Dynamics of holomorphic maps: Resurgence of Fatou coordinates, and Poly-time computability of Julia sets.

by

Artem Dudko

A thesis submitted in conformity with the requirements
for the degree of Doctor of Philosophy
Graduate Department of Mathematics
University of Toronto

Abstract

Dynamics of holomorphic maps:
Resurgence of Fatou coordinates, and
Poly-time computability of Julia sets.

Artem Dudko
Doctor of Philosophy
Graduate Department of Mathematics
University of Toronto
2012

The present thesis is dedicated to two topics in Dynamics of Holomorphic maps. The first topic is dynamics of simple parabolic germs at the origin. The second topic is Polynomial-time Computability of Julia sets.

Dynamics of simple parabolic germs. A simple parabolic germ at the origin has the form

$$F(w) = w + aw^2 + O(w^3).$$

By a linear change of coordinates one can further assume that $a = 1$. Such germs are of a great interest in modern Complex Dynamics (see *e.g.* [19], [20] and [30]). A local description of dynamics of parabolic germs is well-known (see *e.g.* [11], [25]). If we apply the change of coordinates $z = -1/w$ and consider the germ at infinity

$$f(z) = -1/F(-1/z), \text{ then } f(z) = z + 1 + O(z^{-1}).$$

There exist constants $c > 0, \pi > \alpha > \pi/2$ such that in the sectors

$$\{|\arg(z - c)| < \alpha\} \text{ and } \{|\arg(z + c) - \pi| < \alpha\}$$

there exist analytic solutions of the equation

$$\phi(f(z)) = \phi(z) + 1 \qquad (1)$$

with asymptotics

$$\phi(z) = \text{const} + z + A \log z + O(z^{-1})$$

at infinity. These solutions are known as *Fatou coordinates*, and using them, one can completely describe the local picture of the dynamics of $F(w)$.

Fatou coordinates help to answer a fundamental question in dynamics of parabolic germs: the description of analytic conjugacy invariants. In [16] J. Écalle and in [38] S. Voronin independently described a set of analytic invariants which completely determines a conjugacy class of a given germ with a parabolic fixed point. The space of Écalle-Voronin invariants is infinite-dimensional. For the case of a simple parabolic germ these invariants are two infinite sequences of complex numbers which can naturally be interpreted as Taylor's coefficients of two analytic germs H_f defined at $z = 0$ and $z = \infty$. S. Voronin used a dynamical approach to construct these invariants. The approach of J. Écalle relies on so called *Resurgence Theory.* Écalle's Resurgence results are of an independent interest as they give a precise asymptotic description of Fatou coordinates. In [15] and [16] Écalle gave a construction of Resurgence Theory for a particular case of a simple parabolic germ of the form $F(w) = w + w^2 + w^3 + O(w^4)$ and outlined an approach to the general case. In Chapter 1 we give new proofs of J. Écalle's Resurgence results for the general case.

Resurgence Theory is based on the observation that the equation (1) has a formal solution

$$\tilde{\phi}(z) = \text{const} + z + A \log z + \sum_{j=1}^{\infty} b_j z^{-j}, \tag{2}$$

where $\sum b_j z^{-j}$ is a divergent power series. Using Écalle's Resurgence Theory we will show that the right hand side of (2) can be interpreted as the asymptotic expansion of the Fatou coordinates at infinity. Moreover, the Fatou coordinates can be obtained from $\widetilde{\phi}$ using Borel-Laplace summation. In Chapter 1 we also give a new proof of validity of Écalle's construction of analytic conjugacy invariants for a germ with a simple parabolic fixed point.

Computability of Julia sets. Informally, a compact subset of the complex plane is called *computable* if it can be visualized on a computer screen with an arbitrarily high precision. Computer-generated images of mathematical objects play an important role in establishing new results. Among such images, Julia sets of rational functions occupy one of the most prominent position. Recently, it was shown that for a wide class of rational functions their Julia sets can be computed efficiently (see [7], [8], [29]) and yet some of those sets are uncomputable, and so cannot be visualized (see [9], [5]). Also, there are examples of computable Julia sets whose computational complexity is arbitrarily high (see [3]).

One of the natural open questions of computational complexity of Julia sets is how large is the class of rational functions (in a sense of Lebesgue measure on the parameter space) whose Julia set can be computed in a polynomial time. Informally speaking, such Julia sets are *easy* to simulate numerically.

Conjecture. *The class of rational functions of degree $d \geqslant 2$ whose Julia set can be computed in a polynomial time has a full measure in the space of parameters.*

The conjecture would imply, in particular, that typically the Julia set is poly-time computable, even when the Julia set contains critical points, and hence dynamics is not expanding. We make a natural step towards a proof of the conjecture.

Main Theorem. *Let f be a rational function of degree $d \geqslant 2$. Assume that for each critical point $c \in J_f$ the ω-limit set $\omega(c)$ does not contain either recurrent critical point or a parabolic periodic point of f. Then the Julia set J_f is computable in a polynomial time.*

Our results give first examples of poly-time computable Julia sets which contain a critical point.

Acknowledgements

It is my great pleasure to thank my supervisor Michael Yampolsky for suggesting the directions of the thesis, guiding my research and numerous fruitful discussions. It is hard to overestimate his help in preparing this work.

I am grateful to David Sauzin for his mentorship on the first half of this thesis and for generously sharing his knowledge and insights on Resurgence Theory. The results of Chapter 1 are a product of our collaboration with David.

I am grateful to Ida Bulat for being so helpful with all the administrative work.

Finally, I would like to thank to Department of Mathematics at University of Toronto, Ontario Graduate Scholarships, and Connaught Foundation for their generous support.

Contents

Chapter 1

Dynamics of Simple Parabolic Germs via Resurgence Theory

1.1 Introduction

Below we briefly describe the results of Chapter 1 which are obtained jointly with David Sauzin [14].

Let $F(w) \in \mathbb{C}\{w\}$ be a germ at the origin of the form

$$F(w) = w + w^2 + \alpha w^3 + O(w^4) \in \mathbb{C}\{w\}.$$

We will call such a germ *a simple parabolic germ at the origin*. For us it is convenient to work at infinity, in the coordinate $z = -1/w$. Introduce a germ at infinity

$$\begin{gathered} f(z) := -\frac{1}{F(-1/z)} = z + 1 + a(z), \quad \text{where} \\ a(z) = -\rho z^{-1} + O(z^{-2}) \in z^{-1}\mathbb{C}\{z^{-1}\}, \quad \rho = \alpha - 1. \end{gathered} \tag{1.1}$$

Let $T(z) := z + 1$ be the unit translation. Our starting point is the following:

Proposition 1. *There exists a unique formal series without constant term, $\tilde{\varphi}_*(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$ such that the formal transformation*

$$v_*(z) = z + \rho \log z + \tilde{\varphi}_*(z)$$

is a solution of the conjugacy equation

$$v \circ f = T \circ v. \tag{1.2}$$

The general solution of (1.2) in the set $z + \rho \log z + \mathbb{C}[[z^{-1}]]$ is $v = v_* + \text{const}$.

The formal transformations $v_* + \mathrm{const}$ were introduced by Écalle [16] under the name *iterators*, they are the first example in his theory of *resurgent functions* [15, 17]. The series v_* diverges everywhere. However, we will show that using Borel-Laplace summation method one can obtain from v_* two analytic functions v^+, v^- which satisfy equation (1.2). These functions are Fatou coordinates of the germ f with a parabolic fixed point at infinity.

For a formal series

$$\tilde{\varphi}(z) = \sum_{n \geqslant 0} a_n z^{-n}$$

its Borel transform is defined by

$$\hat{\varphi}(\zeta) := \sum_{n \geqslant 0} \frac{a_n \zeta^{n-1}}{(n-1)!}.$$

Definition 2. Let $\tilde{\varphi} \in z^{-1}\mathbb{C}[[z^{-1}]]$ and $\hat{\varphi}$ be its Borel transform. Let $I = (a, b)$, $0 < b - a < \pi$. We will say that $\tilde{\varphi}$ is 1-*summable in the arc of directions* I if the following is true:

3) $\hat{\varphi}$ converges near the origin;

2) $\hat{\varphi}$ extends analytically to a sector $\{\, \zeta = r\,\mathrm{e}^{i\theta} \in \mathbb{C};\ r > 0,\ \theta \in I \,\}$;

3) there exist continuous functions $C, \beta \colon I \to \mathbb{R}^+$ such that $|\hat{\varphi}(r\,\mathrm{e}^{\mathrm{i}\theta})| \leqslant C(\theta)\,\mathrm{e}^{\beta(\theta) r}$ for all $r > 0$ and $\theta \in I$.

The Borel-Laplace sum of such a series $\tilde{\varphi}$ is the analytic function

$$\hat{\mathcal{L}}^I \hat{\varphi}(z) := \int_0^{\mathrm{e}^{i\theta}\infty} \mathrm{e}^{-z\zeta} \hat{\varphi}(\zeta)\,\mathrm{d}\zeta, \quad \theta \in I$$

defined on the set

$$\Sigma(I, \beta) := \{z : Re(ze^{i\theta}) > \beta(\theta) \ \text{ for some } \ \theta \in I\}.$$

We prove the following two results about $\tilde{\varphi}_*$.

Theorem 3. *The series $\tilde{\varphi}_*$ is 1-summable in the arcs of directions $I^+ = (-\pi/2, \pi/2)$ and $I^- = (\pi/2, 3\pi/2)$. Let*

$$\varphi^+ = \hat{\mathcal{L}}^{I^+} \hat{\varphi}_*, \quad \varphi^- = \hat{\mathcal{L}}^{I^-} \hat{\varphi}_*$$

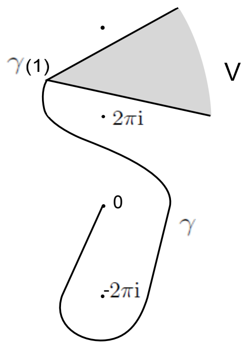

Figure 1.1: Illustration to Theorem 4.

be the corresponding Borel-Laplace sums. Then the functions

$$v^+(z) = z + \rho \log z + \varphi^+(z), \quad v^-(z) = z + \rho \log z + \varphi^-(z)$$

are analytic solutions of the equation (1.2)*.*

Theorem 4. *The germ $\hat{\varphi}_*$ admits an analytic continuation along each path $\gamma : [0,1] \to \mathbb{C}$ such that $\gamma(0) = 0$ and $\gamma((0,1]) \subset \mathbb{C} \setminus 2\pi\mathrm{i}\mathbb{Z}$. Let $a, b \subset I^+ \cup I^-$. Assume that the closed sector*

$$\{\gamma(1) + re^{\mathrm{i}\theta} : \theta \in [a,b], r \geqslant 0\}$$

does not intersect $2\pi\mathrm{i}\mathbb{Z}$. Then there exist constants C, β such that

$$\left| \left(\mathrm{cont}_\gamma \, \hat{\varphi}_* \right) \left(\gamma(1) + r\, \mathrm{e}^{\mathrm{i}\theta} \right) \right| \leqslant C\, \mathrm{e}^{\beta r}. \tag{1.3}$$

for all $r > 0$ and $\theta \in (a,b)$, where $\mathrm{cont}_\gamma \, \hat{\varphi}_*$ stands for the analytic continuation of $\hat{\varphi}_*$ along γ.

Écalle in [16] proved Theorems 3 and 4 for the case $\rho = 0$ and gave a plan for a possible proof in the case $\rho \neq 0$. His proof relies on the so called *mould expansion* of the formal iterator v^*. Here we prove Theorems 3 and 4 for an arbitrary ρ. Our methods are different from Écalle's. In particular, our proof does not involve mould expansions.

Recall a notion of an *asymptotic expansion* (see *e.g.* [28]):

Definition 5. Let V be a sector with the vertex at infinity of the form

$$V = \{z \in \mathbb{C} : |z| > R, \ \arg z \in (a,b)\}, \tag{1.4}$$

where $R > 0$ and $b - a < 2\pi$. Let φ be an analytic function in V and $\tilde{\varphi} = \sum_{n\geqslant 0} a_n z^{-n}$ be a formal series at infinity. We will say that $\tilde{\varphi}$ is an *asymptotic expansion* for φ if for every closed subsector $W \subset V$ and each $n \in \mathbb{N}$ there exists M such that

$$|z|^n \left| f(z) - \sum_{j=0}^{n-1} a_j z^{-j} \right| < M$$

for each $z \in W$. The asymptotic expansion $\tilde{\varphi}$ is said to be *Gevrey-k* if one can take

$$M = M(n) = (n!)^{1/k} A^n,$$

where $A > 0$ does not depend on n.

As a corollary of Theorem 3 we will obtain the following:

Theorem 6. *The formal series $\tilde{\varphi}_*(z)$ is a Gevrey-*1 *asymptotic expansion for both Borel-Laplace sums φ^+ and φ^- in sectors*

$$\{|\arg z| < \alpha, |z| > R\}, \quad \{|\arg z - \pi| < \alpha, |z| > R\}$$

respectively, for all $\pi/2 < \alpha < \pi$ and any R sufficiently large.

The first terms of the asymptotics for Fatou coordinates v^+, v^-:

$$v^\pm(z) = z + \rho \log z + O(z^{-1})$$

are well-known (see *e.g.* [35]). However, the claim that the formal iterator $v_*(z)$ asymptotically approximate Fatou coordinates at infinity was proven only recently by O. Lanford and M. Yampolsky in [20]. Theorem 6 is stronger than the result of O. Lanford and M. Yampolsky, since we obtain that the expansion is Gevrey-1.

Observe that two different analytic functions v^+ and v^- with overlapping domains share the same asymptotic expansion. The fact that the Borel-Laplace sums φ^+ and φ^- of $\tilde{\varphi}_*$ are different on the overlapping of their domains is due to existence of singularities of $\hat{\varphi}_*$ on the imaginary line. In Mathematical Physics, this is known as *Stokes phenomenon.*

One of the main results of Chapter 1 is so called *Bridge equation* formulated by J. Écalle in [16]. He gave a sketch of a proof of this equation. Écalle used the Bridge equation to introduce analytic conjugacy invariants of the germs of the form

$$F(w) = w + w^2 + O(w^3) \in \mathbb{C}\{w\}.$$

at the origin. We will formulate and prove the Bridge equation in Section 1.4.

1.1.1 Dynamics of parabolic germs

In this section we briefly recall description of dynamics of a germ with a parabolic fixed point at the origin. We refer the reader to [11] and [25] for details. Recall some important definitions.

Definition 7. Let $F(z) = \lambda z + O(z^2)$ be a germ with a fixed point at the origin. The origin is called a *parabolic fixed point* of f if $\lambda^q = 1$ for some positive integer q.

Observe that if $\lambda = e^{2\pi i p/q}$, then the iterate of the germ $F^{\circ q}$ has a parabolic fixed point at the origin with the multiplier 1. Replacing F with its iterate, if necessary, we will always assume that $\lambda = 1$. Let

$$F(z) = z(1 + az^n + \ldots), \quad \text{where} \quad a \neq 0.$$

The number n is called the *multiplicity* of the fixed point at the origin.

Definition 8. A complex number ν of modulus 1 is called an *attracting direction* if $a\nu^n < 0$ and a *repelling direction* if $a\nu^n > 0$.

This terminology has the following meaning:

Proposition 9. *If an orbit $\{F^{\circ k}(z)\}_{k \geqslant 0}$ converges to the origin then $F^{\circ k}(z)/|F^{\circ k}(z)|$ converges as $k \to \infty$ to one of the attracting directions.*

Let ν be the attracting direction such that $F^{\circ k}(z)/|F^{\circ k}(z)|$ converges as $k \to \infty$ to ν. Then we will say that the orbit $\{F^{\circ k}(z)\}_{k \geqslant 0}$ converges to the origin *from the direction of* ν.

Observe that a germ $F(z) = z(1 + az^n + \ldots)$ admits a local inverse $G(z)$ near the origin. That is there exists a germ $G(z)$ at the origin such that $F(G(z)) = z = G(F(z))$ for z in some neighborhood of the origin. We define a vector ν to be a *repelling direction* for $F(z)$ if ν is an attracting direction for $G(z)$.

Definition 10. Let ν be an attracting direction for F. A Jordan domain P is called an *attracting Fatou petal* for F if the following is true

- F is injective on P;
- $F(\overline{P} \setminus \{0\}) \subset P$;

- an orbit $\{F^{\circ k}(z)\}_{k\geqslant 0}$ converges to 0 from the direction ν if and only if P contains all but at most finitely many points of this orbit.

Similarly, U is called a *repelling petal* for F if U is an attracting petal for the local inverse F^{-1}.

Theorem 11. *Let $F(z)$ be a germ with a parabolic fixed point of multiplicity $n+1$ at the origin. There exists a collection of n pairwise disjoint attracting petals P_i^a, and n pairwise disjoint repelling petals P_i^r such that the following holds:*

- *the union $(\cup P_i^a)\cup(\cup P_i^r)\cup\{0\}$ forms an open simply connected neighborhood of the origin;*

- *an attracting and a repelling petal intersect if and only if the angle between the corresponding directions is π/n.*

In particular, if $n>1$ then each attracting petal intersects exactly 2 repelling petals and each repelling petal intersects exactly 2 attracting petals. If $n=1$ there is only one attracting petal and only one repelling petal; their intersection consists of two connected components.

To describe the local dynamics of F we will use auxiliary changes of coordinates. First, by a suitable conformal change of coordinates near the origin the germ $F(z)$ can be brought into the form (see *e.g.* [25]):

$$F(z)=z+z^{n+1}+\alpha z^{2n+1}+O(z^{3n+1}).$$

Further, let ν be an attracting direction. Consider the sector between two adjacent repelling directions containing ν. The change of coordinates

$$w=\kappa(z)=-\frac{1}{nz^n}$$

sends this sector bijectively to the complex plane with a cut along the negative real axis. The inverse of this change of coordinates κ is given by

$$\kappa^{-1}(w)=\left(-\frac{1}{nw}\right)^{\frac{1}{n}},$$

where the branch of the n-th root is chosen such that the image of 1 is equal to ν. In the new coordinate w the germ F has the following form:

$$f(w)=\kappa\circ F\circ\kappa^{-1}(w)=w+1+\frac{A}{w}+O\left(\frac{1}{w^2}\right),$$

where
$$A = \frac{1}{n}\left(\frac{n+1}{2} - \alpha\right).$$
In particular, near infinity f is closed to the unit translation. In fact, the following is true (see *e.g.* [11] and [25]):

Proposition 12. *Let P be an attracting or a repelling petal for F. Then there exists a conformal change of coordinates $\Phi : P \to \mathbb{C}$, conjugating F to the unit translation $T(z) = z + 1$:*
$$\Phi \circ F = T \circ \Phi. \tag{1.5}$$

By definition, for an attracting or a repelling petal P a conformal change of coordinates $\Phi : P \to \mathbb{C}$ satisfying (1.5) is called an *attracting* or a *repelling* Fatou coordinate correspondingly. For a given petal P Fatou coordinate is unique up to an additive constant. Namely, if Φ_1, Φ_2 are two Fatou coordinates on P, then $\Phi_1 - \Phi_2 = \text{const}$ on P.

For technical reasons, it is convenient to consider Fatou petals satisfying an additional property.

Definition 13. An attracting Fatou petal P is called *ample* if $\kappa(P)$ contains a sector of the form
$$\{z : \arg(z - a) \in (-\beta, \beta)\} \text{ for some } a > 0, \beta > \pi/2.$$

Similarly, one can define an ample repelling petal.

Lemma 14. *The petals in Theorem 11 can be chosen to be ample. Further, if P is an ample attracting petal, then $\Phi(P)$ contains a right half-plane, if P is an ample repelling petal, then $\Phi(P)$ contains a left half-plane.*

1.1.2 Horn maps and Écalle's analytic invariants

From now on we restrict ourselves to the case of multiplicity $n + 1 = 2$.

Definition 15. Let $F(z) = z + az^2 + \mathcal{O}(z^3)$ $a \neq 0$ be an analytic germ at the origin. Then the origin is called a *simple parabolic fixed point* of F. We will call such a germ F a *simple parabolic germ* at the origin.

Let $F(z) = z + az^2 + \mathcal{O}(z^3)$, $a \neq 0$, be a simple parabolic germ. By a linear change of coordinates we may further assume that $a = 1$. There exist an ample attracting petal P_a and an ample repelling petal P_r such that $P_a \cup P_r \cup \{0\}$ is an open simply connected neighborhood of the origin and $P_a \cap P_r$ consists of two domains. Let Φ_a and Φ_r be corresponding Fatou coordinates. Consider an equivalence relation $\sim$ on P_a given by

$$z_1 \sim z_2 \text{ if either } F^j(z_1) = z_2 \text{ or } F^j(z_2) = z_1 \text{ for some } j \geqslant 0.$$

In other words, two points are equivalent if they are on the same orbit. Then the quotient $\mathcal{C}_A := P_a/_\sim$ is a Riemann surface. Observe that two points $z_1, z_2 \in P_a$ belong to the same orbit if and only if $\Phi_a(z_1) - \Phi_a(z_2) \in \mathbb{Z}$. Since $\Phi_a(P_a)$ contains a right half plane, the quotient space $\Phi_a(P_a)/\mathbb{Z}$ is isomorphic to the cylinder $\mathbb{C}/\mathbb{Z}$. The Fatou coordinate Φ_a define by passage to quotients an isomorphism

$$\widetilde{\Phi}_a : \mathcal{C}_a \to \mathbb{C}/\mathbb{Z}, \text{ where } \widetilde{\Phi}_a(z) \equiv \Phi_a(z) \bmod \mathbb{Z}.$$

Similarly we define a Riemann surface $\mathcal{C}_R := P_r/_\sim$ and an isomorphism $\widetilde{\Phi}_r : \mathcal{C}_r \to \mathbb{C}/\mathbb{Z}$.

Observe that some orbits of F belong to both petals P_a and P_r. Therefore, some points from $\mathcal{C}_r$ and $\mathcal{C}_a$ are naturally identified. This allows us to define an analytic map $\tilde{h} = \widetilde{\Phi}_a \circ \widetilde{\Phi}_r^{-1}$. The domain of definition of this map is $(\Phi_r(P_a \cap P_r) \bmod \mathbb{Z}) \subset \mathbb{C}/\mathbb{Z}$. It contains two regions $\widetilde{W}^+$ and $\widetilde{W}^-$ such that

$$\widetilde{W}^+ \supset \{w \in \mathbb{C}/\mathbb{Z} : \mathrm{Im}(w) > A\}, \ \widetilde{W}^- \supset \{w \in \mathbb{C}/\mathbb{Z} : \mathrm{Im}(w) < A\}$$

for some large $A > 0$.

Proposition 16. *One has*

$$\mathrm{Im}\, \tilde{h}(w) \to \pm\infty \ \textit{ when } \ \mathrm{Im}\, w \to \pm\infty.$$

The cylinder $\mathbb{C}/\mathbb{Z}$ is naturally isomorphic to the punctured complex plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ with an isomorphism given by

$$\mathrm{ixp}(z) \equiv \exp(2\pi \mathrm{i} z).$$

Denote

$$H = \mathrm{ixp} \circ \tilde{h}^{-1} \circ \mathrm{ixp}^{-1}.$$

Then H is analytic in a punctured neighborhood of the origin and a punctured neighborhood of infinity. Moreover, H has removable singularities at 0 and ∞. Set $H(0) = 0$ and $H(\infty) = \infty$. Thus, H defines a pair of analytic germs in a neighborhood of the origin and a neighborhood of infinity. We will use the notation H_f to emphasize the dependence of H on f. Since the Fatou coordinates are defined up to additive constants, the pair of analytic germs H_f is defined up to pre- and postcomposition with mutliplication by a nonzero number. In other words H_f can be replaced with $cH_f(z/d)$, where $c, d \in \mathbb{C}^*$. The analytic germs H_f are called *Horn maps* or *Écalle-Voronin invariants*. The reason for the second name is the following statement due to Voronin and Écalle.

Theorem 17 ([16],[38]). *Let f and g be simple parabolic germs. Then f and g are conjugated by a conformal change of coordinates near the origin if and only if*

$$H_f(z) = cH_g(z/d), \quad \textit{for some } \; c, d \in \mathbb{C}^*.$$

Thus, the conformal conjugacy class of an analytic germ of the form $f(z) = z + z^2 + \mathcal{O}(z^3)$ is completely determined by the corresponding pair of analytic germs H_f. The latter can be selected arbitrarily.

Proposition 18 ([16],[38]). *Let H be a pair of analytic germs at the origin and at infinity such that $H'(0)H'(\infty) \neq 0$. Then there exists an analytic germ of the form*

$$f(z) = z + z^2 + \alpha z^3 + \mathcal{O}(z^4)$$

such that

$$H \equiv H_f.$$

Moreover,

$$H'(0)H'(\infty) = \exp(4\pi^2(1-\alpha)).$$

Further, the Horn maps H_f can be encoded by the Taylor coefficients $\{A_k\}_{k\geqslant 1}$ and $\{A_k\}_{k\geqslant -1}$ of the germs H_f at the origin and at infinity. In fact, Écalle used the name *analytic invariants* for a different sequence of coefficients (see [16]). Namely, consider the map

$$h = \Phi_a \circ \Phi_r^{-1} : \Phi_r(P_a \cap P_r) \to \mathbb{C}.$$

The map $h - \mathrm{Id}$ extends to a 1-periodic map defined on the union $W^+ \cup W^-$, where

$$W^+ = \{w : w \bmod \mathbb{Z} \in \widetilde{W^+}\}, \quad W^- = \{w : w \bmod \mathbb{Z} \in \widetilde{W^-}\}.$$

Let $B_k^+ = B_k^+(f)$ and $B_k^- = B_k^-(f)$ be the Fourier coefficients of the restrictions $(h - \mathrm{Id})_{|W^+}$ and $(h - \mathrm{Id})_{|W^-}$:

$$h(w) = w + \sum_{k\geqslant 0} B_k^+ e^{2\pi k i w}, \;\; \mathrm{Im}\, w \gg 0, \;\; h(w) = w + \sum_{k\geqslant 0} B_k^- e^{-2\pi k i w}, \;\; \mathrm{Im}\, w \ll 0. \quad (1.6)$$

The coefficients $B_k^{\pm}, k \geqslant 0$, are called *Écalle's analytic invariants.* They are related with the Horn maps by the following:

$$H_f(z) = z \operatorname{ixp}\left(\sum_{k\geqslant 0} B_k^+ z^k\right), \quad z \in U(0), \quad H_f(z) = z \operatorname{ixp}\left(\sum_{k\geqslant 0} B_k^- z^{-k}\right), \quad z \in U(\infty),$$

where $U(0)$ is a neighborhood of the origin and $U(\infty)$ is a neighborhood of ∞. In particular, the coefficients $A_j, j \geqslant 1$, can be written as polynomials of the coefficients $B_k^+, k \geqslant 0$, and vice versa. Similar relations hold between the coefficients $A_j, j \leqslant -1$, and $B_k^-, k \geqslant 0$.

Observe that the coefficients $B_k^{\pm}$ are not uniquely defined. For each $\alpha \in \mathbb{C}$ and each $\beta \in \mathbb{C} \setminus \{0\}$ these coefficients can be replaced with the coefficients $\widetilde{B}_k^{\pm}$, where

$$\widetilde{B}_0^{\pm} = \alpha + B_0^{\pm}, \quad \widetilde{B}_k^+ = \beta^k B_k^+, \quad \widetilde{B}_k^- = \beta^{-k} B_k^-.$$

In terms of Écalle's analytic invariants Theorem 17 can be rewritten in the following way:

Theorem 19. *Let f and g be simple parabolic germs. Then f and g are conjugated by a conformal change of coordinates near the origin if and only if*

$$B_0^+(f) - B_0^-(f) = B_0^+(g) - B_0^-(g), \quad B_k^+(g) = \beta^k B_k^+(f), \quad B_k^-(g) = \beta^{-k} B_k^-(f)$$

for all $k \geqslant 1$, where $\beta \neq 0$ is some constant.

In subsequent sections we will describe Écalle's construction of the invariants $B_k^{\pm}$ and give a proof of validity of this construction.

1.1.3 Borel-Laplace summation

In this subsection we explain the Borel-Laplace method of summation. We refer the reader to [12] and [15] for details and proofs. First, let us give a definition and describe properties of the Borel transform.

Borel transform

The formal Borel transform $\hat{\mathscr{B}}$ is defined as the linear isomorphism

$$\hat{\mathscr{B}}\colon\ z^{-1}\mathbb{C}[[z^{-1}]] \to \mathbb{C}[[\zeta]], \qquad \hat{\mathscr{B}}\left(\sum_{n\geqslant 0} c_n z^{-n-1}\right) = \sum_{n\geqslant 0} c_n \frac{\zeta^n}{n!}.$$

For a formal power series $\tilde{\varphi} \in z^{-1}\mathbb{C}[[z^{-1}]]$ we will use the notation $\hat{\varphi} := \hat{\mathscr{B}}\tilde{\varphi}$ to denote its Borel transform. The preimage under $\hat{\mathscr{B}}$ of the subspace $\mathbb{C}\{\zeta\}$ of convergent series at the origin is the subspace of *Gevrey-1* formal series without constant term $z^{-1}\mathbb{C}[[z^{-1}]]_1$ defined by

$$\begin{gathered}\tilde{\varphi}(z) = \sum_{n \geqslant 0} c_n z^{-n-1} \in z^{-1}\mathbb{C}[[z^{-1}]]_1 \quad \Leftrightarrow \\ \exists C, K > 0 \text{ such that } |c_n| \leqslant CK^n n! \; \forall n \geqslant 0.\end{gathered} \tag{1.7}$$

For such a formal series $\tilde{\varphi}(z)$, its Borel image $\hat{\varphi}(\zeta) \in \mathbb{C}\{\zeta\}$ defines a holomorphic function in the disk $D_{1/k}(0) = \{\, \zeta \in \mathbb{C} \mid |\zeta| < 1/K \,\}$.

Recall that for two germs $\hat{\varphi}, \hat{\psi} \in \mathbb{C}\{\zeta\}$ their convolution is a germ at the origin defined by

$$(\hat{\varphi} * \hat{\psi})(\zeta) = \int_0^\zeta \hat{\varphi}(\zeta_1)\hat{\psi}(\zeta - \zeta_1)\,\mathrm{d}\zeta_1. \tag{1.8}$$

Notice that the formula (1.8) gives an analytic germ defined in the intersection of the disks of convergence of $\hat{\varphi}$ and $\hat{\psi}$.

Consider the monomials

$$\hat{I}_k = \hat{\mathscr{B}}(z^{-k}), \quad \hat{I}_k(\zeta) = \frac{\zeta^{k-1}}{(k-1)!}, \quad k \in \mathbb{N}.$$

Integrating by parts one can show that

$$\hat{I}_k * \hat{I}_j = \hat{I}_{k+j} \text{ for each } k, j \in \mathbb{N}.$$

In other words, $\hat{\mathscr{B}}\big(z^{-k-j}\big) = \hat{\mathscr{B}}\big(z^{-k}\big) * \hat{\mathscr{B}}\big(z^{-j}\big)$. More generally, the following is true.

Lemma 20. *Let $\hat{\varphi}, \hat{\psi} \in \mathbb{C}\{\zeta\}$ be the formal Borel transforms of $\tilde{\varphi}, \tilde{\psi} \in z^{-1}\mathbb{C}[[z^{-1}]]_1$. Consider the product series $\tilde{\chi} = \tilde{\varphi}\tilde{\psi} \in \mathbb{C}[[z^{-1}]]_1$. Then its formal Borel transform is given by the convolution*

$$(\hat{\mathscr{B}}\tilde{\chi})(\zeta) = (\hat{\varphi} * \hat{\psi})(\zeta) = \int_0^\zeta \hat{\varphi}(\zeta_1)\hat{\psi}(\zeta - \zeta_1)\,\mathrm{d}\zeta_1. \tag{1.9}$$

Thus, the Borel transform induces an isomorphism from the algebra $(z^{-1}\mathbb{C}[[z^{-1}]]_1, \cdot)$ with the usual multiplication rule to the algebra $(\mathbb{C}\{\zeta\}, *)$ equipped with the convolution product. Observe that the algebras above have no unit, since we did not define the image of the element $1 \in \mathbb{C}[[z^{-1}]]$ by the Borel transform.

As for the differentiation operator $\partial = \frac{\mathrm{d}}{\mathrm{d}z}$, its Borel counterpart is the operator of multiplication by $(-\zeta)$:

$$\hat{\partial} = \hat{\mathscr{B}} \circ \partial \circ (\hat{\mathscr{B}})^{-1}, \quad \hat{\partial}\hat{\varphi}(\zeta) = -\zeta\hat{\varphi}(\zeta).$$

The case of a convergent formal series $\tilde{\varphi}(z)$ is characterized by the following:

$$\tilde{\varphi}(z) \in z^{-1}\mathbb{C}\{z^{-1}\} \quad \Leftrightarrow \quad \hat{\varphi}(\zeta) \text{ extends analytically to } \mathbb{C} \text{ and}$$
$$\exists C, \beta > 0 \text{ such that } |\hat{\varphi}(\zeta)| \leqslant C\,\mathrm{e}^{\beta|\zeta|} \text{ for all } \zeta \in \mathbb{C}. \tag{1.10}$$

Definition 21. For a formal power series $\tilde{\varphi} = \sum c_p z^{-p} \in \mathbb{C}[[z^{-1}]]$ define *order* of $\tilde{\varphi}$ by

$$\operatorname{ord}\big(\tilde{\varphi}\big) = \min\{\, p \mid c_p \neq 0 \,\}, \ \text{ if } \ \tilde{\varphi} \neq 0, \ \text{ and } \ \operatorname{ord}(0) = \infty.$$

Define a distance between formal power series $\tilde{\varphi}, \tilde{\psi} \in \mathbb{C}[[z^{-1}]]$ by

$$d\big(\tilde{\varphi}, \tilde{\psi}\big) := 2^{-\operatorname{ord}(\tilde{\psi}-\tilde{\varphi})}. \tag{1.11}$$

The topology on $\mathbb{C}[[z^{-1}]]$ generated by the distance (1.11) is called *topology of the formal convergence*, or *Krull topology.*

Remark 22. Observe that a sequence $(\tilde{\varphi}_k)_{k\in\mathbb{N}} \subset \mathbb{C}[[z^{-1}]]$ is convergent in the Krull topology if and only if for each p there exists l such that the coefficients at z^p of $\tilde{\varphi}_k$ are equal for all $k \geqslant l$.

Theorem 23. *The distance* (1.11) *makes* $\mathbb{C}[[z^{-1}]]$ *a complete ultrametric space.*

Let $T_s : z \to z + s$ be the translation by $s \in \mathbb{C}$. For a formal power series $\tilde{\varphi} \in \mathbb{C}[[z^{-1}]]$ we define $\tilde{\varphi} \circ T_s$ as a formally convergent series

$$\tilde{\varphi}(z+s) = \sum_{k\geqslant 0} \frac{s^k}{k!} \partial^k \tilde{\varphi}(z),$$

in which each of the terms is a formal power series. Here is another important property of the Borel transform:

$$\hat{\mathscr{B}}\,(\tilde{\varphi} \circ T_s) = e^{-\zeta s} \hat{\mathscr{B}} \tilde{\varphi} \ \text{ for any } \ \tilde{\varphi} \in z^{-1}\mathbb{C}[[z^{-1}]]. \tag{1.12}$$

Laplace transform and Borel-Laplace summation

Recall the definition of the Laplace transform. Denote by $V(\theta, \epsilon)$ a region of the form

$$V(\theta, \epsilon) = \{\zeta \in \mathbb{C} : |\zeta - re^{i\theta}| < \epsilon \ \text{ for some } \ r > 0\} \tag{1.13}$$

where $\theta \in \mathbb{R}, \epsilon > 0$.

Definition 24. Let $\hat{\varphi}$ be a germ at the origin which admits an analytic continuation into a strip $V(\theta, \epsilon)$. Assume that $\hat{\varphi}$ has an exponential type in the strip $V(\theta, \epsilon)$, that is for some constants $C, \beta > 0$

$$\left|\hat{\varphi}(\zeta)\right| \leqslant Ce^{\beta|\zeta|} \text{ for all } \zeta \in V(\theta, \epsilon). \tag{1.14}$$

Then the Laplace transform of $\hat{\varphi}$ in the direction θ is the function

$$\hat{\mathcal{L}}^{\theta}\hat{\varphi}(z) := \int_0^{\mathrm{e}^{i\theta}\infty} \mathrm{e}^{-z\zeta}\hat{\varphi}(\zeta)\,\mathrm{d}\zeta, \quad Re(z\,\mathrm{e}^{i\theta}) > \beta. \tag{1.15}$$

Observe that monomials $\hat{I}_k(\zeta) = \frac{\zeta^{k-1}}{(k-1)!}$, $k \in \mathbb{N}$, admit Laplace transform in any direction θ. One has

$$\hat{\mathcal{L}}^{\theta}\hat{\mathscr{B}}(z^{-k}) = \left(\hat{\mathcal{L}}^{\theta}\hat{I}_k\right)(z) = z^{-k}.$$

Thus, Borel transform can be viewed as a formal inverse of the Laplace transform.

Laplace transform sends a convolution of two germs to the product of the Laplace transforms of these germs.

Lemma 25. *Let $\hat{\varphi}_1, \hat{\varphi}_2$ be two germs at the origin which admit analytic continuation in a region of the form $V(\theta, \epsilon)$ and satisfy* (1.14). *Then $\hat{\varphi}_1 * \hat{\varphi}_2$ is a germ at the origin which admits an analytic continuation in $V(\theta, \epsilon)$ and for each $\beta' > \beta$ one has*

$$\left|\left(\hat{\varphi}_1 * \hat{\varphi}_2\right)(\zeta)\right| \leqslant C' e^{\beta'|\zeta|} \text{ for all } \zeta \in V(\theta, \epsilon),$$

where $C' = C'(\beta') > 0$ is some constant. Moreover,

$$\hat{\mathcal{L}}^{\theta}\left(\hat{\varphi}_1 * \hat{\varphi}_2\right)(z) = \hat{\mathcal{L}}^{\theta}\hat{\varphi}_1(z)\hat{\mathcal{L}}^{\theta}\hat{\varphi}_2(z). \tag{1.16}$$

Let $I = (a, b) \subset \mathbb{R}, 0 < b - a < \pi$. Let $\hat{\varphi}$ be a germ at the origin. Assume that

1) $\hat{\varphi}$ extends analytically to a sector $V := \{\, \zeta = r\,\mathrm{e}^{i\theta} \in \mathbb{C};\ r > 0,\ \theta \in I \,\}$,

2) there exist continuous functions $C, \beta\colon I \to \mathbb{R}^+$ such that $|\hat{\varphi}(r\,\mathrm{e}^{\mathrm{i}\theta})| \leqslant C(\theta)\,\mathrm{e}^{\beta(\theta)r}$ for all $r > 0$ and $\theta \in I$.

Any β as above is called an *estimated type function* for $\hat{\varphi}$. Using Cauchy Theorem one can show that the Laplace transforms $\hat{\mathcal{L}}^{\theta_1}\hat{\varphi}$ and $\hat{\mathcal{L}}^{\theta_2}\hat{\varphi}$, $\theta_1, \theta_2 \in I$, agree on the intersection of their domains. This allows us to introduce the function

$$\hat{\mathcal{L}}^{I}\hat{\varphi}(z) := \hat{\mathcal{L}}^{\theta}\hat{\varphi}(z), \quad z \in \Sigma(I, \beta) := \{z : Re(ze^{i\theta}) > \beta(\theta) \text{ for some } \theta \in I\}.$$

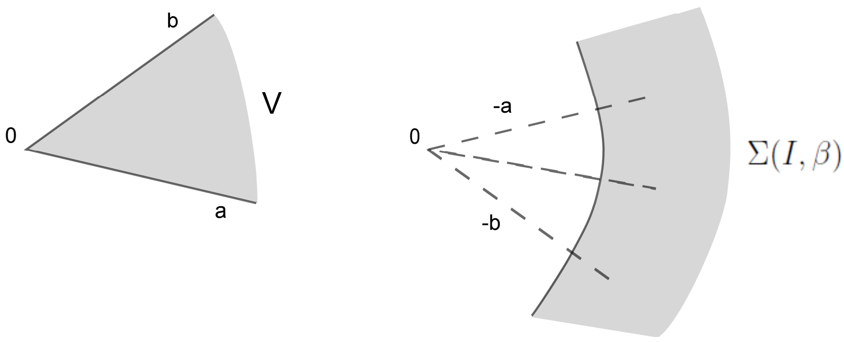

Figure 1.2: The sector V and the set $\Sigma(I, \beta)$.

Definition 26. Let $\tilde{\varphi} \in z^{-1}\mathbb{C}[[z^{-1}]]_1$ and $\hat{\varphi} = \hat{\mathscr{B}}\tilde{\varphi}$. Let $I = (a, b)$, $0 < b - a \leqslant \pi$. We will say that $\tilde{\varphi}$ is 1-*summable in the arc of directions* I if there exists continuous functions $C, \beta \colon I \to \mathbb{R}^+$ such that conditions 1) and 2) above hold. We will call the function $\hat{\mathcal{L}}^I\hat{\varphi}$ the *Borel-Laplace sum* of $\tilde{\varphi}$.

Gathering together properties of Borel and Laplace transforms we obtain that the Borel-Laplace summation commutes with the operations of differentiation and multiplication.

