A New Upper Bound for Diagonal Ramsey Numbers

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Abstract

We prove a new upper bound for diagonal two-colour Ramsey numbers, showing that there exists a constant C such that

$$r(k+1,k+1) \leq k^{-C\frac{\log k}{\log\log k}}\binom{2k}{k}.$$

1. Introduction

The Ramsey number $r(k,l)$ is the smallest natural number n such that, in any red and blue colouring of the edges of the complete graph on n vertices, we are guaranteed to find either a red K_k or a blue K_l.

That these numbers exist is a consequence of Ramsey's original theorem [R30], but the standard upper bound

$$r(k+1,l+1) \leq \binom{k+l}{k}$$

is due to Erdős and Szekeres [ES35].

Very little progress was made on improving this bound until the mid-eighties, when a number of successive improvements were given, showing that, as expected, $r(k+1,l+1) = o(\binom{k+l}{k})$. Firstly, Rödl showed that for some constants c and c' we have

$$r(k+1,l+1) \leq \frac{c\binom{k+l}{k}}{\log^{c'}(k+l)}.$$

This result was never published, but a weaker bound,

$$r(k+1,l+1) \leq \frac{6\binom{k+l}{k}}{\log\log(k+l)}$$

appears in the survey paper concerning Ramsey bounds by Graham and Rödl [GR87].

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Not long after these bounds were proven, Thomason [T88] proved that there was a positive constant A such that, for $k \geq l$,

$$r(k+1, l+1) \leq \exp\left\{-\frac{l}{2k}\log k + A\sqrt{\log k}\right\}\binom{k+l}{k},$$

this being a major improvement on Rödl's bound when k and l are of approximately the same order, implying in particular that

$$r(k+1, k+1) \leq k^{-1/2+A/\sqrt{\log k}}\binom{2k}{k}.$$

In this paper we will show how to improve on Thomason's result, obtaining

THEOREM 1.1. *There exists a constant C such that*

$$r(k+1, k+1) \leq k^{-C\frac{\log k}{\log\log k}}\binom{2k}{k}.$$

In particular we have the following natural extension of Thomason's Theorem:

THEOREM 1.2. *For all $s > 0$ there exists a constant C_s such that*

$$r(k+1, k+1) \leq \frac{C_s}{k^s}\binom{2k}{k}.$$

2. An Outline of the Proof

Our argument (and also Thomason's) begins by assuming that we are trying to prove a bound of the form $r(k+1, l+1) \leq f(k,l)\binom{k+l}{k}$, where $f(k,l)$ is some slowly changing function in k and l. In order to construct an inductive argument we will assume that for some such function we have $r(a+1, b+1) \leq f(a,b)\binom{a+b}{a}$ whenever a is less than k or b is less than l, and that we would like to show that the same holds for $a = k$ and $b = l$.

To this end, let us suppose that $n = \lfloor f(k,l)\binom{k+l}{k}\rfloor = f^*(k,l)\binom{k+l}{k}$, say. Then by the argument that proves the Erdős-Szekeres inequality

$$r(k+1, l+1) \leq r(k, l+1) + r(k+1, l),$$

we see that, within any red/blue colouring of the edges of K_n which does not contain a red K_{k+1} or a blue K_{l+1}, every vertex x can have red degree at most $r(k, l+1) - 1$ and blue degree at most $r(k+1, l) - 1$. Therefore, if d_x is the red degree of the vertex x (so that $n - 1 - d_x$ is the blue degree),

$$\begin{aligned} d_x &< r(k, l+1) \\ &\leq f(k-1, l)\binom{k+l-1}{k-1} \\ &= \frac{f(k-1,l)}{f^*(k,l)}\frac{k}{k+l}n. \end{aligned}$$

Similarly, we may use the fact that $n-1-d_x \leq r(k+1,l)-1$ to show that

$$d_x \geq \left(1 - \frac{f(k,l-1)}{f^*(k,l)} \frac{l}{k+l}\right) n.$$

Now, note that if f were always one, then we would know that d_x was less than $\frac{k}{k+l}n$ for each vertex x and also that it was greater than or equal to $\frac{k}{k+l}n$, a contradiction which is equivalent to the Erdős-Szekeres argument.

If, instead, we allow the size of $f(k,l)$ to change with both k and l, albeit slowly, then we find that for each vertex x the red degree d_x is not much greater than $\frac{k}{k+l}n$ nor much less than it. So we find that the graph is approximately regular in degree, the proximity to true regularity being dependent upon how slowly $f(k,l)$ changes.

This approximate degree-regularity is not however the only structural information that we have about graph colourings which contain neither a red K_{k+1} nor a blue K_{l+1}. We also know, for example, that in such a graph any red edge can lie in at most $r(k-1,l+1)-1$ red triangles, and if the vertices of this red edge are x and y then there are at most $r(k,l)-1$ vertices which are connected to x by a red edge and y by a blue edge. If we let d_{xy} be the number of vertices which are connected to both x and y by a red edge, then these two conditions are enough to tell us that

$$d_{xy} \approx \left(\frac{k}{k+l}\right)^2 n,$$

the exact proximity being again dependent upon the rate at which $f(k,l)$ changes. That is, providing that we don't try and improve too much on the Erdős-Szekeres bound, we can conclude that across any red edge we have approximately the expected number of red triangles (that would be in a random graph formed by choosing red edges with probability $\frac{k}{k+l}$). As a consequence, we see that across any red edge there are approximately the expected number of red C_4s of which the red edge is a diagonal. Importantly, this latter result is not restricted to red edges alone - it is straightforward to use the degree-regularity conditions and the analogous condition that we have approximately the expected number of blue C_4s across a blue edge in order to show that we have approximately the expected number of red C_4s across that edge as well.

At this stage it is appropriate to recall (at least roughly) the definition of quasirandomness: a regular or approximately regular graph is called quasirandom if it contains approximately the expected number of C_4s that would be in a random graph chosen with the edge probability dictated by the density of the graph (see for example [CGW89], [T87]). The standard results of the theory imply that if a graph satisfies this criterion then it also satisfies many of the properties that are expected with high probability of a random graph. For example, and this is what will be important to us, it contains approximately the expected number of any small graph.

The properties that we now know about a colouring of a K_n not containing either a red K_{k+1} or a blue K_{l+1} are enough to tell us that both the red and blue components of our colouring are quasirandom, so we see that in such a colouring we have approximately the expected number of any small graph in either colour. In particular, for any fixed r, we have approximately

$$\left(\frac{k}{k+l}\right)^{\binom{r}{2}} n^r$$

ordered red r-tuples (we find it more convenient to count r-tuples rather than K_rs since we then don't have to worry about multiple counting in our estimates, but it might perhaps be best to think of it in terms of counting K_rs).

