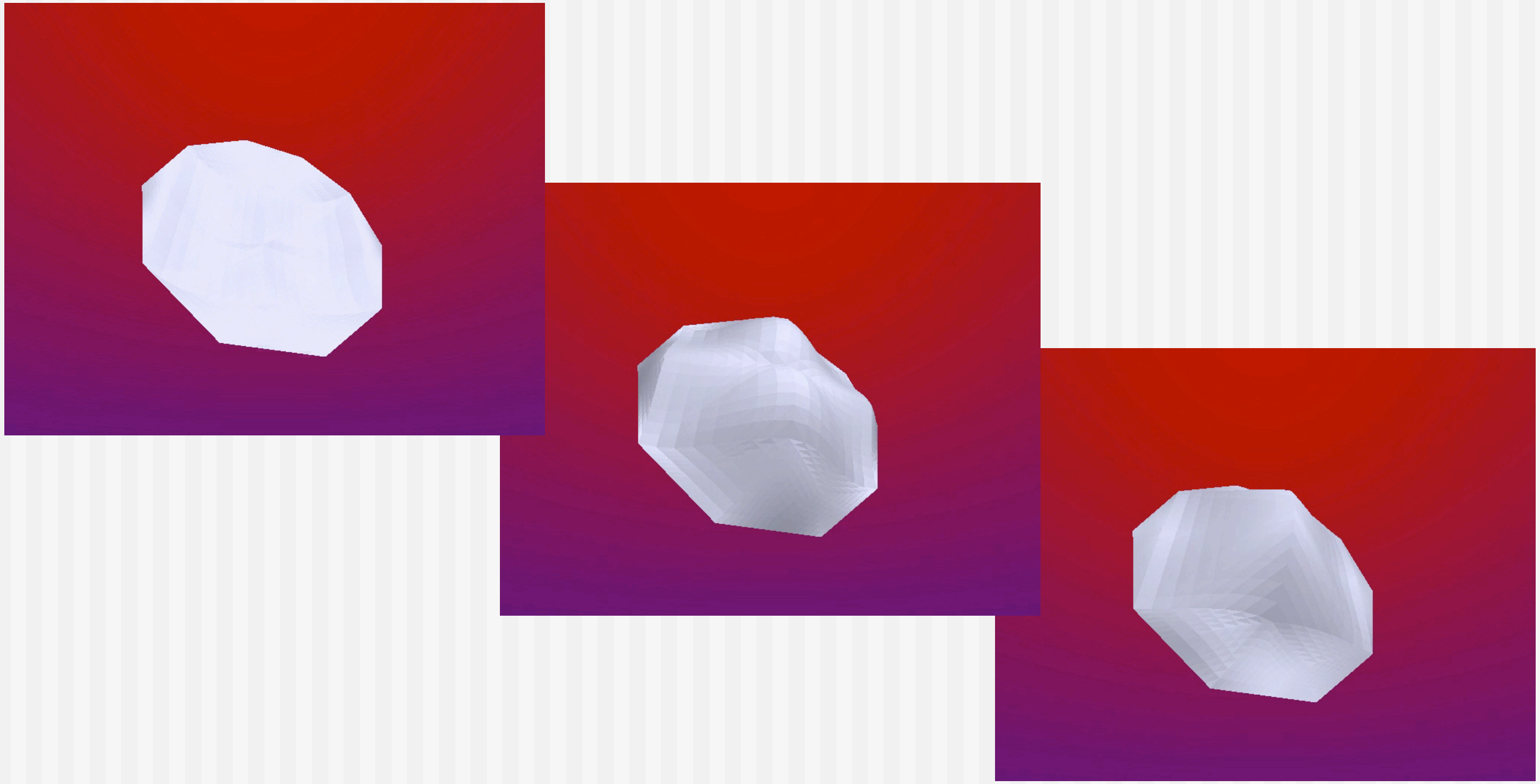
Global Shape Characteristica of Solids

- Volume, area, length, angles, …
- Principal Component Analysis (PCA)
- **Spectrum of Laplace-Beltrami operator**
- **Medial axis**
- **Geodesic structure**
- Curvature lines and umbilics
- …

The Laplace and Laplace-Beltrami Operator

Capturing shape via analysis of acoustical properties

"Can we hear the shape of a drum?"

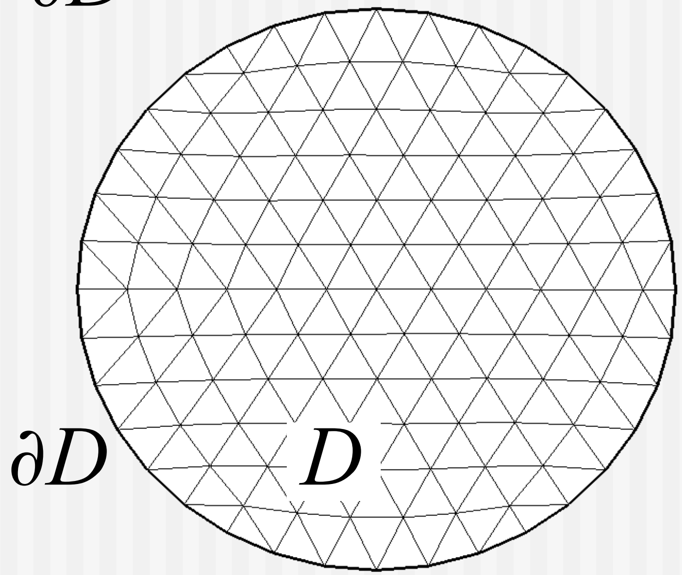


Laplace spectrum

Find $f : D \subset R^2 \rightarrow R$ and λ with

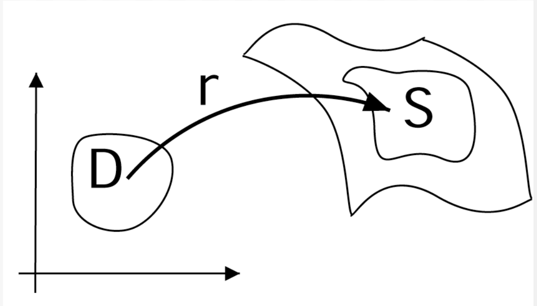
$$\Delta f = \lambda f \, , \, f_{\partial D} = 0$$

where $\Delta f = \frac{\partial^2 f}{\partial^2 u} + \frac{\partial^2 f}{\partial^2 v}$



Laplace-Beltrami operator

$$r : D \subset R^2 \rightarrow S \subset R^3$$



Metric tensor:

$$(g_{ij}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} \qquad g_{ij} = \left\langle \frac{\partial r}{\partial u^i}, \frac{\partial r}{\partial u^j} \right\rangle$$

$$(g^{ij}) := (g_{ij})^{-1}$$

Laplace-Beltrami operator

Define Christoffel symbols Γ_{ij}^{k} via:

$$\frac{\partial^2 r}{\partial u^i \partial u^j} = \sum_{i,j} \Gamma_{ij}^{k} \frac{\partial r}{\partial u^k} + \mathrm{L}_{ij} N$$

In general:

$$\Gamma_{ij}^{k} = \frac{1}{2} \sum_{m} g^{km} \left(\frac{\partial g_{kj}}{\partial u^i} + \frac{\partial g_{ik}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} \right)$$

Laplace-Beltrami operator

Differential operator mapping

$$f : R^2 \to R$$

to

$$\Delta f := \sum_{i,j=1}^{2} \left(\frac{\partial}{\partial u^j} [g^{ij} f_i] + \sum_{k=1}^{2} \Gamma_{jk}^{k} g^{ij} f_i \right)$$

extendable to higher dimensions (hearing the shape of a volume)

Laplace-spectra as fingerprints for shape matching, M. Reuter, F.-E. Wolter, N. Peinecke, SPM'05: Proceedings of the 2005 ACM symposium on solid and physical modeling

Spectral information

- Eigenvalues determine:
  - Volume
  - Volume of boundary
  - Curvature integrals
  - Euler characteristic
  - …
- Spectrum can be used as „Shape-DNA“ for
  - Surfaces
  - Solids
  - Images
  - …

Wolter, F.-E.; Peinecke, N.; Reuter, M., "A Method for the Characterization of Objects (Surfaces, Solids and Images)", German Patent Application, June 2005, US Patent US2009/0169050 A1, July 2, 2009, 2006

Computing of spectra via FEM–methods

Variational formulation

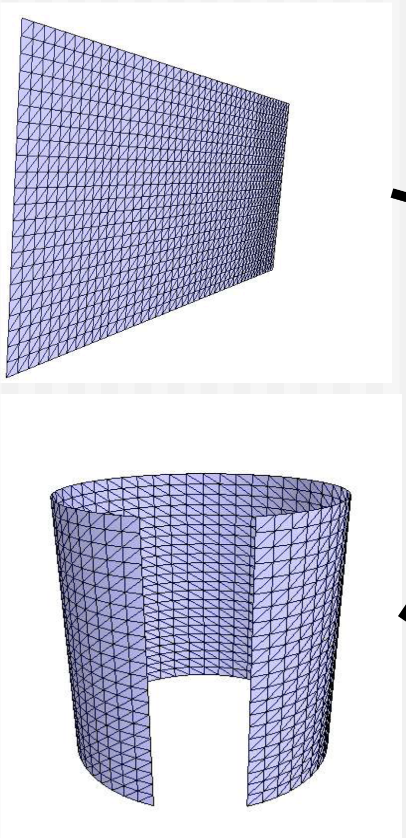
$$\int_S \langle \nabla f, \nabla \varphi \rangle dA = \lambda \int_S f\varphi \, dA \qquad \forall \varphi$$