Proposition 27. *Let $\tilde{\varphi}$ and $\tilde{\psi}$ be* 1*-summable in the arc of directions I with a common estimated type function β. Then $\partial\tilde{\varphi}$ and $\tilde{\varphi}\tilde{\psi}$ are* 1*-summable in the arc of directions I with estimated type function $\beta' = \beta + \varepsilon$ for any positive constant ε. Moreover,*

$$\hat{\mathcal{L}}^I\big(\hat{\partial}\hat{\varphi}\big) = \partial\hat{\mathcal{L}}^I\hat{\varphi}, \quad \hat{\mathcal{L}}^I\big(\hat{\varphi} * \hat{\psi}\big) = \big(\hat{\mathcal{L}}^I\hat{\varphi}\big)\big(\hat{\mathcal{L}}^I\hat{\psi}\big).$$

In the next proposition we assume that $b(z) \in z^{-1}\mathbb{C}\{z^{-1}\}$. By (1.10) the function $\hat{b}$ is entire and there exists $C_0, \beta_0 > 0$ such that

$$|\hat{b}(\zeta)| \leqslant C_0\,\mathrm{e}^{\beta_0|\zeta|} \text{ for all } \zeta \in \mathbb{C}. \tag{1.17}$$

Proposition 28. *Let $\tilde{\varphi}(z)$ be* 1*-summable in an arc of directions I and let β_1 be an estimated type function for $\tilde{\varphi}$. Let $s \in \mathbb{C}, b(z) \in z^{-1}\mathbb{C}\{z^{-1}\}$ and $C_0, \beta_0 > 0$ are as in* (1.17)*. Then $\tilde{\varphi}\circ T_s$ and $\tilde{\varphi}\circ(\mathrm{Id} + b)$ are* 1*-summable in the arc of directions I with estimated type functions $\beta_2(\theta) = \beta_1(\theta) + |s|$ and $\beta_3(\theta) = \max\{\beta_1(\theta) + C_0, \beta_0 + 1\}$ correspondingly. Moreover,*

$$\hat{\mathcal{L}}^I\hat{\mathscr{B}}(\tilde{\varphi} \circ T_s) = \big(\hat{\mathcal{L}}^I\hat{\mathscr{B}}\tilde{\varphi}\big) \circ T_s, \quad \hat{\mathcal{L}}^I\hat{\mathscr{B}}\big(\tilde{\varphi} \circ (\mathrm{Id} + b)\big) = \big(\hat{\mathcal{L}}^I\hat{\mathscr{B}}\tilde{\varphi}\big) \circ (\mathrm{Id} + b).$$

Proof. The formal series $\tilde{\varphi}\circ T_s$ can be defined as a formally convergent series

$$\tilde{\varphi}\circ T_s(z)=\sum_{r\geqslant 0}\frac{s^r}{r!}\partial^r\tilde{\varphi}(z).$$

Let $\hat{\varphi}=\hat{\mathscr{B}}\tilde{\varphi}$. One has:

$$\hat{\mathscr{B}}\big(\tilde{\varphi}\circ T_s\big)=\sum_{r\geqslant 0}\frac{(-s\zeta)^r}{r!}\hat{\varphi}=\mathrm{e}^{-s\zeta}\hat{\varphi},$$

which implies 1−summability of $\tilde{\varphi}\circ T_s$ in the arc of directions I with estimated type function $\beta_2\coloneqq\beta_1+|s|$. Further, applying Laplace transform to $e^{-s\zeta}\hat{\varphi}$ for $z\in\Sigma(I,\beta_2)$ and $\theta\in I$ such that $\mathrm{Re}(ze^{i\theta})>\beta_2(\theta)$ we obtain

$$\big(\hat{\mathcal{L}}^I\big(\mathrm{e}^{-s\zeta}\hat{\varphi}\big)\big)(z)=\int_0^{\mathrm{e}^{i\theta}\infty}\mathrm{e}^{-\zeta z}\mathrm{e}^{-\zeta s}\hat{\varphi}(\zeta)\mathrm{d}\zeta=\int_0^{\mathrm{e}^{i\theta}\infty}\mathrm{e}^{-\zeta(z+s)}\hat{\varphi}(\zeta)\mathrm{d}\zeta=\big(\hat{\mathcal{L}}^I\hat{\varphi}\big)(z+s).$$

Set $\tilde{\chi}=\tilde{\varphi}\big(z+b(z)\big)$. Then

$$\tilde{\chi}(z)=\sum_{r\geqslant 0}\frac{b(z)^r}{r!}\partial^r\tilde{\varphi}(z).$$

Thus, Borel transform of the formal series $\tilde{\chi}$ can be written as a formally convergent sum

$$\begin{gathered}\hat{\chi}=\sum_{r\geqslant 0}\hat{\chi}_r,\quad\text{where }\ \hat{\chi}_r=\Big((-\zeta)^r\hat{\varphi}\Big)*\hat{b}_r\ \text{ and}\\ \hat{b}_r=\frac{1}{r!}\hat{\mathscr{B}}(b^r)=\frac{1}{r!}\hat{b}*\cdots*\hat{b}\ (r\text{ times}).\end{gathered}\tag{1.18}$$

Let us show by induction that for each $r\in\mathbb{N}$

$$\Big|\hat{b}_r(\zeta)\Big|\leqslant\frac{C_0^r|\zeta|^{r-1}}{r!(r-1)!}\mathrm{e}^{\beta_0|\zeta|}\tag{1.19}$$

for all $\zeta\in\mathbb{C}$. The base of induction $r=1$ is true by (1.17). Assume that (1.19) is true for a given r. Then one has

$$\hat{b}_{r+1}(\zeta)=\tfrac{1}{r+1}\hat{b}_r(\zeta)*\hat{b}(\zeta)=\int_0^{\zeta}\hat{b}_r(\zeta-\zeta_1)\hat{b}(\zeta_1)\mathrm{d}\zeta_1,\ \ \text{therefore}$$

$$\big|\hat{b}_{r+1}(\zeta)\big|\leqslant\tfrac{1}{r+1}\int_0^{|\zeta|}\tfrac{C_0^r}{r!(r-1)!}s^{r-1}e^{\beta_0 s}\cdot C_0e^{\beta_0(|\zeta|-s)}\mathrm{d}s=\frac{C_0^{r+1}|\zeta|^r}{(r+1)!r!}\mathrm{e}^{\beta_0|\zeta|},$$

which proves the inductive step. For our purposes the following estimate rougher than (1.19) will be enough:

$$\Big|\hat{b}_r(\zeta)\Big|\leqslant\frac{C_0^r}{r!}\mathrm{e}^{(\beta_0+1)|\zeta|}$$

Let $\zeta \in \mathbb{C}$ such that $\theta := \arg\zeta \in I$. For simplicity we will assume that $C_0+\beta_1(\theta) \neq \beta_0+1$ (the opposite case can be treated similarly). Then one has

$$\left|\hat{\chi}(\zeta)\right| \leqslant \sum_{r\geqslant 0}\left|\hat{\chi}_r(\zeta)\right| \leqslant \sum_{r\geqslant 0}\int\limits_0^{|\zeta|} s^r C(\theta) e^{\beta_1(\theta)s}\frac{C_0^r}{r!}e^{(\beta_0+1)(|\zeta|-s)}\mathrm{d}s =$$

$$C(\theta)\int\limits_0^{|\zeta|} e^{(\beta_1(\theta)+C_0)s}e^{(\beta_0+1)(|\zeta|-s)}\mathrm{d}s = C(\theta)\frac{e^{(\beta_1(\theta)+C_0)|\zeta|}-e^{(\beta_0+1)|\zeta|}}{\beta_1(\theta)+C_0-\beta_0-1}.$$

This proves that $\tilde{\chi}(z) = \tilde{\varphi}\big(z+b(z)\big)$ is 1−summable in the arc of directions I with estimated type function $\beta_3(\theta) = \max\{\beta_1(\theta)+C_0, \beta_0+1\}$. Moreover, by Lebesgue's dominated convergence theorem and Proposition 27, for each $z \in \Sigma(I,\beta_3)$ one has

$$\big(\hat{\mathcal{L}}^I\hat{\chi}\big)(z) = \sum_{r\geqslant 0}\big(\hat{\mathcal{L}}^I\hat{\chi}_r\big)(z) = \sum_{r\geqslant 0}\frac{b^r(z)}{r!}\frac{\mathrm{d}^r}{\mathrm{d}z^r}\big(\hat{\mathcal{L}}^I\hat{\varphi}\big)(z).$$

By Taylor formula, the latter series converges to $\big(\hat{\mathcal{L}}^I\hat{\varphi}\big)(z+b(z))$, which finishes the proof. □

1.1.4 Asymptotic expansions

In this section we recall the notion of an *asymptotic expansion* (see *e.g.* [28]).

Definition 29. Let V be a sector with the vertex at infinity of the form

$$V = \{z \in \mathbb{C} : |z| > R,\ \arg z \in (a,b)\}, \tag{1.20}$$

where $R > 0$ and $b - a < 2\pi$. Let φ be an analytic function in V and $\tilde{\varphi} = \sum_{n\geqslant 0} a_n z^{-n}$ be a formal series at infinity. We will say that $\tilde{\varphi}$ is an *asymptotic expansion* for φ if for every subsector

$$W = \{z \in \mathbb{C} : |z| \geqslant R_1,\ \arg z \in [a_1,b_1]\} \subset V$$

and each $n \in \mathbb{N}$ there exists $M = M(n) > 0$ such that

$$|z|^n\left|f(z) - \sum_{j=0}^{n-1} a_j z^{-j}\right| < M$$

for each $z \in W$. The asymptotic expansion $\tilde{\varphi}$ is called *Gevrey*-1 if one can take $M(n) = (n!)^{1/k}A^n$, where $A > 0$ does not depend on n.

Definition 29 implies the following:

Proposition 30. *Let φ be an analytic function on a sector V of the form* (1.20)*. If $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are asymptotic expansions for φ, then $\tilde{\varphi}_1 = \tilde{\varphi}_2$.*

In this thesis we deal only with Gevrey-1 asymptotic expansions. The following result is classical (see *e.g.* [12], [22] and [33]).

Proposition 31. *Let $\tilde{\varphi} \in z^{-1}\mathbb{C}[[z^{-1}]]_1$ be 1-summable in an arc of directions $I = (a, b)$. Denote by $\varphi = \hat{\mathcal{L}}^I \hat{\varphi}$ the Borel-Laplace sum of $\tilde{\varphi}$. Then for each $\epsilon > 0$ there exists $R > 0$ such that $\tilde{\varphi}$ is a Gevrey-1 asymptotic expansion of φ in the sector*

$$V = \{z \in \mathbb{C} : |z| > R,\ \arg z \in (-b - \pi/2 + \epsilon, -a + \pi/2 - \epsilon)\}. \tag{1.21}$$

For the convenience of the reader we present here a proof of Proposition 31.

Proof. For simplicity assume that $\epsilon < \min\{1, (b-a)/2\}$. Let $\hat{\varphi}$ be the Borel transform of $\tilde{\varphi}$. The definition of 1-summability implies that there exist constants $C, \beta > 0$ such that

$$|\hat{\varphi}(\zeta)| \leqslant C\,\mathrm{e}^{\beta|\zeta|} \ \text{ for all } \ \zeta \in S := \{r e^{\mathrm{i}\theta} : \theta \in (a + \epsilon/4, b - \epsilon/4), r > 0\}. \tag{1.22}$$

Let $\zeta = re^{\mathrm{i}\theta}$, where $\theta \in (a + 3\epsilon/4, b - 3\epsilon/4), r > 0$. Then $D_{\epsilon r/4}(\zeta) \subset S$. In particular,

$$|\hat{\varphi}(\zeta_1)| \leqslant C e^{\beta(1+\epsilon/4)|\zeta|} \ \text{ for all } \ \zeta_1 \in D_{\epsilon r/4}(\zeta).$$

By the Cauchy estimates, for all $j \in \mathbb{N}$ we obtain:

$$|\hat{\varphi}^{(j)}(\zeta)| < CM^j \frac{j! e^{\beta'|\zeta|}}{|\zeta|^j}, \ \text{ where } \ M = 4/\epsilon, \beta' = (1 + \epsilon/4)\beta. \tag{1.23}$$

Fix $\delta > 0$. Set

$$R = (\beta' + \delta)/\sin(\epsilon/4).$$

Let V be as in (1.21). Fix $z \in V$. There exists $\theta \in (a + 3\epsilon/4, b - 3\epsilon/4)$ such that $|\arg(e^{\mathrm{i}\theta} z)| < \pi/2 - \epsilon/4$. Then

$$\mathrm{Re}(e^{\mathrm{i}\theta} z) > \mathrm{Re}(e^{\mathrm{i}(\pi/2 - \epsilon/4)}|z|) > \sin(\epsilon/4)R > \beta' + \delta.$$

By the definition of the Borel-Laplace sum, one has:

$$\varphi(z) = \hat{\mathcal{L}}^\theta \hat{\varphi}(z) := \int_0^{\mathrm{e}^{i\theta}\infty} \mathrm{e}^{-z\zeta} \hat{\varphi}(\zeta) \mathrm{d}\zeta.$$

Let $\tilde{\varphi}(z) = \sum_{j\geqslant 1} a_j z^{-j}$. Then

$$\hat{\varphi}(\zeta) = \sum_{j\geqslant 1} a_j \frac{\zeta^{j-1}}{(j-1)!}.$$

The inequality (1.23) implies that the integrals $\int_0^{\mathrm{e}^{i\theta}\infty} \mathrm{e}^{-z\zeta}\hat{\varphi}^{(j)}(\zeta)\mathrm{d}\zeta$ converges for all $j \in \mathbb{Z}_+$. Moreover, integrating by parts $n-1$ times, we obtain:

$$\varphi(z) = a_1 z^{-1} + z^{-1}\int_0^{\mathrm{e}^{i\theta}\infty} \mathrm{e}^{-z\zeta}\hat{\varphi}'(\zeta)\mathrm{d}\zeta = \ldots = \sum_{j=1}^{n-1} a_j z^{-j} + z^{-n}\int_0^{\mathrm{e}^{i\theta}\infty} \mathrm{e}^{-z\zeta}\hat{\varphi}^{(n)}(\zeta)\mathrm{d}\zeta.$$

It remains to bound the integral $\int_0^{\mathrm{e}^{i\theta}\infty} \mathrm{e}^{-z\zeta}\hat{\varphi}^{(n)}(\zeta)\mathrm{d}\zeta$. Since $\hat{\varphi}$ is a germ at the origin, there exists $K, s > 0$, such that

$$\left|\hat{\varphi}^{(j)}(\zeta)\right| \leqslant j!K^j$$

for all $j \in \mathbb{N}$ and $\zeta \in D_s(0)$. Combining this inequality with (1.23) we obtain

$$\left|\int_0^{\mathrm{e}^{i\theta}\infty} \mathrm{e}^{-z\zeta}\hat{\varphi}^{(n)}(\zeta)\mathrm{d}\zeta\right| \leqslant n!K^n \int\limits_0^s e^{-(\beta'+\delta)t}\mathrm{d}t + CM^n n! \int\limits_s^\infty \tfrac{e^{-\delta t}}{t^n}\mathrm{d}t <$$

$$sK^n n! + \tfrac{C}{\delta}\left(\tfrac{M}{s}\right)^n n! \leqslant C_1 A^n n!,$$

where $C_1 = s + C/\delta,\ A = \max\{K, M/s\}$. This finishes the proof. □

1.1.5 Two examples

In this paragraph on two examples we explain how Borel-Laplace summation can be used to solve analytically and to study the solutions of difference and differential equations.

Example 1: Euler's equation.

Consider the Euler's differential equation

$$t^2F'(t) + F(t) = t. \tag{1.24}$$

For us it is convenient to use a change coordinate $t = 1/z$. In coordinate z the Euler's equation has the following form:

$$f(z) - \partial f(z) = 1/z. \tag{1.25}$$

It is easy to check that this equation has a solution $\tilde{f}(z)$ in terms of formal power series in z^{-1}:

$$\tilde{f}(z) = \sum_{n\geqslant 0} (-1)^n n! z^{-n-1}.$$

This series is divergent everywhere, however it satisfies the Gevrey-1 condition (see (1.7)). Thus, $\hat{f} := \hat{\mathscr{B}}\tilde{f}$ is an analytic germ in a neighborhood of the origin. In fact, we can compute it explicitly:

$$\hat{f}(\zeta) = \sum_{n\geqslant 0} (-1)^n \zeta^n = \frac{1}{\zeta+1}.$$

By Definition 26, for any arc of direction $I \subset (-\pi, \pi)$ and any $\beta > 0$ the formal series $\tilde{f}$ is 1-summable in the arc of directions I with an estimated type function β. The corresponding Borel-Laplace sum is

$$\hat{\mathcal{L}}^I \hat{f}(z) = \int_0^{e^{\mathrm{i}\theta}\infty} \frac{e^{-\zeta z}}{\zeta+1} \mathrm{d}\zeta, \quad \theta \in I, \mathrm{Re}(e^{\mathrm{i}\theta} z) > 0.$$

Proposition 27 implies that the map $\hat{\mathcal{L}}^I \hat{f}(z)$ satisfies the equation (1.25) on the domain of its definition. In particular, for $\theta = 0$ we obtain the classical real analytic solution of the Euler's equation on $\mathbb{R}^+$:

$$F(t) = \hat{\mathcal{L}}^0 \hat{f}(1/t) = \int_0^{\infty} \frac{e^{-x/t}}{x+1} \mathrm{d}x, \quad t > 0.$$

Observe that $\hat{f}$ has a unique singularity at -1. Let $z \in \mathbb{C}$, $\mathrm{Re}\, z < 0$, $\theta_1 \in (\pi/2, \pi)$ and $\theta_2 \in (-\pi, \pi/2)$ such that $\mathrm{Re}(e^{\mathrm{i}\theta_{1,2}} z) > 0$. Then Laplace transforms of $\hat{f}(\zeta)$ in directions θ_1 and θ_2 evaluated at z give different results. Namely, using Cauchy formula, we obtain:

$$\hat{\mathcal{L}}^{\theta_1} \hat{f}(z) - \hat{\mathcal{L}}^{\theta_2} \hat{f}(z) = 2\pi \mathrm{i} e^z.$$

Denote by $\underset{\bullet}{\mathbb{C}}$ the Riemann surface of the logarithm:

$$\underset{\bullet}{\mathbb{C}} = \{\zeta = r\,\mathrm{e}^{\mathrm{i}\theta} \mid r > 0,\ \theta \in \mathbb{R}\}.$$

The function $\hat{\mathcal{L}}^0 \hat{f}(re^{\mathrm{i}\alpha}), \alpha \in (-\pi/2, \pi/2), r > 0$, can be extended to an analytic function f on $\underset{\bullet}{\mathbb{C}}$ such that

- for each arc of directions $I = (a, b) \subset (-\pi, \pi)$ the function $\hat{\mathcal{L}}^I \hat{f}$ coincides with the restriction of f onto $\{re^{\mathrm{i}\alpha} : r > 0, \alpha \in (-b - \pi/2, -a + \pi/2)\}$;

- for each $r > 0$ and $\alpha \in \mathbb{R}$ one has:

$$f(re^{\mathrm{i}(\alpha+2\pi)}) - f(re^{\mathrm{i}\alpha}) = 2\pi\mathrm{i}e^{z}, \quad \text{where} \;\; z = re^{\mathrm{i}\alpha}.$$

Example 2: Abel's equation.

Consider the difference equation

$$\varphi(z+1) - \varphi(z) = a(z), \tag{1.26}$$

where $a(z)$ is a given function. This equation is called *Abel's equation.* We will assume that $a(z) \in z^{-2}\mathbb{C}\{z^{-1}\}$ is an analytic germ at infinity. First, consider (1.26) as an equation in formal series in z^{-1}. Let

$$\tilde{\varphi} \in z^{-1}\mathbb{C}[[z^{-1}]], \;\; \hat{\varphi} = \hat{\mathscr{B}}\tilde{\varphi}, \;\; \hat{a} = \hat{\mathscr{B}}a.$$

To find a formal (and then analytic) solution of equation (1.26) we transform it to an equation in terms of $\hat{\varphi}$. By (1.12) we obtain that $\tilde{\varphi}$ is a formal solution of (1.26) if and only if $\hat{\varphi}$ satisfies to

$$(e^{-\zeta} - 1)\hat{\varphi}(\zeta) = \hat{a}(\zeta).$$

By (1.10), $\hat{a}(\zeta)$ is entire and there exist constants $C_0, \beta_0 > 0$ such that

$$|\hat{a}(\zeta)| \leqslant C_0 e^{\beta_0|\zeta|} \quad \text{for all} \quad \zeta \in \mathbb{C}.$$

Observe that $\hat{a}(0) = 0$. It follows that the function

$$\hat{\varphi}(\zeta) = \hat{a}(\zeta)/(e^{-\zeta} - 1)$$

is analytic in $\mathbb{C} \setminus 2\pi\mathrm{i}\mathbb{Z}^*$, where $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$, and there exists a continuous function $C : (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2) \to \mathbb{R}_+$ such that

$$\left|\hat{\varphi}(re^{\mathrm{i}\theta})\right| \leqslant C(\theta)e^{\beta_0 r} \;\text{ for all }\; r > 0, \theta \in (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2).$$

Define $\tilde{\varphi} = \hat{\mathscr{B}}^{-1}\big(\hat{\varphi}\big)$. We obtain that $\tilde{\varphi}$ is 1-summable in the arcs of directions $I^+ = (-\pi/2, \pi/2)$ and $I^- = (\pi/2, 3\pi/2)$. Consider the Borel-Laplace sums of $\tilde{\varphi}$:

$$\varphi^+ = \hat{\mathcal{L}}^{I^+}\hat{\varphi}, \;\; \varphi^- = \hat{\mathcal{L}}^{I^-}\hat{\varphi}.$$

Using Proposition 28, we obtain that $\varphi^{\pm}$ are solutions of the equation (1.26). The functions $\varphi^{\pm}$ are defined and analytic on the domains

$$\Sigma(I^{\pm}, \beta_0) = \{z \in \mathbb{C} : \mathrm{Re}(ze^{\mathrm{i}\theta}) > \beta_0, \;\; \text{for some} \;\; \theta \in I^{\pm}\}.$$

These domains intersect by two halfplanes:

$$\Sigma(I^+,\beta_0)\cap\Sigma(I^-,\beta_0) = W^{up}\cup W^{low}, \;\; W^{up} = \{z : \operatorname{Im} z > \beta_0\}, \;\; W^{low} = \{z : \operatorname{Im} z < -\beta_0\}.$$

Though φ^+ and φ^- are Borel-Laplace sums of the same formal power series, generally, their values on the intersection of their domains are different. Let $z \in W^{up}$. Let $\theta_1 \in (0,\pi/2)$ and $\theta_2 \in (\pi/2,\pi)$ such that $\operatorname{Re}(e^{\mathrm{i}\theta_{1,2}}z) > \beta_0$. Then, using Residue Formula, we obtain

$$\varphi^+(z) - \varphi^-(z) = \int_0^{e^{\mathrm{i}\theta_1}\infty} \frac{e^{-\zeta z}\hat{a}(\zeta)}{e^{-\zeta}-1}\mathrm{d}\zeta - \int_0^{e^{\mathrm{i}\theta_2}\infty} \frac{e^{-\zeta z}\hat{a}(\zeta)}{e^{-\zeta}-1}\mathrm{d}\zeta =$$

$$2\pi\mathrm{i}\sum_{k\in\mathbb{N}} \operatorname{Res}\left(\frac{e^{-\zeta z}\hat{a}(\zeta)}{e^{-\zeta}-1}, 2\pi k\mathrm{i}\right) = -2\pi\mathrm{i}\sum_{k\in\mathbb{N}} \hat{a}(2\pi k\mathrm{i})e^{-2\pi k\mathrm{i}z}. \quad (1.27)$$

Similarly, for $z \in W^{low}$

$$\varphi^+(z) - \varphi^-(z) = 2\pi\mathrm{i}\sum_{k\in\mathbb{N}} \hat{a}(-2\pi k\mathrm{i})e^{2\pi k\mathrm{i}z}. \quad (1.28)$$

Thus, the difference $\varphi^+(z) - \varphi^-(z)$ is a periodic function of period 1 with Fourier coefficients $\big\{-2\pi\mathrm{i}\hat{a}(-2\pi k\mathrm{i})\big\}_{k\in\mathbb{N}}$ on W^{up} and $\big\{2\pi\mathrm{i}\hat{a}(2\pi k\mathrm{i})\big\}_{k\in\mathbb{N}}$ on W^{low}. Later we will study an equation more general than (1.26) and see more complicated phenomenons than those which occur for the solutions of the Euler's and the Abel's equations.

1.1.6 Resurgent functions

In the example of the Abel's equation the Borel transform of the formal solution extends to an analytic function on $\mathbb{C}\setminus 2\pi\mathrm{i}\mathbb{Z}^*$. The formal iterator $\tilde{\varphi}_*$ satisfies a more complicated equation. As we will prove later, the Borel transform $\hat{\varphi}_* = \hat{\mathscr{B}}\tilde{\varphi}_*$ is convergent near the origin and admits analytic continuation along each path avoiding integer multiples of $2\pi\mathrm{i}$ (see Theorems 3 and 4). However, in general, analytic continuations of $\hat{\varphi}_*$ along paths which are not homotopic in $\mathbb{C}\setminus 2\pi\mathrm{i}\mathbb{Z}$ give different results. It is convenient to state this property of analytic continuation on the appropriate Riemann surface.

Definition 32. Consider the set $\mathscr{P}_0$ of all (continuous) paths $\gamma\colon [0,1] \to \mathbb{C}$ such that either $\gamma\big([0,1]\big) = \{0\}$ or $\gamma(0) = 0$ and $\gamma\big((0,1]\big) \subset \mathbb{C}\setminus 2\pi\mathrm{i}\mathbb{Z}$. We denote by $\mathscr{R}_0$ the set of all equivalence classes of $\mathscr{P}_0$ for the relation of homotopy with fixed endpoints. The map $\gamma \in \mathscr{P}_0 \mapsto \gamma(1) \in \mathbb{C}\setminus 2\pi\mathrm{i}\mathbb{Z}^*$ (here $\mathbb{Z}^* = \mathbb{Z}\setminus\{0\}$) passes to the quotient and defines the "projection" $\pi_0\colon \mathscr{R}_0 \to \mathbb{C}\setminus 2\pi i\mathbb{Z}^*$. We equip $\mathscr{R}_0$ with the unique structure of Riemann surface which turns π_0 into a local biholomorphism.

Let $\gamma \in \mathscr{P}_0$. Then there exists a unique continuous path $\gamma' \subset \mathscr{R}_0$ such that $\gamma'(0) = 0$ and $\pi_0(\gamma'(t)) = \gamma(t)$ for each $t \in [0,1]$. The equivalence class of γ defines the point $\gamma'(1) \in \mathscr{R}_0$.

Observe that the universal cover $\mathscr{R}_1$ of $\mathbb{C}\setminus 2\pi i\mathbb{Z}$ with base point at 1 can be given an analogous definition:

Definition 33. Consider the space $\mathscr{P}_1$ of all paths $\gamma\colon\ [0,1] \to \mathbb{C}$ such that $\gamma(0) = 1$ and $\gamma\big((0,1]\big) \subset \mathbb{C}\setminus 2\pi i\mathbb{Z}$. We denote by $\mathscr{R}_1$ the set of all equivalence classes of $\mathscr{P}_1$ for the relation of homotopy with fixed endpoints. The map $\gamma \in \mathscr{P}_1 \mapsto \gamma(1) \in \mathbb{C}\setminus 2\pi i\mathbb{Z}$ passes to the quotient and defines the "projection" $\pi_1\colon \mathscr{R}_1 \to \mathbb{C}\setminus 2\pi i\mathbb{Z}$. We equip $\mathscr{R}_1$ with the unique structure of Riemann surface which turns π_1 into a local biholomorphism.

Both $\mathscr{R}_0$ and $\mathscr{R}_1$ are connected and simply connected Riemann surfaces. There is a unique point in $\mathscr{R}_0$ which projects onto 0 (the homotopy class of the trivial path $\gamma(t) \equiv 0$). This point is called the origin of $\mathscr{R}_0$ and is denoted by 0. We will call a *sheet of* $\mathscr{R}_0$ (or $\mathscr{R}_1$) any connected component of the preimages of the half-planes $\{\operatorname{Re}\zeta > 0\}$ and $\{\operatorname{Re}\zeta < 0\}$ by π_0 (respectively, π_1).

A holomorphic function of $\mathscr{R}_0$, $\hat{\varphi} \in \mathscr{O}(\mathscr{R}_0)$, naturally identifies itself with a convergent germ at the origin of $\mathbb{C}$ which admits analytic continuation along every path of $\mathscr{P}_0$ (see [15], [30]); we shall use the same symbol $\hat{\varphi}$ for the function and its germ at 0. Using the projection π_0 we pull-back the 1-form $\mathrm{d}\zeta$ to the space $\mathscr{R}_0$. This allows us to integrate holomorphic functions on $\mathscr{R}_0$ over continuous paths in $\mathscr{R}_0$.

Lemma 34. *Let $\hat{\varphi}$ be an entire function and $\hat{\psi} \in \mathscr{O}(\mathscr{R}_0)$. Then $\hat{\chi} = \hat{\varphi} * \hat{\psi} \in \mathscr{O}(\mathscr{R}_0)$. Let $\zeta \in \mathscr{R}_0$ and $\gamma \subset \mathscr{R}_0$ such that $\gamma(0) = 0, \gamma(1) = \zeta$. Then one has*

$$\hat{\chi}(\zeta) = \int_\gamma \hat{\varphi}(\pi_0(\zeta) - \pi_0(\zeta_1))\hat{\psi}(\zeta_1)\,\mathrm{d}\zeta_1. \tag{1.29}$$

Proof. The function $h(\zeta,\zeta_1) = \hat{\varphi}(\pi_0(\zeta) - \pi_0(\zeta_1))\hat{\psi}(\zeta_1)$ is a holomorphic function of two variables on $\mathscr{R}_0 \times \mathscr{R}_0$. Therefore, the integral in (1.29) does not depend on the choice of γ and defines an analytic function on $\mathscr{R}_0$. By definition of the convolution, this integral gives an analytic continuation of $\hat{\varphi} * \hat{\psi}$. $\square$

In fact, for any two functions $\hat{\chi}, \hat{\psi} \in \mathscr{O}(\mathscr{R}_0)$ their convolution $\hat{\chi} * \hat{\psi}$ lies in $\mathscr{O}(\mathscr{R}_0)$, but the proof is more involved. For details we refer the reader to [26] and [32].

1.2 Asymptotic expansion of Fatou coordinates

In this section we introduce the formal iterators of simple parabolic germs and prove their properties formulated in Theorems 3–6.

1.2.1 The formal iterators of a simple parabolic germ

Let

$$F(w) = w + w^2 + \alpha w^3 + O(w^4) \in \mathbb{C}\{w\}.$$

be a simple parabolic germ at the origin. For us it is convenient to work at infinity, in the coordinate $z = -1/w$. Introduce a germ at infinity

$$\begin{gathered} f(z) := -\frac{1}{F(-1/z)} = z + 1 + a(z), \quad \text{where} \\ a(z) = -\rho z^{-1} + O(z^{-2}) \in z^{-1}\mathbb{C}\{z^{-1}\}, \quad \rho = \alpha - 1. \end{gathered} \tag{1.30}$$

Following Écalle [16], we call ρ the *résidu itératif* [1] , or *resiter*, of the germ f. Observe that ρ is a formal conjugacy invariant for the germ f (see [25]).

Lemma 35. *Two germs f, g at infinity of the form* (1.30) *with the resiters ρ_f and ρ_g correspondingly are conjugate by a formal power series:*

$$g = \chi \circ f \circ \chi^{-1}, \quad \textit{where} \quad \chi(z) = z + \sum_{k=1}^{\infty} a_k z^{-k},$$

if and only if $\rho_f = \rho_g$.

Lemma 35 implies that for $\rho \neq 0$ the germ $f(z)$ and the unit translation

$$T(z) := z + 1 \tag{1.31}$$

are not conjugated by a formal power series. However, a conjugacy between f and T can be found in a larger class of formal transformations. Namely, we will prove the following (see *e.g.* [20]):

Proposition 36. *There exists a unique formal series without constant term, $\tilde{\varphi}_*(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$ such that the formal transformation*

$$v_*(z) = z + \rho \log z + \tilde{\varphi}_*(z)$$

[1] Beware that it is $-\rho$ that Buff and Hubbard call résidu itératif in their manuscript [11].

is a solution of the conjugacy equation

$$v \circ f = T \circ v. \tag{1.32}$$

Every formal solution

$$v(z) = z + \rho \log z + \sum_{k \geqslant 0} a_k z^{-k}$$

of (1.32) *is of the form* $v = v_* + \text{const}$.

Here, by $\log z$ we mean a symbol whose composition with f is defined by

$$\log f(z) \coloneqq \log z + \log\big(1 + z^{-1} + z^{-1}a(z)\big),$$

where the last summand is defined by substitution of the formal series $z^{-1} + z^{-1}a(z)$ inside the formal series $\log(1+w) = w - w^2/2 + \cdots$. Thus, the conjugacy equation (1.32) for a formal transformation $v(z) = z + \rho \log z + \tilde{\varphi}(z)$ is equivalent to

$$\tilde{\varphi}\big(z + 1 + a(z)\big) - \tilde{\varphi}(z) = h(z), \tag{1.33}$$

where the right-hand side is

$$h(z) \coloneqq -\rho \log\big(1 + z^{-1} + z^{-1}a(z)\big) - a(z) \in z^{-2}\mathbb{C}\{z^{-1}\}$$

and, in the left-hand side of (1.33), the term $\tilde{\varphi} \circ (\mathrm{Id} + 1 + a)$ can be defined as the formally convergent series of formal series $\sum \frac{1}{r!}(\partial^r \tilde{\varphi}(z))(1 + a(z))^r$.

The formal transformations $v_* + \text{const}$ were introduced by Écalle [16] under the name *iterators*, they are the first example in his theory of resurgent functions [15, 17]. Observe that the iterator v_* is formally invertible with the inverse of the form

$$u_*(z) = z + \sum_{n,m \geqslant 0} C_{n,m} z^{-n} (z^{-1} \log z)^m.$$

The formal series u_* were studied in [15, 17] (see also [30]).

The first important result we prove is the following:

Theorem 37. *The formal series* $\tilde{\varphi}_*(z)$ *of Proposition 36 is* 1*-summable in the arcs of directions* $I^+ = (-\frac{\pi}{2}, \frac{\pi}{2})$ *and* $I^- = (\frac{\pi}{2}, \frac{3\pi}{2})$.

Let $\beta : I^+ \cup I^- \to \mathbb{R}_+$ be an estimated type function for $\tilde{\varphi}_*$. As a consequence of Theorem 37, we can define holomorphic functions

$$\varphi_*^+ = \hat{\mathcal{L}}^{I^+} \hat{\varphi}_*, \qquad \varphi_*^- = \hat{\mathcal{L}}^{I^-} \hat{\varphi}_*,$$

in $\Sigma\big(I^+,\beta\big)$ and $\Sigma\big(I^-,\beta\big)$. Denote by

$$\operatorname{Log}^+ z = \int_1^z \frac{\mathrm{d}z_1}{z_1}$$

the principal branch of the logarithm defined in $\mathbb{C}\setminus\mathbb{R}^-$, and by Log^- its analytic continuation to $\mathbb{C}\setminus\mathbb{R}^+$ obtained by turning anticlockwise around the origin, *i.e.*

$$\operatorname{Log}^- z = \mathrm{i}\pi + \int_{-1}^z \frac{\mathrm{d}z_1}{z_1}.$$

Introduce *sectorial iterators* by the formulas:

$$\begin{aligned} v_*^+(z) &= z + \rho \operatorname{Log}^+ z + \varphi_*^+(z), \qquad && z \in \Sigma\big(I^+,\beta\big), \\ v_*^-(z) &= z + \rho \operatorname{Log}^- z + \varphi_*^-(z), \qquad && z \in \Sigma\big(I^-,\beta\big). \end{aligned} \tag{1.34}$$

Proposition 38. *The sectorial iterators $v_*^+(z)$ and $v_*^-(z)$ are analytic solutions of equation* (1.32) *in the regions*

$$\Sigma\big(I^+,\beta\big) \cap f^{-1}\big(\Sigma\big(I^+,\beta\big)\big) \ \textit{ and } \ \Sigma\big(I^-,\beta\big) \cap f^{-1}\big(\Sigma\big(I^-,\beta\big)\big)$$

correspondingly.

As another Corollary of Theorem 37, using Proposition 31, we obtain the following:

Theorem 39. *For each $\epsilon > 0$ there exists $R > 0$ such that the formal series $\tilde\varphi_*(z)$ is a Gevrey-*1 *asymptotic expansion for φ^+ and φ^- in the sectors*

$$V^+ = \{z : |z| > R, \arg z \in (\pi-\epsilon, \pi+\epsilon)\} \ \textit{ and } \ V^- = \{z : |z| > R, \arg z \in (\epsilon, 2\pi-\epsilon)\}$$

correspondingly.

Observe that the functions $v^\pm$ are Fatou coordinates for the germ f with a parabolic fixed point at infinity. Returning to the initial coordinate $w = -1/z$, we obtain:

Corollary 40. *The maps $\Phi_a(w) = v_*^+(-w^{-1})$ and $\Phi_r(w) = v_*^-(-w^{-1})$ are an attracting and a repelling Fatou coordinates for F correspondingly.*

Also, in this section we prove Theorem 4. Observe that for every sector of the form

$$V = \{\zeta_0 + r e^{\mathrm{i}\theta} : r \geqslant 0, \theta \in [a,b]\} \subset \mathscr{R}_0,$$

there exists a sheet $\mathscr{H}$ of $\mathscr{R}_0$ such that $V \setminus \mathscr{H}$ is bounded. Therefore, Theorem 4 is equivalent to the following statement:

Theorem 41. *The Borel image $\hat{\varphi}_*$ of the formal series $\tilde{\varphi}_*(z)$ belongs to $\mathscr{O}(\mathscr{R}_0)$. Moreover, for each $\gamma \in \mathscr{P}_0$ such that $\gamma(1) = 2\pi\mathrm{i}(k+\frac{1}{2})$ with $k \in \mathbb{Z}$, there exist continuous functions $C, \beta\colon I^+ \cup I^- \to \mathbb{R}^+$ such that*

$$\left|\big(\operatorname{cont}_\gamma \hat{\varphi}_*\big)\big(\gamma(1) + r\,\mathrm{e}^{\mathrm{i}\theta}\big)\right| \leqslant C(\theta)\,\mathrm{e}^{\beta(\theta) r}$$

for all $r > 0$ and $\theta \in I^+ \cup I^-$.

We will use this result in subsequent sections to present the construction of Écalle's analytic invariants.

1.2.2 Existence and uniqueness of the formal iterator

In this subsection we prove Proposition 36. As before, let $T(z) = z+1$ be the unit translation. Define

$$b(z) \coloneqq a(z-1) \in z^{-1}\mathbb{C}\{z^{-1}\}, \qquad m(z) \coloneqq -h(z-1), \tag{1.35}$$

so that equation (1.33) for a formal series is equivalent to

$$\tilde{\varphi} \circ T^{-1} - \tilde{\varphi} \circ (\mathrm{Id} + b) = m. \tag{1.36}$$

Observe that

$$m(z) = \rho \log\left(\frac{1 + z^{-1}b(z)}{1 - z^{-1}}\right) + b(z) \in z^{-2}\mathbb{C}\{z^{-1}\}. \tag{1.37}$$

Since $b(z)$ is convergent near infinity, by (1.10) there exists $C_0, \beta_0 > 0$ such that $\hat{b} = \hat{\mathscr{B}} b$ satisfies

$$\left|\hat{b}(\zeta)\right| \leqslant C_0 e^{\beta_0|\zeta|}. \tag{1.38}$$

Introduce operators D and B on the space $\mathbb{C}[[z^{-1}]]$ by

$$(D\tilde{\varphi})(z) = \tilde{\varphi}(z-1) - \tilde{\varphi}(z), \qquad (B\tilde{\varphi})(z) = \tilde{\varphi}\big(z + b(z)\big) - \tilde{\varphi}(z), \tag{1.39}$$

in other words

$$D = \sum_{r \geqslant 1} \tfrac{(-1)^r}{r!} \partial^r, \qquad B = \sum_{r \geqslant 1} \tfrac{1}{r!} b^r \partial^r, \tag{1.40}$$

where $\partial \coloneqq \frac{\mathrm{d}}{\mathrm{d}z}$ is the operator of differentiation of formal series. Then we can rewrite the equation (1.36) as

$$(D - B)\tilde{\varphi} = m. \tag{1.41}$$

Let $\tilde{\varphi}$ be a non-constant formal power series, $1 \leqslant \operatorname{ord} \tilde{\varphi} = p < \infty$. Then

$$\operatorname{ord}(\partial^r \tilde{\varphi}) = p + r, \quad \text{hence} \quad \operatorname{ord}(D\tilde{\varphi}) = p + 1.$$

On the other hand, $\operatorname{ord}(B\tilde{\varphi}) \geqslant p + 2$. Thus,

$$\operatorname{ord}(D - B)\tilde{\varphi} = p + 1 < \infty.$$

It follows that the kernel of $D - B$ consists of constant formal power series:

$$\ker(D - B) = \mathbb{C}.$$

Now, to prove Proposition 36, it remains to find a solution $\tilde{\varphi} \in z^{-1}\mathbb{C}[[z^{-1}]]$ of the equation (1.41).