If this were in fact precise then it would be inconsistent with the fact that any red $(r-1)$-tuple lies in at most $r(k-r+2, l+1)-1$ red r-tuples, since this gives an upper bound on the number of red r-tuples of

$$r(k-r+2, l+1)\left(\frac{k}{k+l}\right)^{\binom{r-1}{2}} n^{r-1},$$

which, since

$$r(k-r+2, l+1) \leq \frac{f(k-r+1, l)}{f^*(k,l)} \frac{k\cdots(k-r+2)}{(k+l)\cdots(k+l-r+2)} n,$$

is strictly less than the expected number if the rate of change of f is sufficiently small.

It is precisely this contradiction which allows us to prove our result. There are of course several technical caveats, the most interesting of which is that, in order to derive Theorem 1.1, it is not sufficient to know that the graph is simply quasirandom. It is necessary to apply instead our local condition that we have approximately the expected number of red C_4s across any given edge. Theorem 1.2, on the other hand, is derivable from the quasirandomness condition alone (though we will not do this here).

Secondly, the argument as stated above is slightly illusory - in order to derive a useful result it is necessary to take into account the fact that a change in the number of K_{r-1}s will be reflected by a change in the number of K_rs. Without doing this, we would be able to do no better than Thomason's result.

Where, incidentally, do we depart from Thomason's work? His proof is essentially the argument given above in the case $r=3$. He counts, in two different ways, the number of monochromatic triangles within a graph not containing a red K_{k+1} or a blue K_{l+1}, showing that, unless a bound of the form

$$r(k+1, l+1) \leq \exp\left\{-\frac{l}{2k}\log k + A\sqrt{\log k}\right\}\binom{k+l}{k}$$

held, there would be a contradiction. While his method of finding an upper bound for the number of monochromatic triangles is similar to ours above (the number of red triangles across a given red edge is at most $r(k-1,l+1)-1$, and we know, approximately, the number of red edges), his method for finding a lower bound is to apply Goodman's formula

$$T = \frac{1}{2}\left[\sum_x \binom{d_x}{2} + \sum_x \binom{n-1-d_x}{2} - \binom{n}{3}\right],$$

where by d_x we mean the red degree of the vertex x. This formula is only dependent upon the degree sequence, and so, knowing that every degree is approximately what's expected, we can show that the number of monochromatic triangles is approximately what's expected. Our main advance then is to have shown how we can use the quasirandomness conditions to circumvent the fact that there is no Goodman-type formula for $r \geq 4$.

This discussion raises one further question: why, if Thomason's result implies an off-diagonal estimate as well as a diagonal one, and our arguments are the natural extension of Thomason's argument, do we not also have off-diagonal theorems which include Theorem 1.1 and Theorem 1.2 as special cases? The first part of the answer is that for Theorem 1.2 we do, the following theorem being our main result in this case:

THEOREM 2.1. *Let s and ϵ be fixed positive constants with $\epsilon \leq 1$. Then there exists a constant $C_{s,\epsilon}$ such that for $k \geq l \geq \epsilon k$*

$$r(k+1,l+1) \leq C_{s,\epsilon} \exp\left\{-s\frac{l}{k}\log k\right\}\binom{k+l}{k}.$$

However, if we now fix ϵ and let s increase, the best theorem that we can deduce from our knowledge of the growth rate of the $C_{s,\epsilon}$ is

THEOREM 2.2. *For all $\epsilon \leq 1$ there exists a constant C_ϵ such that for all k and l with $k \geq l \geq \epsilon k$,*

$$r(k+1,l+1) \leq \exp\{-C_\epsilon \log^{3/2} k\}\binom{k+l}{k}.$$

Interestingly, as with Theorem 1.2, we only need the ordinary quasirandomness condition and not its local counterpart to derive these results. It is only when we make ϵ tend towards 1 as s increases that our local conditions become genuinely useful. So the reason why we don't have an off-diagonal version of Theorem 1.1 now emerges: our method doesn't allow one.

We will not prove Theorems 2.1 and 2.2 in this paper, concentrating instead on the diagonal results. It should however be clear to the reader how we can go about changing our main results in order to derive them.

We begin the proof proper in the next section by considering, more formally, the various regularity conditions that a graph containing neither a red

K_{k+1} nor a blue K_{l+1} must satisfy, and showing what these conditions imply about such a graph.

3. The Regularity Conditions

The following notation will prove essential to us in what follows:

Definition. Suppose we have a red/blue colouring of the edges of the complete graph on n vertices, and let V be the set of vertices. Then we define the balanced function of the colouring around probability p as the function $g : V \times V \to \mathbb{R}$ with

$$g(x, y) = A(x, y) - p,$$

where $A : V \times V \to \mathbb{R}$ is the characteristic function of red edges, that is $A(x, y)$ is 1 if there is a red edge between x and y and 0 otherwise.

Note that normally one chooses the probability p in such a way as to make $\sum_{x,y} g(x, y) = 0$, but for the sake of simplicity in our exposition, we will be centring around a probability which is not quite the correct balanced probability, but which is very close.

We will also need to introduce two constants, γ and δ, which bound the growth (or rather fall) of $f(k, l)$ with respect to k and l respectively. Our main result in the next section will be an inequality telling us what kind of rate of change of $f(k, l)$ is admissible. More specifically, we will assume that we have two real numbers γ and δ and a natural number $n = \lfloor f(k, l)\binom{k+l}{k}\rfloor = f^*(k, l)\binom{k+l}{k}$, such that for $m = 1, 2$ and $r - 1$, each of the inequalities

$$r(k + 1 - m, l + 1) \leq f(k - m, l)\binom{k - m + l}{k - m},$$

$$r(k + 1, l + 1 - m) \leq f(k, l - m)\binom{k + l - m}{k},$$

$$\frac{f(k - m, l)}{f^*(k, l)} \leq 1 + m\gamma \text{ and } \frac{f(k, l - m)}{f^*(k, l)} \leq 1 + m\delta$$

holds. What we will show (by the counting K_rs argument we discussed in the last section) is that if $k \geq l$, where k and l are sufficiently large numbers of approximately the same magnitude, and if

$$k\gamma + l\delta \leq \frac{r - 3}{2}\frac{l}{k},$$

then

$$r(k + 1, l + 1) \leq f(k, l)\binom{k + l}{k}.$$

The conditions on γ and δ essentially amount to γ and δ being the partial derivatives of $\phi(k, l) = -\log f(k, l)$ with respect to k and l respectively. Thus, if we consider the inequality $k\gamma + l\delta \leq \frac{r-3}{2}\frac{l}{k}$ as a partial differential equation (by putting $\gamma = \frac{\partial\phi}{\partial k}$ and $\delta = \frac{\partial\phi}{\partial l}$), it is easy to see that taking $f(k,l) = \exp\{-\frac{r-3}{2}\frac{l}{k}\log k\}$ for $k \geq l$ works as a potential solution. Indeed a more careful treatment of this argument, taking into account the fact that γ and δ do not quite equal the respective derivatives, is what will allow us to derive our results.

