On the fundamental ideas of measure theory

[Matematičeskiĭ Sbornik (N.S.) 25(67), 107–150 (1949)]

V. A. Rohlin

Translated from

В. А. Рохлин

Об основных понятиях теории меры

[Математический Сборник (Н.С.) 25(67), 107-150 (1949)]

by Edwin Hewitt

First published by the American Mathematical Society in 1952 as Translation Number 71. Reprinted without revision.

The present work is devoted to the axiomatic description of the ordinary measure of Lebesgue or Lebesgue-Stieltjes in terms of the abstract theory of measure and to the study of the *Lebesgue space*, its *homomorphisms*, *measurable decompositions*, and *factor spaces*, which arise when one carries out this axiomatization.

The work consists of four paragraphs. §1 contains general definitions and notations, used throughout the entire work. For the most part, these definitions are not new, and are applicable not only to Lebesgue spaces, but to arbitrary measure spaces. §2 is devoted to the definition and structural properties of Lebesgue spaces. In §3, we set forth the general theory of measurable decompositions, homomorphisms, and factor spaces of Lebesgue spaces. Finally in §4, we give a classification of measurable decompositions and homomorphisms.

The principal definitions and theorems of the first three paragraphs, in particular, the axiomatic description of a Lebesgue space and the theorem on the existence of a canonical system of measures, is taken from my unpublished opus *Unitary rings and dynamical systems* (University of Moscow, June 1940). In connection with the matters set forth in Nos. 1, 2, and 4 of §2, I note that in 1942, P. R. Halmos and J. v. Neumann [1] published a different axiomatic description of the unit interval from the point of view of the abstract theory of measure; their axioms of countability coincide with those which we take in No. 1 of §2, but the place of the axiom of completeness is taken by a different axiom. The theorem on isomorphisms (No. 5 of §2) is due to von Neumann [2], and the proof given here is a simplified version of von Neumann's proof. Nevertheless, the *formulation* of this theorem given in the text is stronger than the formulation of von Neumann, which is insufficient for the aim of the present work (the improvement consists in the fact that the image UM of the space M is not assumed at the beginning to be a Lebesgue space). I am indebted to A. N. Kolmogorov for the idea of a canonical system of measures (No. 1 of §3), who informed me of it in the year 1940. Other results, closely related to the theorem on the existence of canonical systems of measures, were published in [3] and [4].*

The principal results of the present work have appeared in

*Note added in proof. As I learned from [9], the indicated results from [3] and [4] are false.

Doklady Akad. Nauk SSSR [5].

§1. General measures.

No. 1. *Space with a measure. Subspaces.* It is well known that a real-valued function μ, defined on a certain collection Ω_μ of subsets of an arbitrary set M, is called completely additive if Ω_μ is a Borel field of sets; this means that Ω_μ contains, with any two sets, their difference, and with any sequence of sets, their union (and consequently, their intersection); for every sequence of pairwise disjoint sets $A_n \in \Omega_\mu$,

$$\mu\Big(\bigcup_n A_n\Big) = \sum_n \mu A_n.$$

By the term measure, we shall mean throughout this work a non-negative completely additive function μ, have the following two properties: if $A \subset B$ and $\mu B = 0$, then $A \in \Omega_\mu$ (consequently $\mu A = 0$);

$$M \in \Omega_\mu \quad \text{and} \quad \mu M = 1.$$

Under these circumstances, the basic set M is called a space with the measure μ, and its elements are called points. Sets of the collection Ω_μ are said to be measurable. Measurable functions and the Lebesgue integral are defined in the usual fashion.

The outer measure $\mu_e A$ of a set $A \subset M$ is defined as the lower bound of the measures of the measurable sets containing A. Of course, this lower bound is always attained; that is, there always exists a measurable set B containing A for which $\mu B = \mu_e A$. Every such set B is called a measurable hull of the set A. The measurable hull is defined uniquely up to sets of measure zero. If B and B' are two measurable hulls of the set A, then $\mu(\overline{B}B' + B\overline{B}') = 0$.* Indeed, the intersection BB' is again a measurable hull of the set A, and for this reason $\mu(BB') = \mu B = \mu B'$.

The outer measure μ_e in its turn completely defines the measure μ; a subset A of M is measurable if and only if $\mu_e A + \mu_e \overline{A} = 1$; and if A is measurable, then $\mu A = \mu_e A$.

If M is a space with measure, then every subset A of M which is not a set of measure zero ($\mu_e A > 0$) can also be considered as a space of the same kind (a subspace of the space M), if we define in A a measure μ_A, taking the sets of the form $X_A = AX$, where

* The line above indicates the complementary set: $\overline{B} = M - B$.

$X \in \Omega_\mu$, as measurable, and defining μ_A for such sets by the relation

$$\mu_A X_A = \frac{\mu_e X_A}{\mu_e A}. \tag{1}$$

In particular, if A is measurable, then Ω_{μ_A} is simply the collection of those measurable sets of the space M, which lie in A, and the measure μ_A is defined by the formula $\mu_A X = \mu X/\mu A$.

No. 2. *Homomorphisms. Decompositions. Factor-spaces.* A single-valued mapping of the space M into the space M' is called a homomorphic mapping, or simply a homomorphism, if the inverse image of every measurable set is measurable and has the same measure as its image. It is evident that every homomorphism of the space M into the space M' is a homomorphism of M onto a certain subspace of the space M'.

Closely connected with homomorphisms of the space M are its decompositions into disjoint subsets, the so-called elements of the decomposition. Sets which are unions of elements of the decomposition ζ, we call ζ-sets. With every homomorphism H of the space M, we associate a definite decomposition of this space, namely, the decomposition ζ_H whose elements are the inverse images of *points* under the mapping H; ζ_H-sets are the inverse images of sets. Conversely, to every decomposition of the space M, there corresponds a definite homomorphism $H = H_\zeta$ of the space M, for which $\zeta_H = \zeta$. In order to construct this homomorphism, we take the elements C of the decomposition ζ as points of a new space, the factor space of the space M with respect to the composition ζ, which we shall designate as M/ζ, and which becomes a space with a measure, if we introduce into it a measure μ_ζ, defining a set $X \subset M/\zeta$ to be measurable in M/ζ when the ζ-set

$$Z = \bigcup_{C \in X} C$$

is measurable in M, and setting

$$\mu_\zeta X = \mu Z.$$

The homomorphism H_ζ is the mapping of the space M onto M/ζ which assigns to every point of the space M that element of the decomposition ζ in which the point lies. It is obvious that the decomposition ζ_H associated with the homomorphism $H = H_\zeta$, is once again ζ. The definition of the measure μ_ζ is such that for H_ζ, not only is the *inverse image* of every measurable set measurable, but also the *image* of every measurable set is measurable. On account of this fact, it is possible to assert that if the decomposition ζ is generated by some homomorphism H of the space M into a different space M' $(\zeta=\zeta_H)$, then the one-to-one mapping T_H of the factor space M/ζ into M', which carries every point $C \in M/\zeta$ into the point of the space M' into which H carries all of the points of the set $C \in M/\zeta$, is a homomorphism. It is evident that the homomorphisms H, H_ζ and T_H are connected by the relation $H = H_\zeta T_H$.

Starting with the decomposition ζ, we arrive at a collection of ζ-sets. Conversely, every collection of sets lying in M leads to a definite decomposition. Let $\Sigma = \{S_\alpha\}$ by an arbitrary system of subsets of the set M. We shall agree to designate by the symbol R_α one of the two sets $S_\alpha, \overline{S}_\alpha$, and consider all possible sets of the form $C = \bigcap R_\alpha$,
where α runs through *all* possible values. The sets C are disjoint and cover M, that is, they generate a definite decomposition, which we denote by the symbol $\zeta(\Sigma)$. The set of all $\zeta(\Sigma)$-sets coincides with the set of all sets which can be obtained from the sets of the system Σ with the help of all possible set-theoretic operations (including complementation), repeated as many times as desired. This collection of sets is denoted by the symbol $\mathfrak{M}\Sigma$. A particularly important case is that in which $\mathfrak{M}\Sigma$ is the collection of *all* subsets of the space M. This takes place if and only if $\zeta(\Sigma)$ is the decomposition of M into individual points.

The system Σ also induces in every subspace A of the space M a definite system of subsets, namely, the system Σ_A of all sets of the form $S_A = AS$, $S \in \Sigma$. In exactly the same way, there corresponds to every decomposition ζ of the space M a definite decomposition ζ_A of the space A, namely, the decomposition of the space A into the sets $C_A = AC$, where C is an element of the decomposition ζ. Regarding ζ_A-sets of the space A as sets in M, we shall call them ζ-subsets of the set A.

No. 3. *Systems of measurable sets.* A measure μ' defined on a collection of sets $\Omega_{\mu'}$, is called a part of the measure μ'', defined on the collection of sets $\Omega_{\mu''}$, if $\Omega_{\mu'} \subset \Omega_{\mu''}$ and $\mu' A = \mu'' A$ for all $A \in \Omega_{\mu'}$. On the other hand, the transition from μ' to μ'', as well as the measure μ'' itself, is called an extension of the measure μ'.

Let $\Sigma = \{S_\alpha\}$ be an arbitrary system of measurable sets. It is plain that among the parts of the measure μ, defined for all sets S_α, there is always a minimal element: this is the common part of the family of all such measures. We shall denote it by the symbol μ_Σ. The measure μ_Σ can be obtained in the following way. We denote by $\overline{\Sigma}$ the system of sets complementary to the sets of the system Σ. We agree further to denote by δ a finite set of indices α. If the number of indices in the set δ is equal to r, then, we shall sometimes write δ^r instead of δ; in this connection, the value $r = 0$ is admissible: δ^0 is the void set. We set:

$$D(\delta^r) = \bigcap_{\alpha \in \delta^r} S_\alpha, \quad \overline{D}(\delta^r) = \bigcap_{\alpha \in \delta^r} \overline{S}_\alpha; \quad r > 0;$$

$$D(\delta^0) = \overline{D}(\delta^0) = M.$$

Instead of $D(\delta)$ and $\overline{D}(\delta)$, we shall also write $D_\Sigma(\delta)$ and $\overline{D}_\Sigma(\delta)$. The collection of sets of the form $D_\Sigma(\delta)$ we shall denote by the symbol Σ_d. In accordance with this, we shall denote the collection of all sets of the form $D_\Sigma(\delta)\overline{D}_\Sigma(\delta)$ by $(\Sigma+\overline{\Sigma})_d$, and the collection of sets of the form

$$E = \bigcup_{n=1}^{r} E_n, \qquad E_n \in (\Sigma + \overline{\Sigma})_d,$$

which is the *field* of sets generated by the system Σ and the set M, by the symbol $(\Sigma+\overline{\Sigma})_{ds}$. Furthermore, the collection of sets of the form

$$E = \bigcup_{n=1}^{\infty} E_n, \quad E_n \in (\Sigma + \overline{\Sigma})_d, \tag{2}$$

will be designated by the symbol $(\Sigma+\overline{\Sigma})_{d\sigma}$. Finally the collection of all sets of the form

$$E = \bigcap_{n=1}^{\infty} E_n, \qquad E_n \in (\Sigma + \overline{\Sigma})_{d\sigma},$$

is denoted by the symbol $(\Sigma+\overline{\Sigma})_{d\sigma\delta}$. For every set $A \subset M$,

$$(\mu_\Sigma)_e A = \inf \mu E, \quad E \in (\Sigma + \overline{\Sigma})_{d\sigma}, \quad E \supset A, \tag{3}$$

thus the measurable hull of the set A with respect to the measure μ_Σ can always be found in the collection of sets $(\Sigma+\overline{\Sigma})_{d\sigma\delta}$, and hence *a fortiori* in the Borel field $\mathfrak{B}\Sigma$ generated by the system Σ and the set M.

Since, for every set $E \in (\Sigma+\overline{\Sigma})_{d\sigma}$, the representation (2) can be chosen so that the sets E_n are pairwise disjoint, it follows from (3) that the measure μ_Σ, which is not in general defined by its values on Σ, is defined by its values on $(\Sigma+\overline{\Sigma})_d$. Indeed, it is already defined by its values on Σ_d, since if these last values are known, then the values of the measure μ_Σ on $(\Sigma+\overline{\Sigma})_d$ can be calculated from the easily verified formula:

$$\mu(D(\delta)\overline{D}(\overline{\delta}^r)) = \sum_{k=0}^{r} (-1)^k \sum_{\overline{\delta}^k \subset \overline{\delta}^r} \mu D(\delta + \overline{\delta}^k). \tag{4}$$

The numbers $\mu D_\Sigma(\delta)$ defining, in this fashion, the measure μ_Σ, we call them characteristic numbers of the system Σ. We introduce the following special notation for them:

$$\chi(\delta) = \chi_\Sigma(\delta) = \chi_\Sigma(\mu; \delta) = \mu D(\delta).$$

$\chi(\delta)$ is a function defined on the collection Δ of all sets δ. Its values are not entirely arbitrary: if we introduce the function $\omega(\delta, \overline{\delta}) = \omega_\Sigma(\delta, \overline{\delta}) = \omega_\Sigma(\mu; \delta, \overline{\delta})$, setting

$$\omega(\delta, \overline{\delta}^r) = \sum_{k=0}^{r} (-1)^k \sum_{\overline{\delta}^k \subset \overline{\delta}^r} \chi(\delta + \overline{\delta}^k), \tag{5}$$

then, in view of (4), it turns out that

$$\omega(\delta, \overline{\delta}) \geqslant 0. \tag{6}$$

Furthermore,

$$\chi(\delta^0) = 1. \tag{7}$$

The Borel field Ω_{μ_Σ}, on which the measure μ_Σ is defined, we shall denote in the sequel by the symbol $\mathfrak{L}\Sigma$. It contains the Borel field $\mathfrak{B}\Sigma$ and is contained in its turn in the Borel field $\mathfrak{L}'\Sigma$, consisting of all sets of the form

$$A' = A \pm N, \quad A \in \mathfrak{B}\Sigma, \quad \mu N = 0.$$

It is possible that both fields $\mathfrak{C}\Sigma$ and $\mathfrak{C}'\Sigma$ are not contained in $\mathfrak{M}\Sigma$.

No. 4. *The space* M_χ. Let $\{\alpha\}$ be an arbitrary countable set of indices. For M, we take the set of all possible systems $y = (y_\alpha)$, where each of the symbols y_α takes two values x_α and $\bar{x}_\alpha$; for S_{α_0}, we take the set of those systems $y = (y_\alpha)$ for which $y_{\alpha_0} = x_{\alpha_0}$, and for Σ we take the system of all possible S_α. Let χ be an arbitrary real-valued function defined on Δ and satisfying the inequalities (6), where ω is defined by formula (5) and condition (7). We shall show that there exists a measure μ in M, defined in particular on Σ, such that

$$\chi_\Sigma(\mu; \delta) = \chi(\delta). \tag{8}$$

Proof. We define the function μ on the collection of sets $(\Sigma+\bar{\Sigma})_d$ by means of the formula:

$$\mu(D(\delta)\cdot\bar{D}(\bar{\delta})) = \omega(\delta, \bar{\delta}), \tag{9}$$

and extend it further by additivity over the field $(\Sigma+\bar{\Sigma})_{ds}$, consisting of sets of the form

$$E = \overset{\prime}{\bigcup_{n=1}} E_n, \qquad E_n \in (\Sigma + \bar{\Sigma})_d. \tag{10}$$

Such an extension is possible, since for every set $\Sigma \in (\Sigma+\bar{\Sigma})_{ds}$, there exists a representation of the form (10) with pairwise disjoint summands; the fact that the result does not depend upon the choice of the representation (10) follows from the definition (5) of the function ω. The extended function satisfies all of the conditions of the known theorem on the extension of additive functions and therefore can be extended further to a certain measure μ, which will satisfy condition (8).