Definition 42. Define the operators $\hat{E}\colon\ \zeta\mathbb{C}[[\zeta]] \to \mathbb{C}[[\zeta]]$ and $E\colon\ z^{-2}\mathbb{C}[[z^{-1}]] \to z^{-1}\mathbb{C}[[z^{-1}]]$ by the formulas

$$\big(\hat{E}\hat{\varphi}\big)(\zeta) = \frac{\hat{\varphi}(\zeta)}{\mathrm{e}^{\zeta} - 1}, \qquad E = (\hat{\mathscr{B}})^{-1} \circ \hat{E} \circ \hat{\mathscr{B}}_{|z^{-2}\mathbb{C}[[z^{-1}]]}. \tag{1.42}$$

Observe that for each $\tilde{\varphi} \in z^{-2}\mathbb{C}[[z^{-1}]]_1$ one has

$$\hat{\varphi} = \hat{\mathscr{B}}\tilde{\varphi} \in \zeta\mathbb{C}\{\zeta\}, \quad \text{hence} \quad \hat{E}\hat{\varphi} \in \mathbb{C}\{\zeta\} \quad \text{and} \quad E\tilde{\varphi} \in z^{-1}\mathbb{C}[[z^{-1}]]_1.$$

Further, $\hat{\mathscr{B}}(\partial\tilde{\varphi})(\zeta) = -\zeta\hat{\varphi}(\zeta)$ and $\hat{\mathscr{B}}(D\tilde{\varphi})(\zeta) = (\mathrm{e}^{\zeta} - 1)\hat{\varphi}(\zeta)$, therefore

$$D \circ E = \mathrm{Id} \quad \text{on } z^{-1}\mathbb{C}[[z^{-1}]]. \tag{1.43}$$

In fact, for $\tilde{\varphi} \in z^{-2}\mathbb{C}[[z^{-1}]]$, $E\tilde{\varphi}$ is the only preimage of $\tilde{\varphi}$ in $z^{-1}\mathbb{C}[[z^{-1}]]$ for the operator D. Observe that $\operatorname{ord}(E\tilde{\varphi}) = \operatorname{ord}\tilde{\varphi} - 1$.

Now equation (1.41) with the restriction $\tilde{\varphi} \in z^{-1}\mathbb{C}[[z^{-1}]]$ is equivalent to

$$\tilde{\varphi} = Em + EB\tilde{\varphi}.$$

The last equation is a fixed-point equation which has a unique solution, given by the formally convergent series of formal series

$$\tilde{\varphi}_* = \sum_{r \geqslant 0} \tilde{\varphi}_{*,r}, \quad \tilde{\varphi}_{*,r} = (EB)^r Em \tag{1.44}$$

($\operatorname{ord}(B\tilde{\varphi}) \geqslant \operatorname{ord}\tilde{\varphi} + 2$, thus $\operatorname{ord}\tilde{\varphi}_{*,r} \geqslant r + 1$). This completes the proof of Proposition 36.

Remark 43. Using the representation

$$B = \sum_{k\geqslant 1} B_k, \qquad B_k := \frac{1}{k!} b^k \partial^k \tag{1.45}$$

we can define formal series

$$\tilde{\varphi}_*^{\boldsymbol{k}} = EB_{k_1} \cdots EB_{k_r} Em \tag{1.46}$$

for any word $\boldsymbol{k} = k_1 \cdots k_r$ with letters $k_1, \dots, k_r \in \mathbb{N}$ and with any length $r \geqslant 0$ (with the convention $\tilde{\varphi}_*^{\emptyset} = Em$ for the empty word, when $r = 0$). Denote by $\mathscr{N}$ the set of all such words, including the empty word. Then we obtain a representation

$$\tilde{\varphi}_* = \sum_{\boldsymbol{k}\in\mathscr{N}} \tilde{\varphi}_*^{\boldsymbol{k}}.$$

The family $(\tilde{\varphi}_*^{\boldsymbol{k}})$ is an example of *mould* with values in $\mathbb{C}[[z^{-1}]]$, *i.e.* a map defined on the set of all words on a given alphabet (see [17] and [31]).

1.2.3 Convergence of $\hat{\varphi}_*$ near the origin

Denote by $\hat{B}$ the Borel counterpart of the operator B:

$$\begin{gathered} \hat{B} = \hat{\mathscr{B}} \circ B \circ (\hat{\mathscr{B}})^{-1} = \sum_{k\geqslant 1} \hat{B}_k, \quad \hat{B}_k \hat{\chi} = \hat{b}_k * (\hat{\partial}^k \hat{\chi}), \\ \text{where } \hat{b}_k = \frac{1}{r!} \hat{\mathscr{B}}(b^k) = \frac{1}{k!} \hat{b} * \cdots * \hat{b} \ (k \text{ times}). \end{gathered} \tag{1.47}$$

Lemma 44. *Let $\hat{\chi}$ be an analytic function in the disk $D_r(0) = \{z \in \mathbb{C} : |z| < r\}$, where $r \leqslant 1$. Assume that for some $C_0 > 0$ and a function $X : [0,1] \to \mathbb{R}_+$ one has*

$$\left|\hat{b}(\zeta)\right| \leqslant C_0 \text{ and } \left|\hat{\chi}(\zeta)\right| \leqslant X(|\zeta|) \text{ for each } \zeta \in D_r(0).$$

Then $\hat{\psi} := \hat{E}\hat{B}\hat{\chi}$ defines a holomorphic function on $D_r(0)$. Moreover,

$$\left|\hat{\psi}(\zeta)\right| \leqslant \frac{e^{C_0} - 1}{1 - e^{-1}} (1 * X)(|\zeta|) \text{ for each } \zeta \in D_r(0).$$

Proof. Using (1.42), we obtain the following:

$$\hat{\psi}(\zeta) = \sum_{k\geqslant 1} \frac{1}{e^\zeta - 1} \int_0^\zeta \hat{b}_k(\zeta - \zeta_1)(-\zeta_1)^k \hat{\chi}(\zeta_1)\, \mathrm{d}\zeta_1. \tag{1.48}$$

Using induction, one can show that the functions $\hat{b}_k$ (see (1.47)) satisfy the following inequality:

$$\left|\hat{b}_k(\zeta)\right| \leqslant \frac{C_0^k |\zeta|^{k-1}}{k!(k-1)!}. \tag{1.49}$$

Observe that for each $k \in \mathbb{N}$, assuming that $|\zeta| \leqslant 1$, one has:

$$\frac{|\zeta^k|}{|e^\zeta - 1|} \leqslant \frac{|\zeta|}{|e^\zeta - 1|} \leqslant \max_{|\zeta|=1} \frac{1}{|e^\zeta - 1|} = \frac{1}{1 - e^{-1}}.$$

It follows that for $\zeta \in D_r(0)$

$$\left| \frac{1}{e^\zeta - 1} \int_0^\zeta \hat{b}_k(\zeta - \zeta_1)(-\zeta_1)^k \hat{\chi}(\zeta_1) \, \mathrm{d}\zeta_1 \right| \leqslant$$
$$\int_0^{|\zeta|} \frac{C_0^k (|\zeta| - s)^{k-1}}{(1 - e^{-1})k!} X(s) \, \mathrm{d}s \leqslant \frac{C_0^k}{(1 - e^{-1})k!} (1 * X)(|\zeta|).$$

Taking the sum over $k \geqslant 1$ we obtain the inequality of Lemma 44. □

Corollary 45. $\tilde{\varphi}_* \in z^{-1}\mathbb{C}[[z^{-1}]]_1$.

Proof. The germs $\hat{\varphi}_{*,r}$ satisfy the following recurrent relations:

$$\hat{\varphi}_{*,r+1} = \hat{E}\hat{B}\hat{\varphi}_{*,r}, \quad \hat{\varphi}_{*,0} = \hat{E}\hat{m}.$$

Choose a constant C_0 such that $\left|\hat{b}(\zeta)\right| \leqslant C_0, \left|\hat{m}(\zeta)\right| \leqslant C_0|\zeta|$ on the unit disk $\mathbb{D}$. Using Lemma 44, by induction we obtain:

$$\left|\hat{\varphi}_{*,r}(\zeta)\right| \leqslant \frac{M^{r+1}}{r!} |\zeta|^r,$$

for each $\zeta \in \mathbb{D}$, where $M = \frac{e^{C_0} - 1}{1 - e^{-1}}$. This implies that the series $\sum\limits_{r \geqslant 0} \hat{\varphi}_{*,r}$ converges uniformly on $\mathbb{D}$. It follows that $\hat{\varphi}_*$ is analytic on $\mathbb{D}$. Therefore, $\tilde{\varphi}_* \in z^{-1}\mathbb{C}[[z^{-1}]]_1$. □

1.2.4 Auxiliary formal series $\tilde{\varphi}^+_{*,r}$

In the sequel, we will use the formal series $\tilde{\varphi}_{*,r}$ to prove 1-summability of $\tilde{\varphi}_*$ in the arc of directions $(-\frac{\pi}{2}, \frac{\pi}{2})$. In this subsection we construct another variant of the decomposition (1.44) which we will use to prove summability of $\tilde{\varphi}_*$ in the arc of directions $(\frac{\pi}{2}, \frac{3\pi}{2})$.

Recall that we have introduced $b = a \circ T^{-1}$, so that the holomorphic germ whose dynamics we study is

$$f = T + a = (\mathrm{Id} + b) \circ T.$$

Consider now the local inverse of the germ f. One has

$$f^{-1} = (\mathrm{Id} + b^+) \circ T^{-1},$$

where $b^+(z) = f^{-1} \circ T - \mathrm{Id} \in z^{-1}\mathbb{C}[[z^{-1}]]$, so that

$$T \circ (\mathrm{Id} + b^+) = (\mathrm{Id} + b)^{-1} \circ T. \tag{1.50}$$

Composing both sides of equation

$$\tilde{\varphi} \circ T^{-1} - \tilde{\varphi} \circ (\mathrm{Id} + b) = m.$$

by the expression (1.50), we see that $\tilde{\varphi}_*$ is the unique solution in $z^{-1}\mathbb{C}[[z^{-1}]]$ of the equation

$$\tilde{\varphi} \circ T - \tilde{\varphi} \circ (\mathrm{Id} + b^+) = m^+ \tag{1.51}$$

where $m^+ = -m \circ T \circ (\mathrm{Id} + b^+) = h \circ (\mathrm{Id} + b^+) \in z^{-2}\mathbb{C}\{z^{-1}\}$. The equation (1.51) can be written in a form similar to (1.41). Namely, introduce operators

$$D^+ = \mathrm{e}^{\partial} - \mathrm{Id} = -D \circ T \colon\ \tilde{\varphi} \mapsto \tilde{\varphi} \circ T - \tilde{\varphi} \ \text{ and}$$
$$B^+ = \sum_{r \geqslant 1} \frac{1}{r!} (b^+)^r \partial^r \colon\ \tilde{\varphi} \mapsto \tilde{\varphi} \circ (\mathrm{Id} + b^+) - \tilde{\varphi}.$$

We get:

$$(D^+ - B^+)\tilde{\varphi}_* = m^+. \tag{1.52}$$

Similarly to Definition 42 we introduce the operators

$$\hat{E}^+ \colon\ \zeta\mathbb{C}[[\zeta]] \to \mathbb{C}[[\zeta]] \ \text{ and } \ E^+ \colon\ z^{-2}\mathbb{C}[[z^{-1}]] \to z^{-1}\mathbb{C}[[z^{-1}]]$$

by the formulas

$$\big(\hat{E}^+ \hat{\varphi}\big)(\zeta) = \frac{1}{\mathrm{e}^{-\zeta} - 1} \hat{\varphi}(\zeta), \qquad E^+ = (\hat{\mathscr{B}})^{-1} \circ \hat{E}^+ \circ \hat{\mathscr{B}}_{|z^{-2}\mathbb{C}[[z^{-1}]]} \tag{1.53}$$

and, analogously to (1.44), we define

$$\tilde{\varphi}^+_{*,r} = (E^+ B^+)^r E^+ m^+.$$

Then we obtain a representation for $\tilde{\varphi}_*$ different from (1.44):

$$\tilde{\varphi}_* = \sum_{r \geqslant 0} \tilde{\varphi}^+_{*,r}. \tag{1.54}$$

1.2.5 Summability of $\tilde{\varphi}_*$ in the main sheet

In this section we prove Theorem 37. The proof is based on the following technical statement.

Lemma 46. *Let $\tilde{\chi}$ be 1-summable in the arc of directions $(-\pi/2, \pi/2)$. Set $\hat{\chi} = \hat{\mathscr{B}}\tilde{\chi}$. Assume that for some $C_0, \beta_0 > 0$, a continuous function $X : \mathbb{R}_+ \to \mathbb{R}_+$ and continuous functions $C_1, \beta_1 : (-\pi/2, \pi/2) \to \mathbb{R}_+$ one has*

$$\left|\hat{b}(\zeta)\right| \leqslant C_0 e^{\beta_0|\zeta|} \text{ for each } \zeta \in \mathbb{C}, \quad \left|\hat{\chi}(\zeta)\right| \leqslant C_1(\theta) e^{\beta_1(\theta)|\zeta|} X(|\zeta|), \text{ if } \operatorname{Re}\zeta > 0,$$

where $\theta = \arg(\zeta)$. Then $\tilde{\psi} = EB\tilde{\chi}$ is 1-summable in the arc of directions $(-\pi/2, \pi/2)$. Moreover, if $\operatorname{Re}\zeta > 0$, then

$$\left|\hat{\psi}(\zeta)\right| \leqslant \frac{C_0 C_1(\theta)}{\cos\theta} e^{\beta'(\theta)|\zeta|}(1 * X)(|\zeta|),$$

where $\beta'(\theta) = \max\left\{\beta_0 + \frac{C_0}{\cos\theta}, \beta_1(\theta)\right\}$ and $\theta = \arg\zeta$.

Proof. As we have shown before (see (1.19)), the functions $\hat{b}_k = \frac{1}{k!}\hat{b}^{*k}$ under the conditions of the lemma satisfy the following inequality:

$$\left|\hat{b}_k(\zeta)\right| \leqslant \frac{C_0^k |\zeta|^{k-1} e^{\beta_0|\zeta|}}{k!(k-1)!} \tag{1.55}$$

for all $\zeta \in \mathbb{C}$. Observe that for each $k \in \mathbb{N}$, assuming that $\theta = \arg\zeta \in (-\pi/2, \pi/2)$, one has

$$\left|e^{\zeta} - 1\right| \geqslant \left|e^{\cos\theta|\zeta|} - 1\right| \geqslant \frac{1}{k!}(\cos\theta|\zeta|)^k.$$

In particular, we obtain that

$$\left|\frac{\zeta_1^k}{e^{\zeta} - 1}\right| \leqslant \frac{k!}{\cos\theta^k}$$

for each $\zeta_1 \in [0, \zeta]$. Therefore, one has:

$$\left|\frac{1}{e^{\zeta} - 1}\int_0^{\zeta} \hat{b}_k(\zeta - \zeta_1)(-\zeta_1)^k \hat{\chi}(\zeta_1)\, \mathrm{d}\zeta_1\right| \leqslant C_1(\theta)\int_0^{|\zeta|} \frac{C_0^k (|\zeta| - s)^{k-1} e^{\beta_0(|\zeta| - s)}}{(k-1)!\cos\theta^k} e^{\beta_1(\theta)s} X(s)\, \mathrm{d}s.$$

Thus, we obtain

$$\left|\hat{\psi}(\zeta)\right| \leqslant \frac{C_0 C_1(\theta)}{\cos\theta} \sum_{k\geqslant 1} \int_0^{|\zeta|} \frac{C_0^{k-1}(|\zeta| - s)^{k-1}}{(k-1)!\cos\theta^{k-1}} e^{\beta_0(|\zeta| - s) + \beta_1(\theta)s} X(s)\, \mathrm{d}s =$$

$$\frac{C_0 C_1(\theta)}{\cos\theta} \int_0^{|\zeta|} e^{\left(\beta_0 + \frac{C_0}{\cos\theta}\right)(|\zeta| - s) + \beta_1(\theta)s} X(s)\, \mathrm{d}s \leqslant \frac{C_0 C_1(\theta)}{\cos\theta} e^{\beta'(\theta)|\zeta|}(1 * X)(|\zeta|).$$

In particular, this implies 1-summability of $\tilde{\psi}$ in the arc of directions $(-\pi/2, \pi/2)$. $\square$

Now we are ready to prove Theorem 37:

The formal series $\tilde{\varphi}_*$ are 1-summable in the arcs of directions $(-\pi/2, \pi/2)$ and $(\pi/2, 3\pi/2)$.

Proof. Since $b \in z^{-1}\mathbb{C}\{z^{-1}\}$ and $m \in z^{-2}\mathbb{C}\{z^{-1}\}$, by (1.10), there exist $C_0, \beta_0 > 0$ such that

$$\left|\hat{b}(\zeta)\right| \leqslant C_0 e^{\beta_0|\zeta|}, \quad \text{for each } \zeta \in \mathbb{C}, \quad \left|\frac{\hat{m}(\zeta)}{e^{\zeta}-1}\right| \leqslant \frac{C_0 e^{\beta_0|\zeta|}}{\cos\theta}, \text{ if } \operatorname{Re}\zeta > 0, \tag{1.56}$$

where $\theta = \arg\zeta$. Recall that

$$\hat{\varphi}_{*,0}(\zeta) = \frac{\hat{m}(\zeta)}{e^{\zeta}-1}, \quad \hat{\varphi}_{*,r+1} = \hat{E}\hat{B}\hat{\varphi}_{*,r} \text{ for } r \geqslant 0.$$

Using Lemma 46, by induction we obtain that for all ζ with $\operatorname{Re}\zeta > 0$ one has:

$$\left|\hat{\varphi}_{*,r}(\zeta)\right| \leqslant \frac{C_0^{r+1}}{r!\cos\theta^{r+1}}|\zeta|^r e^{\beta'(\theta)|\zeta|},$$

where $\beta'(\theta) = \beta_0 + \frac{C_0}{\cos\theta}$. It follows that the series $\sum_{r\geqslant 0}\hat{\varphi}_{*,r}(\zeta)$ converges and gives analytic continuation of $\hat{\varphi}_*$ along the path $[0,\zeta]$. Moreover, we obtain that

$$\left|\hat{\varphi}_*(\zeta)\right| \leqslant \frac{C_0}{\cos\theta} e^{\left(\beta'(\theta)+\frac{C_0}{\cos\theta}\right)|\zeta|}. \tag{1.57}$$

This shows that $\hat{\varphi}_*$ is 1-summable in the arc of directions $(-\pi/2, \pi/2)$.

To prove 1-summability in the arc of directions $(\pi/2, 3\pi/2)$ we use representation (1.54) for $\tilde{\varphi}_*$. Let $\hat{\varphi}^+_{*,r} = \hat{\mathscr{B}}\left(\tilde{\varphi}^+_{*,r}\right), \hat{b}^+_k = \hat{\mathscr{B}}\left(\frac{1}{k!}(b^+)^k\right)$. Then we obtain

$$\hat{\varphi}^+_{*,r+1}(\zeta) = \sum_{k\geqslant 1}\frac{1}{e^{-\zeta}-1}\int_0^{\zeta}\hat{b}^+_k(\zeta-\zeta_1)(-\zeta_1)^k\hat{\varphi}^+_{*,r}(\zeta_1)\,\mathrm{d}\zeta_1, \quad \hat{\varphi}^+_{*,0}(\zeta) = \frac{\hat{m}^+(\zeta)}{e^{-\zeta}-1}.$$

Using these formulas, we obtain an analog of the inequality (1.57) for $\operatorname{Re}\zeta < 0$. □

1.2.6 Sectorial iterators

In this section we prove Proposition 38. Define $I^+ = (-\frac{\pi}{2}, \frac{\pi}{2}), I^- = (\frac{\pi}{2}, \frac{3\pi}{2})$. By Theorem 37, $\tilde{\varphi}_*$ is 1-summable in the arcs of directions I^+ and I^-. Let $\beta : I^+ \cup I^- \to \mathbb{R}_+$ be an estimated type function for $\tilde{\varphi}_*$. Define holomorphic functions

$$\varphi_*^+ = \hat{\mathcal{L}}^{I^+}\hat{\varphi}_*, \qquad \varphi_*^- = \hat{\mathcal{L}}^{I^-}\hat{\varphi}_*,$$

in $\Sigma\big(I^+, \beta\big)$ and $\Sigma\big(I^-, \beta\big)$. Set

$$\beta_*(\theta) := \max(\beta_1(\theta) + C_0, \beta_0, \beta_1(\theta) + 1).$$

Then, by Proposition 28, the formal series

$$\tilde{\varphi}_* \circ T^{-1} - \tilde{\varphi}_* \circ (\mathrm{Id} + b)$$

is 1-summable and has Borel-Laplace sums holomorphic in $\Sigma\big(I^+, \beta_*\big)$ and $\Sigma\big(I^-, \beta_*\big)$. Moreover, since $\tilde{\varphi}_*$ is a formal solution of (1.36), by Proposition 28 it follows that

$$\varphi_*^{\pm} \circ T^{-1} - \varphi_*^{\pm} \circ (\mathrm{Id} + b) = m.$$

Define

$$\Sigma^+ := T\Big(\Sigma\big(I^+, \beta_*\big)\Big), \qquad \Sigma^- := T\Big(\Sigma\big(I^-, \beta_*\big)\Big).$$

Since $f = (\mathrm{Id} + b) \circ T$ and $h = -m \circ T$, we obtain the following

Corollary 47. *For every $z \in \Sigma^{\pm}$ one has*

$$\varphi_*^{\pm}(f(z)) - \varphi_*^{\pm}(z) = h(z).$$

Now we are ready to prove Proposition 38:

The sectorial iterators $v_*^+(z)$ and $v_*^-(z)$ are analytic solutions of equation (1.2) *in the regions*

$$\Sigma\big(I^+, \beta\big) \cap f^{-1}\big(\Sigma\big(I^+, \beta\big)\big) \ \ \textit{and} \ \ \Sigma\big(I^-, \beta\big) \cap f^{-1}\big(\Sigma\big(I^-, \beta\big)\big)$$

correspondingly.

Proof. There exists $R > 0$ such that $1 + |a(z)| < |z|$ if $|z| > R$. Let $z \in \Sigma^-$ such that $\operatorname{Re} z < 0$ and $|z| > R$. Then

$$\operatorname{Re}(f(z)/z) = \operatorname{Re}\big(1 + \tfrac{1+a(z)}{z}\big) > 0.$$

Therefore, $f(z) \in \mathbb{C} - \mathbb{R}^+$ and

$$\operatorname{Log}^- f(z) = \operatorname{Log}^- z + \operatorname{Log}^+\big(f(z)/z\big).$$

Since $h(z)$ is defined as the convergent series at the origin

$$h(z) = -\rho \operatorname{Log}^+(f(z)/z) - a(z),$$

using Corollary 47, we obtain that

$$v_*^- \circ f(z) = f(z) + \rho \operatorname{Log}^- f(z) + \varphi_*^-(f(z)) =$$
$$z + 1 - h(z) + \rho \operatorname{Log}^- z + \varphi_*^-(f(z)) = v_*^-(z) + 1.$$

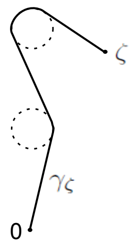

Figure 1.3: An example of a path γ_ζ.

Thus,

$$v_*^- \circ f(z) = v_*^-(z) + 1$$

for all $z \in \Sigma^-$ such that $\operatorname{Re} z < 0$ and $|z| > R$. By analyticity, we obtain that this formula holds at each point where both sides are defined. In particular, it holds on the domain

$$\Sigma\big(I^-, \beta\big) \cap f^{-1}\big(\Sigma\big(I^-, \beta\big)\big).$$

For $v_*^+(z)$ the proof is similar. $\square$

1.2.7 Resurgence of $\tilde{\varphi}_*$ and summability in other sheets

To prove Theorem 41 we need to introduce some auxiliary objects in the space $\mathscr{R}_0$. For $\pi > \delta > 0$ denote by $\mathscr{U}_\delta$ the union of all connected components of the set

$$\{\zeta \in \mathscr{R}_0 : dist(\pi_0(\zeta), 2\pi i\mathbb{Z}) < \delta\}$$

except the connected component which contains the origin. Let $\zeta \in \mathscr{R}_0 \setminus \mathscr{U}_\delta$. Denote by $\gamma_\zeta = \gamma_{\zeta,\delta}$ the shortest path in $\mathscr{R}_0 \setminus \mathscr{U}_\delta$ which connects the origin with ζ. We will assume that γ_ζ is parameterized naturally. In particular, if l_ζ is the length of the path γ_ζ, then

$$\gamma_\zeta(l_\zeta) = \zeta.$$

Clearly, for any $\zeta_1 = \gamma_\zeta(s), s \in (0, l_\zeta]$, the path γ_{ζ_1} coincides with the restriction of γ_ζ onto $[0, s]$. Notice that:

$$|\zeta| \leqslant l_\zeta \ \text{ and } \ |\zeta - \gamma_\zeta(s)| \leqslant l_\zeta - s$$

for each $\zeta \in \mathscr{R}_0 \setminus \mathscr{U}_\delta$ and $0 \leqslant s \leqslant l_\zeta$.

Lemma 48. *Let $\hat{\chi} \in \mathscr{O}(\mathscr{R}_0)$ and $\hat{\psi} = \hat{E}\hat{B}\hat{\chi}$. Then*

$$\hat{\psi} \in \mathscr{O}(\mathscr{R}_0).$$

Let $R > 1 > \delta > 0$ and $C = \max\left\{\left|\hat{b}(\zeta)\right| : |\zeta| \leqslant R\right\}$. Assume that $X : [0, R] \to \mathbb{R}_+$ is such that

$$\left|\hat{\chi}(\zeta)\right| \leqslant X(l_\zeta) \text{ for all } \zeta \in \mathscr{R}_0 \setminus \mathscr{U}_\delta \text{ with } l_\zeta \leqslant R.$$

Then there exists a constant $M = M(C, R, \delta)$ such that

$$\left|\hat{\psi}(\zeta)\right| \leqslant M(1 * X)(l_\zeta) \tag{1.58}$$

for all $\zeta \in \mathscr{R}_0 \setminus \mathscr{U}_\delta$ with $l_\zeta \leqslant R$.

Proof. Formally,

$$\hat{\psi}(\zeta) = \sum_{k=1}^{\infty} \frac{1}{e^\zeta - 1} \int_{\gamma_\zeta} \hat{b}_k(\zeta - \zeta_1)(-\zeta_1)^k \hat{\chi}(\zeta_1)\, \mathrm{d}\zeta_1. \tag{1.59}$$

For each $k \in \mathbb{N}$ the formula

$$\frac{1}{e^\zeta - 1} \int_{\gamma_\zeta} \hat{b}_k(\zeta - \zeta_1)(-\zeta_1)^k \hat{\chi}(\zeta_1)\, \mathrm{d}\zeta_1 \tag{1.60}$$

defines an element of $\mathscr{O}(\mathscr{R}_0)$. For each ζ with $|\zeta| \leqslant R$ one has

$$\left|\hat{b}_k(\zeta)\right| \leqslant \frac{C^k |\zeta|^{k-1}}{(k-1)!k!}.$$

Let $\zeta \in \mathscr{R}_0 \setminus \mathscr{U}_\delta$ such that $l_\zeta \leqslant R$. If ζ is in the main sheet and $|\zeta| < \delta$, then

$$\frac{l_\zeta}{|e^\zeta - 1|} = \frac{|\zeta|}{|e^\zeta - 1|} \leqslant \frac{1}{1 - e^{-\delta}}.$$

Otherwise, $d(\zeta, 2\pi i\mathbb{Z}) \geqslant \delta$ and

$$\frac{l_\zeta}{|e^\zeta - 1|} \leqslant \frac{R}{1 - e^{-\delta}}.$$

It follows that

$$\begin{gathered}\sum_{k=1}^{\infty} \left| \frac{1}{e^\zeta - 1} \int_{\gamma_\zeta} \hat{b}_k(\zeta - \zeta_1)(-\zeta_1)^k \hat{\chi}(\zeta_1)\, \mathrm{d}\zeta_1 \right| \leqslant \\ \sum_{k=1}^{\infty} \frac{(Cl_\zeta)^k}{k!|e^\zeta - 1|} \int_0^{l_\zeta} \frac{(l_\zeta - s)^{k-1}}{(k-1)!} X(s) \mathrm{d}s \leqslant \frac{CRe^{(C+1)R}}{1 - e^{-\delta}} (1 * X)(l_\zeta).\end{gathered} \tag{1.61}$$

The inequality (1.61) implies that the series (1.59) converges uniformly on compact subsets of $\mathscr{R}_0$. Thus, $\hat{\psi} \in \mathscr{R}_0$. $\square$

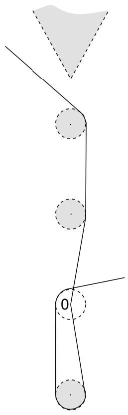

Figure 1.4: Illustration to Definition 49: the set $\mathscr{L}^{\pi/3}_{\pi/2,2}$.

Definition 49. Let $n \in \mathbb{N}$ and $\alpha \in (0, \pi/2)$. Denote by $\mathscr{L}^{\alpha}_{\delta,n}$ the set of all points $\zeta \in \mathscr{R}_0 \setminus \mathscr{U}_\delta$ such that the following is true

1) $|\operatorname{Im}\zeta| \leqslant \tan\alpha\,|\operatorname{Re}\zeta| + (2n+1)\pi$,

2) the projection of the path γ_ζ on $\mathbb{C}$ intersects the imaginary line at most n times,

3) for every $a \in \gamma_\zeta$ with $\operatorname{Re}(a) = 0$ one has $|a| < (2n+1)\pi$.

Item 2) should be understood in the following way: the number of connected components of the set

$$\{\zeta_1 \in \gamma_\zeta : \operatorname{Re}\zeta_1 = 0\}$$

is less or equal to n. Observe that this number is at least 1 for each ζ, since $\gamma_\zeta(0) = 0$. Also, notice that for each $\zeta \in \mathscr{L}^{\alpha}_{\delta,n}$ one has $\gamma_\zeta \subset \mathscr{L}^{\alpha}_{\delta,n}$. Set

$$\mathscr{L}^{\alpha,+}_{\delta,n} = \{\zeta \in \mathscr{L}^{\alpha}_{\delta,n} : \operatorname{Re}\zeta \geqslant -1\}. \tag{1.62}$$

Lemma 50. *For each $\alpha \in (0, \pi/2), \delta \in (0,1), n \in \mathbb{N}$ there exists a constant $C_1 = C_1(\alpha, \delta, n) > 0$ such that for each $k \in \mathbb{N}$ and $\zeta \in \mathscr{L}^{\alpha,+}_{\delta,n}$ one has*

$$\frac{l^k_\zeta}{k!|e^\zeta - 1|} \leqslant C^k_1.$$

Proof. From the definitions of $\mathscr{L}^{\alpha}_{\delta,n}$ and γ_ζ it follows that there exists a constant $M = M(\alpha, \delta, n) > 0$ such that

$$l_\zeta \leqslant M|\zeta| \quad \text{for all } \zeta \in \mathscr{L}^{\alpha}_{\delta,n}. \tag{1.63}$$

Let $\zeta \in \mathscr{L}_{\delta,n}^{\alpha,+}$. First, assume that $\operatorname{Re}\zeta \in [-1,1]$. Then

$$|\zeta| \leqslant M_0 = (2n+1)\pi + \tan\alpha + 1.$$

The set $\{\omega \in \mathscr{L}_{\delta,n}^{\alpha,+} : \operatorname{Re}\omega \in [-1,1]\}$ projects to a compact subset of $\mathbb{C} \setminus 2\pi i\mathbb{Z}^*$. It follows that $|\zeta|/|e^\zeta - 1|$ is bounded from above by a constant $M_1 = M_1(\alpha,\delta,n)$ on this set. Also,

$$\frac{|\zeta|^{k-1}}{k!} < e^{M_0},$$

which finishes the proof of Lemma 50 in the case $\operatorname{Re}\zeta \in [-1,1]$.

Let $\operatorname{Re}\zeta > 1$. One can show that $\arg\zeta \leqslant \theta_0 = \arctan((2n+1)\pi + \tan\alpha)$. It follows that for each $k \in \mathbb{N}$

$$\left|e^\zeta - 1\right| \geqslant \left|e^{\cos\theta_0|\zeta|} - 1\right| \geqslant \frac{(\cos\theta_0|\zeta|)^k}{k!}.$$

Using (1.63), we obtain the desired inequality. $\square$

Proposition 51. *Let $\hat{\chi} \in \mathscr{O}(\mathscr{L}_{\delta,n}^{\alpha,+})$ and $\hat{\psi} = \hat{E}\hat{B}\hat{\chi}$. Let $\alpha \in (0,\pi/2), \delta \in (0,1), n \in \mathbb{N}$. Assume that for each $\zeta \in \mathscr{L}_{\delta,n}^{\alpha,+}$ one has*

$$\left|\hat{\chi}(\zeta)\right| \leqslant e^{\beta_1 l_\zeta} X(l_\zeta),$$

where $\beta_1 > 0$ and $X : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function. Then for each $\zeta \in \mathscr{L}_{\delta,n}^{\alpha,+}$ one has

$$\left|\hat{\psi}(\zeta)\right| \leqslant C_0 C_1 e^{\beta' l_\zeta} (1 * X)(l_\zeta),$$

where $\beta' = \max\{\beta_0 + C_0C_1, \beta_1\}$, C_0, β_0 are as in Lemma 46 and $C_1 = C_1(\alpha,\delta,n)$ is as in Lemma 50.

Proof. For each $\zeta \in \mathscr{L}_{\delta,n}^{\alpha,+}$ by (1.55) and Lemma 50 one has

$$\left|\frac{1}{e^\zeta - 1}\int_{\gamma_\zeta} \hat{b}_k(\zeta - \zeta_1)(-\zeta_1)^k \hat{\chi}(\zeta_1)\,\mathrm{d}\zeta_1\right| \leqslant (C_0C_1)^k \int_0^{l_\zeta} \frac{(l_\zeta - s)^{k-1}}{(k-1)!} e^{\beta_0(l_\zeta - s) + \beta_1 s} X(s)\mathrm{d}s.$$

It follows that

$$\left|\hat{\psi}(\zeta)\right| \leqslant \sum_{k=1}^{\infty} (C_0C_1)^k \int_0^{l_\zeta} \frac{(l_\zeta - s)^{k-1}}{(k-1)!} e^{\beta_0(l_\zeta - s) + \beta_1 s} X(s)\mathrm{d}s \leqslant C_0C_1 e^{\beta' l_\zeta}(1 * X)(l_\zeta)$$

for all $\zeta \in \mathscr{L}_{\delta,n}^{\alpha,+}$. $\square$

Now, we will obtain Theorem 41 as corollary of Proposition 51. Recall that we set $I^+ = (-\frac{\pi}{2}, \frac{\pi}{2}), I^- = (\frac{\pi}{2}, \frac{3\pi}{2})$.

Corollary 52. *The Borel image* $\hat{\varphi}_*$ *of the formal series* $\tilde{\varphi}_*(z)$ *belongs to* $\mathscr{O}(\mathscr{R}_0)$. *For each* $\gamma \in \mathscr{P}_0$ *such that* $\gamma(1) = 2\pi\mathrm{i}(k+\frac{1}{2})$ *with* $k \in \mathbb{Z}$, *there exist upper semi-continuous functions* $C, \beta \colon I^+ \cup I^- \to \mathbb{R}^+$ *such that*

$$\left|\left(\operatorname{cont}_\gamma \hat{\varphi}_*\right)\left(\gamma(1) + r\,\mathrm{e}^{\mathrm{i}\theta}\right)\right| \leqslant C(\theta)\,\mathrm{e}^{\beta(\theta)r}. \tag{1.64}$$

for all $r > 0$ *and* $\theta \in I^+ \cup I^-$.

Proof. Let $\alpha \in (0, \pi/2), \delta \in (0,1)$ and $n \in \mathbb{N}$. Using Proposition 51, by induction we obtain that for any $\zeta \in \mathscr{L}^{\alpha,+}_{\delta,n}$ one has:

$$\left|\hat{\varphi}_{*,r}(\zeta)\right| \leqslant \frac{1}{r!} C_0^{r+1} C_1^r l_\zeta^r e^{\beta' l_\zeta},$$

where $C_0 = C_0(\delta), \beta_0 > 0$ are such that

$$\left|\hat{b}(\zeta)\right| \leqslant C_0 e^{\beta_0|\zeta|}, \text{ for each} \zeta \in \mathbb{C}, \quad \left|\frac{\hat{m}(\zeta)}{e^\zeta - 1}\right| \leqslant C_0 e^{\beta_0|\zeta|}, \text{ for each } \zeta \in \mathscr{R}_0 \setminus \mathscr{U}_\delta,$$

C_1 is the constant from Lemma 50 and $\beta' = \beta_0 + C_0 C_1$. Thus, the series $\hat{\varphi}_*(\zeta) = \sum_{r\geqslant 0} \hat{\varphi}_{*,r}(\zeta)$ converges uniformly on compact subsets of the set $\mathscr{L}^{\alpha,+}_{\delta,n}$. Moreover,

$$\left|\hat{\varphi}_*(\zeta)\right| \leqslant C_0 e^{(\beta' + C_0 C_1) l_\zeta} \tag{1.65}$$

for $\zeta \in \mathscr{L}^{\alpha,+}_{\delta,n}$.

Now, let γ be as in the statement of Corollary 52 and $\theta \in (-\alpha, \alpha)$. Let $\delta < \pi\cos\alpha$ and n be such that $\gamma \in \mathscr{L}^\alpha_{\delta,n}$. Without loss of generality we may assume that $\gamma \subset \mathscr{L}^{\alpha,+}_{\delta,n}$. Then for each $r \geqslant 0$ from (1.65) we obtain:

$$\left|\left(\operatorname{cont}_\gamma \hat{\varphi}_*\right)\left(\gamma(1) + r\,\mathrm{e}^{\mathrm{i}\theta}\right)\right| \leqslant C_0 e^{(\beta' + C_0 C_1)(l(\gamma) + r)},$$

where $l(\gamma)$ is the length of the path γ. Thus, for each α there exist some numbers $C_2 = C_2(\alpha), \beta_2 = \beta_2(\alpha)$ such that

$$\left|\left(\operatorname{cont}_\gamma \hat{\varphi}_*\right)\left(\gamma(1) + r\,\mathrm{e}^{\mathrm{i}\theta}\right)\right| \leqslant C_2 e^{\beta_2 r}$$

for all $r > 0, \theta \in (-\alpha, \alpha)$. It is not hard to see that one can construct two continuous functions $C, \beta : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}_+$ such that for each $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ there exists $\alpha \in (|\theta|, \frac{\pi}{2})$ for which

$$C(\theta) > C_2(\alpha), \quad \beta(\theta) > \beta_2(\alpha).$$

Then (1.64) is satisfied for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. The case $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$ can be considered similarly, using the representation

$$\hat{\varphi}_*(\zeta) = \sum_{r \geqslant 0} \hat{\varphi}^+_{*,r}(\zeta), \quad \text{where} \quad \hat{\varphi}^+_{*,r} = \hat{\mathscr{B}} \tilde{\varphi}^+_{*,r}.$$

$\square$

1.3 General singularities

In this section we enlarge the framework of resurgent functions. As we will see later, to obtain Écalle's analytic invariants we need to study analytic continuations of $\hat{\varphi}_*$ to the points near $2\pi \mathrm{i}\mathbb{Z}$. As we have shown in Theorem 41, generally, $\hat{\varphi}_*$ has branched singularities at $2\pi k\mathrm{i}, k \in \mathbb{Z}^*$. This motivates introducing and studying germs with branched singularity at the origin. We give here a brief account of the formalism of *singularities*, referring the reader to [15], [30] for proofs and details.

1.3.1 General singularities, integrable singularities

Let us denote by

$$\underset{\bullet}{\mathbb{C}} = \{\zeta = r\,\mathrm{e}^{\mathrm{i}\theta} \mid r > 0,\ \theta \in \mathbb{R}\}$$

the Riemann surface of the logarithm. Observe that $\underset{\bullet}{\mathbb{C}}$ is the universal cover of $\mathbb{C} \setminus \{0\}$; it can be given a definition analogous to Definition 33, with $2\pi\mathrm{i}\mathbb{Z}$ replaced with $\{0\}$. Define ANA to be the space of the germs of functions analytic in a *spiralling neighbourhood of the origin*, *i.e.* a domain of the form

$$\{r\,\mathrm{e}^{\mathrm{i}\theta} \mid 0 < r < h(\theta),\ \theta \in \mathbb{R}\} \subset \underset{\bullet}{\mathbb{C}}$$

with a continuous $h\colon \mathbb{R} \to \{t \in \mathbb{R} : t > 0\}$. We will view the space $\mathbb{C}\{\zeta\}$ of regular germs at the origin as a subspace of ANA.

Definition 53. Let SING $=$ ANA $/\mathbb{C}\{\zeta\}$. The elements of this space are called *singularities*. Denote by sing_0 the canonical projection:

$$\mathrm{sing}_0\colon \begin{cases} \mathrm{ANA} \to \mathrm{SING} \\ \quad \check{\varphi} \quad \mapsto \overset{\triangledown}{\varphi} = \mathrm{sing}_0\big(\check{\varphi}(\zeta)\big). \end{cases}$$

Any representative $\overset{\vee}{\varphi}$ of a singularity $\overset{\triangledown}{\varphi}$ is called a *major* of this singularity. Define the variation map as follows:

$$\mathrm{var}\colon \begin{cases} \qquad \mathrm{SING} \;\to\; \mathrm{ANA} \\ \overset{\triangledown}{\varphi} = \mathrm{sing}_0\big(\overset{\vee}{\varphi}\big) \mapsto \hat{\varphi}(\zeta) = \overset{\vee}{\varphi}(\zeta) - \overset{\vee}{\varphi}(\zeta\,\mathrm{e}^{-2\pi\mathrm{i}}). \end{cases}$$

The germ $\hat{\varphi} = \mathrm{var}\,\overset{\triangledown}{\varphi}$ is called the *minor* of the singularity $\overset{\triangledown}{\varphi}$.

