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Robust Replication of Default Contingent Claims

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Abstract

We show how to replicate the payoffs to a class of default-contingent claims by taking static positions in a continuum of credit default swaps (CDS) of different maturities. Although we assume deterministic interest rates and a constant recovery rate on the CDS, the replication is otherwise robust in that we make no assumptions on the process triggering default. In particular, we can robustly replicate the payoff to an Arrow Debreu security paying one dollar at a fixed date if a given entity survives to that date. As a consequence, we can determine risk-neutral survival probabilities from an arbitrarily given yield curve and CDS curve.

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1 Introduction

The volume in credit default swaps continues to grow. According to the August 31, 2006 issue of the Wall Street Journal, the underlying notional now exceeds $17 trillion. Much effort has been spent developing models that can determine what the CDS spread of a given maturity should be. For example, one might be interested in determining what the 5 year CDS spread should be given some model for default frequency, some assessment of market risk aversion, and the term structure of interest rates. A second line of research consists of developing the entire CDS spread curve when one or more points along the curve are given. For example, one might try to infer all CDS spreads for 0 to 10 years, given just the 5 year quote, the term structure of interest rates, and some assumptions on smoothness.

In this paper, we explore a third line of research. We take the entire CDS curve as given along with the term structure of interest rates. We do not use any other information. We then try to extract unique arbitrage-free prices of a set of default-contingent claims, making as few probabilistic restrictions as possible. The set of default-contingent claims that we focus on include any claim which promises to make fixed payments over time according to a pre-set schedule. Even though the target claims have payoffs closely related to those of a CDS, we find that some probabilistic restrictions are needed. For example, the payoffs to just the premium leg of a CDS is in our class. Yet in order to extract the initial value of the premium leg of a CDS, we will be assuming deterministic interest rates and constant recovery rates in this paper.

A second objective of this paper is to indicate the positions needed to replicate the payoffs of the target claims. To the extent that institutions wish to offer the target claims that we focus on, the proper hedging of the ensuing liabilities requires exact knowledge of positions required in the hedging instruments. The hedging instruments that we use are a cash account and CDS of all maturities. The positions in the CDS are static. To hedge the payoff of a target claim in our class, the hedger initially assumes a position in zero cost CDS of all maturities. The hedger also sets up a bank account whose initial balance matches the theoretical value of the target claim. As calendar time evolves, CDS values deviate from zero and as a result, the bank account balance deviates from the theoretical value of the target claim. Prior to any default by the reference entity, the positions in the CDS and in the target claim require cash inflows or outflows which are financed out of the bank account. The static nature of our proposed hedge implies that the initial CDS positions are never adjusted afterwards. This feature cuts down on transaction and monitoring costs, but requires initial liquidity in more CDS maturities than a more standard model-based dynamic hedge.

While we assume deterministic interest rates and a constant recovery rate on the CDS, our static hedging strategy is otherwise robust in that we make no assumptions on the process triggering default. In particular, our results are consistent with both reduced form and structural models of default. We solve the replication problem by deriving a new terminal value problem which determines the survival-contingent bank balance when replicating a claim. This problem consists of a simple linear second order differential equation coupled with two terminal conditions. The solution of this problem also determines the CDS holdings.

A target payoff of particular interest for us is a defaultable annuity. This claim pays one dollar per year continuously until the earlier of a random default time and a fixed maturity date. Our interest in the pricing and hedging of this claim arises for at least three reasons. First, the product of the initial value of a defaultable annuity and the known co-terminal CDS rate gives the initial value of the premium leg of a CDS. As a result, the initial value of a defaultable annuity gives the DV01 of a CDS, i.e. the impact of a one basis point shift in the CDS spread on the value of the premium leg of a CDS. Thus, the initial value of a defaultable annuity can be interpreted as this risk measure and for brevity, we will frequently refer to it as such. Second, suppose for each fixed expiry date, a trader tries to keep the net notional in the protection legs of its CDS positions as close to zero as possible. This strategy often arises for a market maker in CDS or for an active trader betting on CDS spread levels, rather than on the default time. When the net notional in the protection leg is zero, the market value of the CDS position is then given by the value of the net notional in the co-terminal premium legs. As the CDS rate fluctuates over time, the net fixed payment is non-zero. Thus, for such a CDS trader, the mark-to-market of an accumulated CDS position amounts to determining the value of a position in defaultable annuities from the current CDS spread curve. Third, we show that the solution of the replication problem for a defaultable annuity can be used to determine the portfolio weights when replicating the payoff of an arbitrary default-contingent claim. In particular, we show that the survival-contingent bank balance that arises when replicating the payoff of a defaultable annuity is a Green's function, or fundamental solution, for the determination of the survival-contingent bank balance when replicating other default-contingent claims in our class. We also show that the required CDS positions become determined once this bank balance is known.

Our identification of the survival-contingent bank balance as a fundamental solution further implies that the function relating the initial value of a defaultable annuity to its maturity solves an initial value problem. This initial value problem also consists of a simple linear second order differential equation, but now coupled with two initial conditions. Our initial value problem can be numerically solved to efficiently calculate the term structure of DVO1's from the given CDS spread curve and yield curve. Once one has solved for this term structure, other financially relevant quantities are quickly determined. For example, the present values of either leg of a CDS is obtained by simply multiplying the DV01 by the known CDS rate. We show that one can also quickly determine the risk-neutral probability density function of the default time.

