A Radon-Nikodym Theorem for Completely Positive Maps

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Abstract

The aim of this paper is to generalize a noncommutative Radon-Nikodym theorem to the case of completely positive (CP) map. By only assuming absolute continuity with respect to another CP map the existence of a Hermitian-positive density as the unique "Radon-Nikodym derivative" is proved in the commutant of the Steinspring representation of the reference CP map.

1 Preliminaries and definitions

Let $\mathcal{A}$ be a C*-normed algebra, and let $B(\mathfrak{h})$ denote the algebra of all bounded operators on a Hilbert space $\mathfrak{h}$. In this paper we will obtain a positive self-adjoint density operator ϱ for a completely positive map κ from $\mathcal{A}$ into $B(\mathfrak{h})$ strongly absolutely continuous with respect to another such map ϕ given, say, by a faithful weight or trace φ as $\phi = \mathbf{1}\varphi$. It will be uniquely defined as a noncommutative generalization of Radon-Nikodym derivative κ_ϕ in the Hilbert space $\mathcal{H}$ of Steinspring representation of ϕ.

To this end, we first recall the definition of compete positivity. If $\mathcal{A}$ and $\mathcal{B}$ are C*-algebras, $M(n)\,(n \geq 1)$ the algebra of $n \times n$ complex matrices and κ is a linear map from $\mathcal{A}$ to $\mathcal{B}$, we shall say that κ is n-positive if the map

$$\kappa_n : \mathcal{A} \otimes M(n) \to \mathcal{B} \otimes M(n),$$
$$\kappa_n(a \otimes m) = \kappa(a) \otimes m, \quad a \in \mathcal{A}, m \in M(n),$$

is positive. The map κ is called completely positive if it is n-positive for all integers n.

The completely positive maps play an important role in the description of quantum channels and time evolutions of open quantum systems [2].

Let us consider two quantum systems described in terms of C*-algebras $\mathcal{A}$ and $\mathcal{B}$. It can be easily shown that if the Heisenberg dynamics of the compound system is described by a $*$-endomorphism γ of $\mathcal{A}\otimes\mathcal{B}$, then the reduced dynamics as conditional expectation ϵ of γ corresponding to an independent state on $\mathcal{B}$ is described by a completely positive identity preserving maps $\mu : \mathcal{A} \to \mathcal{A}$ (such $\mu = \epsilon \circ \gamma$ is usually called a dynamical map on $\mathcal{A}$). The complete positivity of a reduced dynamics was first pointed out by Kraus [4] in the context of state changes produced by quantum measurements.

If a C*-algebra $\mathcal{A}$ describes an open physical system subject to completely positive dynamics, then any dynamical map of this system, considered in a representation ι, is a completely positive map of norm one $\kappa : \mathcal{A} \to B(\mathfrak{h})$, where $\kappa = \iota \circ \mu$.

Let us recall that the condition of complete positivity of κ can be written [5] in the form

$$\sum_{i,k=1}^{n} \left\langle \eta_i | \kappa(a_i^* a_k) \eta_k \right\rangle \geq 0, \quad \forall \eta_j \in \mathfrak{h}, \forall a_j \in \mathcal{A}, j = 1, \ldots, n, \forall n \in \mathbb{N}.$$

The condition of normalization of κ can be expressed in the form $\kappa(1) = \mathbf{1}$ if $1 \in \mathcal{A}$ and $\mathbf{1}$ stand for identities in $\mathcal{A}$ and $B(\mathfrak{h})$, respectively.

According to the famous results of Stinespring [5] any (normalized) completely positive map $\kappa : \mathcal{A} \to B(\mathfrak{h})$ can be represented in the form

$$\kappa(a) = F_\kappa^* \pi_\kappa(a) F_\kappa,$$

where $\pi_\kappa : \mathcal{A} \to B(\mathcal{H}_\kappa)$ is a representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\kappa$ and F_κ is a bounded (isometric) linear operator from $\mathfrak{h}$ into $\mathcal{H}_\kappa$. Such a representation of a completely positive map will be called spatial. The normalization condition for a dynamical map implies the isometricity $F_\kappa^* F_\kappa = \mathbf{1}$.

