Braids, Galois Groups, and Some Arithmetic Functions

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This lecture is about some new relations among the classical objects of the title. The study of such relations was started by [B_1, G, De, Ih_1] from independent motivations, and was developed in [A_3, C_3, A-I, IKY, Dr_2, O, N], etc. It is still a very young subject, and there are several different approaches, each partly blocked by its own fundamental conjectures! But it is already allowing one to glimpse some new features of the classical "monster" $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, and providing a bridge connecting $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ even with such "modern" objects as the quantum groups [Dr_2]. I will not try to "explain" any general philosophies that are still in the air, but to draw a few lines sketching the concretely visible features of the subject.

§1. Introduction

The absolute Galois group over the rational number field $\mathbb{Q}$, denoted by $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, is one of the classical mathematical objects which we really need to understand better. It is primarily a (huge) topological group; the automorphism group of the field $\bar{\mathbb{Q}}$ of all algebraic numbers in $\mathbb{C}$, equipped with the Krull topology. It is moreover equipped with arithmetic structure, i.e., the system of conjugacy classes of embeddings $G_{\mathbb{Q}_p} \to G_{\mathbb{Q}}$ of all local absolute Galois groups into $G_{\mathbb{Q}}$. Through the arithmetic structure of $G_{\mathbb{Q}}$, each "natural" representation of $G_{\mathbb{Q}}$ provides the set of all prime numbers with a "natural" additional structure. For example, the character $\chi_N : G_{\mathbb{Q}} \to (\mathbb{Z}/N)^{\times}$ defining the action of $G_{\mathbb{Q}}$ on the group of N-th roots of unity gives rise to the classification of prime numbers modulo N. Study of the group $G_{\mathbb{Q}}$ and its natural representations has an *ultimate* goal to understand the "total structure" of the set of all prime numbers.

Now if X is an algebraic variety over $\mathbb{Q}$, the fundamental group $\pi_1(X(\mathbb{C}), b)$, when suitably completed, is equipped with a natural action of $G_{\mathbb{Q}}$. Here, b is a $\mathbb{Q}$-rational base point of X. Already in some basic cases, this action gives rise to a very big and interesting representation of $G_{\mathbb{Q}}$. What this action amounts to is the following. (We may, and will, assume X to be geometrically connected.) As is well known, all finite (topological) coverings of $X(\mathbb{C})$ and morphisms between two such coverings are algebraic and defined over $\bar{\mathbb{Q}}$. The above action contains all information related to the "field of definition" and the $G_{\mathbb{Q}}$-conjugations of

Proceedings of the International Congress
of Mathematicians, Kyoto, Japan, 1990

these coverings, morphisms and the points above b. In particular, if they are all defined over some Galois extension $\Omega/\mathbb{Q}$ $(\Omega \subset \bar{\mathbb{Q}})$, then $G_{\mathbb{Q}}$ acts via its factor group $\mathrm{Gal}(\Omega/\mathbb{Q})$. (The converse is also valid if the completion of π_1 has trivial center.) Choice of completions of π_1 depends on the family of finite coverings one wants to consider. If this is all the finite coverings, the choice should be the profinite completion $\hat{\pi}_1 = \varprojlim(\pi_1/N)$, the projective limit of all finite factor groups of π_1. It is also important to consider the pronilpotent completion π_1^{nil}, where one allows only those π_1/N (the covering groups) that are nilpotent.

If there is no family of finite coverings of X having a remarkable common field of definition, then the representation of $G_{\mathbb{Q}}$ on $\hat{\pi}_1(X(\mathbb{C}), b)$ would not be so interesting. But already when $X = \mathbb{P}^1 - \{0, 1, \infty\}$, there are several natural families with various remarkable arithmetic features in their fields of definition (see (4) below). One of the reasons why this type of representations is interesting lies in that these different arithmetic features can be viewed simultaneously from a certain height, the representation of $G_{\mathbb{Q}}$ on $\hat{\pi}_1$. Other reasons and motivations will also be explained below, with names of the main contributors. Before this we shall give a sequence of basic examples of X starting with $\mathbb{P}^1 - \{0, 1, \infty\}$. For $n \geq 4, X_n$ is the moduli space of ordered n-tuples $(x_1, \dots, x_n)$ of distinct points on the projective line $\mathbb{P}^1$. The corresponding fundamental group $P_n = \pi_1(X_n(\mathbb{C}), b)$ is the quotient modulo center $(\cong \mathbb{Z}/2)$ of the pure sphere braid group on n strings. Note that $X_4 \simeq \mathbb{P}^1 - \{0, 1, \infty\}$, $P_4 \simeq F_2$ (free, rank 2).

Recently, several substantial works have been done in connection with these "big Galois representations." They include the following (1)~(6):

(1) Already for more than a decade, this type of Galois representations has been *used* effectively to construct finite Galois extensions over $\mathbb{Q}$ with given Galois groups, for various cases of finite simple groups (G. V. Belyĭ, M. Fried, B. Matzat, K. Y. Shih, J. G. Thompson, ...). By the Hilbert irreducibility theorem, it suffices to construct a Galois extension over some rational function field $\mathbb{Q}(t_1, \dots, t_m)$ $(m \geq 1)$ having a given Galois group. Many such extensions have been detected among the (function fields of) coverings of X_{m+3} (etc.) by combining (i) only the basic knowledge on how $G_{\mathbb{Q}}$ acts on $\hat{\pi}_1(X(\mathbb{C}), b)$, with (ii) a deep knowledge on specific finite groups and their characters. As there is a distinguished report on this topic in the last ICM [B$_2$] (see also [Ma] for the later development), we shall only recall and stress the following:

Belyĭ proved, among other things in [B$_1$] that the canonical representation

$$\varphi_X : G_{\mathbb{Q}} \to \mathrm{Out}\, \hat{\pi}_1(X(\mathbb{C}))$$

for $X = \mathbb{P}^1 - \{0, 1, \infty\}$ is *injective*. This sets in evidence the importance of the problem to characterize the image. Here, $\mathrm{Out}\, \hat{\pi}_1$ is the outer automorphism group, and φ_X is induced from the $G_{\mathbb{Q}}$-action on $\hat{\pi}_1(X(\mathbb{C}), b)$. φ_X is "independent" of b in the obvious sense.

(2) A. Grothendieck made some basic proposals related to the study of the $G_{\mathbb{Q}}$-action on $\hat{\pi}_1(X(\mathbb{C}), b)$ [G]. One of them is as follows. Take $X = X_4 = \mathbb{P}^1 - \{0, 1, \infty\}$. Then it has the obvious S_3-symmetry but there is no S_3-invariant choice of a

base point b. This already indicates that the use of the fundamental *groupoid* on a suitable S_3-stable set B of base points would be better. (The natural action of S_4 on X_4 factors through its quotient $\simeq S_3$.) He also suggests using all possible combinatorial relations among the $X_n(\mathbb{C})$ $(n = 4, 5, \ldots)$, and that, in a certain sense, the two cases of n with $\dim X_n = 1, 2$ (i.e., $n = 4, 5$) would be basic. In particular, understanding and using full relationship between $X_4(\mathbb{C})$ and $X_5(\mathbb{C})$ would give, presumably, all the crucial non-obvious information on the image of φ_X for $X = X_4$.

We shall describe more about these in the main text; the action of $G_{\mathbb{Q}}$ on the completed fundamental groupoids (§2), Deligne's tangential base points (which will serve as B) (§2.3), Drinfeld's new information on the image of φ_X (§3), and the Lie version of the study of the $G_{\mathbb{Q}}$-action on $\{\pi_1^{\mathrm{nil}}(X_n(\mathbb{C}))\}_n$ (§5).

(3) π_1^{nil} *as a new test case for motivic philosophy.*

In [De], P. Deligne develops a motivic theory of nilpotent quotients of fundamental groups in algebraic geometry. The first main point is that π_1^{nil} is not just a topological group with a $G_{\mathbb{Q}}$-action but is a limit of objects with more structures –: To each $\pi_1 = \pi_1(X(\mathbb{C}), b)$, one can associate, via Malcev, some projective system $\{U^m\pi_1\}_{m\geq 1}$ of linear unipotent algebraic groups $U^m\pi_1$ over $\mathbb{Q}$. One may assume that the group of $\mathbb{Z}$-valued points $(U^m\pi_1)(\mathbb{Z})$ is the quotient modulo torsion of $\pi_1/\pi_1(m+1)$, where $\pi_1(m+1)$ is the $(m+1)$-th member of the lower central series of π_1. The Galois group $G_{\mathbb{Q}}$ acts on the profinite groups $(U^m\pi_1)(\widehat{\mathbb{Z}})$ $(m \geq 1)$, from which the $G_{\mathbb{Q}}$-action on π_1^{nil} can be almost recovered. Under some assumptions on X, the underlying vector space over $\mathbb{Q}$ of the Lie algebra $\mathrm{Lie}(U^m\pi_1)$ has mixed Hodge structure (J. Morgan, D. Sullivan, K. T. Chen, R. Hain, ...). Deligne adds more structures to $\mathrm{Lie}(U^m\pi_1)$ (and accordingly, on $U^m\pi_1$), the mixed motif structure, in terms of various *realizations.*

When $X = \mathbb{P}^1 - \{0, 1, \infty\}$, the motivic $\mathrm{Lie}(U^m\pi_1)$ is a successive extension of Tate's motives "$\mathbb{Q}(k)$" $(1 \leq k \leq m)$. Knowledge on the motivic extension of "$\mathbb{Q}$" by "$\mathbb{Q}(k)$" turns out to be crucial in order to understand a certain portion of the Galois action on $(U^m\pi_1)(\widehat{\mathbb{Z}})$. Deligne constructs a basic extension of "$\mathbb{Q}$" by "$\mathbb{Q}(k)$" ($k \geq 3$, odd), describes a certain portion of the Galois action on the double commutator quotient of π_1^{nil}, and under the hypothesis "$\mathrm{Ext}^1(\mathbb{Q}, \mathbb{Q}(k))$ *for such k is one-dimensional*", derives further desirable consequences related to the size and the "$\mathbb{Q}$-rationality" of the Galois image. (See §5.4 below.)