The specifics of this must, however, wait until later sections. The task at hand is show what we can say about large graphs not containing either red K_{k+1}s or blue K_{l+1}s. We begin by writing our various regularity conditions as constraints on the size of certain products of the balanced function:

LEMMA 3.1. *Let k and l be natural numbers, let γ and δ be real numbers and let $n = \lfloor f(k,l)\binom{k+l}{k}\rfloor = f^*(k,l)\binom{k+l}{k}$. Suppose that for $m = 1$ and $m = 2$ each of the inequalities*

$$r(k+1-m, l+1) \leq f(k-m,l)\binom{k-m+l}{k-m},$$

$$r(k+1, l+1-m) \leq f(k,l-m)\binom{k+l-m}{k},$$

$$\frac{f(k-m,l)}{f^*(k,l)} \leq 1 + m\gamma \text{ and } \frac{f(k,l-m)}{f^*(k,l)} \leq 1 + m\delta$$

holds.

Then, in any red/blue colouring of K_n not containing either a red K_{k+1} or a blue K_{l+1}, the balanced function $g(x,y)$ of the colouring around $p = \frac{k}{k+l}$ satisfies

$$-\frac{l\delta}{k+l}n \leq \sum_y g(x,y) \leq \frac{k\gamma}{k+l}n$$

for all x, and

$$\sum_y g(x,y)g(y,z) \leq 2\frac{\max(k,l)}{(k+l)^2}(k\gamma + l\delta)n + 1$$

for all x and z with $x \neq z$.

Proof. The first part of the lemma follows from our observation in Section 2 that for any vertex x in our colouring we have

$$\left(1 - \frac{f(k,l-1)}{f^*(k,l)}\frac{l}{k+l}\right)n \leq d_x \leq \frac{f(k-1,l)}{f^*(k,l)}\frac{k}{k+l}n.$$

Noting that $d_x = \sum_y A(x,y)$, $A(x,y) = p + g(x,y)$, and applying our assumptions on the growth rate of f gives the required result. To prove the upper bound, for example, note that

$$\begin{aligned}\frac{k}{k+l}n + \sum_y g(x,y) = d_x \\ \leq \frac{f(k-1,l)}{f^*(k,l)}\frac{k}{k+l}n \\ \leq (1+\gamma)\frac{k}{k+l}n.\end{aligned}$$

Subtracting $\frac{k}{k+l}n$ from either side then gives the required bound.

For the second part of the lemma note that no red edge (x,z) can lie in more than $r(k-1,l+1)-1$ red triangles. This implies that

$$\sum_y A(x,y)A(y,z) \leq r(k-1,l+1)-1.$$

If we split up the left-hand side we then get, by using the conditions of the theorem, that

$$p^2n + p\sum_y g(x,y) + p\sum_y g(y,z) + \sum_y g(x,y)g(y,z) \leq p^2(1+2\gamma)n,$$

and hence by the first part of the lemma

$$\sum_y g(x,y)g(y,z) \leq 2\frac{k}{(k+l)^2}(k\gamma + l\delta)n.$$

The result follows similarly for blue edges, although we need to be a little bit careful, since we get two extra degenerate "triangles" (those for which $y = x$ or $y = z$). □

In counting the number of red K_rs in a given colouring, we will use the following notation:

Notation. Fix a red/blue colouring on K_n and let $g(x,y)$ be the balanced function of the colouring around probability p. Suppose also that K_r is the complete graph on the r vertices $v_1, v_2, \cdots, v_r$, with $r \leq n$. Then, for every subgraph H of K_r,we write

$$g_H = \sum_{x_1,\cdots,x_r} \prod_{(v_i,v_j)\in E(H)} g(x_i,x_j),$$

where the sum is taken over all r-tuples of vertices in K_n (including degenerate terms where two or more of the x_i are the same).

By rights this is a function of n and r as well as H, but we will be almost universally consistent about counting K_rs within K_ns, so these labels are essentially redundant.

Given this notation, the number of red K_rs (or rather red r-tuples) in a colouring of K_n is given by

$$\sum_{x_1,\cdots,x_r} \prod_{(v_i,v_j)\in E(K_r)} A(x_i,x_j) = \sum_{x_1,\cdots,x_r} \prod_{(v_i,v_j)\in E(K_r)} (p+g(x_i,x_j))$$
$$= \sum_{H\subset K_r} p^{\binom{r}{2}-e(H)} g_H,$$

where, again, the sum is taken over all r-tuples of vertices in K_n. So in order to estimate the number of K_rs we will need to be able to estimate g_H for each and every subgraph H of K_r. Almost all of the estimates we will need are encapsulated in the next lemma, which shows how we may use our local quasirandomness condition to obtain estimates on products of the balanced function.

Utilising the information provided by the previous lemma, we shall now assume that we have $\sum_y g(x,y)g(y,z) \leq \nu n$ for all x and z with $x \neq z$, where ν is some positive constant. The next lemma tells us that if H has a vertex of degree d then (to the highest order in n) $|g_H| \leq \sqrt{2}\nu^{d/2}n^r$. Within the statement of the lemma, we will make the simple assumption that $\nu \leq 1$. This is not strictly necessary but tidies up the form of the lemma, and as we shall see later is trivially satisfied for k and l large.

LEMMA 3.2. *Suppose that the balanced function $g(x,y)$ of a red/blue colouring of a graph on n vertices satisfies*

$$\sum_y g(x,y)g(y,z) \leq \nu n$$

for all x and z with $x \neq z$, and some fixed positive real ν. Then, provided that $\nu \leq 1$,

$$|\sum_y \sum_{x_1,\cdots,x_{c+d}} g(y,x_1)\cdots g(y,x_d)h(x_1,\cdots,x_{c+d})| \leq \sqrt{2}\nu^{d/2}n^{c+d+1} + \frac{1}{\sqrt{2}\nu^{d/2+1}}n^{c+d},$$

for any function h of $c+d$ vertices which is bounded above in absolute value by 1.