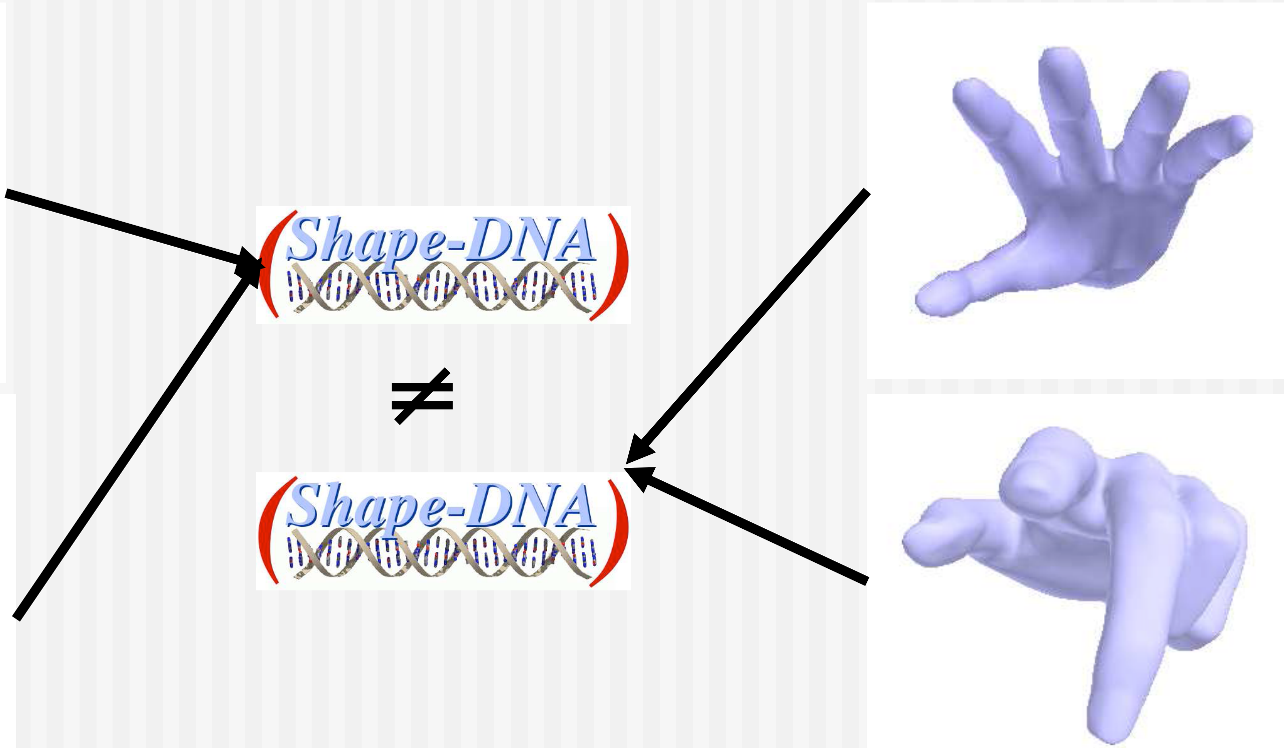
Galerkin Discretization $\mathrm{f} = \sum_{\mathrm{i}} \mathrm{f}_{\mathrm{i}} \psi_i$:

$$\int_S \langle \nabla \psi_i, \nabla \psi_j \rangle f_i \, dA = \lambda \int_S \psi_i \, \psi_j \, dA \quad \forall j$$

Using Eigenvalues for Shape Matching

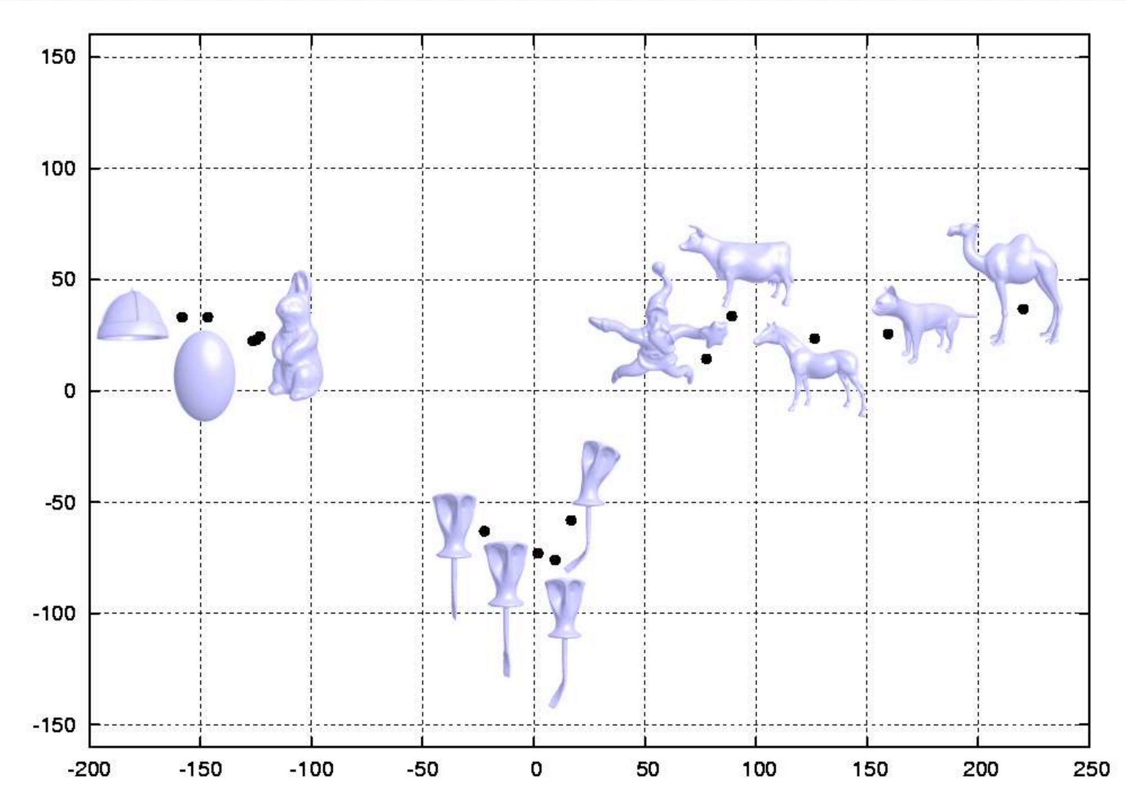
- **Theorem**: Isometric shapes => Same spectra





Reuter, M.; Wolter, F.-E.; Peinecke, N., "Laplace-Beltrami Spectra as Shape DNA of Surfaces and Solids", Computer-Aided Design 4 (2006), no. 38, 342-366 (Most Cited Paper Award in 2009)

Clusterings via Shape DNA for Surfaces



Niethammer, M.; Reuter, M, Wolter, F.- E., e.a. , “Global medical shape analysis using the Laplace-Beltrami spectrum”, MICCAI (1) 2007: 850-857

M. Reuter, M. Niethammer, F. –E. Wolter, e.a., “Global Medical Shape Analysis Using the Volumetric Laplace Spectrum“, Lect Notes Comput Sci. 2007 Oct 24;2007:417-426. http://doi.ieeecomputersociety.org/10.1109/CW.2007.42; http://www.ncbi.nlm.nih.gov/pubmed/20046537

M. Reuter, F.-E. Wolter, e.a. „Laplace-Beltrami eigenvalues and topological features of eigenfunctions for statistical shape analysis". Computer-Aided Design 41(10): 739-755 (2009)

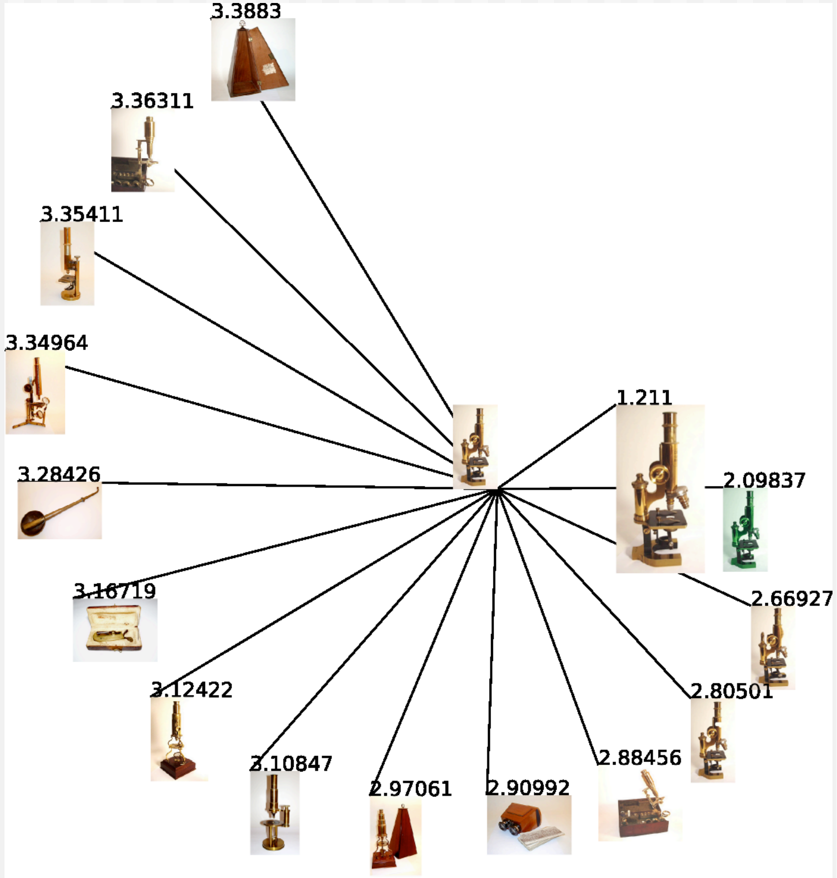
Interpret images as manifolds

- Grayscale image v as height function

$$\left(u^1, u^2\right) \longrightarrow \begin{bmatrix} u^1 \\ u^2 \\ v(u^1, u^2) \end{bmatrix}$$

