CHROMATIC POLYNOMIALS OF PLANE TRIANGULATIONS

1. Basic results. Throughout this survey, G will denote a multigraph with n vertices, m edges, c components and b blocks, and m' will denote the smallest number of edges whose deletion from G leaves a simple graph. The corresponding numbers for G_i will be denoted by n_i, m_i, c_i, b_i and m_i'.

Let $P(G,t)$ denote the number of different (proper vertex-) t-colourings of G. Anticipating a later result, we call $P(G,t)$ the **chromatic polynomial** of G. It was introduced by G. D. Birkhoff (1912), who proved many of the following basic results.

Proposition 1. (Examples.) Here T_n denotes an arbitrary tree with n vertices, F_n denotes an arbitrary forest with n vertices and c components, and R_n denotes the graph of an arbitrary triangulated polygon with n vertices: that is, a plane n-gon divided into triangles by $n-3$ noncrossing chords.

(a) $P(\bar{K}_n, t) = t^n$,

(b) $P(T_n, t) = t(t-1)^{n-1}$,

(c) $P(R_n, t) = t(t-1)(t-2)^{n-2}$,

(d) $P(K_n, t) = t(t-1)(t-2)\dots(t-n+1)$,

(e) $P(F_n, t) = t^c(t-1)^{n-c}$,

(f) $P(C_n, t) = (t-1)^n + (-1)^n(t-1)$

$$= (-1)^n t(t-1)[1+(1-t)+(1-t)^2+\dots+(1-t)^{n-2}].$$

Proposition 2. (a) If the components of G are $G_1,\dots,G_c$, then

$$P(G,t) = P(G_1,t)\dots P(G_c,t).$$

(b) If $G = G_1 \cup G_2$ where $G_1 \cap G_2 = K_r$, then

$$P(G,t) = \frac{P(G_1,t)\,P(G_2,t)}{P(K_r,t)}.$$

Proposition 3. (The **deletion-contraction formula**.) For each edge e of G,

$$P(G,t) = P(G-e,t) - P(G/e,t).$$

(When this result is rewritten in the form

$$P(G-e,t) = P(G,t) + P(G/e,t),$$

it is sometimes called the **addition-identification formula**.)

Proposition 4. If G has a loop, then $P(G,t)$ is identically zero. Otherwise, $P(G,t)$ is a monic polynomial in t of degree n, the smallest power of t with a nonzero coefficient is t^c, the coefficient of t^{n-1} is $m'-m$, and the powers of t between t^c and t^n all have nonzero coefficients, which alternate in sign. Thus if G is simple then

$$P(G,t) = t^n - mt^{n-1} + a_{n-2}t^{n-2} - \dots + (-1)^{n-c}a_c t^c,$$

where $a_c,\dots,a_{n-2}$ are all positive.

In view of this result, it makes sense to evaluate $P(G,t)$ at noninteger values of t, and to make statements like $P(G,\sqrt{5}) = -\frac{1}{2}(1+\sqrt{5})$. Of course, this does not mean there are $-\frac{1}{2}(1+\sqrt{5})$ ways of colouring G with $\sqrt{5}$ colours; it is purely a statement about the value of a certain polynomial function at a particular value of its argument. For the record, R. P. Stanley (1973) has found a combinatorial interpretation of $P(G, t)$ whenever t is a *negative* integer; in particular, $P(G, -1)$ is $(-1)^n$ times the number of ways in which one can direct all the edges of G without creating any directed circuits. As far as I know, nobody has discovered a combinatorial interpretation of $P(G, t)$ for any noninteger value of t.