Proof. For d odd, we have

$$\begin{aligned}
&|\sum_y \sum_{x_1,\cdots,x_{c+d}} g(y,x_1)g(y,x_2)\cdots g(y,x_d)h(x_1,\cdots,x_{c+d})|^2\\
&\le n^{c+d} \sum_{x_1,\cdots,x_{c+d}} |\sum_y g(y,x_1)g(y,x_2)\cdots g(y,x_d)h(x_1,\cdots,x_{c+d})|^2\\
&\le n^{c+d} \sum_{x_1,\cdots,x_{c+d}} |\sum_y g(y,x_1)g(y,x_2)\cdots g(y,x_d)|^2\\
&= n^{2c+d} \sum_{y,y'} (\sum_x g(y,x)g(x,y'))^d\\
&\le \nu^d n^{2c+2d+2} + n^{2c+2d+1},
\end{aligned}$$

where the remainder comes from the degenerate terms. Since this is less than the square of

$$\nu^{d/2} n^{c+d+1} + \frac{1}{2\nu^{d/2}} n^{c+d},$$

we are done in this case.

For d even, the proof is the same until we reach the second last line, when we need to estimate

$$\sum_{y,y'} (\sum_x g(y,x)g(x,y'))^d.$$

To do this we split our sum into two pieces, a set P of edges (y,y') where $\sum_x g(y,x)g(x,y')$ is positive and a similar set N where $\sum_x g(y,x)g(x,y')$ is negative. Then the proof in the odd case tells us, since a sum of squares is positive, that

$$\sum_{(y,y')\in P} (\sum_x g(y,x)g(x,y'))^{d+1} \ge - \sum_{(y,y')\in N} (\sum_x g(y,x)g(x,y'))^{d+1}$$

which implies

$$\begin{aligned}
\sum_{y,y'} |\sum_x g(y,x)g(x,y')|^{d+1} &\le 2 \sum_{(y,y')\in P} (\sum_x g(y,x)g(x,y'))^{d+1}\\
&\le 2\nu^{d+1} n^{d+3} + 2n^{d+2}.
\end{aligned}$$

Finally, applying the power mean inequality, we get

$$\begin{aligned}
\sum_{y,y'} (\sum_x g(y,x)g(x,y'))^d &\le n^{\frac{2}{d+1}} (\sum_{y,y'} |\sum_x g(y,x)g(x,y')|^{d+1})^{\frac{d}{d+1}}\\
&\le n^{\frac{2}{d+1}} (2\nu^{d+1} n^{d+3} + 2n^{d+2})^{\frac{d}{d+1}}\\
&\le 2\nu^d n^{d+2} + \frac{2}{\nu} n^{d+1},
\end{aligned}$$

so we are done in this case as well. □

The result mentioned before the lemma now follows from taking y to be a vertex within H of degree d. The function h is then what remains, i.e. a certain product of balanced functions, and so satisfies the requirement of the lemma.

Ultimately, as we shall see in the next section, we would like to show that as many g_H terms as possible vanish to more than the first order in γ and δ. While the above results are sufficient to show that this is so when the graph H has maximum degree 3 or more, it still leaves a large collection of graphs of maximum degree 2 for which we have not reached this bound. The next lemma shows, however, that if we use the degree-regularity condition as well as the quasirandomness condition, then we have the required bounds except in the two cases where H is a K_2 or a K_3.

LEMMA 3.3. *Suppose that the balanced function $g(x, y)$ of a red/blue colouring of a graph on n vertices satisfies*

$$|\sum_y g(x,y)| \leq \mu n$$

for all x, and

$$\sum_y g(x,y)g(y,z) \leq \nu n$$

for all x and z with $x \neq z$, and some fixed positive constants μ and ν with $\nu \leq 1$. Then, for $l \geq 3$,

$$|\sum_{x_1,\cdots,x_l} g(x_1,x_2)g(x_2,x_3)\cdots g(x_{l-1},x_l)| \leq 2\mu^{l+1-2\lfloor l/2\rfloor}\nu^{\lfloor l/2\rfloor-1}n^l + \frac{2\mu^{l+1-2\lfloor l/2\rfloor}}{\nu^3}n^{l-1}$$

and

$$|\sum_{y_1,\cdots,y_l} g(y_1,y_2)g(y_2,y_3)\cdots g(y_l,y_1)| \leq 2\nu^{\lfloor l/2\rfloor}n^l + \frac{2}{\nu}n^{l-1}.$$

Proof. For the first part we simply apply the Hölder inequality:

$$|\sum_{x_1,\cdots,x_l} g(x_1,x_2)\cdots g(x_{l-1},x_l)|^{\lceil l/2\rceil}$$

$$= \left|\sum_{x_2,x_4,\cdots}\left(\sum_{x_1} g(x_1,x_2)\right)\left(\sum_{x_3} g(x_2,x_3)g(x_3,x_4)\right)\cdots\right|^{\lceil l/2\rceil}$$

$$\leq \left(\sum_{x_2,x_4,\cdots}|\sum_{x_1} g(x_1,x_2)|^{\lceil l/2\rceil}\right)\left(\sum_{x_2,x_4,\cdots}|\sum_{x_3} g(x_2,x_3)g(x_3,x_4)|^{\lceil l/2\rceil}\right)\cdots$$

$$\leq \left(\mu^{\lceil l/2\rceil}n^l\right)^{l+1-2\lfloor l/2\rfloor}\left(2\nu^{\lceil l/2\rceil}n^l + \frac{2}{\nu}n^{l-1}\right)^{\lfloor l/2\rfloor-1}$$

$$\leq \left(2\mu^{l+1-2\lfloor l/2\rfloor}\nu^{\lfloor l/2\rfloor-1}n^l\right)^{\lceil l/2\rceil}\left(1+\frac{1}{\nu^{\lceil l/2\rceil+1}n}\right)^{\lceil l/2\rceil},$$

which implies the result. The second part follows similarly. □

4. The Fundamental Lemma

In this section we will prove an extension of a lemma due to Thomason [T88] which gives an inequality telling us how quickly our function $f(k,l)$ may change. The main idea of our proof is one that we have already seen. Instead of counting the number of monochromatic triangles as Thomason did, we will count the number of monochromatic K_rs (or rather a certain weighted sum of the number of red K_rs and the number of blue K_rs), showing, using the fact that our graph must be random-like if it does not contain the required cliques, that this is approximately what is expected. On the other hand, we can again bound the number of monochromatic K_rs above using the following further generalisation of the Erdős-Szekeres condition: in a graph not containing a red K_{k+1} or a blue K_{l+1} any red K_{r-1} is contained in at most $r(k-r+2,l+1)-1$ red K_rs and any blue K_{r-1} is contained in at most $r(k+1,l-r+2)-1$ blue K_rs. Then since the number of K_{r-1}s can also be estimated (as approximately the expected number) we have an upper bound which we can balance against our lower bound.