The minimal measure μ, satisfying condition (8), we shall designate in the sequel by μ_χ, and the space M with the measure μ_χ, we shall designate by M_χ.

No. 5. *Types*. A homomorphic mapping of the space M into the space M' is called an isomorphism if it is one-to-one and if the inverse mapping is homomorphic. An isomorphism is also called an isomorphic mapping. If there exists an isomorphic mapping of the space M *onto* M', then the spaces M and M' are also said to be isomorphic. This definition carries over also to a wider class of objects: to systems of spaces and systems of subsets defined thereon, decompositions, mappings, and functions. Two objects S and S', defined in the systems $\{M_\alpha\}$ and $\{M'_\alpha\}$ of spaces with measure, are called isomorphic, if there exists a system of isomorphisms U_α, connecting the spaces M_α and M'_α, which carries S into S'. For example, two homomorphisms, a homomorphic mapping H of the space M_1 onto the space M_2 and a homomorphic mapping H' of the space M'_1 onto the space M'_2 are isomorphic if there exists an isomorphism U_1 of the space M_1 onto the space M'_1 and an isomorphism U_2 of the space M_2 onto the space M'_2 such that $H' = U_2 H U_1^{-1}$. Another example: a decomposition ζ of the space M is isomorphic to a decomposition ζ' of the space M', if there exists an isomorphic mapping U of the first space onto the second such that for every element C of the decomposition ζ, the set $C' = UC$ is an element of the decomposition ζ'. It is evident that if the homomorphisms H and H' are isomorphic, then the corresponding decompositions ζ_H and $\zeta_{H'}$ are also isomorphic. In turn, isomorphism between two decompositions ζ and ζ' implies an isomorphism of the corresponding factor spaces M/ζ and M'/ζ'.

To every *specified* isomorphism of the space M into the space M', there corresponds an imbedding of the first into the second. Thus we designate the identification of points of M and M', carrying M into a subspace of the space M'.

For the theory of measure, the principal concept is not that of an isomorphism, but the concept of an isomorphism modulo zero: we speak of an isomorphism modulo zero, if upon removing from the corresponding spaces appropriate sets of measure zero, we obtain an isomorphism. Here we have encountered an expression which we shall constantly use throughout this work. Consider two objects S and S', defined in systems $\{M_\alpha\}$ and $\{M'_\alpha\}$ of spaces with measure. We shall say that S and S' are identified modulo zero if it is possible to make S and S' identical by the removal of appropriate sets of measure zero from the spaces M_α and M'_α. In general, the expression modulo zero (mod 0) in assertions concerning the object S means that this assertion is true for some object S' which is identical with S mod 0.

We shall say that objects which are isomorphic mod 0 are of one and the same type. We shall designate the type of the object S by the symbol $\tau(S)$. Properties of an object which are at the same time properties of all objects isomorphic to it mod 0, that is, properties of the *type*, are called invariant.

§2. Lebesgue measure.

No. 1. *Separability*. We shall say that the space M is separable, if there exists a countable system Γ of measurable sets having the following two properties:

($\mathfrak{L}$) For every measurable set $A \subset M$, there exists a set B such that $A \subset B \subset M$, B is identical with A mod 0, and B is an element of the Borel field $\mathfrak{B}\Gamma$ generated by Γ; in other words, $\mathfrak{L}\Gamma = \Omega_\mu$ (see No. 3 of §1).

($\mathfrak{M}$) For every pair of points $x, y, \in M$, there exists a set $G \in \Gamma$ such that either $x \in G$, $y \notin G$, or $x \in G$, $y \notin G$; in other words, $\zeta(\Gamma)$ is a decomposition of the space M into individual points, and $\mathfrak{B}\Gamma$ is the collection of all subsets of the space M (see No. 2 of §1).

Every countable system Γ of measurable sets satisfying conditions (L) and (M) will be called a basis of the space M.

As an example of a separable space, we may present the space M_χ (No. 4 of §1). The system Σ serves as a basis for this space.

If A is a subspace of the space M and Γ is a basis in M, then Γ_A (No. 2 of §1) is a basis in A; consequently, subspaces of separable spaces are separable.

One can infer in a trivial fashion from ($\mathfrak{L}$):

($\mathfrak{L}'$) For every measurable set $A \subset M$, there exists a set $B \subset M$ identical with A mod 0 which belongs to the Borel field $\mathfrak{B}\Gamma$ generated by Γ; in other words, $\mathfrak{L}'\Gamma = \Omega_\mu$. (See No. 3 of §1.)

If M is simply a space with measure, then condition ($\mathfrak{L}'$) is essentially weaker than condition ($\mathfrak{L}$); we shall show, however, that *in a separable space, every countable system of measurable sets satisfying condition* ($\mathfrak{L}'$) *is a basis mod* 0.

Indeed, let Γ' be an arbitrary countable system of measurable sets, satisfying condition ($\mathfrak{L}'$). We take any basis at all in M, call it Γ, and construct, for every set $G \in \Gamma$, a set $G' \in \mathfrak{B}\Gamma'$ which is identical with G mod 0. We then remove from M the set $\bigcup (\bar{G}G' + G\bar{G}')(G \in \Gamma)$, which evidently has measure 0. Then M is converted into a new space, and Γ' becomes a system of sets in this space which, as one can easily see, is a basis.

No. 2. *Completeness*. Let M be a separable space, and let $\mathrm{B} = B_\beta$ be an arbitrary basis in M. We agree to let the symbol A_β stand for one of the two sets $B_\beta, \bar{B}_\beta$; in particular, we shall let $A_\beta(a)$ stand for that one of these sets which contains the point a. Since an intersection of the form

$$\bigcap A_\beta \tag{1}$$

(β runs through all possible values) cannot contain more than one point, it follows that the intersection

$$\bigcap A_\beta(a)$$

consists exactly of the point a. Consequently, every set consisting of one point, and therefore all finite and countable sets, are measurable.

If *all* intersections of the form (1) are non-void, then we say that the space M is complete with respect to the basis B. In this case, the formula

$$A_\beta = A_\beta(a) \tag{2}$$

establishes a one-to-one correspondence between the points of the space M and the systems $\{A_\beta\}$. For example, the space M_χ (No. 4 of §1) is complete with respect to its basis Σ.

If the space M is not complete with respect to its basis B, the question arises as to completing it. A space $\widetilde{M}$ with a specified basis $\widetilde{\mathrm{B}}$, with respect to which $\widetilde{M}$ is complete is called the completion of the space M with respect to the basis B, if M is a subspace of the space $\widetilde{M}$ with outer measure $1 (\widetilde{\mu}_e M = 1$, where $\widetilde{\mu}$ is the measure in the space $\widetilde{M}$), and if the basis $\widetilde{\mathrm{B}}$ induces the basis B in $M (\mathrm{B} = \widetilde{\mathrm{B}}_M)$. Since the systems (A_β) are in one-to-one correspondence with the systems $(\widetilde{A}_\beta)$, where $\widetilde{A}_\beta = \widetilde{B}_\beta$, $\widetilde{M} - \widetilde{B}_\beta$, and consequently with the points of the space $\widetilde{M}$, and since the characteristic numbers of the basis $\widetilde{\mathrm{B}}$ are equal to the corresponding characteristic numbers of the basis B, it follows that the space $\widetilde{M}$ with basis $\widetilde{\mathrm{B}}$ can be completely described in terms of the space M and its basis B. In this fashion, *the completion is unique* in the sense that if there is given another completion of M with respect to B, let us say the space $\widetilde{M}'$ with basis $\widetilde{\mathrm{B}}'$, then there exists an isomorphic mapping of the space $\widetilde{M}$ onto $\widetilde{M}'$, carrying $\widetilde{\mathrm{B}}$ into $\widetilde{\mathrm{B}}'$ and leaving the points of M fixed. On the other hand, *the completion always exists*. As a matter of fact, we set $x_\beta = B_\beta$, $\overline{x}_\beta = \overline{B}_\beta$, and construct the space M_χ, taking the characteristic numbers of the basis Σ equal to the corresponding characteristic numbers of the basis B. The space M_χ, defined in this fashion by means of the space M with the basis B, will be denoted as $\widetilde{M} = \widetilde{M}(\mathrm{B})$, its measure as $\widetilde{\mu}$, and its basis Σ as $\widetilde{\mathrm{B}}$. Formula (2) defines a one-to-one mapping of the space M onto a certain part of the space $\widetilde{M}$. Going in turn from the collections B_d and $\widetilde{\mathrm{B}}_d$ (No. 3 of §1) to the collections $(\mathrm{B}+\overline{\mathrm{B}})_d$ and $(\widetilde{\mathrm{B}}+\overline{\widetilde{\mathrm{B}}})_d$, $(\mathrm{B}+\overline{\mathrm{B}})_{d\sigma}$ and $(\widetilde{\mathrm{B}}+\overline{\widetilde{\mathrm{B}}})_{d\sigma}$, Ω_μ and $\Omega_{\widetilde{\mu}}$, we assure ourselves that this mapping is an isomorphism, and in this fashion we produce an *imbedding* of the space M in $\widetilde{M}(\mathrm{B})$. Under this imbedding, the basis $\widetilde{\mathrm{B}}$ induces the basis B in M, and $\widetilde{\mu}_e(M) = 1$, that is, the space $\widetilde{M}(\mathrm{B})$ with basis $\widetilde{\mathrm{B}}$ is indeed the completion of the space M with respect to B.

In view of the equality $\widetilde{\mu}_e M = 1$, only two cases are possible: either M covers all of $\widetilde{M}$ mod 0, or M is non-measurable in $\widetilde{M}$. In the first case, the pair M, B is identical mod 0 with the pair $\widetilde{M}$, $\widetilde{\mathrm{B}}$, that is, M *is complete* mod 0 with respect to B. We shall show that in the second case, M cannot be complete mod 0 with respect to B. In fact, let M' be such a space and let $\mathrm{B}' = \{B'_\beta\}$ be a basis of M' such that M' is complete with respect to B' and the pair M', B' is identical mod 0 with the pair M, B. Then the mapping

$$B_\beta \to B'_\beta$$

of the system B onto the system B' generates an isomorphic mapping of the space M onto the space M', carrying the basis $\widetilde{\mathrm{B}}$ into the basis B' and leaving fixed the points which belong to both spaces. But M covers all of M' mod 0; consequently, M covers all of $\widetilde{M}$ mod 0.

If the space M is complete mod 0 *with respect to some basis, then it is complete* mod 0 *with respect to every other basis.*

Proof. Let $\mathrm{B} = \{B_\beta\}$ and $\Gamma = \{G_\gamma\}$ be two arbitrary bases in M. We unite them into a new basis $\Pi = \mathrm{B} + \Gamma = \{B_\beta, G_\gamma\}$ and we agree to denote the points of the spaces $\widetilde{M}(\mathrm{B})$ and $\widetilde{M}(\Gamma)$, respectively, by the symbols $\widetilde{\alpha}$ and $\widetilde{f}$:

$$\widetilde{\alpha} = (A_\beta) \quad (A_\beta = B_\beta, \overline{B}_\beta), \quad \widetilde{f} = (F_\gamma) \quad (F_\gamma = G_\gamma, \overline{G}_\gamma).$$

Then points of the space $\widetilde{M}(\Pi)$ are represented in the form:

$$(A_\beta, F_\gamma) = (\widetilde{\alpha}, \widetilde{f}). \tag{3}$$

Like every measurable subset of M, G_γ is the intersection of the set M with a certain measurable subset G'_γ of $\widetilde{M}(\mathrm{B})$:

$$G_\gamma = MG'_\gamma. \tag{4}$$

Choosing sets G'_γ in some fixed fashion which satisfy the equalities (4), we combine the systems $\widetilde{\mathrm{B}}\{\widetilde{B}_\beta\}$ and $\Gamma' = \{G'_\gamma\}$; then we obtain the system $\Pi' = \widetilde{\mathrm{B}} + \Gamma' = \{B_\beta, G'_\gamma\}$, which is a basis in the space $M' = \widetilde{M}(\mathrm{B})$ with exactly the same characteristic numbers as the basis Π in M. Consequently, the mapping

$$\widetilde{B}_\beta \to B_\beta, \quad G'_\gamma \to G_\gamma$$

of the system Π' onto the system Π defines an isomorphism of the space $\widetilde{M}'(\Pi')$ onto the space $\widetilde{M}(\Pi)$. Under this isomorphism, the points of the set M remain invariant, and the set M' goes into the set L_B of all points (3), for which all of the sets F'_γ, corresponding, in view of (4), to sets $F_\gamma (F'_\gamma = G'_\gamma$ if $F_\gamma = G_\gamma$ and $F'_\gamma = \overline{G}'_\gamma (=M'-G'_\gamma)$ if $F_\gamma = \overline{G}_\gamma)$, contain the point $\widetilde{\alpha}$. In fact, the set M' consists precisely of those points $(\widetilde{A}_\beta, F'_\gamma)$ of the space $M'(\Pi')$ for which the point $\widetilde{\alpha}$ belongs to all of the sets F'_γ.

We denote by D_γ the element of the basis $\widetilde{\Pi}$ of the space $\widetilde{M}(\Pi)$ corresponding to the element G_γ of the basis $\Pi (D_\gamma$ is the set of those points (3), for which $F_\gamma = G_\gamma)$ and by E_γ the set of those points (3) for which $\widetilde{\alpha} \in G'_\gamma$. Of course, the set D_γ is measurable in $\widetilde{M}(\Pi)$; but also the set E_γ is measurable in $\widetilde{M}(\Pi)$. In fact, since the basis $\widetilde{\mathrm{B}} = \{\widetilde{B}_\beta\}$ of the space $\widetilde{M}(\mathrm{B})$ and the subsystem of the basis $\widetilde{\Pi}$ of the space $\widetilde{M}(\Pi)$ corresponding to it have the same characteristic numbers, it follows that the outer measure of the set E_γ in $\widetilde{M}(\Pi)$ is equal to the outer measure of the set G'_γ in $\widetilde{M}(B)$; and the same is true of the complements of these sets; hence, measurability of the set G'_γ in $\widetilde{M}(B)$ implies measurability of the set E_γ in $\widetilde{M}(\Pi)$.