Thus a contribution of this paper is to show that the replication and pricing problems for an arbitrary default-contingent claim can be expressed in terms of the solution of the corresponding problems for a defaultable annuity. Both problems just require solving a simple linear second order differential equation coupled with two boundary conditions. In general, both evolution equations must be solved numerically, but we indicate important special cases in which they can be solved in closed form.

The remainder of the paper is organized as follows. The next section lays out our assumptions and notation. The subsequent section indicates how the survival-contingent bank balance and the CDS hedge weights are determined by a terminal value problem. The section afterwards gives the financial interpretation of the Green's function for this problem. The following section indicates how this interpretation can be used to generate the term structure of DV01's by solving an initial value problem. The next section gives two expressions for the risk-neutral probability density function of the default time and shows that the initial value problem arises by equating them. The next two sections focus on the hedging and pricing of a unit recovery claim and a survival claim respectively. The penultimate section presents closed form solutions to our evolution problems that arise when either the yield curve or the CDS curve are flat. We conclude in the final section with a summary and some further possible extensions.

2 Assumptions and Notation

We consider a finite time horizon $t \in [0, T]$ where time $t = 0$ is the valuation time and time $t = T$ is a positive finite constant. We assume deterministic interest rates for simplicity and we let $r(t), t \geq 0$ be the spot interest rate at time t. One dollar invested in the money market account at time 0 grows to $e^{\int_0^t r(u)du}$ dollars with certainty at time t. To apply the present deterministic interest rate theory, one can equate the future spot rate $r(t)$ to the forward interest rate $f_0(t)$. The fact that the argument t of $r(t)$ is continuous is tantamount to assuming that there exists a complete term structure of default-free bonds out to T.

Besides the default-free bonds, we assume that credit default swaps for a given reference name also trade. As is well known, a CDS costs zero to enter and provides protection against the loss arising from a default over a specified time period. In return for this protection, the long side of the CDS makes fixed payments periodically. For simplicity, we assume that the payments on the premium leg of the CDS occur continuously over time, rather than discretely. In practice, the fixed payments are made quarterly, but an accrued interest payment is made whenever default occurs between one of the quarterly payment dates. As a result, the continuous payment assumption has little impact on the analysis, while permitting the use of differential calculus rather than the difference calculus.

Throughout the paper, we assume that the recovery rate on the insured bond is a known constant. In reality, recovery rates are random and the impact of covariance between the recovery rate and default incidence is the major concern of an emerging line of research. Our primary justification for the constant recovery rate assumption is simplicity. The difference between unity and the recovery rate is called the loss given default, denoted by $L \in (0, 1]$. If default occurs at some random time τ, then all CDS maturing after τ have value L at time τ. Once the liquidating payment L is made at τ, the ex-dividend CDS have zero value afterwards.

We assume that at time 0, an investor can also take positions of any size in CDS of any maturity up to a fixed horizon T. In practice, only several discrete maturities are available, but market makers are willing to quote on any maturity out to some fixed horizon. Our assumption that a continuum of maturities are available for trading again facilitates the use of differential calculus rather than the difference calculus, while having little qualititative impact on the conclusions. We let $S_0(u)$ denote the initially observed CDS spread for all maturities $u \in [0, T]$.

We consider the problem of replicating a claim which promises to pay an infinitesimal cash flow $c(t)dt$ continuously for all $t \in [0, T]$. For example, the promised coupon rate might be $c(t) = 1_{t \in [0,u]}$, for $u \in [0, T]$, i.e. one dollar per year until the maturity date $u \leq T$. For ease of exposition, we do not allow discrete cash flows to be promised at any time, but we will show how to extend our analysis so as to deal with such claims.

We allow for the possibility that the issuer of the target claim may default at any time. If default occurs at some random time $\tau \in [0,T]$, then the promised cash flows stop and the claim is worth its recovery value $R(\tau)$, where the recovery function $R(t), t \in [0,T]$ is known at time 0. The specification of the promised coupon rate $c(t)$ and the recovery function $R(t)$ on the domain $t \in [0,T]$ completely determines the payoffs to the target claim. When we refer to arbitrary default-contingent claims in the sequel, we have in mind the ensemble of all claims generated by any choice of continuous functions $c(t)$ and $R(t)$.

For example, if $c(t) = 1_{t\in(0,u)}, u \in [0,T]$ and $R(t) = 0$ on $t \in (0,T)$, then the target claim is a *defaultable annuity* maturing at u, i.e. a claim that pays one dollar per year until the earlier of the default time τ and its fixed maturity date $u \in [0,T]$. If default occurs at some random time τ before u, then the defaultable annuity becomes worthless at τ. Thus, the actual cash flow from the defaultable annuity at each instant $t \in (0,T)$ is $1_{t\in(0,\tau\wedge u)}dt$.

One reason for our focus on the replication of defaultable annuities is that the payoffs from the premium leg of a CDS are proportional to the payoff from a defaultable annuity. In fact, a static position in $S_0(T)$ units of the defaultable annuity has the same payoffs as the premium leg of a CDS of maturity T, when payments are made continuously. It follows that the initial value of this premium leg is just the product of the initial value of the defaultable annuity and the initial CDS spread.

The payments arising from the protection leg of a T maturity CDS also arise in our framework by setting $c(t) = 0$ and $R(t) = L \in [0,1)$ for $t \in [0,T]$. By the definition of the CDS rate, the initial value of this protection leg matches the initial value of the premium leg. Thus, the payoffs from a defaultable annuity, the premium leg, and the protection leg all arise in our framework. While it might at first appear that at least one of these example payoffs arises from a position in just a CDS of maturity T, this appearance is misleading. In fact, the results of the next section imply that the replication of each of the three payoffs requires positions in cash and CDS of all maturities *up to* T.