- RGB Color image as a surface in R^5

Clusterings via Spectral DNA for Images



- Peinecke, N.; Wolter, F.-E.; Reuter, M., "Laplace-Spectra as Fingerprints for Image Recognition", Computer-Aided Design 6 (2007), no. 39, 460-476
- Niklas Peinecke, Franz-Erich Wolter, "Mass Density Laplace-Spectra for Image Recognition", IEEE, Proceedings, cw, pp. 409-416, 2007 International Conference on Cyberworlds (CW'07), 2007

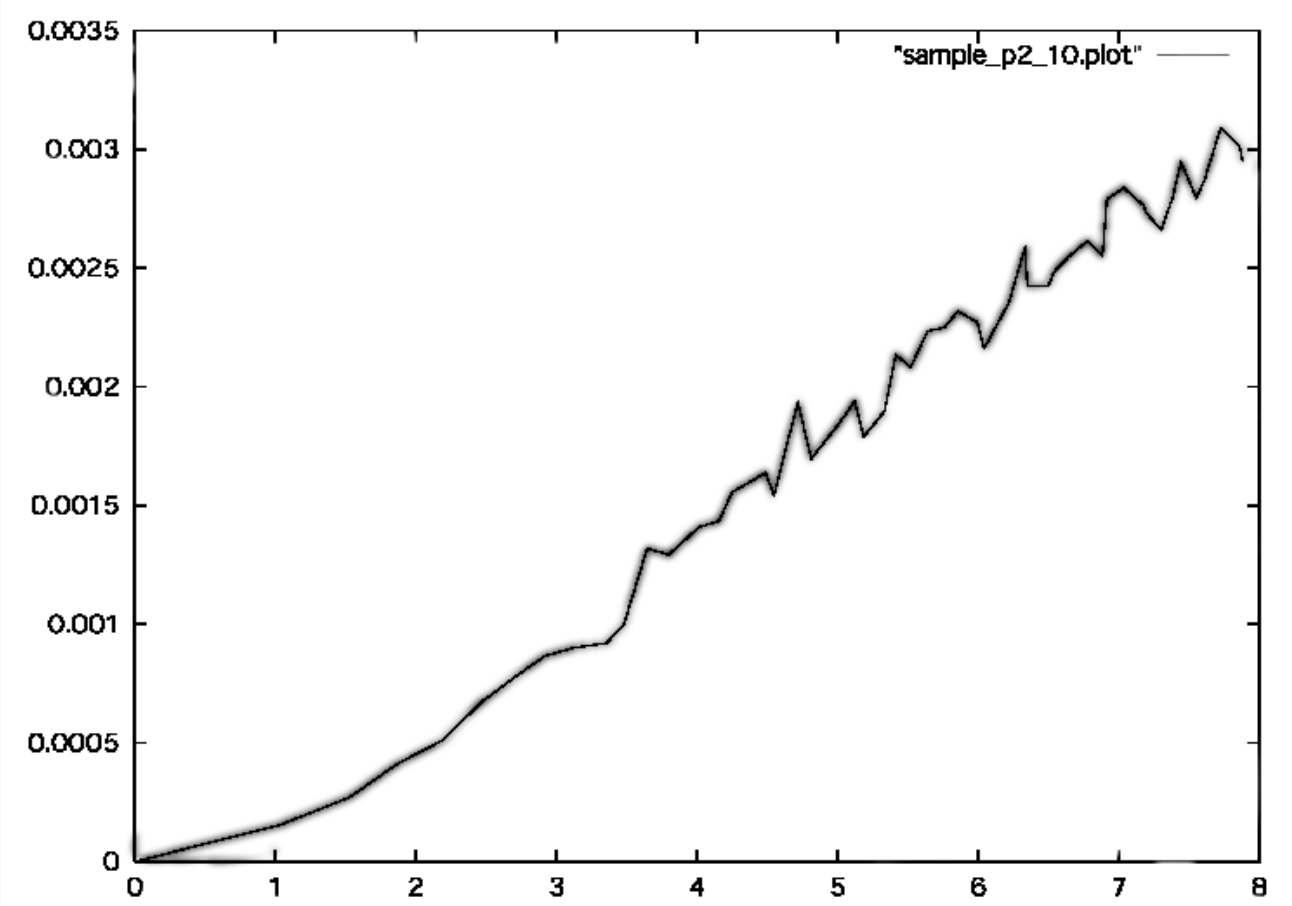
Influence of noise

original

10%

20%

40%



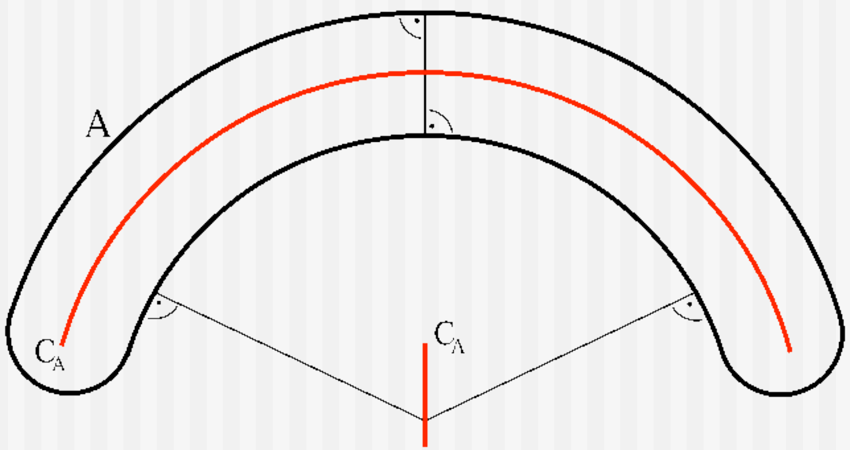
Reconstruction, Modification Design and Compression of Shape

Medial Axis Transform as Basis for Shape Interrogation, Compression and Intuitive Design

Definition: Cut Locus

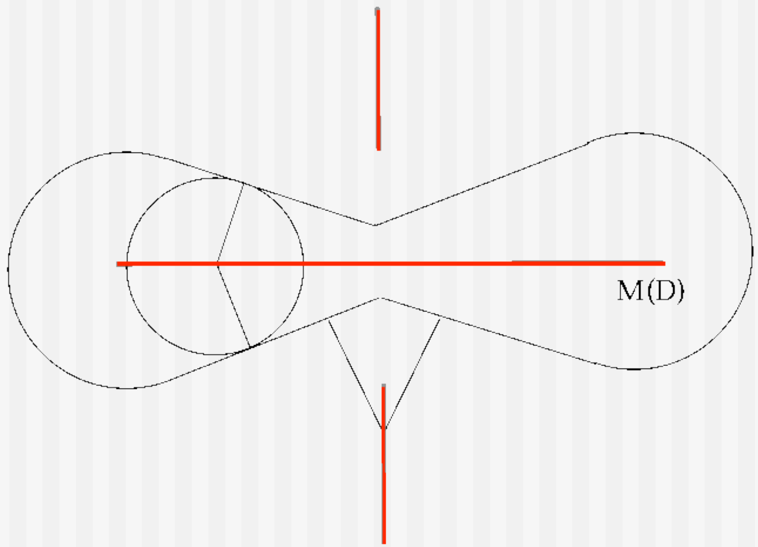
Cut Locus C_A of a closed set $A \subset R^n$:

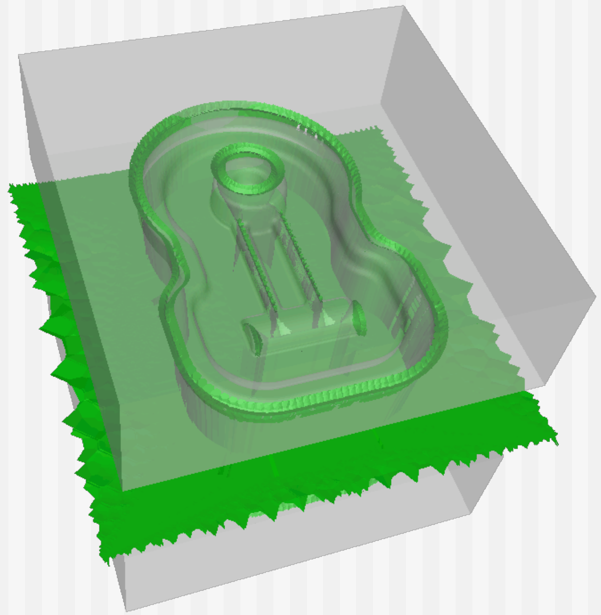
$$C_A = \{q \in R^n \mid q \text{ has at least two shortest paths to } A\}$$