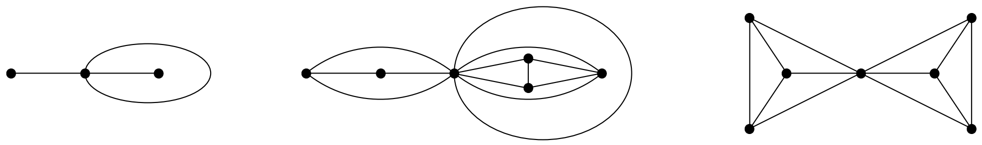
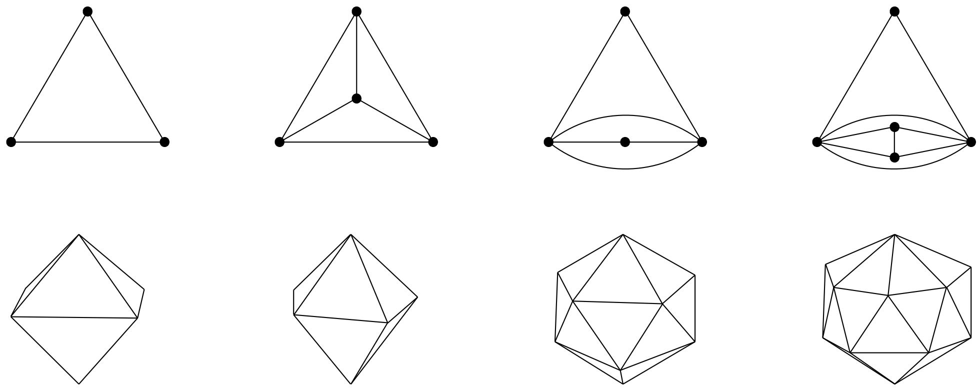


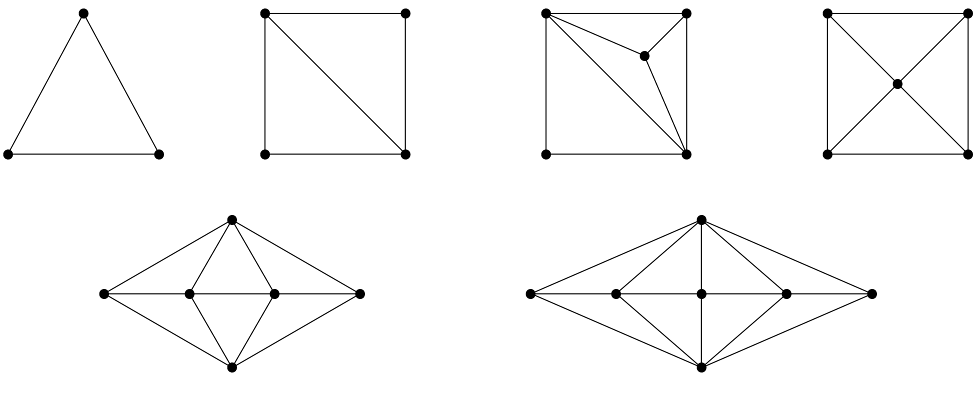
Fig. 1. Neither triangulations nor near-triangulations



$\bar{K}_2 + C_4$ $\bar{K}_2 + C_5$ Icosahedron

Octahedron

Fig. 2. Eight triangulations



$\bar{K}_2 + P_4$ $\bar{K}_2 + P_5$

Fig. 3. Six near-triangulations

2. Inequalities, zeros and plane triangulations. A loopless plane multigraph is a **triangulation** if every region is bounded by exactly three edges, and a **near-triangulation with k-face** if every region but one is bounded by exactly three edges and the exceptional region (usually drawn as the outside region) is bounded by a circuit of $k \geqslant 3$ edges. It is easy to see that triangulations and near-triangulations are 2-connected. A **separating circuit** in a plane graph is a circuit that has at least one vertex inside it and at least one vertex outside it.

For a positive integer t, G is t-colourable if and only if $P(G,t) \neq 0$. Thus it is natural to examine the distribution and nature of zeros of chromatic polynomials. The 4-colour theorem says that 4 is not a zero of $P(G,t)$ if G is a planar graph. It is easy to see that this is true for all planar graphs if and only if it is true for plane triangulations. And it turns out that the chromatic polynomials of plane triangulations have a more regular structure than the chromatic polynomials of planar graphs in general. Thus it is natural to concentrate on proving results about the chromatic polynomials of plane triangulations.

Incidentally, Woodall (1977) showed that the chromatic polynomials of complete bipartite graphs, whose largest integer zero is 1, can have arbitrarily large real zeros. Hence there is no general upper bound on the size of the largest real zero in terms of the largest integer zero—although there is such a bound for planar graphs (see Theorem 5(e) below).

Theorem 1.
(a) If $t < 0$, then $P(G,t)$ is nonzero with the sign of $(-1)^n$.
(b) At 0, $P(G,t)$ has a zero of multiplicity c (hence, a simple zero if G is connected).
(c) If $0 < t < 1$, then $P(G,t)$ is nonzero with the sign of $(-1)^{n-c}$.
(d) At 1, $P(G,t)$ has a zero of multiplicity b (hence, a simple zero if G is 2-connected).
(e) If $1 < t \leqslant \frac{32}{27}$, then $P(G,t)$ is nonzero with the sign of $(-1)^{n-c-b}$.