It follows from the definitions of the sets L_B, D_γ, and E_γ that

$$\bar{L}_B = \bigcup_\gamma (\bar{D}_\gamma E_\gamma + D_\gamma \bar{E}_\gamma), \tag{5}$$

where all of the complements are taken in the space $\widetilde{M}(\Pi)$. But, evidently

$$MD_\gamma = G_\gamma,$$

and, in view of (4), we also have

$$ME_\gamma = G_\gamma;$$

consequently, the measurable sets $\bar{D}_\gamma E_\gamma + D_\gamma \bar{E}_\gamma$ do not intersect with the set $M \subset \widetilde{M}(\Pi)$, which has outer measure 1 in $\widetilde{M}(\Pi)$. Therefore all of these sets have measure 0. This means, in view of (5), that *the set L_B covers all of the space $\widetilde{M}(\Pi)$* mod 0.

The proof is now quickly brought to its conclusion: from the assertion that M is complete mod 0 with respect to B, we conclude in turn that M is identical mod 0 with $\widetilde{M}(B)$, with L_B (in view of the isomorphism constructed above between $\widetilde{M}(B)$ and L_B), with $\widetilde{M}(\Pi)$, with the set L_Γ, which is obtained in place of L_B if we interchange the places of the bases B and Γ, and finally, with $\widetilde{M}(\Gamma)$ (in view of the isomorphism between $\widetilde{M}(\Gamma)$ and L_Γ, analogous to the isomorphism constructed above between $M(B)$ and L_B). This implies of course that M is complete mod 0 with respect to Γ.

Separable spaces which are complete mod 0 with respect to their bases are called Lebesgue spaces; and the corresponding measures are called Lebesgue measures.

No. 3. *Completeness and measurability.* We have seen, that if the space M is separable, then all of its subspaces A are also separable. We shall show that *if M is a Lebesgue space, then the subspace A is a Lebesgue space if and only if A is measurable in M.*

Proof. We shall show first, that if A is measurable in M, then A is a Lebesgue space. We take an arbitrary basis $\Gamma = \{G_\alpha\}$ in M and take the system Γ_A as a basis in A (No. 2 of §1). There exists a natural mapping of the basis Γ onto the basis Γ_A:

$$G_\alpha \to AG_\alpha,$$

and to this mapping, there corresponds a one-to-one mapping of the space $\widetilde{M}(\Gamma)$ onto the space $\widetilde{A}(\Gamma_A)$. We denote this mapping by U. Evidently,

$$\tilde{\omega}_{\Gamma_A}(\mu_A;\ \delta,\ \bar{\delta}) \leqslant \frac{1}{\mu A}\,\omega_\Gamma(\mu;\ \delta,\ \bar{\delta})$$

(see No. 3 of §1). From this, it easily follows that

$$(\tilde{\mu}_A)_e\,(UX) \leqslant \frac{1}{\mu A}\,\tilde{\mu}_e X,$$

and, consequently, sets of measure zero go into sets of measure zero. Furthermore, the collection $\widetilde{\mathfrak{A}}$ (No. 3 of §1) goes into the collection $\widetilde{\mathfrak{A}}_A$; and this implies that U carries every measurable set into a measurable set. Since M is a Lebesgue space and since A is measurable in M, A is measurable in $\widetilde{M}(\Gamma)$, and the image UA of the set A is measurable in $\widetilde{A}(\Gamma_A)$. But $UA = A$; consequently, A is measurable in $\widetilde{A}(\Gamma_A)$, that is, A is a Lebesgue space.

We shall now prove that if A is a Lebesgue space, then A is measurable in M. We may assume without loss of generality that $\mu_e A = 1$, for one may always replace M by the measurable hull of the set A, and if A is measurable in this hull, then A will be measurable in M as well. Let Γ be a basis in M. We have:

$$A \subset M \subset \widetilde{M}(\Gamma),$$

and since $\mu_e A = 1$ and $\tilde{\mu}_e M = 1$, it follows that

$$\tilde{\mu}_e A = 1.$$

The space $\widetilde{M}(\Gamma)$ is complete with respect to the basis $\widetilde{\Gamma}$; consequently, $\widetilde{M}(\Gamma)$ is the completion of the space A with respect to the basis induced in A by the basis $\widetilde{\Gamma}$ (it coincides with Γ_A). But A is a Lebesgue space; consequently, A is measurable in $\widetilde{M}(\Gamma)$, and hence in M also.

The theorem just proved shows that *the property of a set to* *be measurable does not depend upon its accidental occurence in a Lebesgue space but is actually an intrinsic property of the set itself.*

We note, that the proof given above that A is measurable in M if A is a Lebesgue space retains its validity even if M is not a Lebesgue space but merely a separable space. Thus, a Lebesgue space is a measurable subset of every separable space in which it is imbedded, and we obtain the following theorem:

In order for a separable space to be a Lebesgue space, it is necessary and sufficient that it be absolutely measurable, that is, measurable in every separable space containing it.

No. 4. *Construction of a Lebesgue space.* Let M be a Lebesgue space. Since $\mu M = 1$, there cannot be more than $n - 1$ points with measures exceeding $1/n$, for every n. Consequently, the sets consisting of one point which positive measure form a no more than countably infinite collection and can be numbered in a sequence

$$P_1, P_2, \ldots, \tag{5'}$$

for which $\mu P_1 \geq \mu P_2 \geq \cdots$.

We set

$$m_n(M) = \mu P_n \quad (n = 1, 2, \ldots).$$

if the sequence (5') is infinite, and

$$m_n(M) = \begin{cases} \mu P_n & \text{for } n \leqslant p, \\ 0 & \text{for } n > p \end{cases}$$

if it contains only p members. The numbers $m_n(M)$ are of course invariants of the space M.

If

$$\sum_{n=1}^{\infty} m_n(M) = 1,$$

that is, if the space M consists mod 0 of points of positive measure, then we call the measure μ discrete. In this trivial case, all sets are measurable, and for every $A \subset M$,

$$\mu A = \sum_{P_n \subset A} \mu P_n.$$

If

$$m_n(M) = 0 \quad (n = 1, 2, \ldots), \tag{6}$$

that is, if there are no points of positive measure, then we call the measure μ continuous. In this case, as we shall show, *the space M has the type of the unit interval with ordinary Lebesgue measure.*

First of all, we translate (6) into the language of characteristic numbers. We agree to take the natural numbers $1, 2, \ldots$ as the indices α and consider the sequence of pairs

$$\pi_n = (\delta_n, \bar{\delta}_n) \quad (n = 0, 1, 2, \ldots).$$

(see No. 3 of §1), in which $\delta_0 = \bar{\delta}^0 = \delta^0$, and the pair π_n is obtained from the pair π_{n-1} by the process of adding the number n to one of the sets δ_{n-1}, $\bar{\delta}_{n-1}$. It is evident that in such an admissible sequence, the sets δ_n and $\bar{\delta}_n$ do not intersect, and their union is the set of the first n natural numbers; furthermore, if $n_1 < n_2$, then $\delta_{n_1} \subset \delta_{n_2}$ and $\bar{\delta}_{n_1} \subset \bar{\delta}_{n_2}$. We now set

$$\omega_\Gamma(\mu; \pi_n) = \omega_\Gamma(\mu; \delta_n, \bar{\delta}_n).$$

If the measure μ is continuous, then, for every basis $\Gamma = \{G_n\}$, we have the following limit property:

$$\lim_{n\to\infty} \omega_\Gamma(\mu; \pi_n) = 0, \tag{7}$$

for an arbitrary admissible sequence $\pi_1, \pi_2, \ldots$. Conversely, if the relation (7) holds for a certain basis Γ, and for arbitrary admissible sequences $\pi_1, \pi_2, \ldots$, then the measure μ is continuous.

For the proof, it suffices to refer to the formula

$$\omega_\Gamma(\mu; \pi_n) = \mu(D_\Gamma(\delta_n)\bar{D}_\Gamma(\bar{\delta}_n)) \quad (\pi_n = (\delta_n, \bar{\delta}_n))$$

(see No. 3 of §1) and to the sequence of inclusions

$$D_\Gamma(\delta_1)\bar{D}_\Gamma(\bar{\delta}_1) \supset D_\Gamma(\delta_2)\bar{D}_\Gamma(\bar{\delta}_2) \supset \cdots.$$

From these, it follows that the left side of relation (7) is equal to the measure of the intersection

$$\bigcap_{n=1}^{\infty} (D_\Gamma(\delta_n)\overline{D}_\Gamma(\overline{\delta}_n)),$$

which is either void or consists of a single point.

Let $\chi(\delta)$ be an arbitrary function defined on the collection Δ of all finite sets of natural numbers and satisfying, in addition to conditions (6) and (7) of No. 3, §1, the condition

$$\lim_{n\to\infty} \omega(\pi_n) = 0 \tag{8}$$

for every admissible sequence $\pi_1, \pi_2, \ldots$. We shall designate by L the half-open interval $(0, 1]$ of the real line and by λ ordinary Lebesgue measure on L. (We use curved parentheses if the endpoint is not included and square brackets if the endpoint is included.) We construct a basis $\Lambda = \Lambda_\chi = \{L_n\}$ for L such that

$$\chi_\Lambda(\lambda;\ \delta) = \chi(\delta). \tag{9}$$

As previously, we shall denote by the symbol π_n pairs $(\delta_n, \overline{\delta}_n)$ for which the intersection of the sets $\delta_n, \overline{\delta}_n$ is void and for which the union is one set of the first n natural numbers. We first assign to each pair π_n a certain half-open interval $J(\pi_n) = (\alpha(\pi_n), \beta(\pi_n)]$ in L. Only one pair π_0 corresponds to the value $n = 0$, and we shall take $J(\pi_0) = L$, that is, $\alpha(\pi_0) = 0$, $\beta(\pi_0) = 1$. If all of the half-open intervals $J(\pi_{n-1})$ have already been constructed, then we construct the half-open interval $J(\pi_n) = (\alpha(\pi_n), \beta(\pi_n)]$, corresponding to an arbitrary pair $\pi_n = (\delta_n, \overline{\delta}_n)$, in the following fashion. We consider the pair π_{n-1} into which the pair π_n is transformed when we remove from it the number n; and we define

$$\alpha(\pi_n) = \alpha(\pi_{n-1}), \quad \beta(\pi_n) = \alpha(\pi_{n-1}) + \omega(\pi_n),$$

if $n \in \delta_n$, and

$$\alpha(\pi_n) = \beta(\pi_{n-1}) - \omega(\pi_n), \quad \beta(\pi_n) = \beta(\pi_{n-1}),$$

if $n \in \overline{\delta}_n$. From the sets $J(\pi_n)$ which have been defined in this manner, we define the sets L_n by means of the formula:

$$L_n = \bigcup_{\delta_n \ni n} J(\pi_n).$$

Obviously,

$$J(\pi_n) = D_\Lambda(\delta_n)\overline{D}_\Lambda(\overline{\delta}_n),$$

and

$$\lambda J(\pi_n) = \omega(\pi_n);$$

consequently, the formula

$$\omega_\Lambda(\lambda;\ \delta, \overline{\delta}) = \omega(\delta, \overline{\delta})$$

is clearly valid for those pairs $(\delta, \overline{\delta})$ which we have agreed to designate as π_n. But then it is also valid for arbitrary pairs $(\delta, \overline{\delta})$, for if the largest of the numbers belonging to the sets δ, $\overline{\delta}$, is equal to n, then

$$D_\Lambda(\delta)\overline{D}_\Lambda(\overline{\delta}) = \bigcup_{\substack{\delta_n \supset \delta,\ \overline{\delta}_n \supset \overline{\delta} \\ (\delta_n, \overline{\delta}_n) = \pi_n}} D_\Lambda(\delta_n)\overline{D}_\Lambda(\overline{\delta}_n);$$

in particular, relation (9) is valid.

It follows from condition (8) that Λ is a basis of the space L. We shall show that L *is complete* mod 0 *with respect to* Λ. Indeed, every intersection

$$\bigcup_{n=1}^{\infty} K_n; \quad K_n = L_n,\ \overline{L}_n$$

can be represented in the form

$$\bigcap_{n=1}^{\infty} J(\pi_n), \tag{10}$$

where $\pi_1, \pi_2, \ldots$ is a certain admissible sequence. The intersection (10) is void only when the sequence of half-open intervals $J(\pi_1) \supset J(\pi_2) \supset \cdots$ converges to a point which serves as the common left-hand endpoint of all of these half-open intervals, beginning with a certain one of them. That is, the intersection (10) is void for a countable set of sequences $K_1, K_2, \ldots$. But this means that the space $\widetilde{L}(\Lambda)$ differs from L only on a countable set of points; hence L is measurable in $\widetilde{L}(\Lambda)$, that is, L is complete mod 0 with respect to Λ.

We can now summarize the results obtained in the following fashion: *the unit interval (with the usual measure) is a Lebesgue space in which there exist bases with arbitrary characteristic numbers, satisfying condition (8).*

Since every one-to-one correspondence between two bases, under which the corresponding characteristic numbers are equal, produces an isomorphic correspondence mod 0 between the points of the corresponding Lebesgue spaces, it follows that we have proved the theorem stated at the beginning of No. 4: every Lebesgue space with a continuous measure is isomorphic mod 0 to the unit interval. From this theorem it evidently follows that the numbers $m_n(M)$ form a complete system of invariants of the space M: if

$$m_n(M') = m_n(M'') \quad (n = 1, 2, \ldots),$$

then

$$\tau(M') = \tau(M'').$$

Therefore, *the Lebesgue space M is isomorphic* Mod 0 *to the space consisting of an interval of length*

$$m_0(M) = 1 - \sum_{n=1}^{\infty} m_n(M)$$

with ordinary Lebesgue measure and a sequence of points with measure $m_n(M)$ $(n=1, 2, \ldots)$.

No. 5. *Minimal properties of Lebesgue measure.*

Lemma. *Let U be a one-to-one mapping of the Lebesgue space M into the separable space M'. Suppose that under this mapping, the inverse image of every measurable set is measurable, and that the inverse image of a set of measure zero always has measure zero, and that the inverse image of a set of positive measure always has positive measure. Then M' is a Lebesgue space, and the image of every measurable set is measurable; in particular, the image UM of the space M covers* mod 0 *the entire space M'.*

Proof. We shall first prove this lemma under the assumption that M' a Lebesgue space. Since U obviously establishes a one-to-one correspondence between the points of the spaces M and M' which have positive measure, we can remove these points at the very beginning, and, in view of the results of the preceding No., we may suppose that M and M' *are intervals with ordinary Lebesgue measure.* Let Γ' be the set of all intervals in M' having rational endpoints. The inverse image $J = U^{-1}J'$ of the interval $J' \in \Gamma'$ is a measurable set, and hence there exists a *Borel* set B_J which contains J and is identical with J mod 0. The set

$$\bigcup_J (B_J - J) \quad (UJ = J' \in \Gamma'),$$

has measure zero and therefore can be included in a *Borel* set of measure zero, say B_0. Let us denote by V the mapping induced by the mapping U on the Borel set $B = M - B_0$. If B is regarded as a new space, then V is a mapping of B into M' which is identical mod 0 with U.