Observe that the kernel of $\mathrm{var}\colon\ \mathrm{SING} \to \mathrm{ANA}$ is isomorphic to the space of entire functions of $\frac{1}{\zeta}$ without constant term. The simplest examples of singularities are poles

$$\delta := \mathrm{sing}_0\left(\frac{1}{2\pi\mathrm{i}\zeta}\right), \qquad \delta^{(n)} := \mathrm{sing}_0\left(\frac{(-1)^n n!}{2\pi\mathrm{i}\zeta^{n+1}}\right), \qquad n \geqslant 0,$$

which belong to ker(var), and logarithmic singularities with regular variation, for which we use the notation

$$^\flat\hat{\varphi} := \mathrm{sing}_0\left(\frac{1}{2\pi\mathrm{i}}\hat{\varphi}(\zeta)\log\zeta\right), \qquad \hat{\varphi}(\zeta) \in \mathbb{C}\{\zeta\}. \tag{1.66}$$

The last example is a particular case of *integrable singularity.*

Definition 54. An *integrable minor* is a germ $\hat{\varphi} \in \mathrm{ANA}$ which is uniformly integrable at the origin in any sector, in the sense that for each $\theta_1 < \theta_2$ there exists a Lebesgue integrable function $f\colon\ (0, r^*) \to \mathbb{R}^+$ such that

$$\theta_1 < \arg\zeta < \theta_2 \text{ and } |\zeta| \leqslant r^* \;\;\Rightarrow\;\; |\hat{\varphi}(\zeta)| \leqslant f(|\zeta|),$$

where $r^* > 0$ is small enough so as to ensure that $\hat{\varphi}(\zeta)$ be defined. The corresponding subspace of ANA is denoted by $\mathrm{ANA}^{\mathrm{int}}$. An *integrable singularity* is a singularity $\overset{\triangledown}{\varphi} \in$ SING which admits a major $\overset{\vee}{\varphi}$ such that $\zeta\overset{\vee}{\varphi}(\zeta) \to 0$ as $\zeta \to 0$ uniformly in any sector $\theta_1 \leqslant \arg\zeta \leqslant \theta_2$ and $\mathrm{var}\,\overset{\triangledown}{\varphi} \in \mathrm{ANA}^{\mathrm{int}}$. The corresponding subspace of SING is denoted by $\mathrm{SING}^{\mathrm{int}}$.

For example, the formulas

$$\overset{\triangledown}{I}_\sigma = \mathrm{sing}_0\big(\overset{\vee}{I}_\sigma\big), \qquad \overset{\vee}{I}_\sigma(\zeta) = \frac{\zeta^{\sigma-1}}{(1-\mathrm{e}^{-2\pi\mathrm{i}\sigma})\Gamma(\sigma)}, \qquad \sigma \in \mathbb{C}\setminus\mathbb{Z}_+ \tag{1.67}$$

define a family of singularities[2] among which the integrable singularities correspond to $\mathrm{Re}\,\sigma > 0$. Another example is provided by polynomials of $\log\zeta$, which can be viewed as integrable minors, and also as majors of integrable singularities.

[2] In view of the poles of the Gamma function, $\overset{\triangledown}{I}_\sigma$ is well-defined for $\sigma = -n \in -\mathbb{N}$ and $\overset{\triangledown}{I}_{-n} = \delta^{(n)}$.

By restriction, the variation map induces a linear isomorphism $\mathrm{var}^{\mathrm{int}}\colon\ \mathrm{SING}^{\mathrm{int}} \to \mathrm{ANA}^{\mathrm{int}}$ (see *e.g.* [30]). The inverse map is denoted by

$$\hat{\varphi} \in \mathrm{ANA}^{\mathrm{int}} \mapsto {}^{\flat}\hat{\varphi} \in \mathrm{SING}^{\mathrm{int}}\,.$$

Regular minors are particular cases of integrable minors and (1.66) provides their preimage for $\mathrm{var}^{\mathrm{int}}$). When the integrable minor $\hat{\varphi}$ is not regular at the origin, one can obtain its preimage by $\mathrm{var}^{\mathrm{int}}$ using a Cauchy integral. Namely, pick any point λ in the domain of analyticity of $\hat{\varphi}$ and define

$$\check{\varphi}(\zeta) = \frac{1}{2\pi\mathrm{i}} \int_0^{\lambda} \frac{\hat{\varphi}(\zeta_1)}{\zeta - \zeta_1}\,\mathrm{d}\zeta_1, \qquad \arg\lambda - 2\pi < \arg\zeta < \arg\lambda, \tag{1.68}$$

to obtain a major of ${}^{\flat}\hat{\varphi}$.

Observe that for every germ $\alpha(\zeta) \in \mathbb{C}\{\zeta\}$ at the origin multiplication of majors by α passes to the quotient:

$$\overset{\triangledown}{\varphi} = \mathrm{sing}_0\,\check{\varphi} \to \alpha\overset{\triangledown}{\varphi} = \mathrm{sing}_0\,\big(\alpha\check{\varphi}\big) \in \mathrm{SING}\,.$$

In particular, multiplication by $-\zeta$ and $e^{\zeta}-1$ define the operators

$$\overset{\triangledown}{\partial}, \overset{\triangledown}{D}\colon\ \mathrm{SING} \to \mathrm{SING},\ \ \overset{\triangledown}{\partial}\overset{\triangledown}{\varphi} = \mathrm{sing}_0\,\big(-\zeta\check{\varphi}\big),\ \ \overset{\triangledown}{D}\overset{\triangledown}{\varphi} = \mathrm{sing}_0\,\big((e^{\zeta}-1)\check{\varphi}\big).$$

The operator $\overset{\triangledown}{\partial}$ is a derivation on the algebra $(\mathrm{SING}, *)$.

1.3.2 Convolution of general singularities

The notion of convolution (1.8) can be generalized for general singularities (see [15], [30]), making SING an algebra with the convolution product. In the present thesis we deal mostly with convolution of an integrable singularity with a general singularity.

First, consider the case of two integrable singularities. Let $\overset{\triangledown}{\varphi}, \overset{\triangledown}{\psi} \in \mathrm{SING}^{\mathrm{int}}$. Then $\hat{\varphi} = \mathrm{var}\,\overset{\triangledown}{\varphi}, \hat{\psi} = \mathrm{var}\,\overset{\triangledown}{\psi} \in \mathrm{ANA}^{\mathrm{int}}$. The formula

$$\hat{\chi}(\zeta) = \int_0^{\zeta} \hat{\varphi}(\zeta_1)\hat{\psi}(\zeta-\zeta_1)\,\mathrm{d}\zeta_1$$

defines an element $\hat{\chi} \in \mathrm{ANA}^{\mathrm{int}}$. We set

$$\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi} := {}^{\flat}\hat{\chi}. \tag{1.69}$$

Now let us introduce convolution of a general singularity with an integrable singularity. For a point $\lambda \in \underset{\bullet}{\mathbb{C}}$ and $r > 0$ we denote by $D_r(\lambda)$ the set of points which can be reached from λ be a straight path $[\lambda, \lambda+\zeta]$ of the length $|\zeta| < r$.

Lemma 55. *Let $\overset{\triangledown}{\varphi}$ be an arbitrary singularity and $\overset{\triangledown}{\chi}$ be an integrable singularity. Let $\overset{\wedge}{\chi} := \operatorname{var}\overset{\triangledown}{\chi}$ and $\overset{\vee}{\varphi}$ be a major of $\overset{\triangledown}{\varphi}$. Fix a point $\lambda \in \underset{\bullet}{\mathbb{C}}$ and $r > 0$ such that $D_r(0)$ belongs to the domain of analyticity of $\overset{\wedge}{\chi}$ and $D_r(\lambda)$ belongs to the domain of analyticity of $\overset{\vee}{\varphi}$. Define*

$$\overset{\vee}{\psi}_\lambda(\zeta) = \int_\lambda^\zeta \overset{\vee}{\varphi}(\zeta_1)\overset{\wedge}{\chi}(\zeta - \zeta_1)\,\mathrm{d}\zeta_1, \ \ \zeta \in D_r(\lambda).$$

Then $\overset{\vee}{\psi}_\lambda$ admits an analytic continuation to a spiralling neighborhood of the origin. Moreover, the singularity

$$\overset{\triangledown}{\psi} := \operatorname{sing}_0 \overset{\vee}{\psi}_\lambda$$

does not depend either on the choice of λ or on the choice of the major $\overset{\vee}{\varphi}$.

For $\overset{\triangledown}{\chi}, \overset{\triangledown}{\varphi}, \overset{\triangledown}{\psi}$ from Lemma 55 we write

$$\overset{\triangledown}{\psi} = \overset{\triangledown}{\chi} * \overset{\triangledown}{\varphi}$$

and call $\overset{\triangledown}{\psi}$ the convolution of $\overset{\triangledown}{\chi}$ and $\overset{\triangledown}{\varphi}$. For the definition of a convolution of two arbitrarily singularities we refer the reader to [15] and [30].

Remark 56. The element $\delta = \operatorname{sing}_0\left(\frac{1}{2\pi \mathrm{i}\zeta}\right)$ is a unit of the algebra $(\mathrm{SING}, *)$:

$$\delta * \overset{\triangledown}{\varphi} = \overset{\triangledown}{\varphi} * \delta = \overset{\triangledown}{\varphi} \ \text{ for all } \ \overset{\triangledown}{\varphi} \in \mathrm{SING}\,.$$

Define

$$\overset{\triangledown}{I}_n = {}^\flat\big(\tfrac{\zeta^{n-1}}{(n-1)!}\big), \ n \in \mathbb{N}, \ \ \overset{\triangledown}{I}_0 = \delta. \tag{1.70}$$

Then (1.67) together with (1.70) give a family of singularities $\overset{\triangledown}{I}_\sigma$ parameterized by $\sigma \in \mathbb{C}$. The family $\overset{\triangledown}{I}_\sigma, \sigma \in \mathbb{C}$ is a group under the convolution product:

$$\overset{\triangledown}{I}_{\sigma_1} * \overset{\triangledown}{I}_{\sigma_2} = \overset{\triangledown}{I}_{\sigma_1+\sigma_2} \ \text{ for each } \ \sigma_1, \sigma_2 \in \mathbb{C}. \tag{1.71}$$

1.3.3 Resurgent singularities

Identify the half-sheet

$$\mathscr{H}^+ := \{\zeta = r\,\mathrm{e}^{\mathrm{i}\theta} \mid r > 0, \ \theta \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2})\}$$

of $\underset{\bullet}{\mathbb{C}}$ with the connected component of $\pi_1^{-1}\big(\{\operatorname{Re}\zeta > 0\}\big)$ which contains the class of the trivial path $\gamma \equiv 1$. Then we can view $\mathscr{O}(\mathscr{R}_1)$ as a subspace of ANA.

Definition 57. We call a singularity $\overset{\triangledown}{\varphi}$ *resurgent* if it admits a major $\overset{\vee}{\varphi} \in \mathscr{O}(\mathscr{R}_1)$, *i.e.* a major which extends to the universal cover of $\mathbb{C} \setminus 2\pi\mathrm{i}\mathbb{Z}$ with base point at 1. Denote the space of resurgent singularities by

$$\mathrm{RES} = \mathrm{sing}_0\left(\mathscr{O}(\mathscr{R}_1)\right).$$

The half-sheet $\mathscr{H}^+$ can be identified also with the connected component of $\pi_0^{-1}\big(\{\mathrm{Re}\,\zeta > 0\}\big)$ which contains the origin of $\mathscr{R}_0$ in its closure. Then we can write

$$\mathscr{O}(\mathscr{R}_0) = \mathscr{O}(\mathscr{R}_1) \cap \mathbb{C}\{\zeta\}, \qquad \mathrm{RES} = \mathscr{O}(\mathscr{R}_1)/\mathscr{O}(\mathscr{R}_0).$$

Clearly, var maps RES into $\mathscr{O}(\mathscr{R}_1)$, *i.e.* the minor of a resurgent singularity extends analytically to $\mathscr{R}_1$. For this reason we sometimes call the elements of $\mathscr{O}(\mathscr{R}_1)$ *resurgent minors* and the elements of $\mathscr{O}(\mathscr{R}_0)$ *regular resurgent minors.*

In Theorem 41 and Proposition 51, we showed that

$$\hat{\varphi}_* \in \mathscr{O}(\mathscr{R}_0) \ \text{ and } \ \hat{\varphi}_{*,r} \in \mathscr{O}(\mathscr{R}_0) \ \text{ for each } \ r \geqslant 0.$$

Introduce the corresponding integrable singularities:

$$\overset{\triangledown}{\varphi}_* := {}^\flat\hat{\varphi}_*, \qquad \overset{\triangledown}{\varphi}_{*,r} := {}^\flat\hat{\varphi}_{*,r}. \tag{1.72}$$

Using the formalism of resurgent singularities we will define other useful singularities and study the analytic continuations of the germs $\hat{\varphi}_*$ and $\hat{\varphi}_{*,r}$ to the spiralling neighborhoods of the points from $2\pi\mathrm{i}\mathbb{Z}$.

Now, let us define the convolution of a resurgent singulary with a singularity of the form ${}^\flat\hat{\chi}$, where $\hat{\chi}$ is an entire function.

Proposition 58. *Let $\hat{\chi}$ be an entire function and $\overset{\triangledown}{\varphi}$ be a resurgent singularity with a major $\overset{\vee}{\varphi}$. Let $\zeta \in \mathscr{R}_1$ and γ be a path in $\mathscr{R}_1$, starting at 1 and terminating at ζ. Then the following formula*

$$\overset{\vee}{\psi}(\zeta) = \int_\gamma \overset{\vee}{\varphi}(\zeta_1)\hat{\chi}(\pi_1(\zeta) - \pi_1(\zeta_1))\mathrm{d}\zeta_1$$

defines an element of $\mathscr{O}(\mathscr{R}_1)$. The resurgent singularity $\overset{\triangledown}{\psi} = \mathrm{sing}_0\,\overset{\vee}{\psi}$ does not depend on the choice of the major $\overset{\vee}{\varphi}$.

As we mentioned in subsection 1.3.1, the notion of convolution can be extended to the case of arbitrary singularities $\overset{\triangledown}{\varphi}, \overset{\triangledown}{\psi} \in \mathrm{SING}$. In fact, RES is stable under the convolution:

Proposition 59. *For any* $\overset{\triangledown}{\varphi}, \overset{\triangledown}{\psi} \in \mathrm{RES}$ *one has* $\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi} \in \mathrm{RES}$.

See [17] for the details.

Definition 60. Let $\overset{\triangledown}{\varphi}_r \in \mathrm{RES}, r \in \mathbb{N}$ be a collection of singularities. We will say that the series $\sum_{r\geqslant 1} \overset{\triangledown}{\varphi}_r$ converges if there exist majors $\overset{\vee}{\varphi}_r$ such that the series $\sum_{r\geqslant 1} \overset{\vee}{\varphi}_r$ converges uniformly on compact subsets of $\mathscr{R}_1$ to some function $\overset{\vee}{\varphi} \in \mathrm{RES}$. We will call

$$\overset{\triangledown}{\varphi} := \operatorname{sing}_0 \overset{\vee}{\varphi}$$

the sum of the series $\sum_{r\geqslant 1} \overset{\triangledown}{\varphi}_r$.

Lemma 61. *The singularity* $\overset{\triangledown}{\varphi}$ *in Definition 60 does not depend on the choice of the majors* $\overset{\vee}{\varphi}_r$.

Proof. Let $\overset{\vee}{\varphi}_{r,0} \in \mathrm{RES}$ be other majors of $\overset{\triangledown}{\varphi}_r$. Then

$$\overset{\vee}{\varphi}_r - \overset{\vee}{\varphi}_{r,0} \in \mathscr{O}(\mathscr{R}_0) \ \text{ for each } \ r.$$

Moreover, the series

$$\sum_{r\in\mathbb{N}} \left(\overset{\vee}{\varphi}_r(\zeta) - \overset{\vee}{\varphi}_{r,0}(\zeta)\right) \tag{1.73}$$

converges uniformly on compact subsets of $\mathscr{R}_1$ to some function $\overset{\vee}{\psi} \in \mathscr{O}(\mathscr{R}_1)$. Since the functions $\overset{\vee}{\varphi}_r - \overset{\vee}{\varphi}_{r,0}$ are analytic near the origin, using Cauchy Theorem we obtain that the series (1.73) converges uniformly on a neighborhood of the origin. It follows that

$$\overset{\vee}{\psi} \in \mathscr{O}(\mathscr{R}_0), \ \text{ hence } \ \operatorname{sing}_0\left(\overset{\vee}{\psi}\right) = 0.$$

□

1.3.4 Laplace transforms of summable singularities

For an arc of directions $I = (\theta_1, \theta_2)$ such that $0 < \theta_2 - \theta_1 < \pi$ and a continuous function β on I denote by $\mathrm{SING}^{I,\beta}$ the space of singularities $\overset{\triangledown}{\varphi}$ such that

1) $\operatorname{var} \overset{\triangledown}{\varphi} \in \mathrm{ANA}$ extends analytically to the sector $\{\, \zeta = r\,\mathrm{e}^{\mathrm{i}\theta} \in \underset{\bullet}{\mathbb{C}} : \ r > 0, \ \theta \in I \,\}$;

2) for every $\epsilon > 0$ there exists a continuous function $C : I \to \mathbb{R}_+$ such that

$$|\hat{\varphi}(r\,\mathrm{e}^{\mathrm{i}\theta})| \leqslant C(\theta)\,\mathrm{e}^{(\beta(\theta)+\epsilon)r}$$

for all $r > 0$ and $\theta \in I$, where $\hat{\varphi} = \operatorname{var} \overset{\triangledown}{\varphi}$.

We will call elements of $\mathrm{SING}^{I,\beta}$ *summable* singularities. By analogy with (1.15) we introduce Laplace transform of singularities.

Definition 62. Let $\overset{\triangledown}{\varphi} \in \mathrm{SING}^{I,\beta}$ with a major $\overset{\vee}{\varphi}$ and the minor $\hat{\varphi}$. Fix $\theta \in I$. Let $a > 0$ be small enough such that $\overset{\vee}{\varphi}$ is defined on an open neighborhood of the set $\{\zeta = re^{it} : \theta-2\pi \leqslant t \leqslant \theta,\ 0 < r \leqslant a\}$. Denote by γ_a the curve $\gamma_a(t) = ae^{it}, t \in [\theta-2\pi, \theta]$. Then we define the Laplace transform of $\overset{\triangledown}{\varphi}$ in the direction θ by

$$\overset{\triangledown}{\mathcal{L}}{}^{\theta}\overset{\triangledown}{\varphi}(z) := \int_{\gamma_a} \mathrm{e}^{-z\zeta}\overset{\vee}{\varphi}(\zeta)\,\mathrm{d}\zeta + \int_{\mathrm{e}^{\mathrm{i}\theta}a}^{\mathrm{e}^{\mathrm{i}\theta}\infty} \mathrm{e}^{-z\zeta}\hat{\varphi}(\zeta)\,\mathrm{d}\zeta, \ \ \mathrm{Re}(z\,\mathrm{e}^{\mathrm{i}\theta}) > \beta(\theta). \tag{1.74}$$

Observe that by Cauchy Theorem the value $\overset{\triangledown}{\mathcal{L}}{}^{\theta}\overset{\triangledown}{\varphi}(z)$ does no depend either on the choice of $\overset{\vee}{\varphi}$ or on the choice of a. If the integrals $\int_{\mathrm{e}^{\mathrm{i}\theta}a}^{\mathrm{e}^{\mathrm{i}\theta}\infty} \mathrm{e}^{-z\zeta}\overset{\vee}{\varphi}(\zeta)\,\mathrm{d}\zeta$ and $\int_{\mathrm{e}^{\mathrm{i}(\theta-2\pi)}a}^{\mathrm{e}^{\mathrm{i}(\theta-2\pi)}\infty} \mathrm{e}^{-z\zeta}\overset{\vee}{\varphi}(\zeta)\,\mathrm{d}\zeta$ converge then by definition of the majors and minors one has:

$$\overset{\triangledown}{\mathcal{L}}{}^{\theta}\overset{\triangledown}{\varphi}(z) = \int_{\Gamma_a} \mathrm{e}^{-z\zeta}\overset{\vee}{\varphi}(\zeta)\,\mathrm{d}\zeta,$$

where $\Gamma_a = [\mathrm{e}^{\mathrm{i}(\theta-2\pi)}\infty, \mathrm{e}^{\mathrm{i}(\theta-2\pi)}a] \cup \gamma_a \cup [\mathrm{e}^{\mathrm{i}\theta}a, \mathrm{e}^{\mathrm{i}\theta}\infty]$. As in the case of Laplace transform of regular germs we denote by $\overset{\triangledown}{\mathcal{L}}{}^{I}\overset{\triangledown}{\varphi}(z)$ the analytic function on

$$\Sigma(I,\beta) = \{\zeta : \ \mathrm{Re}(z\,\mathrm{e}^{\mathrm{i}\theta}) > \beta(\theta) \ \text{ for some } \ \theta \in I\}$$

given by

$$\overset{\triangledown}{\mathcal{L}}{}^{I}\overset{\triangledown}{\varphi}(z) = \overset{\triangledown}{\mathcal{L}}{}^{\theta}\overset{\triangledown}{\varphi}(z), \ \text{ where } \ \theta \in I \ \text{ such that } \ \mathrm{Re}(z\,\mathrm{e}^{\mathrm{i}\theta}) > \beta(\theta).$$

If $\overset{\triangledown}{\varphi} \in \mathrm{SING}^{I,\beta}$ is an integrable singularity, choosing a major $\overset{\vee}{\varphi}$ as in Definition 54, when a goes to 0 we obtain

$$\overset{\triangledown}{\mathcal{L}}{}^{I}\overset{\triangledown}{\varphi}(z) = \hat{\mathcal{L}}^{I}\hat{\varphi}(z) = \int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \mathrm{e}^{-z\zeta}\hat{\varphi}(\zeta)\,\mathrm{d}\zeta, \ \ \mathrm{Re}(z\,\mathrm{e}^{\mathrm{i}\theta}) > \beta(\theta).$$

Thus, the definition of Laplace transform for singularities is consistent with the definition of Laplace transform for minors. We will mostly deal with resurgent singularities. Therefore, we set $\mathrm{RES}^{I,\beta} = \mathrm{SING}^{I,\beta} \cap \mathrm{RES}$. Here is an example of Laplace transform of a non-integrable singularity (case $\mathrm{Re}\,\sigma < 0$)

$$\overset{\triangledown}{\mathcal{L}}{}^{\theta}\overset{\triangledown}{I}_{\sigma}(z) = z^{-\sigma}, \ \ \arg z \in (\theta - \pi/2, \theta + \pi/2),\ \sigma \in \mathbb{C}, \tag{1.75}$$

where $z^{-\sigma}$ is considered as a multivalued function with the branch point at the origin.

Proposition 63. *Let $\overset{\triangledown}{\varphi}$, $\overset{\triangledown}{\varphi}_1$, $\overset{\triangledown}{\varphi}_2 \in \mathrm{SING}^{I,\beta}$. Then one has:*

$$\overset{\triangledown}{\varphi}_1 * \overset{\triangledown}{\varphi}_2,\ \overset{\triangledown}{\partial}\overset{\triangledown}{\varphi} \in \mathrm{SING}^{I,\beta} \quad \textit{and} \quad \overset{\triangledown}{D}\overset{\triangledown}{\varphi} \in \mathrm{SING}^{I,\beta+1}\,.$$

Moreover, the following holds:

$$\begin{aligned} \overset{\triangledown}{\mathcal{L}}^I\big(\overset{\triangledown}{\varphi}_1 * \overset{\triangledown}{\varphi}_2\big) = \overset{\triangledown}{\mathcal{L}}^I\overset{\triangledown}{\varphi}_1 \cdot \overset{\triangledown}{\mathcal{L}}^I\overset{\triangledown}{\varphi}_2,\ \ \overset{\triangledown}{\mathcal{L}}^I\Big(\overset{\triangledown}{\partial}\overset{\triangledown}{\varphi}\Big) &= \frac{\mathrm{d}}{\mathrm{d}z}\Big(\overset{\triangledown}{\mathcal{L}}^I\overset{\triangledown}{\varphi}\Big), \\ \overset{\triangledown}{\mathcal{L}}^I\left(\overset{\triangledown}{D}\overset{\triangledown}{\varphi}\right) &= \left(\overset{\triangledown}{\mathcal{L}}^I\overset{\triangledown}{\varphi}\right) \circ (T^{-1} - \mathrm{Id}). \end{aligned} \tag{1.76}$$

Observe that the map

$$\overset{\triangledown}{\mathscr{B}} : z^{-1}\mathbb{C}[[z^{-1}]]_1 \to {}^\flat\mathbb{C}\{\zeta\} \subset \mathrm{SING},\ \ \overset{\triangledown}{\mathscr{B}}\tilde{\varphi} := {}^\flat\overset{\wedge}{\mathscr{B}}\tilde{\varphi} \tag{1.77}$$

is an isomorphism. The map $\overset{\triangledown}{\mathscr{B}}$ can be extended to a larger domain including non-integer powers of z and powers of $\log z$ using the formulas:

$$\overset{\triangledown}{\mathscr{B}}(z^{-\sigma}) = \overset{\triangledown}{I}_\sigma,\ \ \sigma \in \mathbb{C}, \quad \overset{\triangledown}{\mathscr{B}}(\log z) = -\frac{\log\zeta}{2\pi\mathrm{i}\zeta}. \tag{1.78}$$

The extended operator $\overset{\triangledown}{\mathscr{B}}$ can be considered as generalized Borel transform. It is used to define Borel-Laplace summation of formal series involving non-integer powers of z and powers of $\log z$. However, we will use the operator $\overset{\triangledown}{\mathscr{B}}$ restricted to the domain $\mathbb{C}\log z + \mathbb{C}[[z^{-1}]]_1$.

1.4 Bridge equation and Écalle's analytic invariants

1.4.1 The formal series $\tilde{\psi}_{*,t}$

In section 1.2 we studied the formal iterator in the form $v_*(z) = z + \rho\log z + \tilde{\varphi}_*(z)$, where $\tilde{\varphi}_* \in z^{-1}\mathbb{C}[[z^{-1}]]$. To investigate the iterator further it will be more convenient for us to write v_* in the form

$$v_*(z) = z + \tilde{\psi}_*(z), \tag{1.79}$$

where $\tilde{\psi}_*(z)$ is a unique solution of the equation

$$(D - B)\tilde{\psi} = b \tag{1.80}$$

such that $\tilde{\psi} \in \rho\log z + z^{-1}\mathbb{C}[[z^{-1}]]$.

More generally, let $t \in \mathbb{C}$. Set $f_t(z) = z + 1 + ta(z) \in \mathbb{C}\{z^{-1}\}$. Consider an equation

$$v \circ f_t = T \circ v. \tag{1.81}$$

Proposition 36 implies that there exists a unique formal power series

$$\tilde{\psi}_{*,t} \in t\rho \log z + z^{-1}\mathbb{C}[[z^{-1}]]$$

such that $v_{*,t} = z + \tilde{\psi}_{*,t}(z)$ is a solution of the equation (1.81). In particular, $\tilde{\psi}_{*,1} = \tilde{\psi}_*$. The series $\tilde{\psi}_{*,t}$ is a solution of the equation

$$\psi(z-1) - \psi(z + tb(z)) = tb(z).$$

This equation can be rewritten as follows:

$$(D - B_t)\tilde{\psi}_{*,t} = tb,$$

where B_t can be obtained from (1.39) by replacing $b(z)$ with $tb(z)$:

$$(B_t\tilde{\varphi})(z) = \tilde{\varphi}\big(z + tb(z)\big) - \tilde{\varphi}(z), \ B_t = \sum_{r\geqslant 1} \frac{1}{r!}(tb)^r \partial^r. \tag{1.82}$$

Similarly to formula (1.44) we obtain

$$\tilde{\psi}_{*,t} = \sum_{r\geqslant 0} \tilde{\psi}_{*,r,t}, \tag{1.83}$$

with

$$\begin{aligned} \tilde{\psi}_{*,0,t} &= t\rho \log z + E\big[tb(z) - t\rho\log(1 - z^{-1})\big] \in t\rho\log z + z^{-1}\mathbb{C}[[z^{-1}]], \\ \tilde{\psi}_{*,r,t} &= (EB_t)^r \tilde{\psi}_{*,0,t} \in z^{-1}\mathbb{C}[[z^{-1}]]. \end{aligned} \tag{1.84}$$

Here the first term $\tilde{\psi}_{*,0,t} = t\rho\log z + tE\big[b(z) - \rho D \log z\big]$ is a preimage of tb for the operator D; there are no logarithms in the other terms since

$$B_t(EB_t)^{r-1}\tilde{\psi}_{*,0,t} \in z^{-2}\mathbb{C}[[z^{-1}]]$$

for $r \geqslant 1$. Denote $\tilde{\psi}_{*,r} = \tilde{\psi}_{*,r,1}$, so that $\tilde{\psi}_* = \sum_{r\geqslant 0} \tilde{\psi}_{*,r}$. The relationship between the sequences $\tilde{\psi}_{*,r}$ and $\tilde{\varphi}_{*,r}$ is

$$\tilde{\psi}_{*,r} - \tilde{\varphi}_{*,r} = \rho(EB)^r(\log z - EB\log z).$$

Introduce an operator

$$B_t^+ = \sum_{r\geqslant 1} \frac{1}{r!}(tb^+)^r \partial^r : \ \tilde{\varphi} \mapsto \tilde{\varphi} \circ (\mathrm{Id} + tb^+) - \tilde{\varphi}.$$

Similarly to (1.52) we obtain an equation

$$\left(D^+ - B_t^+\right) \tilde{\psi}_t = tb^+.$$

By analogy with the formula (1.54), we obtain a representation for $\tilde{\psi}_{*,t}$ distinct from (1.83):

$$\tilde{\psi}_{*,t} = \sum_{r\geqslant 0} \tilde{\psi}^+_{*,r,t}, \tag{1.85}$$

where

$$\begin{aligned} &\tilde{\psi}^+_{*,0,t} = t\rho \log z + E^+\big[tb^+(z) - t\rho\log(1+z^{-1})\big] \in t\rho\log z + z^{-1}\mathbb{C}[[z^{-1}]], \\ &\tilde{\psi}^+_{*,r,t} = (E^+B_t^+)^r \tilde{\psi}^+_{*,0,t} \in z^{-1}\mathbb{C}[[z^{-1}]]. \end{aligned} \tag{1.86}$$

1.4.2 Summability and resurgence of $\tilde{\psi}_{*,t}$

For $r \in \mathbb{N}$ set

$$\hat{\psi}_{*,r,t} = \hat{\mathscr{B}}(\tilde{\psi}_{*,r,t}),$$

where $\tilde{\psi}_{*,r,t}$ are defined in (1.84). To define $\hat{\psi}_{*,0,t}$ we need to introduce an element $\hat{\mathscr{B}}(\log z)$. In view of the following observations:

$$\partial \log z = z^{-1}, \ \ \hat{\mathscr{B}}(\partial\tilde{\chi}) = -\zeta\hat{\chi}(\zeta) \ \text{ for all } \ \tilde{\chi} \in z^{-1}\mathbb{C}[[z^{-1}]], \ \ \hat{\mathscr{B}}(z^{-1}) = 1,$$

it is natural to set

$$\hat{\mathscr{B}}(\log z) = -\frac{1}{\zeta} \ \text{ (compare with (1.78))}$$

and

$$\hat{\psi}_{*,0,t} = -\frac{t\rho}{\zeta} + \hat{\mathscr{B}}\left(E\big[tb(z) - t\rho\log(1-z^{-1})\big]\right). \tag{1.87}$$

Since $\hat{\mathscr{B}}\log(1-z^{-1}) = \frac{e^\zeta - 1}{\zeta}$, we obtain that:

$$\hat{\psi}_{*,0,t} = \frac{t\hat{b}(\zeta)}{e^\zeta - 1} \in -\frac{t\rho}{\zeta} + \mathbb{C}\{\zeta\}.$$

An analogue of Corollary 52 holds for the family of functions $\hat{\psi}_{*,t} = \hat{\mathscr{B}}\big(\tilde{\psi}_{*,t}\big) = \sum_{r\geqslant 0}\hat{\psi}_{*,r,t}$:

Proposition 64. *For any $t \in \mathbb{C}$ the Borel image $\hat{\psi}_{*,t}$ of the formal series $\tilde{\psi}_{*,t}(z)$ belongs to $-\frac{t\rho}{\zeta} + \mathscr{O}(\mathscr{R}_0)$. Let $R > 1$, $k \in \mathbb{Z}$ and $\gamma \in \mathscr{P}_0$ such that $\gamma(1) = 2\pi \mathrm{i}(k + \frac{1}{2})$. There exist continuous functions $C, \beta\colon I^+ \cup I^- \to \mathbb{R}^+$ such that for any t with $|t| \leqslant R$ one has*

$$\left|\left(\operatorname{cont}_\gamma \hat{\psi}_{*,t}\right)\left(\gamma(1) + r\,\mathrm{e}^{\mathrm{i}\theta}\right)\right| \leqslant C(\theta)\,\mathrm{e}^{\beta(\theta) r}. \tag{1.88}$$

for all $\theta \in I^+ \cup I^-$ and $r > 0$.

Proof. Fix $\alpha \in (0, \pi/2), \delta \in (0,1)$ and $n \in \mathbb{N}$. Observe that

$$\hat{\psi}_{*,1,t} = t\hat{\mathscr{B}}\left(E\sum_{k\geqslant 1}\frac{t^k b^k}{k!}\partial^k E b\right) = t\hat{E}\sum_{k\geqslant 1}\left(t^k \hat{b}_k * \hat{\partial}^{k-1}\hat{\chi}\right), \quad \text{where } \hat{\chi}(\zeta) = \frac{-\zeta\hat{b}(\zeta)}{e^\zeta - 1}. \tag{1.89}$$

Fix $t \in \mathbb{C}$ with $|t| \leqslant R$. Let $C_0, C_1, \beta_0 > 0$ be constants such that:

$$\left|\hat{b}(\zeta)\right| \leqslant C_0 e^{\beta_0|\zeta|} \text{ for each } \zeta \in \mathbb{C}, \quad \frac{l_\zeta^k}{k!|e^\zeta - 1|} \leqslant C_1^k \text{ for each } k \in \mathbb{N} \text{ and } \zeta \in \mathscr{L}_{\delta,n}^{\alpha,+}$$

(see (1.62) and Definition 49). Let $\zeta \in \mathscr{L}_{\delta,n}^{\alpha,+}$. Then

$$\left|\hat{\chi}(\zeta)\right| \leqslant C_0 C_1 e^{\beta_0|\zeta|}, \quad \left|\hat{\psi}_{*,0,t}(\zeta)\right| \leqslant \frac{C_0 C_1 e^{\beta_0 l_\zeta}}{l_\zeta}.$$

Using the same technique as we used in the proof of Proposition 51, we obtain:

$$\begin{aligned} \left|\hat{\psi}_{*,1,t}(\zeta)\right| \leqslant \frac{1}{|e^\zeta - 1|}\sum_{k\geqslant 1}\int_0^{l_\zeta}\frac{|t|^{k+1}C_0^k(l_\zeta - s)^{k-1}e^{\beta_0(l_\zeta - s)}}{k!(k-1)!}C_0 C_1 s^{k-1}e^{\beta_0 s}\mathrm{d}s \leqslant \\ \frac{|t|^2 C_0^2 C_1^2 e^{(\beta_0+1)l_\zeta}}{l_\zeta}\int_0^{l_\zeta} e^{|t|C_0(l_\zeta - s)}\mathrm{d}s \leqslant R^2 C_0^2 C_1^2 e^{(RC_0+\beta_0+1)l_\zeta}. \end{aligned} \tag{1.90}$$

Let $C_2 = R^2 C_0^2 C_1^2, \beta_2 = \max\{RC_0 + \beta_0 + 1, \beta_0 + C_0 C_1\}$. Using Proposition 51, by induction we obtain that

$$\left|\hat{\psi}_{*,r,t}(\zeta)\right| \leqslant \frac{C_2(C_0C_1)^{r-1}l_\zeta^{r-1}}{(r-1)!}e^{\beta_2 l_\zeta}$$

for all $r \in \mathbb{N}$. It follows that

$$\sum_{r\geqslant 0}\left|\hat{\psi}_{*,r,t}(\zeta)\right| \leqslant C_2\left(1 + \tfrac{1}{l_\zeta}\right)e^{(\beta_2 + C_0C_1)l_\zeta}. \tag{1.91}$$

The rest of the proof is similar to the Corollary 52. $\square$

Remark 65. In Proposition 64 we have proved convergence of the series $\sum_{r\geqslant 0}\hat{\psi}_{*,r,t}$ only for $\operatorname{Re}\zeta \geqslant -1$. However, in fact, using Lemma 48 we obtain that the series

$$\sum_{r\geqslant 0}\hat{\psi}_{*,r,t}$$

converges uniformly for (t,ζ) from compact subsets of $\mathbb{C}\times(\mathscr{R}_0\setminus\{0\})$.

Corollary 66. *The function*

$$\mathbb{C}\times(\mathscr{R}_0\setminus\{0\})\to\mathbb{C},\quad (t,\zeta)\to\hat{\psi}_{*,t}(\zeta)$$

is a holomorphic function of two variables.

Proof. For simplicity, for a complex manifold M denote by $\mathscr{O}(\mathbb{C}\times M)$ the space of functions $\chi(t,\zeta)$ on $\mathbb{C}\times M$ which are holomorphic in two variables (t,ζ). Consider the elements $\hat{\psi}_{*,r,t}$ as functions of two variables (t,ζ). Observe that (1.87) implies that the function $\hat{\psi}_{*,0,t}$ lies in $\mathscr{O}(\mathbb{C}\times(\mathbb{C}\setminus\{2\pi i\mathbb{Z}\}))$. Since the series in (1.89) converges uniformly on compact subsets of $\mathbb{C}\times\mathscr{R}_0$ (see (1.90)), the function $\hat{\psi}_{*,1,t}$ lies in $\mathscr{O}(\mathbb{C}\times\mathscr{R}_0)$. Moreover, for $r\geqslant 1$ the element $\hat{\psi}_{*,r+1,t}$ can be written as the series

$$\hat{\psi}_{*,r+1,t}=\sum_{k\geqslant 1}t^k\hat{b}_k*\hat{\partial}^k\hat{\psi}_{*,r,t}. \tag{1.92}$$

Assume that $\hat{\psi}_{*,r,t}\in\mathscr{O}(\mathbb{C}\times\mathscr{R}_0)$. Using Lemma 34 we obtain that $t^k\hat{b}_k*\hat{\partial}^k\hat{\psi}_{*,r,t}\in\mathscr{O}(\mathbb{C}\times\mathscr{R}_0)$ for each $k\in\mathbb{N}$. Inequality (1.61) implies that the series (1.92) converges uniformly for (t,ζ) from compact subsets of $\mathbb{C}\times\mathscr{R}_0$. It follows that $\hat{\psi}_{*,r+1,t}\in\mathscr{O}(\mathbb{C}\times\mathscr{R}_0)$. Thus, by induction we obtain that $\hat{\psi}_{*,r,t}\in\mathscr{O}(\mathbb{C}\times\mathscr{R}_0)$ for all $r\in\mathbb{N}$. By (1.91), the series $\sum_{r\geqslant 0}\hat{\psi}_{*,r,t}$ converges uniformly on compact subsets of $\mathbb{C}\times(\mathscr{R}_0\setminus\{0\})$. Therefore,

$$\hat{\psi}_{*,t}=\sum_{r\geqslant 0}\hat{\psi}_{*,r,t}$$

is holomorphic in (t,ζ) on $\mathbb{C}\times(\mathscr{R}_0\setminus\{0\})$. $\square$

1.4.3 The τ-normalization

Let I be an arc of directions and $\beta\colon I\to\mathbb{R}^+$ be a continuous function. We pick a point $\tau\in\Sigma(I,\beta)$ and denote by $\mathrm{SING}^{I,\beta}_\tau$ the subspace of $\mathrm{SING}^{I,\beta}$ consisting of the singularities $\overset{\triangledown}{\varphi}$ such that

$$\overset{\triangledown}{\mathcal{L}}{}^I\overset{\triangledown}{\varphi}(\tau)=0.$$

Set $\mathrm{RES}^{I,\beta}_\tau:=\mathrm{SING}^{I,\beta}_\tau\cap\mathrm{RES}$.

Lemma 67. *Let I be an arc of directions such that $I \cap (\pi\mathbb{Z} + \pi/2) = \varnothing$. For any $\overset{\triangledown}{\varphi} \in \mathrm{SING}^{I,\beta}$, there is a unique $\overset{\triangledown}{\psi} \in \mathrm{SING}^{I,\beta}_{\tau}$ such that*

$$\overset{\triangledown}{D}\overset{\triangledown}{\psi} = \overset{\triangledown}{\varphi}. \tag{1.93}$$

We will use the notation $\overset{\triangledown}{\psi} = \overset{\triangledown}{E}_{\tau}\overset{\triangledown}{\varphi}$.