Let ϕ and κ denote completely positive maps from $\mathcal{A}$ into $B(\mathfrak{h})$ and let $\left\{(a_{jm})_m,\, j = 1, \ldots, n\right\}$ be a family of sequences in $\mathcal{A}$. Such a family will be called a (ϕ, κ) family of sequences if for any $n \in \mathbb{N}$

$$\begin{aligned} &\lim_{m\to\infty} \sum_{i,k=1}^{n} \left\langle \eta_i | \phi(a_{im}^* a_{km}) \eta_k \right\rangle \\ = &\lim_{m,r\to\infty} \sum_{i,k=1}^{n} \left\langle \eta_i | \kappa\big((a_{im} - a_{ir})^* (a_{km} - a_{kr})\big) \eta_k \right\rangle = 0 \qquad (1.1) \\ &\forall \eta_j \in \mathfrak{h}, \quad j = 1, \ldots, n. \end{aligned}$$

Now we generalize various forms and strengthened forms of the concept of absolute continuity [3] in the case of completely positive maps.

Definition 1 *A completely positive map κ is called*

(1) completely absolutely continuous *with respect to a completely positive map* ϕ *if for any* $n \in \mathbb{N}$

$$\inf_m \sum_{i,k=1}^{n} \langle \eta_i | \phi(a_{im}^* a_{km}) \eta_k \rangle = 0$$

for any increasing family $\{A_m\}$ *of matrices* $A_m = [a_{im}^* a_{km}]$ *implies*

$$\inf_m \sum_{i,k=1}^{n} \langle \eta_i | \kappa(a_{im}^* a_{km}) \eta_k \rangle = 0, \quad \forall \eta_j \in \mathfrak{h},\ j = 1, \ldots, n,$$

(2) strongly completely absolutely continuous *with respect to* ϕ *if for any* (ϕ, κ) *family of sequences* $\{(a_{jm})_m, j = 1, \ldots, n\}$ *we have for any* $n \in \mathbb{N}$

$$\lim_{m\to\infty} \sum_{i,k=1}^{n} \langle \eta_i | \kappa(a_{im}^* a_{km}) \eta_k \rangle = 0, \quad \forall \eta_j \in \mathfrak{h}, \quad j = 1, \ldots, n,$$

(3) completely dominated *by* ϕ *if there exists a* $\lambda > 0$ *such that for any* $n \in \mathbb{N}$

$$\sum_{i,k=1}^{n} \langle \eta_i | \kappa(a_i^* a_k) \eta_k \rangle \leq \lambda \sum_{i,k=1}^{n} \langle \eta_i | \phi(a_i^* a_k) \eta_k \rangle,$$
$$\forall \eta_j \in \mathfrak{h}, \quad \forall a_j \in \mathcal{A}, \quad j = 1, \ldots, n.$$

It is rather obvious that (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

In the particular case $\phi(a) = \varphi(a)\mathbf{1}$, where $\varphi : \mathcal{A} \to \mathbb{C}$ denotes a positive functional on $\mathcal{A}$ (e.g. a reference state, or trace), we shall say that κ is completely absolutely continuous or strongly completely absolutely continuous or completely dominated by the functional φ. If completely positive maps are of the form $\phi(a) = \varphi(a)\mathbf{1}$, $\kappa(a) = \varkappa(a)\mathbf{1}$, where $\varphi, \varkappa$ are positive functionals on $\mathcal{A}$, then one can easily verify that our forms of absolute continuity (1)-(3) imply that $\varkappa$ is ($1'$) φ-absolutely continuous, ($2'$) strongly φ-absolutely continuous, ($3'$) φ-dominated, respectively in the sense of Gudder [3].

2 A Radon-Nikodym theorem for completely positive maps

Theorem 2 *Let* ϕ *and* κ *be a bounded completely positive maps form* $\mathcal{A}$ *into* $B(\mathfrak{h})$ *and let* $\mathcal{H}$ *be a Hilbert space of a representation* $\pi : \mathcal{A} \to B(\mathcal{H})$ *in which* ϕ *is spatial, that is*

$$\phi(a) = F^* \pi(a) F, \quad \forall a \in \mathcal{A}, \tag{2.1}$$

where F *is assumed to be bounded operator* $\mathfrak{h} \to \mathcal{H}$. *Then*

(a) *κ is completely absolutely continuous with respect to ϕ if and only if it has a spatial representation $\kappa(a) = K^*\pi(a)K$ with $\pi(a)K = \vartheta\pi(a)F$, where ϑ is a densely defined operator in the minimal $\mathcal{H}$, commuting with $\pi(\mathcal{A}) = \{\pi(a), a \in \mathcal{A}\}$ on the lineal $\mathcal{D} = \left\{\sum_j \pi(a_j)F\eta_j\right\}$.*

(b) κ is strongly completely absolutely continuous with respect to ϕ if and only if κ is spatial in $(\pi, \mathcal{H})$ and there exists a positive self-adjoint operator ϱ, uniquely defined on $\mathcal{D}$, affiliated with the commutant $\pi(\mathcal{A})'$ and such that

$$\kappa(a) = F^*\varrho\pi(a)F = (\varrho^{1/2}F)^*\pi(a)(\varrho^{1/2}F), \quad \forall a \in \mathcal{A}, \tag{2.2}$$

(c) *κ is completely dominated by ϕ if and only if* (2.2) *holds and ϱ is bounded.*

Proof. Let us first sketch the prove the part (a) given in [1].