Again, as we mentioned in the outline, it will be necessary in the proof to take into account the fact that the number of K_rs and the number of K_{r-1}s are not independent of one another, being composed almost entirely of like terms, although in different proportions. While most of these terms may be reduced to $o(1)$ factors at the outset as being quite unimportant to the argument, the terms coming from single edges and triangles, which are the highest order, and hence the critical, terms, will be left unestimated until after we have balanced the number of red K_rs against $r(k-r+2,l+1)$ times the number of red K_{r-1}s. Doing this allows us to reduce the error term coming from the single edges from being of the order of $r^2\sum_{x,y} g(x,y)$ to being $r\sum_{x,y} g(x,y)$, since the single edge terms which occur in counting the number of K_{r-1}s cancel out most of the like terms which we get in counting the number of K_rs. Without this care, our result would yield no improvement over the old bound.

Before we begin, we need to present a few more remarks in order to illuminate some of the assumptions of the lemma. What we will prove is that if k and l are sufficiently large depending on r, $k \geq l \geq \left(1-\frac{1}{r}\right)k$ and

$$k\gamma + l\delta \leq \frac{r-3}{2}\frac{l}{k},$$

then (given the obvious induction hypothesis), we have

$$r(k+1,l+1) \leq f(k,l)\binom{k+l}{k}.$$

Now, as at the start of Section 3, we see that with an inequality of this form, we expect $f(k,l)$ to be roughly of the form

$$\exp\left\{-\frac{r-3}{2}\frac{l}{k}\log k\right\},$$

or some multiple thereof. One result of this is that we expect both $|\gamma|$ and $|\delta|$ to be bounded by $\frac{r-3}{2}\frac{\log k}{k}$. Since our eventual hope is to prove that $f(k,l)$ has such a form we will in the course of our forthcoming proof, in order to simplify the final form of the result, make the assumptions that $f(k,l)$ is at the smallest equal to $\exp\{-r\frac{l}{k}\log k\}$ and that both $|\gamma|$ and $|\delta|$ are smaller than $r\frac{\log k}{k}$. There is no deep mystery to our using r rather than $\frac{r-3}{2}$ here. It's just neater, and makes the lemma look slightly more digestible.

We are now ready to begin the formalities.

Lemma 4.1. *Let r be a natural number, let γ and δ be real numbers and let $n = \lfloor f(k,l)\binom{k+l}{k}\rfloor = f^*(k,l)\binom{k+l}{k}$. Suppose that, for $m=1$, $m=2$ and $m=r-1$, each of the inequalities*

$$r(k+1-m,l+1) \le f(k-m,l)\binom{k-m+l}{k-m},$$

$$r(k+1,l+1-m) \le f(k,l-m)\binom{k+l-m}{k},$$

$$\frac{f(k-m,l)}{f^*(k,l)} \le 1+m\gamma \text{ and } \frac{f(k,l-m)}{f^*(k,l)} \le 1+m\delta$$

holds. Suppose also that

1. $k \ge l \ge \left(1-\frac{1}{r}\right)k$,
2. $|\gamma|$ *and* $|\delta|$ *are both smaller than* $r\frac{\log k}{k}$ *and*
3. $f(k,l) \ge \exp\{-r\frac{l}{k}\log k\}$.

Then there exists a constant c such that if k and l are both greater than r^{cr}, and

$$k\gamma + l\delta \le \frac{r-3}{2}\frac{l}{k},$$

the inequality

$$r(k+1,l+1) \le n \le f(k,l)\binom{k+l}{k}$$

holds.

Proof. To begin, note that, from Lemma 3.1, in a colouring avoiding red K_{k+1}s and blue K_{l+1}s, we must have that the balanced function $g(x,y)$ satisfies

$$-\frac{l\delta}{k+l}n \le \sum_y g(x,y) \le \frac{k\gamma}{k+l}n$$

for all x, and therefore, using assumption 2 of the lemma, we have that

$$|\sum_y g(x,y)| \leq r\frac{\log k}{k}n.$$

Also from Lemma 1, note that, since $k\gamma + l\delta \leq \frac{r-3}{2}$, we have that

$$\sum_y g(x,y)g(y,z) \leq \frac{r-3}{k+l}n$$

for all x and z with $x \neq z$ (we may subsume the O(1) term into the n term for k and l larger than some fixed constant - it is in performing this kind of estimate that we will use assumption 3 of the lemma).

For later brevity we will use the notation that $\sum_{x,y} g(x,y) = \frac{s}{k+l}n^2$, and we will also write $\sum_{x,y,z} g(x,y)g(y,z)g(z,x) = \frac{t}{k+l}n^3$. Moreover, we will denote the quantity $\frac{r-3}{k+l}$ by ν so that

$$\sum_y g(x,y)g(y,z) \leq \nu n,$$

noting that, for $k+l \geq r$, we have $\nu \leq 1$.

Recall that the number of r-tuples spanning a red clique is given by

$$\sum_{H \subset K_r} p^{\binom{r}{2}-e(H)} g_H.$$

Our first aim will be to show that the contribution of all terms in this sum other than the main term (corresponding to the null set), the edge terms and the triangle terms can be made smaller in absolute value than $\frac{1}{2^{\binom{r}{2}} r^{dr} k}$ for any fixed d, by taking k and l to be larger than r^{cr} for some appropriately large c.

Let us denote by S the set of subgraphs of K_r other than the null graph, the edges, and the triangles. We will split this set into two further subsets, S', the set of all subgraphs with maximum degree greater than or equal to 3, and S'', the complement of this set in S.

For graphs in S', Lemma 3.2 tells us that, for k and l greater than r^{cr},

$$\begin{aligned}
|g_H| &\leq \sqrt{2}\nu^{\Delta/2}n^r + \frac{1}{\sqrt{2}\nu^{\Delta/2+1}}n^{r-1} \\
&\leq \sqrt{2}\left(\frac{r}{k}\right)^{\Delta/2} n^r + \frac{1}{\sqrt{2}}\left(\frac{k}{r}\right)^{\Delta/2+1} n^{r-1} \\
&\leq \frac{1}{r^{c_1 \Delta r} k} n^r,
\end{aligned}$$

where Δ is the maximum degree of H, and where c_1 depends on and grows with c.

Now, every graph in S'' either contains a path of length two or a cycle of length 4, in which case we have, from Lemma 3.3, and using our bounds on

$\mu = \max_x |\sum_y g(x,y)|$ and ν, that

$$|g_H| \leq 2r^2 \frac{\log^2 k}{k^2} n^r + 2\left(\frac{k}{r}\right)^3 n^{r-1}$$
$$\leq \frac{1}{r^{c_2 r} k} n^r,$$

or is a product of single edges and triangles, in which case

$$|g_H| \leq 4r^2 \frac{\log^2 k}{k^2} n^r + 8n^{r-1} + 4\left(\frac{k}{r}\right)^2 n^{r-2}$$
$$\leq \frac{1}{r^{c_2 r} k} n^r,$$

where again c_2 is just some constant that grows with c.