Parts (a)–(c) of Theorem 1 are due to W. T. Tutte (1974), part (d) to D. R. Woodall (1977) and (independently) E. G. Whitehead and L.-C. Zhao (1984), and part (e) to B. Jackson (1993). Jackson showed that the figure $\frac{32}{27}$ in (e) cannot be increased, and C. Thomassen (1997) went further and showed that the zeros of chromatic polynomials are dense in the interval $(\frac{32}{27}, \infty)$.

Even without the last sentence, it is easy to see that the pattern of Theorem 1 (a)–(d) cannot continue: it follows from these results that if G is any 2-connected bipartite graph with an odd number of vertices, then $P(G,t)$ is negative just to the right of 1, and positive at 2, and so it has a zero between 1 and 2. For example, we could take G to be $K_{2,3}$, which is also planar. There are also nonbipartite planar examples, such as . However, for near-triangulations of the plane, the pattern does continue a bit further:

Theorem 2. (G. D. Birkhoff and D. C. Lewis, 1946.) Let G be a plane near-triangulation.
(a) If $1 < t < 2$, then $P(G,t)$ is nonzero with the sign of $(-1)^n$.
(b) At 2, $P(G,t)$ has a zero of multiplicity at least $m'+1$, with equality if G is a triangulation; thus $P(G,t)$ has a simple zero at 2 if G is a 3-connected triangulation.

It seems possible to make the following conjectured extensions of Theorem 2; the graphs in part (a) need not be planar.

Conjecture. (a) (B. Jackson, 1993.) The conclusion of Theorem 2 (a) holds also for all 3-connected nonbipartite graphs (or, perhaps, for all 3-connected graphs that are not bipartite with an odd number of vertices?).
(b) The chromatic polynomial of a 3-connected nonbipartite planar graph has a simple zero at 2.

In connection with (b), there are nonplanar 3-connected graphs whose chromatic polynomials have multiple zeros at 2; it does not seem to be known whether there are examples that are more highly connected than this.

Theorem 1 (a)–(d) and Theorem 2 follow from the following two theorems. Write $A \geqslant_x B$ if A and B can be expressed as polynomials in x and, when this is done, each coefficient in A is at least as large as the corresponding coefficient in B.

Theorem 3. (D. R. Woodall, 1992a.) Let G be a simple graph and write $\gamma := m-n+c$ and $\mu := n-c-b$. Define $q(G,t)$ by

$$P(G,t) = (-1)^{\mu} t^{c} (t-1)^{b} q(G,t).$$

Then $q(G,t)$ is a polynomial in t and

$$q(G,t) \geqslant_s 1+\gamma s+\gamma s^2+\ldots+\gamma s^{\mu-1}+s^{\mu}$$

where $s := 1-t$. Thus $q(G,t) \geqslant 1$ if $t \leqslant 1$. (Equality holds if G is a forest, or is unicyclic, and for some other graphs with two or three circuits.)

Theorem 4. (G. D. Birkhoff and D. C. Lewis, 1946.) Let G be a near-triangulation with k-face F. Define $q(G,t)$ by

$$P(G,t) = (-1)^{n-3-m'} t(t-1)(t-2)^{m'+1} q(G,t).$$

Then $q(G,t)$ is a polynomial in t and

$$q(G,t) \geqslant_r r^{k-3}(1+r)^{n-k-m'}$$

where $r := 2-t$. Thus $q(G,t) \geqslant (2-t)^{k-3}(3-t)^{n-k-m'}$ if $t \leqslant 2$.

Theorem 5. Let G be a plane triangulation with n vertices and let $\tau := \frac{1}{2}(1+\sqrt{5}) = 1.6180339\ldots$, the golden ratio (see the next section).