We shall first show that *for the mapping V,* the image of every measurable set is measurable. Since $B_J - J \subset B_0$, the intersection JB coincides with the intersection B_JB and is hence a Borel set; in other words, the inverse images under V of all intervals $J' \in \Gamma'$ are Borel sets. But in this case, V is a Baire function,* and the *images* of all Borel sets are also Borel sets.* Thus, if A is a *Borel* set, then its image VA is measurable. If A is a set *of measure zero,* then it can be included in a Borel set of measure zero, say B_A. The image VB_A of this latter set, according to what we have proved, is measurable; furthermore, VB_A has measure zero, since its inverse image B_A has measure zero. However, $VA \subset VB_A$; consequently, the set VA is measurable. Finally if A is *an arbitrary measurable set,* then it can be represented as the union of a certain Borel set A_1 and a certain set A_0 of measure zero, and we have: $VA = VA_1 + VA_0$. Hence in this case also, VA is measurable.

We return now to the mapping U. Since $M' - UB = M' - VB$, and the image VB of the space B is measurable, it follows that *the set $B_0' = M' - UB$ is measurable in M; furthermore, its measure is equal to zero,* for its inverse image $U^{-1}B_0'$ coincides with B_0. From this it follows that for every set $A \subset M$, the first member of the decomposition

$$UA = U(B_0A) + V(BA) \tag{11}$$

is always measurable $(U(B_0A) \subset B_0')$. But, if A is measurable in M, it follows that BA is measurable in B, and by what has been proved above, the second member of the dissection (11) is also measurable; hence the entire set UA is measurable.

We have proved that the image of every measurable set is measurable. The fact that UM covers mod 0 all of M' has already

*See [6], 39.

tacitly been proved by us; we have seen, indeed, that UB covers mod 0 all of M'.

Suppose now that M' is an arbitrary *separable* space. We construct an arbitrary completion $\widetilde{M}'$ of the space M' and shall consider U as a mapping of the space M into $\widetilde{M}'$. It is not difficult to see that this mapping satisfies all of the conditions of the lemma. Consequently, the image of every measurable set is measurable in $\widetilde{M}'$, and therefore, measurable in M'. Furthermore, UM covers mod 0 all of $\widetilde{M}'$. But $UM \subset M' \subset \widetilde{M}'$. Consequently, M' covers mod 0 all of $\widetilde{M}'$, that is, M' is a Lebesgue space. At the same time, we see that UM covers mod 0 all of the space M'.

As an immediate consequence of the lemma just proved, we have:

Theorem on isomorphisms. A one-to-one homomorphism of a Lebesgue space into a separable space is an isomorphism.

In its turn, the following theorem is easily obtained from the theorem on isomorphisms:

A proper part of a Lebesgue measure cannot be the measure on a separable space. A proper extension of the measure on a separable space cannot be a Lebesgue measure.

Proof. Let M be a Lebesgue space with measure μ, and M' a separable space into which M is changed if its measure is replaced by a certain part thereof, μ'. Evidently, the identical mapping of M onto M' satisfies all of the conditions of the theorem on isomorphisms. Consequently, $\Omega_\mu = \Omega_{\mu'}$ and $\mu' = \mu$.

Finally, from this theorem follows:

Theorem on bases. Every countable system Γ of measurable subsets of a Lebesgue space which satisfies property ($\mathfrak{M}$) (No. 1 of §2) is a basis.

Proof. The measure induced by the measure μ on the collection $\mathfrak{C}\Gamma$ (No. 1 of §2) is a separable part of the measure μ. Consequently, $\mathfrak{C}\Gamma = \Omega_\mu$, and Γ is a basis.

No. 6. *The metric structure associated with a Lebesgue space.* We set $\rho(A,B) = \mu(\bar{A}B + A\bar{B})$ for any two sets $A,\ B \subset M$ which are measurable. The function ρ satisfies the axiom of symmetry and the triangle axiom and will satisfy the axiom of identity, if we agree to understand identity as identity mod 0. Going from the collection Ω_μ to the collection $\underline{\Omega}(M)$ of *classes* of measurable sets which are identical mod 0, we thus obtain a metric space. Union and intersection of measurable sets lead to the same operations on elements of the collection $\underline{\Omega}(M)$. The collection $\underline{\Omega}(M)$ with the metric ρ defined in it and with the operations of union and intersection of elements is a metric structure, which we regard as being associated with the space M.

We shall not list all of the properties of a metric structure $\underline{\Omega}(M)$ which are necessary for its independent axiomatic definition, and merely note that the space $\underline{\Omega}(M)$ is complete and separable in the sense of the theory of metric spaces. Its completeness is a simple consequence of the Riesz-Fischer theorem, and separability of the space M: if Γ is any basis in M, then the field $(\Gamma+\bar{\Gamma})_{ds}$ generated by Γ is countable and is everywhere dense in $\underline{\Omega}(M)$.

By definition, two metric structures $\underline{\Omega}(M)$ and $\underline{\Omega}(M')$ are isomorphic to each other, if there exists a one-to-one correspondence carrying the first of them onto the second -- an isomorphic mapping -- which leaves invariant the metric ρ and the operations of union and intersection of elements. To every mapping of the space M onto the space M' which is an isomorphism mod 0, there corresponds a natural isomorphic mapping of the metric structure $\underline{\Omega}(M)$ onto the metric structure $\underline{\Omega}(M')$. It turns out that *every isomorphism of the metric structure $\underline{\Omega}(M)$ onto the metric structure $\underline{\Omega}(M')$ is generated in this sense by a mapping of the space M onto the space M' which is an isomorphism* mod 0.

Proof. Let $\underline{U}$ be an arbitrary isomorphic mapping of the metric structure $\underline{\Omega}(M)$ onto the metric structure $\underline{\Omega}(M')$. We choose any basis Γ in M and consider the system of elements of the structure $\underline{\Omega}(M)$ which correspond to the sets in Γ. The images of these elements in the structure $\underline{\Omega}(M')$ under the mapping $\underline{U}$ are classes of measurable sets of the space M'. In each of these classes, we select a certain set; in this way, we obtain a certain countable system Γ' of measurable sets of the space M', satisfying, evidently, condition ($\mathfrak{L}'$) and therefore being a basis mod 0 (No. 1 of §2). Corresponding characteristic numbers of the systems Γ and Γ' are equal; consequently, the correspondence between the systems Γ and Γ' corresponds to an isomorphism mod *0* of the space M onto the space M' (No. 2 of §2), and it is easy to convince one's self that the isomorphism of $\Omega(M)$ onto $\Omega(M')$ engendered thereby is actually the original isomorphism $\underline{U}$.

The argument set forth above shows at the same time that *any two isomorphisms* mod *0* *of the space* M *onto the space* M'*, generating one and the same isomorphic mapping of the structure* $\Omega(M)$ *onto* $\Omega(M')$ *are identical* mod *0*.

No. 7. *Measures in metric spaces.* The goal of the present No. is to show that all of the most important measures are actually Lebesgue measures. Its content tends in the direction of the abstract theory of measure, and its results will not be used in the present work.

Let R be a metric space, separable and complete in the sense of the theory of metric spaces. Let us suppose that there is introduced into R a certain measure μ. We shall show that *if all open sets are measurable with respect to, and form a system satisfying condition* ($\mathfrak{C}$) *of* No. 1, §2, *then* μ *is a Lebesgue measure.*

Proof. We suppose first that R is the unit interval with its usual metric, that is, that μ is a certain Lebesgue-Stieltjes measure. Evidently, we are at liberty to assume that there are no points of positive measure. We designate by the symbol I_x the closed interval with endpoints $0, x$ and set $f(x) = \mu I_x$. The function f produces an isomorphism mod *0* of the space R with measure μ onto the unit closed interval with ordinary Lebesgue measure. Consequently, μ is a Lebesgue measure.

In order to prove the theorem in the general case, we shall now show that every measure of the type under consideration is isomorphic mod *0* to a certain Lebesgue-Stieltjes measure. For this, we note that according to a well-known theorem of the descriptive theory of sets*, R can be represented as the image of a Borel subset of the closed unit interval under a certain one-to-one and continuous mapping. With the help of this mapping, which we designate by U, we transfer the measure μ from R onto the closed unit interval. Then we obtain a measure μ', isomorphic mod *0* to the measure μ, and the proof will be completed, if we make it clear that μ' is a Lebesgue-Stieltjes measure, that is, that all open subsets of the closed unit interval are measurable

*See [6], 35.

with respect to μ' and form a system satisfying condition ($\mathfrak{C}$) of No. 1, 2. But their measurability follows from the fact that U carries every Borel set into a Borel set*, and condition ($\mathfrak{C}$) is fulfilled in view of the fact that U^{-1} carries every open set into an open set.

§3. Measurable decompositions.

No. 1. *Canonical system of measures.* Let ζ be an arbitrary decomposition of the Lebesgue space M. Let us suppose that by means of the introduction of certain measures μ_C, the elements C of this decomposition themselves are turned into spaces with measure. We shall say that the system $\{\mu_C\}$ is canonical with respect to ζ, if

1) μ_C is a Lebesgue measure for every mod *0* point C of the factor-space M/ζ;

2) for every measurable set $A \subset M$, a) the set AC is measurable in its space C for every mod *0* point $C \in M/\zeta$, b) $\mu_C(AC)$ is a measurable function of the point $C \in M/\zeta$, and c)

$$\mu A = \int_{M/\zeta} \mu_C(AC)\, d\mu_\zeta.$$

It is immediately obvious that *if the system* $\{\mu_C\}$ *is canonical with respect to* ζ*, then, for every basis* Γ *of the space* M*, the system* Γ_C*, consisting of the sets* $G_C = GC$*,* $G \in \Gamma$*, serve as a basis in the space* C*, for every* mod *0* *point* $C \in M/\zeta$. This is an immediate consequence of Conditions 1) and 2a) and the theorem on bases (No. 5 of §2), for every system Γ_C evidently satisfies condition ($\mathfrak{M}$) in its own space. From this theorem, it is not difficult to infer that *a canonical system of measures is defined essentially uniquely by its decomposition* ζ, that is, if any two systems of measures $\{\mu_C\}$ and $\{\mu'_C\}$ are canonical with respect to ζ, then $\mu_C = \mu'_C$ for all mod *0* points $C \in M/\zeta$. Indeed, designating by Z the inverse image of the measurable set $X \subset M/\zeta$ under the homomorphism H_ζ (No. 2 of §1), we shall have, for every measurable set $A \subset M$:

$$\int_X \mu_C(AC)\, d\mu_\zeta = \int_{M/\zeta} \mu_C(AZC)\, d\mu_\zeta = \mu(AZ) =$$
$$= \int_{M/\zeta} \mu'_C(AZC)\, d\mu_\zeta = \int_X \mu'_C(AC)\, d\mu_\zeta,$$

*See [6], 35.

from which it follows that $\mu_C(AC) = \mu'_C(AC)$ for all mod 0 points $C \in M/\zeta$. In order to convince one's self of the truth of our assertion, it is now sufficient to have A run through the collection Γ_d (see No. 3 of §1), generated by an arbitrary basis Γ of the space M.

Having established in this way the *uniqueness* of a canonical system of measures, we now turn to the problem of its *existence*.

We agree to call a decomposition ζ measurable, if there exists a *countable* system Σ of measurable sets -- a basis of the decomposition ζ-- such that $\zeta(\Sigma) = \zeta$. As an example of a measurable decomposition, we offer the decomposition of M into the inverse images of points under the mapping defined on M by a measurable real function or a finite or countably infinite system of measurable real functions. As a basis of such a decomposition, one can take, for example, the inverse images of open intervals with rational endpoints. A different example: the decomposition ζ_H (No. 2 of §1), corresponding to the homomorphism H of the space M into a different Lebesgue space M', is always measurable, since the inverse image of an arbitrary basis of the space M' is certainly a basis of the decomposition ζ_H.

In order for the decomposition ζ to possess a canonical system of measures, it is necessary and sufficient that it be measurable.

Proof of necessity. Let $\{\mu_C\}$ be a system of measures which is canonical with respect to ζ. Let us take in M a basis Γ and, for an arbitrary set A of the collection Γ_d (see No. 3 of §1) and an arbitrary point $x \in C$:

$$\varphi_A(x) = \mu_C(AC).$$

Since each of the functions $\mu_C(AC)$ is defined for all mod 0 points $C \in M/\zeta$ and since the set of these functions is countable, it follows that, ignoring a certain ζ-set of measure zero, we can consider that all of the functions ϕ_A are defined on the entire space M. Let ζ' be the measurable decomposition generated by these functions. We shall show (and by this the necessity of our condition will be proved) that $\zeta = \zeta'$.

Let us assume that $\zeta \neq \zeta'$. Since the functions ϕ_A are constant on each element of the decomposition ζ, it follows from this that there exists two different elements C_1 and C_2 of the decomposition ζ, on which $\mu_{C_1}(AC_1) = \mu_{C_2}(AC_2)$ for every set $A \in \Gamma_d$. But this means that the bases Γ_{C_1} and Γ_{C_2}, which are induced by the basis Γ in the spaces C_1 and C_2, have the same characteristic numbers, in virtue of which the spaces C_1 and C_2 themselves are connected by a natural isomorphism mod 0. The corresponding elements of these spaces are obviously distinct, since they lie in different sets of the decomposition ζ, and furthermore, every set $A \in \Gamma$ which contains one of them contains the other. The absurdity of this conclusion shows that $\zeta = \zeta'$.

We note that the reasoning used above does not use property 2c) of canonical systems of measures. Therefore *every decomposition of a Lebesgue space which satisfies conditions* 1), 2a), *and* 2b) *is measurable.*

Proof of sufficiency. We shall divide this proof into several parts.

A) Let Γ be a certain basis in M, and let $A \in \Gamma_d$. We denote by Z the inverse image of a measurable set $X \subset M/\zeta$ under the mapping H_ζ. Since, for fixed A, the function $\mu(AZ)$ is a completely additive function of the set X, which vanishes along with $\mu_\zeta X$, there exists a measurable function ϕ_A defined on M/ζ such that for every measurable subset X of M/ζ,

$$\mu(AZ) = \int_X \varphi_A(C)\, d\mu_\zeta \tag{1}$$

(See [7], p. 168). This function is defined by the set A in an essentially unique fashion.

We choose a certain fixed function ϕ_A corresponding to the set $A \in \Gamma_d$, and we set

$$\varphi_A(C) = \nu_C A.$$

ν_C is a set function defined for all $A \in \Gamma_d$. We wish to extend it to a Lebesgue measure.

B) Let us first suppose that M is complete with respect to Γ. Then the problem of extending the function ν_C to a Lebesgue measure is none other than the problem of constructing the measure μ_χ according to the function $\chi(\delta) = \chi^C(\delta) = \nu_C D(\delta)$ (see No.

4 of §1 and No. 2 of §2), and, therefore, is solvable if conditions (6) and (7) of §1 are satisfied. But these conditions are satisfied for all mod 0 points $C \in M/\zeta$, for, if we denote the function ω corresponding to $\chi = \chi^C$ in virtue of formula (5) of §1 by the symbol ω^C and if we use the linearity of the formula referred to, then we obtain:

$$\int_X \omega^C(\delta, \overline{\delta})\, d\mu_\zeta = \mu(D(\delta)\cdot\overline{D}(\overline{\delta})\cdot Z) \geqslant 0.$$

Besides this,

$$\int_X \chi^C(\delta^0)\, d\mu_\zeta = \int_X \nu_C(M)\, d\mu_\zeta = \mu(MZ) = \mu Z.$$

Thus, if M is complete with respect to Γ, then the extension of the function ν_C to a Lebesgue measure is possible for every mod 0 point $C \in M/\zeta$. But this result retains its validity also in the general case. It is not difficult to verify this, going from the space M with the basis Γ to the space $\widetilde{M}(\Gamma)$ with the basis $\widetilde{\Gamma}$ and from the system Σ to the system $\widetilde{\Sigma}$, consisting of all sets of the system Σ and the set M.