Proof. Denote by $P = P_\tau$ the projector

$$\overset{\triangledown}{\varphi} \in \mathrm{SING}^{I,\beta} \mapsto \big(\overset{\triangledown}{\mathcal{L}}{}^{I}\overset{\triangledown}{\varphi}(\tau)\big)\delta \in \mathbb{C}\,\delta.$$

Then

$$\mathrm{SING}^{I,\beta}_{\tau} = \ker P.$$

Let $\overset{\triangledown}{\varphi} \in \mathrm{SING}^{I,\beta}$. Choose a major $\check{\varphi}$. Observe that the equation $\overset{\triangledown}{D}\overset{\triangledown}{\psi} = \overset{\triangledown}{\varphi}$ in $\mathrm{SING}^{I,\beta}$ is equivalent to having a major $\check{\psi}$ such that

$$(\mathrm{e}^{\zeta} - 1)\check{\psi} - \check{\varphi} \in \mathbb{C}\{\zeta\}.$$

The latter condition can be rewritten as:

$$\check{\psi}(\zeta) = \frac{\check{\varphi}(\zeta)}{\mathrm{e}^{\zeta} - 1} + \frac{c}{2\pi\mathrm{i}\zeta} + \text{any regular germ},$$

where $c \in \mathbb{C}$ is arbitrary. Thus, the solutions of (1.93) are given by

$$\overset{\triangledown}{\psi} = \mathrm{sing}_0\left(\frac{\check{\varphi}(\zeta)}{\mathrm{e}^{\zeta} - 1}\right) + c\,\delta$$

where $c \in \mathbb{C}$ is arbitrary and $\delta = \mathrm{sing}_0\left(\frac{1}{2\pi\mathrm{i}\zeta}\right)$. Among these solutions there exists a unique $\overset{\triangledown}{\psi} \in \ker P$, namely:

$$\overset{\triangledown}{\psi} = (\mathrm{Id} - P)\,\mathrm{sing}_0\left(\frac{\check{\varphi}(\zeta)}{\mathrm{e}^{\zeta} - 1}\right). \tag{1.94}$$

□

Remark 68. Beware that $\mathrm{sing}_0\left(\frac{\check{\varphi}(\zeta)}{\mathrm{e}^{\zeta}-1}\right)$ depends on the choice of the major $\check{\varphi}$ (not only on the singularity $\overset{\triangledown}{\varphi}$), but the singularity in (1.94) does not. Indeed, suppose that $\check{\varphi}_1$ is another major of $\overset{\triangledown}{\varphi}$. Then

$$\check{\chi} = \check{\varphi}_1 - \check{\varphi} \in \mathbb{C}\{\zeta\}, \quad \text{hence} \quad \mathrm{sing}_0\left(\frac{\check{\varphi}_1(\zeta) - \check{\varphi}(\zeta)}{\mathrm{e}^{\zeta} - 1}\right) = 2\pi\mathrm{i}\check{\chi}(0)\delta \in \ker \mathrm{Id} - P.$$

Thus, Lemma 67 defines an operator $\overset{\triangledown}{E}_\tau\colon\ \mathrm{SING}^{I,\beta} \to \mathrm{SING}^{I,\beta}_\tau$ which satisfies

$$\overset{\triangledown}{D} \circ \overset{\triangledown}{E}_\tau = \mathrm{Id}$$

on the whole space $\mathrm{SING}^{I,\beta}$.

Remark 69. Observe that in Definition 42 we used a different kind of normalization to invert the analog of $\overset{\triangledown}{D}$ in the space of formal series.

Let I and β be as above. As before, let $\beta_0 \in \mathbb{R}$ be such that

$$\left|\hat{b}(\zeta)\right| \leqslant C_0 e^{\beta_0|\zeta|} \ \text{ for all } \ \zeta \in \mathbb{C}.$$

1.4.4 The singularity $\overset{\triangledown}{\psi}_{\tau,t}$ and the iterator υ_τ.

In this subsection, we will transport the formulas (1.84) to the space of general singularities. Let I stand either for $I^+ = (-\frac{\pi}{2}, \frac{\pi}{2})$ or for $I^- = (-\frac{\pi}{2}, \frac{\pi}{2})$. To fix the ideas, we assume that $I = I^+$. The case $I = I^-$ can be treated in a similar way. Let $R > 0$. Till the end of this subsection we will assume that $t \in \mathbb{C}$ is a parameter such that $|t| \leqslant R$. Introduce singularities

$$\overset{\triangledown}{\psi}_{*,r,t} \coloneqq \overset{\triangledown}{\mathscr{B}}\tilde{\psi}_{*,r,t}, r \geqslant 0, \quad (\text{see } \ (1.77) \ \text{ and } \ (1.78)).$$

By (1.66) and (1.87), for any r the element $\overset{\triangledown}{\psi}_{*,r,t}$ has a major of the form:

$$\check{\psi}_{*,r,t}(\zeta) = \frac{\log\zeta}{2\pi i}\hat{\psi}_{*,r,t}(\zeta). \tag{1.95}$$

From the proof of Proposition 64 (see (1.91)) we obtain that there exists a continuous function $\beta : I \to \mathbb{R}_+$ such that for all $t \in D_R(0)$

$$\overset{\triangledown}{\psi}_{*,r,t} \in \mathrm{SING}^{I,\beta} \ \text{ for each } \ r \in \mathbb{Z}_+, \ \ \overset{\triangledown}{\psi}_{*,t} \in \mathrm{SING}^{I,\beta}, \tag{1.96}$$

and for each $\tau \in \Sigma(I, \beta)$ one has:

$$\overset{\triangledown}{\mathcal{L}}{}^I \overset{\triangledown}{\psi}_{*,t}(\tau) = \sum_{r \geqslant 0} \overset{\triangledown}{\mathcal{L}}{}^I \overset{\triangledown}{\psi}_{*,r,t}(\tau),$$

where the convergence is absolute. Without loss of generality we may assume that $\beta(\theta) > \beta_0$ for all $\theta \in I$. Introduce τ-normalized singularities:

$$\overset{\triangledown}{\psi}_{\tau,r,t} = \overset{\triangledown}{\psi}_{*,r,t} - \overset{\triangledown}{\mathcal{L}}{}^I \overset{\triangledown}{\psi}_{*,r,t}(\tau)\delta \in \mathrm{RES}^{I,\beta}_\tau, \ \ \overset{\triangledown}{\psi}_{\tau,t} = \overset{\triangledown}{\psi}_{*,t} - \overset{\triangledown}{\mathcal{L}}{}^I \overset{\triangledown}{\psi}_{*,t}(\tau)\delta \in \mathrm{RES}^{I,\beta}_\tau. \tag{1.97}$$

We want to write the resurgent singularities $\overset{\triangledown}{\psi}_{\tau,r,t}$ in a form similar to (1.84). For this reason, introduce the operators

$$\overset{\triangledown}{B}_k\colon\ \overset{\triangledown}{\varphi}\in\mathrm{SING}\mapsto \overset{\triangledown}{b}_k * \big(\overset{\triangledown}{\partial}{}^k\overset{\triangledown}{\varphi}\big),\quad \text{where}\ \ \overset{\triangledown}{b}_k := \frac{1}{k!}\overset{\triangledown}{b}{}^{*k},\ \ \overset{\triangledown}{b} := {}^\flat\hat{b}.$$

Observe that $\overset{\triangledown}{b}_k = {}^\flat\hat{b}_k$.

Lemma 70. *Let* $\overset{\triangledown}{\chi}\in\mathrm{RES}$ *and* $s\in\mathbb{C}$. *Then the series of singularities*

$$\sum_{k\in\mathbb{N}} s^k\overset{\triangledown}{B}_k\overset{\triangledown}{\chi}$$

converges as of Definition 60.

Proof. We will generalize some of the techniques which we used in Subsection 1.2.7 to the case of resurgent singularities. Set

$$\overset{\triangledown}{\phi}_k = \overset{\triangledown}{B}_k\overset{\triangledown}{\chi}.$$

Fix a major $\overset{\vee}{\chi}$ of $\overset{\triangledown}{\chi}$ such that $\overset{\vee}{\chi}\in\mathscr{O}(\mathscr{R}_1)$. By Proposition 58, the singularity $\overset{\triangledown}{\phi}_k$ has a major of the form:

$$\overset{\vee}{\phi}_k(\zeta) := \int_\gamma \overset{\vee}{\chi}(\zeta_1)\hat{b}_k\big(\pi_1(\zeta)-\pi_1(\zeta_1)\big)\mathrm{d}\zeta_1,\quad \zeta\in\mathscr{R}_1, \tag{1.98}$$

where γ is any path starting at 1 and terminating at ζ.

Let $0<\delta<\pi$. Set

$$\mathscr{U}_\delta = \pi_1^{-1}(\{\zeta\in\mathbb{C} : d(\zeta,2\pi\mathrm{i}\mathbb{Z})<\delta\}).$$

For a point $\zeta\in\mathscr{R}_1\backslash\mathscr{U}_\delta$ denote by Γ_ζ the shortest path in $\mathscr{R}_1$ starting at 1 and terminating at ζ. Let L_ζ stand for the length of Γ_ζ. Fix $R>0$ and set

$$M = \max\big\{\big|\overset{\vee}{\chi}(\zeta)\big| : \zeta\in\mathscr{R}_1\setminus\mathscr{U}_\delta, L_\zeta\leqslant R\big\},\quad C=\max\big\{\big|\hat{b}(\zeta)\big| : \zeta\in\mathbb{C}, |\zeta|\leqslant R\big\}.$$

Then one has:

$$\big|\hat{b}_k(\zeta)\big|\leqslant\frac{C^kR^{k-1}}{k!(k-1)!}\ \text{ for each }\ \zeta\in\mathbb{C}\ \text{ such that }\ |\zeta|\leqslant R\ \text{ (see (1.49))}.$$

Let $\zeta\in\mathscr{R}_1\setminus\mathscr{U}_\delta$ such that $L_\zeta\leqslant R$. Using $\gamma=\Gamma_\zeta$ in the formula (1.98), we obtain:

$$\big|\overset{\vee}{\phi}_k(\zeta)\big|\leqslant\frac{MC^kR^k}{k!(k-1)!}.$$

It follows that for each $s \in \mathbb{C}$ one has:

$$\sum_{k\in\mathbb{N}} \left|s^k \overset{\vee}{\phi}_k(\zeta)\right| \leqslant M \exp(|s|CR).$$

This proves that the series $\sum s^k \overset{\vee}{\phi}_k(\zeta)$ converges uniformly on compact subsets of $\mathscr{R}_1$. By Definition 60, the series of singularities

$$\sum_{k\in\mathbb{N}} s^k \overset{\triangledown}{\phi}_k$$

is convergent. $\square$

Now, introduce a family of operators $\overset{\triangledown}{B}_s : \mathrm{RES} \to \mathrm{RES},\ s \in \mathbb{C}$, by the formula

$$\overset{\triangledown}{B}_s \overset{\triangledown}{\chi} := \sum_{k\in\mathbb{N}} s^k \overset{\triangledown}{B}_k \overset{\triangledown}{\chi}, \quad \overset{\triangledown}{\chi} \in \mathrm{RES}\,.$$

Lemma 71. *For each $r \geqslant 0$ one has*

$$\overset{\triangledown}{B}_t \left(\overset{\triangledown}{E}_\tau \overset{\triangledown}{B}_t\right)^r \overset{\triangledown}{E}_\tau \overset{\triangledown}{b} \in \mathrm{RES}^{I,\beta}_\tau\,.$$

Moreover,

$$\overset{\triangledown}{\psi}_{\tau,r,t} = t \left(\overset{\triangledown}{E}_\tau \overset{\triangledown}{B}_t\right)^r \overset{\triangledown}{E}_\tau \overset{\triangledown}{b}. \tag{1.99}$$

Proof. First, let us show that $\overset{\triangledown}{\psi}_{\tau,0,t} = t\overset{\triangledown}{E}_\tau \overset{\triangledown}{b}$. By (1.87) and (1.95), the element $\overset{\triangledown}{\psi}_{*,0,t}$ has a major of the form

$$\overset{\vee}{\psi}_{*,0,t} = \frac{t \log\zeta}{2\pi i}\left(-\frac{\rho}{\zeta} + \hat{\mathscr{B}}\left(E\big[b(z) - \rho\log(1-z^{-1})\big]\right)\right).$$

It follows that the element $\overset{\triangledown}{D}\overset{\triangledown}{\psi}_{*,0,t}$ has a major of the form

$$(e^\zeta - 1)\overset{\vee}{\psi}_{*,0,t} = \frac{t\log\zeta}{2\pi i}\left(-\frac{\rho(e^\zeta-1)}{\zeta} + \hat{\mathscr{B}}\left(\big[b(z) - \rho\log(1-z^{-1})\big]\right)\right) = \frac{t\log\zeta}{2\pi i}\hat{b}\,.$$

Here we used the identity $\hat{\mathscr{B}}(\log(1-z^{-1})) = -\frac{e^\zeta-1}{\zeta}$, which can be verified straightforwardly. Thus, by (1.97), we obtain

$$\overset{\triangledown}{D}\overset{\triangledown}{\psi}_{\tau,0,t} = \overset{\triangledown}{D}\overset{\triangledown}{\psi}_{*,0,t} = t\overset{\triangledown}{b}.$$

Recall that $\overset{\triangledown}{\psi}_{\tau,0,t} \in \mathrm{RES}^{I,\beta}_\tau$. By Lemma 67, we obtain the desired identity: $\overset{\triangledown}{\psi}_{\tau,0,t} = t\overset{\triangledown}{E}_\tau\overset{\triangledown}{b}$.

Further, let $r \geqslant 0$. Then, since $\overset{\triangledown}{\partial}\delta = 0 \in \mathrm{SING}$, one has

$$\overset{\triangledown}{B}_t \overset{\triangledown}{\psi}_{\tau,r,t} = \overset{\triangledown}{B}_t \overset{\triangledown}{\psi}_{*,r,t}.$$

On the other hand, the element $\overset{\triangledown}{\psi}_{*,r+1,t}$ has a major of the form

$$\overset{\vee}{\psi}_{*,r+1,t}(\zeta) = \frac{\log \zeta}{2\pi i} \overset{\wedge}{\psi}_{*,r+1,t}(\zeta).$$

Observe that $B_t \tilde{\psi}_{*,r,t} \in z^{-2}\mathbb{C}[[z^{-1}]]_1$. Therefore,

$$\overset{\wedge}{\psi}_{*,r+1,t}(\zeta) = \overset{\wedge}{\mathscr{B}}\left(EB_t \tilde{\psi}_{*,r,t}\right) = \frac{1}{e^{\zeta}-1} \overset{\wedge}{\mathscr{B}}\left(B_t \tilde{\psi}_{*,r,t}\right).$$

It follows that $\overset{\triangledown}{D}\overset{\triangledown}{\psi}_{*,r+1,t}$ has a major of the form

$$\frac{\log \zeta}{2\pi i} \overset{\wedge}{\mathscr{B}}\left(B_t \tilde{\psi}_{*,r,t}\right).$$

This implies that $\overset{\triangledown}{D}\overset{\triangledown}{\psi}_{*,r+1,t} = \overset{\triangledown}{\mathscr{B}}\left(B_t \tilde{\psi}_{*,r,t}\right) = \overset{\triangledown}{B}_t \overset{\triangledown}{\psi}_{*,r,t}$. Thus,

$$\overset{\triangledown}{D}\overset{\triangledown}{\psi}_{\tau,r+1,t} = \overset{\triangledown}{D}\overset{\triangledown}{\psi}_{*,r+1,t} = \overset{\triangledown}{B}_t \overset{\triangledown}{\psi}_{*,r,t} = \overset{\triangledown}{B}_t \overset{\triangledown}{\psi}_{\tau,r,t}.$$

By Lemma 67, we obtain that $\overset{\triangledown}{\psi}_{\tau,r+1,t} = \overset{\triangledown}{E}_{\tau} \overset{\triangledown}{B}_t \overset{\triangledown}{\psi}_{\tau,r,t}$. Using an induction by r we complete the proof of Lemma 71. □

Formally, we have:

$$\overset{\triangledown}{\psi}_{\tau,t} = \sum_{r \geqslant 1} \overset{\triangledown}{\psi}_{\tau,r,t}. \tag{1.100}$$

Observe that the singularity $\overset{\triangledown}{\psi}_{\tau,r,t}$ has a major of the form

$$\overset{\vee}{\psi}_{\tau,r,t} = \frac{\log \zeta}{2\pi i} \overset{\wedge}{\psi}_{*,r,t} - \overset{\triangledown}{\mathcal{L}}^{I} \overset{\triangledown}{\psi}_{*,r,t}(\tau) \frac{1}{2\pi i \zeta}.$$

By Proposition 64 and Remark 65 we obtain that the series (1.100) converges (see Definition 60). By Lemma 71 we obtain the following:

Corollary 72. *The singularity $\overset{\triangledown}{\psi}_{\tau,t}$ is a solution of the equation*

$$\left(\overset{\triangledown}{D} - \overset{\triangledown}{B}_t\right) \overset{\triangledown}{\psi} = t\overset{\triangledown}{b}.$$

Set

$$\overset{\triangledown}{v}_{\tau} := \delta^{(1)} + \overset{\triangledown}{\psi}_{\tau}, \quad \text{where} \quad \delta^{(1)} = \overset{\triangledown}{I}_{-1} = \operatorname{sing}_0\left(-\frac{1}{2\pi \mathrm{i} \zeta^2}\right).$$

Define sectorial iterators by the formulas:

$$v_{\tau}^{\pm} = \overset{\triangledown}{\mathcal{L}}^{I_{\pm}} \overset{\triangledown}{v}_{\tau} = \mathrm{Id} + \overset{\triangledown}{\mathcal{L}}^{I_{\pm}} \overset{\triangledown}{\psi}_{\tau}.$$

We call v_{τ} the τ-normalized sectorial iterator, since it is the unique sectorial iterator of f such that $v_{\tau}^{+}(\tau) = \tau$. In fact, $v_{\tau} = v_{*} + \tau - v_{*}^{+}(\tau)$.

1.4.5 Algebra $\mathrm{RES}[[\mathfrak{t}]]$

In this section we will consider both formal power series and convergent power series with coefficients in RES. To distinguish different kinds of series we will use the symbol t for convergent power series and the symbol $\mathfrak{t}$ for formal power series. We introduce the space of formal series in $\mathfrak{t}$ with coefficients from RES:

$$\mathrm{RES}[[\mathfrak{t}]] = \left\{ \sum_{n \geqslant 0} \mathfrak{t}^n \overset{\triangledown}{\chi}_n : \overset{\triangledown}{\chi}_n \in \mathrm{RES} \right\}.$$

Notice that in the series above we treat $\mathfrak{t}$ as a formal symbol and do not require convergence. Similarly, define spaces $\mathrm{RES}^{I,\beta}[[\mathfrak{t}]]$ and $\mathrm{RES}^{I,\beta}_{\tau}[[\mathfrak{t}]]$.

The space $\mathrm{RES}[[\mathfrak{t}]]$ forms an algebra under the convolution

$$\sum_{n \geqslant 0} \mathfrak{t}^n \overset{\triangledown}{\chi}_n * \sum_{k \geqslant 0} \mathfrak{t}^k \overset{\triangledown}{\phi}_k = \sum_{n \geqslant 0} \mathfrak{t}^n \sum_{0 \leqslant k \leqslant n} \overset{\triangledown}{\chi}_k * \overset{\triangledown}{\phi}_{n-k}.$$

By the exponent of a series of singularities $\overset{\triangledown}{\chi}_{\mathfrak{t}} = \sum\limits_{n \geqslant 1} \mathfrak{t}^n \overset{\triangledown}{\chi}_n$ we mean the following formal series

$$\overset{\triangledown}{\exp} \left(\overset{\triangledown}{\chi}_{\mathfrak{t}} \right) := \sum_{k \geqslant 0} \frac{1}{k!} \overset{\triangledown}{\chi}_{\mathfrak{t}}^{*k}.$$

The latter series is formally convergent since $\overset{\triangledown}{\chi}_{\mathfrak{t}}^{*k} \in \mathfrak{t}^k\, \mathrm{RES}[[\mathfrak{t}]]$ for each k.

Any sequence of operators $\overset{\triangledown}{A}_n : \mathrm{RES} \to \mathrm{RES}, n \geqslant 0$, defines an operator

$$\overset{\triangledown}{A}_{\mathfrak{t}} := \sum_{n \geqslant 0} \mathfrak{t}^n \overset{\triangledown}{A}_n : \mathrm{RES}[[\mathfrak{t}]] \to \mathrm{RES}[[\mathfrak{t}]], \quad \overset{\triangledown}{A}_{\mathfrak{t}} \overset{\triangledown}{\chi}_{\mathfrak{t}} = \sum_{n \geqslant 0} \mathfrak{t}^n \sum_{0 \leqslant k \leqslant n} \overset{\triangledown}{A}_k \overset{\triangledown}{\chi}_{n-k}.$$

1.4.6 An auxiliary singularity $\overset{\triangledown}{\mathcal{W}}_{\omega,\tau,\mathfrak{t}}$

Fix $\omega \in 2\pi \mathrm{i} \mathbb{Z}$. Let us introduce operators

$$\overset{\triangledown}{B}_{\omega,k} \colon \ \overset{\triangledown}{\varphi} \in \mathrm{RES} \mapsto \frac{1}{k!} \overset{\triangledown}{b}^{*k} * \left(\left(-\omega + \overset{\triangledown}{\partial} \right)^k \overset{\triangledown}{\varphi} \right), \quad \overset{\triangledown}{B}_{\omega,\mathfrak{t}} = \sum_{k \geqslant 1} \mathfrak{t}^k \overset{\triangledown}{B}_{\omega,k} \colon \ \mathrm{RES}[[\mathfrak{t}]] \to \mathrm{RES}[[\mathfrak{t}]],$$

where $\overset{\triangledown}{\partial}$ is the operator of multiplication by $(-\zeta)$. To study the singularities at ω of the branches of the minors $\hat{\psi}_{\tau,t}$, we define an auxiliary singularity

$$\overset{\triangledown}{\mathcal{W}}_{\omega,\tau,\mathfrak{t}} = \mathfrak{t} \sum_{r \geqslant 0} \left(\overset{\triangledown}{E}_{\tau} \overset{\triangledown}{B}_{\omega,\mathfrak{t}} \right)^r \overset{\triangledown}{E}_{\tau} \overset{\triangledown}{b}. \tag{1.101}$$

By Proposition 63 and the definition of operator $\overset{\triangledown}{E}_{\tau}$ (see Lemma 67), since $\beta(\theta) > \beta_0$ for all $\theta \in I$, we obtain that $\overset{\triangledown}{\mathcal{W}}_{\omega,\tau,\mathfrak{t}} \in \mathrm{RES}^{I,\beta}[[\mathfrak{t}]]$.

In this section together with the element $\overset{\triangledown}{\psi}_{\tau,t} \in \mathrm{RES}$ we will consider the formal series $\overset{\triangledown}{\psi}_{\tau,\mathfrak{t}} \in \mathrm{RES}[[\mathfrak{t}]]$ obtained by substituting $\mathfrak{t}$ for t:

$$\overset{\triangledown}{\psi}_{\tau,r,\mathfrak{t}} := \mathfrak{t}\left(\overset{\triangledown}{E}_\tau \overset{\triangledown}{B}_\mathfrak{t}\right)^r \overset{\triangledown}{E}_\tau \overset{\triangledown}{b} = \left(\overset{\triangledown}{E}_\tau \overset{\triangledown}{B}_\mathfrak{t}\right)^{r+1} \delta^{(1)}.$$

We prove the following:

Proposition 73. *In* $\mathrm{RES}[[\mathfrak{t}]]$ *the following identity holds:*

$$\overset{\triangledown}{\mathcal{W}}_{\omega,\tau,\mathfrak{t}} = \overset{\triangledown}{\exp}\left(-\omega \overset{\triangledown}{\psi}_{\tau,\mathfrak{t}}\right). \tag{1.102}$$

Notice that in Proposition 73 we do not state that the series from (1.102) are convergent. We only state that the identity (1.102) holds coefficientwise. That is, for each k the coefficients at $\mathfrak{t}^k$ in the left hand side and the right hand side of (1.102) coincide.

Denote an operator $\overset{\triangledown}{C}_\mathfrak{t} : \mathrm{RES}[[\mathfrak{t}]] \to \mathrm{RES}[[\mathfrak{t}]]$ by the formula:

$$\overset{\triangledown}{C}_\mathfrak{t} = \mathrm{Id} + \overset{\triangledown}{B}_\mathfrak{t} = \mathrm{Id} + \sum_{k\geqslant 1} \mathfrak{t}^k \overset{\triangledown}{B}_k.$$

Set $\overset{\triangledown}{T} = \overset{\triangledown}{D} + \mathrm{Id} = \exp\left(-\overset{\triangledown}{\partial}\right)$. The operator $\overset{\triangledown}{T}$ acts by multiplication by e^{ζ}.

Remark 74. In the space of formal power series in $\mathfrak{t}$ with coefficients from $\mathbb{C}[[z^{-1}]]$, the operator $\overset{\triangledown}{C}_\mathfrak{t}$ corresponds to taking composition with $\mathrm{Id} + \mathfrak{t}b$ and the operator $\overset{\triangledown}{T}$ corresponds to taking composition with $\mathrm{Id}+1$.

Lemma 75. *For every* $\overset{\triangledown}{\chi}_\mathfrak{t}, \overset{\triangledown}{\psi}_\mathfrak{t} \in \mathrm{RES}[[\mathfrak{t}]]$ *one has:*
1) $\overset{\triangledown}{C}_\mathfrak{t}\left(\overset{\triangledown}{\chi}_\mathfrak{t} * \overset{\triangledown}{\psi}_\mathfrak{t}\right) = \overset{\triangledown}{C}_\mathfrak{t}\overset{\triangledown}{\chi}_\mathfrak{t} * \overset{\triangledown}{C}_\mathfrak{t}\overset{\triangledown}{\psi}_\mathfrak{t}$, $\overset{\triangledown}{T}\left(\overset{\triangledown}{\chi}_\mathfrak{t} * \overset{\triangledown}{\psi}_\mathfrak{t}\right) = \overset{\triangledown}{T}\overset{\triangledown}{\chi}_\mathfrak{t} * \overset{\triangledown}{T}\overset{\triangledown}{\psi}_\mathfrak{t}$;
2) $\overset{\triangledown}{C}_\mathfrak{t}\overset{\triangledown}{\exp}\left(\overset{\triangledown}{\chi}_\mathfrak{t}\right) = \overset{\triangledown}{\exp}\left(\overset{\triangledown}{C}_\mathfrak{t}\overset{\triangledown}{\chi}_\mathfrak{t}\right)$, $\overset{\triangledown}{T}\overset{\triangledown}{\exp}\left(\overset{\triangledown}{\chi}_\mathfrak{t}\right) = \overset{\triangledown}{\exp}\left(\overset{\triangledown}{T}\overset{\triangledown}{\chi}_\mathfrak{t}\right)$.

Proof. Observe that 1) implies 2) by linearity of the operators $\overset{\triangledown}{C}_\mathfrak{t}$ and $\overset{\triangledown}{T}$. Since both of the operators commute with multiplication by $\mathfrak{t}$, to prove 1) it is enough to show that

$$\overset{\triangledown}{C}_\mathfrak{t}\left(\overset{\triangledown}{\chi} * \overset{\triangledown}{\psi}\right) = \overset{\triangledown}{C}_\mathfrak{t}\overset{\triangledown}{\chi} * \overset{\triangledown}{C}_\mathfrak{t}\overset{\triangledown}{\psi},\ \overset{\triangledown}{T}\left(\overset{\triangledown}{\chi} * \overset{\triangledown}{\psi}\right) = \overset{\triangledown}{T}\overset{\triangledown}{\chi} * \overset{\triangledown}{T}\overset{\triangledown}{\psi}$$

for any $\overset{\triangledown}{\chi}, \overset{\triangledown}{\psi} \in \mathrm{RES}$. Since $\overset{\triangledown}{\partial}$ is a differentiation on SING, using commutativity of the convolution, we obtain:

$$\overset{\triangledown}{C}_\mathfrak{t}\left(\overset{\triangledown}{\chi} * \overset{\triangledown}{\psi}\right) = \sum_{k\geqslant 0}\left(\frac{\mathfrak{t}^k}{k!}\overset{\triangledown}{b}^{*k} * \overset{\triangledown}{\partial}^k\left(\overset{\triangledown}{\chi} * \overset{\triangledown}{\psi}\right)\right) =$$

$$\sum_{k\geqslant 0}\left(\frac{\mathfrak{t}^k}{k!}\overset{\triangledown}{b}^{*k} * \sum_{m=0}^{k}\left(\frac{k!}{m!(k-m)!}\overset{\triangledown}{\partial}^m\overset{\triangledown}{\chi} * \overset{\triangledown}{\partial}^{k-m}\overset{\triangledown}{\psi}\right)\right) =$$

$$\sum_{k\geqslant 0}\sum_{m=0}^{k}\left(\frac{\mathfrak{t}^m}{m!}\overset{\triangledown}{b}^{*m} * \overset{\triangledown}{\partial}^m\overset{\triangledown}{\chi}\right) * \left(\frac{\mathfrak{t}^{k-m}}{(k-m)!}\overset{\triangledown}{b}^{k-m} * \overset{\triangledown}{\partial}^{k-m}\overset{\triangledown}{\psi}\right) = \overset{\triangledown}{C}_\mathfrak{t}\overset{\triangledown}{\chi} * \overset{\triangledown}{C}_\mathfrak{t}\overset{\triangledown}{\psi}.$$

The corresponding identity for $\overset{\triangledown}{T}$ can be proven in a similar way. □

Proof of Proposition 73. Introduce an operator $\overset{\triangledown}{H}_{\mathfrak{t}} : \mathrm{RES}^{I,\beta}[[\mathfrak{t}]] \to \mathrm{RES}^{I,\beta}[[\mathfrak{t}]]$ as follows:

$$\overset{\triangledown}{H}_{\mathfrak{t}}\overset{\triangledown}{\chi}_{\mathfrak{t}} = \overset{\triangledown}{E}_{\tau}\left(\overset{\triangledown}{\exp}\left(-\omega\mathfrak{t}\overset{\triangledown}{b}\right) * \left(\overset{\triangledown}{C}_{\mathfrak{t}}\overset{\triangledown}{\chi}_{\mathfrak{t}}\right) - \overset{\triangledown}{\chi}_{\mathfrak{t}}\right).$$

Observe that the operator $\overset{\triangledown}{H}_{\mathfrak{t}}$ increases the discrete valuation

$$val\left(\overset{\triangledown}{\chi}_{\mathfrak{t}}\right) = \min\left\{k \geqslant 0 : \overset{\triangledown}{\chi}_k \neq 0\right\}$$

on $\mathrm{RES}[[\mathfrak{t}]]$. It follows that the following equation

$$\left(\mathrm{Id} - \overset{\triangledown}{H}_{\mathfrak{t}}\right)\overset{\triangledown}{\chi}_{\mathfrak{t}} = \delta \tag{1.103}$$

has a unique solution in $\mathrm{RES}[[\mathfrak{t}]]$. This solution can be written as formally convergent series $\sum\limits_{r=0}^{\infty}\overset{\triangledown}{H}{}^{r}_{\mathfrak{t}}\overset{\triangledown}{\chi}_{\mathfrak{t}}$. We will show that both sides of (1.102) are solutions of the equation (1.103). This will prove Proposition 73.

For any $\overset{\triangledown}{\chi} \in \mathrm{RES}$ one has

$$\overset{\triangledown}{E}_{\tau}\overset{\triangledown}{B}_{\omega,\mathfrak{t}}\overset{\triangledown}{\chi} = \sum_{k\geqslant 1}\mathfrak{t}^k\overset{\triangledown}{E}_{\tau}\overset{\triangledown}{B}_{\omega,k}\overset{\triangledown}{\chi} = \overset{\triangledown}{E}_{\tau}\sum_{k\geqslant 1}\left(\frac{\mathfrak{t}^k}{k!}\overset{\triangledown}{b}{}^{*k} * \left(\left(-\omega + \overset{\triangledown}{\partial}\right)^k\overset{\triangledown}{\chi}\right)\right) =$$

$$\overset{\triangledown}{E}_{\tau}\sum_{l,m\geqslant 0}\left(\frac{(-\omega\mathfrak{t})^l\overset{\triangledown}{b}{}^{*l}}{l!} * \frac{1}{m!}\left(\mathfrak{t}^m\overset{\triangledown}{b}{}^{*m} * \overset{\triangledown}{\partial}{}^m\overset{\triangledown}{\chi}\right)\right) - \overset{\triangledown}{E}_{\tau}\overset{\triangledown}{\chi} = \overset{\triangledown}{E}_{\tau}\left(\overset{\triangledown}{\exp}\left(-\omega\mathfrak{t}\overset{\triangledown}{b}\right) * \overset{\triangledown}{C}_{\mathfrak{t}}\overset{\triangledown}{\chi}\right) - \overset{\triangledown}{E}_{\tau}\overset{\triangledown}{\chi}. \tag{1.104}$$

It follows that the following identity holds in $\mathrm{RES}[[\mathfrak{t}]]$:

$$\overset{\triangledown}{\mathcal{W}}_{\omega,\tau,\mathfrak{t}} = \sum_{r\geqslant 0}\overset{\triangledown}{H}{}^{r}_{\mathfrak{t}}\delta.$$

Further, similarly to the identity (1.104), by (1.99), we obtain that for all $r \geqslant 0$

$$\overset{\triangledown}{\psi}_{\tau,r+1,\mathfrak{t}} = \overset{\triangledown}{E}_{\tau}\left(\overset{\triangledown}{C}_{\mathfrak{t}} - \mathrm{Id}\right)\overset{\triangledown}{\psi}_{\tau,r,\mathfrak{t}}.$$

It follows that

$$\overset{\triangledown}{\psi}_{\tau,\mathfrak{t}} = \mathfrak{t}\sum_{r\geqslant 0}\left(\overset{\triangledown}{E}_{\tau}\left(\overset{\triangledown}{C}_{\mathfrak{t}} - \mathrm{Id}\right)\right)^r\overset{\triangledown}{E}_{\tau}\overset{\triangledown}{b}.$$

Applying the operator $\overset{\triangledown}{D}\left(\mathrm{Id} - \left(\overset{\triangledown}{E}_{\tau}\left(\overset{\triangledown}{C}_{\mathfrak{t}} - \mathrm{Id}\right)\right)\right) = \overset{\triangledown}{D} + \mathrm{Id} - \overset{\triangledown}{C}_{\mathfrak{t}} = \overset{\triangledown}{T} - \overset{\triangledown}{C}_{\mathfrak{t}}$ to this equation we obtain:

$$\overset{\triangledown}{T}\overset{\triangledown}{\psi}_{\tau,\mathfrak{t}} = \overset{\triangledown}{C}_{\mathfrak{t}}\overset{\triangledown}{\psi}_{\tau,\mathfrak{t}} + \mathfrak{t}\overset{\triangledown}{b}.$$

Now, by Lemma 75, we have:

$$\overset{\triangledown}{D}\left(\mathrm{Id}-\overset{\triangledown}{H}_{\mathfrak{t}}\right)\overset{\triangledown}{\exp}\left(-\omega\overset{\triangledown}{\psi}_{\tau,\mathfrak{t}}\right)=\overset{\triangledown}{T}\overset{\triangledown}{\exp}\left(-\omega\overset{\triangledown}{\psi}_{\tau,\mathfrak{t}}\right)-\overset{\triangledown}{\exp}\left(-\omega\mathfrak{t}\overset{\triangledown}{b}\right)*\left(\overset{\triangledown}{C}_{\mathfrak{t}}\exp\left(-\omega\overset{\triangledown}{\psi}_{\tau,\mathfrak{t}}\right)\right)=$$
$$\overset{\triangledown}{\exp}\left(-\omega\overset{\triangledown}{T}\overset{\triangledown}{\psi}_{\tau,\mathfrak{t}}\right)-\overset{\triangledown}{\exp}\left(-\omega\overset{\triangledown}{C}_{\mathfrak{t}}\overset{\triangledown}{\psi}_{\tau,\mathfrak{t}}-\omega\mathfrak{t}\overset{\triangledown}{b}\right)=0.$$

This implies that

$$\left(\mathrm{Id}-\overset{\triangledown}{H}_{\mathfrak{t}}\right)\overset{\triangledown}{\exp}\left(-\omega\overset{\triangledown}{\psi}_{\tau,\mathfrak{t}}\right)=K(\mathfrak{t})\delta$$

for some $K(\mathfrak{t})=\sum_{n\geqslant 0}K_n\mathfrak{t}^n\in\mathbb{C}[[\mathfrak{t}]]$. Observe that by definition of $\overset{\triangledown}{E}_{\tau}$,

$$\left(\overset{\triangledown}{\mathcal{L}}^I\left(\mathrm{Id}-\overset{\triangledown}{H}_{\mathfrak{t}}\right)\overset{\triangledown}{\exp}\left(-\omega\overset{\triangledown}{\psi}_{\tau,\mathfrak{t}}\right)\right)(\tau)=\left(\overset{\triangledown}{\mathcal{L}}^I\overset{\triangledown}{\exp}\left(-\omega\overset{\triangledown}{\psi}_{\tau,\mathfrak{t}}\right)\right)(\tau)=1.$$

Therefore, $K_0=1$ and $K_n=0$ for $n\geqslant 1$. Thus, $\overset{\triangledown}{\exp}\left(-\omega\overset{\triangledown}{\psi}_{\tau,\mathfrak{t}}\right)$ is a solution of the equation (1.103). This finishes the proof of Proposition 73. □

1.4.7 Alien operators and the singularity $\overset{\triangledown}{\psi}_{\tau,\mathfrak{t}}$

Let $\hat{\chi}\in\mathscr{O}(\mathscr{R}_1)$. For a point $\eta\in\mathscr{R}_1$ such that $\pi_1(\eta)=2\pi ik+1, k\in\mathbb{Z}$, introduce the following germ in a neighborhood of 1:

$$S_\eta\hat{\chi}(1+\zeta):=\hat{\chi}(\eta+\zeta),$$

where $\zeta\in\mathbb{C}$ and $|\zeta|<1$. Clearly, the germ $S_\eta\hat{\chi}$ defines an element of $\mathscr{O}(\mathscr{R}_1)$.

Definition 76. Let $\omega=2\pi ik$ for some $k\in\mathbb{Z}$ and $\eta\in\mathscr{R}_1$ such that $\pi_1(\eta)=\omega+1$. Define the alien operator $A_\eta:\mathrm{RES}\to\mathrm{RES}$ by the formula:

$$A_\eta\overset{\triangledown}{\chi}=\mathrm{sing}_0\left(S_\eta\hat{\chi}\right),$$

where $\overset{\triangledown}{\chi}\in\mathrm{RES}$ and $\hat{\chi}=\mathrm{var}\,\overset{\triangledown}{\chi}$. In addition, if η is in the main sheet, we will use the notation $\Delta^+_\omega:=A_\eta$.

As before, we fix $R>0$ and assume that $t\in\mathbb{C}$ is a parameter such that $|t|\leqslant R$. From the proof of Proposition 64 (see (1.91)) we obtain that there exist continuous functions $\beta_{R,\eta}(\theta)$ on $I=I^+$ such that for all $\tau,z\in\Sigma(I,\beta_{R,\eta})$ all the values

$$\left(\overset{\triangledown}{\mathcal{L}}^I\overset{\triangledown}{\psi}_{\tau,r,t}\right)(z),\ \left(\overset{\triangledown}{\mathcal{L}}^I\overset{\triangledown}{\psi}_{\tau,t}\right)(z),\ \left(\overset{\triangledown}{\mathcal{L}}^I A_\eta\overset{\triangledown}{\psi}_{\tau,r,t}\right)(z),\ \left(\overset{\triangledown}{\mathcal{L}}^I A_\eta\overset{\triangledown}{\psi}_{\tau,t}\right)(z)$$

are defined and the following series converge absolutely

$$\left(\mathcal{L}^I\overset{\triangledown}{\psi}_{\tau,t}\right)(z) = \sum_{r\geqslant 0}\left(\mathcal{L}^I\overset{\triangledown}{\psi}_{\tau,r,t}\right)(z), \quad \left(\mathcal{L}^I A_\eta\overset{\triangledown}{\psi}_{\tau,t}\right)(z) = \sum_{r\geqslant 0}\left(\mathcal{L}^I A_\eta\overset{\triangledown}{\psi}_{\tau,r,t}\right)(z). \tag{1.105}$$

Similar statement is true for I^- by performing the Borel-Laplace summation of the series $\tilde{\psi}_{\tau,t} = \sum\tilde{\psi}^+_{\tau,r,t}$. Thus, we will assume that $\beta_{R,\eta}$ is defined on the whole $J = I^-\cup I^+$.

The main result of this section is the following

Theorem 77. *Let $\tau\in\Sigma(I,\beta_{R,\eta})$. There exists a unique scalar formal series $R_{\eta,\tau,\mathfrak{t}} = \sum_{r\geqslant 0}\mathfrak{t}^r R_{\eta,\tau,r}$ such that*

$$A_\eta\overset{\triangledown}{\psi}_{\tau,\mathfrak{t}} = R_{\eta,\tau,\mathfrak{t}}\overset{\triangledown}{\mathcal{W}}_{\omega,\tau,\mathfrak{t}}.$$

Let us study the operators A_η. The definition of A_η directly implies that for any entire function $\hat{\chi}$ and any $\overset{\triangledown}{\varphi}\in\mathrm{RES}$ one has

$$A_\eta{}^\flat\hat{\chi} = 0, \quad A_\eta\left(\hat{\chi}\overset{\triangledown}{\varphi}\right) = \left(T_\omega\hat{\chi}\right)\cdot A_\eta\overset{\triangledown}{\varphi}, \tag{1.106}$$

where $\omega = \pi_1(\eta)-1$ and $\left(T_\omega\hat{\chi}\right)(\zeta) = \hat{\chi}(\zeta+\omega)$.