Definition: Medial Axis

$$M(D) = \{q \in R^n \mid q \text{ center of any maximal disc in } D\}$$

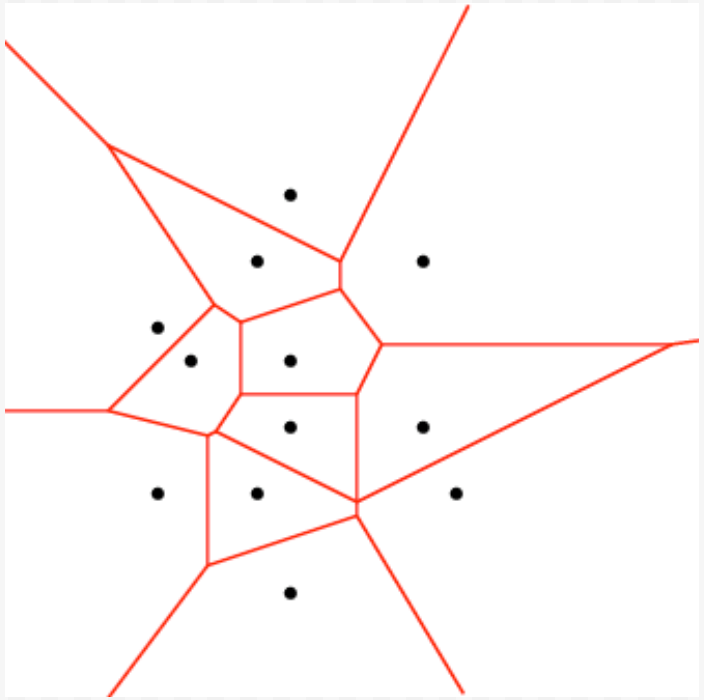


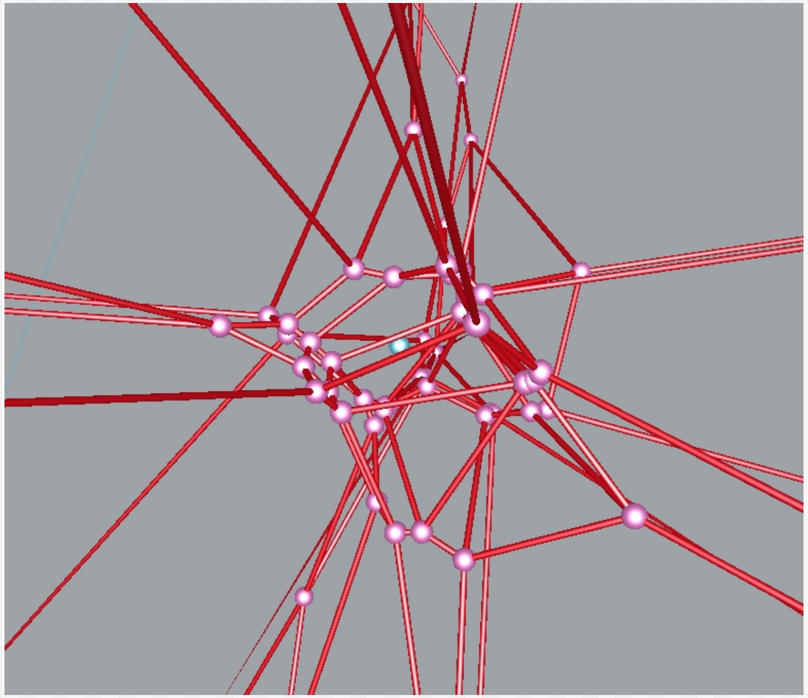


$$M(D) = C_{\partial D} \cap D$$

Definition: Voronoi Diagram

Voronoi Diagram = Cut Locus of point set





Medial Axis Transform as a Tool to Capture and Model 2D and 3D objects

The Medial Axis Transform (MAT)

Compute for a shape D:

- **medial axis M(D)**
  centers of all maximal balls that fit inside the body
- **the *radius function*** $r: M(D) \rightarrow R$
  radius of the associated maximal balls

MA contains shape topology

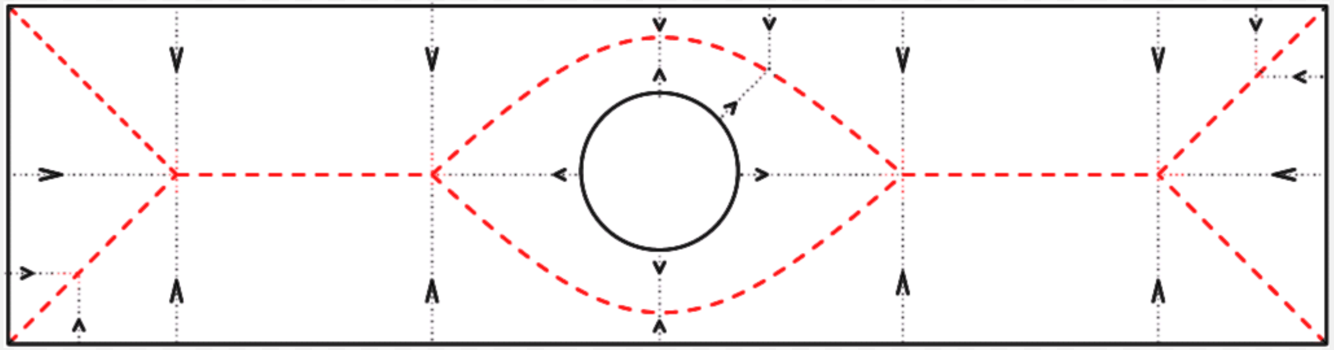
D has a benign boundary

$\Rightarrow M(D)$ is a deformation retract of D.

$\Rightarrow M(D)$ has the **homotopy type** of D.

•Wolter, F.-E., "Cut Locus & Medial Axis in Global Shape Interrogation & Representation”, MIT Design Laboratory Memorandum 92-2 and MIT Sea Grant Report, 1992. (US National Sea Grant Library)

Deformation Retract



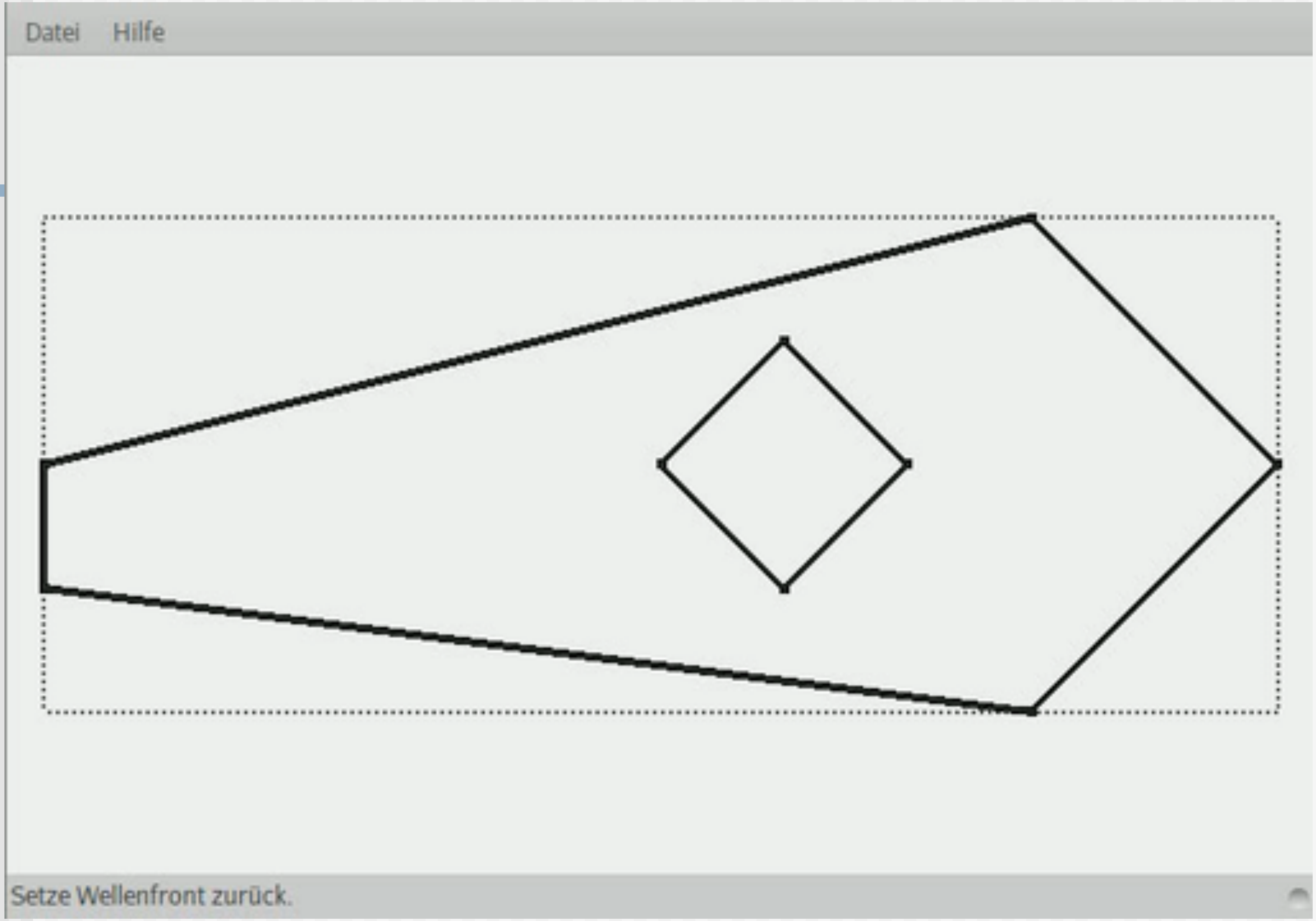
Homotopy $H(x,t):(D\setminus\partial D)\times[0,1]\rightarrow(D\setminus\partial D)$

$$H(x,0) = x\forall x\in D\setminus\partial D$$

$$H(x,t) = x\forall x\in M(D)\setminus\partial D$$

$$H(x,1) = R(x) \text{ with } R: D\setminus\partial D\rightarrow M(D)\setminus\partial D$$