(4) Some new objects in number theory, such as adelic analogues of beta and gamma functions, and higher circular l-units, have been constructed and used to give explicit comparisons of the $G_{\mathbb{Q}}$-actions, on π_1^{nil} of $\mathbb{P}^1 - \{0, 1, \infty\}$, on torsion points of Fermat Jacobians, and on higher circular l-units (G. Anderson, R. Coleman, the author, ...). This series of work was started in [Ih_1] and was developed in [A_3, C_3, IKY, Ih_i, A-$\mathrm{I}_{1,2}$], etc., by combining with other ideas [A_1, C_1, C_2], ...

The following special towers of coverings of $\mathbb{P}^1 - \{0, 1, \infty\}$ are relevant; (i) *the meta-abelian tower,* (ii) *the nilpotent tower,* and (iii) *the genus* 0 *tower.* Here, (i) corresponds to the double commutator quotient $\hat{\pi}_1/\hat{\pi}_1''$ of $\hat{\pi}_1$, (ii) to π_1^{nil}, and (iii) is also quite big – what it generates is bigger than (ii). As for the common field of definition, (i) is related to abelian extensions over the cyclotomic field $\mathbb{Q}(\mu_\infty)$, (ii) to a very natural sequence of Galois extensions over $\mathbb{Q}$ that are nilpotent over $\mathbb{Q}(\mu_\infty)$, and (iii) to (the field generated by) higher circular l-units.

As for (i), the conclusion is that the Galois actions on $\hat{\pi}_1/\hat{\pi}_1''$, on torsion points of Fermat Jacobians, and on roots of circular units, can be compared with each other in terms of *explicit universal formulas*. The size of the Galois image is also measured explicitly. The theory developed to that of Anderson's hyperadelic gamma function, which also plays the role of a bridge connecting Gauss sums with circular units.

As for (ii) and (iii), the main conclusion is that the action of $\sigma \in G_\mathbb{Q}$ on π_1^{nil} can be expressed explicitly $\bmod l^n$, for any $n \geq 1$, in terms of its action on the group of higher circular l-units.

For more of these, see §6.

(5) V. G. Drinfeld more recently discovered a striking connection between (what is expected to be) the automorphism group of the tower $\{\pi_1(X_n(\mathbb{C}))\}_{n\geq 4}$ and a "universal braid transformation group"

$$(A, \Delta, \varepsilon, \phi, R) \to (A, \Delta, \varepsilon, \phi', R')$$

acting on the structures of quasi-triangular quasi-Hopf algebras [Dr2]. These groups become non-trivial after suitable *completions* of π_1, etc. The Galois image in $\operatorname{Aut}\hat{\pi}_1(X_4(\mathbb{C}))$ is contained in these (essentially the same) groups. (See §4.)

(6) There are several other important works on this subject including the following.

(i) Grothendieck raises a question as to whether the intertwiners of two Galois representations in $\hat{\pi}_1$ arising from two algebraic varieties always correspond to algebraic morphisms in the "non-abelian" situation. In the special case of $\mathbb{P}^1$ minus finitely many points, H. Nakamura [N] gives an affirmative answer.

(ii) T. Oda and Y. Matsumoto each gives, from different viewpoints, a non-abelian analogue of the Néron-Ogg-Šafarevič criterion for good reduction of curves, using π_1^{nil} instead of H_1^{et}; cf. [O].

Due to the lack of time, these will not be included in this lecture.

We shall start by defining the Galois action on the completed fundamental groupoids.

§2. The Basic Definitions

2.1 Let X be an algebraic variety defined over $\mathbb{Q}$, and $X(\mathbb{C})$ be the set of $\mathbb{C}$-rational points of X equipped with the usual (complex analytic) topology. We assume X to be geometrically connected, which is equivalent with $X(\mathbb{C})$ being connected. For $a, b \in X(\mathbb{C})$, call $\pi_1(X(\mathbb{C}); a, b)$ the set of homotopy classes (rel. to a, b) of paths from a to b on $X(\mathbb{C})$. When $a = b$, it is denoted by $\pi_1(X(\mathbb{C}), b)$. The composition rule

$$\pi_1(X(\mathbb{C}); b, c) \times \pi_1(X(\mathbb{C}); a, b) \longrightarrow \pi_1(X(\mathbb{C}); a, c) \tag{2.1.1}$$

gives the system $\{\pi_1(X(\mathbb{C});a,b)\}_{a,b\in B}$ structure of groupoid, the fundamental groupoid of $X(\mathbb{C})$ with base point set B ($B \subset X(\mathbb{C})$). When $B = \{b\}$, $\pi_1(X(\mathbb{C}),b)$ is the fundamental group.

We shall recall the definitions of the profinite completion $\hat{\pi}_1(X(\mathbb{C});a,b)$ and the $G_{\mathbb{Q}}$-action on $\hat{\pi}_1(X(\mathbb{C});a,b)$ when $a,b \in X(\mathbb{Q})$ (the $\mathbb{Q}$-rational points of X).

First, $\hat{\pi}_1(X(\mathbb{C});a,b)$ is the completion of $\pi_1(X(\mathbb{C});a,b)$ with respect to the following topology. Two elements p,p' of the latter set are sufficiently close to each other if the associated "round trip" $p'^{-1}\cdot p$ belongs to a sufficiently small subgroup of $\pi_1(X(\mathbb{C}),a)$ with finite index. The completed set $\hat{\pi}_1(X(\mathbb{C});a,b)$ is then profinite, i.e., compact and totally disconnected topological space. The composition (2.1.1) is continuous and carries over to the completion. Each path class $p \in \pi_1(X(\mathbb{C});a,b)$ gives rise to a "compatible" system of fiber bijections $p_f : f^{-1}(a) \xrightarrow{\sim} f^{-1}(b)$, where f runs over all finite coverings of $X(\mathbb{C})$, and p_f is induced by tracing above a path representing p. Here, "compatible" means that for any f,f', p_f and $p_{f'}$ are compatible with the fiber projections induced from each element of $\mathrm{Hom}(f,f')$. This procedure $p \to \{p_f\}_f$ induces the bijection

$$\hat{\pi}_1(X(\mathbb{C});a,b) \approx \left\{\begin{array}{l}\text{compatible systems of fiber}\\ \text{bijections}\quad \hat{p}_f : f^{-1}(a) \xrightarrow{\sim} f^{-1}(b)\end{array}\right\}_f .$$

This is because $X(\mathbb{C})$ is locally arcwise connected and locally simply connected, being an underlying space of an analytic space.

Since all finite coverings of $X(\mathbb{C})$ are algebraic and defined over $\overline{\mathbb{Q}}$ (the generalized Riemann existence theorem), we may assume that f runs over all finite etale coverings of $X \otimes \overline{\mathbb{Q}}$. If $a,b \in X(\mathbb{Q})$, each $\sigma \in G_{\mathbb{Q}}$ induces the fiber bijections $f^{-1}(a) \xrightarrow{\sim} (\sigma f)^{-1}(a)$, $f^{-1}(b) \xrightarrow{\sim} (\sigma f)^{-1}(b)$; hence σ acts on $\hat{\pi}_1(X(\mathbb{C});a,b)$ by

$$\{\hat{p}_f\} \longrightarrow \{\sigma \circ \hat{p}_{\sigma^{-1}f} \circ \sigma^{-1}\} .$$

It is easy to see that this action is compatible with the composition induced from (2.1.1) by completion. Therefore, $G_{\mathbb{Q}}$ acts as an automorphism group of the completed fundamental groupoid of $X(\mathbb{C})$ with base point set $B \subset X(\mathbb{Q})$; in particular, as that of the completed fundamental group with base point $b \in X(\mathbb{Q})$. These actions are compatible with the groupoid homomorphisms induced from any algebraic morphisms $X' \to X$ over $\mathbb{Q}$.

Example. Take $X = \mathbb{P}^1 - \{0,\infty\}$ and $b = 1$. Then $\pi_1(X(\mathbb{C}),1) \simeq \mathbb{Z}$, being generated by the class p of the loop $\tau \to \exp(2\pi i\tau)$ $(0 \le \tau \le 1)$, and $\hat{\pi}_1(X(\mathbb{C}),1) = \hat{\mathbb{Z}} = \varprojlim(\mathbb{Z}/N)$. If f_N is the cyclic covering $t \to t^N$ of X $(N = 1,2,\dots)$, then

$$f_N^{-1}(1) = \{1,\zeta_N,\dots,\zeta_N^{N-1}\}, \quad \zeta_N = \exp(2\pi i/N) .$$

Write $\sigma(\zeta_N) = \zeta_N^{\chi(\sigma)_N}$ $(\sigma \in G_{\mathbb{Q}}, \chi(\sigma)_N \in \mathbb{Z}/N)$. Then p_{f_N} acts on $f_N^{-1}(1)$ by $\theta \longrightarrow \zeta_N\theta$, and σ acts on $f_N^{-1}(1)$ by $\theta \longrightarrow \theta^{\chi(\sigma)_N}$ $(\theta \in f_N^{-1}(1))$. Therefore,

$$\sigma \circ p_{f_N} \circ \sigma^{-1} = p_{f_N}^{\chi(\sigma)_N} . \tag{2.1.2}$$

Call $\chi(\sigma) = \varprojlim \chi(\sigma)_N \in \widehat{\mathbb{Z}}^\times$. Then χ is a (continuous) homomorphism

$$\chi : G_{\mathbb{Q}} \longrightarrow \widehat{\mathbb{Z}}^\times \ , \tag{2.1.3}$$

called the cyclotomic character. By (2.1.2), σ acts on $\hat{\pi}_1 = \widehat{\mathbb{Z}}$ via $\chi(\sigma)$-multiplication. (Here and in the following, for any associative ring A with unit element, $A^\times$ denotes the group of invertible elements of A.)