(a) (D. R. Woodall, 1992b.) If $2 < t < 2.5466023\ldots$, then $q(G,t)$ (defined in Theorem 4) is positive and so $P(G,t)$ is nonzero with the sign of $(-1)^{n-m'-1}$; here $2.5466023\ldots$ is the unique real zero of the polynomial $t^3-9t^2+29t-32$, which is a factor of the chromatic polynomial of the octahedron $\bar{K}_2+C_4$ (see Fig. 2).
(b) (W. T. Tutte, 1970a.) $|P(G,\tau+1)| \leqslant \tau^{5-n}$.
(c) (W. T. Tutte, 1970b.) $P(G,\tau+2) = (\tau+2)\tau^{3n-10}P(G,\tau+1)^2$.
(d) (The Four-Colour Theorem.) $P(G,4) > 0$.
(e) (G. D. Birkhoff and D. C. Lewis, 1946.) If $t \geqslant 5$, then $P(G,t)$ and all its derivatives (up to the nth) are strictly positive.

Birkhoff and Lewis conjectured that (e) holds if $t \geqslant 4$, and there is overwhelming evidence to support this; if true, this would then follow for all planar graphs. If G is a 3-colourable (i.e., Eulerian) plane triangulation then it seems very likely that (e) holds whenever $t \geqslant 3$. But (a) is not even true for all near-triangulations: using the addition-identification formula one sees that the near-triangulation $\bar{K}_2+P_4$ in Fig. 3 has chromatic polynomial

$$t(t-1)(t-2)(t-3)^3+t(t-1)(t-2)^3 = t(t-1)(t-2)(t^3-8t^2+23t-23),$$

which has a zero at about $2.4301597\ldots$. By analogy with (a), which I conjectured in 1977, I also conjectured that $P(G,t) \neq 0$ if $2.6778146\ldots < t < 3$, where $2.6778146\ldots$ is a zero of the polynomial $t^3-6t^2+30t-35$, which is a factor of the chromatic polynomial of the pentagonal double pyramid $\bar{K}_2+C_5$ (see Fig. 2). It now seems that this is false; indeed, it seems that the zeros of chromatic polynomials of plane triangulations are very probably dense in this interval. However, all known (4-connected) counterexamples have vertices with degree less than 5, and I have made the following replacement conjecture.

Conjecture. (D. R. Woodall, 2001.) The chromatic polynomial of a 4-connected plane triangulation with minimum degree 5 cannot have a zero (a) between 2 and 2.6180317… or (b) between 2.6181972… and 3.

The noninteger appearing in (b) is a zero of the chromatic polynomial of the icosahedron, and the one in (a) is a zero of the chromatic polynomial of the 16-vertex polyhedron shown to the right of the icosahedron in Fig. 2. Part (b) of the conjecture is particularly appealing by analogy with Theorem 5 (a), involving the octahedron. A proof of (b) (with the expected sign) would also establish that a 4-connected triangulation with minimum degree 5 *does* have a zero between 2.5466023… and 2.6181972…, ‘close’ to $\tau+1 = 2.6180339\ldots$ where the polynomial is small by Theorem 5 (b). (The truth of (a) and (b) together would prove the existence of a zero even closer to $\tau+1$.)

3. The Beraha numbers, and other developments. In this section we will consider only plane triangulations. Berman and Tutte (1969) used a computer to plot the zeros of the chromatic polynomials of hundreds of plane triangulations. They discovered that there always seems to be a real zero at about 2.61803..., and there is often another near 3.246..., the closeness of approximation to these values seeming to increase as the number of vertices in the triangulation increases. Since the well known golden ratio has the value $\tau = 1.61803\ldots$, Berman and Tutte suggested that the ‘true value’ of 2.61803... is the larger root of the equation $t^2-3t+1 = 0$, which is $\frac{1}{2}(3+\sqrt{5})$, or $\tau+1$. They referred to the zero that usually occurs near this value as the ‘golden root’ of the chromatic polynomial. D. W. Hall suggested that the ‘true value’ of 3.246... might be the largest root of the equation $t^3-5t^2+6t-1 = 0$, since this polynomial features prominently in various calculations involving chromatic polynomials. Tutte suggested that the zero that often occurs near this value should be called the ‘silver root’ of the chromatic polynomial.

n	b_n	n	b_n
2	0	7	3.2469796…
3	1	8	3.4142135… $(= 2+\sqrt{2})$
4	2	9	3.5320888…
5	2.6180339… $(= \tau+1)$	10	3.6180339… $(= \tau+2)$
6	3	…	…

S. Beraha pointed out that $\tau+1 = 2+2\cos 2\pi/5$, and the largest root of the above cubic equation is $2+2\cos 2\pi/7$. This suggests that one should look at what are now called the **Beraha numbers**, defined by the equation $b_n := 2+2\cos 2\pi/n$ (see table). For $n = 5$ and $n = 7$ we get the two values already referred to. For $n = 2$, 3 and 4, we get simply 0, 1 and 2, where there are always zeros, and $n = 6$ gives 3, where there is a zero unless the triangulation is 3-colourable, which happens if and only if it is Eulerian. For larger values of n we get a sequence of numbers between 3 and 4, and $\lim_{n\to\infty} b_n = 4$. This last fact may well be significant: it suggests that there may be some way of using the Beraha numbers in order to prove the four-colour theorem.