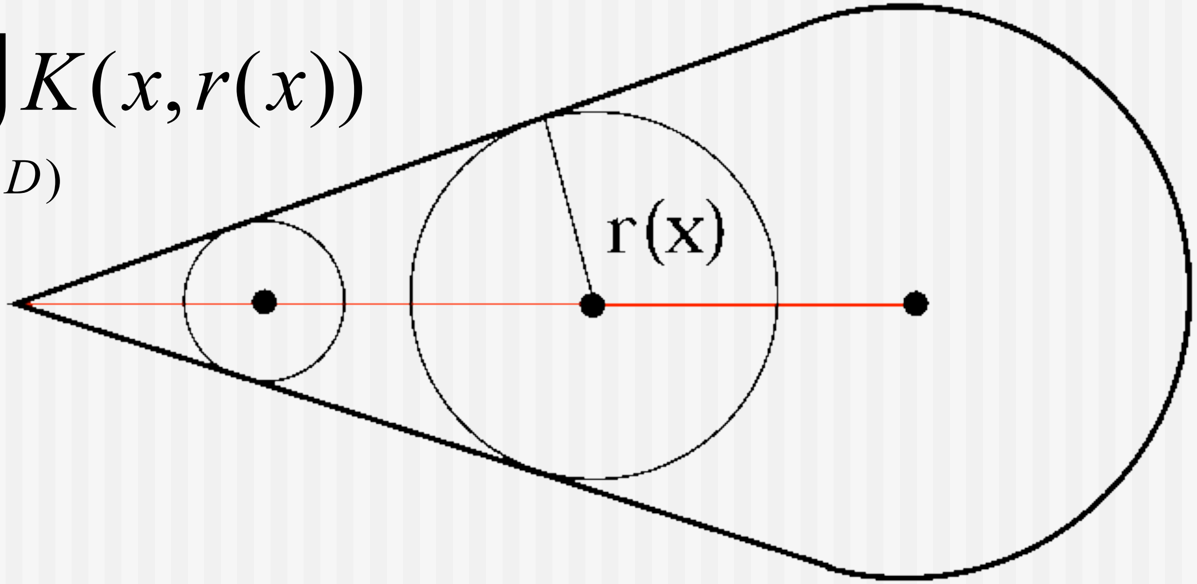
•Wolter, F.-E., "Cut Locus & Medial Axis in Global Shape Interrogation & Representation", MIT Design Laboratory Memorandum 92-2 and MIT Sea Grant Report, 1992.


Datei
Hilfe
Setze Wellenfront zurück.

Inverse MAT: Reconstruction Theorem

Reconstruct D from MAT :

$$D = \bigcup_{x \in M(D)} K(x, r(x))$$

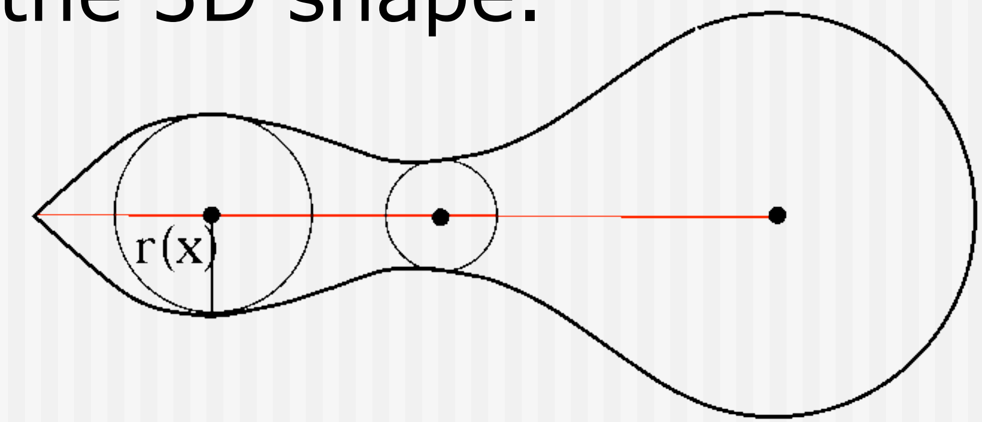


- Wolter, F.-E., "Cut Locus & Medial Axis in Global Shape Interrogation & Representation", MIT Design Laboratory Memorandum 92-2 and MIT Sea Grant Report, 1992.
- Wolter, F.-E.; Friese, K.-I., "Local & Global Geometric Methods for Analysis Interrogation, Reconstruction, Modification & Design of Shape", Proceedings of Computer Graphics International 2000, pp. 137-151

Inverse MAT: Shape design

Modify the value of the radius function *r(x)*:

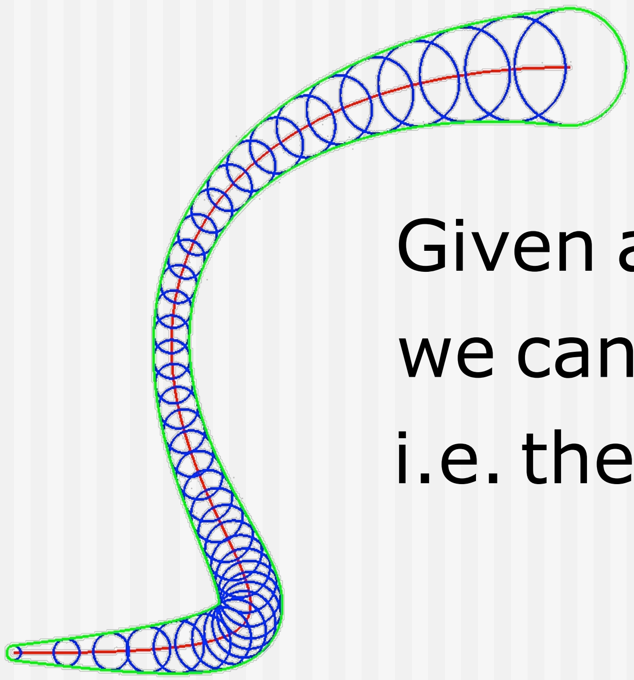
- natural global **thickening**
- and **thinning** of the 3D shape.



•Wolter, F.-E.; Reuter, M.; Peinecke, N., "Geometric Modeling for Engineering Applications", in Encyclopedia of Computational Mechanics. E. Stein, R. de Borst & T.J.R. Hughes (eds.), John Wiley & Sons, 2007.

•Wolter, F.-E.; Friese, K.-I., "Local & Global Geometric Methods for Analysis Interrogation, Reconstruction, Modification & Design of Shape", Proceedings of Computer Graphics International 2000, pp. 137-151

Inverse MAT: Computation

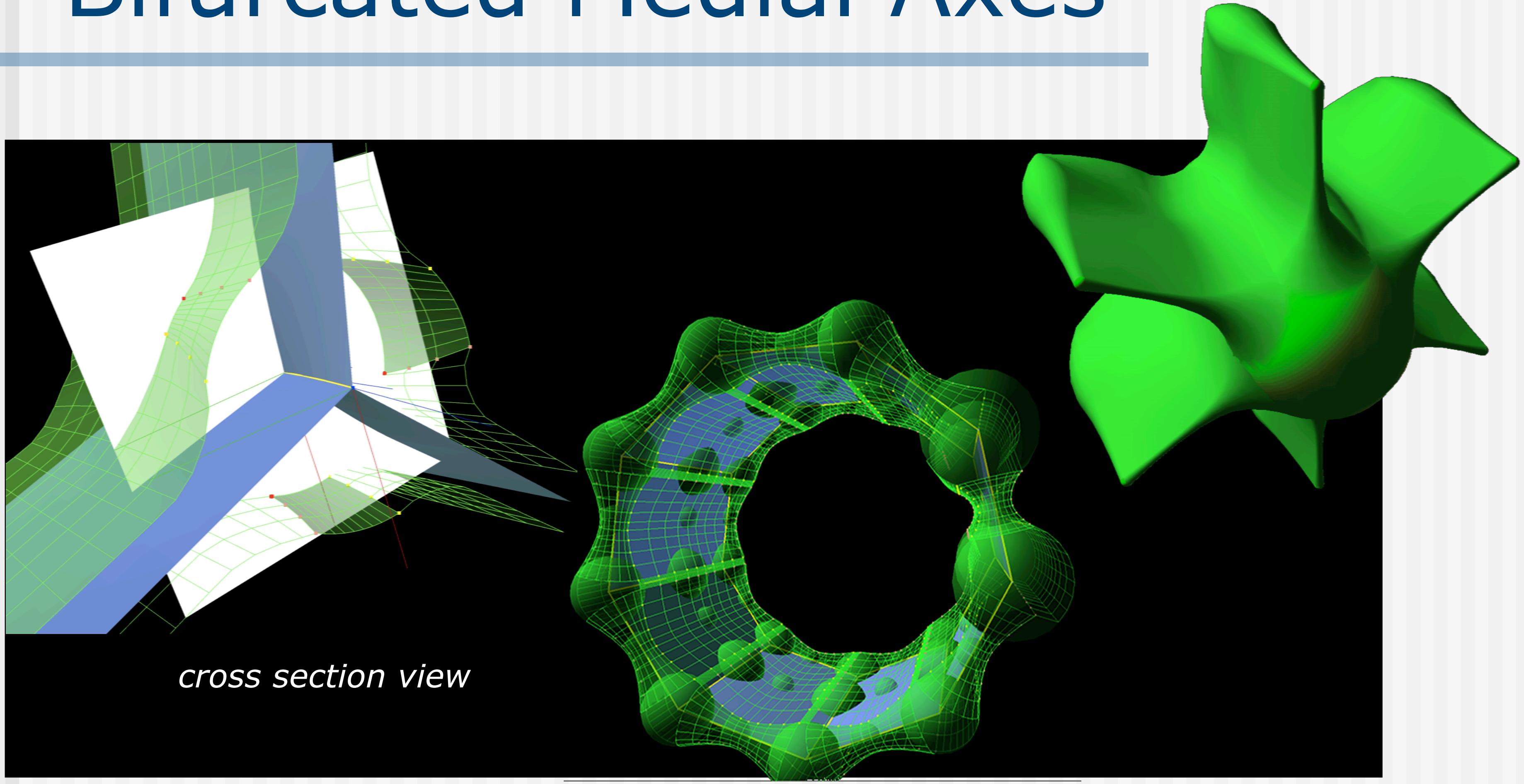


Given a parametrization of the MAT, we can rapidly compute the boundary, i.e. the envelopping curve/surface ∂D.