2.2 We return to the general case. The Galois group $G_{\mathbb{Q}}$ thus acts on the completed fundamental groupoid $\{\hat{\pi}_1(X(\mathbb{C}); a, b)\}_{a,b\in B}$ $(B \subset X(\mathbb{Q}))$ and in particular on $\hat{\pi}_1(X(\mathbb{C}), b)$ $(b \in X(\mathbb{Q}))$ as an automorphism group:

$$\varphi_{X,b} : G_{\mathbb{Q}} \longrightarrow \operatorname{Aut} \hat{\pi}_1(X(\mathbb{C}), b) \ .$$

The induced homomorphism obtained by forgetting the role of the base point is also of interest:

$$\varphi_X : G_{\mathbb{Q}} \longrightarrow \operatorname{Out} \hat{\pi}_1(X(\mathbb{C}))$$

$(\operatorname{Out}(\hat{\pi}_1) = \operatorname{Aut}(\hat{\pi}_1)/\operatorname{Int}(\hat{\pi}_1)$, where $\operatorname{Int}(\hat{\pi}_1)$ is the inner automorphism group). Replacing $\hat{\pi}_1$ by the pronilpotent completion π_1^{nil}, we obtain $\varphi_{X,b}^{\mathrm{nil}}$, etc.

As for X, we shall mainly consider the varieties

$$X = X_n = \{(x_1, \ldots, x_n) \in (\mathbb{P}^1)^n; \quad x_i \neq x_j \quad \text{for} \quad i \neq j\}/PGL(2)$$

$(n \geq 4)$, where $PGL(2) = \operatorname{Aut} \mathbb{P}^1$ acts on X diagonally. Note that

$$X_4 \simeq \mathbb{P}^1 - \{0, 1, \infty\},$$
$$X_5 \simeq (X_4)^2 - \Delta \quad (\Delta : \text{the diagonal}),$$

$P_n = \pi_1(X_n(\mathbb{C}), *)$ is the quotient modulo center $(= \pi_1(PGL(2, \mathbb{C})) \simeq \mathbb{Z}/2)$ of the pure sphere braid group on n strings, and that

$$P_4 \simeq F_2 \quad \text{(free, rank 2)}$$
$$P_5 = F_2 \ltimes F_3 \quad \text{(semi-direct, } F_2 \text{ acts on } F_3) \ .$$

Now let $X = X_4 = \mathbb{P}^1 - \{0, 1, \infty\}$, so that $\pi_1(X(\mathbb{C})) \simeq F_2$. As reviewed in §1, Belyǐ proved that φ_X is then injective. This focuses light on the following

Question 1. What can one say about the images of $\varphi_X, \varphi_X^{\mathrm{nil}}$, etc.?

If one can characterize the image of φ_X explicitly, then one is led to a completely different description of $G_{\mathbb{Q}}$.

Question 2. What can one say about the "elementwise description" of $\varphi_X, \varphi_X^{\mathrm{nil}}$, etc.?

Actually, this *Q 2* is not well posed at this stage of the development of mathematics, as we know no good system of NAMES for elements of $G_{\mathbb{Q}}$. As far as the author knows, no element of $G_{\mathbb{Q}}$ other than the identity element and the complex conjugation (remember that $\overline{\mathbb{Q}} \subset \mathbb{C}$ in our formulation) has an explicit name to identify itself. Nor is it so for $\widehat{F}_2$. Before asking to give an explicit description of the homomorphism φ_X, we must ask ourselves the possibility of giving good names to elements of $G_{\mathbb{Q}}, \widehat{F}_r (r \geq 2)$. If, however, we replace $\widehat{F}_r$ by the nilpotent completion F_r^{nil}, each element of F_r^{nil} then has a good name to identify itself which is obtained from the non-commutative Taylor expansion in r variables with coefficients in $\widehat{\mathbb{Z}}(\cong \prod_l \mathbb{Z}_l)$. Thus, at present, the sense of *Q 2* is to ask to *give* a new system of names for elements of $G_{\mathbb{Q}}$ using φ_X^{nil} *and describe* any interplay in terms of these names. (φ_X^{nil} is no longer injective; hence it concerns with a certain quotient of $G_{\mathbb{Q}}$ which is "big and small".)

We shall mainly discuss:

1) Works of Deligne, Drinfeld, and the author on *Q 1* (§§3, 4, 5), and

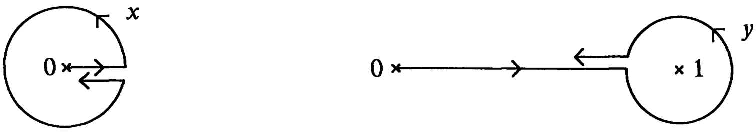
2) Works of Anderson, Coleman, and the author on *Q 2* (§6).

Among them, 1) is closely related to $\varphi_{X,B}$ for $X = X_5$.

2.3 Let $X = \mathbb{P}^1 - \{0, 1, \infty\}$, and F_2 be the free group of rank 2 on two letters x, y. Our first goal is to show that for each $\sigma \in G_{\mathbb{Q}}$, $\varphi_X(\sigma)$ is determined by two "coordinates", $\chi(\sigma)$ and f_σ, where $\chi : G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^\times$ is the cyclotomic character and f_σ is an element of $\widehat{F}_2' = (\widehat{F}_2, \widehat{F}_2)$, the commutator subgroup of $\widehat{F}_2$. This can also be done relying more on group-theoretic normalization as in [B$_1$], [Ih$_1$], but we proceed more "conceptually" using Deligne's tangential base points [De].

Let $\mathbb{B}$ be the set of "arrows" $\overrightarrow{ij}$ with $i, j \in \{0, 1, \infty\}, i \neq j$. Thus, $\mathbb{B}$ has six elements and the symmetric group S_3 acts simply transitively on $\mathbb{B}$. For $a, b \in \mathbb{B}$, Deligne defines $\pi_1 = \pi_1(X(\mathbb{C}); a, b)$ and the $G_{\mathbb{Q}}$-action on its profinite completion. Topologically, it is clear what π_1 should mean when, in general, a, b are *simply connected subspaces* of $X(\mathbb{C})$. The base point $\overrightarrow{ij}$ plays the same role as the open interval I_{ij} on $\mathbb{R}$ bounded by i, j *and* not containing the third point k from $\{0, 1, \infty\}$. For a finite etale covering $Y \to X$ over $\overline{\mathbb{Q}}$, the fiber above $\overrightarrow{ij}$ consists of points $P \in \overline{Y}(\overline{\mathbb{Q}})$ above i given together with a "topological branch" (i.e., a lifting of I_{ij}) at each P. Here $\bar{f} : \overline{Y} \to \mathbb{P}^1$ is the compactification of f. In order to define the $G_{\mathbb{Q}}$-action on $\hat{\pi}_1(X(\mathbb{C}); a, b)$ for $a, b \in \mathbb{B}$, it suffices to give an *algebraic* interpretation of the branches at P. One way to put it is as follows. (This device proved to be useful [A–I$_1$].) Let t_{ij} be the linear fractional function $\mathbb{P}^1 \to \mathbb{P}^1$ which maps i, j, k to $0, 1, \infty$, respectively. Then a branch at P is a local embedding of the local ring of $\overline{Y}$ at P into the ring of *Puiseux series* in t_{ij} which extends: (i) the obvious embedding of the local ring of $\mathbb{P}^1$ at i into the ring of power series in t_{ij}, and (ii) the residue field embedding determined by the geometric point P. The corresponding topological branch is obtained by the principle to choose "the positive real root for $t_{ij}^{1/e}$, on I_{ij}". The group $G_{\mathbb{Q}}$ acts on the fibers above $\overrightarrow{ij}$ via its action on the Puiseux coefficients $\in \overline{\mathbb{Q}}$. One may prefer to reinterpret this in terms of the normalization of the fiber product of f with $\operatorname{Spec} \mathbb{C}[t_{ij}^{1/e}]$ (e: the ramification index).

Now we consider $\pi_1(X(\mathbb{C}); \overrightarrow{01}, \overrightarrow{10})$, $\pi_1(X(\mathbb{C}), \overrightarrow{01})$ and the Galois action on their completions. The first set contains an obvious element defined from the interval (0,1). Call it p. The second group contains a small positive loop around 0, called x, and $y = p^{-1} \circ x' \circ p$, where x' is the transform of x by $t \to 1-t$ ($t = t_{01}$). This group is free on x, y.



Now for each $\sigma \in G_{\mathbb{Q}}$, put

$$f_\sigma = p^{-1} \circ \sigma(p) \in \hat{\pi}_1(X(\mathbb{C}), \overrightarrow{01}) \ . \tag{2.3.1}$$

Then σ acts on the generators of $\hat{\pi}_1(X(\mathbb{C}), \overrightarrow{01}) = \widehat{F}_2$ as

$$x \longrightarrow x^{\chi(\sigma)}, \quad y \longrightarrow f_\sigma^{-1} \cdot y^{\chi(\sigma)} \cdot f_\sigma \ . \tag{2.3.2}$$

It follows easily that $f_\sigma \in \widehat{F}_2'$, and that (2.3.2) with this requirement characterizes f_σ. When σ is the complex conjugation, $\chi(\sigma) = -1$, $f_\sigma = 1$.