Lemma 78. *Let $\hat{\chi}$ be an entire function and $\overset{\triangledown}{\varphi}\in\mathrm{RES}$. Then one has:*

$$A_\eta\left({}^\flat\hat{\chi}*\overset{\triangledown}{\varphi}\right) = {}^\flat\hat{\chi}*A_\eta\overset{\triangledown}{\varphi}.$$

Proof. Let $\zeta\in\mathscr{R}_1$ be a point in the main sheet such that $|\pi_1(\zeta)-1|<1$. In particular, ζ can be reached from 1 by a straight path $\Gamma_\zeta := [1,\zeta]\subset\mathscr{R}_1$. Then for any η such that $\pi_1(\eta) = 2\pi i k+1, k\in\mathbb{Z}$ the point $\eta+(\zeta-1)$ is well defined. Let $\check{\varphi}\in\mathscr{O}(\mathscr{R}_1)$ be a major of $\overset{\triangledown}{\varphi}$. By definition, the element $\overset{\triangledown}{\psi} = {}^\flat\hat{\chi}*\overset{\triangledown}{\varphi}$ has a major $\check{\psi}$ such that

$$\check{\psi}(\zeta) = \int_{\Gamma_\zeta}\check{\varphi}(\zeta_1)\hat{\chi}\left(\pi_1(\zeta)-\pi_1(\zeta_1)\right)\mathrm{d}\zeta_1$$

for ζ as above (see Proposition 58). Let γ be the path in $\mathscr{R}_1$, starting at 1, terminating at $e^{-2\pi i}$ and going around the origin by a circle of radius 1 centered at the origin. Denote by Γ'_ζ the path starting at $e^{-2\pi i}$, which projects to the same path in $\mathbb{C}$ as Γ_ζ. Then the path Γ'_ζ terminates at the point $\zeta' = e^{-2\pi i}+(\zeta-1)$. By Proposition 58 and definition of operators S_η, one has:

$$S_{e^{-2\pi i}}\check{\psi}(\zeta) = \check{\psi}\left(e^{-2\pi i}+(\zeta-1)\right) = \int_{\gamma\cup\Gamma'_\zeta}\check{\varphi}(\zeta_1)\hat{\chi}\left(\pi_1(\zeta)-\pi_1(\zeta_1)\right)\mathrm{d}\zeta_1.$$

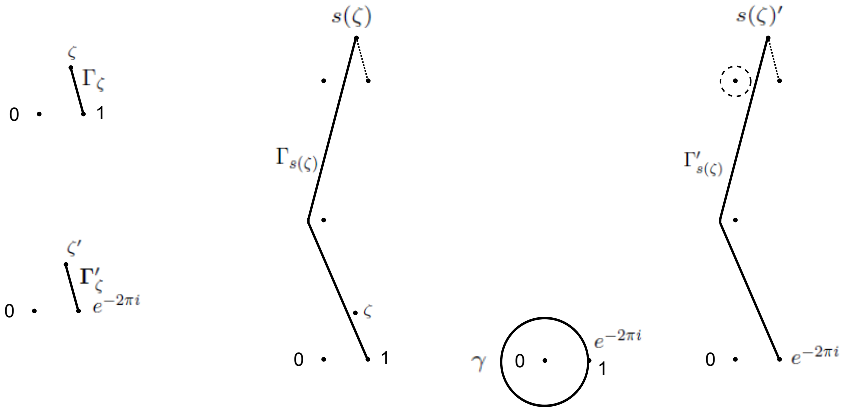

Figure 1.5: Illustration to Lemma 78.

Denote by P_ζ the path starting at ζ', going by $\gamma \cup \Gamma'_\zeta$ in the reverse direction, following Γ_ζ from 1 and terminating at ζ. It follows that

$$\operatorname{var}\left({}^\flat\hat{\chi} * \overset{\triangledown}{\psi}\right)(\zeta) = \check{\psi}(\zeta) - S_{e^{-2\pi i}}\check{\psi}(\zeta) = \int\limits_{P_\zeta} \check{\varphi}(\zeta_1)\hat{\chi}\left(\pi_1(\zeta) - \pi_1(\zeta_1)\right) \mathrm{d}\zeta_1. \tag{1.107}$$

Let $\pi_1(\eta) = 2\pi k i + 1$. Set

$$s(\zeta) = \eta + (\zeta - 1).$$

That is, $s(\zeta)$ and η are in the same half-sheet, and $s(\zeta)$ projects to

$$\pi_1(\eta) + \pi_1(\zeta) - 1 = 2\pi k i + \pi_1(\zeta) \in \mathbb{C}.$$

Let $\Gamma_{s(\zeta)}$ be a path starting at 1 and terminating at $s(\zeta)$. Denote by $\Gamma'_{s(\zeta)}$ be the path starting at $e^{-2\pi i}$ such that $\Gamma'_{s(\zeta)}$ projects to the same path in $\mathbb{C}$ as Γ_ζ (see Figure 4). Denote by $P_{s(\zeta)}$ the path starting at $s(\zeta)'$, going by $\gamma \cup \Gamma'_{s(\zeta)}$ in the reverse direction, following $\Gamma_{s(\zeta)}$ from 1 and terminating at $s(\zeta)$. Consider the analytic continuation of the germs in the formula (1.107) along the path $\Gamma_{s(\zeta)}$. By definition 76, the element $\overset{\triangledown}{\phi} = A_\eta\left({}^\flat\hat{\chi} * \overset{\triangledown}{\psi}\right)$ has a major $\check{\phi}$ such that

$$\check{\phi}(\zeta) = \operatorname{var}\left({}^\flat\hat{\chi} * \overset{\triangledown}{\psi}\right)(s(\zeta)) = \int\limits_{P_{s(\zeta)}} \check{\varphi}(\zeta_1)\hat{\chi}\left(2\pi k i + \pi_1(\zeta) - \pi_1(\zeta_1)\right) \mathrm{d}\zeta_1.$$

Denote by $s(\zeta)'$ the end point of the path $\Gamma'_{s(\zeta)}$ and let η' be the closest point to $s(\zeta)'$

which projects to $2\pi ik+1$. Using Cauchy Theorem we obtain that

$$\overset{\vee}{\psi}(\zeta)=\int\limits_{P_{s(0)}}\overset{\vee}{\varphi}(\zeta_1)\hat{\chi}\left(2\pi ki+\pi_1(\zeta)-\pi_1(\zeta_1)\right)\mathrm{d}\zeta_1+$$

$$\int\limits_{[\eta,s(\zeta)]}\overset{\vee}{\varphi}(\zeta_1)\hat{\chi}\left(2\pi ki+\pi_1(\zeta)-\pi_1(\zeta_1)\right)\mathrm{d}\zeta_1-\int\limits_{[\eta',s(\zeta)']}\overset{\vee}{\varphi}(\zeta_1)\hat{\chi}\left(2\pi ki+\pi_1(\zeta)-\pi_1(\zeta_1)\right)\mathrm{d}\zeta_1,$$

The first of the integrals in the formula above is a regular germ at the origin. Therefore, one has:

$$\overset{\vee}{\psi}(\zeta)=\mathrm{reg}(\zeta)+\int\limits_{[1,\zeta]}\overset{\vee}{\chi}(\eta+(\zeta_1-1))\hat{\varphi}\left(\pi_1(\zeta)-\pi_1(\zeta_1)\right)\mathrm{d}\zeta_1-$$
$$\int\limits_{[1,\zeta]}\overset{\vee}{\chi}(\eta'+(\zeta_1-1))\hat{\varphi}\left(\pi_1(\zeta)-\pi_1(\zeta_1)\right)\mathrm{d}\zeta_1$$

where $\mathrm{reg}(\zeta)$ is a regular germ at the origin. Using similar considerations as before, one can show that the latter formula denotes a major of ${}^\flat\hat{\chi}*A_\eta\overset{\triangledown}{\varphi}$. This proves that $A_\eta\left({}^\flat\hat{\chi}*\overset{\triangledown}{\varphi}\right)={}^\flat\hat{\chi}*A_\eta\left(\overset{\triangledown}{\varphi}\right)$. □

Formula (1.106) together with Lemma 78 imply the following:

Lemma 79. *One has* $A_\eta\overset{\triangledown}{B}_{\mathfrak{t}}=\overset{\triangledown}{B}_{\omega,\mathfrak{t}}A_\eta$, *where* $\omega=\pi_1(\eta)-1$.

Denote by P_τ the projector

$$\overset{\triangledown}{\chi}\in\mathrm{SING}^{I,\beta}\mapsto\left(\overset{\triangledown}{\mathcal{L}}{}^I\overset{\triangledown}{\chi}(\tau)\right)\delta\in\mathbb{C}\,\delta,$$

Proposition 80. *Let* $\eta\in\mathscr{R}_1$ *such that* $\pi_1(\eta)=2\pi ik+1, k\in\mathbb{Z}$. *Fix* $\tau\in\Sigma(I,\beta)$. *Then for any* $\overset{\triangledown}{\chi}\in\mathrm{RES}^{I,\beta}$ *such that* $A_\eta\overset{\triangledown}{\chi}\in\mathrm{RES}^{I,\beta}$ *one has:*

$$\left(A_\eta\overset{\triangledown}{E}_\tau-\overset{\triangledown}{E}_\tau A_\eta\right)\overset{\triangledown}{\chi}=P_\tau A_\eta\overset{\triangledown}{E}_\tau\overset{\triangledown}{\chi}.$$

Proof. Let $\overset{\vee}{\chi}$ be a major of the singularity $\overset{\triangledown}{\chi}$. Then, by the definition of $\overset{\triangledown}{E}_\tau$ (see Lemma 67) the element $\overset{\triangledown}{E}_\tau\overset{\triangledown}{\chi}$ has a major of the form

$$\frac{\overset{\vee}{\chi}(\zeta)}{e^\zeta-1}+\frac{C_1}{2\pi i\zeta},$$

where $C_1\in\mathbb{C}$. Thus, $\mathrm{var}\,\overset{\triangledown}{E}_\tau\overset{\triangledown}{\chi}=\frac{\hat{\chi}(\zeta)}{e^\zeta-1}$, where $\hat{\chi}=\mathrm{var}\,\overset{\triangledown}{\chi}$. It follows that $\overset{\triangledown}{\varphi}=A_\eta\overset{\triangledown}{E}_\tau\overset{\triangledown}{\chi}$ has a major $\overset{\vee}{\varphi}$ such that

$$\overset{\vee}{\varphi}(\zeta)=\frac{\hat{\chi}(\eta+(\zeta-1))}{e^\zeta-1}$$

for all ζ in the main sheet with $|\pi_1(\zeta)-1|<1$. Similarly, one can show that the element $\overset{\triangledown}{\phi} = \overset{\triangledown}{E}_\tau A_\eta \overset{\triangledown}{\chi}$ has a major $\overset{\vee}{\phi}$ such that

$$\overset{\vee}{\phi}(\zeta) = \frac{\hat{\chi}(\eta + (\zeta - 1))}{e^\zeta - 1} + \frac{C_2}{2\pi i \zeta}$$

for all ζ in the main sheet with $|\pi_1(\zeta)-1|<1$, where $C_2 \in \mathbb{C}$. It follows that

$$\left(A_\eta \overset{\triangledown}{E}_\tau - \overset{\triangledown}{E}_\tau A_\eta\right) \overset{\triangledown}{\chi} = C\delta, \quad C \in \mathbb{C}.$$

Applying operator P_τ to this equality we obtain that $C\delta = P_\tau A_\eta \overset{\triangledown}{E}_\tau \overset{\triangledown}{\chi}$. $\square$

Proof of Theorem 77. The singularities $\overset{\triangledown}{\psi}_{\tau,r,\mathfrak{t}}$ satisfy the following recurrent relations:

$$\overset{\triangledown}{\psi}_{\tau,r+1,\mathfrak{t}} = \overset{\triangledown}{E}_\tau \overset{\triangledown}{B}_\mathfrak{t} \overset{\triangledown}{\psi}_{\tau,r,\mathfrak{t}}, \quad \overset{\triangledown}{\psi}_{\tau,0,\mathfrak{t}} = \mathfrak{t} \overset{\triangledown}{E}_\tau \overset{\triangledown}{b}.$$

By Lemma 79 and Proposition 80, we arrive at the following identities:

$$A_\eta \overset{\triangledown}{\psi}_{\tau,r+1,\mathfrak{t}} = P_\tau A_\eta \overset{\triangledown}{\psi}_{\tau,r+1,\mathfrak{t}} + \overset{\triangledown}{E}_\tau \overset{\triangledown}{B}_{\omega,\mathfrak{t}} A_\eta \overset{\triangledown}{\psi}_{\tau,r,\mathfrak{t}}, \quad A_\eta \overset{\triangledown}{\psi}_{\tau,0,\mathfrak{t}} = P_\tau A_\eta \overset{\triangledown}{\psi}_{\tau,0,\mathfrak{t}} \in \mathbb{C}\delta.$$

By induction we obtain that for any $r \in \mathbb{N}$ one has:

$$A_\eta \overset{\triangledown}{\psi}_{\tau,r,\mathfrak{t}} = \sum_{j=0}^{r} \left(\overset{\triangledown}{E}_\tau \overset{\triangledown}{B}_{\omega,\mathfrak{t}}\right)^j P_\tau A_\eta \overset{\triangledown}{\psi}_{\tau,r-j,\mathfrak{t}}.$$

Summing the last identity over $r \geqslant 0$ we obtain

$$A_\eta \overset{\triangledown}{\psi}_{\tau,\mathfrak{t}} = R_{\eta,\tau,\mathfrak{t}} \overset{\triangledown}{\mathcal{W}}_{\omega,\tau,\mathfrak{t}}, \quad \text{where} \quad R_{\eta,\tau,\mathfrak{t}}\delta = P_\tau A_\eta \overset{\triangledown}{\psi}_{\tau,\mathfrak{t}}. \tag{1.108}$$

$\square$

1.4.8 The Bridge equation

As before, let $I = I^+ = (-\pi/2, \pi/2)$. Combining Proposition 73 with Theorem 77, we obtain that for all $\tau \in \Sigma(I, \beta_{R,\eta})$ one has

$$A_\eta \overset{\triangledown}{\psi}_{\tau,\mathfrak{t}} = R_{\eta,\tau,\mathfrak{t}} \overset{\triangledown}{\exp}\big(-\omega \overset{\triangledown}{\psi}_{\tau,\mathfrak{t}}\big). \tag{1.109}$$

Since $\overset{\triangledown}{\psi}_{*,\mathfrak{t}} - \overset{\triangledown}{\psi}_{\tau,\mathfrak{t}} = \overset{\triangledown}{\mathcal{L}}^I \overset{\triangledown}{\psi}_{*,\mathfrak{t}}(\tau)\delta \in \mathbb{C}\delta$, the following holds.

Theorem 81. *There exists a sequence of constants $R_{\eta,r}$ and a formal series $R_{\eta,\mathfrak{t}} = \sum_{r\geqslant 1} \mathfrak{t}^r R_{\eta,r}$ such that*

$$A_\eta \overset{\triangledown}{\psi}_{*,\mathfrak{t}} = R_{\eta,\mathfrak{t}} \overset{\triangledown}{\exp}\big(-\omega \overset{\triangledown}{\psi}_{*,\mathfrak{t}}\big). \tag{1.110}$$

The relation between $R_{\eta,\tau,\mathfrak{t}}$ and $R_{\eta,\mathfrak{t}}$ is:

$$R_{\eta,\mathfrak{t}} = \exp\left(\omega \overset{\triangledown}{\mathcal{L}}{}^{I} \overset{\triangledown}{\psi}_{*,\mathfrak{t}}(\tau)\right) R_{\eta,\tau,\mathfrak{t}}.$$

We will refer to formulas (1.109) and (1.110) as Bridge equation. It was first formulated by Écalle [16]. Both of the formulas state that for each n the coefficients in front of $\mathfrak{t}^n$ on the left hand side and on the right hand side of the corresponding formula are equal. For instance, for $n = 1$ and $n = 2$ from (1.109) we obtain that

$$A_\eta \overset{\triangledown}{E}_\tau \overset{\triangledown}{b} = R_{\tau,\eta,1}\delta, \quad A_\eta \overset{\triangledown}{E}_\tau \left(\overset{\triangledown}{b} * \overset{\triangledown}{\partial} \overset{\triangledown}{E}_\tau \overset{\triangledown}{b}\right) = -\omega R_{\tau,\eta,1} \overset{\triangledown}{E}_\tau \overset{\triangledown}{b} + R_{\tau,\eta,2}\delta.$$

Observe that by the choice of $\beta_{R,\eta}$ the series

$$R_{\eta,\tau,t} = \sum_{r\geqslant 1} t^r R_{\eta,\tau,r} \;\text{ and }\; R_{\eta,t} = \sum_{r\geqslant 1} t^r R_{\eta,r}$$

are uniformly convergent on the disk $D_R = \{t \in \mathbb{C} : |t| \leqslant R\}$ (see (1.105) and (1.108)). In fact one can prove that the formal series $\overset{\triangledown}{\exp}\big(-\omega \overset{\triangledown}{\psi}_{\tau,t}\big)$ (and $\overset{\triangledown}{\exp}\big(-\omega \overset{\triangledown}{\psi}_{*,t}\big)$) converges for all $t \in \mathbb{C}$ to an element of RES so that the identities (1.109) and (1.110) become identities between elements of RES. However, proof of convergence of $\overset{\triangledown}{\exp}\big(-\omega \overset{\triangledown}{\psi}_{\tau,t}\big)$ is very technical and we do not give it here.

Let $J = I^- \cup I^+ = (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2)$. Let I be either I^- or I^+.

Proposition 82. *Let $\omega \in 2\pi i\mathbb{Z}$ and $\eta \in \mathscr{R}_1$ such that $\pi_1(\eta) = 1+\omega$. For all $z \in \Sigma(J, \beta_{R,\eta})$ one has*

$$\left(\overset{\triangledown}{\mathcal{L}}{}^{I} A_\eta \overset{\triangledown}{\psi}_{*,t}\right)(z) = R_{\eta,t} \exp\left(-\omega\left(\overset{\triangledown}{\mathcal{L}}{}^{I} \overset{\triangledown}{\psi}_{*,t}\right)(z)\right). \tag{1.111}$$

Proof. Let $z \in \Sigma(J, \beta_{R,\eta})$. By Theorem 81 we obtain the following identity in $\mathbb{C}[[\mathfrak{t}]]$:

$$\left(\overset{\triangledown}{\mathcal{L}}{}^{I} A_\eta \overset{\triangledown}{\psi}_{*,\mathfrak{t}}\right)(z) = R_{\eta,\mathfrak{t}}\left(\overset{\triangledown}{\mathcal{L}}{}^{I} \overset{\triangledown}{\exp}\left(-\omega \overset{\triangledown}{\psi}_{*,\mathfrak{t}}\right)\right)(z) = R_{\eta,\mathfrak{t}} \exp\left(-\omega\left(\overset{\triangledown}{\mathcal{L}}{}^{I} \overset{\triangledown}{\psi}_{*,\mathfrak{t}}\right)(z)\right). \tag{1.112}$$

Moreover, using Corollary 66 we obtain that the formal series

$$\left(\overset{\triangledown}{\mathcal{L}}{}^{I} A_\eta \overset{\triangledown}{\psi}_{*,\mathfrak{t}}\right)(z) \;\text{ and }\; \exp\left(-\omega\left(\overset{\triangledown}{\mathcal{L}}{}^{I} \overset{\triangledown}{\psi}_{*,\mathfrak{t}}\right)(z)\right) \in \mathbb{C}[[\mathfrak{t}]]$$

converges absolutely to analytic functions on D_R when plugging $\mathfrak{t} = t \in D_R$. This proves Proposition 82. $\square$

1.4.9 Relation with the Horn maps.

To obtain analytic conjugacy invariants consider the difference $v_*^+(z) - v_*^-(z)$, where $z \in \Sigma^+ \cap \Sigma^-$ (see (1.34)). To fix our ideas, let $\operatorname{Im} z > 0$. Observe that the sets $\Sigma^\pm$ can be written in the form

$$\Sigma^+ = \Sigma\big(I^+, \beta\big), \qquad \Sigma^- = \Sigma\big(I^-, \beta\big). \tag{1.113}$$

where β is a continuous function on $I^+ \cup I^-$. Let $\theta_1 \in (-\pi/2, 0), \theta_2 \in (\pi, 3\pi/2)$ such that $\operatorname{Re}(z\,\mathrm{e}^{\mathrm{i}\theta_1}) > \beta(\theta_1)$ and $\operatorname{Re}(z\,\mathrm{e}^{\mathrm{i}\theta_2}) > \beta(\theta_2)$. Then the values $\hat{\mathcal{L}}^{\theta_1}\hat{\varphi}_*(z)$ and $\hat{\mathcal{L}}^{\theta_2}\hat{\varphi}_*(z)$ are well defined. Therefore,

$$v_*^+(z) - v_*^-(z) = \varphi_*^+(z) - \varphi_*^-(z) = \int_\Gamma e^{-\zeta z}\hat{\varphi}_*(\zeta)\mathrm{d}\zeta,$$

where $\Gamma = (\infty e^{i\theta_2}, 0] \cup [0, \infty e^{i\theta_1})$. Let $1 > a > 0$. Denote by $\gamma \subset \mathbb{C}$ the path

$$(e^{\imath\theta_2}\infty, e^{\imath\theta_2 a}] \cup \{e^{it}a : t \in (\theta_2, \theta_2 - 2\pi)\} \cup [e^{\imath\theta_2 a}, e^{\imath\theta_2}\infty).$$

For $k \in \mathbb{N}$ denote by γ_k (respectively Γ_k) the unique path in $\mathscr{R}_0$ which projects to the path $\gamma - 2\pi ik \subset \mathbb{C}$ (respectively $\Gamma - (2\pi k + \pi)i$) and has a non-empty intersection with the quadrant $\{re^{i\theta} : r > 0, \theta \in (-\pi/2, 0)\} \subset \mathscr{R}_0$. From Corollary 52 using Cauchy Theorem one can deduce that

$$\int_\Gamma e^{-\zeta z}\hat{\varphi}_*(\zeta)\mathrm{d}\zeta = \left(\int_{\gamma_1} + \ldots + \int_{\gamma_n} + \int_{\Gamma_n}\right) e^{-\zeta z}\hat{\varphi}_*(\zeta)\mathrm{d}\zeta.$$

It is not hard to see that

$$\int_{\Gamma_n} e^{-\zeta z}\hat{\varphi}_*(\zeta)\mathrm{d}\zeta = o(e^{2\pi inz}),$$

when $\operatorname{Re} z$ is bounded and $\operatorname{Im} z \to \infty$. Let η be a point in the main sheet of $\mathscr{R}_0$ such that $\pi_1(\eta) = 1 - 2\pi k\mathrm{i}$. Set

$$R_k = R_{\eta,1}.$$

As a consequence of the Bridge equation we obtain the following result.

Proposition 83. *For each $k \in \mathbb{N}$ one has:*

$$\int_{\gamma_k} e^{-\zeta z}\hat{\varphi}_*(\zeta)\mathrm{d}\zeta = -R_k e^{2\pi ikv^-(z)} \tag{1.114}$$

for all $z \in \Sigma^+ \cup \Sigma^-$.

Proof. By the definition of the operator S_η, one has

$$\int_{\gamma_k} e^{-\zeta z}\hat{\varphi}_*(\zeta)\mathrm{d}\zeta = e^{2\pi ikz}\int_{\gamma_0} e^{-\zeta z}\left(S_\eta\hat{\varphi}_*\right)(\zeta)\mathrm{d}\zeta,$$

where $\eta \in \mathscr{R}_1$ is the point in the main sheet such that $\pi_1(\eta) = 1 - 2\pi ik$. Now, definition 76 implies that:

$$\operatorname{sing}_0 S_\eta\hat{\varphi}_* = A_\eta\overset{\triangledown}{\varphi}_*.$$

Recall that

$$\overset{\triangledown}{\psi}_* - \overset{\triangledown}{\varphi}_* = \rho\overset{\triangledown}{\mathscr{B}}(\log z) = \rho\operatorname{sing}_0\left(-\frac{\log\zeta}{2\pi i\zeta}\right), \quad \text{where } \overset{\triangledown}{\psi}_* = \overset{\triangledown}{\psi}_{*,1}.$$

Therefore,

$$A_\eta\overset{\triangledown}{\varphi}_* = A_\eta\overset{\triangledown}{\psi}_*.$$

Fix $R > 1$. Set $R_\eta = R_{\eta,1}$. Let $I = I^- = (\pi/2, 3\pi/2)$. By Proposition 82, for $z \in \Sigma(I, \beta_{R,\eta})$ we have:

$$\left(\overset{\triangledown}{\mathcal{L}}^I A_\eta\overset{\triangledown}{\psi}_*\right)(z) = R_\eta\exp\Big(-\omega\left(\overset{\triangledown}{\mathcal{L}}^I\overset{\triangledown}{\psi}_*\right)(z)\Big).$$

Therefore, using the definition of the Laplace transform, we obtain:

$$\int_{\gamma_0} e^{-\zeta z}\left(S_\eta\hat{\varphi}_*\right)(\zeta)\mathrm{d}\zeta = -\overset{\triangledown}{\mathcal{L}}^I A_\eta\overset{\triangledown}{\psi}_*(z) =$$
$$-R_\eta\exp\Big(-\omega\overset{\triangledown}{\mathcal{L}}^I\overset{\triangledown}{\psi}_*(z)\Big) = -R_\eta\exp(2\pi ki(v_*^-(z) - z)),$$

which finishes the proof. $\square$

As a corollary we obtain

$$v_*^+(z) = v_*^-(z) - \sum_{k=1}^n R_k e^{2\pi kiv_*^-(z)} + o(e^{2\pi kiz}),$$

where $z \in \Sigma^+ \cup \Sigma^-$ and $\operatorname{Im} z \to \infty$. Similar relation is true when $\operatorname{Im} z \to -\infty$:

$$v_*^+(z) = v_*^-(z) - 2\pi\mathrm{i}\rho - \sum_{k=1}^n R_{-k}e^{-2\pi kiv_*^-(z)} + o(e^{-2\pi kiz}).$$

As a consequence we obtain:

Corollary 84. *The coefficients* $\{B_k^\pm\}_{k=1}^\infty$ *given by*

$$B_k^+ = -R_k,\ \ B_k^- = -R_{-k},\ \ \textit{for}\ \ k \in \mathbb{N},\ \ B_0^+ = 0,\ \ B_0^- = -2\pi\mathrm{i}\rho,$$

form analytic conjugacy invariants (1.6).

Chapter 2

Computability of the Julia set. Nonrecurrent critical orbits

2.1 Introduction

This chapter is organized as follows. In Section 2.1 we give all necessary preliminaries in Computability and Complex Dynamics, state the results of Chapter 2 and discuss possible generalizations. To illustrate the results of Chapter 2 on a simple case in Section 2.2 we prove that for every subhyperbolic rational function the Julia set is computable in a polynomial time. In Section 2.3 we prove the main result under a simplifying assumption that the rational map f does not have any parabolic periodic points. In Section 2.4 we complete a proof of the main result.

2.1.1 Preliminaries on computability

In this section we give a brief introduction to computability and complexity of functions and sets. The notion of computability relies on the concept of a Turing Machine (TM). A precise definition of a Turing Machine is quite technical and we do not give it here. For the definition and properties of a Turing Machine we refer the reader to [27] and [37]. The computational power of a Turing Machine is equivalent to that of a RAM computer with infinite memory. One can generally think about a Turing Machine as a formalized definition of an algorithm or a computer program.

There exist several different definitions of computability of sets. For discussion of different approaches to computability we refer the reader to [10]. In this thesis we use the notion of computability related to complexity of drawing pictures on a computer screen. Roughly speaking, a subset S is called computable in time $t(n)$ if there is a computer program which takes time $t(n)$ to decide whether to draw a given $2^{-n} \times 2^{-n}$ square pixel in a picture of S on a computer screen, which is accurate up to one pixel size. Before giving a rigorous definition of a computable Julia set we need to introduce some notations.

First we give the classical definitions of a computable function and a computable number.

Definition 85. Let S, N be countable subsets of $\mathbb{N}$. A function $f : S \to N$ is called computable if there exists a TM which takes x as an input and outputs $f(x)$.

Note that Definition 85 can be naturally extended to functions on arbitrary countable sets, using a convenient identification with $\mathbb{N}$.

Definition 86. A real number α is called computable if there is a computable function $\phi : \mathbb{N} \to \mathbb{Q}$, such that for all n

$$|\alpha - \phi(n)| < 2^{-n}.$$

The set of computable reals is denoted by $\mathbb{R}_{\mathcal{C}}$.

In other words, α is called computable if there is an algorithm which can approximate α with any given precision. The set $\mathbb{R}_{\mathcal{C}}$ is countable, since there are only countably many algorithms. The set of computable complex numbers is defined by $\mathbb{C}_{\mathcal{C}} = \mathbb{R}_{\mathcal{C}} + i\mathbb{R}_{\mathcal{C}}$. Note that both $\mathbb{R}_{\mathcal{C}}$ and $\mathbb{C}_{\mathcal{C}}$ considered with usual multiplication and addition form fields. Moreover, it is easy to see that $\mathbb{C}_{\mathcal{C}}$ is algebraically closed.

Let $d(\cdot, \cdot)$ stand for the Euclidian distance between points or sets in $\mathbb{R}^2$. Recall the definition of the Hausdorff distance between two sets:

$$d_H(S, T) = \inf\{r > 0 : S \subset U_r(T),\ T \subset U_r(S)\},$$

where $U_r(T)$ stands for the r-neighborhood of T:

$$U_r(T) = \{z \in \mathbb{R}^2 : d(z, T) < r\}.$$

We call a set T a 2^{-n} approximation of a bounded set S if $S \subset T$ and $d_H(S, T) \leqslant 2^{-n}$. When we try to draw a 2^{-n} approximation T of a set S using a computer program, it

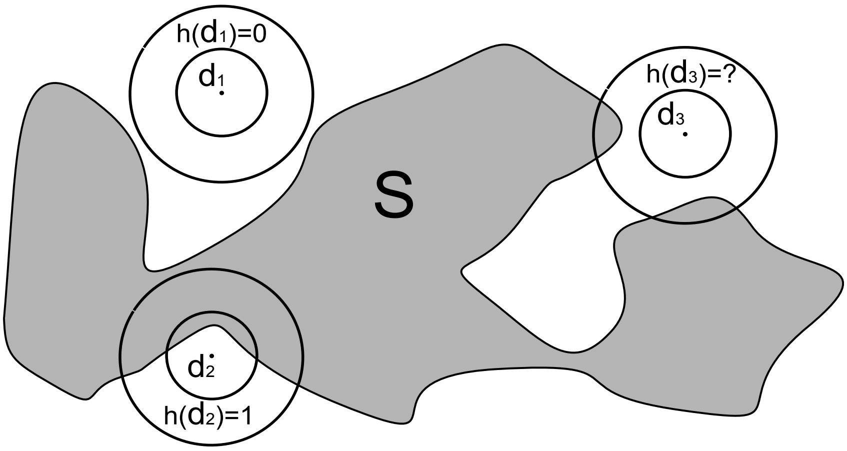

Figure 2.1: Values of the function h_S.

is convenient to let T be a finite collection of disks of radius 2^{-n-2} centered at points of the form $(i/2^{n+2}, j/2^{n+2})$ for $i, j \in \mathbb{Z}$. Such T can be described using a function

$$h_S(n, z) = \begin{cases} 1, & \text{if } d(z, S) \leqslant 2^{-n-2}, \\ 0, & \text{if } d(z, S) \geqslant 2 \cdot 2^{-n-2}, \\ 0 \text{ or } 1 & \text{otherwise}, \end{cases} \tag{2.1}$$

where $n \in \mathbb{N}$ and $z = (i/2^{n+2}, j/2^{n+2}),\ i, j \in \mathbb{Z}$.

Using this function, we define computability and computational complexity of a set in $\mathbb{R}^2$ in the following way.

Definition 87. A bounded set $S \subset \mathbb{R}^2$ is called computable in time $t(n)$ if there is a TM, which computes values of a function $h(n, \bullet)$ of the form (2.1) in time $t(n)$. We say that S is poly-time computable if there exists a polynomial $p(n)$ such that S is computable in time $p(n)$.

Similarly, one can define computability and computational complexity of subsets of $\mathbb{R}^k$. Moreover, definition 87 naturally extends to subsets of $\hat{\mathbb{C}}$ (see [10] Section 2.1). The Riemann sphere $\hat{\mathbb{C}}$ is homeomorphic to the unit sphere

$$S^2 = \{x : \|x\| = 1\} \subset \mathbb{R}^3.$$

Consider the stereographic projection

$$P : S^2 \setminus \{\text{North Pole}\} \to \mathbb{C}.$$

The inverse of this projection is given by

$$P^{-1} : z \to \left(\frac{2\mathrm{Re}(z)}{|z|^2+1}, \frac{2\mathrm{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right);\ P^{-1}(\infty) = (0,0,1).$$

Observe that P^{-1} induces the spherical metric on $\hat{\mathbb{C}}$, given by the formula

$$ds = \frac{dz}{1+|z|^2}.$$

Definition 88. A subset $K \subset \hat{\mathbb{C}}$ is called computable in time $t(n)$ if $P^{-1}(K) \subset \mathbb{R}^3$ is computable in time $t(n)$.

Proposition 89. *Let $K \subset \mathbb{C}$ be a bounded subset. Then K is computable as a subset of $\hat{\mathbb{C}}$ if and only if it is computable as a subset of $\mathbb{R}^2$. Similarly, K is poly-time computable as a subset of $\hat{\mathbb{C}}$ if and only if it is poly-time computable as a subset of $\mathbb{R}^2$.*

In this chapter we discuss computability of the Julia sets of rational functions. For simplicity, consider the case of quadratic polynomials $f_c(z) = z^2 + c$. By computing the Julia set J_c of the map f_c, $c \in \mathbb{C}$, we mean the following problem:

given the parameter c compute a function h of the form (2.1) for $S = J_c$.

However, an algorithm M, computing h, can handle only a finite amount of information. In particular, it can not read or store the entire input c if $c \notin \mathbb{C}_{\mathcal{C}}$. Instead, it may request this input with an arbitrary high precision. In other words, the machine M has a command $READ(m)$ which for any integer m requests the real and the imaginary part of a number $\phi(m)$ such that $|\phi(m) - c| < 2^{-m}$. It can be formalized using the notion of an *oracle*. Let

$$\mathbb{D} = \left\{\frac{k}{2^l} : k \in \mathbb{Z}, l \in \mathbb{N}\right\}$$

be the set of all dyadic numbers. Denote

$$\mathbb{D}^n = \{(x_1, x_2, \dots, x_n) : x_j \in \mathbb{D},\ \ 1 \leqslant\ j \leqslant n\}$$

the n-th Cartesian power of $\mathbb{D}$. We will say that the open disk $U_d(c)$ (respectively closed disk $\overline{U_d(c)}$) is dyadic, if d and c are dyadic. Denote by $\mathcal{C}$ the set of all subsets $U \subset \mathbb{C}$ such that U can be represented as a finite union of dyadic disks.

Definition 90. A function $\phi : \mathbb{N} \to \mathbb{D}^n$ is called an oracle for an element $x \in \mathbb{R}^n$ if $\|\phi(m) - x\| < 2^{-m}$ for all $m \in \mathbb{N}$, where $\|\cdot\|$ stands for the Euclidian norm in $\mathbb{R}^n$. An *oracle Turing Machine* M^ϕ is a TM, which can query $\phi(m)$ for any $m \in \mathbb{N}$.

The oracle ϕ is not a part of the algorithm, but rather enters as a parameter. In case of a computer program the role of the oracle is usually played by the user, who enters the parameters of the program. We should note also that an algorithm using an oracle ϕ require m time units to read $\phi(m)$.

Now we are ready to define computability and computational complexity of the Julia set of a rational map.

Definition 91. Let f be a rational map. The Julia set J_f is called computable in time $t(n)$ if there is a Turing Machine with an oracle for the coefficients of f, which computes values of a function $h(n, \bullet)$ of the form (2.1) for $S = J_f$ in time $t(n)$. We say that J_f is poly-time computable if there exists a polynomial $p(n)$ such that J_f is computable in time $p(n)$.

2.1.2 Hyperbolic maps

A rational map f is called hyperbolic if there is a Riemannian metric μ on a neighborhood of the Julia set J_f in which f is strictly expanding:

$$\|Df_z(v)\|_\mu > \|v\|_\mu$$

for any $z \in J_f$ and any tangent vector v (see [25]). It follows that for a hyperbolic map f there is a neighborhood U of J_f on which the metric d_μ induced by μ is strictly expanding:

$$d_\mu(f(x), f(y)) \geqslant k d_\mu(x, y), \quad \text{for any } \ x, y \in U, \quad \text{where } \ k > 1.$$

Let d_S be the spherical metric on the Riemann sphere. Then d_μ and d_S are equivalent on a compact neighborhood of J_f. One can deduce that there exists $N \in \mathbb{N}$, $K > 1$ and a neighborhood V of J_f such that

$$d_S\left(f^N(x), f^N(y)\right) \geqslant K d_S(x, y), \quad \text{for any } \ x, y \in V.$$

Moreover, the last property is equivalent to the definition of hyperbolicity. Hyperbolic maps have the following topological characterization (see [25]).

Proposition 92. *A rational map f is hyperbolic if and only if every critical orbit of f either converges to an attracting (or a super-attracting) cycle, or is periodic.*

Braverman in [7] and Rettinger in [29] have independently proven the following result.

Theorem 93. *For any $d \geqslant 2$ there exists a Turing Machine with an oracle for the coefficients of a rational map of degree d which computes the Julia set of every hyperbolic rational map in polynomial time.*

To explain why this is an important result we would like to mention the following. It is known that hyperbolicity is an open condition in the space of coefficients of rational maps of degree $d \geqslant 2$. The famous conjecture of Fatou states that the set of hyperbolic parameters is dense in this space. This conjecture is known as the Density of Hyperbolicity Conjecture. It is the central open question in Complex Dynamics. Lyubich [21] and Graczyk-Swiatek [18] independently showed that this conjecture is true for the real quadratic family. Namely, the set of real parameters c, for which the map $f(z) = z^2 + c$ is hyperbolic, is dense in $\mathbb{R}$.

2.1.3 Subhyperbolic maps

A rational map f is called subhyperbolic if it is expanding on a neighborhood of the Julia set J_f in some orbifold metric (see [25]). An orbifold metric is a conformal metric $\gamma(z)dz$ with a finite number of singularities of the following form. For each singularity a there exists an integer index $\nu = \nu_a > 1$ such that for the branched covering $z(w) = a + w^\nu$ the induced metric

$$\gamma(z(w)) \left| \frac{dz}{dw} \right| dw$$

in w-plane is smooth and nonsingular in a neighborhood of the origin. Douady and Hubbard proved the following (see [25])

Proposition 94. *A rational map is subhyperbolic, if and only if every critical orbit of f is either finite or converges to an attracting (or a super-attracting) cycle.*

For a subhyperbolic map, the corresponding orbifold has singularities only at post-critical points, which lie in J_f. In presence of these singularities the algorithm, which works in the case of hyperbolic maps, can not be applied directly for subhyperbolic maps. We show how to change this algorithm to compute the Julia set of a subhyperbolic map in a polynomial time. One of the results of Chapter 2 is the following theorem.

Theorem 95. *There is a TM M^ϕ with an oracle for the coefficients of a rational map f such that for every subhyperbolic map f the machine M^ϕ computes J_f in polynomial time, given some finite non-uniform information about the orbits of critical points of f.*

We will specify later, which non-uniform information does M^ϕ use.

2.1.4 The main results: nonrecurrent critical orbits

Let f be a rational map. For a point $z \in \mathbb{C}$ denote by

$$\mathcal{O}(z) = \{f^n(z) : n \geqslant 1\}$$

the forward orbit of z. Let $\omega(z) = \overline{\mathcal{O}(z)} \setminus \mathcal{O}(z)$ be the ω-limit set of z. Denote C_f the set of critical points of f which lie in J_f. Put

$$\Omega_f = \bigcup_{c \in C_f} \omega(c).$$

Our main result is devoted to the class of rational maps without recurrent critical points. Namely, we prove here the following:

Theorem 96. *Let f be a rational map. Assume that f has no parabolic periodic points and Ω_f does not contain any critical points. Then J_f is poly-time computable by a TM with an oracle for the coefficients of f.*

Although in this case there is no guarantee that any kind of expansion holds on the whole Julia set of f, the following well-known theorem of Mañé (see [23],[36]) implies that there is an expansion on the closure of the postcritical set of f.

Theorem 97. *Let f be a rational map. Let $M \subset J_f$ be a compact invariant set such that M does not contain any critical points of f or parabolic periodic points and $M \cap \omega(c) = \varnothing$ for any recurrent critical point of f. Then there exists $N \in \mathbb{N}$, such that $\left|Df^N(z)\right| > 1$ for any $z \in M$.*

As we see, the theorem of Mañé gives an expansion only near the points, whose forward orbits are isolated from the parabolic periodic points. The question arises: is it possible to generalize Theorem 96 for the maps f with parabolic periodic points? It is known that the algorithms, used for computing hyperbolic Julia sets, require exponential time in the presence of parabolic points (see [25], app. H). However, in the paper [8] (see also [10]) Braverman proved that for any rational function f such that every critical orbit of f converges either to an attracting or to a parabolic orbit, the Julia set is poly-time computable. Combining the algorithm used in [8] with the algorithm, which we use in here to prove Theorem 96, we obtain the following generalization of Theorem 96:

Theorem 98. *Let f be a rational map. Assume that Ω_f does not contain neither critical points nor parabolic periodic points. Then J_f is poly-time computable by a TM with an oracle for the coefficients of f.*

2.1.5 Possible generalizations

In this subsection we discuss possible generalizations of our results. First we would like to mention that the algorithm which we use to prove Theorem 98 cannot be applied to compute the Julia set of a rational map f such that Ω_f contains a parabolic periodic point. However, we believe that the following statement is true.

Conjecture 99. Let f be a rational map. Assume that Ω_f does not contain any critical points. Then J_f is poly-time computable by a TM with an oracle for the coefficients of f.

Another important class of rational maps is the class of Collet-Eckmann maps.