In fact, I know of only two relevant theorems about Beraha numbers, both proved by Tutte in 1970. The first, Theorem 5 (b), says that $|P(G,b_5)| \leqslant \tau^{5-n}$, which is very small. This does not prove that there is a zero nearby—indeed, there may not be, if G is not 4-connected (or even, in one case, if it is)—but it is not surprising that there usually is. The second, Theorem 5 (c), relates $P(G,b_{10})$ to $P(G,b_5)$. It does not show that $P(G,b_{10})$ is small; what it does show, much more interestingly, is that it is always *positive*. Unfortunately, there are chromatic polynomials of plane triangulations that, although positive at b_{10} and 4 (by the 4-colour theorem), are negative at points in between, and it is now known that there are no other Beraha numbers at which the chromatic polynomial is always positive; this follows from work of G. F. Royle (2005), who has shown that there are plane triangulations whose chromatic polynomials have real zeros arbitrarily close to 4. The Beraha numbers were a very exciting discovery, but nobody has been able to see quite how to make use of them.

Another idea that looks suggestive, but has not yet been tested, is to transform the chromatic polynomial. Put $u := t-3$ and $x := u^{-1}$. If G is a plane triangulation, define $Q(G,u)$ and $R(G,x)$ by the equations

$$\begin{aligned} P(G,t) &= t(t-1)(t-2)(t-3)\,Q(G,t-3) \\ &= (u+3)(u+2)(u+1)uQ(G,u) \end{aligned}$$

and

$$R(G,x) = u^{4-n}Q(G,u),$$

so that

$$P(G,t) = t(t-1)(t-2)(t-3)^{n-3}R(G,x).$$

$Q(G,u)$ is called the **Q-chromial** of G. It is a polynomial unless G is Eulerian, in which case G is 3-colourable and so $(t-3)$ is not a factor of $P(G,t)$ and $Q(G,u)$ has a term in u^{-1}. The coefficients in $Q(G,u)$ are usually much smaller than those in $P(G,t)$, and so the published tables tend to be of Q-chromials rather than the chromatic polynomials themselves. $R(G,x)$, however, is always a polynomial. Its advantages over $P(G,t)$ are threefold.

Firstly, the coefficients of $R(G,x)$ are the same as those of $Q(G,u)$ but in the reverse order, and so the polynomials $R(G,x)$ can be read off from the published tables of Q-chromials. Here are some small examples.

	n	$Q(G,u)$	$R(G,x)$
K_4	4	1	1
octahedron	6	$u^2+0u+2+u^{-1}$	$1+0x+2x^2+x^3$
$\bar{K}_2+C_5$	7	u^3+0u^2+3u+1	$1+0x+3x^2+x^3$
	8	$u^4+0u^3+4u^2-u-1$	$1+0x+4x^2-x^3-x^4$
	8	$u^4+0u^3+4u^2+3u+3+u^{-1}$	$1+0x+4x^2+3x^3+3x^4+x^5$

Secondly, some equations look simpler for $R(G,x)$ than for $P(G,t)$. For example, if G' is obtained from G by inserting a new vertex of degree 3 in a triangular face, then $P(G',t) = (t-3)P(G,t)$, but $R(G',x) = R(G,x)$. And Tutte's 'golden identity' (Theorem 5 (c)) takes the particularly simple form $R(G,\tau) = R(G,-\tau^2)^2$.

Thirdly, $R(G,x)$ unifies some of the known results about $P(G,t)$. Note the following table of corresponding values.

t:	3	$\tau+1 = \tau^2$	$\mathbf{2\frac{1}{2}}$	**2**	1	0	-1	-2	$\boldsymbol{\pm\infty}$	6	**5**	**4**	$\tau+2$	3
x:	$\pm\infty$	$-\tau^2$	$\mathbf{-2}$	$\mathbf{-1}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$	$-\frac{1}{5}$	**0**	$\frac{1}{3}$	$\mathbf{\frac{1}{2}}$	**1**	τ	$\pm\infty$

Theorem 6. If G is a plane triangulation then $R(G,x) > 0$ when $-1 < x \leqslant \frac{1}{2}$, and also when $-2 \leqslant x \leqslant -1$ if G is simple.