Remark. By (3.1.1)(I),(II) below, it follows also that $z = (xy)^{-1}$ (a loop around ∞) is mapped to $g_\sigma^{-1} z^{\chi(\sigma)} g_\sigma$, where $g_\sigma = f_\sigma(x,z) x^{\frac{1}{2}(1-\chi(\sigma))}$.

Although $\widehat{F}_2$ contains much more than the free words on x, y, we shall express an element of this group conveniently as $f(x,y)$, because it will then make sense to speak of $f(\xi,\eta)$ for any elements $\xi, \eta \in G$ of any profinite group G; the image of f under the unique homomorphism $\widehat{F}_2 \to G$ mapping x, y to ξ, η respectively.

§3. The Galois Action (Profinite)

3.1 So what is the image of the mapping $G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^\times \times \widehat{F}_2'$ defined by $\sigma \mapsto (\chi(\sigma), f_\sigma)$? The known equations satisfied by $\lambda = \chi(\sigma)$, $f = f_\sigma$ are as follows.

$$\begin{aligned} &\text{(I)} && f(x,y)f(y,x) = 1; \\ &\text{(II)} && f(z,x)z^m f(y,z)y^m f(x,y)x^m = 1, \\ & && \qquad \text{if} \quad xyz = 1, \quad m = \tfrac{1}{2}(\lambda - 1); \end{aligned} \tag{3.1.1}$$

(III) (Drinfeld) Let $P_5 = \pi_1(X_5(\mathbb{C}), \mathscr{B}_5)$ and $x_{ij} \in P_5$

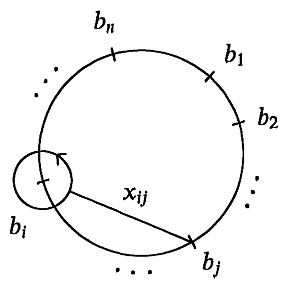
($1 \le i, j \le 5$) be as defined below. Then in $\widehat{P}_5$,

$$f(x_{12}, x_{23}) f(x_{34}, x_{45}) f(x_{51}, x_{12}) f(x_{23}, x_{34}) f(x_{45}, x_{51}) = 1 \ .$$

Remark. Drinfeld's formula given in [Dr$_2$] is in terms of *plane* braid group on 4 strings and is non-cyclic. The above formula is equivalent to his. The author previously wrote down more complicated formulas as 4 transposition relations (w.r.t. (1 2),(2 3),(3 4),(4 5) in S_5 instead of (12345)) in [Ih$_6$]. Drinfeld thinks this type of formula may very well be known to Grothendieck.

As for the more obvious question: "use of X_n $(n \geq 6) \dots$?", see §3.3.

The definitions of $\mathscr{B}_n$ and x_{ij}. Let $\widetilde{\mathscr{B}}_n$ be the space of all n-tuples $(b_1, \dots, b_n)$ of distinct points of $\mathbb{R}^{\cup}(\infty)$ satisfying the condition: b_{i+1} *is next to* b_i *in the positive direction for all* i $(1 \leq i \leq n-1)$ *including the case of passing through* ∞. Then $PGL_2^+(\mathbb{R})$, the real projective linear group of degree two with positive determinant, acts on $\widetilde{\mathscr{B}}_n$ diagonally, and the quotient space $\mathscr{B}_n = \widetilde{\mathscr{B}}_n / PGL_2^+(\mathbb{R})$ is naturally embedded into $X_n(\mathbb{C})$. The space $\mathscr{B}_n$ is simply connected and hence it makes sense to speak of the fundamental group $P_n = \pi_1(X_n(\mathbb{C}), \mathscr{B}_n)$. It is generated by the elements x_{ij} $(1 \leq i, j \leq n)$ shown below:



3.2 *Sketch of proof of (3.1.1).* (I) Apply the automorphism $\theta : t \to 1-t$ to both sides of (2.3.1).

(II) Let r be the element of $\pi_1(X(\mathbb{C}); \overrightarrow{10}, \overrightarrow{1\infty})$ corresponding to the rotation of argument π at the point 1:

Then it is easy to see that

$$r^{-1} \cdot \sigma(r) = \theta(x)^{\frac{1}{2}(\chi(\sigma)-1)} \in \hat{\pi}_1(X(\mathbb{C}), \overrightarrow{10}) \ .$$

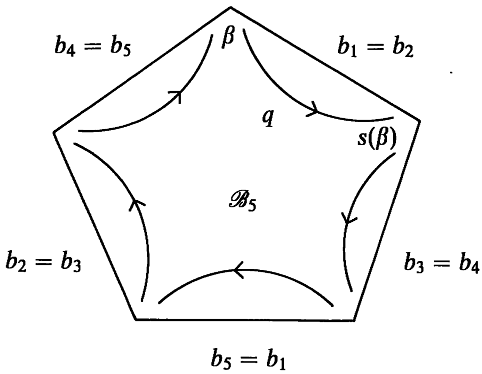
Therefore, if $q = r \circ p$, we have $\sigma(q) = qy^{\frac{1}{2}(\chi(\sigma)-1)} f_\sigma(x, y)$. Let ω be the automorphism $t \to (1-t)^{-1}$ of X. Then $\omega^2(q)\omega(q)q = 1$. Apply σ on this to obtain (II).

(III) The symmetric group S_5 acts on X_5 from the left by substitution of coordinates, as $(x_i) \to (x_{s^{-1}i})$ $(s \in S_5)$. Let $s = (13524)$. Then for a certain "tangential base point"β and a path class $q \in \pi_1(X_5(\mathbb{C}); \beta, s(\beta))$, one can naturally identify $\pi_1(X_5(\mathbb{C}), \beta)$ with $\pi_1(X_5(\mathbb{C}), \mathscr{B}_5) = P_5$ and show that

$$\sigma(q) = q \cdot f_\sigma(x_{45}, x_{34}) \ , \tag{3.2.1}$$

$$s^4(q) \cdot s^3(q) \cdot s^2(q) \cdot s(q) \cdot q = 1 \ . \tag{3.2.2}$$

The equation (III) follows directly from these. A topological illustration:



(3.2.3)

The author used a double Puiseux series expansion to define the algebraic interpretation of β.

3.3 As in [Dr2], define $\widehat{GT}$ (the "Grothendieck-Teichmüller group") to be the subgroup of $\mathrm{Aut}\,\widehat{F}_2$ consisting of all automorphisms of the form

$$x \longrightarrow x^\lambda, \quad y \longrightarrow f^{-1} \cdot y^\lambda \cdot f \qquad (\lambda \in \widehat{\mathbb{Z}}^\times,\ f \in \widehat{F}_2') \ ,$$

where λ and f satisfy (3.1.1)(I),(II),(III). (It forms a subgroup!) Then by definition, the image of $\varphi_{X,b}$ ($b = \overrightarrow{01}$) is contained in $\widehat{GT}$. As φ_X is injective (Belyĭ), so is $\varphi_{X,b}$. Therefore, $\varphi_{X,b}$ induces an inclusion

$$\varphi_{X,b} : G_{\mathbb{Q}} \hookrightarrow \widehat{GT} \ . \tag{3.3.1}$$

Two questions arise:

Question 3.3.2 Can one still obtain new relations using X_n for higher n's ?

Question 3.3.3 Can one characterize the image of $G_{\mathbb{Q}}$ using these types of relations (or to ask more strongly, is (3.3.1) already a bijection) ?

If one believes a strict analogy with conformal field theory, then no more essentially new relations would be obtained by using X_n for higher n ([M-S, G, T-K, Dr_2]). Drinfeld's new quantum group theoretic interpretation of $\widehat{GT}$ (see §4 below) gives a very impressive version of this philosophical reasoning.

See §5.3 for the Lie version of these questions.

§4. Connection with Transformation of Structure of "Complete" Quasi-Triangular Quasi-Hopf Algebras

4.1 Drinfeld introduced the concept of quasi-triangular quasi-Hopf algebra (abbrev. *qtqH* algebra) [Dr_1, Dr_2]. It is more general than Hopf algebra in that the coassociativity and the cocommutativity assumptions are weakened (equalities replaced by conjugations). Thus, a *qtqH* algebra over a field k is a quintuple $(A, \Delta, \varepsilon, \phi, R)$, where (i) A is an associative k-algebra with unit element 1, (ii) $\Delta : A \to A \otimes A$ (comultiplication) and $\varepsilon : A \to k$ (counit) are k-algebra homomorphisms mapping 1 to 1 ($\otimes$: the tensor product over k), such that

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id} \ , \tag{4.1.1}$$

(iii) $\phi \in (A \otimes A \otimes A)^\times$ and $R \in (A \otimes A)^\times$ satisfy:

(A) The basic conjugacy relations:

$$(\mathrm{id} \otimes \Delta)(\Delta(a)) = \phi \cdot (\Delta \otimes \mathrm{id})(\Delta(a)) \cdot \phi^{-1} \ , \tag{4.1.2}$$

$${}^t\Delta(a) = R \cdot \Delta(a) \cdot R^{-1} \ , \tag{4.1.3}$$

for all $a \in A$, where $b \to {}^t b$ ($b \in A \otimes A$) denotes the transposition of two factors. It is *not* assumed that ${}^tR \cdot R = 1$.

(B) The compatibility equalities:

$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})\phi = 1 \ , \tag{4.1.4}$$

(*the pentagon relation*) (4.1.5)

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)\phi \cdot (\Delta \otimes \mathrm{id} \otimes \mathrm{id})\phi = (1 \otimes \phi) \cdot (\mathrm{id} \otimes \Delta \otimes \mathrm{id})\phi \cdot (\phi \otimes 1) \ ,$$

(*two hexagon relations*) omitted (cf. [Dr_2](I·6, ab)). (4.1.6)

Finally, the existence of an antipode is assumed.