C) We denote by M_C the space into which M is changed when we replace the measure μ by the measure ν_C. M_C is a Lebesgue space for every mod 0 point $C \in M/\zeta$. We shall show that if A is a measurable subset of M, then

a′) A is measurable in M_C for every mod 0 point $C \in M/\zeta$,

b′) $\nu_C A$ is a measurable function of the point $C \in M/\zeta$, and

c′) for every measurable set $X \subset M/\zeta$,

$$\mu(AZ) = \int_X \nu_C A\, d\mu_\zeta.$$

Assertions a′), b′), and c′) are evidently true, if $A \in \Gamma_d$; in view of the linearity of formula (4) of No. 3 of §1, they are hence valid also if $A \in (\Gamma+\overline{\Gamma})_d$. If $A \in (\Gamma+\overline{\Gamma})_{d\sigma}$, then A can be represented in the form:

$$A = \bigcup_{n=1}^{\infty} A_n, \quad A_n \in (\Gamma+\overline{\Gamma})_d,$$

where the sets A_n are pairwise disjoint, and assertion b′) follows from the formula

$$\nu_C A = \sum_{n=1}^{\infty} \nu_C A_n, \tag{2}$$

while c′) follows from (2) and the relations

$$\mu(AZ) = \sum_{n=1}^{\infty} \mu(A_n Z) = \sum_{n=1}^{\infty} \int_X \nu_C A_n\, d\mu_\zeta = \int_X \Big(\sum_{n=1}^{\infty} \nu_C A_n\Big)\, d\mu_\zeta.$$

If $A \in (\Gamma+\overline{\Gamma})_{d\sigma\delta}$, then A can be represented in the form

$$A = \bigcap_{n=1}^{\infty} A_n, \quad A_n \in (\Gamma+\overline{\Gamma})_{d\sigma},$$

where $A_1 \supset A_2 \supset \cdots$, and b′) follows from the relation

$$\nu_C A = \lim_{n\to\infty} \nu_C A_n, \tag{3}$$

and c′) follows from (3) and the relations

$$\mu(A_n Z) = \int_X \nu_C A_n\, d\mu_\zeta, \quad 0 \leqslant \nu_C A_n \leqslant 1,$$

the latter of which holds for all mod 0 points $C \in M/\zeta$. Finally, if A is an arbitrary measurable set in M, then there exist sets $B_1 \in (\Gamma+\overline{\Gamma})_{d\sigma\delta}$ and $B_2 \in (\Gamma+\overline{\Gamma})_{d\sigma\delta}$ such that

$$B_1 \supset A, \quad \mu B_1 = \mu A; \quad B_2 \supset \overline{A}, \quad \mu B_2 = \mu\overline{A},$$

and we thus have:

$$\int_X (\nu_C B_1 + \nu_C B_2)\, d\mu_\zeta = \int_X \nu_C B_1\, d\mu_\zeta + \int_X \nu_C B_2\, d\mu_\zeta = \mu(B_1 Z) + \mu(B_2 Z) = \mu Z.$$

For this reason, we have, for all mod 0 points $C \in M/\zeta$,

$$\nu_C B_1 + \nu_C B_2 = 1,$$

that is, a′) is valid. We see also, that for all mod 0 points $C \in M/\zeta$,

$$\nu_C A = \nu_C B_1,$$

from which the validity of b′) and c′) also follows.

D) The canonical system of measures μ_C which we wish to construct is obtained from the family of measures ν_C in an extremely simple fashion: μ_C is the measure induced by the measure ν_C in the subspace C of the space M_C. Evidently, the proof of the present theorem will be completed if we show that fou every mod 0 point $C \in M/\zeta$, the set C is measurable in the space M_C and

$\nu_C C = 1$.

Let $A \in (\Sigma+\bar{\Sigma})_d$ and let $A' = H_\zeta A$. Since the set A is measurable with respect to μ, it is also measurable with respect to all mod 0 measures ν_C, and we have:

$$\mu(AZ) = \int_X \chi_{A'}(C)\, d\mu_\zeta,$$

where $\chi_{A'}$ is the characteristic function of the set A'. Consequently, for all mod 0 points $C \in M/\zeta$,

$$\nu_C(A) = \chi_{A'}(C), \qquad (4)$$

and since the collection $(\Sigma+\bar{\Sigma})_d$ is countable, it follows that *all* relations (4), corresponding to different sets $A \in (\Sigma+\bar{\Sigma})_d$, are valid for all mod 0 points $C \in M/\zeta$. Furthermore, every set C can be represented in the form:

$$C = \bigcap_{n=1}^{\infty} A_n, \quad A_n \in (\Sigma+\bar{\Sigma})_d,$$

where $A_1 \supset A_2 \supset \cdots$. Consequently, if we discard a set of measure zero, it appears, that the set C is measurable with respect to ν_C and

$$\nu_C C = \lim_{n\to\infty} \nu_C A_n = 1,$$

for, in view of (4), $\nu_C A_n = 1$ for all $n = 1, 2, \cdots$.

No. 2. *Factor-spaces and homomorphisms of Lebesgue spaces.* Let Σ be an arbitrary basis of the measurable decomposition ζ. We shall show that *the system Σ', into which the system Σ goes under the homomorphism H* (see No. 2 of §1), *is a basis of the factor space M/ζ*. It is plain that the system Σ' satisfies condition $(\mathfrak{M})$ of No. 1 of §2. In order to show that it also satisfies condition $(\mathfrak{C})$, we denote by μ' the part of the measure μ_ζ, defined on the collection $\mathfrak{C}\Sigma'$, that is, we set $\mu' = (\mu_\zeta)_{\Sigma'}$ (see No. 3 of §1). μ' is just that measure into which the homomorphism H_ζ carries the measure μ_Σ, and $\Omega_{\mu'} = \mathfrak{C}\Sigma'$ is just that collection into which H_ζ carries the collection $\mathfrak{C}\Sigma$. In the space M/ζ, we replace the measure μ_ζ by the measure μ', and obtain a space M', for which the system Σ' serves as a basis. We shall establish (and by this our assertion will be proved) that $\mu' = \mu_\zeta$.

For this purpose, we return to the proof, set forth in the previous No., that canonical systems of measures exist. It is not

difficult to verify that this proof remains valid if we replace it in the measure μ_ζ by the measure μ'. Consequently, for every measurable subset A of M, the function $\mu_C(AC)$ is measurable with respect to μ'. But if A is a ζ-set, then the function $\mu_C(AC)$ obviously coincides with the characteristic function of the set $H_\zeta A$, in view of which this set itself must be measurable with respect to the measure μ'. Thus, all sets of the form $H_\zeta A$, where A is a measurable ζ-set, are measurable with respect to μ', and this means that $\mu' = \mu_\zeta$.

We shall now show that *the factor space of a Lebesgue space with respect to a measurable decomposition is a Lebesgue space.* We have already proved that M/ζ is a separable space. As a basis for M/ζ, one may take the image of any basis Σ of the decomposition ζ under the homomorphism H_ζ. We extend the system Σ by adding some countable system of measurable sets, obtaining in this way a basis Γ of the space M, and denote by $\tilde{\Sigma}$ the part of the basis $\tilde{\Gamma}$ of the space $\tilde{M} = \tilde{M}(\Gamma)$ which corresponds to the system Σ (see No. 2 of §2). Furthermore, we consider the decomposition $\tilde{\zeta} = \tilde{\zeta}(\tilde{\Sigma})$ of the space $\tilde{M}$ (see No. 2 of §1) and the factor-space corresponding thereto, $\tilde{M}/\tilde{\zeta}$. The system $\tilde{\Sigma}'$, into which the system $\tilde{\Sigma}$ is carried by the homomorphism $H_{\tilde{\zeta}}$, is a basis of the space $\tilde{M}/\tilde{\zeta}$, and it is evident that $\tilde{M}/\tilde{\zeta}$ is complete with respect to $\tilde{\Sigma}'$. Thus, $\tilde{M}/\tilde{\zeta}$ is a Lebesgue space. But, the decomposition induced in M by the decomposition $\tilde{\zeta}$, is precisely ζ, and if we carry the element C of the decomposition ζ into that element $\tilde{C}$ of the decomposition $\tilde{\zeta}$ for which $C = M\tilde{C}$, then we obtain an *imbedding* of the space M/ζ into $\tilde{M}/\tilde{\zeta}$. This imbedding has the property that $H_{\tilde{\zeta}} M = M/\zeta$, and since M covers mod 0 all of $\tilde{M}$, it follows that M/ζ covers mod 0 all of $\tilde{M}/\tilde{\zeta}$. Consequently, M/ζ is a Lebesgue space.

Suppose now that H is an arbitrary homomorphism of the Lebesgue space M into the *separable* space M'. We have:

$$H = H_{\zeta_H} T_H,$$

where T_H is a one-to-one homomorphism of the factor space M/ζ into M' (No. 2 of §1). Since the decomposition ζ_H is measurable, M/ζ_H is a Lebesgue space, and, applying the theorem on isomorphisms (No. 5 of §2), we obtain the following theorem on homomorphisms:

If H is a homomorphism of a Lebesgue space into a separable space, then T_H is an isomorphism, so that the homomorphisms H and H_{ζ_H} are isomorphic.

In particular:

A homomorphic image of a Lebesgue space in a separable space is a Lebesgue space, and

under a homomorphic mapping H of a Lebesgue space into a separable space, the images of measurable ζ_H-sets are measurable.

No. 3. *Operations on decompositions.* Let M be an arbitrary set and let ζ and ζ' be decompositions of M. We agree to write

$$\zeta<\zeta', \quad \zeta'>\zeta, \tag{5}$$

if $\zeta \neq \zeta'$ and ζ' is a subdecomposition of the decomposition ζ (that is, all elements of the decomposition ζ, and consequently all ζ-sets, are ζ'-sets). The relation $\gtrless$ defined in this way makes the collection of all decompositions of the set M into a partially ordered set, which is a *complete structure* and which we denote by the symbol Z. In this structure, we shall call the least upper bound of a system of decompositions its product, and the greatest lower bound, as usual, its intersection. The product is denoted by the symbol Π, and the intersection by the symbol $\cap$. If $\{\zeta_\alpha\}$ is an arbitrary system of decompositions, the elements of the decomposition $\Pi\zeta_\alpha$ are sets of the form

$$C = \bigcap_\alpha C_\alpha,$$

where C_α is an element of the decomposition ζ_α. With regard to the decomposition $\cap\zeta_\alpha$, we can say that two points x and x' lie in the same element if and only if there exists a finite sequence $x_1, x_2, \ldots, x_n$ of points such that in the chain

$$x,\ x_1,\ x_2,\ \ldots,\ x_n,\ x'$$

every neighboring pair of points belong to one and the same elements of one of the decompositions ζ_α.

The structure Z possesses both a zero and a unit element: zero is the decomposition whose sole element is the set M, and the unit is the decomposition of the set M into individual points.

We now suppose that M is a *Lebesgue space*, and denote by Z_M the set of all *measurable* decompositions of the space M. Being a subset of the set Z, Z_M is also a partially ordered set; however, generally speaking, Z_M is not a substructure of the structure Z. For, the product of a finite or countably infinite system of measurable decompositions is again a measurable decomposition (in order to obtain a basis of the decomposition $\Pi\zeta_\alpha$, it is sufficient to unite bases of the decompositions ζ_α); but the intersection of even two measurable decompositions can be non-measurable. The set Z_M is not a structure even if we forget about the operations Π and $\cap$ in the structure Z and try to introduce new operations in Z_M, using the fact that Z_M is a partially ordered set, for two elements of this partially ordered set may fail to have a greatest lower bound. In order to obtain a structure from Z_M, one must go over *to the classes of decompositions which are identical* mod 0.

We shall designate the class of a measurable decomposition by the same letter as the decomposition itself, but with underlining. The set of all such classes will be denoted by the symbol $\underline{Z}_M$. We agree to write

$$\underline{\zeta}<\underline{\zeta}', \quad \underline{\zeta}'>\underline{\zeta},$$

if $\underline{\zeta} \neq \underline{\zeta}'$ and there exist decompositions $\zeta \in \underline{\zeta}$, $\zeta' \in \underline{\zeta}'$, such that (5) holds. In this way, we make $\underline{Z}_M$ into a partially ordered set. We shall show that *$\underline{Z}_M$ is a complete structure.*

We have already seen that the product of an arbitrary finite or countably infinite system of measurable decompositions is a measurable decomposition. Therefore every finite or countably infinite system of classes $\underline{\zeta}_\alpha$ possesses a least upper bound: this is the class of the product Π_{ζ_α}, constructed from arbitrary decompositions ζ_α of the classes $\underline{\zeta}_\alpha$. But one can assert more: *every non-void system of classes ζ_α has a least upper bound in $\underline{Z}_M$.* For the proof, we select in each class $\underline{\zeta}_\alpha$ a certain decomposition α_ζ, denote by P the collection of all ζ_α-sets corresponding to all possible decompositions ζ_α, choose in P, using the separability of the metric structure $\underline{\Omega}_M$ (No. 6 of §2), an arbitrary countable system Σ dense in P, and set: $\zeta = \zeta(\Sigma)$. The decomposition ζ is measurable, and we assert that its class $\underline{\zeta}$ is the least upper bound of the system of classes $\underline{\zeta}_\alpha$. That is,

1) $\underline{\zeta} \geq \underline{\zeta}_\alpha$ for all α;

2) if $\zeta' \geq \zeta_\alpha$ for all α, then $\underline{\zeta}' \geq \underline{\zeta}$.

In order to prove assertion 1), we consider an arbitrary basis $\Sigma_\alpha = \{S_{\alpha\beta}\}$ of the decomposition ζ_α. Since the system Σ is dense in P, it is possible to find, for every set $S_{\alpha\beta}$, a set $S'_{\alpha\beta}$ of the Borel field $\mathfrak{B}\Sigma$ which is identical with it mod 0. It is evident that the decompositions $\zeta'_\alpha = \zeta(\Sigma'_\alpha)$, where $\Sigma'_\alpha = \{S'_{\alpha\beta}\}$, belongs to the same class $\underline{\zeta}_\alpha$ as ζ_α. (The decomposition ζ'_α can differ from ζ_α only on the set

$$\bigcup_\beta (\bar{S}_{\alpha\beta} S'_{\alpha\beta} + S_{\alpha\beta} \bar{S}'_{\alpha\beta}),$$

which has measure zero.) But $\zeta \geq \zeta'_\alpha$; therefore $\underline{\zeta} \geq \underline{\zeta}_\alpha$.