Proof. By Theorem 5(e), if $t \geqslant 5$, that is, $0 < x \leqslant \frac{1}{2}$, then $P(G,t) > 0$, and so $R(G,x) > 0$. $R(G,0) = 1$ since $R(G,x)$ always has constant term 1. By Theorem 4, if $t \leqslant 2$, that is, $-1 \leqslant x < 0$, then $R(G,x) = (-1)^{n-3-m'}(t-2)^{m'}(t-3)^{3-n}q(G,t)$ where $q(G,t) > 0$. Thus $R(G,x) > 0$ if $t < 2$, and also if $t = 2$ when G is simple ($m' = 0$). Finally, by Theorem 5 (a), if $2 < t \leqslant 2\frac{1}{2}$, that is, $-2 \leqslant x < -1$, then the same inequality holds, and so again $R(G,x) > 0$. ☐

The conjecture of Birkhoff and Lewis mentioned after Theorem 5 can be tied in with Theorem 6 to give:

Conjecture. If G is a simple plane triangulation then $R(G,x) > 0$ whenever $-2 \leqslant x \leqslant 1$.

EXERCISES

1 Prove the formulae

$$\begin{aligned} P(C_n,t) &= (t-1)^n + (-1)^n(t-1) \\ &= (-1)^n t(t-1)[1+(1-t)+(1-t)^2+\ldots+(1-t)^{n-2}] \end{aligned}$$

in Proposition 1.

2 Calculate the chromatic polynomials of (at least) the triangulations $\bar{K}_2+C_4$ and $\bar{K}_2+C_5$ in Fig. 2 and the last two near-triangulations in Fig. 3.

3 Prove Proposition 2(b), that if $G = G_1 \cup G_2$ where $G_1 \cap G_2 = K_r$, then

$$P(G,t) = \frac{P(G_1,t)P(G_2,t)}{P(K_r,t)}.$$

Would the analogous result be true with $\bar{K}_r$ in place of K_r?

4 Prove Proposition 4, that if G is simple then

$$P(G,t) = t^n - mt^{n-1} + a_{n-2}t^{n-2} - \ldots + (-1)^{n-c}a_c t^c,$$

where $a_c,\ldots,a_{n-2}$ are all positive.

5 Prove that, for any graph G and any complex number t, $|P(G,t)| \leqslant |t|^{n-m}(|t|+1)^m$.

6 Prove that, if G is a simple graph and $t \leqslant 1$, then

$$|P(G,t)| \geqslant |t^c(t-1)^{n-c-1}|(m-n+c+1-t) \geqslant |t^c(t-1)^{n-c}|.$$

(Hint: Prove the result directly for forests. For other graphs, apply the deletion-contraction formula to an edge in a circuit.)

7 Prove Theorem 1 parts (a)–(c) directly, without using Theorem 3. (Hint: Use the fact that the chromatic polynomial is multiplicative over components. Prove the results directly for trees, proving also that the coefficient of t is nonzero and has the sign of $(-1)^{n-1}$. For other connected graphs, apply the deletion-contraction formula to an edge in a circuit.)

8 Prove Theorem 1(d) directly by the following method. Split the graph apart at any cut-vertex; for 2-connected graphs, use the deletion-contraction formula to prove that the derivative of $P(G,t)$ at 1 is nonzero and has the sign of $(-1)^n$.

9 Let $e = v_1v_2$ be an edge in a 2-connected graph G.
(a) Prove that $G-e$ is 2-connected if and only if there is a circuit in $G-e$ containing v_1 and v_2.
(b) Prove that G/e is 2-connected if and only if v_2 is not a cut-vertex of $G-v_1$.
(c) Deduce that, if $G \neq K_3$, then G contains an edge e such that G/e is 2-connected.

10 Prove that Tutte's 'golden identity' (Theorem 5(c)) is the same as $R(G,\tau) = R(G,-\tau^2)^2$. (Hint: $\tau^2-\tau-1 = 0$, so $\tau+1 = \tau^2$, $\tau-1 = \tau^{-1}$ and $\tau-2 = -\tau^{-2}$.)

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