The condition of absolute continuity means that κ is normal in the minimal spatial representation of ϕ with the support orthoprojector P_κ majorised by the support P_ϕ of ϕ. Therefore it is spatial, with the operator $K : \mathfrak{h} \to \mathcal{H}$ uniquely defining the operator $\vartheta = \pi'(K)$ on $\mathcal{D}$ by

$$\pi'(K)\pi(a)F\eta = \pi(a)K\eta, \ \ \forall A \in \mathcal{A}, \eta \in \mathfrak{h}.$$

such that it commutes with $\pi(\mathcal{A})$. The reverse is obvious.

Let us now prove the part (b) of our theorem.

($\Rightarrow$) Let π_κ be a representation of a C*-algebra $\mathcal{A}$ in the Hilbert space $\mathcal{H}_\kappa$ generated by the algebraic tensor product $\mathcal{A} \otimes \mathfrak{h}$ with respect to a positive Hermitian bilinear form

$$\left\langle \sum_i a_i \otimes \eta_i \middle| \sum_k a_k \otimes \eta_k \right\rangle_\kappa = \sum_{i,k=1}^n \langle \eta_i | \kappa(a_i^* a_k)\eta_k \rangle \tag{2.3}$$

defined by the equality

$$\pi_\kappa(a) \left| \sum_j a_j \otimes \eta_j \right\rangle_\kappa = \left| \sum_j a a_j \otimes \eta_j \right\rangle_\kappa \tag{2.4}$$

(for details see [5]).

Let us denote by F_κ the bounded operator $\mathcal{H}_\kappa \to \mathfrak{h}$,

$$F_\kappa : \eta \mapsto |1 \otimes \eta\rangle_\kappa, \tag{2.5}$$

(a canonical isometry $\mathfrak{h} \to \mathcal{H}_\kappa$ if κ is normalized). Then we have [5]

$$\kappa(a) = F_\kappa^* \pi_\kappa(a) F_\kappa. \tag{2.6}$$

Define an operator I_κ in $\mathcal{H}$ into $\mathcal{H}_\kappa$ by the formula

$$I_\kappa : \sum_j \pi(a_j)F\eta_j \mapsto \left| \sum_j a_j \otimes \eta_j \right\rangle_\kappa = \sum_j \pi_\kappa(a_j)F_\kappa\eta_j. \tag{2.7}$$

This is a consistent definition of a linear operator on the lineal $\mathcal{D} \subseteq \mathcal{H}$ because condition (2) implies (1) from which, taking into account (2.1), (2.5) and (2.6), we obtain the condition

$$\Big(\langle \sum_k \pi(a_k)F\eta_k | \sum_j \pi(a_j)F\eta_j \rangle = 0\Big) \Rightarrow \Big(\langle \sum_k a_k \otimes \eta_k | \sum_j a_j \otimes \eta_j \rangle_\kappa = 0\Big).$$

Obviously, we have $F_\kappa = I_\kappa F$.

To prove that I_κ is closable let us first note that any sequence of elements of $\Big\{| \sum_j \pi(a_j)F\eta_j\rangle\Big\}$ can be expressed in the form $\Big(| \sum_j \pi(a_{jm}F\eta_j\rangle\Big)_m$ be any sequence such that $\Big(| \sum_j \pi(a_{jm})F\eta_j\rangle\Big)_m \to 0$ and $\Big(| \sum_j a_{jm} \otimes \eta_j\rangle_\kappa\Big)_m$ is convergent. Then for any set η_j, $j = 1, \dots, n$

$$\begin{aligned}
&\lim_{m\to\infty} \sum_{i,k} \langle \eta_i | \phi(a^*_{im} a_{km})\eta_k\rangle \\
&= \lim_{m\to\infty} \sum_{i,k} \langle \eta_i | F^* \pi(a_{im})^* \pi(a_{km}) F\eta_k\rangle \\
&= \lim_{m\to\infty} \sum_{i,k} \langle \pi(a_{im})F\eta_i | \pi(a_{km})F\eta_k\rangle \\
&= \lim_{m\to\infty} \Big\| | \sum_i \pi(a_{im})F\eta_i\rangle \Big\|^2 = 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\lim_{m,r\to\infty} \sum_{i,k} \big\langle \eta_i | \kappa\big((a_{im} - a_{ir})^*(a_{km} - a_{kr})\big)\eta_k\big\rangle \\
&= \lim_{m,r\to\infty} \Big\langle \sum_i (a_{im} - a_{ir}) \otimes \eta_i | \sum_k (a_{km} - a_{kr}) \otimes \eta_k \Big\rangle_\kappa \\
&= \lim_{m,r\to\infty} \Big\| | \sum_i (a_{im} - a_{ir}) \otimes \eta_i \rangle_\kappa \Big\|^2 = 0,
\end{aligned}$$