MA and Envelopping curve

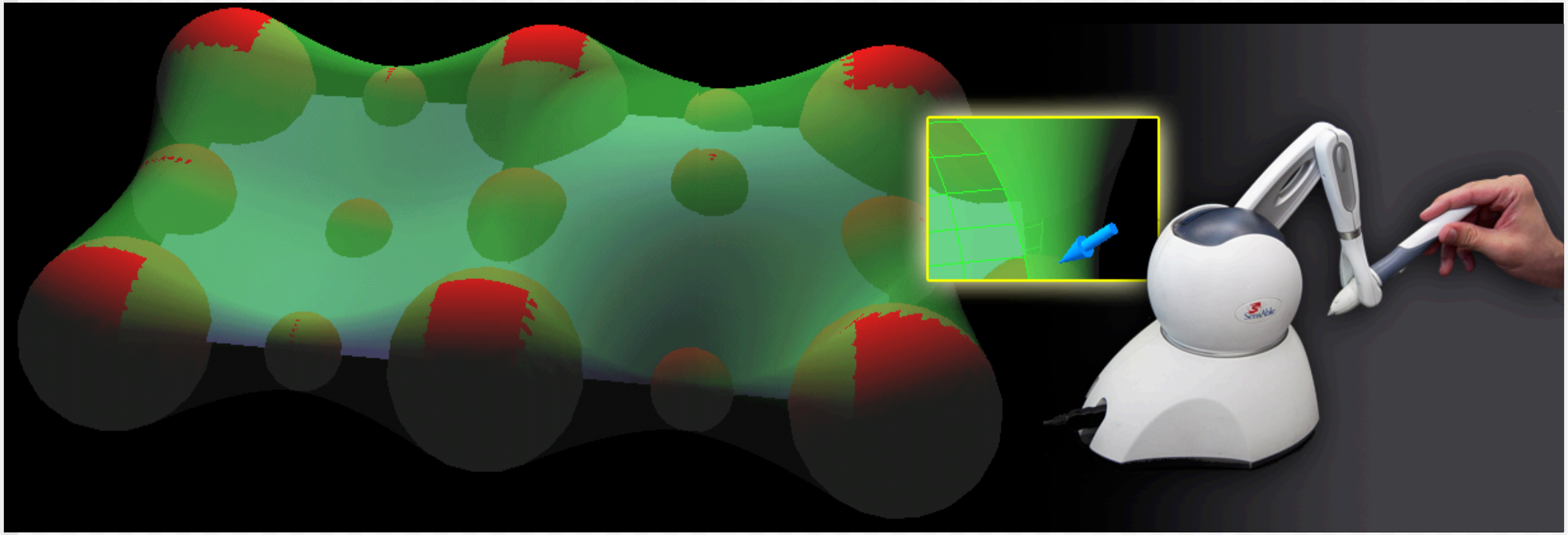
•Wolter, F.-E.; Friese, K.-I., "Local & Global Geometric Methods for Analysis Interrogation, Reconstruction, Modification & Design of Shape", Proceedings of Computer Graphics International 2000, pp. 137-151

Modeling Complex Shapes by Bifurcated Medial Axes



•Wolter, F.-E.; Reuter, M.; Peinecke, N., "Geometric Modeling for Engineering Applications", in Encyclopedia of Computational Mechanics. E. Stein, R. de Borst & T.J.R. Hughes (eds.), John Wiley & Sons, 2007.

Haptic Deformation of Medial Surfaces



Generalization of Medial Axis to Curved Surfaces and Riemannian Spaces

Geodesic Medial Axis and
Geodesic Voronoi Diagrams

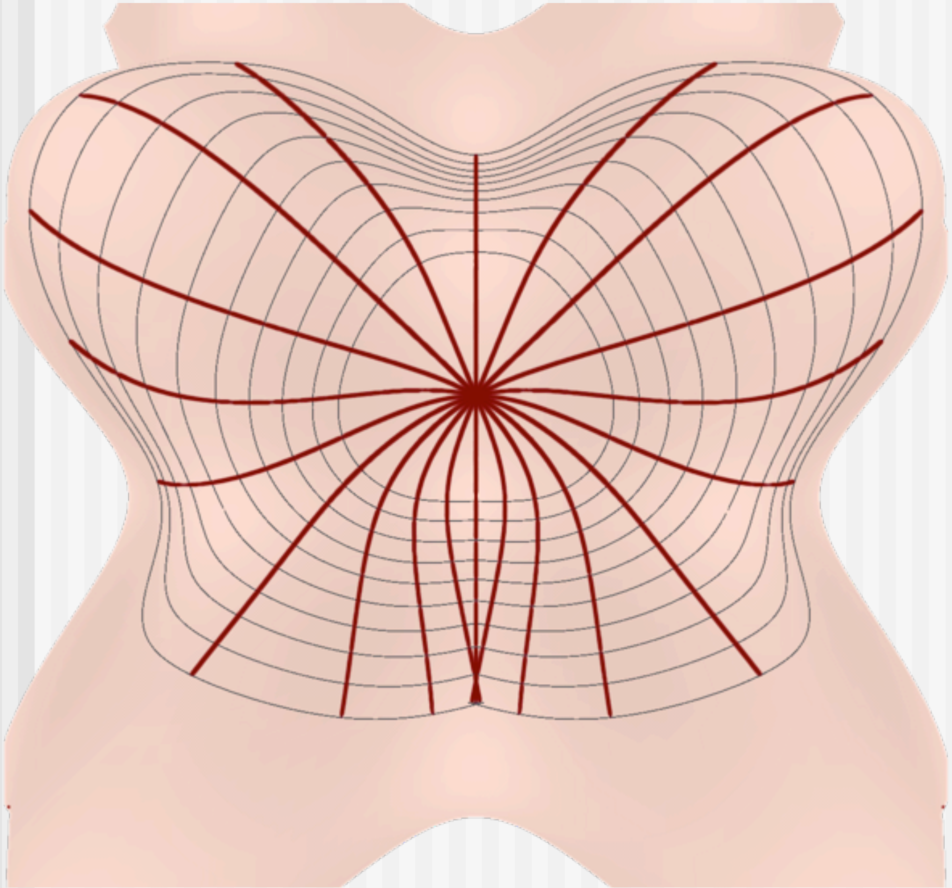
Tools for computational geometry in curved spaces

- Geodesic differential equations
- Geodesic polar coordinates
- Variation of geodesic diff. equation
- Geodesic medial differential equation
- Geometrically motivated homotopy methods for computing geodesic connections

- Rausch, T.; Wolter, F.-E.; Sniehotta, O. , "Computation of Medial Curves in Surfaces", Conference on the Mathematics of Surfaces VII, Institute of Mathematics and its Applications, IMA Conference Series, pp. 43-68, 1997

- Naß, H.; Wolter, F.-E.; Dogan, Cem; Thielhelm, H., "Computation of Geodesic Voronoi Diagrams in 3-Space using Medial Equations", in IEEE, Computer Society Press, Conference Proceedings of NASAGEM, ISBN 0-7695-3005-2, pp. 376-385, 2007