3 Replication Via Backward Equation

Recall that in our setting, the payoffs from an arbitrary default-contingent claim with maturity T are determined by the promised coupon rate $c(t), t \in [0,T]$ and a recovery function $R(t), t \in [0,T]$. The actual cash flow received at each instant of time $t \in [0,T]$ is random and given by $[c(t)1(t > \tau) + R(t)\delta(t-\tau)]dt$, where τ is the random default time. In this section, we focus on determining the replicating strategy in cash and CDS for this target claim. Let $M(t)$ be the amount of money kept in the money market account at time $t \in [0,T]$, given no default prior to t. For brevity, we refer to $M(t)$ as the survival-contingent bank balance as of the future time t. Let $Q(u)du$ be the notional amount in CDS of maturity $u \in [0,T]$ that the investor *writes* at time 0. If $Q(u)$ is positive for some maturity u, then the investor is selling protection at that maturity. Our sign convention results in the two controls M and Q both being nonnegative when the target functions c and R are both nonnegative. Accordingly, the language used to convey intuition for our mathematical results will accord with all quantities being nonnegative, but the mathematical results extend to all quantities being real.

We interpret the controls M and Q as generalized functions. This interpretation allows impulse controls to be applied at a fixed time such as T when a claim matures. We now show that the two controls $M(t)$ and $Q(t)$ are uniquely determined by two equations. We refer to the first of these equations as the *recovery matching condition*:

$$M(t) - L\int_t^T Q(u)du = R(t), \qquad t \in [0,T]. \tag{1}$$

If default occurs at a candidate future time t, then (1) says that the the two controls $M(\cdot)$ and $Q(\cdot)$ are initially chosen so that the value at t in the money market account and from all of the outstanding CDS sum to the target recovery value $R(t)$. Differentiating (1) w.r.t. t implies:

$$M'(t) = R'(t) - LQ(t), \qquad t \in [0,T]. \tag{2}$$

Thus, the change in the survival-contingent bank balance increases due to any increase in the required recovery amount and decreases as the liability from the expiring CDS rolls off. Solving (2) for $Q(t)$ implies:

$$Q(t) = \frac{1}{L}[R'(t) - M'(t)], \qquad t \in [0,T]. \tag{3}$$

Since the loss given default L and the recovery function $R(t), t \in [0,T]$ of the target claim are both given, (3) indicates that the rate $Q(u)$ at which CDS are written at each maturity u is determined once one first determines the survival-contingent bank balance $M(t)$.

The second equation governing the two controls is the *self-financing condition*:

$$M'(t) = r(t)M(t) + \int_t^T S_0(u)Q(u)du - c(t), \qquad t \in [0,T]. \tag{4}$$

In words, the change in value of the survival-contingent bank balance arises from interest earned on the previous balance, premium inflows due to all of the outstanding short positions in CDS, less the withdrawal required to finance the promised coupon rate of the target claim. Differentiating (4) w.r.t. t implies:

$$M''(t) = r(t)M'(t) + r'(t)M(t) - S_0(t)Q(t) - c'(t), \qquad t \in [0,T]. \tag{5}$$

Substituting (3) in (5) and re-arranging implies that $M(t)$ solves the following linear second order ordinary differential equation (ODE):

$$\mathcal{L}M \equiv M''(t) - \left[r(t) + \frac{S_0(t)}{L}\right] M'(t) - r'(t)M(t) = f(t), \qquad t \in [0,T], \tag{6}$$

where the forcing function $f(t)$ is given by:

$$f(t) \equiv -c'(t) - \frac{S_0(t)}{L}R'(t), \qquad t \in [0,T]. \tag{7}$$

For default-contingent claims maturing at T, a unique solution for M arises on the domain $t \in (0,T)$ by imposing two terminal conditions:

$$M(T) = 0, \tag{8}$$

and

$$\lim_{t \uparrow T} M'(t) = -c(T). \tag{9}$$

The homogeneous terminal condition (8) arises due to our interest only in claims that have continuous cash flows over time. The left slope condition (9) arises from letting $t \uparrow T$ in (4) and substituting in (8). For arbitrarily given yield curves and CDS curves, one must numerically solve the terminal value problem (6) to (9) for $M(t), t \in [0, T)$ using finite differences. However, we later show that the problem can be solved in closed form if either the forward rate curve or the CDS curve is initially flat.

Whether it is obtained by finite differences or a formula, the scalar $M(0)$ has the important financial interpretation as the initial value of the target claim with maturity T, because all of the CDS used in the replicating portfolio are initially costless. Furthermore, (3) gives the rate at which CDS are initially written for each maturity $t \in (0, T)$. Thus, we have a complete solution to the problem of replicating a target claim of maturity T using a money market account and CDS of all maturities up to T.