As is explained in [Dr_2] the assumptions in (iii) can be understood more conceptually in terms of isomorphisms between tensor products of A-modules. If V, W are (left) A-modules (w.r.t. the k-algebra structure of A), then their tensor product $V \otimes W$ over k is natually an $(A \otimes A)$-module, and via Δ, again an A-module. If V_1, V_2, V_3 are three A-modules, then the $(A \otimes A \otimes A)$-module $V_1 \otimes V_2 \otimes V_3$ can be regarded as an A-module in two different ways, and (4.1.2) imposes that the ϕ-multiplication induce an A-isomorphism

$$(V_1 \otimes V_2) \otimes V_3 \xrightarrow{\sim} V_1 \otimes (V_2 \otimes V_3) \ .$$

The pentagon relation requires the diagram

$$\begin{tikzcd}
& ((V_1 \otimes V_2) \otimes V_3) \otimes V_4 \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\
(V_1 \otimes (V_2 \otimes V_3)) \otimes V_4 \arrow[d, "\sim"'] & & (V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \arrow[d, "\sim"] \\
V_1 \otimes ((V_2 \otimes V_3) \otimes V_4) \arrow[rr, "\sim"] & & V_1 \otimes (V_2 \otimes (V_3 \otimes V_4))
\end{tikzcd} \tag{4.1.7}$$

to be commutative. In the diagrams for hexagon relations, the choice of directions of arrows related to "R" is delicate. The two diagrams (3.2.3),(4.1.7) are related to each other via Knizhnik–Zamolodchikov equations ([Dr$_2$]§2)

4.2 Now take $m \in \mathbb{Z}$ and $f(x, y) \in F_2$. If we replace ϕ, R by

$$\begin{cases} \phi' &= \phi \cdot f(R^{21}R^{12},\ \phi^{-1}R^{32}R^{23}\phi) \ , \\ R' &= R \cdot ({}^tRR)^m \ , \end{cases} \tag{4.2.1}$$

respectively, then the 2 equalities in (A) are still satisfied. Here, if $R = \sum a_i \otimes b_i$, $R^{23} = \sum 1 \otimes a_i \otimes b_i$, $R^{32} = \sum 1 \otimes b_i \otimes a_i$, etc. Drinfeld shows that $(A, \Delta, \varepsilon, \phi', R')$ *satisfies* (B) *if and only if* (λ, f) *satisfies the* 3 *equalities of* (3.1.1), *where* $m = \frac{1}{2}(\lambda - 1)$. Actually, the only pairs of λ and f satisfying (3.1.1) are $(\lambda, f) = (\pm 1, 1)$. But if $\mathbb{Z}$ and F_2 are replaced by suitable *completions*, then there are many solutions. It suffices to recall that for $\lambda \in \widehat{\mathbb{Z}}^\times$ and $f \in \widehat{F}_2'$, (3.1.1) was the system of defining equations for the group $\widehat{GT}$, and that $\widehat{GT}$ contains $G_{\mathbb{Q}}(!)$. The above two explicit solutions correspond to the identity element and the complex conjugation of $G_{\mathbb{Q}}$. Drinfeld considers some types of *complete qtqH* algebras over the ring of formal power series $k[\![h]\!]$ over a field k of characteristic 0, called *quantized universal enveloping algebras* (*abbrev*. QUE-algebras). In this case, $\otimes$ is replaced by the completed tensor product $\widehat{\otimes}$. For such *qtqH* algebras, the transformation (4.2.1) will make sense if λ, f are elements of $k^\times$, $F_2^{\mathrm{nil}}(k)$ (the "k-nilpotent completion" of F_2) respectively, and if R is sufficiently close to 1. Thus, one may define the group $GT(k)$ which is still "big" and has a meaning as the group of transformations of structures of QUE-algebras over $k[\![h]\!]$. (The group law for $\widehat{GT}$ corresponds to the reverse of the composition of transformations (4.2.1).) Roughly speaking, one may consider $\widehat{GT}$ as the universal group of transformations of structures of *qtqH* algebras.

Remark. There is another type of transformations of structures of *qtqH* algebras called *twists* [Dr$_2$]. The two types of transformations commute with each other, and the twists do not change the structure (as quasi-tensor category) of the category of A-modules.

Question 4.2.2. Is there any "profinite" $(A, \Delta, \varepsilon, \phi, R)$ over some ring related to $\mathbb{Q}$ on which $G_{\mathbb{Q}}$ acts "naturally" and for which the natural action of σ^{-1} coincides with the transformation (4.2.1) obtained by $(\chi(\sigma), f_\sigma)$?

§5. The Galois Action (Pronilpotent)

5.1 For $\pi_1 = \pi_1(X(\mathbb{C}), *)$, instead of $\hat{\pi}_1$ we may also consider the pronilpotent completion π_1^{nil}, the projective limit of all finite nilpotent factor groups of π_1. It is the direct product of its l-Sylow subgroups $\pi_1^{(l)}$, the pro-l completion of π_1, where l runs over all prime numbers. For a topological group G, we denote by

$$G = G(1) \supseteq G(2) \supseteq \cdots \supseteq G(m+1) \supseteq \cdots$$

its lower central series defined by $G(m+1) = (G, G(m))$ $(m \geq 1)$. Here, (,) is the closure of the algebraic commutator. Note that

$$\bigcap_m G(m) = \{1\}, \quad \text{for} \quad G = \pi_1^{\mathrm{nil}},\ \pi_1^{(l)} .$$

The representations $\varphi_X, \varphi_{X,b}$ of $G_{\mathbb{Q}}$ induce the quotient representations $\varphi_X^{\mathrm{nil}}, \varphi_X^{(l)}$, etc. There are various advantages of passage to these quotients: (i) For a "canonical" choice of X such as $\mathbb{P}^1 - \{0, 1, \infty\}$, φ_X^{nil} gives rise to a natural filtration of some quotient of $G_{\mathbb{Q}}$ which is of arithmetic interest. (ii) Each element of π_1^{nil}, for $\mathbb{P}^1 - \{0, 1, \infty\}$ etc., allows an explicit presentation as formal non-commutative power series over $\hat{\mathbb{Z}}$. So, we may ask more explicit questions about the projection of $f_\sigma(x, y)$ on π_1^{nil} (than about $f_\sigma(x, y)$ itself). (iii) For π_1^{nil}, we may use Lie algebra techniques. (iv) The nilpotent quotients of the fundamental group π_1 are, in some sense, close to cohomology groups and are known to have additional structures. Under some assumptions on X, the Lie algebra of $\pi_1/\pi_1(m+1)$ has *mixed Hodge structure* (J. Morgan, D. Sullivan, K. T. Chen, R. Hain), and further, *mixed motif structure* (Deligne).

We shall start by explaining (i).

5.2 For each prime number l, there is a canonical sequence

$$\mathbb{Q} \subseteq \mathbb{Q}^{(l)}(1) \subseteq \cdots \subseteq \mathbb{Q}^{(l)}(m) \subseteq \cdots \subseteq \mathbb{Q}^{(l)}(\infty) = \bigcup \mathbb{Q}^{(l)}(m) \tag{5.2.1}$$

of (infinite) Galois extensions over $\mathbb{Q}$, starting with $\mathbb{Q}^{(l)}(1) = \mathbb{Q}(\mu_{l^\infty})$ (μ_{l^∞} : the group of roots of unity of l-power order), and an associated graded Lie algebra $\mathfrak{g}^{(l)}$, defined as follows. For each $m \geq 1, \mathbb{Q}^{(l)}(m)$ is the field corresponding to the kernel of the representation

$$G_{\mathbb{Q}} \longrightarrow \mathrm{Out}(F_2^{(l)}/F_2^{(l)}(m+1))$$

induced from $\varphi_X^{(l)}$ for $X = X_4 = \mathbb{P}^1 - \{0, 1, \infty\}$. This kernel will not change if X_4 is replaced by X_n ($n \geq 5$) [Ih5]. The union field $\mathbb{Q}^{(l)}(\infty)$ corresponds to the kernel of $\varphi_X^{(l)}$ for $X = X_n$, for any $n \geq 4$. It is a pro-l (non-abelian) extension over $\mathbb{Q}(\mu_{l^\infty})$ unramified outside l. For each $m \geq 1$, the Galois group $\mathrm{Gal}(\mathbb{Q}^{(l)}(m+1)/\mathbb{Q}^{(l)}(m))$ is a free $\mathbb{Z}_l$-module of finite rank (call it $r^{(l)}(m)$). It is centralized by $\mathrm{Gal}(\mathbb{Q}^{(l)}(m+1)/\mathbb{Q}^{(l)}(1))$, and as $\mathrm{Gal}(\mathbb{Q}^{(l)}(1)/\mathbb{Q})$-module, has the Tate twist m. The graded Lie algebra $\mathfrak{g}^{(l)}$ is the direct sum of its m-th graded pieces

$$\mathrm{Gal}(\mathbb{Q}^{(l)}(m+1)/\mathbb{Q}^{(l)}(m)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \quad (m = 1, 2, 3, \ldots),$$

each of which is a $\mathbb{Q}_l$-module ($\mathbb{Q}_l$: the l-adic number field); cf. [Ih4]. This graded Lie algebra $\mathfrak{g}^{(l)}$ over $\mathbb{Q}_l$ is a standard "approximation" of the filtered Galois group $\mathrm{Gal}(\mathbb{Q}^{(l)}(\infty)/\mathbb{Q})$.