In order to prove assertion 2), we choose an arbitrary decomposition ζ' in the class $\underline{\zeta}'$. Since $\underline{\zeta}' \geq \underline{\zeta}_\alpha$ for all α, it follows that one can find for every set $S \in \Sigma$ a ζ'-set S' which is identical with it mod 0. It is evident that the decomposition $\zeta'_1 = \zeta(\Sigma')$, where Σ' is the system of all sets S', belongs to the same class $\underline{\zeta}$ as ζ (ζ'_1 can differ from ζ only on the set $\bigcup(\bar{S}S' + S\bar{S}')$, which has measure zero.) But $\zeta' \geq \zeta'_1$; consequently $\underline{\zeta}' \geq \underline{\zeta}$.

We have proved that in $\underline{Z}_M$, every non-void system of elements $\underline{\zeta}_\alpha$ has a least upper bound. From this, it is easy to show that every non-void system of elements $\underline{\zeta}_\alpha$ possesses also a *greatest lower* bound in $\underline{Z}_M$. Indeed, we consider the collection of elements ζ, which satisfy the relation $\zeta \leq \zeta_\alpha$ for all α. This collection of elements is evidently non-void, for it contains the null class (corresponding to the null decomposition of the structure Z). Consequently, it possesses a least upper bound in $\underline{Z}_M$, which will be the greatest lower bound of the system $\{\underline{\zeta}_\alpha\}$. Thus, $\underline{Z}_M$ is a genuine complete structure.

The structure $\underline{Z}_M$ has a zero (the null class) and a unit (the class of the unit decomposition, which is measurable in view of the separability of the space M). Two measurable decompositions ζ_1 and ζ_2 whose product is identical mod 0 with the unit decomposition will be referred to as mutually complementary. It is plain that the decompositions ζ_1 and ζ_2 are mutually complementary if and only if the space M has a basis mod 0 (that is, a countable system of measurable sets, enjoying property ($\mathfrak{L}'$) (No. 1 of §3)), which consists of ζ_1-sets and ζ_2-sets.

Now, let ζ be any decomposition *whatsoever* of the space M. We consider the set of all measurable decompositions ζ' that satisfy the relation $\zeta' \geq \zeta$. The collection of the corresponding classes has a greatest lower bound in $\underline{Z}_M$, which we denote by $\underline{\zeta}_1$. A decomposition ζ_1 belonging to the class $\underline{\zeta}_1$ can be characterized mod 0 as the finest of all the measurable decompositions for which ζ is a subdecomposition. We call it the measurable hull of the decomposition ζ. The measurable hull of a decomposition ζ is defined in essentially a unique manner. From the foregoing exposition, it follows that it can be found by the formula:

$$\zeta_1 = \zeta(\Sigma),$$

where Σ is an arbitrarily chosen countable system of measurable ζ-sets which is dense in the collection of all measurable ζ-sets.

No. 4. *The product of Lebesgue spaces.* Let M_1 and M_2 be Lebesgue spaces with measures μ_1 and μ_2. We consider an arbitrary function $\phi(A_1, A_2)$ of sets $A_1 \in \Omega_{\mu_1}$ and $A_2 \in \Omega_{\mu_2}$, which enjoys the following properties:

a) ϕ is non-negative;

b) ϕ is completely additive with respect to both of its arguments;

c) $\phi(A_1, M_2) = \mu_1 A_1$, $\phi(M_1, A_2) = \mu_2 A_2$.

We denote by M the set-theoretic product of the sets M_1 and M_2:

$$M = M_1 \times M_2,$$

and by $\mathbf{M}$ the collection of sets of the form

$$A = A_1 \times A_2, \quad A_1 \in \Omega_{\mu_1}, \quad A_2 \in \Omega_{\mu_2}.$$

We shall show that there exists a measure μ for M, defined in particular on $\mathbf{M}$ and satisfying the relation

$$\mu(A_1 \times A_2) = \varphi(A_1, A_2) \tag{6}$$

for all pairs of sets $A_1 \in \Omega_{\mu_1}$ and $A_2 \in \Omega_{\mu_2}$.

The process of constructing the measure μ differs in no way from the process of constructing the ordinary measure in the product of two spaces. Although this ordinary measure is obtained from our general definition only for $\phi(A_1, A_2) = \mu_1 A_1 \cdot \mu_2 A_2$, nevertheless, the well-known proofs which its construction demands employ only properties a), b), and c), of the function ϕ, and not the special form of ϕ indicated above. This circumstance releases us from the necessity of carrying out detailed proofs.

We consider the field $(\mathrm{M}+\overline{\mathrm{M}})_{ds}$; this is the collection of sets of the form

$$A = \overset{\prime}{\bigcup_{n=1}} A^{(n)}, \quad A^{(n)} \in \mathrm{M},$$

and we extend the function μ, defined on M by formula (6), over $(\mathrm{M}+\overline{\mathrm{M}})_{ds}$ by the additive law (this is possible, since there exists a representation in the form (7) for every $A \in (\mathrm{M}+\overline{\mathrm{M}})_{ds}$ with pairwise disjoint set $A^{(n)}$; the uniqueness of the extension follows from the additivity of the function ϕ with respect to its arguments). The extended function satisfies the conditions of the known theorem on the extension of additive functions and therefore can be extended in its turn to form a certain minimal measure, which we shall also designate as μ. We shall call the space M with the measure μ the product of the spaces M_1 and M_2 with respect to the function ϕ. In particular, the ordinary product, corresponding to the function $\phi(A_1, A_2) = \mu_1 A_1 \cdot \mu_2 A_2$, will be called the direct product.

These definitions can be generalized to the case of an arbitrary finite or infinite system of spaces M_α with measures μ_α.* In doing this, it is assumed that ϕ is a function of sets $A_\alpha \in \Omega_{\mu_\alpha}$, defined only for such systems $\{A_\alpha\}$ that the sets A_α which differ from the corresponding sets M_α are present only in finite numbers, and satisfying conditions a) and b), and condition c) in the following formulation:

c) if the system $\{A_\alpha\}$ is such that $A_\alpha \neq M_\alpha$ only for $\alpha = \beta$, then $\phi(\{A_\alpha\}) = \mu A_\beta$.

M is the set-theoretic product of the spaces M_α:

$$M = \prod_\alpha M_\alpha,$$

M is the collection of sets of the form

$$A = \prod_\alpha A_\alpha,$$

*Compare [8], III, 4.

where only a finite number of the sets A_α are distinct from the corresponding sets M_α, and μ is the minimum of all measures ν for which $\Omega_\nu \supset \mathrm{M}$ and

$$\nu\left(\prod_\alpha A_\alpha\right) = \varphi(\{A_\alpha\}). \tag{8}$$

If we wish that the product M of the spaces M_α should be a Lebesgue space, then we must first of all postulate that the set of these spaces should be countable, for if it is uncountable, then, barring extremely trivial cases, the space M will not be separable. In the case of a countable set of spaces M_α, the space M is always separable, for if Γ_α are bases of the spaces M_α, then the union Γ of systems of sets

$$A_\alpha \times \prod_{\beta \neq \alpha} M_\beta \quad (A_\alpha \in \Gamma_\alpha)$$

is a basis of the space M. In this connection, it is not difficult to see that from the completeness of the spaces M_α with respect to their bases Γ_α, one can infer that M is complete with respect to Γ. In this way, *the product of a finite or countably infinite system of Lebesgue spaces is a Lebesgue space*. Furthermore, *the measure μ is the only Lebesgue measure defined for all sets of the collection* M *and satisfying relation* (8). As a matter of fact, let μ' be a different Lebesgue measure defined for all sets of the collection M and satisfying relation (8). Since the collection $\Omega_{\mu'}$ contains M, it contains the system Γ which has just been constructed; and since Γ enjoys property ($\mathfrak{M}$) of No. 1 of §2 in the collection M, it follows from the theorem on bases (No. 5 of §2) that μ' is completely determined by its values on the collection $\Gamma_d \subset \mathrm{M}$. That is, μ' is determined by property (8).

Now let

$$x = \prod x_\alpha, \quad x_\alpha \in M_\alpha,$$

be an arbitrary point of the space M. We set

$$H_\alpha x = x_\alpha.$$

H_α is a homomorphism of the space M onto the space M_α, and the corresponding decomposition ζ_{H_α} is the decomposition of the space M into sets of the form

$$x_\alpha \times \prod_{\beta \neq \alpha} M_\beta.$$

By No. 2 of §3, T_{H_α} is an isomorphism, and we may *identify* the space M_α with the corresponding factor space M/ζ_α and suppose that M is the product of all of its factor-spaces M/ζ_α. This product is defined by the relations:

$$\prod_\alpha C_\alpha = \bigcap_\alpha C_\alpha, \quad \varphi(\{X_\alpha\}) = \mu\left(\bigcap_\alpha Z_\alpha\right),$$

where C_α is understood on the left to be a point of the space M/ζ_α and on the right to be a set in the space M, and $X_\alpha = H_{\zeta_\alpha} Z_\alpha$.

We now reverse the question. Let M be a Lebesgue space and let $Z = \{\zeta_\alpha\}$ be a finite or countably infinite system of measurable decompositions of the space M; under what conditions can M be considered to be the product of the factor-spaces M/ζ_α in the sense just described? From the discussion above, it follows that this will be the case if and only if all intersections

$$\bigcap C_\alpha, \tag{9}$$

where C_α is an arbitrary element of the decomposition ζ_α, consist of exactly one point; that is, if

a) all of the intersections (9) are non-void; this condition we shall express by saying that the system Z is crossed;

b) $\Pi\zeta_\alpha$ is the decomposition of M into individual points.

If we are concerned not with an *exact* resolution into a product, but merely with a resolution mod 0, that is, if we demand only that the system should be identical mod 0 with a certain system $Z' = \{\zeta'_\alpha\}$, which resolves its space M' into the product of factor-spaces M'/ζ'_α, then we can obtain the following stronger result:

Every finite or countably infinite system Z, which satisfies mod 0 *the condition* b), *resolves* mod 0 *the space M into the product of factor-spaces* M/ζ_α. For the proof, it is sufficient to observe that *in a Lebesgue space, every finite or countably infinite system of measurable decompositions is crossed* mod 0. Indeed, let Σ_α be any basis at all of the decomposition ζ_α. We form the union of all of the systems Σ_α and extend the countable system of measurable sets obtained in this way to a basis Γ of the space M. Let $\tilde{\Sigma}_\alpha$ be the part of the basis $\tilde{\Gamma}$ of the space $\tilde{M} = \tilde{M}(\Gamma)$ which corresponds to the system Σ_α; we set $\tilde{\zeta}_\alpha = \zeta(\tilde{\Sigma}_\alpha)$ and $\tilde{Z} = \{\zeta_\alpha\}$. It is evident that the system $\tilde{Z}$ is crossed in $\tilde{M}$ and that it induces the system Z in the space M. Since M is a Lebesgue space it follows that M covers mod 0 all of $\tilde{M}$, and, consequently, the systems Z and $\tilde{Z}$ are identical mod 0.

We shall call measurable decompositions ζ_1 and ζ_2 independent* if, for every measurable ζ_1-set Z_1 and every measurable ζ_2-set Z_2,

$$\mu(Z_1 Z_2) = \mu Z_1 \cdot \mu Z_2.$$

Since, on the one hand,

$$\mu(Z_1 Z_2) = \int_{M/\zeta_1} \mu_{C_1}(C_1 Z_1 Z_2)\, d\mu_{\zeta_1} = \int_{H_\zeta Z_1} \mu_{C_1}(C_1 Z_2)\, d\mu_{\zeta_1}$$

(C_1 is an element of the decomposition ζ_1), and, on the other hand,

$$\mu Z_1 \cdot \mu Z_2 = \int_{H_\zeta Z_1} \mu Z_1\, d\mu_{\zeta_1},$$

it follows that the condition of independence can be formulated in the following fashion also: every measurable set ζ_2-set Z_2 intersects all mod 0 spaces C_1 in sets having the same measure, which is equal to the measure of the set Z_2 in the space M:

$$\mu_{C_1}(C_1 Z_2) = \mu Z_2. \tag{10}$$

For a arbitrary measurable decomposition ζ and an arbitrary measurable set A, we set:

$$d_\zeta A = \operatorname*{vrai\,min}_{C \in M/\zeta} \mu_C(CA), \quad D_\zeta A = \operatorname*{vrai\,max}_{C \in M/\zeta} \mu_C(CA).$$

We call $d_\zeta A$ the inner diameter and $D_\zeta A$ the outer diameter, or simply the diameter, of the set A with respect to ζ. If $d_\zeta A = D_\zeta A$, then we say that A is a set of constant width mod 0 with respect to ζ. It follows from the foregoing that decompositions ζ_1 and ζ_2 are independent if and only if every measurable ζ_2-set is a set of constant width mod 0 with respect to ζ_1. For this, it is clearly sufficient that the sets of the collection $(\Sigma_2)_d$ should have constant width mod 0 with respect to ζ_1 (see No. 3 of §1), where $(\Sigma_2)_d$ is any basis at all of the decomposition ζ_2.

If the decompositions ζ_1 and ζ_2 are not only independent, but also mutually complementary (No. 3 of §3), or, as we shall say, independent complements of each other, then there corresponds to them a resolution of the space M into the direct product of the

*This concept is borrowed from the theory of probability; see [8], VI, §1.

factor-spaces M/ζ_1 and M/ζ_2. In this case, letting the set Z_2 run through the collection $(\Sigma_2)_d$, we concluded from the relations (10) that the homomorphism H_{ζ_1} induces in all mod 0 spaces C_1 isomorphic mappings of these spaces onto the factor-space M/ζ_1. (In the case of a resolution mod 0, these mappings are isomorphisms mod 0.)

§4. Construction of a measurable decomposition.

No. 1. *Formulation of results.* In this paragraph, we shall give a complete classification of measurable decompositions of a Lebesgue space. In view of the theorem on homomorphisms (No. 2 of §3), this will give at the same time a complete classification of the homomorphisms of these spaces.

Let ζ be an arbitrary measurable decomposition of the Lebesgue space M. Every mod 0 element C of this decomposition, regarded as a Lebesgue space, (No. 1 of §3), posseses its own invariants $m_1(C), m_2(C), \ldots$ (No. 4 of §2). We can consider m_n as a function defined on the factor space M/ζ. In view of their definition, they satisfy mod 0 the following inequalities:

$$m_n \geqslant 0, \quad m_n \geqslant m_{n+1}, \quad \sum_{n=1}^{\infty} m_n \leqslant 1. \tag{1}$$

We shall designate the sequence $m_1, m_2, \ldots$ corresponding in this way to the decomposition ζ by the symbol $\mathfrak{m}_\zeta$. It is clear that the type $\tau(\mathfrak{m}_\zeta)$ of the sequence $\mathfrak{m}_\zeta$* is an invariant of the decomposition ζ: if $\tau(\zeta') = \tau(\zeta'')$, then $\tau(\mathfrak{m}_{\zeta'}) = \tau(\mathfrak{m}_{\zeta''})$. The principal result of the present paragraph is that:

(I) *the functions m_n are measurable;*

(II) *if $\tau(\mathfrak{m}_{\zeta'}) = \tau(\mathfrak{m}_{\zeta''})$, then $\tau(\zeta') = \tau(\zeta'')$;*

(III) *for every sequence $\mathfrak{m}$ of measurable functions m_n defined on a certain Lebesgue space and satisfying* mod 0 *the inequalities* (1), *there exists a measurable decomposition ζ (of a certain other Lebesgue space) such that $\tau(\mathfrak{m}_\zeta) = \tau(\mathfrak{m})$.*

In other words, *the formula*

*In accordance with the general definition of No. 5 of §1, the sequence $\mathfrak{m}'$ of functions m_n', defined on the Lebesgue space M', and the sequence $\mathfrak{m}''$ of functions m_n'', defined on the Lebesgue space M'', belong to a single type:

$$\tau(\mathfrak{m}') = \tau(\mathfrak{m}''),$$

if there exists an isomorphism mod 0, $x'' = Ux'$, of the space M' onto M'', such that for all n and for all mod 0 points $x' \in M'$,

$$m_n'(x') = m_n''(x'').$$

$$\tau(\zeta) \to \tau(\mathfrak{m}_\zeta)$$

establishes a one-to-one correspondence between types of measurable decompositions and types of sequences of measurable functions satisfying mod 0 *the inequalities* (1).

