Exercises:

2. Find the primitive Pythagorean triples that arise from applying this method to the odd squares 25 and 49.

3. Explain how this method shows that there are infinitely many primitive Pythagorean triples.

An 1844 edition of a book originally by Jacques Ozanam, [O 1844] republished by Montucla, and then translated from French by one "Dr. Hutton," then revised by Edward Riddle gives another way to generate infinitely many Pythagorean triples:

Another Method.—Take a progression of whole and fractional numbers, as $1\frac{1}{3}$, $2\frac{2}{5}$, $3\frac{3}{7}$, $4\frac{4}{9}$, &c., the properties of which are; 1st, The whole numbers are those of the common series, and have unity for their common difference. 2nd, The numerators of the fractions, annexed to the whole numbers, are also the natural numbers. 3rd, The denominators of these fractions are the odd numbers 3, 5, 7, &c.

Take any term of this progression, for example $3\frac{3}{7}$, and reduce it to an improper fraction, by multiplying the whole number 3 by 7, and adding to 21, the product, the numerator 3, which will give $\frac{24}{7}$. The numbers 7 and 24 will be the sides of a right-angled triangle, the hypothenuse of which may be found by adding together the squares of these two numbers, viz. 49 and 576, and extracting the square root of the sum. The sum in this case being 625, the square root of which is 25, this number will be the hypothenuse required. The sides therefore of the triangle produced by the above term of the generating progression, are 7, 24, 25.

In like manner, the first term $1\frac{1}{3}$ will give the rightangled triangle 3, 4, 5.

The second term $2\frac{2}{5}$ will give 5, 12, 13.

The fourth $4\frac{4}{9}$ will give 9, 40, 41. All these triangles have the ratio of their sides different; and they all possess this property, that the greatest side and the hypothenuse differ only by unity.

This is surprising, and nobody we've asked has ever seen it before. It says to start with the sequence of mixed fractions:

$$1\tfrac{1}{3},\ 2\tfrac{2}{5},\ 3\tfrac{3}{7},\ 4\tfrac{4}{9},\ 5\tfrac{5}{11},\ \text{etc.}, \text{ the general term of which is } n+\tfrac{n}{2n+1}$$

Then take any term of this and convert it to an "improper fraction" in lowest terms. The numerator and denominator of this improper fraction will be two sides of a Pythagorean triple. The hypotenuse can be found using the Pythagorean theorem.

If, for example, we take the fifth term, $5\frac{5}{11}$, we get the improper fraction $5\frac{5}{11}=\frac{60}{11}$. According to the rule, 60 and 11 are two sides of a Pythagorean triple, and the hypotenuse will be $\sqrt{60^2+11^2}=\sqrt{3600+121}=\sqrt{3721}=61$. Hence, 60, 11 and 61 form a Pythagorean triple.

Exercises:

4. Show why this works.

5. Find a Pythagorean triple that this method does *not* generate.

6. The 1844 edition goes on to say, "The progression $1\frac{7}{8}$, $2\frac{11}{12}$, $3\frac{15}{16}$, $4\frac{19}{20}$, &c., is of the same kind as the preceding." Check that the first three triples, (8, 15, 17), (12, 35, 37) and (16, 63, 65) are all Pythagorean.

   Find the general term of the progression.

   Show why this works.

   (Hard) Find another similar progression with the same properties.

7. Different editions of Ozanam's book are quite different. The French edition of 1778 does not have the sequence of fractions, but it does have the claim at the right. It translates as:

   "If two numbers are such that their squares, added together, makes a square, then the product of these two numbers is divisible by six."

   Si deux nombres ſont tels, que leurs quarrés ajoutés enſemble faſſent un quarré, le produit de ces deux nombres eſt diviſible par ſix.

   Prove this assertion.

We mentioned above that there is a fast and powerful way to generate all Pythagorean triples. It is the way found in most modern number theory texts. Because most texts tend to take a straight line to their goals, they seldom have the time step off the beaten path. As a result, almost nothing of what we've described above is in any modern number theory text. Instead, we find the following theorem:

Theorem: Let m and n be distinct positive integers, with $m > n$,

1. Then the three numbers, $2mn$, $m^2 - n^2$ and $m^2 + n^2$ form a Pythagorean triple.
2. If m and n are relatively prime, and if one is odd and the other is even, then the resulting Pythagorean triple is primitive.
3. Every primitive Pythagorean triple can be found in this manner in exactly one way.

A few examples are given in the table below.

m	n	a	b	c
2	1	4	3	5
3	2	12	5	13

4	1	8	15	17

We won't prove this theorem, though we will note that part 1 is rather easy, part 2 a bit harder, and part 3 is a good deal of work.

Exercises:

8. In the theorem, what goes wrong if we take *m* and *n* not relatively prime? What goes wrong if we take them both to be odd?

9. Which of the exercises 1 to 7 become really easy if you know this theorem?

[F] Fibonacci (Leonardo of Pisa), *Leonardo Pisano Fibonacci: The Book of Squares*, an annotated translation into Modern English by L. E. Sigler, Academic Press, Boston, 1987.

[O 1778] Ozanam, Jacques, *Récreations mathématiques et physiques*, vol. 1, Jombert, Paris, 1778. Available online at Gallica, http://gallica.bnf.fr.

[O 1844] Ozanam, Jacques, *Science and Natural Philosophy: Dr. Hutton's Translation of Montucla's edition of Ozanam*, revised by Edward Riddle, Thomas Tegg, London, 1844. Available online at The Cornell Library Historical Mathematics Monographs, http://historical.library.cornell.edu/math/index.html.