Here is a set of very basic open questions.

Question 5.2.2 (i) What is the structure of $\mathfrak{g}^{(l)}$ *? (ii) Does there exist a natural graded Lie algebra* $\mathfrak{g}$ *over* $\mathbb{Q}$ *such that* $\mathfrak{g}^{(l)} \simeq \mathfrak{g} \otimes \mathbb{Q}_l$ *for all* l *? (iii) In particular, is the rank* $r^{(l)}(m)$ *for each* m *independent of* l *?*

We shall discuss two approaches to *Q 5.2.2 (ii).*

5.3 We shall define a candidate for the Lie algebra $\mathfrak{g}$ in question (cf. [Ih5]). As before, let $P_n = \pi_1(X_n(\mathbb{C}), \mathscr{B}_n)$ ($n \geq 4$) (see §3.1), and let $\mathfrak{P}_n$ be the graded Lie algebra over $\mathbb{Q}$ associated with the lower central series of P_n. Thus, the m-th graded piece $\mathrm{gr}^m \mathfrak{P}_n$ of $\mathfrak{P}_n$ is the $\mathbb{Q}$-module

$$(P_n(m)/P_n(m+1)) \otimes \mathbb{Q} \quad (m \geq 1) \ .$$

If X_{ij} ($1 \leq i, j \leq n$) denotes the element of $\mathfrak{P}_n$ of degree 1 represented by x_{ij} (§3.1), then $\mathfrak{P}_n$ is the graded Lie algebra over $\mathbb{Q}$ generated by the X_{ij}'s which satisfy the fundamental relations

$$X_{ii} = 0, \quad X_{ij} = X_{ji}, \quad \sum_{k=1}^{n} X_{ik} = 0 \qquad (1 \leq i, j \leq n)$$
$$[X_{ij}, X_{kl}] = 0 \qquad \text{if } \{i, j\} \cap \{k, l\} = \phi$$

([K, Ih5]). A *special derivation* of $\mathfrak{P}_n$ of degree m (≥ 1) is a derivation D of $\mathfrak{P}_n$ into itself such that $D(X_{ij}) = [T_{ij}, X_{ij}]$ with some $T_{ij} \in \mathrm{gr}^m \mathfrak{P}_n$ ($1 \leq i, j \leq n$). Let $\mathscr{D}_n$ be the graded Lie algebra over $\mathbb{Q}$ whose m-th graded piece is the $\mathbb{Q}$-module of all *symmetric special outer* derivations of $\mathfrak{P}_n$ of degree m ($m \geq 1$). Here, "symmetric" refers to the invariance with respect to the obvious S_n-action on $\mathfrak{P}_n$, and "outer" refers to considering modulo inner derivations.

The sense of considering such a Lie algebra $\mathscr{D}_n$ is that the $G_{\mathbb{Q}}$-action on $P_n^{(l)}$ gives rise to a degree-preserving Lie algebra embedding

$$\mathfrak{g}^{(l)} \hookrightarrow \mathscr{D}_n \otimes \mathbb{Q}_l \quad (n \geq 4) \tag{5.3.1}$$

for each prime number l. Now, when n increases, $\mathscr{D}_n$ gets *smaller*. More precisely, the projection $\mathfrak{P}_n \to \mathfrak{P}_{n-1}$, defined by letting $X_{ij} \to 0$ for i or $j = n$, induces a Lie homomorphism $\mathscr{D}_n \to \mathscr{D}_{n-1}$. We know that this is *injective* for $n \geq 5$. Thus, these embeddings give rise to an infinite chain

$$\mathscr{D}_\infty = \bigcap_{n\geq 4} \mathscr{D}_n \subseteq \cdots \subseteq \mathscr{D}_n \subseteq \cdots \subseteq \mathscr{D}_5 \subseteq \mathscr{D}_4 \ , \tag{5.3.2}$$

of graded Lie algebras over $\mathbb{Q}$. By (5.3.1) and some compatibility,

$$\mathfrak{g}^{(l)} \subseteq \mathscr{D}_\infty \otimes \mathbb{Q}_l \ . \tag{5.3.3}$$

Now we ask the following Lie versions of *Q 3.3.2, Q 3.3.3*:

Question 5.3.4 (i) $\mathscr{D}_\infty = \mathscr{D}_5$? (ii) $\mathfrak{g}^{(l)} = \mathscr{D}_\infty \otimes \mathbb{Q}_l$ *for all* l ?

Remark. From (5.3.2) (5.3.3), it follows that

$$r^{(l)}(m) \leq \dim \mathrm{gr}^m \mathscr{D}_\infty \leq \cdots \leq \dim \mathrm{gr}^m \mathscr{D}_5 \leq \dim \mathrm{gr}^m \mathscr{D}_4$$

for each $m \geq 1$. It is interesting to see which of the $\leq$ are equalities. For $m < 7$, $r^{(l)}(m) = \dim \mathrm{gr}^m \mathscr{D}_4$. But $r^{(l)}(7) = 1$ and $\dim \mathrm{gr}^7 \mathscr{D}_4 = 2$. Terada-Ihara and Drinfeld independently verified that $\dim \mathrm{gr}^7 \mathscr{D}_5 = 1$ ([Ih$_7$, Dr$_2$]), which is in favor of the affirmative aspect of *Q 5.3.4*. See [Ih$_3$, Ih$_4$] for more about these ranks.

Problem 5.3.5. *Construct elements of* $\mathscr{D}_\infty$ *by algebraic or topological means.*

5.4 Deligne constructs a basic motivic extension of the Tate motives, "$\mathbb{Q}$" by "$\mathbb{Q}(m)$" ($m \geq 3$, odd) [De]. Moreover, assuming his conjecture (*loc. cit.* (8.1)) which asserts that such extensions form a 1-dimensional space generated by his extension, and using [So$_{1,2}$], he proves the following.

The Galois representation in $\pi_1^{(l)}/\pi_1^{(l)}(m+1)$ *is "*$\mathbb{Q}$*-rational" in the following sense (for* $X = \mathbb{P}^1 - \{0, 1, \infty\}$, $\pi_1 = \pi_1(X(\mathbb{C}), \overrightarrow{01})$*): There exists a linear algebraic group* Del_m *over* $\mathbb{Q}$ *and a short exact sequence*

$$1 \longrightarrow U\mathrm{Del}_m \longrightarrow \mathrm{Del}_m \longrightarrow GL(1) \longrightarrow 1 \ ,$$

with $U\mathrm{Del}_m$ *unipotent, all independent of* l*, such that for each* l *the Galois representation in* $\pi_1^{(l)}/\pi_1^{(l)}(m+1)$ *factors through a representation* $G_\mathbb{Q} \to \mathrm{Del}_m(\mathbb{Q}_l)$ *which has an open image at least if* $l > 2$*. Moreover, the abelianization of* $U\mathrm{Del}_m$*, with the* $GL(1)$*-action, decomposes as*

$$(U\mathrm{Del}_m)^{\mathrm{ab}} \simeq \bigoplus_{3\leq k\leq m, odd} \mathbb{A}^1(k)$$

($\mathbb{A}^1$ *: the affine line,* k *: the Tate twist). Finally,* $\{\mathrm{Del}_m\}_{m\geq 1}$ *forms a* $G_\mathbb{Q}$*-compatible projective system.*

In particular, if one assumes the above conjecture on the extension of "$\mathbb{Q}$" by "$\mathbb{Q}(m)$", then *Q 5.2.2 (ii)* will have an affirmative answer with an additional information: the graded Lie algebra $\mathfrak{g}$ is generated by some subset of the form $\{s_m\}_m$, where m runs over *odd* integers ≥ 3 and s_m is of degree m.

§6. Arithmetic Aspects

6.1 In §6, we shall review some works of Anderson, Coleman, the author, ..., on the "elementwise description" and the arithmetic study of the representation φ_X^{nil} for $X = \mathbb{P}^1 - \{0, 1, \infty\}$.

Let $X = \mathbb{P}^1 - \{0, 1, \infty\}$ and, as in §2.3, identify the fundamental group $\pi_1(X(\mathbb{C}), \overrightarrow{01})$ with the free group F_2 on x, y. The Galois group $G_{\mathbb{Q}}$ acts on $\hat{\pi}_1(X(\mathbb{C}), \overrightarrow{01})$, and hence on $\widehat{F}_2$ and $F_2^{\mathrm{nil}} = \prod_l F_2^{(l)}$. Take any $\sigma \in G_{\mathbb{Q}}$. Then the action of σ on $\widehat{F}_2$ can be expressed by two coordinates $\chi(\sigma)$ and f_σ ($\chi(\sigma) \in \widehat{\mathbb{Z}}^\times, f_\sigma \in \widehat{F}_2'$) (see §2.3). Therefore, its action on F_2^{nil} can be expressed by $\chi(\sigma)$ and the projection f_σ^{nil} of f_σ on F_2^{nil}. Call $\mathscr{A} = \widehat{\mathbb{Z}} \ll \xi, \eta \gg$ the *non-commutative* power series algebra in two variables over $\widehat{\mathbb{Z}}$. Then F_2^{nil} can be embedded into $\mathscr{A}^\times$ via $x \to 1 + \xi$, $y \to 1 + \eta$, and f_σ^{nil} can be regarded as an element of $\mathscr{A}^\times$. Each coefficient of the power series f_σ^{nil} may be regarded as *a new invariant* of σ. One is motivated to compare it with other invariants of σ, the old ones (§§6.2, 6.3) or the newly constructed ones (§6.5).