The decomposition ζ assumes an especially simple form when the functions m_n are mod 0 *constant*, that is, all mod 0 spaces C are isomorphic mod 0 among themselves. As an example of such a decomposition, one may take the decomposition of the direct product

$$M = M_1 \times M_2 \tag{2}$$

into sets $C = x_1 \times M_2$, where $x_1 \in M_1$. In view of (II), *every measurable decomposition with functions m_n which are constant* mod 0 *assumes this form*. In other words, if the functions m_n are constant mod 0, then the decomposition ζ possesses an independent complement. Since the converse is obvious, we can assert: in order for the measurable decomposition ζ to admit *an independent complement, it is necessary and sufficient that all* mod 0 *spaces C should be of one and the same type.*

The most important case is that in which the measures being considered -- the measure μ_ζ and all mod 0 measures μ_C -- are *continuous*. Then, as spaces M_1 and M_2 in the product (2), we can simply take closed intervals, and we obtain the following theorem:

if the measure μ_ζ and all mod 0 *measures μ_C are continuous, then the decomposition ζ is isomorphic* mod 0 *to a decomposition of the unit square into closed intervals parallel to one of its sides.*

In the general case, a measurable decomposition has only a slightly more complex form. We can consider the functions m_n as being defined on a set L of points of the axis x, consisting of the interval $(0, l_0)$ and a certain sequence of points $x_1, x_2, \ldots$, let us say $x_k = 1 + 1/k$. The measure is defined in the space L as ordinary Lebesgue measure on the interval $(0, l_0)$, and the measure of the point x_k is l_k, where $l_k \geq l_{k+1}$ and $l_0 + \Sigma_{k+1}^{\infty} l_k = 1$. We designate as M_{00} the set of points of the plane xy, bounded on the left and right by the lines $x = 0$ and $x = l_0$, below by the line $y = 0$ and above by the curve

$$y = m_0(x) = 1 - \sum_{n=1}^{\infty} m_n(x).$$

We designate as M_{0k} ($k=1, 2, \ldots$) the interval on the line $x = 1 + 1/k$ included between the line $y = 0$ and the curve $y = m_0(x)$. Next, we set:

$$M_0 = M_{00} + \bigcup_{k=1}^{\infty} M_{0k}.$$

Next, we denote by M_n ($n=1, 2, \ldots$) the set of points of the line $y = 1 + 1/n$ which lie over points of the set L. Our space M is obtained by uniting the set M_0 with the sets $M_1, M_2, \ldots$:

$$M = M_0 + \bigcup_{n=1}^{\infty} M_n. \tag{3}$$

In order to define the measure μ, we defined by μ_{00} the ordinary plane Lebesgue measure, and by μ_{0k} ($k=1, 2, \ldots$) the ordinary linear Lebesgue measure on the line $x = 1 + 1/k$, and for a set $A_0 \subset M_0$, we set:

$$\mu_0 A_0 = \mu_{00} (M_{00} A_0) + \sum_{k=1}^{\infty} \mu_{0k} (M_{0k} A_0).$$

We consider A_0 as measurable with respect to μ_0 if and only if all of the intersections $M_{0k}A_0$ are measurable with respect to their measures μ_{0k}. Furthermore, we designate as μ_n ($n=1, 2, \ldots$) the Lebesgue-Stieltjes measure defined by the formula:

$$\mu_n I_n = \int_I m_n(x)\, d\lambda,$$

where I_n is an interval on the line $y = 1 + 1/n$, and I is the interval on the axis of x which lies under I_n. Finally, we agree to consider those sets $A \subset M$ as measurable whose intersections with all of the spaces M_n ($n=0, 1, 2, \ldots$) are measurable with respect to the appropriate measures, and we define

$$\mu A = \mu_0 (M_0 A) + \sum_{n=1}^{\infty} \mu_n (M_n A).$$

It is not difficult to convince one's self that the space M with the measure μ is a Lebesgue space. Our decomposition is a decomposition of the space M into sets C_a, lying on the vertical lines $x = a$. This decomposition is obviously measurable. The space C_a consists of the interval of the line $x = a$ contained between the line $y = 0$ and the curve $y = m_0(x)$, with ordinary linear Lebesgue measure, and the sequence of points $y_n = 1 + 1/n$ of this same line ($n=1, 2, \ldots$) with measures $m_n(a)$. We can identify the factor-

space M/ζ with the space L. In this way, the sequence m_ζ corresponding to the decomposition ζ is given by the sequence $m_1, m_2, \ldots$.

Since the numbers l_n and the functions m_n were given arbitrarily, *the present construction proves Theorem* (III). At the same time, in the light of Theorem (II), it provides us with a measurable decomposition of the most general kind.

No. 2. *Removal of one-sheeted sets of positive measure.* The sets $M_1, M_2, \ldots$ which served us in the preceding No. for constructing the decomposition ζ, have the property that they intersect every element of the decomposition ζ in not more than one point. Sets of this kind will be referred to as one-sheeted with respect to ζ. We commence with the construction of a resolution of the form (3) for an *arbitrary* decomposition ζ.

Lemma. *Among the measurable sets which are one-sheeted with respect to ζ, there exists a set of maximal measure.*

Proof. We denote by α the least upper bound of measures of one-sheeted measurable sets; let $A_1, A_2, \ldots$ be a sequence of sets for which

$$\lim_{n \to \infty} \mu A_n = \alpha. \tag{4}$$

We wish to replace $A_1, A_2, \ldots$ by another sequence, also enjoying property (4), but, besides this, converging. This replacement is carried out inductively: we set: $A'_1 = A_1$, and if the set A'_{n-1} is already defined ($n=2, 3, \ldots$), then we denote by X_{n-1} the set of those $C \in M/\zeta$, for which $\mu_C(CA_n) > \mu_C(CA'_{n-1})$, (5)
and by Z_{n-1} the inverse image of the set X_{n-1} under the homomorphism H_ζ. We define the set A'_n by the formula: $A'_n = \overline{Z}_{n-1}A'_{n-1} + Z_{n-1}A_n$. All of the sets A'_n are, clearly enough, one-sheeted measurable sets. For all mod 0 points $C \in \overline{X}_{n-1}$, we have: $CA'_n = CA'_{n-1}$, and in consequence, $\mu_C(CA'_n) = \mu_C(CA'_{n-1})$; if $C \in X_{n-1}$, then $CA'_n = CA_n$, and this means that $\mu_C(CA'_n) = \mu_C(CA_n) > \mu_C(CA'_{n-1})$. Therefore for all mod 0 points $C \in M/\zeta$, $\mu_C(CA'_n) \geq \mu_C(CA'_{n-1})$; (6)
the sets $\overline{Z}_{n-1}$ and Z_{n-1} are characterized mod 0 by the fact that for $C \subset \overline{Z}_{n-1}$, equality obtains in relation (6), while for $C \subset Z_{n-1}$, the strict inequality holds.

$CA'_1, CA'_2, \ldots$ is a sequence of sets of the space C each consisting of not more than one point, and it is plain from (6) that their measures for a non-decreasing sequence. But, under these conditions, the sequence $CA'_1, CA'_2, \ldots$ can contain a finite number of different members; consequently, there exists a natural number $n(C)$ such that for $n > n(C)$,

$$CA'_n = CA'_{n(C)}.$$

We set

$$M_1 = \bigcup_C CA'_{n(C)}.$$

M_1 is a one-sheeted set and also, mod 0, is the set-theoretic limit of the sequence $A'_1, A'_2, \ldots$. Consequently, it is measurable, and

$$\mu M_1 = \lim_{n \to \infty} \mu A'_n. \tag{7}$$

Since for all mod 0 points $C \in \overline{X}_{n-1}$,

$$\mu_C(CA'_n) = \mu_C(CA'_{n-1}) \geqslant \mu_C(CA_n),$$

and for all $C \in X_{n-1}$,

$$\mu_C(CA'_n) = \mu_C(CA_n),$$

it follows that for all mod 0 points $C \in M/\zeta$,

$$\mu_C(CA'_n) \geqslant \mu_C(CA_n),$$

so that

$$\mu A'_n = \int_{M/\zeta} \mu_C(CA'_n) d\mu_\zeta \geqslant \int_{M/\zeta} \mu_C(CA_n) d\mu_\zeta = \mu A_n .$$

But, $\mu A'_n \leq \alpha$; consequently,

$$\mu A_n \leqslant \mu A'_n \leqslant \alpha, \tag{8}$$

Now, combining (8) with (4) and (7), we obtain:

$$\mu M_1 = \alpha,$$

that is, M_1 is a one-sheeted set of maximal measure. The lemma is proved.

We now set $N_0 = M$ and $N_1 = \overline{M}_1$. If $\mu N_1 > 0$, then N_1 can be considered as a subspace of the space M, and the decomposition induces a definite decomposition ζ_1 in N_1. We can apply our lemma again to ζ_1; among the measurable sets which are one-sheeted with respect to ζ_1, there exists a set M_2 of maximal measure. Plainly enough M_2 is at the same time a one-sheeted measurable set with respect to ζ, and also it has maximal measure if one considers only subsets of N_1.

Continuing this process, we obtain sets $N_2 = N_1 - M_2, M_3$, and so on. If one of the sets $M_1 M_2, \ldots$ turns out to be a set of measure zero, then all of the succeeding sets will also be of measure zero. This occurs if one of the sets $N_0, N_1, \ldots$ contains no one-sheeted set of positive measure, for example, if the set itself has measure zero. In any case, upon admitting sets of measure zero, we can define *all* of the sets $M_1, M_2, \ldots$. Finally, we set

$$M_0 = \bigcap_n N_n = \overline{\bigcup_{n=1}^{\infty} M_n}.$$

M_0 has no one-sheeted subsets of positive measure. This is evident if there are only a *finite number* of sets of positive measure among the sets $M_1, M_2, \ldots$. Now, if *all* of the sets M_n have positive measure, then $\lim_{n \to \infty} \mu M_n = 0$ (for the sets M_n are pairwise disjoint), and since, for every one-sheeted set $A \subset M_0$, we must have $\mu A \leq \mu M_n$ for all n, it follows that $\mu A = 0$.

We summarize the results obtained above in the following way:

To every measurable decomposition ζ of the space M, there corresponds at least one decomposition of the form

$$M = M_0 + \bigcup_{n=1}^{\infty} M_n, \tag{9}$$

where $M_1, M_2, \ldots$ are measurable sets which are one-sheeted with respect to ζ, where M_n is a set of maximal measure among all measurable one-sheeted subsets of the set $N_{n-1} = \overline{\bigcup_{k=1}^{n-1} M_k}$, and M_0 contains no subset of positive measure which is one-sheeted with respect to ζ.

No. 3. *Decompositions without one-sheeted sets of positive measure.* In this number, we shall prove the following theorem:

Every measurable decomposition without one-sheeted sets of positive measure admits an independent complement.

The proof is based on a series of Lemmas, in which ζ always denotes a measurable decomposition without one-sheeted sets of positive measure.

Lemma 1. *If A is a set of positive measure, then among its measurable subsets, there exists at least one which is not identical mod 0 with any of its ζ-subsets* (No. 2 of §1).

Proof. Such a subset can be found among the elements of an arbitrary basis $\Gamma = \{G_\alpha\}$ of the set A (considered as a subspace of the space M). In fact, in the opposite case, one can find for every set G_α a ζ-subset Z_α of the set A which is identical mod 0 with G_α, and, removing from A the set $\bigcup(\overline{G}_\alpha Z_\alpha + G_\alpha \overline{Z}_\alpha)$, the measure of which is equal to zero, we find a one-sheeted set of positive measure.

Lemma 2. *Every measurable set of positive measure admits measurable subsets of positive measure with arbitrarily small diameter* (No. 4 of §3).

Proof. Suppose that the lemma is false; let A be a measurable set of positive measure such that

$$\inf D_\zeta B > 0;\ B \subset A,\ \mu B > 0. \tag{10}$$

We denote the left side of the inequality (10) by ε. Then, on the one hand, for every measurable set $B \subset A$ of positive measure,

$$D_\zeta B \geqslant \varepsilon, \tag{11}$$

while on the other hand, there exists a measurable set $B_0 \subset A$ of positive measure such that

$$D_\zeta B_0 < 2\varepsilon.$$

Let B_1 be a measurable subset of the set B_0 which is not identical mod 0 with any of its ζ-subsets (Lemma 1). We set $B_2 = B_0 - B_1$ and denote by X_0 the set of those $C \in M/\zeta$, for which the two inequalities

$$\mu_C(CB_1) > 0, \quad \mu_C(CB_2) > 0 \tag{12}$$

are valid. In view of the choice of the set B_1, X_0 is a set of positive measure, and since for all mod 0 points $C \in M/\zeta$,

$$\mu_C(CB_1) + \mu_C(CB_2) = \mu_C(CB_0) < 2\varepsilon,$$

there exists a set $X \subset X_0$ of positive measure such that if

$$\mu_C(CB_1) < \varepsilon, \tag{13}$$

does not hold for all $C \in X$, then

$$\mu_C(CB_2) < \varepsilon \tag{14}$$

does hold for all $C \in X$. We denote by Z the inverse image of the set X under the homomorphism H_ζ. In case (13), we set $B = ZB_1$, in case (14), we set $B = ZB_2$, and in both cases, we have:

$$D_\zeta B < \varepsilon,$$

which contradicts relation (11), for, in view of (12),

$$\mu B = \int_X \mu_C(CB)\, d\mu_\zeta > 0.$$

Lemma 3. *For every measurable set A and every real number satisfying the inequalities*

$$D_\zeta A \leqslant \theta \leqslant 1, \tag{15}$$

there exists a measurable set B which contains A and which has mod 0 constant width θ.

Proof. We construct a transfinite sequence of measurable sets B_α having the following properties:

a) $B_1 = A$;

b) for $\alpha'' > \alpha'$, we always have $B_{\alpha''} \supset B_{\alpha'}$ and $\mu B_{\alpha''} > \mu B_{\alpha'}$;

c) $D_\zeta B_\alpha \leq \theta$.