Geodesic polar coordinates

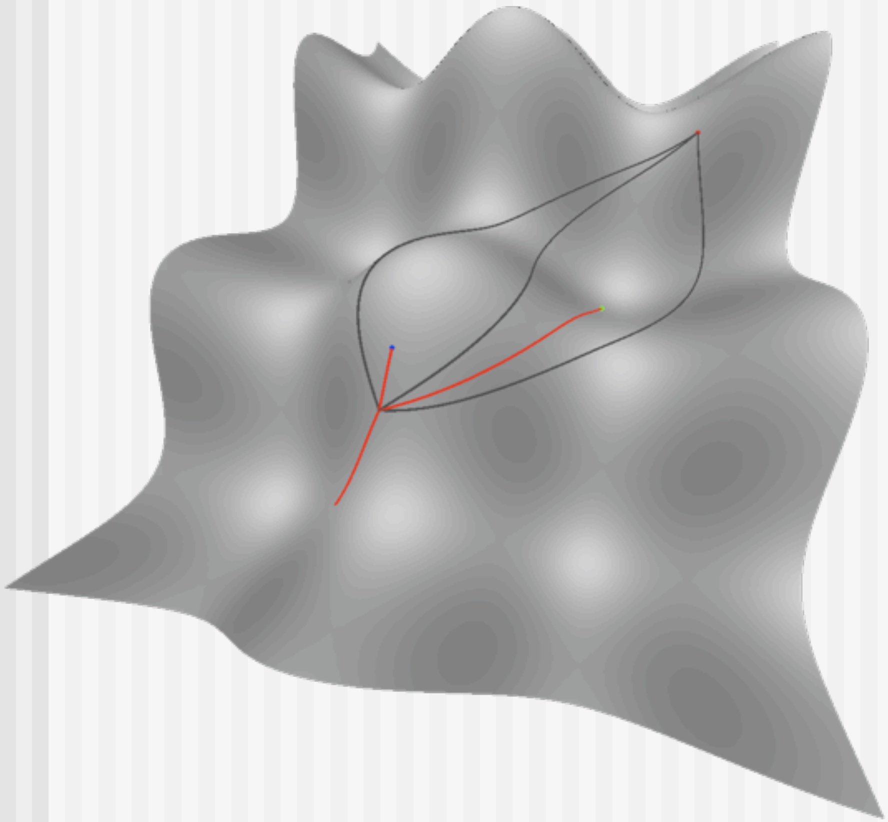


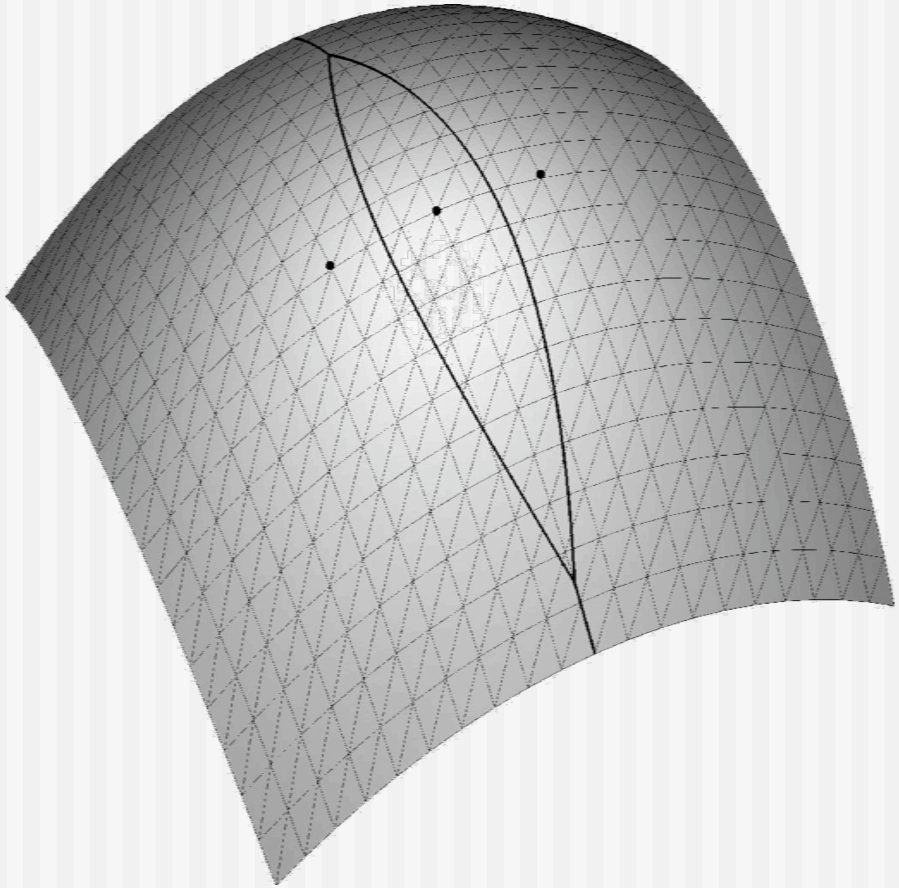
Difficulty:

Geodesic connections are not necessarily unique

⇒new topological structures

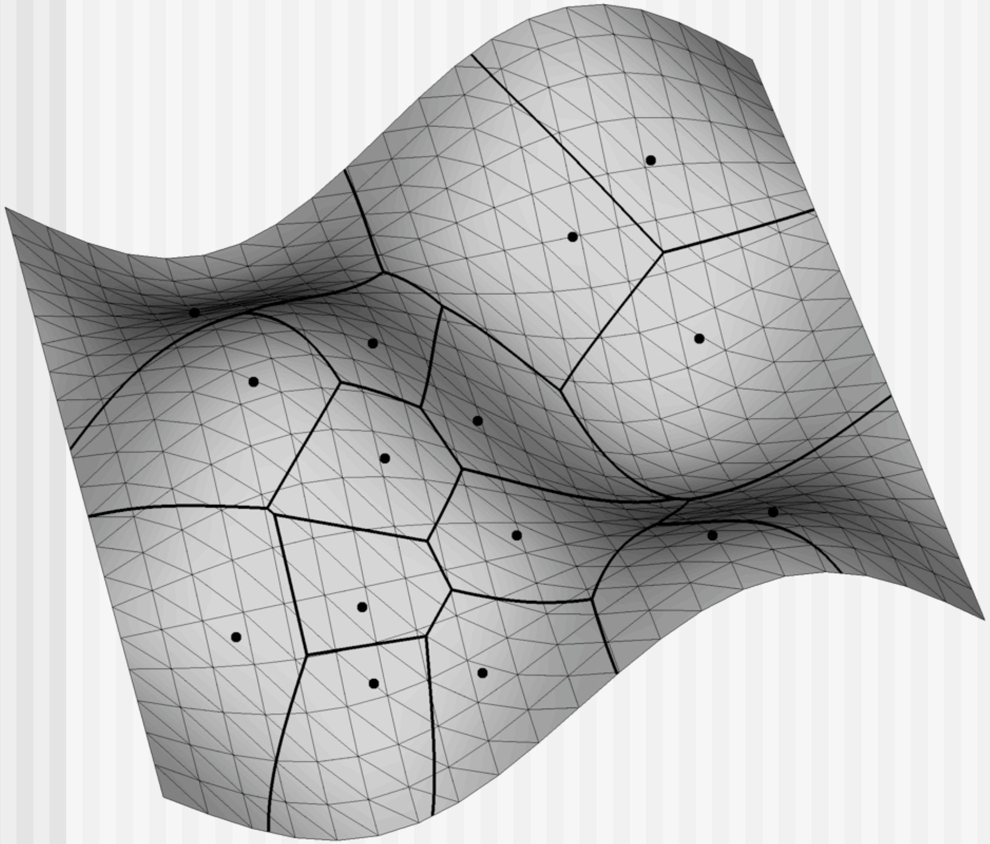
Topological difficulties

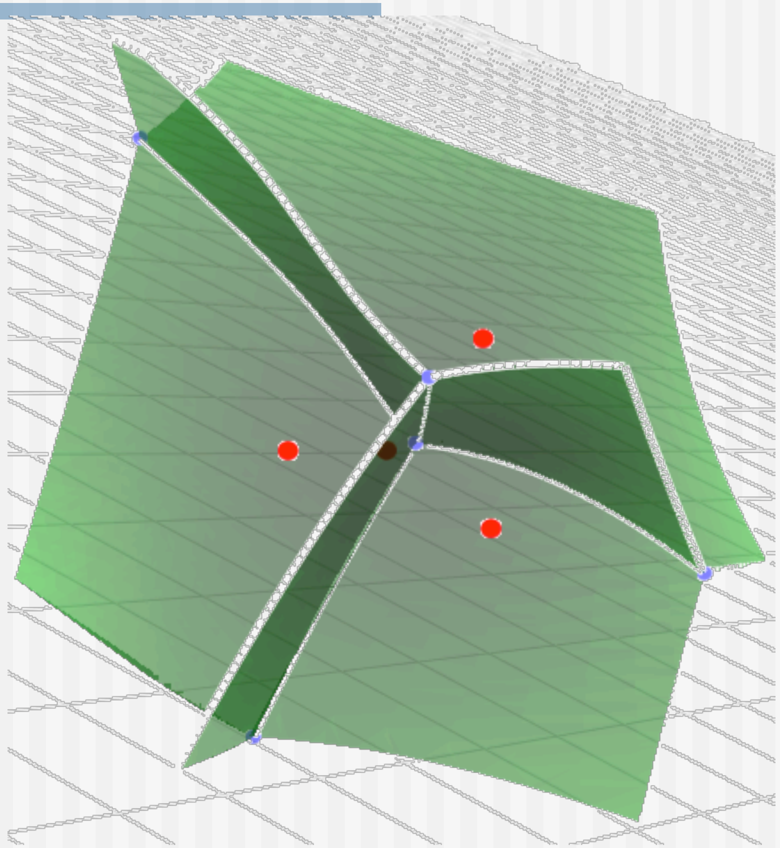




•Kunze, R.; Wolter, F.-E.; Rausch T. , "Geodesic Voronoi Diagrams on Parametric Surfaces", CGI '97, IEEE, Computer Society Press Conference Proceedings, pp. 230-237, June 1997.