To illustrate our solution procedure, consider the terminal value problem that arises when replicating a defaultable annuity of maturity T. Let $M_a(t; T)$ denote the survival-contingent bank balance at $t \in [0, T]$ when replicating this claim. Setting $c(t) = 1_{t \in (0,T)}$ and $R(t) \equiv 0$ in (6) to (9) implies that $M_a(t; T)$ solves the homogeneous ODE:

$$\mathcal{L} M_a(t; T) = 0, \qquad t \in (0, T), \tag{10}$$

subject to the terminal conditions:

$$M_a(T; T) = 0, \tag{11}$$

and

$$\lim_{t \uparrow T} \frac{\partial}{\partial t} M_a(t; T) = -1. \tag{12}$$

Since T is fixed, $M_a(0; T)$ is the scalar indicating the initial value of the defaultable annuity maturing at T. From (3), the rate at which CDS are initially written is:

$$Q_a(t; T) \equiv -\frac{\partial}{\partial t} \frac{M_a(t; T)}{L} \tag{13}$$

for each maturity $t \in [0, T]$. So long as interest rates are nonnegative, it can be shown that $\frac{\partial}{\partial t} M_a(t; T) \leq 0$ for each $t \in [0, T]$. As a result, we have $Q_a(t; T) \geq 0$ for each $t \in [0, T]$, which explains our sign convention.

4 Green's Function

We return to the general problem of replicating a claim with promised coupon rate $c(t)$ and recovery function $R(t)$ entering through the forcing term $f(t)$ defined in (7). It is well known that the general solution can be expressed in terms of the *Green's function* $g(t; u)$, defined as a function solving:

$$\mathcal{L} g(t; u) = \delta(t - u), \qquad t \in [0, T], \tag{14}$$

where $u \in [0, T]$. Since $\mathcal{L}$ is a second order differential operator, we also need two boundary conditions to uniquely determine a Green's function. For this purpose, we choose:

$$g(T; u) = 0, \text{ and } \lim_{t \uparrow T} \frac{\partial}{\partial t} g(t; u) = 0. \tag{15}$$

These homogeneous terminal conditions cause $g(t; u)$ to vanish for $t \in (u, T)$.

We note that the ODE (14) arises from (6) by setting $c(t) = 1_{t \in (0,u)}$ and $R(t) \equiv 0$ for $t \in [0, T]$. Thus, the Green's function $g(t; u)$ is connected with the replication of a defaultable annuity maturing at u. In fact, a solution to the terminal value problem consisting of the ODE (14) and the terminal conditions (15) is:

$$g(t; u) = \begin{cases} M_a(t; u) & \text{if } t \in [0, u] \\ 0 & \text{if } t \in (u, T). \end{cases} \tag{16}$$

Thus, for fixed u, $g(t; u)$ is the survival-contingent bank balance at $t \in [0, u]$ when replicating the u maturity defaultable annuity. Like M, the function g is in general only obtained numerically, but it can be obtained in closed form in special cases. In contrast to M, we seek the dependence of $g(t; u)$ on u rather than t. Once $g(t; u)$ is known as a function of u for some t, then the survival-contingent bank balance which arises when replicating an arbitrary default-contingent claim can be expressed in terms of it:

$$M(t) = \int_t^T f(u) g(t; u) du, \qquad t \in [0, T]. \tag{17}$$

It follows from (3) and (11) that the rate at which CDS are written at each maturity can also be expressed in terms of the Green's function:

$$Q(t) = \frac{1}{L} \left[R'(t) - \int_t^T f(u) \frac{\partial}{\partial t} g(t; u) du \right], \qquad t \in [0, T]. \tag{18}$$

Recall that all of the CDS used in the replicating portfolio are initially costless. Hence, evaluating (17) at $t = 0$ relates the initial value of the target claim with maturity T to the Green's function $g(0; u)$.

$$M(0) = \int_0^T f(u) g(0; u) du, \qquad t \in [0, T]. \tag{19}$$

Thus, the valuation of a target claim reduces to determining $g(0; u)$ for all $u \in [0, T]$. Given the above definition of g, it might seem that the ODE (14) has to be numerically solved once for each level of u. Fortunately, the next section shows that another ODE governs g when it is considered as a function of u. As a result, only a single ODE needs to be numerically solved to obtain the arbitrage-free price of an arbitrary default-contingent claim.

5 Pricing via the Forward Equation

The last section showed that $g(t; u)$ is the survival-contingent bank balance at $t \in [0, u]$ when replicating the u maturity defaultable annuity. Thus, for each fixed u, the number $g(0; u)$ is the initial value of the defaultable annuity of maturity $u \in [0,T]$. This section shows that this latter observation can be used to efficiently determine the whole term structure of initial values of a defaultable annuity, $g(0;u), u \in [0,T]$, from the given initial CDS curve $S_0(u), u \in [0,T]$ and the given initial forward rate curve $r(u), u \in [0,T]$.

The appendix proves the well known result (see eg. Stakgold[2] page 200), that g satisfies the adjoint ODE when considered as a function of its second variable u, i.e.:

$$\mathcal{L}^* g(t;u) = \delta(t-u), \tag{20}$$

where $\mathcal{L}^*$ is the following linear differential operator:

$$\mathcal{L}^* \equiv \mathcal{D}_u^2 + \left[r(u) + \frac{S_0(u)}{L} \right] \mathcal{D}_u^1 + \frac{S_0'(u)}{L} \mathcal{D}_u^0, \tag{21}$$

and $\mathcal{D}_u$ denotes differentiation w.r.t. u. The domain for the inhomogeneous ODE (20) is the square $(t,u) \in [0,T] \times [0,T]$.