hence $\Big(| \sum_i a_{im} \otimes \eta_i\rangle_\kappa\Big)_m$ is Cauchy by assumption. Hence $\big\{(a_{jm})_m,\, j = 1, \dots, n\big\}$ form a (ϕ, κ) family of sequences. Then from the strong complete absolute continuity of κ with respect to ϕ we have for any $n \in \mathbb{N}$

$$\begin{aligned}
0 &= \lim_{m\to\infty} \sum_{i,k} \langle \eta_i | \kappa(a^*_{im} a_{km})\eta_k\rangle \\
&= \lim_{m\to\infty} \Big\langle \sum_i a_{im} \otimes \eta_i | \sum_k a_{km} \otimes \eta_k \Big\rangle_\kappa \\
&= \lim_{m\to\infty} \Big\| | \sum_i a_{im} \otimes \eta_i \rangle_\kappa \Big\|^2.
\end{aligned}$$

This proves that I_κ is closable.

Denote by $\bar{I}_\kappa$ its closure. Then there exists an adjoint operator I_κ^* defined on the lineal $\Big\{ \big| \sum_j a_j \otimes \eta_j \big\rangle_\kappa \Big\}$, dense in $\mathcal{H}_\kappa$, by the equality

$$\Big\langle \sum_j a_j \otimes \eta_j \big| \sum_k a_k \otimes \eta_k \Big\rangle_\kappa = \Big\langle \sum_j \pi(a_j) F \eta_j \big| I_\kappa^* \sum_k a_k \otimes \eta_k \Big\rangle. \tag{2.8}$$

The positive self-adjoint operator $\varrho = I_\kappa^* \bar{I}_\kappa$ on the lineal $\big\{ |\pi(a_j) F \eta_j\rangle \big\}$ is affiliated with $\pi(\mathcal{A})'$ because on the domains of I_κ and I_κ^* we have

$$\pi_\kappa(a) I_\kappa = I_\kappa \pi(a), \quad I_\kappa^* \pi_\kappa(a) = \pi(a) I_\kappa^*, \tag{2.9}$$

Let us verify the first of the equalities (2.9). Taking into account (2.7) and (2.4) we have

$$\begin{aligned} \pi_\kappa(a) I_\kappa \Big| \sum_j \pi(a_j) F \eta_j \Big\rangle &= \pi_\kappa(a) \Big| \sum_j a_j \otimes \eta_j \Big\rangle_\kappa \\ &= \Big| \sum_j a a_j \otimes \eta_j \Big\rangle_\kappa \\ &= I_\kappa \Big| \sum_j \pi(a a_j) F \eta_j \Big\rangle \\ &= I_\kappa \pi(a) \Big| \sum_j \pi(a_j) F \eta_j \Big\rangle. \end{aligned}$$

Taking into account that $F_\kappa = I_\kappa F$, we obtain

$$\begin{aligned} \kappa(a) &= F_\kappa^* \pi_\kappa(a) F_\kappa = F^* \varrho \pi(a) F \\ &= (\varrho^{1/2} F)^* \pi(a) (\varrho^{1/2} F) = K^* \pi(a) K, \end{aligned}$$

where $K = \varrho^{1/2} F$.

($\Leftarrow$) Let $\big\{ (a_{jm})_m,\ j = 1, \ldots, n \big\}$ be a family of (ϕ, κ) sequences. Then $\Big(\big| \sum_i \pi(a_{im}) F \eta_i \big\rangle \Big)_m \to 0$ and moreover

$$\begin{aligned} 0 &= \lim_{m,r\to\infty} \sum_{i,k} \big\langle \eta_i | \kappa\big((a_{im} - a_{ir})^* (a_{km} - a_{kr}) \big) \eta_k \big\rangle \\ &= \lim_{m,r\to\infty} \sum_{i,k} \big\langle \eta_i | (\varrho^{1/2} F)^* \pi(a_{im} - a_{ir})^* \pi(a_{km} - a_{kr}) (\varrho^{1/2} F) \eta_k \big\rangle \\ &= \lim_{m,r\to\infty} \Big\langle \varrho^{1/2} \sum_i \pi(a_{im} - a_{ir}) F \eta_i \big| \varrho^{1/2} \sum_k \pi(a_{km} - a_{kr}) F \eta_k \Big\rangle \\ &= \lim_{m,r\to\infty} \Big\| \varrho^{1/2} \big| \sum_i \pi(a_{im}) F \eta_i \Big\rangle - \varrho^{1/2} \big| \sum_i \pi(a_{ir}) F \eta_i \big\rangle \Big\|^2. \end{aligned}$$