Geodesic Voronoi Diagrams

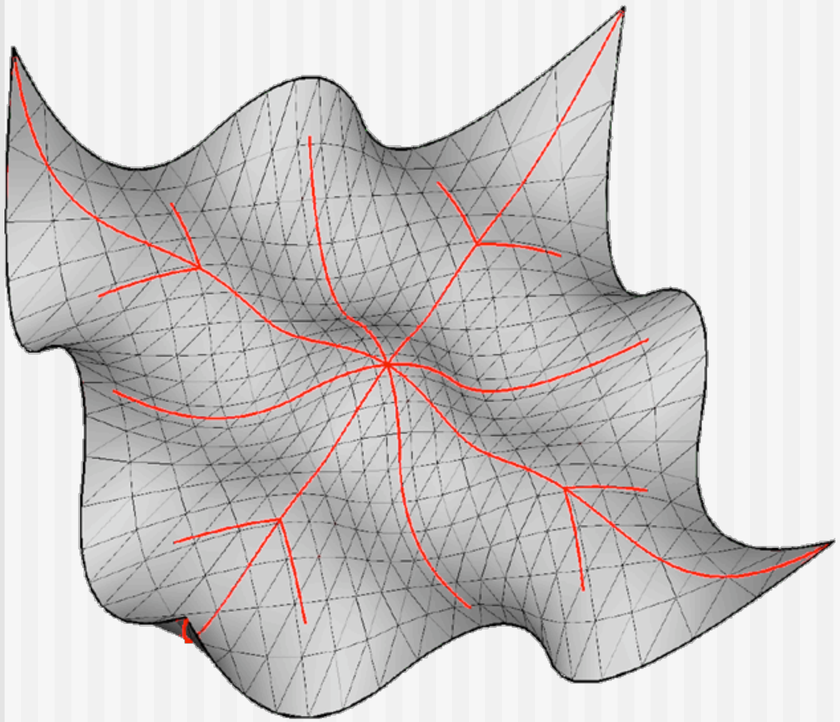


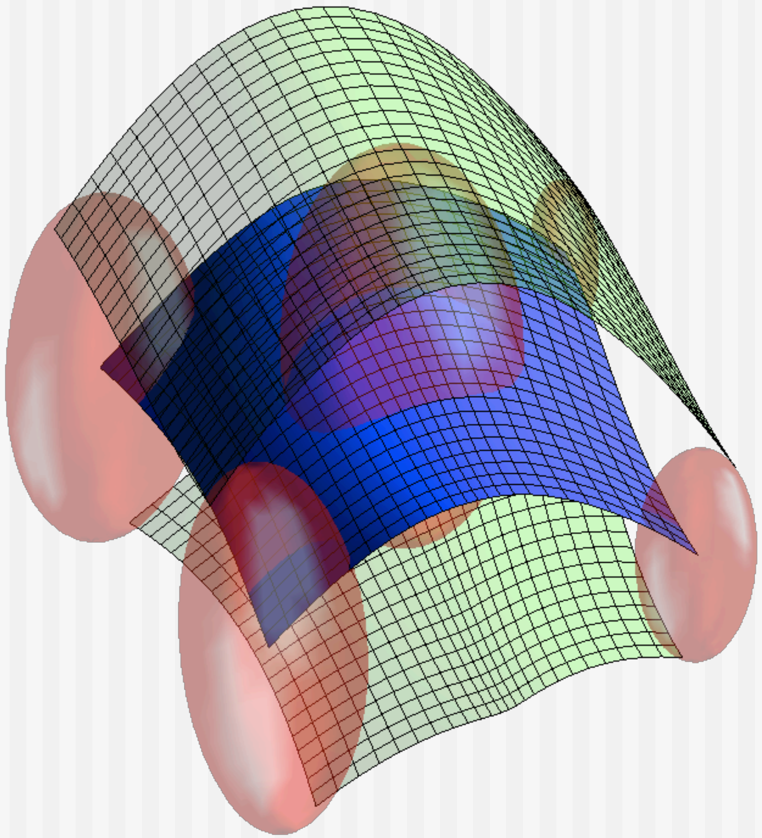


- Kunze, R.; Wolter, F.-E.; Rausch T. , "Geodesic Voronoi Diagrams on Parametric Surfaces", CGI '97, IEEE, Computer Society Press Conference Proceedings, pp. 230-237, June 1997.

- Naß, Henning; Wolter, Franz-Erich; Dogan, Cem; Thielhelm, Hannes, "Computation of Geodesic Voronoi Diagrams in 3-Space using Medial Equations", IEEE, Computer Society Press Conference Proceedings of NASAGEM, pp. 386-385, 2007

Geodesic Medial Axis

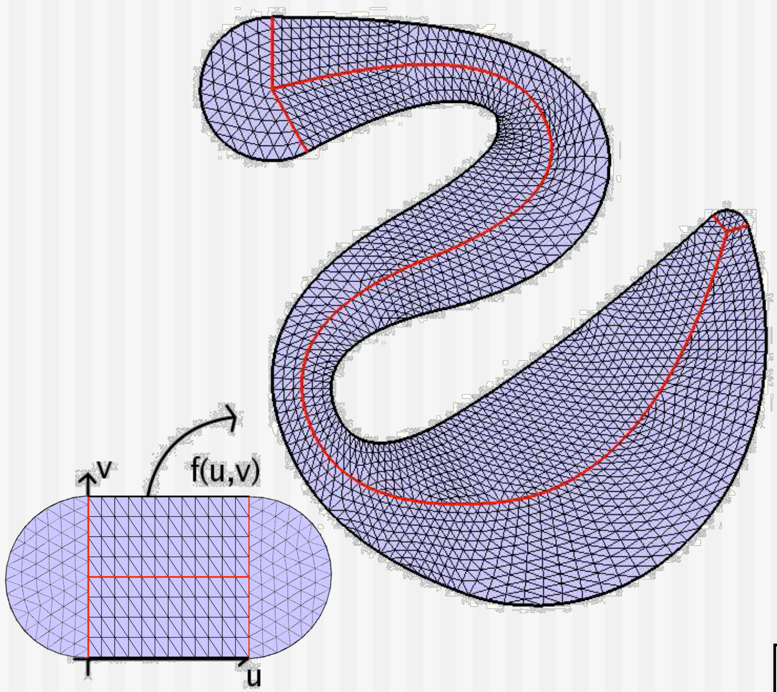




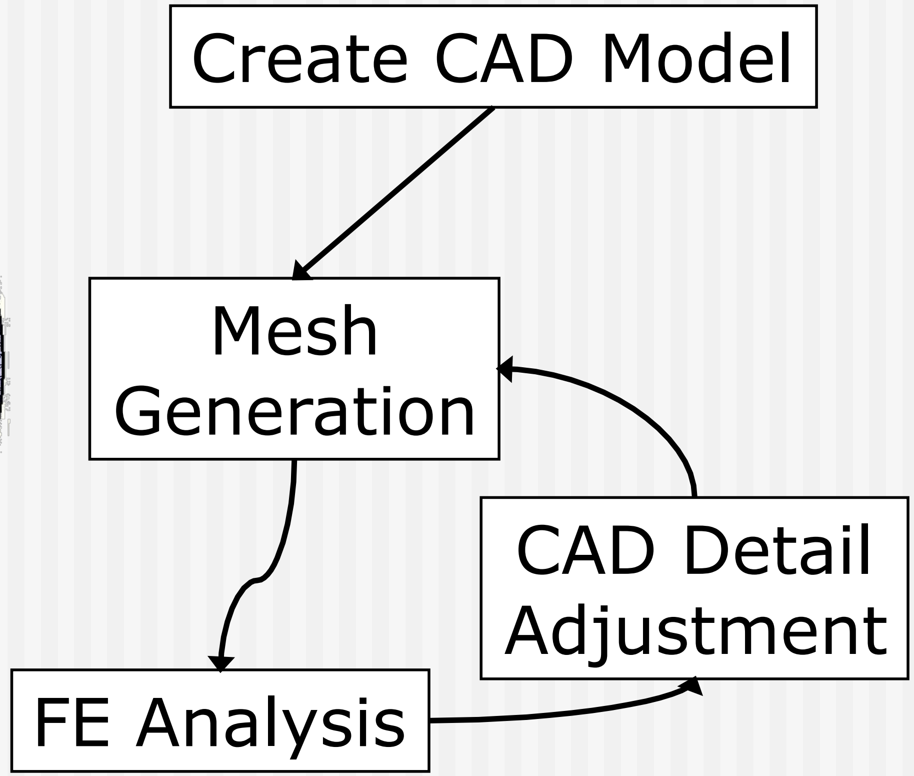
• Wolter, F.-E.; Friese, K.-I., "Local & Global Geometric Methods for Analysis Interrogation, Reconstruction, Modification & Design of Shape", Proceedings of Computer Graphics International 2000, pp. 137-151, Geneva, Switzerland, IEEE Computer Society

• Naß, H.; Wolter, F.-E.; Dogan, C.; Thielhelm, H. , "Medial Axis Inverse Transform in 3-Dimensional Riemannian Complete Manifolds", in IEEE Proceedings of NASAGEM, pp. 386-395, 2007

Application:
FE shape optimization



2D IMAT triangulation



- Wolter, F.-E.; Reuter, M.; Peinecke, N., "Formoptimierung und effiziente FEM-Berechnung mit Hilfe der Medialen Achse / Shape Optimization and Efficient FEM Computation Employing the Medial Axis", International Patent Application, pending, 2006.
- Wolter, F.-E.; Reuter, M.; Peinecke, N., "Geometric Modeling for Engineering Applications", in Encyclopedia of Computational Mechanics. E. Stein, R. de Borst & T.J.R. Hughes (eds.), John Wiley & Sons, 2007.

Conclusions

- Laplace-Beltrami Spectra are useful for shape and image classification
- MAT and IMAT are useful for shape modeling and compression, (also in Riemannian worlds)
- Parametrization induced by MAT can be used in numerical shape optimization problems (also in Riemannian worlds)

All papers referenced in the slides of this lecture can be downloaded at:

http://www.welfenlab.de/en/research/publikationen/

http://www.welfenlab.de/en/research/publikationen/welfenlab_reports

Thank you for your attention!