Equations (16) and (20) imply that the survival-contingent bank balance solves the homogeneous ODE:

$$\mathcal{D}_u^2 M_a(t;u) + \left[r(u) + \frac{S_0(u)}{L} \right] \mathcal{D}_u M_a'(t;u) + \frac{S_0'(u)}{L} M_a(t;u) = 0, \tag{22}$$

on the domain $u \in (t,T)$ and imposing the initial conditions:

$$M_a(t;t) = 0, \tag{23}$$

and:

$$\lim_{u \downarrow t} \frac{\partial}{\partial u} M_a(t;u) = 1. \tag{24}$$

In general, the initial value problem (22) to (24) must be solved numerically using finite differences. However, we later show that the problem can be solved in closed form if either the forward rate curve or the CDS curve is initially flat.

Recall that for a finite-lived claim, the ODE (6) was solved by propagating the terminal conditions (8) and (9) backwards in calendar time t from the terminal time T. In contrast, (22) is solved by propagating the initial conditions (23) and (24) forward in the maturity date u from time t. As a result, we refer to (6) as the backward ODE and we refer to (22) as the forward ODE. The backward ODE governs the survival-contingent bank balance as a function of t and it applies to a wide class of payoffs. Hence its solution at time $t = 0$ yields the initial price of a wide class of claims. In contrast, the forward ODE governs the survival-contingent bank balance as a function of u and it applies only when replicating a defaultable annuity. The reason for the greater scope of the backward ODE is that the application of any linear operator in the parameter u leaves the backward equation unchanged, while it changes the forward equation. It is the application of the integral operator with f as kernel that causes the backward ODE to hold for any claim.

Recall that for any target claim, the initial bank balance when replicating the payoff with cash and CDS is just the initial value of the target claim. Applying this result to a defaultable annuity, let

$A_0(u) \equiv M_a(0;u)$ denote the initial value (DV01) of the defaultable annuity maturing at $u \in [0,T]$. Evaluating (22) to (24) at $t=0$ implies that $A_0(u)$ solves the homogeneous ODE:

$$A_0''(u) + \left[r(u) + \frac{S_0(u)}{L} \right] A_0'(u) + \frac{S_0'(u)}{L} A_0(u) = 0, \tag{25}$$

on the domain $u \in (0,T)$ and imposing the initial conditions:

$$A_0(0) = 0, \tag{26}$$

and:

$$\lim_{u \downarrow 0} A_0'(u) = 1. \tag{27}$$

6 Financial Derivation of Forward Equation

In the last section, the Green's function $g(t;u)$ was used to derive a forward equation (22) governing the survival contingent bank balance when replicating a defaultable annuity. By considering this equation at $t=0$, we developed a forward equation for the initial value A_0 of a defaultable annuity considered as a function of its maturity date u. This section shows that this forward ODE arises directly as a consequence of no arbitrage.

We assume that the default time τ has a probability density function (PDF) and our approach is to find two equivalent expressions for it. We start from the observation that $S_0(u)A_0(u)$ is the spot value of the premium leg of a CDS of maturity u. By the definition of $S_0(u)$, $S_0(u)A_0(u)$ is also the spot value of the protection leg of a CDS of maturity u. Dividing this value by L produces the term structure of initial values for a claim paying one dollar at the default time τ if $\tau < u$ and zero otherwise. We refer to this default-contingent claim as a unit recovery claim. The next section shows how to *replicate* the payoff of this unit recovery claim. In this section, we just observe that $\frac{S_0(u)A_0(u)}{L}$ is the spot value of the unit recovery claim. Differentiating w.r.t. u implies that $\frac{\partial}{\partial u}\frac{S_0(u)A_0(u)}{L}$ is the spot value of a claim with the payoff $\delta(\tau - u)$ at u. Future valuing this spot value gives the default time PDF:

$$\mathbb{Q}\{\tau \in du\} = e^{\int_0^u r(v)dv} \frac{\partial}{\partial u} \frac{S_0(u)A_0(u)}{L}. \tag{28}$$

A second observation is that that $A_0'(u)$ is the spot value of a survival claim, i.e. a claim that pays one dollar at u if the reference entity survives until then. As a result, the complementary distribution function of the default time is its forward price:

$$\mathbb{Q}\{\tau \geq u\} = e^{\int_0^u r(v)dv} A_0'(u). \tag{29}$$

Differentiating both sides w.r.t. u and negating implies that:

$$\mathbb{Q}\{\tau \in du\} = -r(u) e^{\int_0^u r(v)dv} A_0'(u) - e^{\int_0^u r(v)dv} A_0''(u). \tag{30}$$

Equations (28) and (30) both express the risk-neutral PDF of the default time in terms of the ex ante unknown annuity value. Equating them yields the forward equation (25) governing the initial value $A_0(u)$ of a defaultable annuity, considered as a function of its maturity date u:

$$A_0''(u) + \left[r(u) + \frac{S_0(u)}{L}\right] A_0'(u) + \frac{S_0'(u)}{L} A_0(u) = 0, \qquad u \in [0, T]. \tag{31}$$

Subjecting the function $A_0(u)$ to the two initial conditions (26) and (27) uniquely determines it. Either of (28) or (30) then implies that the risk-neutral PDF of τ is also determined. It follows that any claim that pays $f(\tau)$ at time τ can be uniquely priced relative to the initial yield curve and CDS curve. The claim can be finite-lived or even perpetual, so long as the payoff does not lead to infinite value. While we have assumed deterministic interest rates and a constant recovery rate on the CDS, the determination of the default time PDF is otherwise robust in that we have made no assumptions on the process triggering default. In particular, our results are consistent with both reduced form and structural models. Equations (28) and (30) can be regarded as analogs of the Breeden and Litzenberger[1] result linking the risk-neutral PDF of the underlying stock price to the second strike derivative of a call. Both equations link the risk-neutral PDF of the default time to partial derivatives of observables, with the main difference being that $A_0(u)$ only becomes observable once an initial value problem is solved. While our result is not presently as explicit as the BL result, we will later show that our result can be made completely explicit if either the CDS curve or yield curve is initially flat.