Definition 100. Let f be a rational map. Assume that there exist constants $C, \gamma > 0$ such that the following holds: for any critical point c of f whose forward orbit does not contain any critical points one has:

$$|Df^n(f(c))| \geqslant Ce^{\gamma n} \text{ for any } n \in \mathbb{N}. \tag{2.2}$$

Then we say that the map f is Collet-Eckmann (CE). The condition (2.2) is called the Collet-Eckmann condition.

In [2] Avila and Moreira showed that for almost every real parameter c the map $f_c(z) = z^2 + c$ is either Collet-Eckmann or hyperbolic. In [1] Aspenberg proved that the set of Collet-Eckmann maps has positive Lebesgue measure in the parameter space of all rational maps of fixed degree $d \geqslant 2$. Moreover, there is a conjecture that for almost all rational maps f in this space the following is true:

1) the forward orbit of every critical point $c \notin J_f$ either is finite or converges to an attracting periodic orbit;

2) for any critical point $c \in J_f$ either there exists $\gamma, C > 0$ such that $|Df^n(f(c))| \geqslant Ce^{\gamma n}$ for any $n \in \mathbb{N}$ or the forward orbit of c contains another critical point.

By the theorem of Mañé 97, every rational map f such that Ω_f does not contain neither critical points nor parabolic periodic points is Collet-Eckmann. We believe that the following generalization of Theorem 98 is true.

Conjecture 101. Let f be a rational map such that the conditions 1) and 2) above are satisfied. Then J_f is poly-time computable by a TM with an oracle for the coefficients of f.

Thus, we conjecture that

Conjecture 102. For almost all rational maps f of degree $d \geqslant 2$ J_f is poly-time computable.

2.2 Poly-time computability for subhyperbolic maps

In this section we prove Theorem 95. Namely, we construct an algorithm A which for every subhyperbolic rational map f computes J_f in polynomial time. The algorithm A uses the coefficients of the map f and some non-uniform information which we will specify in the following subsection.

2.2.1 Preparatory steps and non-uniform information

In this work we actively use the classical Koebe distortion theorem (see [13]). Let us state it here. For $\delta > 0, z \in \mathbb{C}$ set

$$U_\delta(z) = \{w \in \mathbb{C} : |w - z| < \delta\}.$$

Theorem 103. *Let $f : U_r(a) \to \mathbb{C}$ be a univalent function. Then for any $z \in U_r(a)$ one has:*

$$\frac{(1 - |z-a|/r)|f'(a)|}{(1 + |z-a|/r)^3} \leqslant |f'(z)| \leqslant \frac{(1 + |z-a|/r)|f'(a)|}{(1 - |z-a|/r)^3} \tag{2.3}$$

$$\frac{|z-a||f'(a)|}{(1 + |z-a|/r)^2} \leqslant |f(z) - f(a)| \leqslant \frac{|z-a||f'(a)|}{(1 - |z-a|/r)^2} \tag{2.4}$$

The statement (2.4) of the Koebe distortion theorem can be reformulated the following way. Let $1 > \alpha > 0$, $r_1 = \frac{\alpha|f'(a)|r}{(1+\alpha)^2}$, $r_2 = \frac{\alpha|f'(a)|r}{(1-\alpha)^2}$. Then

$$U_{r_1}(f(a)) \subset f(U_{\alpha r}(a)) \subset U_{r_2}(f(a)). \tag{2.5}$$

We will also use Koebe One-Quarter Theorem, which can be derived from Koebe Distortion Theorem.

Theorem 104. *Suppose $f : U_r(z) \to \mathbb{C}$ is a univalent function. Then the image $f(U_r(z))$ contains the disk of radius $\frac{1}{4}r|f'(z)|$ centered at $f(z)$.*

Fatou and Julia proved the following fundamental result.

Theorem 105. *Let f be a rational map of degree $d \geqslant 2$. Then the immediate basin of each attracting cycle contains at least one critical point. In particular, the number of attracting periodic orbits is finite and do not exceed the number of critical points.*

In our proof of Theorem 95 we will use the following general fact (see e.g. [40]).

Proposition 106. *Let $h(z)$ be a complex polynomial. There exists a TM M^ϕ with an oracle for the coefficients of $h(z)$ and a natural number n as an input such that M^ϕ outputs a finite sequence of complex dyadic numbers $\beta_1, \beta_2, \ldots, \beta_k$ for which:*
1) *each β_i lies at a distance not more than 2^{-n} from some root of $h(z)$;*
2) *each root of $h(z)$ lies at a distance not more than 2^{-n} from one of β_i.*

Denote $N_F(f)$ and $N_J(f)$ the number of critical points of f which lie in the Fatou set and the Julia set of f correspondingly. The algorithm computing the Julia set of a subhyperbolic map will use the numbers

$$N_F(f),\ N_J(f) \tag{2.6}$$

as the non-uniform information.

Observe that the assertions $N_F(f) = 0$ and $J_f = \hat{\mathbb{C}}$ are equivalent. Indeed, by Proposition 94 if $N_F(f) > 0$ then there is at least one attracting (or superattracting) periodic point and the Fatou set is nonempty. On the other hand if the Fatou set is nonempty, then there is at least one attracting (or superattracting) periodic point. The basin of this periodic point contains a critical point. Thus if $N_F(f) = 0$, then the problem of computing J_f is trivial. Therefore, we will assume that $N_F(f) \neq 0$ and $J_f \neq \hat{\mathbb{C}}$.

Also, without loss of generality we may assume that $\infty \notin J_f$. Indeed if $\infty \in J_f$, then using Proposition 106 we can find a dyadic point z_0 which lie in the Fatou set of f. Let $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a Möbius map such that $h(\infty) = z_0$. Consider the map $h^{-1} \circ f \circ h$ instead of f.

As a corollary of Proposition 106 one can obtain the following result (see [10], Proposition 3.3).

Proposition 107. *Let f be a subhyperbolic rational map. There exists a Turing Machine M^ϕ with an oracle for the coefficients of f such that the following is true. Given the number $N_F(f)$ of critical points $c \notin J_f$ M^ϕ outputs a dyadic set B such that*
1) *all the attracting and super attracting orbits of f belong to B,*
2) *for any $z \in B$ the orbit of z converges to an attracting periodic orbit,*
3) *$f(B) \Subset B$.*

Proof. The algorithm works as follows. Initially, let $\mathcal{B}$ be an empty collection of dyadic disks. At m-th step, $m \in \mathbb{N}$, do the following. By Proposition 106 there is an algorithm which finds all periodic points of f of period at most m with precision 2^{-m-3}. Let p_i be an approximate position of a periodic point of period k_i. The corresponding periodic point z_i belongs to the disk $U_{2^{-m-3}}(p_i)$. Consider the disk $U_{2^{-m/2}}(p_i)$. If $U_{2^{-m/2}}(p_i)$ does not intersect neither one of the disks from $\mathcal{B}$, then approximate the image $f^{k_i}(U_{2^{-m/2}}(p_i))$ by a dyadic set with precision 2^{-m-1}. Namely, find a set $W_i \in \mathcal{C}$ such that

$$d_H\left(W_i, f^{k_i}(U_{2^{-m/2}}(p_i))\right) < 2^{-m-1}.$$

Verify if

$$U_{2^{-m}}(W_i) \subset U_{2^{-m/2}}(p_i). \tag{2.7}$$

This would imply that

$$f^{k_i}(U_{2^{-m/2}}(p_i)) \subset U_{2^{-m-1}}(W_i) \subset U_{2^{-m/2}-2^{-m-1}}(p_i). \tag{2.8}$$

In this case, compute dyadic sets B_j such that $B_0 = U_{2^{-m/2}}(p_i)$ and for each $j = 0, 1, \ldots, k_i - 1$, $f(B_j) \Subset B_{j+1}$, where $j+1$ is taken modulo k_i. Add dyadic sets B_j to the collection $\mathcal{B}$.

Next, calculate approximations s_i of the images $f^m(c_i)$ of critical points of f such that $|s_i - f^m(c_i)| < 2^{-m-1}$. If there are $N_F(f)$ of the points s_i, such that

$$U_{2^{-m-1}}(s_i) \subset B = \bigcup_{S \in \mathcal{B}} S,$$

then we stop the algorithm and output B. Otherwise, go to step $m+1$.

Let us show that the algorithm eventually stops and outputs a set B, satisfying the conditions 1)−3) of Proposition 107. Let z be an attracting (or super-attracting) periodic point of period k with multiplier λ. Let $|\lambda| < r < 1$. Then for small enough $\varepsilon > 0$ one has

$$f^k(U_\varepsilon(z)) \subset U_{r\varepsilon}(z).$$

It follows that for some $m > k$ the corresponding approximation p_i of z and the set W_i satisfy the property (2.7). On the other hand if (2.8) holds, then by Schwartz Lemma $U_{2^{-m/2}}(p_i)$ contains an attracting periodic point, whose basin contains $U_{2^{-m/2}}(p_i)$. Therefore if the algorithm runs sufficient amount of steps, the union B of dyadic sets from $\mathcal{B}$ satisfies the condition 1) of the Proposition 107.

By Proposition 94, the orbit of each critical point of f which does not lie in J_f converges to an attracting periodic orbit. Thus, for some m there will be $N_F(f)$ of the points s_i, belonging to B. This implies that the algorithm stops. By Theorem 105, we obtain that B contains all attracting periodic orbits of f. Notice that conditions 2) and 3) of the Proposition 107 for this set B are satisfied by construction. $\square$

Denote CF_f the set of critical points of f which lie in the Fatou set of f. Put

$$PF_f = \bigcup_{j \geqslant 0} f^j(CF_f).$$

The next statement is a subhyperbolic analog of Proposition 3.7 from [10].

Proposition 108. *There exists an algorithm which, given the coefficients of a subhyperbolic rational map f of degree $d \geqslant 2$ and the number $N_F(f)$, outputs a planar domain $U \in \mathcal{C}$ such that:*

(1) $U \Subset f(U)$,

(2) $f(U) \cap PF_f = \varnothing$,

(3) $J_f \Subset U \subset U_1(J_f)$.

Proof. First use the algorithm from Proposition (107) to find a dyadic set B satisfying to the conditions 1) – 3) of Proposition 107. Let $m \in \mathbb{N}$ be large enough so that

$$\mathbb{C} \setminus f^{1-m}(B) \subset U_1(J_f).$$

We can algorithmically construct a dyadic set W such that

$$f^{-m}(B) \Supset W \Supset f^{1-m}(B).$$

Compute a dyadic number $d > 0$ such that

$$U_d\,(W) \Subset f^{-1}(W).$$

Also, compute dyadic approximations s_i of critical points c_i of f such that

$$|s_i - c_i| \leqslant d.$$

By Theorem 94, for any critical point $c_i \in CF_f$ we will eventually have:

$$s_i \in U_d(c_i) \subset U_d\,(W) \Subset f^{-1}(W).$$

Therefore, for large enough m, the set W will contain $N_F(f)$ of the points s_i. Take such m. Then for any $s_i \in W$ one has

$$c_i \in U_d(s_i) \subset U_d(W) \subset f^{-1}(W).$$

Thus, $CF_f \subset f^{-1}(W)$. Compute a dyadic set $\widetilde{W}$ such that

$$f^{-3}(W) \Supset \widetilde{W} \Supset f^{-2}(W).$$

Clearly, for the set $U = \mathbb{C} \setminus \widetilde{W}$ conditions $(1) - (3)$ hold. □

2.2.2 Construction of the subhyperbolic metric

Here we give the construction of a subhyperbolic metric (see [25]). We modify the construction from [25] to be able to write it as an algorithm. First we recall the definition and basic properties of an orbifold. We refer the reader to [25] for details.

Definition 109. An orbifold (S, ν) is a Riemann surface together with a function $\nu : S \to \mathbb{N}$ such that the set $\{z \in S : \nu(z) \neq 1\}$ is discrete. Points z for which $\nu(z) \neq 1$ are called branch points.

Recall that an orbifold metric on a Riemann surface is a conformal metric $\gamma(z)dz$ with a finite number of singularities of the following form. For each singularity a there exists an integer index $\nu = \nu_a > 1$ such that for the branched covering $z(w) = a + w^\nu$ the induced metric

$$\gamma(z(w)) \left| \frac{dz}{dw} \right| dw$$

in w-plane is smooth and nonsingular in a neighborhood of the origin.

Definition 110. Let f be a subhyperbolic rational map. An orbifold metric μ on a neighborhood U of J_f is called subhyperbolic if f is strictly expanding on U with respect to μ:

$$\|Df(z)\|_\mu > \lambda > 1$$

for any $z \in f^{-1}(U)$ except the branch points.

Let $p : S' \to S$ be a regular branched covering. Then for every $z \in S$ the local degree of p at a point $w \in p^{-1}(z)$ does not depend on w. One can define the weight function $\nu : S \to \mathbb{N}$ of the covering p assigning $\nu(z)$ the local degree of p at $w \in f^{-1}(z)$.

Definition 111. Let (S,ν) be an orbifold. A regular branched covering

$$p : S' \to S$$

with the weight function ν such that S' is simply connected is called a universal covering of the orbifold (S,ν). We will use the notation $\widetilde{S}_\nu \to (S,\nu)$ for a universal covering of this orbifold.

Proposition 112. *Let (S,ν) be an orbifold. The universal covering*

$$\widetilde{S}_\nu \to (S,\nu)$$

exists and unique up to conformal isomorphism, except in the following two cases:
1) $S \approx \widehat{\mathbb{C}}$ *(the Riemann sphere) and S has only one branch point;*
2) $S \approx \widehat{\mathbb{C}}$ *and S has two branch points a_1, a_2 such that $\nu(a_1) \neq \nu(a_2)$.*

The Euler characteristic of an orbifold (S,ν) is the number

$$\chi(S,\nu) = \chi(S) - \sum_{z\in S}\left(1 - \frac{1}{\nu(z)}\right),$$

where $\chi(S)$ is the Euler characteristic of the Riemann surface S. Since the set of branch points of S is discrete the last sum contains at most countable number of nonzero terms. If S contains infinitely many branch points then we set $\chi(S,\nu) = -\infty$. The orbifold (S,ν) is called hyperbolic if $\chi(S,\nu) < 0$.

Lemma 113. *If (S,ν) is a hyperbolic orbifold then $\widetilde{S}_\nu$ conformally isomorphic to the unit disk.*

Let f be a subhyperbolic rational map. Let U be the set from Proposition 108. Construct an orbifold (U,ν) in the following way. Put $S = U$. Denote by CJ_f the set of the critical points of f which lie in J_f. As the set of branch points of U take $BP = \{f^j(c), c \in CJ_f, j \in \mathbb{N}\}$. Since f is subhyperbolic, BP is finite. Put $\nu(z) = 1$ for all $z \in U \setminus BP$. Denote by $n(f,z)$ the local degree of f at z. Define numbers $\nu(a), a \in BP$ such that the following condition holds:

$$\text{for any } \ z \in U \ \ \nu(f(z)) \ \text{ is a multiple of } \ \nu(z)n(f,z). \tag{2.9}$$

If the orbifold (U,ν) which we obtained is not hyperbolic, take any repelling orbit $\{z_j\}$

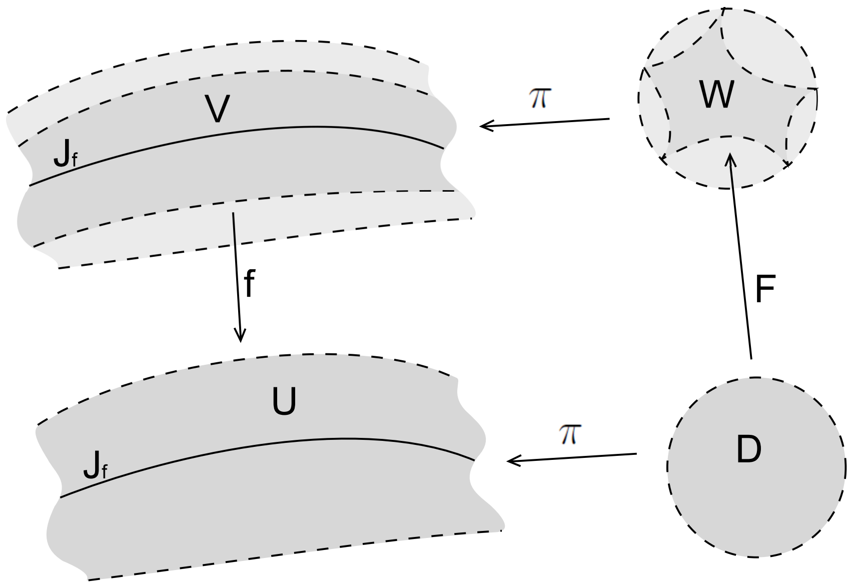

Figure 2.2: Construction of the subhyperbolic metric

of f of the length at least 5 and replace $\nu(z_j)$ by $2\nu(z_j)$ for all z_j from this orbit. The new orbifold will be hyperbolic and satisfying the condition (2.9).

By Proposition 112, there exists a universal covering

$$\widetilde{U}_\nu \xrightarrow{\pi} (U, \nu).$$

By Lemma 113, without loss of generality we may assume that $\widetilde{U}_\nu = \mathbb{U}$ is the open unit disc. Since $V = f^{-1}(U) \subset U$, Condition (2.9) guarantee that the map f^{-1} lifts to a holomorphic map

$$F : \mathbb{U} \to \mathbb{U}.$$

Note that $W = F(\mathbb{U}) = \pi^{-1}(V)$ is strictly contained in $\mathbb{U}$. Denote $\rho_\mathbb{U}$ the Poincaré metric on $\mathbb{U}$. By the Schwartz-Pick Theorem, the map F is strictly decreasing in the metric $\rho_\mathbb{U}$. Let μ be the projection of $\rho_\mathbb{U}$ onto U. For any $w \in \mathbb{U}$ and $z = \pi(F(w))$ one has:

$$\|Df(z)\|_\mu = \|DF(w)\|^{-1}. \tag{2.10}$$

It follows that the map f is strictly expanding with respect to the norm induced by μ. To show that the metric μ is subhyperbolic we need to prove that μ is uniformly strictly expanding. First we will prove an auxiliary lemma.

Lemma 114. *There exist a constant $C > 0$ such that for any $z \in U$ except the branch points of the orbifold (U, ν) one has:*

$$C^{-1} < \left|\frac{\mu(z)}{dz}\right| < C \max\left\{|z - a_j|^{\frac{1}{\nu(a_j)} - 1}\right\},$$

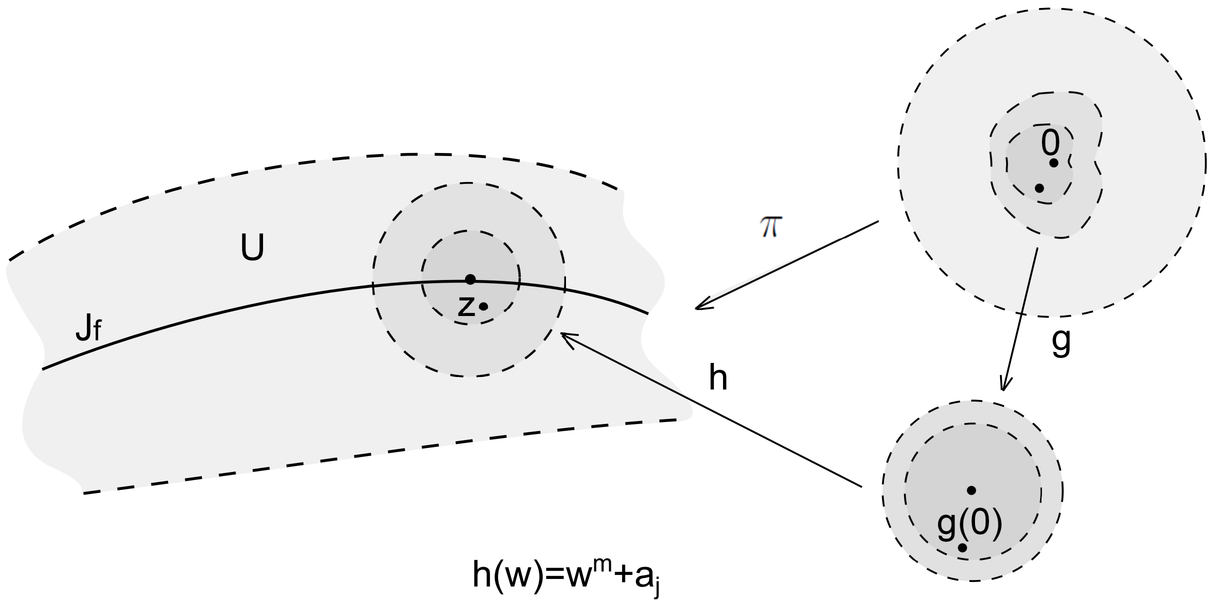

Figure 2.3: Illustration to Lemma 114

where a_j are the branch points. The constant C can be obtained constructively.

Proof. Let $z \in U$. Assume that z is not a branch point. To estimate $\mu(z)$ without loss of generality we may assume that $z = \pi(0)$. Recall that the Poincaré metric on the unit disk is of the form:

$$|\rho_{\mathbb{U}}(w)| = \frac{2|dw|}{1 - |w|^2}.$$

Therefore one has:

$$\left|\frac{\mu(z)}{dz}\right| = \frac{2}{|D\pi(0)|}.$$

We can construct a dyadic number R such that U is contained in a disk of radius R. By Schwartz Lemma,

$$|D\pi(0)| \leqslant R.$$

Thus,

$$\left|\frac{\mu(z)}{dz}\right| \geqslant 2R^{-1}.$$

Recall that the set BP of branch points consists of critical points which lie in J_f and possibly one repelling periodic orbit. It follows from Proposition 106 that we can construct a dyadic number $\varepsilon > 0$ such that disks $U_{2\varepsilon}(a_j)$ are pairwise disjoint and all belong to U. Assume first that $z \notin \bigcup U_\varepsilon(a_j)$. Considering the branch

$$\phi : U_\varepsilon(z) \to \mathbb{U},$$

of π^{-1} such that $\phi(z) = 0$, by Schwartz Lemma we obtain

$$|D\pi(0)| \geqslant \varepsilon.$$

Let $z \in U_\varepsilon(a_j)$ for some j. Then in a neighborhood $V(0)$ of 0 the map π can be written in the form

$$\pi(w) = g(w)^m + a_j,$$

where $g(w)$ is a one to one map from $V(0)$ onto $U_\delta(0)$ with $\delta = (2\varepsilon)^{1/m}$. In $V(0)$ one has

$$D\pi(w) = mg(w)^{m-1}Dg(w). \tag{2.11}$$

Consider the map $\chi = g^{-1} : U_\delta(0) \to V(0)$. Since

$$|g(0)| = |z - a_j|^{1/m} \leqslant 2^{-1/m}\delta,$$

by Schwartz Lemma we get:

$$|D\chi(g(0))| \leqslant K = K(\varepsilon),$$

where constant $K > 0$ can be obtained constructively. Now (2.11) implies that

$$|D\pi(0)| \geqslant mK^{-1}|z - a_j|^{1-1/m},$$

which finishes the proof. □

Proposition 115. *There exists a constant $\lambda > 1$ such that*

$$\|Df(z)\|_\mu > \lambda$$

for any $z \in V = f^{-1}(U)$. The constant λ can be constructed algorithmically.

Proof. For a map $g : U_1 \to U_2$ between two hyperbolic Riemann surfaces denote by

$$\|Dg(z)\|_{U_1,U_2}$$

the magnitude of the derivative of g computed with respect to the two Poincaré metrics. Denote dist_μ the distance in the metric μ on U and $\text{dist}_{\mathbb{U}}$ the distance in the Poincaré metric on $\mathbb{U}$. Let

$$i : W \to \mathbb{U}$$

be the inclusion map (see figure 2). Then we have:

$$\|DF(w)\|_{\mathbb{U},\mathbb{U}} = \|DF(w)\|_{\mathbb{U},W}\|Di(F(w))\|_{W,\mathbb{U}} \leqslant \|Di(F(w))\|_{W,\mathbb{U}}.$$

Using Lemma 114 it is not hard to show that we can construct a dyadic constant $R > 0$ such that

$$\operatorname{dist}_\mu(z, U \setminus V) < R \text{ for any } z \in V.$$

Let $z \in V$. Then there exists $w \in W$ and $\zeta \in \mathbb{U} \setminus W$ such that

$$\pi(w) = z \text{ and } \operatorname{dist}_{\mathbb{U}}(w, \zeta) < R.$$

A suitable fractional linear transformation sends ζ to 0 and w to $x > 0$. Explicit calculations show that

$$d = d_{\mathbb{U}}(x, 0) = \log \frac{1+x}{1-x}, \text{ so that } x = \frac{e^d - 1}{e^d + 1} \leqslant \frac{e^R - 1}{e^R + 1}. \tag{2.12}$$

Now, by Schwartz-Pick Theorem, one has:

$$\|Di(w)\|_{W,\mathbb{U}} \leqslant \|Di(x)\|_{\mathbb{U}\setminus\{0\},\mathbb{U}}. \tag{2.13}$$

The right hand side of the last inequality can be estimated explicitly. It is equal to

$$a(x) = \frac{2|x \log x|}{1 - x^2} < 1.$$

Note that $a(x)$ increases with x. By (2.12) and (2.13), the value $\|Di(w)\|_{W,\mathbb{U}}$ is bounded from above by $a(X) < 1$ for $X = (e^R - 1)/(e^R + 1)$. By (2.10) we obtain that $\|Df(z)\|_\mu$ for $z \in V$ is bounded from below by $1/a(X)$. □

Corollary 116. *The metric μ is subhyperbolic.*

Lemma 114 and Proposition 115 together make a subhyperbolic analog of Proposition 3.6 from [10].

2.2.3 The algorithm

Denote $V_k = f^{-k}(V)$. Notice that

$$J_f \Subset V_{k+1} \Subset V_k \Subset V$$

for any $k \in \mathbb{N}$. Let us prove the following auxiliary statement.

Proposition 117. *There is an algorithm computing two dyadic constants $K_1, K_2 > 0$ such that for any $z \in V_3 \setminus J_f$ and any $k \in \mathbb{N}$ if $f^k(z) \in V_1 \setminus V_3$ then one has*

$$\frac{K_1}{|Df^k(z)|} \leqslant d(z, J_f) \leqslant \frac{K_2}{|Df^k(z)|}.$$

Proof. First construct a dyadic number R such that

$$0 < R < \min\{d(\mathbb{C} \setminus V_3, J_f), d(V_1, \mathbb{C} \setminus V)\}.$$

Then for any z, k, satisfying the conditions of the proposition, the open disk $U_R(f^k(z))$ does not intersect neither J_f nor a forward orbit of a critical point of f. Let

$$\phi : U_R(f^k(z)) \to \mathbb{C}$$

be the branch of $(f^k)^{-1}$ such that $\phi(f^k(z)) = z$. Then, by Koebe Quarter Theorem, the image $\phi(U_R(f^k(z)))$ contains a disk or radius $\frac{1}{4}R|D\phi(f^k(z))|$. Since J_f is invariant under f, it follows that

$$d(z, J_f) \geqslant \frac{R}{4|Df^k(z))|}.$$

Set $K_1 = R/4$.

Further, consider

$$PJ_f = \{f^j(c) : c \in CJ_f, j \geqslant 0\}. \tag{2.14}$$

Notice that PJ_f is finite. Recall that we can algorithmically construct approximate positions of all critical points of f which lie in J_f with any given precision. Thus, we can approximate PJ_f. We can also algorithmically construct positions of some points in J_f which lie apart from PJ_f with any given precision. For instance, using the algorithm from Proposition 106 we can calculate an approximate position of a repelling periodic orbit. Using the above we can construct a finite number of pairs of simply connected dyadic sets $W_k \Subset U_k$ such that the following is true

1) $W_j \cap J_f \neq \varnothing$ for any j;

2) $\bigcup W_j \supset V_1 \setminus V_3$;

3) $\bigcup U_j \subset V \setminus PJ_f$.

For each j fix the Riemann mapping $\psi_j : U_j \to \mathrm{D} = \{z : |z| < 1\}$. Assume now that z, k satisfy the conditions of the proposition. Then $f^k(z) \subset W_j$ for some j. Let $\phi : U_j \to U(z)$ be the branch of $(f^k)^{-1}$ such that $\phi(f^k(z)) = z$. Consider the map

$$\phi \circ \psi_j^{-1} : \mathrm{D} \to U(z).$$

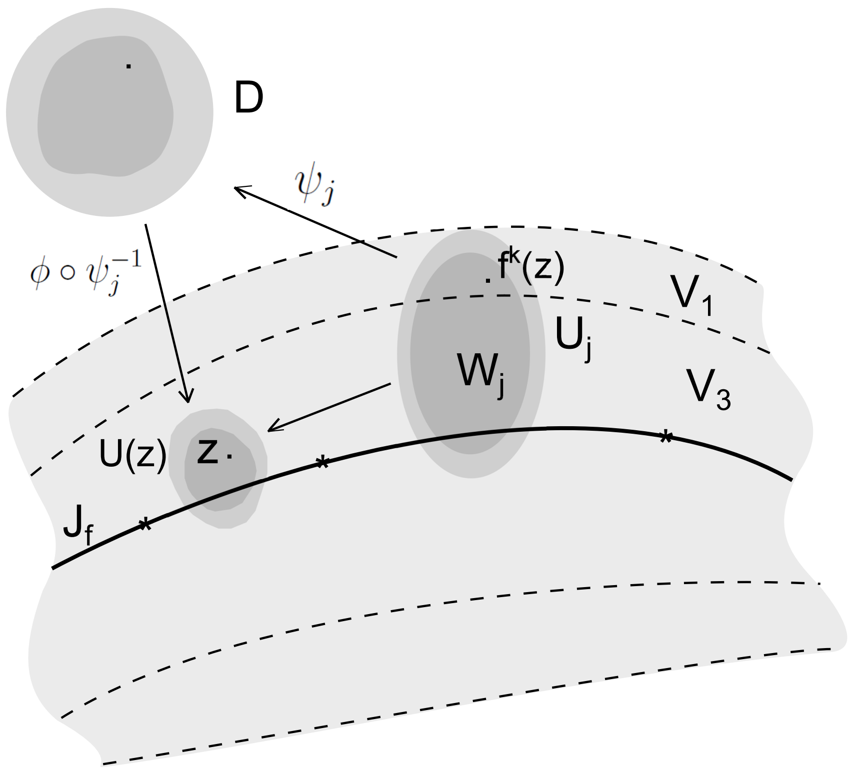

Figure 2.4: Illustration to Proposition 117.

Notice that $\psi_j(W_j) \Subset \mathrm{D}$. Applying both parts of Theorem 103 to the map $\phi \circ \psi_j^{-1}$, we can construct a dyadic number $r_j > 0$ not depending either on z or on k such that $\phi(W_j)$ is contained in the disk of radius $r_j |D\phi(f^k(z))|$ centered at z. It follows that

$$d(z, J_f) \leqslant \frac{r_j}{|Df^k(z)|}.$$

Set $K_2 = \max\{r_j\}$. $\square$

As a preparatory step construct a dyadic set W_2 such that

$$V_3 \Subset W_2 \Subset V_2.$$

Compute dyadic numbers $1 > s > 0, \varepsilon > 0$ such that

$$\varepsilon < \min\{d(W_2, \mathbb{C} \setminus V_2), d(V_3, \mathbb{C} \setminus W_2)\}, \quad V_1 \subset U_s(J_f). \tag{2.15}$$

Let $m = \max\{\nu(a_j)\}$. Lemma 114 implies that for any $z \in V$

$$C^{-1} d(z, J_f) \leqslant \mathrm{dist}_\mu(z, J_f) \leqslant mCd(z, J_f)^{1/m}. \tag{2.16}$$

Let log stand for the logarithm with base 2. We will use the standard notation $[x]$ for the integer part of a real number x.

Assume that we would like to verify that a dyadic point z is 2^{-n-1} close to J_f. Consider first points z which lie outside V_3. Construct a dyadic set W_3 such that

$$J_f \subset W_3 \Subset V_3.$$

Then we can approximate the distance from a point $z \notin W_3$ to J_f by the distance form z to W_3 up to a constant factor.

Now assume that $z \in V_3$. Consider the following subprogram:

$i := 1$

while $i \leqslant [\log(mC^2 s^{1/m})/\log\lambda + (n+1)/\log\lambda] + 1$ **do**

(1) Compute dyadic approximations

$$p_i \approx f^i(z) = f(f^{i-1}(z)) \text{ and } d_i \approx \left|Df^i(z)\right| = \left|Df^{i-1}(z) \cdot Df(f^{i-1}(z))\right|$$

with precision $\min\{2^{-n-1}, \varepsilon\}$.

(2) Check the inclusion $p_i \in W_2$:

- if $p_i \in W_2$, go to step (5);
- if $p_i \notin W_2$, proceed to step (3);

(3) Check the inequality $d_i \geqslant K_2 2^{n+1} + 1$. If true, output 0 and exit the subprogram, otherwise

(4) output 1 and exit subprogram.

(5) $i \to i + 1$

end while

(6) Output 0 end exit.

end

The subprogram runs for at most $L = [\log(mC^2 s^{1/m})/\log\lambda + (n+1)/\log\lambda] + 1 = O(n)$ number of while-cycles each of which consists of a constant number of arithmetic operations with precision $O(n)$ dyadic bits. Hence the running time of the subprogram can be bounded by $O(n^2 \log n \log\log n)$ using efficient multiplication.

Proposition 118. *Let $f(n,z)$ be the output of the subprogram. Then*

$$f(n,z) = \begin{cases} 1, & \text{if } d(z,J_f) > 2^{-n-1}, \\ 0, & \text{if } d(z,J_f) < K2^{-n-1}, \\ \text{either } 0 \text{ or } 1, & \text{otherwise}, \end{cases} \tag{2.17}$$

where $K = \frac{K_1}{K_2+1}$,

Proof. Suppose first that the subprogram runs the while-cycle L times and exits at the step (6). This means that $p_i \in W_2$ for $i = 1, \ldots, L$. In particular, $p_L \in W_2$. It follows that $f^L(z) \in V_1$. By (2.15) and (2.16) we obtain:

$$d(z,J_f) \leqslant C\mathrm{dist}_\mu(z,J_f) \leqslant C\lambda^{-L}\mathrm{dist}_\mu(f^L(z),J_f) \leqslant$$
$$\lambda^{-L}mC^2 d(f^L(z),J_f)^{1/m} \leqslant \lambda^{-L}mC^2 s^{1/m} \leqslant 2^{-n-1}.$$

Thus if $d(z,J_f) > 2^{-n-1}$, then the subprogram exits at a step other than (6).

Now assume that for some $i \leqslant L$ the subprogram falls into the step (3). Then

$$p_{i-1} \in W_2 \ \text{ and } \ p_i \notin W_2.$$

By (2.15), $f^i(z) \in V_1 \setminus V_3$. Now if $d_i \geqslant K_2 2^{n+1} + 1$, then $|Df^i(z)| \geqslant K_2 2^{n+1}$. By Proposition 117,

$$d(z,J_f) \leqslant 2^{-n-1}.$$

Otherwise, $|Df^i(z)| \leqslant K_2 2^{n+1} + 2 \leqslant (K_2+1)2^{n+1}$. In this case Proposition 117 implies that

$$d(z,J_f) \geqslant \frac{K_1}{K_2+1} 2^{-n-1}.$$

$\square$

Now, to distinguish the case when $d(z,J_f) < 2^{-n-1}$ from the case when $d(z,J_f) > 2^{-n}$ we can partition each pixel of size $2^{-n} \times 2^{-n}$ into pixels of size $(2^{-n}/K) \times (2^{-n}/K)$ and run the subprogram for the center of each subpixel. This would increase the running time at most by a constant factor.

2.3 Maps without recurrent critical orbits and parabolic periodic points

In this section we will prove Theorem 96. Throughout this section let f stand for a rational map without parabolic periodic points such that Ω_f does not intersect the set of critical points of f.

2.3.1 Preparatory steps and nonuniform information

As in the case of subhyperbolic map, without loss of generality we will assume that $\infty \notin J_f$. For any $c \in CJ_f$ put

$$N_0(c) = \max\{n : f^n(c) \in CJ_f\} + 1.$$

Put $N_0 = \max\limits_{c \in CJ_f} N_0(c)$. Denote

$$\widetilde{C} = \{f^n(c) : c \in CJ_f, 0 \leqslant n < N_0(c)\}, \ \ M = \overline{\{f^n(c) : c \in CJ_f, n \geqslant N_0(c)\}}.$$

By our assumptions, there are no either recurrent critical orbits or parabolic periodic points of f, the set M is invariant and does not contain critical points of f. Thus, the set M satisfies the conditions of Mañé's Theorem 97. The following result is classical (see [25]).

Theorem 119. *Let g be a rational map. Then the boundary of each cycle of Siegel disks and each cycle of Herman rings belongs to $\overline{PJ_g}$ (see (2.14)).*

Lemma 120. *There are no either Siegel disk cycles or Herman ring cycles in the Fatou set of f.*

Proof. Assume for simplicity that there is a Siegel disk Δ. By replacing f with an iterate if necessary, we can assume that $f(\Delta) = \Delta$. Then the boundary $\partial\Delta$ of the Siegel disk is forward invariant under f and belongs to $\overline{PJ_f}$. It follows that $\partial\Delta$ does not contain any critical point of f. Thus, $\partial\Delta$ satisfies the condition of Mañé's Theorem 97. Therefore, there exists N such that f^N is expanding on a neighborhood of $\partial\Delta$. This is impossible, since f is conjugated to a rotation inside Δ. The other case can be treated similarly. $\square$

To compute the Julia set the algorithm will use the following **non-uniform information**:

N1. $N_F(f)$, $N_J(f)$ and degrees $m_1, \ldots, m_{N_J(f)}$ of the critical points of f which lie in J_f;

N2. N_0, $N \in \mathbb{N}$, dyadic numbers $\delta, \delta' > 0, q > 1$ and a dyadic set $U \Supset M$ such that

$$U_{\delta/2}(M) \supset f^N(U), \ \ U_{\delta/2}(M) \supset U \supset U_{\delta'}(M), \ \ U_{3\delta/2}(U) \cap \widetilde{C} = \varnothing$$

and for any $z \in U_{3\delta/2}(U)$ one has

$$|Df^N(z)| > q.$$

In this section we will prove the following theorem.

Theorem 121. *Let f be a rational map such that f has no parabolic periodic points and Ω_f does not contain any critical points. There exists a Turing Machine which, given an oracle for the coefficients of the map f and the non-uniform information $(N1, N2)$, computes J_f in a polynomial time.*

Now we prove several auxiliary lemmas.

Lemma 122. *For any $z \in U$ and any $z_1, z_2 \in U_{3\delta/2}(z)$ one has*

$$\left|f^N(z_1) - f^N(z_2)\right| \geqslant q|z_1 - z_2|.$$

Notice that, in particular, the restriction of f^N on $U_{3\delta/2}(z)$ is one to one for any $z \in U$.

Proof. Using Lagrange formula

$$g(z_1) - g(z_2) = (z_1 - z_2)Dg(\lambda z_1 + (1-\lambda)z_2), \quad \lambda \in [0,1], \tag{2.18}$$

for $g(z) = f^N(z)$ we obtain

$$\left|f^N(z_1) - f^N(z_2)\right| \geqslant q|z_1 - z_2|.$$

□

For simplicity set $F = f^N$. From $N2$ one can deduce that

$$d(F(z), J_f) \geqslant qd(z, J_f) \text{ for each } z \in U_\delta(U). \tag{2.19}$$

Indeed, let $\zeta_0 \in J_f$ be such that

$$d(F(z), J_f) = |F(z) - \zeta_0|.$$

Let I be a preimage of the straight segment $[\zeta_0, F(z)]$ by F such that z is one of the ends of the curve I. Denote by z_0 the other end of I. If $I \subset U_{3\delta/2}(U)$, then $|DF(w)| > q$ for all $w \in I$, therefore (2.19) is true. Otherwise, let z' be the first intersection of I with $\partial U_{3\delta/2}(U)$ and I' be the part of I starting at z_0 and terminating at z'. One has:

$$|F(z) - \zeta_0| \geqslant |F(z) - F(z')| \geqslant ql(I') \geqslant 3\delta q/2 \geqslant qd(z, J_f),$$

where $l(I')$ is the length of the path I'.