Now as a $\widehat{\mathbb{Z}}$-module, $\mathscr{A}$ is the direct sum

$$\mathscr{A} = \widehat{\mathbb{Z}} \cdot 1 \oplus \mathscr{A} \cdot \xi \oplus \mathscr{A} \cdot \eta \ , \tag{6.1.1}$$

and $(f_\sigma^{\mathrm{nil}})^{-1}$ decomposes as

$$(f_\sigma^{\mathrm{nil}})^{-1} = 1 + A_1\xi + A_2\eta \quad (A_1, A_2 \in \mathscr{A}) \ .$$

Put $\psi_\sigma(\xi, \eta) = 1 + A_1\xi$. Then it follows easily from (3.1.1)(I) that

$$f_\sigma^{\mathrm{nil}} = \psi_\sigma(\eta, \xi) \cdot \psi_\sigma(\xi, \eta)^{-1} \ .$$

Thus, *knowing ψ_σ is equivalent to knowing f_σ^{nil}*. Moreover, ψ_σ is an anti 1-cocycle

$$\psi_{\sigma\tau} = \sigma(\psi_\tau) \cdot \psi_\sigma \qquad (\sigma, \tau \in G_{\mathbb{Q}})$$

with respect to the action of $G_{\mathbb{Q}}$ on $\mathscr{A}$ extending that on F_2^{nil}, and is more convenient for describing the σ-action on abelian subquotients of $\hat{\pi}_1$ ([Ih$_2$]; cf. also [A-I$_2$§2]).[1]

Our first subject now is an explicit formula for the *commutative* power series $\psi_\sigma^{\mathrm{ab}}$ obtained from ψ_σ by letting ξ and η commute. Let $\mathscr{A}^{\mathrm{ab}} = \widehat{\mathbb{Z}}[[\xi, \eta]]$ be the *commutative* formal power series algebra, with the induced $G_{\mathbb{Q}}$-action $1 + \xi \to (1 + \xi)^{\chi(\sigma)}$, $1 + \eta \to (1 + \eta)^{\chi(\sigma)}$ ($\sigma \in G_{\mathbb{Q}}$). Then $G_{\mathbb{Q}} \to (\mathscr{A}^{\mathrm{ab}})^\times$ ($\sigma \mapsto \psi_\sigma^{\mathrm{ab}}$) is a

[1] In these papers, the base point is $\overrightarrow{\infty 1}$ and x, y are loops around 0, 1, respectively. So, the definitions are slightly different. See *Remark* at the end of §6.5.

1-cocycle, and it turns out that each coefficient of $\psi_\sigma^{\mathrm{ab}}$ can be expressed in terms of "old invariants" of σ which we now recall.

6.2 The Old Invariants. Fix a prime number l.

(i) *The l-adic cyclotomic character* $\chi^{(l)}$ is the l-component of χ, i.e., $\chi(\sigma) = (\chi^{(l)}(\sigma))$, $\chi^{(l)}(\sigma) \in \mathbb{Z}_l^\times$.

(ii) *The cyclotomic elements* (Soulé, Deligne). These are certain continuous mappings

$$\kappa_m^{(l)} : G_{\mathbb{Q}} \longrightarrow \mathbb{Z}_l \quad (m \geq 1, odd) ,$$

satisfying the 1-cocycle relation

$$\kappa_m^{(l)}(\sigma\tau) = \kappa_m^{(l)}(\sigma) + \chi^{(l)}(\sigma)^m \kappa_m^{(l)}(\tau) \quad (\sigma, \tau \in G_{\mathbb{Q}}) . \tag{6.2.1}$$

[Construction] Let $n \geq 1$ (but $n \geq 2$ if $l = 2$). Put $\zeta_n = \exp\left(\frac{2\pi i}{l^n}\right)$ and

$$\varepsilon_{m,n} = \prod_a (\zeta_n^a - 1)^{\langle a^{m-1} \rangle} ,$$

where the product is over all integers a such that $0 < a < l^n$ and $(a, l) = 1$; $\langle a^{m-1} \rangle$ is the smallest positive integer congruent to a^{m-1} mod l^n. Note that $\varepsilon_{m,n}$ is totally real and totally positive (because m is *odd*). It is easy to see that each of $\varepsilon_{m,n+1}/\varepsilon_{m,n}$ and $\sigma(\varepsilon_{m,n})/\varepsilon_{m,n}^b$ is an l^n-th power of a totally positive element of $\mathbb{Q}(\mu_{l^\infty})$, where $\sigma \in G_{\mathbb{Q}}$, $b \in \mathbb{Z}$, $b \equiv \chi^{(l)}(\sigma)^{1-m} (\mathrm{mod}\, l^n)$. Hence there is a unique $\kappa_m^{(l)}(\sigma) \in \mathbb{Z}_l$ such that

$$\sigma((\varepsilon_{m,n})^{1/l^n}) = (\sigma(\varepsilon_{m,n}))^{1/l^n} \cdot \zeta_n^{\chi^{(l)}(\sigma)^{1-m} \cdot \kappa_m^{(l)}(\sigma)}$$

holds for all $n \geq 2$. Moreover, $\kappa_m^{(l)}$ satisfies (6.2.1). Here, for any positive real number c, c^{1/l^n} denotes its *positive real* root. By (6.2.1), $\kappa_m^{(l)}$ factors through $\mathrm{Gal}(\mathbb{Q}(\mu_{l^\infty})^{\mathrm{ab}}/\mathbb{Q})$, $\mathbb{Q}(\mu_{l^\infty})^{\mathrm{ab}}$ being the maximal *abelian* extension of $\mathbb{Q}(\mu_{l^\infty})$. Moreover, by Soulé [So1,2], these 1-cocycles $\kappa_m^{(l)}$ do not vanish at least if $l > 2$.

6.3 The following *explicit formula* for the coefficients of $\psi_\sigma^{\mathrm{ab}}$ is due to the contributions of Anderson, Coleman, Deligne, the author, Kaneko and Yukinari; cf. [A3, C3, IKY]. (See also [Ich] for a simplification of [IKY].)

For each $\sigma \in G_{\mathbb{Q}}$, define $\kappa_m^*(\sigma) \in \hat{\mathbb{Z}}^\times = \prod_l \mathbb{Z}_l^\times$ by

$$\kappa_m^*(\sigma) = ((l^{m-1} - 1)^{-1} \kappa_m^{(l)}(\sigma))_l .$$

Theorem [A3, C3, IKY]. *The commutative power series $\psi_\sigma^{\mathrm{ab}}(\xi, \eta)$ can be expressed explicitly as follows.*

$$\psi_\sigma^{\mathrm{ab}}(\xi, \eta) = \exp\left\{ \sum_{m \geq 3, odd} \frac{\kappa_m^*(\sigma)}{m!} ((X+Y)^m - X^m - Y^m) \right\}$$
$$\times \exp\left\{ -\frac{1}{2} \sum_{m \geq 2, even} \frac{b_m(1 - \chi(\sigma)^m)}{m!} ((X+Y)^m - X^m - Y^m) \right\} ,$$

where $X = \log(1+\xi)$, $Y = \log(1+\eta)$, *and the constants* b_m *are defined by*

$$\log\left(\frac{1-e^{-t}}{t}\right) = \sum_{m\geq 1} \frac{b_m}{m!} t^m \ .$$

(mb_m *is the m-th Bernoulli number.)*

Note that $\psi_\sigma^{\mathrm{ab}}(\xi,\eta)$ is of the form

$$\gamma_\sigma(\xi)\gamma_\sigma(\eta)\gamma_\sigma((1+\xi)(1+\eta)-1)^{-1} \ .$$

This power series γ_σ is closely related to Anderson's hyperadelic gamma function Γ_σ.

The image of $\sigma \to \psi_\sigma^{\mathrm{ab}}(\xi,\eta)$ was studied closely by Coleman [C3], Ichimura and Kaneko [IK]. The expected image can be figured out via Coleman theory [C1, C2], and the difference from the expected image can be measured in terms of the "Vandiver gap".

6.4 We now explain another aspect of $\psi_\sigma^{\mathrm{ab}}$. It is a connection with the action of σ on the *double* commutator quotient $\widehat{F}_2/\widehat{F}_2''$ of $\widehat{F}_2$, or equivalently, on torsion points of Fermat Jacobians. Put $\mathscr{F} = \widehat{F}_2$, and consider the abelianizations $\mathscr{F}^{\mathrm{ab}} = \mathscr{F}/\mathscr{F}'$ and $\mathscr{F}'^{\mathrm{ab}} = \mathscr{F}'/\mathscr{F}''$ of $\mathscr{F}$ and $\mathscr{F}'$ first as (additive) $\widehat{\mathbb{Z}}$-modules. Then $\mathscr{F}^{\mathrm{ab}} = \widehat{\mathbb{Z}}\underline{x} \oplus \widehat{\mathbb{Z}}\underline{y}$ on which σ acts via $\chi(\sigma)$-multiplication (because of (2.3.2)). Here $\underline{x}, \underline{y}$ are the classes of x, y. Now $\mathscr{F}^{\mathrm{ab}}$ acts on $\mathscr{F}'^{\mathrm{ab}}$ by conjugation. Therefore, $\mathscr{F}'^{\mathrm{ab}}$ may be regarded as a module over the completed group algebra $\widehat{\mathbb{Z}}[[\mathscr{F}^{\mathrm{ab}}]]$. But one can show that *this module is free of rank* 1 *generated by the class* θ' *of* $(x,y) = xyx^{-1}y^{-1}$. As σ acts semi-linearly on $\mathscr{F}'^{\mathrm{ab}}$, this action is presented by the unique element B'_σ of $\widehat{\mathbb{Z}}[[\mathscr{F}^{\mathrm{ab}}]]$ such that

$$\sigma(\theta') = B'_\sigma \cdot \theta' \ . \tag{6.4.1}$$

Now define $B_\sigma \in \widehat{\mathbb{Z}}[[\mathscr{F}^{\mathrm{ab}}]]$ by the formula

$$B'_\sigma = \left(\frac{\underline{x}^{\chi(\sigma)}-1}{\underline{x}-1} \cdot \frac{\underline{y}^{\chi(\sigma)}-1}{\underline{y}-1}\right) B_\sigma \ . \tag{6.4.2}$$