7 Replicating the Payoff of a Unit Recovery Claim

The last section showed that $\frac{S_0(u)A_0(u)}{L}$ is the spot value of the unit recovery claim. In this section, we develop the replicating strategy. Recall that a defaultable annuity maturing at T produces a continuous cash flow at each $t \in [0, T]$ of $1(\tau > t)dt$, where τ is the random default time. Also recall that this payoff is replicated by keeping the survival-contingent bank balance at $M_a(t; T)$ for each $t \in [0, T]$ and also shorting $Q_a(t)dt$ CDS of each maturity $t \in [0, T]$. For each fixed T, the function M_a of t is determined by the terminal value problem (10) to (12), while the function Q_a of t is determined by (13).

Consider scaling these holdings by $\frac{S_0(T)}{L}$. Since the original holdings produce a continuous cash flow at each $t \in [0, T]$ of $1(\tau > t)dt$, the scaled holdings produce a continuous cash flow at each $t \in [0, T]$ of $\frac{S_0(T)}{L} 1(\tau > t)dt$. In contrast, a long position in one CDS of maturity T produces a continuous cash flow at each $t \in [0, T]$ of $[L\delta(\tau - t) - S_0(T)1(\tau > t)]dt$. Hence, if we now add a a long position in $\frac{1}{L}$ CDS of maturity T to the scaled holdings, then the combined position produces a continuous cash flow at each $t \in [0, T]$ of $\delta(\tau - t)dt$. This is the payoff of a unit recovery claim, i.e., a claim paying one dollar at the default time τ if $\tau < T$ and zero otherwise. Letting $M_u(t; T)$ denote the survival-contingent bank balance at time $t \in [0, T]$ when replicating a unit recovery claim, we conclude that:

$$M_u(t; T) = \frac{S_0(T)}{L} M_a(t; T), \qquad t \in [0, T]. \tag{32}$$

Letting $Q_u(t;T)$ denote the rate at which CDS are initially written for maturity $t \in [0,T]$, when replicating a unit recovery claim, we conclude that:

$$Q_u(t;T) = \frac{\delta(t-T)}{L} + \frac{S_0(T)}{L} Q_a(t;T) 1_{t\in[0,T]}, \qquad t \geq 0. \tag{33}$$

It can be shown that this replicating strategy arises from setting $c(t) = 0$ and $R(t) = 1_{t\in[0,T]}$ in the backward ODE (6).

8 Replicating and Pricing Survival Claims

Given no default up to time t, the value at time t of a defaultable annuity maturing at T can be represented as:

$$V_t^{da}(T) = M_a(t;T) + \int_t^T Q_a(u;T) V_t^{cds}(u; S_0(u)) du, \tag{34}$$

where $V_t^{cds}(u;k)$ is the unknown value at time t of a CDS, maturing at $u \in (t,T)$ and with fixed payment rate k. Recall that the survival claim pays off one dollar at its maturity if the firm survives until then, and pays zero otherwise. Differentiating the payoff of a defaultable annuity w.r.t. its maturity T produces the payoff of a survival claim with maturity T. Differentiating (34) w.r.t. T implies:

$$\frac{\partial}{\partial T} V_t^{da}(T) = \frac{\partial}{\partial T} M_a(t;T) + Q_a(T;T) V_t^{cds}(T; S_0(T)) + \int_t^T \frac{\partial}{\partial T} Q_a(u;T) V_t^{cds}(u; S_0(u)) du. \tag{35}$$

The LHS of (35) is the value at time t of a survival claim, given no default up to time t. Since (35) holds for all times $t \in (0,T)$ the RHS represents the survival-contingent value at t of the replicating portfolio. In particular, the survival-contingent bank balance at time t when replicating a survival claim is given by $\frac{\partial}{\partial T} M_a(t;T)$. Evaluating (13) as $t \uparrow T$ implies that:

$$\lim_{t\uparrow T} Q_a(t;T) = -\lim_{t\uparrow T} \frac{M_a'(t;T)}{L} = \frac{1}{L},$$

from (12). Hence, replicating the payoff of a survival claim requires going short $\frac{1}{L}$ CDS of maturity T and also writing $\frac{\partial}{\partial T} Q_a(u) du$ CDS of maturity u for each maturity $u \in (0,T)$. The term structure of survival claim values is given by $A_0'(T)$ for $T \geq 0$.

9 Closed Form Solutions

When replicating the defaultable annuity, the survival-contingent bank balance satisfies the backward ODE (6), while the initial value of the defaultable annuity satisfies the forward ODE (25). This section shows that both ODE's have closed form solutions when either the initial CDS curve $S_0(T), T \geq 0$ is flat in T or the initial forward rate curve $r(T), T \geq 0$ is flat in T. As a result, we also get closed form solutions for the Green's function, and hence the price and hedge of any default-contingent claim.