Hence $\varrho^{1/2} \Big(\big| \sum_i \pi(a_{im}) F \eta_i \big\rangle \Big)_m$ is Cauchy, and since $\varrho^{1/2}$ is closed,

$$\varrho^{1/2} \big(\big| \sum_i \pi(a_{im}) F \eta_i \big\rangle \big)_m \to 0.$$

Then we have

$$\lim_{m\to\infty}\sum_{i,k}\langle\eta_i|\kappa(a_{im}^*a_{km})\eta_k\rangle = \lim_{m\to\infty}\sum_{i,k}\langle\eta_i|(\varrho^{1/2}F)^*\pi(a_{im})^*\pi(a_{km})(\varrho^{1/2}F)\eta_k\rangle$$
$$= \lim_{m\to\infty}\Big\|\varrho^{1/2}|\sum_i\pi(a_{im})F\eta_i\rangle\Big\|^2 = 0.$$

This means that κ is strongly completely absolutely continuous with respect to ϕ. This completes the proof of part (a).

Let us prove part (c) of our theorem.

($\Rightarrow$) Suppose κ to be completely dominated by ϕ. As the condition (3) implies (2), therefore (a) holds. It remains to prove that ϱ is bounded. The boundedness of ϱ follows from the following calculations:

$$\Big\|\varrho^{1/2}|\sum_i\pi(a_i)F\eta_i\rangle\Big\|^2 = \Big\langle\varrho^{1/2}\sum_i\pi(a_i)F\eta_i\big|\varrho^{1/2}\sum_k\pi(a_k)F\eta_k\Big\rangle$$
$$= \sum_{i,k}\big\langle\eta_i|\kappa(a_i^*a_k)\eta_k\big\rangle \leq \lambda\sum_{i,k}\big\langle\eta_i|\phi(a_i^*a_k)\eta_k\big\rangle$$
$$= \lambda\Big\langle\sum_i\pi(a_i)F\eta_i|\sum_k\pi(a_k)F\eta_k\Big\rangle$$
$$= \lambda\Big\||\sum_i\pi(a_i)F\eta_i\rangle\Big\|^2.$$

($\Leftarrow$) Suppose that $\varrho^{1/2}$ is bounded, then

$$\sum_{i,k}\big\langle\eta_i\big|\kappa(a_i^*a_k)\eta_k\big\rangle = \sum_{i,k}\big\langle\eta_i\big|(\varrho^{1/2}F)^*\pi(a_i^*a_k)(\varrho^{1/2}F)\eta_k\big\rangle$$
$$= \Big\langle\varrho^{1/2}\sum_i\pi(a_i)F\eta_i\big|\varrho^{1/2}\sum_k\pi(a_k)F\eta_k\Big\rangle$$
$$= \Big\|\varrho^{1/2}|\sum_i\pi(a_i)F\eta_i\rangle\Big\|^2 \leq \|\varrho^{1/2}\|^2\Big\||\sum_i\pi(a_i)F\eta_i\rangle\Big\|^2$$
$$= \|\varrho^{1/2}\|^2\sum_{i,k}\big\langle\eta_i\big|\phi(a_i^*a_k)\eta_k\big\rangle.$$

Hence κ is completely dominated by ϕ.

The uniqueness of ϱ can be assured by choosing the smallest Hilbert space $\mathcal{H}_\phi$ in which $\phi(a)$ has the Steinspring form $\phi(a) = F^*\pi_\phi(a)F$. Note that, if $\phi = \mathbf{1}\varphi$, $\mathcal{H}_\phi = \mathfrak{h}\otimes\mathcal{H}_\varphi$, $\pi_\phi(a) = \mathbf{1}\otimes\pi_\varphi(a)$ and $F = \mathbf{1}\otimes f$, where $\mathcal{H}_\varphi \ni f$ is the space of the cyclic representation $\varphi(a) = f^*\pi_\varphi(a)f$ of a positive functional φ on $\mathcal{A}$. ■

The formulation of complete absolute continuity for CP maps belongs to VPB, and the Main Theorem in the formulation of Parts (a) and (b) was originally given in [1].

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