Before we proceed with our estimate we also need to note firstly that the number of graphs with maximum degree Δ or less is at most $r^{\Delta r}$ and also that the maximum number of edges in such a graph is Δr (we may of course divide by a 2 here but this is not necessary for our estimates).

We now have, using the fact that p is between 1/3 and 1/2, that

$$\sum_{H \in S} p^{\binom{r}{2} - e(H)} |g_H| \leq \left(\frac{1}{2}\right)^{\binom{r}{2}} \sum_{H \in S'} \frac{3^{e(H)}}{r^{c_1 \Delta r} k} n^r + \sum_{H \in S''} \frac{3^{e(H)}}{r^{c_2 r} k} n^r$$
$$\leq \left(\frac{1}{2}\right)^{\binom{r}{2}} \sum_{\Delta=3}^{r} \frac{(3r)^{\Delta r}}{r^{c_1 \Delta r} k} n^r + \frac{(3r)^{2r}}{r^{c_2 r} k} n^r$$
$$\leq \frac{1}{2^{\binom{r}{2}} r^{dr} k} n^r$$

for c chosen sufficiently large depending on d.

So, getting back to our original intentions, we see that the number of r-tuples spanning a red K_r is greater than or equal to

$$\frac{k^{\binom{r}{2}}}{(k+l)^{\binom{r}{2}}} n^r + \binom{r}{2} \frac{k^{\binom{r}{2}-1}}{(k+l)^{\binom{r}{2}}} sn^r + \binom{r}{3} \frac{k^{\binom{r}{2}-3}}{(k+l)^{\binom{r}{2}-2}} tn^r - \frac{1}{2^{\binom{r}{2}} r^{dr} k} n^r.$$

On the other hand we have that the number of r-tuples with a red K_r across it is less than the number of $(r-1)$-tuples with a red K_{r-1} across it times $r(k+2-r, l+1)$. So we have that the number of K_rs is at most

$$\left(\frac{k^{\binom{r-1}{2}}}{(k+l)^{\binom{r-1}{2}}} n^{r-1} + \binom{r-1}{2} \frac{k^{\binom{r-1}{2}-1}}{(k+l)^{\binom{r-1}{2}}} sn^{r-1} + \binom{r-1}{3} \frac{k^{\binom{r-1}{2}-3}}{(k+l)^{\binom{r-1}{2}-2}} tn^{r-1} \right.$$
$$\left. + \frac{1}{(r-1)^{d(r-1)} k} n^{r-1} \right) \times \left(\frac{k(k-1)\cdots(k-r+2)}{(k+l)(k+l-1)\cdots(k+l-r+2)} \right) (1+(r-1)\gamma)n.$$

Now we are going to subtract the lower bound from the upper bound, and divide through by n^r to get an inequality which must hold if the graph on n

vertices contains neither a red K_{k+1} or a blue K_{l+1}. In so doing it is necessary to use the fact that

$$\left(\frac{k}{k+l}\right)^{r-2} - \left(\frac{(k-1)\cdots(k-r+2)}{(k+l-1)\cdots(k+l-r+2)}\right)$$
$$\geq \binom{r-1}{2}\frac{lk^{r-3}}{(k+l)(k+l-1)\cdots(k+l-r+2)} - \frac{2^r}{(k+l)^2}$$
$$\geq \binom{r-1}{2}\frac{lk^{r-3}}{(k+l)^{r-1}} - \frac{2^r}{(k+l)^2}$$

so that all of the second order terms arising from the use of $\left(\frac{(k-1)\cdots(k-r+2)}{(k+l-1)\cdots(k+l-r+2)}\right)$ instead of $\left(\frac{k}{k+l}\right)^{r-2}$, other than that coming from the main term, will in fact be $1/k^2$ terms or smaller still. They can therefore be subsumed into the remainder term. This yields

$$\binom{r-1}{2}\frac{lk^{\binom{r}{2}-1}}{(k+l)^{\binom{r}{2}+1}} + (r-1)\frac{k^{\binom{r}{2}-1}}{(k+l)^{\binom{r}{2}}}s + \binom{r-1}{2}\frac{k^{\binom{r}{2}-3}}{(k+l)^{\binom{r}{2}-2}}t$$
$$-(r-1)\frac{k^{\binom{r}{2}}\gamma}{(k+l)^{\binom{r}{2}}} \leq \frac{1}{2^{\binom{r}{2}}r^{Dr}k},$$

where D is again just some constant which grows with c.

Similarly, if we count blue K_rs, though we have to be a little bit careful about degenerate terms and the fact that $1-p$ may be slightly bigger than $1/2$, we get

$$\binom{r-1}{2}\frac{kl^{\binom{r}{2}-1}}{(k+l)^{\binom{r}{2}+1}} - (r-1)\frac{l^{\binom{r}{2}-1}}{(k+l)^{\binom{r}{2}}}s - \binom{r-1}{2}\frac{l^{\binom{r}{2}-3}}{(k+l)^{\binom{r}{2}-2}}t$$
$$-(r-1)\frac{l^{\binom{r}{2}}\delta}{(k+l)^{\binom{r}{2}}} \leq \frac{1}{2^{\binom{r}{2}}r^{Dr}k}.$$

Now we take the weighted sum of these inequalities in such a way as to make the triangle terms disappear, by adding $k^{\binom{r}{2}-3}$ times the first inequality to $l^{\binom{r}{2}-3}$ times the second, to get

$$\binom{r-1}{2}\frac{(kl)^{\binom{r}{2}-2}}{(k+l)^{\binom{r}{2}}}+(r-1)\frac{(k^2-l^2)(kl)^{\binom{r}{2}-3}}{(k+l)^{\binom{r}{2}}}s-(r-1)\frac{(k^3\gamma+l^3\delta)(kl)^{\binom{r}{2}-3}}{(k+l)^{\binom{r}{2}}} \leq \frac{2k^{\binom{r}{2}-4}}{2^{\binom{r}{2}}r^{Dr}}.$$

Finally, provided that $l \geq \left(1-\frac{1}{r}\right)k$ and c has been chosen large enough, we see that we can subsume the error term on the right hand side into the first term on the left hand side, the first term on the left hand side being then larger than

$$\binom{r-1}{2}\frac{k^{\binom{r}{2}-4}}{e^{2r}2^{\binom{r}{2}}}$$

(here we have used the fact that $1-\frac{1}{r}\geq e^{-2/r}$ for $r\geq 2$).