9.1 Flat CDS Curve

In this subsection, we assume that the initial CDS spread curve is flat at S_0. In this case, the backward ODE (10) governing the survival-contingent bank balance $M_a(t;T)$ when replicating a defaultable annuity of maturity T simplifies to:

$$\frac{\partial^2}{\partial t^2}M_a(t;T) - \left[r(t) + \frac{S_0}{L}\right]\frac{\partial}{\partial t}M_a(t;T) - r'(t)M_a(t;T) = 0, \qquad t \in (0,T). \tag{36}$$

The ODE is integrable since the LHS integrates:

$$\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial t}M_a(t;T) - \left[r(t) + \frac{S_0}{L}\right]M_a(t;T)\right\} = 0, \qquad t \in (0,T). \tag{37}$$

Subjecting (36) to the terminal conditions (11) and (12) and solving implies that the closed form solution for the survival-contingent bank balance $M_a(t;T)$ is given by:

$$M_a(t;T) = \int_t^T e^{-\left[y(t;t') + \frac{S_0}{L}\right](t'-t)}dt', \qquad t \in [0,T], \tag{38}$$

where:

$$y(t;t') \equiv \frac{\int_t^{t'} r(v)dv}{t'-t} \tag{39}$$

is the yield to maturity at time $t \in [0,T]$ of a default-free bond with maturity $u \in [t,T]$. Substituting (38) in (13) implies that the closed form solution for the rate $Q_a(t;T)$ at which CDS are initially written when replicating a defaultable annuity is given by:

$$Q_a(t;T) = \frac{1 - [r(t) + \frac{S_0}{L}]e^{-\left[y(t;t') + \frac{S_0}{L}\right](t'-t)}}{L}, \qquad t \in [0,T]. \tag{40}$$

When the CDS curve is flat, (16) and (38) imply that the Green's function $g(t;u)$ is given by:

$$g(t;u) = \begin{cases} \int_t^u e^{-\int_t^{t'}\left[r(v) + \frac{S_0}{L}\right]dv}dt', & \text{if } t \in [0,u] \\ 0 & \text{if } t \in (u,T). \end{cases} \tag{41}$$

Evaluating (38) at $t = 0$ implies that if the initial CDS spread curve is flat at S_0, then the initial value of a defaultable annuity of maturity T is given by:

$$A_0(T) \equiv M_a(0;T) = \int_0^T e^{-\left[y(0;t') + \frac{S_0}{L}\right]t'}dt'. \tag{42}$$

Thus, the defaultable annuity is simply valued by discounting each dollar received at time u back to time 0. The rate used for discounting a dollar received at some time $u \in (0,T)$ is the sum of the yield to maturity $y(0;u)$ and the recovery-adjusted CDS spread $\frac{S_0}{L}$. Equation (42) explains why each fixed payment rate S_0 is often referred to as a spread.

When the CDS spread is constant, the forward ODE (25) for $A_0(u)$ simplifies to:

$$A_0''(u) + \left[r(u) + \frac{S_0}{L}\right] A_0'(u) = 0, \qquad u \in [0, T]. \tag{43}$$

The solution of (43) subject to the initial conditions (26) and (27) is given by (42) evaluated at $T = u$. Hence, (29) implies that the risk-neutral survival probability is exponential:

$$\mathbb{Q}\{\tau \geq u\} = e^{-\frac{S_0}{L}u}. \tag{44}$$

Thus, the risk-neutral probability of default is the same as would arise in a reduced form model of default with the hazard rate constant at $\frac{S_0}{L}$.

9.2 Flat Forward Rate Curve

In this subsection, we revert to a deterministic CDS spread curve $S_0(u), u \geq 0$, but now we assume that the initial yield curve is flat at r. It follows that the initial forward rate curve is flat at r. We assume no arbitrage and as we have already assumed deterministic interest rates, it follows that the spot rate is constant over time at r. Under a constant spot rate, the backward ODE (10) governing the survival-contingent bank balance $M_a(t)$ when replicating a defaultable annuity simplifies to:

$$\frac{\partial^2}{\partial t^2} M_a(t;T) - \left[r + \frac{S_0(t)}{L}\right] \frac{\partial}{\partial t} M_a(t;T) = 0, \qquad t \in (0, T). \tag{45}$$

The solution of (45) subject to the terminal conditions (11) and (12) is:

$$M_a(t;T) = \int_t^T e^{-\int_u^T \left[r + \frac{S_0(v)}{L}\right] dv} du, \qquad t \in [0, T]. \tag{46}$$

Substituting (46) in (13) implies:

$$Q_a(t;T) = -\frac{e^{-\int_t^T \left[r + \frac{S_0(v)}{L}\right] dv}}{L}, \qquad t \in [0, T]. \tag{47}$$

When the spot rate is constant at r, (16) implies that the Green's function $g(t;u)$ is given by:

$$g(t;u) = \begin{cases} \int_t^u e^{-\int_{t'}^u \left[r + \frac{S_0(v)}{L}\right] dv} dt', & \text{if } t \in [0, u] \\ 0 & \text{if } t \in (u, T). \end{cases} \tag{48}$$