This B_σ is connected with $\psi_\sigma^{\mathrm{ab}}$ as follows. Consider the projection

$$\mathrm{pr} : \widehat{\mathbb{Z}}[[\mathscr{F}^{\mathrm{ab}}]] \longrightarrow \widehat{\mathbb{Z}}[[\xi,\eta]] = \mathscr{A}^{\mathrm{ab}} \quad \begin{cases} \underline{x} \mapsto 1+\xi, \\ \underline{y} \mapsto 1+\eta. \end{cases} \tag{6.4.3}$$

Then

$$\mathrm{pr}\,(B_\sigma) = \psi_\sigma^{\mathrm{ab}} \ . \tag{6.4.4}$$

The projection pr has a big kernel $\mathscr{K}$. In fact, $\mathscr{K} \cdot \theta'$ $(\subset \mathscr{F}'/\mathscr{F}'')$ is the kernel of $\mathscr{F}/\mathscr{F}'' \to F_2^{\mathrm{nil}}/(F_2^{\mathrm{nil}})''$. *Thus,* $\psi_\sigma^{\mathrm{ab}}$ *is the power series which describes the* σ*-action on* $(F_2^{\mathrm{nil}})'/(F_2^{\mathrm{nil}})''$ *universally.*

This power series was first treated in [Ih$_1$], including its connections with the l-power torsion points of Fermat Jacobians of l-power degree and Jacobi sums. We shall explain this briefly keeping in sight its generalized and refined version due to Anderson [A$_3$]. First, we make the following identification

$$\mathscr{F}^{\mathrm{ab}} = \mathrm{Hom}((\mathbb{Q}/\mathbb{Z})^2, \mu_\infty) \tag{6.4.5}$$

(μ_∞ : the group of roots of unity in $\mathbb{C}$), with $\underline{x}$ (resp. $\underline{y}$) corresponding to $(s,t) \to \exp(2\pi i s)$ (resp. $\exp(2\pi i t)$); $s,t \in \mathbb{Q}/\mathbb{Z}$. Note that the $G_{\mathbb{Q}}$-action on $\mathscr{F}^{\mathrm{ab}}$ is recovered from its action on μ_∞ via (6.4.5). Through (6.4.5) we may regard each element of $\hat{\mathbb{Z}}[[\mathscr{F}^{\mathrm{ab}}]]$ as a function

$$(\mathbb{Q}/\mathbb{Z})^2 \mapsto \hat{\mathbb{Z}} \otimes \mathbb{Q}(\mu_\infty) \ . \tag{6.4.6}$$

The above B_σ, considered as a function (6.4.6), is the *adelic beta function* $B_\sigma(s,t)$ ($s,t \in \mathbb{Q}/\mathbb{Z}$). It is strikingly analogous to the classical beta function [A$_3$]. Very roughly speaking, B_σ plays the role of (the classical beta)$^{\sigma-1}$.

Now, the abelian covering of $X = \mathbb{P}^1 - \{0,1,\infty\}$ corresponding to $\mathscr{F}^{\mathrm{ab}}/(N)$ ($N = 1,2,\ldots$) is the Fermat curve

$$Y_N : \{u^N + v^N = 1; \quad uv \neq 0\} \ .$$

The covering map is given by $(u,v) \to u^N$. The abelian coverings over Y_N are controlled by the group of torsion points of the Jacobian of $\overline{Y}_N$ (the compactification of Y_N). Thus the action of σ on $\mathscr{F}/\mathscr{F}''$ is directly tied to that on the group of these torsion points ([Ih$_1$, A$_{2,3}$]). As the Frobenius elements act on the latter group by multiplication of Jacobi sums (etc.), these collected together in terms of $\psi_\sigma^{\mathrm{ab}}$ (or B_σ) give a universal expression of Jacobi sums. For $\psi_\sigma^{\mathrm{ab}}$, it is:

Theorem [Ih$_1$]. *Let l be a prime number, $n \geq 1$, $\mathfrak{p}$ be a prime ideal of $\mathbb{Q}(\mu_{l^n})$ not lying above l, and $\sigma = \sigma_\mathfrak{p}$ be a Frobenius element of $\mathfrak{p}$. Then for any l^n-torsion points s,t of $\mathbb{Q}/\mathbb{Z}$ with $s, t,\ s+t \neq 0$, the special value of the l-component of $\psi_\sigma^{\mathrm{ab}}$ ($\in \mathbb{Z}_l[[\xi,\eta]]$) at $\xi = \exp(2\pi i s) - 1$, $\eta = \exp(2\pi i t) - 1$, is essentially the Jacobi sum (w.r.t. $\mathfrak{p}, l^n, s, t$).*

More generally, the values of $B_\sigma(s,t)$ ($s,t \in \mathbb{Q}/\mathbb{Z}$) are related to Jacobi sums and also Gauss sums [A$_3$]. Note that the two theorems of §6.3, §6.4, combined, give a direct connection between the circular units and the Jacobi sums.

Anderson [A$_3$, A$_4$] defined the hyperadelic gamma function

$$\Gamma_\sigma : \mathbb{Q}/\mathbb{Z} \longrightarrow \text{(some arithmetic ring)}$$

which factors B_σ just as γ_σ factors $\psi_\sigma^{\mathrm{ab}}$. It interpolates Gauss sums, and its "logarithmic derivative" can be given explicitly in terms of circular units so that it forms a bridge connecting Gauss sums and circular units. The last connection was partly established independently by a different method by Miki [Mi]. See Coleman [C$_3$] for connections with and applications to other aspects of cyclotomy.

6.5 $\psi_\sigma(\xi,\eta)$ and Higher Circular l-Units (Anderson-Ihara [AI$_{1,2}$]). This connection arises from a comparison of the tower of nilpotent coverings and of genus 0 coverings.

Call a finite subset $S \subset \mathbb{P}^1(\mathbb{C})$ *l-elementary*, if S is obtained from $S_0 = \{0,1,\infty\}$ by finite number of operations of the form $S \to S^{1/l}$ (all l-th roots), $S \to T_{a,b,c}(S)$. Here, $a,b,c \in S$ (distinct), and $T_{a,b,c}$ is the projective linear transformation of $\mathbb{P}^1$ that maps a,b,c to $0,1,\infty$ respectively.

Definition 6.5.1. $E^{(l)}$ is the subgroup of $\mathbb{C}^\times$ generated by the constituents of $S-\{0,\infty\}$, where S runs over all l-elementary subsets of $\mathbb{P}^1(\mathbb{C})$.

It is easy to see that elements of $E^{(l)}$ are l-units, i.e., element of $\overline{\mathbb{Q}}$ which, together with its reciprocal, is integral over $\mathbb{Z}[1/l]$. They are called *higher circular l-units*. The group $E^{(l)}$ contains such elements as

$$1-\zeta_n, \quad (1-(1-\zeta_n)^{1/l^n})^{1/l^n}, \dots ,$$

where $\zeta_n = \exp(2\pi i/l^n)$ $(n=1,2,\dots)$.

Theorem [A-I$_2$]. *Each coefficient of $\psi_\sigma(\xi,\eta)$ (mod l^n) can be expressed explicitly in terms of the σ-action on $E^{(l)}$.*

A key dialogue : "How to distinguish different l-th roots in $E^{(l)}$ intelligibly?" : "In terms of a natural structure of "forest" with vertices in $E^{(l)}$."

Corollary [A-I$_1$]

$$\mathbb{Q}^{(l)}(\infty) = \mathbb{Q}(E^{(l)}) \ .$$

We shall conclude this lecture with two additional open questions.

Question 6.5.2. i) *Is $\mathbb{Q}^{(l)}(\infty)$ the maximal pro-l extension over $\mathbb{Q}(\mu_{l^\infty})$ unramified outside l ?*

ii) *How big is $E^{(l)}$ and $E^{(l)} \cap \mathbb{Q}(\mu_{l^\infty})$?*

Remark. The non-commutative (resp. commutative) l-adic power series $\psi(\sigma) \in \mathbb{Z}_l \ll u,v \gg$ of [Ih$_2$] §1 (D) Ex. 1 (resp. $F_\sigma \in \mathbb{Z}_l[\![u,v]\!]$ of [Ih$_1$] §2) are related to the above ψ_σ (resp. $\psi_\sigma^{\mathrm{ab}}$) as follows. Write ξ,η instead of u,v respectively. Then $\psi(\sigma)$ and F_σ are the l-components of

$$f_\sigma(y,z)y^{\frac{1}{2}(\chi(\sigma)-1)} \cdot \psi_\sigma(\eta,\xi)$$

and

$$\frac{\underline{x}^{\chi(\sigma)}-1}{\underline{x}-1}\cdot\frac{\underline{y}^{\chi(\sigma)}-1}{\underline{y}-1}\cdot \underline{y}^{\frac{1}{2}(\chi(\sigma)-1)}\cdot\psi_\sigma^{\mathrm{ab}}(\xi,\eta) \ ,$$

respectively, where $z=(xy)^{-1}$. Note that $f_\sigma^{\mathrm{ab}}=1$ and that $\psi_\sigma^{\mathrm{ab}}$ is symmetric in ξ and η.

Added in proof. The author found later that Question 5.3.4 (i) has an affirmative answer. The proof is based on [Dr_2, Ih_5].

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