Therefore, subsuming this term, we see that

$$\frac{(r-1)(r-3)}{2}\frac{(kl)^{\binom{r}{2}-2}}{(k+l)^{\binom{r}{2}}}+(r-1)\frac{(k^2-l^2)(kl)^{\binom{r}{2}-3}}{(k+l)^{\binom{r}{2}}}s-(r-1)\frac{(k^3\gamma+l^3\delta)(kl)^{\binom{r}{2}-3}}{(k+l)^{\binom{r}{2}}}<0.$$

Simplifying gives

$$\frac{(r-3)}{2}kl+(k^2-l^2)s-(k^3\gamma+l^3\delta)<0.$$

Recall now that

$$s=\frac{(k+l)\sum_{x,y}g(x,y)}{n^2}\geq -l\delta,$$

so therefore, since $k\geq l$,

$$k\gamma+l\delta>\frac{r-3}{2}\frac{l}{k}.$$

This contradicts the assumptions of the lemma, and so we are done. □

5. Using the Inequality

All that now remains to be done is to find a function that satisfies the conditions of Lemma 4.1. The basic idea is to note that if we choose a continuously differentiable function $\alpha:[0,\infty)\to[0,\infty)$, then the function

$$f(k,l)=\exp\{-\alpha(l/k)\log(k+l)\}$$

satisfies the equation

$$k\gamma'+l\delta'=\alpha(l/k),$$

where by γ' and δ' we mean the derivatives of $-\log f(k,l)$ with respect to k and l.

To use this fact we will choose a function α_r which is everywhere less than or equal to the function κ_r, where

$$\kappa_r(x)=\begin{cases}0 & \text{if } 0\leq x<1-\frac{1}{r};\\ \frac{r-3}{2}x & \text{if } 1-\frac{1}{r}\leq x\leq 1;\\ \kappa_r(1/x) & \text{if } x\geq 1.\end{cases}$$

and which, moreover, is twice-differentiable. The specific function, if it is chosen appropriately, will then be such that the true γ and δ differ by very little from γ' and δ' when k and l are quite large, and this will allow us to conclude, for a suitably chosen α_r, that

$$k\gamma+l\delta\leq\frac{r-3}{2}\kappa_r(l/k).$$

It is easy then to check that for some large multiple of the function α_r the conditions of Lemma 4.1 are satisfied.

The first step in formalising this argument is to define an appropriate collection of functions α_r, which we do as follows:

Notation. Let $r \geq 4$ be a positive integer. We write $\beta_r : [0,1] \to [0,\infty)$ for the polynomial function given by

$$\beta_r(z) = 6z^5 - 15z^4 + 10z^3,$$

and $\alpha_r : [o,\infty) \to [0,\infty)$ for the function given by

$$\alpha_r(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1 - \frac{1}{2r}; \\ \frac{r-4}{4}\beta_r(2rx - (2r-1)) & \text{if } 1 - \frac{1}{2r} \leq x \leq 1; \\ \alpha_r(1/x) & \text{if } x \geq 1. \end{cases}$$

This slightly bizarre looking set of functions is chosen just so as to satisfy the following simple lemma:

LEMMA 5.1. *For all $r \geq 4$, α_r is a twice-differentiable function such that:*

1. *for $0 \leq x \leq 1$, $0 \leq \alpha_r(x) \leq \frac{r-4}{2}x$;*
2. *$|\alpha_r'(x)| \leq r^2$ and $|\alpha_r''(x)| \leq 20r^3$ for all x.*

Before we start into the next lemma, we will again need some notation:

Notation. Suppose that $r \geq 4$ is a fixed positive integer. We then write

$$\phi_r(k,l) = \alpha_r(l/k)\log(k+l).$$

Our aim now is to show that $f_r = \exp(-\phi_r)$ (or rather some large multiple of it) is an admissible function. The first step towards this is contained in the following lemma (note that this is essentially the same as Lemma 4 in [T88]):

LEMMA 5.2. *For k and l greater than or equal to $200r^{10}$, the inequalities*

$$\exp\{\phi_r(k,l) - \phi_r(k-m,l)\} \leq 1 + m\Gamma$$

and

$$\exp\{\phi_r(k,l) - \phi_r(k,l-m)\} \leq 1 + m\Delta,$$

where

$$\Gamma = \alpha_r(l/k)\frac{1}{k+l} - \alpha_r'(l/k)\frac{l\log(k+l)}{k^2} + \frac{1}{4(k+l)}$$

$$\Delta = \alpha_r(l/k)\frac{1}{k+l} + \alpha_r'(l/k)\frac{\log(k+l)}{k} + \frac{1}{4(k+l)}$$

hold for $m = 1, 2$ and $r - 1$.

Proof. If we regard $\phi_r(k,l)$ as a function of k with l fixed, then we have, using Taylor's Theorem and the fact that ϕ_r is twice differentiable, that

$$\phi_r(k,l) - \phi_r(k-m,l) = m\frac{\partial \phi_r}{\partial k}(k,l) - \frac{m^2}{2}\frac{\partial^2 \phi_r}{\partial k^2}(k-\theta m, l)$$

for some θ between 0 and 1. Now we have that

$$\frac{\partial \phi_r}{\partial k} = \alpha_r(l/k)\frac{1}{k+l} - \alpha_r'(l/k)\frac{l\log(k+l)}{k^2},$$

and

$$\frac{\partial^2 \phi_r}{\partial k^2}(k,l) = -\alpha_r(l/k)\frac{1}{(k+l)^2} - 2\alpha_r'(l/k)\frac{l}{k^2(k+l)} + 2\alpha_r'(l/k)\frac{l\log(k+l)}{k^3}$$
$$+\alpha_r''(l/k)\frac{l^2\log(k+l)}{k^4}.$$

Now note (by using part 2 of Lemma 5.1) that, for k and l both greater than or equal to $200r^{10}$, $|\frac{\partial^2\phi_r}{\partial k^2}(k,l)|$ is less than or equal to $\frac{1}{4r(k+l)}$.

Therefore, in this case, we have that

$$\phi_r(k,l) - \phi_r(k-m,l) \le m\left(\alpha_r(l/k)\frac{1}{k+l} - \alpha_r'(l/k)\frac{l\log(k+l)}{k^2} + \frac{1}{8(k+l)}\right).$$

For brevity let's call the right hand side mx, noting that for k and l greater than or equal to $200r^{10}$, $mx \le 1$.