Evaluating (46) at $t = 0$ implies that under a constant interest rate r, the initial value of a defaultable annuity of maturity T is given by:

$$A_0(T) \equiv M_a(0;T) = \int_0^T e^{-[\hat{y}(u;T) + r](T-u)} du, \tag{49}$$

where:

$$\hat{y}(u,T) \equiv \frac{\int_u^T \frac{S_0(v)}{L} dv}{T-u}. \tag{50}$$

Curiously, (49) indicates that under a constant interest rate, the defaultable annuity is now valued by discounting each dollar received *forward* to T rather than backward to 0. The rate used for discounting a dollar received at some time $u \in (0,T)$ is the sum of the constant interest rate r and $\hat{y}(u;T)$ defined in (50). To understand why this curious result holds, we note that when the spot riskfree rate is constant at r, the forward ODE (25) for $A_0(u)$ simplifies to:

$$A_0''(u) + \left[r + \frac{S_0(u)}{L}\right] A_0'(u) + \frac{S_0'(u)}{L} A_0(u) = 0, \qquad u \in [0,T], \tag{51}$$

or equivalently:

$$\frac{\partial}{\partial u}\left\{\frac{\partial}{\partial u} A_0(u) + \left[r + \frac{S_0(u)}{L}\right] A_0(u)\right\} = 0, \qquad u \in [0,T]. \tag{52}$$

Suppose we consider a dual economy in which time $\hat{t}$ runs backward from T, i.e. $\hat{t} \equiv T - t$ and in which the CDS spread is constant at r, while the spot interest rate at time $\hat{t}$ is deterministically given by $\frac{S_0(T-t)}{L}$. In such an economy, the yield to maturity at the initial time $\hat{t} = 0$ with term $T-u$ is given by $\hat{y}(u;T)$. By comparing (37) and (52), we see that the latter equation is recognized as the backward ODE in the dual economy. Hence, the solution (49) to (52) is just the solution (42) to (37) with time running backwards from T and the two curves switched. The solution of (51) subject to the initial conditions (26) and (27) is given by (49) evaluated at $T = u$. Differentiating (49) w.r.t. T implies that the initial value of a survival claim of maturity T is:

$$\begin{aligned} A_0'(T) &= 1 - \left(r + \frac{S_0(T)}{L}\right) A_0(T), \\ &= 1 - rA_0(T) - U_0(T), \end{aligned} \tag{53}$$

where $U_0(T) \equiv \frac{S_0(T)}{L} A_0(T)$ is the spot value of a unit recovery claim. Future valuing (53) and using (29) implies that the risk-neutral survival probability is given by:

$$\mathbb{Q}\{\tau \geq T\} = e^{rT}[1 - rA_0(T) + U_0(T)]. \tag{54}$$

10 Flat Forward and CDS Curve

The Green's function in (48) simplifies if both the forward rate curve and the CDS spread are flat:

$$g(t;u) = \begin{cases} \frac{1 - e^{-\left[r + \frac{S_0}{L}\right](u-t)}}{r + \frac{S_0}{L}}, & \text{if } t \in [0,u] \\ 0 & \text{if } t \in (u,T). \end{cases} \tag{55}$$

11 Affine Forward and CDS Curve

In this section, we suppose that the intial forward rate curve and the initial CDS curve are both affine in term. We show that the Green's function can be expressed in terms of confluent hypergeometric functions.

12 Summary and Future Research

We showed how to replicate the payoffs to a class of default-contingent claims by taking static positions in a continuum of CDS of different maturities. Although we assumed deterministic interest rates and a constant recovery rate on the CDS, the replication was otherwise robust in that we made no assumptions on the process triggering default. We also derived a new terminal value problem which determines CDS holdings and we illustrated its specification for defaultable annuities, unit recovery claims, and survival claims. We furthermore derived a new initial value problem which determined the DVO1 of a CDS. In general, the backward and forward differential equations in these problems must be solved numerically, but we indicated important special cases in which they can be solved in closed form.

Future research can proceed in at least four directions. First, one can look at other specifications of the forward rate curve and CDS spread curve that lead to closed form solutions of either the backward or forward equation. For example, if the CDS curve and the forward rate curve are both affine in the term, then one can transform either evolution equation into Kummer's ODE. Second, one can explore whether the results of the current analysis extend to the case where the loss given default, L, is deterministic rather than constant. Third, one can try to address stochastic interest rates. Finally, one can consider calibrating to other instruments such as off market CDS or defaultable zero coupon corporate bonds. When a bond is insured by a CDS, the recovery rate can be allowed to be stochastic and unknown since it drops out of the combined payoff. In the interests of brevity, these extensions are best left for future research.

Appendix

Recall from (6) and (14) that the Green's function $g(u;v)$ solves:

$$\mathcal{L}g(u;v) = \delta(u-v), \tag{56}$$

where:

$$\mathcal{L} \equiv \mathcal{D}_u^2 - \left[r(t) + \frac{S_0(t)}{L}\right]\mathcal{D}_u^1 - r'(t)\mathcal{D}_u^0 \tag{57}$$

is a second order linear differential operator. Multiplying (56) by $g(t;u)$ and integrating u from 0 to T implies:

$$\int_0^T g(t;u)\mathcal{L}g(u;v)du = g(t;v), \tag{58}$$

by the sifting property of delta functions. By the definition of the adjoint operator:

$$\int_0^T \mathcal{L}^* g(t;u)g(u;v)du = \int_0^T \delta(u-t)g(u;v)du, \tag{59}$$

where $\mathcal{L}^*$ is given in (21) and the sifting property of delta functions has been used on the right hand side. It follows that:

$$\int_0^T [\mathcal{L}^* g(t;u) - \delta(u-t)]g(u;v)du = 0, \tag{60}$$

for all v. But this can only be true if:

$$\mathcal{L}^* g(t;u) = \delta(u-t) \tag{61}$$

for all $t, u \in [0, T]$.

References

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[2] Stakgold I., 1979, Green's Functions and Boundary Value Problems, John Wiley & Sons, New York.