Lemma 123. *For any $z \in U$ and $k \leqslant \min\{j \in \mathbb{N} : F^j(z) \notin U\}, k \in \mathbb{N}$, one has*

$$\begin{aligned} \tfrac{1}{4}\left|DF^k(z)\right| d(z, J_f) \leqslant d(F^k(z), J_f) \leqslant \tfrac{9}{4}\left|DF^k(z)\right| d(z, J_f), \\ d(F^k(z), M) \leqslant \tfrac{9}{4}\left|DF^k(z)\right| d(z, M). \end{aligned} \tag{2.20}$$

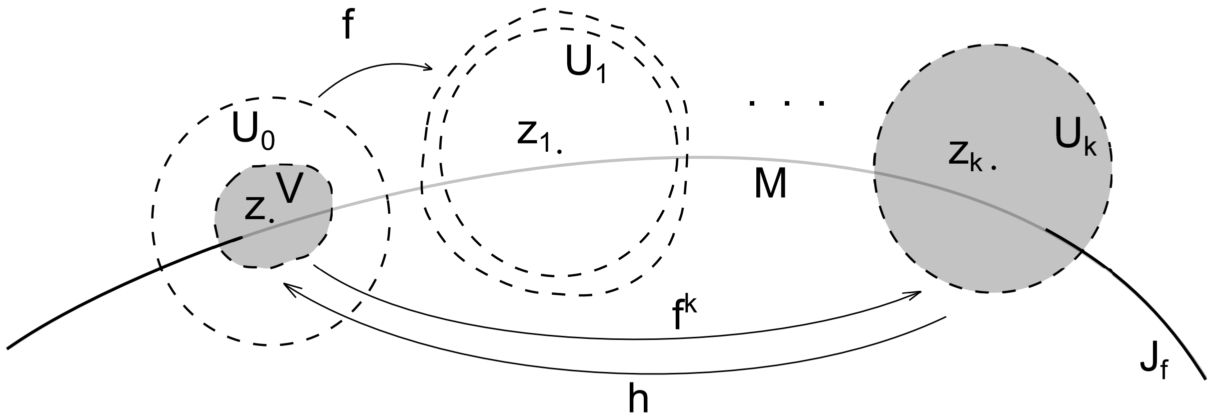

Figure 2.5: Illustration to Lemma 123

Proof. Let z, k satisfy the conditions of Lemma 123. Put

$$z_j = F^j(z), U_j = U_\delta(z_j), j = 0, 1, \dots, k.$$

By Lemma 122, F is univalent on U_j for each $j = 0, 1, \dots, k-1$ and

$$F(U_j) \supset U_{j+1}.$$

It follows that there exists $V \subset U_0$ such that

$$F^k(V) = U_k$$

and F^k is univalent on V. Lemma 122 implies that

$$V \subset U_{\delta/q}(z_0).$$

Let $h : U_k \to V$ such that $h \circ F^k$ is identical on V. Put

$$r_0 = d(z_k, J_f).$$

Then $r_0 \leqslant \frac{\delta}{2}$. Using Koebe Distortion Theorem in the form (2.5) for $r = 2r_0$ and $\alpha = 1/2$, we obtain:

$$U_{\frac{4}{9}r_0|h'(z_k)|}(z) \subset h\left(U_{r_0}(z_k)\right) \subset U_{4r_0|h'(z_k)|}(z). \tag{2.21}$$

Since J_f is invariant under f, it follows that

$$\tfrac{4}{9}r_0|h'(z_k)| \leqslant d(z, J_f) \leqslant 4r_0|h'(z_k)|. \tag{2.22}$$

Since $Dh(z_k) = \frac{1}{DF^k(z)}$, (2.22) is equivalent to the first part of (2.20). Similarly, one can prove the second part of (2.20). Observe that we can not prove a lower bound for $d(F^k(z), M)$ in this way since M is not necessarily invariant under F^{-1}. $\square$

Till the end of this section in each of the preceding statements we assume that the Turing Machine has an access to the non-uniform information $(N1, N2)$ and an oracle for the coefficients of the map f.

Lemma 124. *There exists a TM which computes dyadic numbers $\varepsilon_1 > 0, K > 0$, such that for any critical point c of f which lie in J_f one has*

$$d(f(z), J_f) \geqslant K d(z, J_f)|z - c|^{m-1}$$

for any $z \in U_{\varepsilon_1}(c)$, where m is the degree of f at c.

Proof. By Proposition 106 we can find approximate positions of all critical points of f. Notice that the orbit of every critical point of f which lie in the Fatou set converges to an attracting cycle. Using the ideas from the proof of Proposition 107 we can distinguish the critical points which belong to the Fatou set from the critical points which belong to the Julia set. Let $c \in J_f$ be a critical point of degree m. Then we can approximate the coefficient a_m at $(z - c)^m$ in the Taylor expansion of f at c and compute $\gamma > 0$ such that

$$\left| \frac{f(z) - f(c)}{a_m (z - c)^m} - 1 \right| < 1$$

for any $z \in U_\gamma(c)$. Let ω be any m-th root of a_m. Then there exists a unique holomorphic map ψ from $U_\gamma(c)$ to a neighborhood of the origin such that

$$f(z) = \psi(z)^m + f(c) \text{ on } U_\gamma(c) \text{ and } \psi'(c) = \omega.$$

Make γ smaller if necessary in order to have

$$|\psi'(z)| \geqslant |\omega|/2 \text{ for any } z \in U_\gamma(c).$$

The image $f(U_\gamma(c))$ contains a disk $U_\alpha(f(c))$, where $\alpha > 0$ can be algorithmically constructed. Let $0 < \varepsilon_1 < \gamma$ such that

$$f(U_{\varepsilon_1}(c)) \subset U_{\alpha/2}(f(c)).$$

Take $z \in U_{\varepsilon_1}(c)$. Observe that

$$d(f(z), J_f) \leqslant d(f(z), f(c)) \leqslant \alpha/2.$$

Let $\zeta_0 \in J_f$ such that

$$d(f(z), J_f) = |f(z) - \zeta_0|.$$

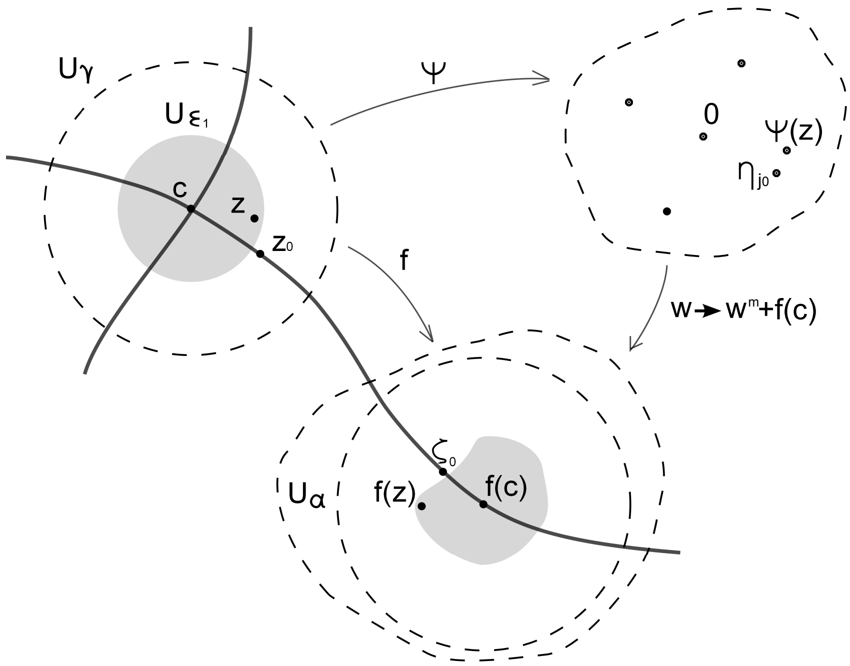

Figure 2.6: Illustration to Lemma 124

Then $\zeta_0 \in U_\alpha(f(c))$. There exists $z_0 \in J_f \cap U_\gamma(c)$ such that $f(z_0) = \zeta_0$. One has

$$f(z) - f(z_0) = \psi(z)^m - \psi(z_0)^m = \prod_{j=0}^{m-1} (\psi(z) - \eta_j), \quad \text{where } \eta_j = e^{2\pi ij/m}\psi(z_0).$$

Let $0 \leqslant j_0 \leqslant m-1$ be the number for which η_{j_0} takes the closest value to $\psi(z)$. From simple geometric observations it follows that

$$|\psi(z) - \eta_j| \geqslant |\psi(z)| \sin(\pi/m) \geqslant \frac{2|\psi(z)|}{m} \geqslant \frac{|\omega|}{m}|z-c|$$

for any $j \neq j_0$. Since $\psi^{-1}(\eta_{j_0}) \in J_f$, it follows that

$$d(f(z), J_f) \geqslant \left(\frac{|\omega||z-c|}{m}\right)^{m-1} |\psi(z) - \eta_{j_0}| \geqslant K|z-c|^{m-1} d(z, J_f),$$

where $K = m^{1-m}|\omega|^m/2$. □

Using the definition of the dyadic set U and Proposition 106 one can prove the following statement.

Lemma 125. *There exists a TM which computes a dyadic number $\varepsilon_2 > 0$ such that*

$$\overline{U_{\varepsilon_2}\left(\widetilde{C}\right)} \cap \overline{f^N(U)} = \varnothing \text{ and } f^{N_0}\left(\overline{U_{\varepsilon_2}\left(\widetilde{C}\right)}\right) \subset U. \tag{2.23}$$

Proposition 126. *There exists a TM which computes a dyadic number* $0 < \gamma < \min\{\varepsilon_1, \varepsilon_2\}$ *such that the following is true. For any* $z \in U_\gamma\left(\widetilde{C}\right)$ *and* $k = \min\{j \in \mathbb{N} : f^{Nj+N_0}(z) \notin U\}$ *one has*

$$d(f^{kN+N_0}(z), J_f) \geqslant q d(z, J_f).$$

Proof. First put $\gamma = \min\{\varepsilon_1, \varepsilon_2\}$. Let $z \in U_\gamma\left(\widetilde{C}\right) \setminus J_f$. Set

$$w = f^{N_0}(z), k = \min\{j \in \mathbb{N} : F^j(w) \notin U\},$$

where $F = f^N$. Using Lemma 123, we obtain:

$$\frac{d(F^k(w), J_f)}{d(w, J_f)} \geqslant \frac{d(F^k(w), M)}{9d(w, M)}. \tag{2.24}$$

Let $c \in \widetilde{C}$. Put $h = f^{N_0}$. Let m_c be the degree of h at $c \in \widetilde{C}$. Then we can algorithmically construct a dyadic number $L_c > 0$ such that

$$|h(z) - h(c)| \leqslant L_c |z - c|^{m_c}$$

for any $z \in U_\gamma(c)$. Then

$$d(h(z), M) \leqslant d(h(z), h(c)) \leqslant L_c |z - c|^{m_c}. \tag{2.25}$$

Lemma 124 implies that we can algorithmically construct dyadic numbers $\alpha_c > 0, K_c > 0$ such that

$$d(h(z), J_f) \geqslant K_c d(z, J_f) |z - c|^{m_c - 1} \tag{2.26}$$

for any $z \in U_{\alpha_c}(c)$. Combining (2.25) with (2.26), we get

$$\frac{d(h(z), J_f)}{d(z, J_f)} \geqslant K_c L_c^{-1} \frac{d(h(z), M)}{|z - c|} \tag{2.27}$$

for any $z \in U_{\min\{\alpha_c, \gamma\}}(c) \setminus J_f$.

Take γ such that $\gamma \leqslant \min\left\{\frac{\delta' K_c L_c^{-1}}{9q}, \alpha_c\right\}$ for any $c \in \widetilde{C}$. Recall that in (2.24) F stands for f^N and w stands for $f^{N_0}(z)$, in (2.27) h stands for f^{N_0}. Now (2.27) implies that

$$\frac{d(w, J_f)}{d(z, J_f)} \geqslant 9q \frac{d(w, M)}{\delta'} \tag{2.28}$$

for any $z \in U_\gamma\left(\widetilde{C}\right)$. Combining (2.24) and (2.28), taking into account that $U \supset U_{\delta'}(M)$, we obtain the statement of Lemma 126. $\square$

An analog of the Proposition 108 holds for the maps f, which satisfy the conditions of Theorem 121.

Proposition 127. *Let f be a rational map such that f does not have any parabolic periodic points and Ω_f does not contain any critical points of f. There exists a TM which given an oracle for the coefficients of the map f and the non-uniform information outputs a planar domain $V \in \mathcal{C}$ such that:*

(1) $V \Subset f(V)$,

(2) $f^2(V) \cap PF_f = \varnothing$,

(3) $J_f \Subset V$.

The proof is similar to the proof of Theorem 108. Denote

$$\Theta = CJ_f \cup \bigcup_{c \in CJ_f} \overline{\mathcal{O}(c)} = \widetilde{C} \cup M, \quad W = V \setminus \Theta.$$

If $CJ_f = \varnothing$ then f is hyperbolic. Since for hyperbolic maps poly-time computability of the Julia set is well known (see [7] and [29]), we will assume that $CJ_f \neq \varnothing$. Then $f(\Theta)$ is strictly smaller than Θ. It follows that $f(W)$ is strictly larger than W. Let $\|\cdot\|_W$ stands for the hyperbolic norm associated with W. Then

$$\|Df(z)\|_W > 1 \text{ for every } z \in W.$$

Let d_W be the metric on W induced by $\|\cdot\|_W$. It follows that:

$$d_W(f(z), J_f) \geqslant d_W(z, J_f) \text{ for every } z \in f^{-1}(W). \tag{2.29}$$

Notice that

$$\inf_{z \in J_f \setminus \left(U_\gamma(\widetilde{C})\right)} |f'(z)| > 0.$$

One can construct a dyadic subset V_1 such that

$$f^{-1}(V) \Subset V_1 \Subset V.$$

Denote $W_1 = V_1 \setminus \Theta$. Using ideas from Section 2.2 and standard considerations one can show the following:

Lemma 128. *One can algorithmically construct dyadic numbers $r \in (0, 1], t > 1, R > 0, \epsilon > 0$, such that*

(1) $d_W(f(z), J_f) \geqslant t\, d_W(z, J_f)$ *for any* $z \in W \setminus \left(U_\gamma\left(\widetilde{C}\right) \cup U\right)$*;*

(2) $d(f(z), J_f) \geqslant r d(z, J_f)$ *for any* $z \in W \setminus U_\gamma\left(\widetilde{C}\right)$*;*

(3) $D_R(0) \supset V \supset V_1 \supset U_\epsilon(J_f)$.

Notice that the Euclidian metric d and the hyperbolic metric d_W are equivalent on any compact subset of W. The following lemma can be proven similarly to Lemma 114.

Lemma 129. *One can algorithmically construct a dyadic constant* $C > 0$ *such that*

$$C^{-1}d(z, J_f) \leqslant d_W(z, J_f) \leqslant Cd(z, J_f) \tag{2.30}$$

for any $z \in W_1 \setminus \left(U_\gamma\left(\widetilde{C}\right) \cup U\right)$.

Lemma 130. *There exists a TM which computes dyadic number* $S > 0$ *and* $m \in \mathbb{N}$ *such that for any* $l \in \mathbb{N}$ *and* $z \in V_1$ *one has* $d\left(f^l(z), J_f\right) \geqslant Sd(z, J_f)^m$.

Proof. It follows from Lemma 124 that one can compute dyadic numbers $1 \geqslant \eta > 0$, $m \geqslant 1$ such that

$$d\left(f^j(z), J_f\right) \geqslant \eta d\left(z, J_f\right)^m \tag{2.31}$$

for any $z \in U_\gamma\left(\widetilde{C}\right)$ and $j = 1, 2, \ldots, N_0 - 1$. Let $z_0 \in V_1 \setminus J_f$ and $l \in \mathbb{N}$. If $f^j(z_0) \notin V_1$ for some $1 \leqslant j \leqslant l$, then by Lemma 128 one has:

$$d(f^l(z_0), J_f) \geqslant \epsilon, \quad \delta \geqslant d(z_0, J_f), \quad \text{and thus,} \quad d(f^l(z_0), J_f) \geqslant \tfrac{\epsilon}{\delta} d(z_0, J_f).$$

Assume that

$$f^j(z_0) \in V_1 \ \text{ for all } \ 1 \leqslant j \leqslant l.$$

Put $l_0 = 0$. Define inductively numbers l_i and points $z_i = f^{l_i}(z_0)$ as follows:

$$l_{i+1} = \begin{cases} l_i + 1, & \text{if } \ z_i \notin U_\gamma\left(\widetilde{C}\right) \cup U, \\ l_i + N, & \text{if } \ z_i \in U \text{ and } l_i + N \leqslant l, \\ l_i + N_0 + kN, & \text{if } \ z_i \in U_\gamma\left(\widetilde{C}\right) \text{ and } l_i + N_0 + kN \leqslant l, \\ l, & \text{otherwise.} \end{cases} \tag{2.32}$$

while $l_i < l$, where

$$k = \min\{j \in \mathbb{N} : f^{Nj+N_0}(z_i) \notin U\}.$$

Now by (2.29), Lemma 128 and Proposition 126 the following is true:

1) $d_W(z_{i+1}, J_f) \geqslant d_W(z_i, J_f)$ for any i;

2) if $z_i \notin U_\gamma\left(\widetilde{C}\right) \cup U$, then

$$d_W(z_{i+1}, J_f) \geqslant t\, d_W(z_i, J_f) \text{ and } d(z_{i+1}, J_f) \geqslant rd(z_i, J_f);$$

3) if $z_i \in U$ and $l_i + N \leqslant l$, then

$$d(z_{i+1}, J_f) \geqslant qd(z_i, J_f);$$

4) if $z_i \in U$ and $l_i + N > l$, then

$$d(z_{i+1}, J_f) \geqslant r^{N-1} d(z_i, J_f);$$

5) if $z_i \in U_\gamma\left(\widetilde{C}\right)$ and $l_i + N_0 + kN \leqslant l$, then

$$d(z_{i+1}, J_f) \geqslant d(z_i, J_f);$$

6) if $z_i \in U_\gamma\left(\widetilde{C}\right)$ and $l_i + N_0 + kN > l$, then

$$d\left(z_{i+1}, J_f\right) \geqslant \eta r^{N-1} d\left(z_i, J_f\right)^m.$$

The items 1) – 5) are direct corollaries of the choice of the dyadic constants, formula (2.29), Lemma 128 and Proposition 126. Let us prove 6). Let

$$z_i \in U_\gamma\left(\widetilde{C}\right) \text{ and } l_i + N_0 + kN > l.$$

If $l_i + N_0 > l$, then 6) follows directly from (2.31). Otherwise let n be the remainder of $l - l_i - N_0$ modulo N. Then by (2.31) and (2.19)

$$d\left(z_{i+1}, J_f\right) \geqslant r^{N-1} d\left(f^{l-l_i-n}(z_i), J_f\right) \geqslant$$
$$r^{N-1} d\left(f^{N_0}(z_i), J_f\right) \geqslant r^{N-1} \eta d\left(z_i, J_f\right)^m.$$

Let s be the number of i-th, for which

$$z_i \notin U_\gamma\left(\widetilde{C}\right) \cup U.$$

If $s \neq 0$ then let I_1 be the minimal and I_2 be the maximal such i.

case 1: $s = 0$. Then one of the following two possibilities holds.

a) $z_0 \in U_\gamma\left(\widetilde{C}\right)$. Then $l = l_1$. By 6),

$$d\left(f^l(z_0), J_f\right) \geqslant \eta r^{N-1} d\left(z_0, J_f\right)^m.$$

b) $z_0 \in U$. Then by 3) and 4),

$$d\left(f^l(z_0), J_f\right) \geqslant r^{N-1} d\left(z_0, J_f\right).$$

case 2: $s \neq 0$**.** There are two possibilities.
a) $l_{I_1} = l$. Then $I_1 = I_2$. As in the case 1,

$$d\left(f^l(z_0), J_f\right) \geqslant \eta r^{N-1} d\left(z_0, J_f\right)^m.$$

b) $l_{I_1} < l$. Then, by 3) and 5), $d(z_{I_1}, J_f) \geqslant d(z_0, J_f)$. By (2.30),

$$d(z_{I_2}, J_f) \geqslant C^{-1} d_W(z_{I_2}, J_f) \geqslant C^{-1} d_W(z_{I_1}, J_f) \geqslant C^{-2} d(z_0, J_f). \tag{2.33}$$

Among the numbers $I_2 + 1, I_2 + 2, \ldots, l$ there is at most one number j such that

$$z_j \in U_\gamma \left(\widetilde{C}\right)$$

and at most one number j such that

$$z_j \in U, \text{ but } l_j + N > l.$$

It follows that

$$d(f^l(z_0), J_f) \geqslant r^{m+N} \eta d\left(z_{I_2}, J_f\right)^m \geqslant S d(z_0, J_f)^m, \tag{2.34}$$

where $S = r^{m+N} \eta C^{-2m}$.

Thus, in both case 1 and case 2

$$d(f^l(z_0), J_f) \geqslant S d(z_0, J_f)^m,$$

which finishes the proof. □

Theorem 131. *There exists a TM which computes the coefficients of a polynomial $p(n)$ such that for any z with $d(z, J_f) > 2^{-n}$ one has*

$$f^{p(n)}(z) \notin V.$$

Proof. Let $z = z_0 \in V_1$ and $d(z_0, J_f) > 2^{-n}$. Let l be the maximal number such that

$$f^j(z_0) \in V_1 \text{ for all } j = 0, 1, \ldots, l.$$

Construct l_i, z_i as in Lemma 130. Then properties 1) – 6) hold. Let $i_1, \dots i_s$ be the sequence of indexes for which

$$z_i \notin U_\gamma\left(\widetilde{C}\right) \cup U.$$

Then, by Lemma 129 and properties 1) and 2) from Lemma 130,

$$\delta \geqslant d(f^l(z_0), J_f) \geqslant C^{-2} t^s d(z_0, J_f) \geqslant C^{-2} t^s 2^{-n}.$$

It follows that

$$s \leqslant K_1(n) = [n \log_t 2 + 2 \log_t C + \log_t \delta].$$

Let i be an index such that $z_i \in U$. Let

$$k_i = \min\{j \in \mathbb{N} : f^{Nj}(z_i) \notin U\}.$$

It follows from Lemma 130 that

$$\delta \geqslant d(f^{Nk_i}(z_i), J_f) \geqslant q^{k_i} d(z_i, J_f) \geqslant q^{k_i} S 2^{-mn}.$$

Therefore,

$$k_i \leqslant K_2(n) = [mn \log_q 2 + \log_q \delta - \log_q S].$$

Now it is easy to see that among any $K_2(n) + 1$ consecutive integer numbers between 0 and l there exists i such that

$$z_i \notin U_\gamma\left(\widetilde{C}\right) \cup U.$$

Let J be the number of elements in the sequence l_i. We obtain that

$$J \leqslant (K_1(n) + 1)(K_2(n) + 1).$$

Observe that for each $0 \leqslant j \leqslant J - 1$ one has:

$$l_{j+1} - l_j \leqslant N K_2(n) + N_0.$$

It follows that

$$l \leqslant (K_1(n) + 1)(K_2(n) + 1)(N K_2(n) + N_0).$$

Since $f^{-1}(V) \Subset V_1 \Subset V$, this finishes the proof. $\square$

2.3.2 The algorithm

Similarly to Proposition 117 one can prove the following result.

Proposition 132. *There is an algorithm computing two dyadic constants $K_1, K_2 > 0$ such that for any $z \in V$ and any $k \in \mathbb{N}$ if $f^k(z) \in f(V) \setminus f^{-1}(V)$ then one has*

$$\frac{K_1}{|Df^k(z)|} \leqslant d(z, J_f) \leqslant \frac{K_2}{|Df^k(z)|}.$$

Now we are ready to describe the algorithm. The steps of the algorithm are analogous to the steps of the corresponding algorithm for a subhyperbolic map. Compute a dyadic number $\varepsilon > 0$ such that

$$\varepsilon < \min\{d(V_1, \mathbb{C} \setminus V), d(f^{-1}(V), \mathbb{C} \setminus V_1)\}. \tag{2.35}$$

Assume that we would like to verify that a dyadic point z is 2^{-n-1} close to J_f. If $z \notin V_1$, we can approximate the distance from z to J_f by the distance form z to V_1 up to a constant factor.

Now assume that $z \in V_1$. Let $p(n)$ be the polynomial from Proposition 131. Similarly to the case of a subhyperbolic map, consider the following subprogram:
$i := 1$
while $i \leqslant p(n+1)$ **do**
(1) Compute dyadic approximations

$$p_i \approx f^i(z) = f(f^{i-1}(z)) \text{ and } d_i \approx \left|Df^i(z)\right| = \left|Df^{i-1}(z) \cdot Df(f^{i-1}(z))\right|$$

with precision $\min\{2^{-n-1}, \varepsilon\}$.
(2) Check the inclusion $p_i \in V_1$:

- if $p_i \in V_1$, go to step (5);
- if $p_i \notin V_1$, proceed to step (3);

(3) Check the inequality $d_i \geqslant K_2 2^{n+1} + 1$. If true, output 0 and exit the subprogram, otherwise
(4) output 1 and exit subprogram.
(5) $i \to i + 1$
end while

(6) Output 0 end exit.

end

The subprogram runs for at most $p(n+1)$ number of while-cycles each of which consist of a constant number of arithmetic operations with precision $O(n)$ dyadic bits. The following proposition is proved in the same way as Proposition 118.

Proposition 133. *Let $f(n,z)$ be the output of the subprogram. Then*

$$f(n,z) = \begin{cases} 1, & \textit{if } d(z,J_f) > 2^{-n-1}, \\ 0, & \textit{if } d(z,J_f) < K2^{-n-1}, \\ \textit{either}\, 0 \textit{ or}\, 1, & \textit{otherwise}, \end{cases} \tag{2.36}$$

where $K = \frac{K_1}{K_2+1}$.

2.4 Maps with parabolic periodic points

In this section we will sketch the proof of Theorem 98. Let f be a rational map such that Ω_f does not contain either critical points or parabolic periodic points. We will assume that the periods and multipliers of the parabolic periodic points are given as a part of the non-uniform information. Replacing f with some iteration of f if necessary, we may assume that the multiplier and the period of each of the parabolic periodic points is equal to 1. Then at each parabolic periodic point (in fact, a fixed point) p the map f can be written in the form

$$f(z) = z + c_p(z-p)^{n_p+1} + O(z^{n_p+2}),$$

where $n_p \in \mathbb{N}$. We will assume that the numbers n_p are also given as a part of the non-uniform information. Observe that some parabolic fixed points of f may belong to the postcritical set of f.

In this section we denote by CJ_f the set of critical points $c \in CJ_f$ such that the forward orbit $\mathcal{O}(z)$ does not contain any parabolic periodic points and by CJ_f^P the set of critical points $c \in J_f$ such that an iteration of c hits a parabolic fixed point. Let $N_0(c), N_0, M, \widetilde{C}$ be the same as in section 2.3. For numbers $\alpha, \beta > 0$, a direction $\nu \in [0, 2\pi]$ and a point $w \in \mathbb{C}$ introduce a sector

$$V_w^\nu(\beta,\alpha) = \{z \in U_\alpha(w) : \arg(z-w) \in (\nu-\beta, \nu+\beta)\}. \tag{2.37}$$

Let $p_1, \ldots, p_s$ be all parabolic fixed points of f. The algorithm will use the same **non-uniform information** $(N1, N2)$ which was described in subsection 2.3.1 and, in

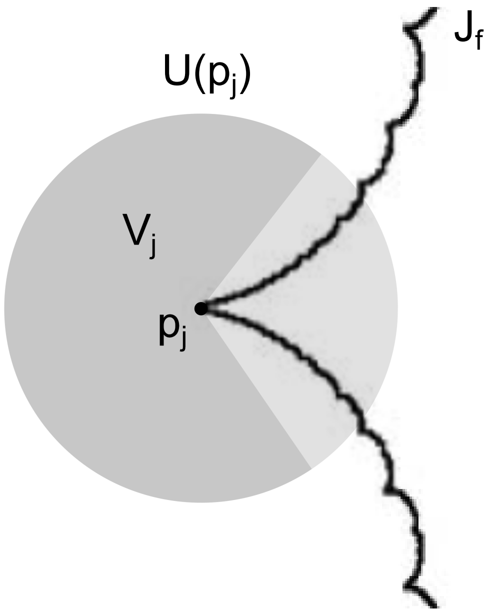

Figure 2.7: Illustration to non-uniform information $N3$.

addition,

N3. approximate positions a_j of p_j with a dyadic precision $\alpha > 0$ and a number $\beta > 0$ such that for each j:

1) p_j is a unique fixed point of f in $U_{2\alpha}(a_j)$ (and thus p_j can be approximated efficiently using Newton method);

2) for each critical point c which lie in J_f one has

$$U_{2\alpha}(a_j) \setminus \{p_j\} \cap \mathcal{O}(c) = \varnothing;$$

3) for each attracting direction $\nu \in [0, 2\pi]$ at p_j the sector

$$V_j^\nu = V_{p_j}^\nu(\beta, 2\alpha)$$

belongs to an attracting Fatou petal at p_j.

Observe that for a point z in the sector V_j^ν the distance $d(z, J_f)$ up to a constant factor can be approximated by $|z - p_j|$. We will use the following result from [8] (see Lemma 8).

Lemma 134. *Let $g(z) = z + c_{n+1}z^{n+1} + c_{n+2}z^{n+2} + \ldots$ be given as a power series with radius of convergence $R > 0$. There is an algorithm which given a point z with $|z| < 1/m < R$ computes $l = [m^n/C]$-th iteration of z and the derivative $Dg^l(z)$ with* *precision 2^{-s} in time polynomial in s and $\log m$. Here C is some dyadic constant which can be algorithmically constructed. The algorithm uses an oracle for the coefficients c_n.*

Lemma 134 implies that we can compute "long" iterations of points close to parabolic periodic points efficiently.

By the assumptions on the map f one can algorithmically construct a dyadic number $\varepsilon > 0$ such that the following is true. Let a point z be ε-close to the Julia set. Assume that some iteration $f^k(z)$ belongs to $U_\varepsilon(p_j)$. Consider the neighborhood $V = U_\varepsilon(f^k(z))$. Let

$$V_0 = U(z), V_1, \ldots, V_k = V$$

be the pullbacks of V under f along the orbit of z, where $U(z)$ is a neighborhood of z. Then the number of n for which V_n contains a critical point $c \in J_f$ is bounded from above by a constant independent from k and z. In a similar fashion as Proposition 117 using ideas of Lemma 124 we can prove the following statement.

Proposition 135. *One can algorithmically construct dyadic numbers $K_1, K_2, \varepsilon > 0$ such that for each $z \in U_\varepsilon(J_f)$ if k satisfies the following*

$$f^n(z) \in U_\varepsilon(J_f) \ \text{ for } \ n = 1, \ldots, k, \quad \text{and} \ \ f^k(z) \in U_\varepsilon(p_j)$$

for some parabolic fixed point p_j then

$$\frac{K_1 d(f^k(z), J_f)}{|Df^k(z)|} \leqslant d(z, J_f) \leqslant \frac{K_2 d(f^k(z), J_f)}{|Df^k(z)|}.$$

From the description of the dynamics near a parabolic point using Koebe Theorem 103 we can obtain the following result.

Proposition 136. *One can algorithmically construct numbers $K_3, L > 0$ such that for each $z \in U_\varepsilon(p_j) \setminus (\cup V_j^\nu)$ and each $n \in \mathbb{N}$ if*

$$f^i(z) \in U_\varepsilon(p_j) \ \text{for all} \ i = 0, 1, \ldots, l = 2^{[Ln]}$$

then one has $|f^l(z) - p_j| \geqslant 2^n |z - p_j|$ and

$$K_3^{-1} \frac{|f^l(z) - p_j|}{|z - p_j|} \leqslant \frac{d(f^l(z), J_f)}{d(z, J_f)} \leqslant K_3 \frac{|f^l(z) - p_j|}{|z - p_j|}.$$

Now we briefly explain how to adopt the algorithm from Paragraph 2.3.2 to prove Theorem 98. Assume that we want to verify if a point z is 2^{-n}-close to J_f. Construct a sequence $z_i = f^{l_i}(z)$ in a way analogues to the construction from Lemma 130 (see (2.32)). We will define l_{i+1} in a different way from (2.32) only if $z_i \in U_\varepsilon(p_j)$ for some parabolic fixed point p_j. In this case we do the following.

1) If $z_i \in V_j^\nu$ for some attracting direction ν then we stop. We can find the distance from z_i to J_f up to a constant factor. Using Proposition 135 we can estimate the distance from z to J_f up to a constant factor.

2) If $z_i \in U_\varepsilon(p_j) \setminus (\cup V_j^\nu)$ then consider the points $f^{k_r}(z_i)$, where $k_r = 2^{[Lr]}$. Observe that by Lemma 134 we can find a 2^{-n} approximation of $f^{k_r}(z_i)$ in time polynomial in r and n. Let r be the minimal nonnegative integer number such that

$$|f^{k_r}(z_i) - p_j| \geqslant \varepsilon.$$

Set $z_{i+1} = f^{k_r}(z_i)$.

Observe that a direct analog of Theorem 131 is not true in a presence of parabolic periodic points, even if we assume that z does not belong to an attracting basin of a parabolic periodic point. For a point 2^{-n} close to a parabolic fixed point in a repelling petal it takes exponential time in n to escape an ε-neighborhood of the parabolic fixed point. However, using Lemma 134 and Proposition 136 we can prove the following:

Proposition 137. *There is an algorithm computing coefficients of a polynomial $p(n)$ such that if*

$$d(z, J_f) > 2^{-n} \ \text{ and } \ z_i \notin \cup V_j^\nu \ \text{ for each } \ i = 0, 1, \ldots, p(n),$$

then $d(z_{p(n)}, J_f) > \varepsilon$. Moreover, we can compute 2^{-n-2}-approximations of $z_i = f^{l_i}(z)$ and $Df^{l_i}(z)$ for $i = 0, \ldots, p(n)$ in time polynomial in n.

Now we briefly describe the algorithm computing the Julia set J_f in a polynomial time in n. Let $z \in U_\varepsilon(J_f)$. Assume that we want to verify that z is 2^{-n} close to J_f. Without loss of generality we may assume that $2^{1-n} < \varepsilon$. Compute approximate values a_i of z_i and d_i of $Df^{l_i}(z_i)$ with precision 2^{-n-2}, $i = 1, \ldots, p(n)$.

1) If $z_i \in \cup V_j^\nu$ for some i such that

$$d(z_j, J_f) \leqslant \varepsilon/2 \ \text{ for } \ j = 1, \ldots, i$$

then we can find approximate distance from z_i to J_f. By Proposition 135, we can find $d(z, J_f)$ up to some constant.

2) If $d(z_i, J_f) \geqslant \varepsilon/2$ for some i, then we can find the distance $d(z_i, J_f)$ up to a constant

factor. Using Koebe Theorem (103) we can find the distance from z to J_f up to a constant factor.
3) If neither 1) nor 2) holds then by Proposition 137

$$d(z, J_f) \leqslant 2^{-n}.$$

In conclusion, let us mention an alternative numerical method to calculate iterations of points close to parabolic periodic orbits. This method has been implemented in recent literature. To illustrate the method we consider a map with a simple parabolic point at the origin:

$$h(z) = z + z^2 + \sum_{k=3}^{\infty} a_k z^k.$$

Let $j(z) = -1/z$ and $F(z)$ be the germ at infinity given by

$$F(z) = j \circ f \circ j(z) = -1/f(-1/z) = z + 1 + a(z), \quad \text{where } \; a(z) = O(z^{-1}).$$

Denote by T the unit shift: $T(z) = z + 1$. In [15] and [16] Ecalle has shown the following

Theorem 138. *The equation $\Psi \circ F(z) = T \circ \Psi(z)$ has a unique formal solution in terms of the series (generally, divergent)*

$$\widetilde{\Psi}(z) = \rho \log z + \sum_{k=1}^{\infty} c_k z^{-k}. \tag{2.38}$$

The series $\widetilde{\Psi}$ gives an asymptotic expansion for an attracting and a repelling Fatou coordinates of the map F, Ψ_a and Ψ_r correspondingly.

For survey on divergent series and asymptotic expansions we refer the reader to [28]. Theorem 138 means that the Fatou coordinates Ψ_a and Ψ_r near infinity can be approximated by finite sums of the series (2.38). To approximate n-th iteration of the map f near the origin one can use the following formula:

$$f^n(z) = j \circ \Psi^{-1} \circ T^n \circ \Psi \circ j,$$

where Ψ stands for either Ψ_a or Ψ_r depending on whether z belongs to an attracting or a repelling Fatou petal of f. We would like to emphasize that this is not a rigorous method since we do not know how many terms of the series (2.38) to take to obtain the desired precision. However, empirically, the asymptotic expansion (2.38) approximates the Fatou coordinates $\Psi_a(z), \Psi_b(z)$ with a very high precision. For instance, in [20] Lanford and Yampolsky used the series (2.38) in their computational scheme for the fixed point of the parabolic renormalization operator. Our work in progress [14] give us a reason to hope that the described method can be made rigorous.

Bibliography

[1] M. Aspenberg, *The Collet-Eckmann condition for rational functions on the Riemann sphere*, Ph.D. Thesis, KTH, Sweden, 2004.

[2] A. Avila, C.G. Moreira, *Statistical properties of unimodal maps: the quadratic family*, Annals of Mathematics, 161, no. 2 (2005), 831-881.

[3] I. Binder, M. Braverman, M. Yampolsky, *On computational complexity of Siegel Julia sets*, Commun. Math. Phys. **264** (2006), no. 2, 317-334

[4] I. Binder, M. Braverman, M. Yampolsky, *Filled Julia sets with empty interior are computable*, Journ. of FoCM, **7** (2007), 405-416.

[5] I. Binder, M. Braverman, M. Yampolsky, *Constructing locally connected non-computable Julia sets*, Commun. Math. Phys. **291** (2009), 513-532.

[6] E. Borel, *Mémoire sur les séries divergentes.* Ann. Sc. École Norm. Sup., Paris (3), 16 (1899).

[7] M. Braverman, *Computational complexity of Euclidian sets: Hyperbolic Julia sets are poly-time computable*, Master's thesis, University of Toronto, 2004.

[8] M. Braverman, *Parabolic Julia sets are polynomial time computable*, Nonlinearity **19**, (2006), no.6, 1383-1401.

[9] M. Braverman, M. Yampolsky, *Non-computable Julia sets*, Journ. Amer. Math. Soc. **19** (2006), no. 3, 551-578.

[10] M. Braverman, M. Yampolsky, *Computability of Julia sets*, Series: Algorithms and Computation in Mathematics, Vol. 23, Springer, 2008.

[11] X. Buff, J. Hubbard, *Dynamics in One Comples Variable*, Manuscript in preparation.

[12] B. Candelpergher, J.-C. Nosmas and F. Pham. *Approche de la résurgence.* Actualités Math., Hermann, Paris, 1993.

[13] J. B. Conway, *Functions of one complex variable II*, Springer-Verlag, New York, New York, 1995.

[14] A. Dudko, D. Sauzin, *Ecalle-Voronin invariants via resurgence*, in preparation.

[15] J. Écalle. *Les fonctions résurgentes, Vol. 1.* Publ. Math. d'Orsay 81-05, 1981.

[16] J. Écalle. *Les fonctions résurgentes, Vol. 2.* Publ. Math. d'Orsay 81-06, 1981.

[17] J. Écalle. *Les fonctions résurgentes, Vol. 3.* Publ. Math. d'Orsay 85-05, 1985.

[18] J. Graczyk, G. Swiatek, *Generic hyperbolicity in the logistic family*, Ann. of Math., v. 146 (1997), 1-52.

[19] H. Inou, M. Shishikura, *The renormalization for parabolic fixed points and their perturbation*, Preprint, 2008.

[20] Oscar Lanford III, M. Yampolsky, *The fixed point of the parabolic renormalization operator*, arXiv:1108.2801.

[21] M. Lyubich, *Dynamics of quadratic polynomials*, I-II. Acta Math., 178 (1997), 185-297.

[22] B. Malgrange. Resommation des séries divergentes. *Exp. Math.* **13** (1995), 163–222.

[23] R. Mañé, *On a Theorem of Fatou*, Bol. Soc. Bras. Mat., Vol. 24, no. 1, 1993, pp. 1-11.

[24] P.D. Miller, *Applied Asymptotic Analysis*, Providence, RI: American Mathematical Society, (2006), pp. 467

[25] J. Milnor, *Dynamics in one complex variable. Introductory lectures*, 3rd ed., Princeton University Press, 2006.

[26] Y. Ou, *On the stability by convolution product of a resurgent algebra,* Annales de la faculté des sciences de Toulouse Sér. 6, 19 no. 3-4 (2010), 687–705.

[27] C. M. Papadimitriou, *Computational complexity*, Addision-Wesley, Reading, Massachusetts, 1994.

[28] J.-P. Ramis. Séries divergentes et théories asymptotiques, Panoramas et Synthèses (1994).

[29] R. Rettinger, *A fast algorithm for Julia sets of hyperbolic rational functions*, Electr. Notes Theor. Comput. Sci. **120** (2005), 145-157.

[30] D. Sauzin, *Resurgent functions and splitting problems*, RIMS Kokyuroku **1493** (2005), 48–117.

[31] D. Sauzin, *Mould expansions for the saddle-node and resurgence monomials,* in Renormalization and Galois theories. Selected papers of the CIRM workshop, Luminy, France, March 2006, p. 83–163, A. Connes, F. Fauvet, J.-P. Ramis (eds.), IRMA Lectures in Mathematics and Theoretical Physics 15, Zrich: European Mathematical Society, 2009.

[32] D. Sauzin, *On the stability under convolution of resurgent functions*, arXiv:1201.1073.

[33] D. Sauzin, *Introduction to 1-summability and the resurgence theory*, in preparation.

[34] V. Shatalov, B. Sternin, *Borel-Laplace Transform and Asymptotic Theory. Introduction to Resrgent Analysis.* Boca Raton, FL : CRC Press, 1996.

[35] M. Shishikura, *The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets.* Ann. of Math. (2) **147** (1998), no. 2, 225-267.

[36] M. Shishikura, Tan Lei, *An alternative proof of Mane's theorem on non-expanding Julia sets*, in "The Mandelbrot set, Theme and Variations", Ed. Tan Lei, London Math. Soc. Lect. Note Ser. 274, Cambridge Univ. Press, 2000, p.265-279.

[37] M. Sipser, *Introduction to the theory of computation, second edition*, BWS Publishing Company, Boston, 2005.

[38] S. Voronin, *Analytic classification of germs of conformal mappings* $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$. (Russian) Funktsional. Anal. i Prilozhen. 15 (1981), no. 1, 1–17, 96. (English translation: Functional Anal. Appl. 15 (1981), no. 1, 1–13.)

[39] G. N. Watson, *The harmonic functions associated with the parabolic cylinder*, Proceedings of the London Mathematical Society **2** no.17 (1918), 116148 .

[40] H. Weyl, *Randbemerkungen zu Hauptproblemen der Mathematik, II, Fundamentalsatz der Algebra and Grundlagen der Mathematik*, Math. Z. **20**(1924), 131-151.