Therefore, using the fact that $e^z \le 1 + z + z^2$ for $|z| \le 1$, we see that

$$\exp\{\phi_r(k,l) - \phi_r(k-m,l)\} \le 1 + mx + m^2x^2.$$

Note then that, as for the second derivative, by taking k and l larger than $200r^{10}$, we can make rx^2 smaller than $\frac{1}{8(k+l)}$. Therefore, adding everything together, we see that

$$x + mx^2 \le \alpha_r(l/k)\frac{1}{k+l} - \alpha_r'(l/k)\frac{l\log(k+l)}{k^2} + \frac{1}{4(k+l)},$$

which yields the required result. The result follows similarly for l. □

We are now ready to tie together everything we have learned in the preceding sections to prove a theorem improving the general upper bound for Ramsey numbers. This theorem is as follows:

THEOREM 5.1. *Let $r \ge 4$ be a fixed positive integer. Then there exists a constant c such that*

$$r(k+1, l+1) \le r^{cr^2}\exp\{-\phi_r(k,l)\}\binom{k+l}{k}.$$

Proof. To begin let us suppose that f_r is a function of the form $f_r(a,b) = C\exp\{-\phi_r(a,b)\}$ for some fixed constant C, and let $n = \lfloor f_r(k,l)\binom{k+l}{k}\rfloor = f_r^*(k,l)\binom{k=l}{k}$, say. Suppose also that $\kappa_r : [0,\infty) \to [0,\infty)$ is the function given by

$$\kappa_r(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 - \frac{1}{r}; \\ \frac{r-3}{2}x & \text{if } 1 - \frac{1}{r} \leq x \leq 1; \\ \kappa_r(1/x) & \text{if } x \geq 1. \end{cases}$$

Then, by Lemma 5.1, since $\alpha_r(x) \leq \frac{r-4}{2}x$, we see that, for $1 \geq x \geq 1 - \frac{1}{r}$, we have

$$\kappa_r(x) \geq \alpha_r(x) + \frac{1}{2}.$$

If we now choose k and l to both be greater than $200r^{10}$, we can apply Lemma 5.2 to see that

$$\frac{f_r(k-m,l)}{f_r(k,l)} = \exp\{\phi_r(k,l) - \phi_r(k-m,l)\} \leq 1 + m\Gamma,$$

where

$$\Gamma \leq \alpha_r(l/k)\frac{1}{k+l} - \alpha_r'(l/k)\frac{l\log(k+l)}{k^2} + \frac{1}{4(k+l)}.$$

Furthermore, we have that

$$\frac{f_r(k-m,l)}{f_r^*(k,l)} \leq \left(1 + \frac{1}{n}\right)\frac{f_r(k-m,l)}{f_r(k,l)} \leq 1 + m\gamma,$$

where

$$\gamma \leq \alpha_r(l/k)\frac{1}{k+l} - \alpha_r'(l/k)\frac{l\log(k+l)}{k^2} + \frac{1}{2(k+l)}.$$

Similarly, we have that, for k and l both larger than $200r^{10}$,

$$\frac{f_r(k,l-m)}{f_r^*(k,l)} \leq \left(1 + \frac{1}{n}\right)\frac{f_r(k,l-m)}{f_r(k,l)} \leq 1 + m\delta,$$

where

$$\delta \leq \alpha_r(l/k)\frac{1}{k+l} + \alpha_r'(l/k)\frac{\log(k+l)}{k} + \frac{1}{2(k+l)}.$$

Note therefore that

$$k\gamma + l\delta \leq \alpha_r(l/k) + \frac{1}{2} \leq \kappa_r(l/k),$$

provided that $1 \geq \min(\frac{l}{k}, \frac{k}{l}) \geq 1 - \frac{1}{r}$. For $\min(\frac{l}{k}, \frac{k}{l}) < 1 - \frac{1}{r}$, we have, provided k and l are large (again $200r^{10}$ will easily suffice), that $f_r(a,b)$ is constant (equal to 1) close to $(a,b) = (k,l)$ and so we again have

$$k\gamma + l\delta \leq \kappa_r(l/k).$$

Finally, choose k and l to be sufficiently large, greater than r^{cr}, for some appropriate c, such that Lemma 4.1 holds in the following form: suppose that for $m = 1, 2$ and $r-1$, each of the inequalities

$$r(k+1-m, l+1) \leq f(k-m, l)\binom{k-m+l}{k-m},$$

$$r(k+1, l+1-m) \leq f(k, l-m)\binom{k+l-m}{k},$$

$$\frac{f(k-m,l)}{f^*(k,l)} \leq 1 + m\gamma \text{ and } \frac{f(k,l-m)}{f^*(k,l)} \leq 1 + m\delta$$

holds. Suppose also that $|\gamma|$ and $|\delta|$ are both smaller than $r\frac{\log k}{k}$ and that $f(k,l) \geq \exp\{-r\frac{l}{k}\log k\}$. Then, provided that

$$k\gamma + l\delta \leq \kappa_r(l/k),$$

we have that

$$r(k+1, l+1) \leq f(k,l)\binom{k+l}{k}.$$

To conclude, suppose that $N > \max(200r^{10}, r^{cr}) = r^{cr}$, for c chosen large enough, and consider the function $f_r(a,b) = (2N)^r \exp\{-\phi_r(a,b)\}$. For either a or b less than or equal to N we have straightforwardly that for $a \geq b$ with $b \leq N$,

$$f_r(a,b) \geq \frac{(2N)^r}{(a+b)^{rb/a}} \geq 1,$$

using the fact that $(a+b)^{b/a}$ is a decreasing function in a. Now, both γ and δ defined above are less than or equal to $r\frac{\log k}{k}$, and $f(k,l)$ is certainly larger than $\exp\{-r\frac{l}{k}\log k\}$. Finally, we have by the construction of ϕ_r and the choice of N that

$$k\gamma + l\delta \leq \kappa_r(l/k).$$

Consequently our induction holds good with this function f_r. □

Theorem 1.2 is now a straightforward consequence of this theorem. The simple proof is in fact contained in the following proof of Theorem 1.1:

Proof of Theorem 1.1. From Theorem 5.1, we know that, for integers $r \geq 5$,

$$\begin{aligned} r(k+1, k+1) &\leq r^{cr^2} \exp\{-\phi_r(k,k)\}\binom{2k}{k} \\ &\leq r^{cr^2} \exp\{-\frac{r-4}{4}\log(2k)\}\binom{2k}{k} \\ &\leq \frac{r^{cr^2}}{k^{dr}}\binom{2k}{k} \end{aligned}$$

for some fixed constants c and d.

If now, for any sufficiently large k, we take $r = \lfloor \frac{d \log k}{2c \log \log k} \rfloor$ (a value which is close to that which minimises $\frac{r^{cr^2}}{k^{dr}}$), we see that for some constant C we have

$$r(k+1, k+1) \leq k^{-C \frac{\log k}{\log \log k}} \binom{2k}{k},$$

as required. □

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