In order to satisfy condition a), we must set $B_1 = A$; in view of inequality (15), condition c) will be satisfied for $\alpha = 1$ if we do this. Let us suppose that sets B_α which satisfy conditions a), b), and c) have already been constructed for all $\alpha < \beta$.

1) If β is a transfinite index of the *first* kind and $B_{\beta-1}$ *is not a set of constant width* θ mod 0, then there evidently exists a set $X \subset M/\zeta$ of positive measure and a positive number ϵ such that for all $C \in X$,

$$\mu_C(CB_{\beta-1}) \leqslant \theta - \varepsilon. \tag{16}$$

We denote by Z the inverse image of the set X under the homomorphism H_ζ. In view of (16),

$$\mu(ZB_{\beta-1}) = \int_X \mu_C(CB_{\beta-1})\, d\mu_\zeta \leqslant (\theta - \varepsilon)\mu_\zeta X < \mu_\zeta X = \mu Z.$$

But this means that $\mu(Z\overline{B}_{\beta-1}) > 0$; in view of Lemma 2, one can find in $Z\overline{B}_{\beta-1}$ a measurable subset E_β of positive measure, the diameter of which is not larger than ϵ, and we set:

$$B_\beta = B_{\beta-1} + E_\beta.$$

2) If β is a transfinite index of the *second* class, then we set:

$$B_\beta = \bigcup_{\alpha<\beta} B_\alpha.$$

Under constructions 1) and 2), conditions b) and c) evidently remain satisfied. In view of b), the transfinite sequence of numbers $\mu B_1, -B_2, \ldots$ is strictly monotone and therefore is no more than countably infinite. Consequently, the process must come to a halt at some transfinite number β of the second number class. But it can come to a halt only if β is a number of the first kind and $B_{\beta-1}$ is a set of constant width θ mod 0. Thus $B = B_{\beta-1}$ is a set of the kind whose existence is asserted by Lemma 3.

Lemma 3′. *For every measurable set A and every real number θ satisfying the inequality*

$$0 \leqslant \theta \leqslant d_\zeta A,$$

there exists a measurable set B contained in A of constant width θ mod 0.

Proof. Since

$$d_\zeta A = 1 - D_\zeta \bar{A},$$

by a passage to the complement, this lemma leads to Lemma 3.

Lemma 4. *For every measurable set A and every natural number r, one can find r measurable ζ-sets $Z_1, Z_2, \ldots, Z_r$ (which can always be chosen pairwise disjoint) and r measurable sets $Z^*_1, Z^*_2, \ldots, Z^*_r$ of constant width* mod 0 *with respect to ζ, such that the set*

$$B = \bigcup_{i=1}^{r} Z_i Z_i^*$$

contains A and

$$\mu(B - A) \leqslant \frac{1}{r}.$$

Proof. We define Z_1 as the sum of those sets C for which

$$0 \leqslant \mu_C(CA) \leqslant \frac{1}{r},$$

and Z_i, and $1 < i \leq r$, as the sum of those sets C for which

$$\frac{i-1}{r} < \mu_C(CA) \leqslant \frac{1}{r}.$$

In order to construct the sets Z^*_i, we shall consider those of the sets Z_i which have positive measure as subspaces of the space M. Let ζ_i be the decomposition induced in Z_i by the decomposition ζ, and let $A_i = Z_i A$. Since, obviously,

$$D_{\zeta_i} A_i \leqslant \frac{1}{r},$$

it follows from Lemma 3 that there exists a measurable set $Z^*_{i,i}$ containing A_i and of constant width i/r mod 0 with respect to ζ_i. Starting with $Z^*_{i,i}$ and using Lemma 3, we construct in turn the sets

$$Z^*_{i,i+1} \supset Z^*_{i,i}, \quad Z^*_{i,i+2} \supset Z^*_{i,i+1}, \ldots, Z^*_{i,r} \supset Z^*_{i,r-1},$$

and using Lemma 3′, sets

$$Z^*_{i,i-1} \subset Z^*_{i,i}, \quad Z^*_{i,i-2} \subset Z^*_{i,i-1}, \ldots, Z^*_{i,1} \subset Z^*_{i,2},$$

where $Z^*_{i,j}$ is a set of constant width j/r mod 0 with respect to ζ_i. Of course, this construction takes place only for those values of i such that $\mu Z_i > 0$; if $\mu Z_i = 0$, then by definition, $Z^*_{i,j} = Z_i (j = 1, 2, \ldots, r)$. Finally, we set

$$Z_j^* = \bigcup_{i=1}^{r} Z^*_{i,j}.$$

Since $Z^*_{i,1} \subset Z^*_{i,2} \subset \cdots \subset Z^*_{i,r}$, it follows that $Z^*_1 \subset Z^*_2 \subset \ldots \subset Z^*_r$, and since $Z_i Z^*_i = Z^*_{i,i}$ and $Z^*_{i,i}$ A_i, it follows that

$$B = \bigcup_{i=1}^{r} Z_i Z_i^* = \bigcup_{i=1}^{r} Z^*_{i,i} \supset \bigcup_{i=1}^{r} A_i = A$$

and that

$$\mu(B - A) = \sum_{i=1} \mu(Z_{i,i} - A_i) =$$

$$= \sum_{i=1}^{r} \int_{H_\zeta Z_i} \mu_C [C(Z^*_{i,i} - A_i)]\, d\mu_\zeta \leqslant \sum_{i=1}^{r} \frac{1}{r} \mu Z_i = \frac{1}{r}.$$

We are now in a position to prove the theorem stated at the beginning of this No. Let Γ be any basis at all of the space M. We consider all possible pairs (G, p), where $G \in (\Gamma + \bar{\Gamma})_d$ (See No. 3 of §1) and p is a natural number. We arrange these pairs in a sequence $\Pi_1 = (G_1, p_1)$, $\Pi_2 = (G_2, p_2), \ldots$. We next construct a sequence $\Lambda_1, \Lambda_2, \ldots$ of finite systems of measurable sets (the union $\bigcup_{n=1}^{\infty} \Lambda_n$ being denoted by Σ) and a sequence $\Lambda^*_1, \Lambda^*_2, \ldots$ of finite systems of measurable sets (the union $\bigcup_{n=1}^{\infty} \Lambda^*_n$ being denoted by Σ^*) such that

a) the system Σ^*_d (see No. 3 of §1) consists of sets which have mod 0 constant width with respect to ζ;

c) the systems Λ_n and Λ^*_n consist of one and the same numbers of sets; if these sets are respectively $Z_{n,1}, Z_{n,2}, \ldots, Z_{n,q_n}$ and

$Z^*_{n,1}, Z^*_{n,2}, \ldots, Z^*_{n,q_n}$, then the set

$$B_n = \bigcup_{k=1}^{q_n} Z_{n,k} Z^*_{n,k}$$

contains G_n, and

$$\mu(B_n - G_n) < \frac{1}{p_n}.$$

The construction is carried out inductively.

1) The systems Λ_1 and Λ^*_1 are constructed on the basis of Lemma 4; it is necessary to set $A = G_1$, $r = p_1$ and then $Z_{1,k} = Z_k$ and $Z^*_{1,k} = Z^*_k$.

2) If the systems $\Lambda_1, \Lambda_2, \ldots, \Lambda_{n-1}$ and $\Lambda^*_1, \Lambda^*_2, \ldots, \Lambda^*_{n-1}$ are already constructed, then the systems Λ_n and Λ^*_n are constructed in the following fashion. Let $S^1, S^2, \ldots, S^{s_n}$ be the elements of the decomposition $\zeta(\Sigma^*_n)$, generated by the union $\Sigma^*_n = \bigcup_{j=1}^{n-1} \Lambda^*_j$ of the systems $\Lambda^*_1, \Lambda^*_2, \ldots, \Lambda^*_{n-1}$. The sets S^i which have positive measure can be considered as subspaces of the space M. Let ζ^i be the decomposition induced by the decomposition ζ on S^i and let $G^i = S^i G_n$. Letting $A = G^i$, $r = p_n$, and applying Lemma 4, we obtain a system of measurable ζ^i-sets $\tilde{Z}^i_1, \tilde{Z}^i_2, \ldots, \tilde{Z}^i_{p_n}$ and a system of measurable sets $Z^{*i}_1 \subset Z^{*i}_2 \subset \cdots \subset Z^{*i}_{p_n}$ of constant width mod 0 with respect to ζ^i, such that the set

$$B^i = \bigcup_{j=1}^{p_n} \tilde{Z}^i_j Z^{*i}_j$$

contains G^i and

$$\mu^i(B^i - G^i) \leqslant \frac{1}{p_n},$$

where μ^i is the measure in the space S^i. Of course, this construction is vacuous for $\mu S^i = 0$; in this case, we set $\tilde{Z}^i_j = Z^{*i}_j = S (j=1, 2, \ldots, p_n)$. We denote by Z^i_j the union of all elements C of the decomposition ζ, which intersect $\tilde{Z}^i_j$, if $\mu S^i > 0$, and the entire space M, if $\mu S^i = 0$ (for $\mu S^i > 0$, the measurability of the set Z^i_j follows from the fact that it coincides with the sum of those sets C for which $\mu_C(C\tilde{Z}^i_j) > 0$). We now set:

$$\left.\begin{aligned} Z_{n,k} &= Z^i_j \\ Z^*_{n,k} &= Z^{*i}_j \end{aligned}\right\} \quad \begin{aligned} &i = 1, 2, \ldots, s_n;\ j = 1, 2, \ldots, p_n; \\ &k = (i-1)p_n + j;\ k = 1, 2, \ldots, q_n = s_n p_n. \end{aligned}$$

It is evident that the sequences Λ_n, Λ^*_n which we have constructed enjoy properties (a) and (b), We shall show that they

also enjoy property (c). Since $B^i \supset G^i$, we have

$$B_n = \bigcup_{k=1}^{q_n} Z_{n,k} Z^*_{n,k} = \bigcup_{i=1}^{s_n} \left(\bigcup_{j=1}^{p_n} Z^i_j Z^{*i}_j \right) = \bigcup_{i=1}^{s_n} \left(\bigcup_{j=1}^{p_n} \tilde{Z}^i_j Z^{*i}_j \right) = \bigcup_{i=1}^{s_n} B^i \supset \bigcup_{i=1}^{s_n} G^i = G_n,$$

and since

$$\mu(B^i - G^i) = \mu S^i \cdot \mu^i (B^i - G^i) \leqslant \frac{1}{p_n} \mu S^i,$$

it follows that

$$\mu(B_n - G_n) = \sum_{i=1}^{s_n} \mu(B^i - G^i) \leqslant \frac{1}{p_n} \sum_{i=1}^{s_n} \mu S^i = \frac{1}{p_n}.$$

We set $\zeta^* = \zeta(\Sigma^*)$. We shall show that ζ^* is *the independent complement of the decomposition* ζ. In fact, in view of (b), the decompositions ζ and ζ^* are independent (No. 4 of §3), and they are also complements of each other, for, in view of c), the union of the systems Σ and Σ^* enjoys property ($\mathfrak{L}$) (see No. 3 of §3).

The theorem is proved.

From the fundamental formula 2c) of No. 1 of §3, it follows that a decomposition ζ evidently does not admit one-sheeted sets of positive measure if all mod 0 measures μ_C are continuous. Since every mod 0 element C^* of the independent complement ζ^* is a set which is one-sheeted with respect to ζ, the measure of which for all mod 0 elements C is equal to the measure $\mu_C(CC')$ of the set CC' (which consists of exactly one point), the converse is also true. That is, if there are no one-sheeted sets of positive measure, the theorem just proved shows that all mod 0 measures μ_C must be continuous. In this way, we see that *a measurable decomposition ζ does not admit any one-sheeted sets of positive measure if and only if all* mod 0 *measures μ_C are continuous.*

Assembling the results obtained, we see that *two decompositions with continuous measures μ_C are isomorphic* mod 0 *if and only if the corresponding factor-spaces are isomorphic* mod 0. But this assertion is plainly nothing but Theorem (II) (See No. 1) stated for decompositions with continuous measures μ_C.

No. 4. *Measurable decompositions of general type.* In the general case, the proof of Theorems (I) and (II) is based upon properties of the decomposition (9). We break up this proof into several parts.

A) *On M_0 all* mod *0 measures μ_C are continuous; on $M_1, M_2, \ldots$, they satisfy for all* mod *0 points $C \in M/\zeta$ the inequalities*

$$\mu_C(CM_1) \geqslant \mu_C(CM_2) \geqslant \ldots.$$

We shall denote by ζ_0 the decomposition induced in the subspace M_0 of the space M by the decomposition (we assume that $\mu M_0 > 0$) and by C_0 the element of the decomposition ζ_0 corresponding to the element C of the decomposition $\zeta (C_0 = M_0 C)$. The decomposition ζ_0 has associated with it its canonical system of measures μ_{C_0}, which, as one can easily see, is connected with the system $\{\mu_C\}$ by the relation:

$$\mu_C(X_0) = \mu_C(C_0) \cdot \mu_{C_0}(X_0) \quad (X_0 \subset C_0). \tag{17}$$

Since the set M_0 contains no one-sheeted set of positive measure, it follows from the results of the previous No. that all mod 0 measures μ_{C_0} are continuous. Consequently, all mod 0 measures μ_C are continuous on M_0. By this, the first of our assertions is proved. In order to prove the second, we denote by X_n the set of those $C \in M/\zeta$ for which $\mu_C(CM_n) < \mu_C(CM_{n+1})$, and by Z_n the inverse image of the set X_n under the homomorphism H_ζ and set

$$M'_n = \bar{Z}_n M_n + Z_n M_{n+1}.$$

M'_n is a measurable one-sheeted set lying in the set $N_{n-1} = \overline{\bigcup_{k=1}^{n-1} M_k}$. If $\mu_\zeta X_n > 0$, then

$$\int_{M/\zeta} \mu_C(CM'_n)\, d\mu_\zeta = \int_{\bar{X}_n} \mu_C(CM_n)\, d\mu_\zeta + \int_{X_n} \mu_C(CM_{n+1})\, d\mu_\zeta >$$
$$> \int_{\bar{X}_n} \mu_C(CM_n)\, d\mu_\zeta + \int_{X_n} \mu_C(CM_n)\, d\mu_\zeta = \int_{M/\zeta} \mu_C(CM_n)\, d\mu_\zeta,$$

or,

$$\mu M'_n > \mu M_n,$$

which is impossible. Hence $\mu_\zeta X_n = 0$.

B) *For all* mod *0 points $C \in M/\zeta$,*

$$\mu_C(CM_n) = m_n(C) \quad (n = 0, 1, \ldots). \tag{18}$$

Indeed, in view of A), for every mod 0 point $C \in M/\zeta$, the sequence $CM_1, CM_2, \ldots$ consists of sets each containing not more than one point, and so arranged that their measures form a non-increasing sequence; furthermore, this sequence contains every set of positive measure which contains only one point. But this means that the equalities (18) are valid.

C) *The functions m_n are measurable* (Theorem (I) *of* No. 1 *of* §4) *and if Z is the inverse image of a set $X \subset M/\zeta$ under the homomorphism H_ζ, then*

$$\mu(M_n Z) = \int_X m_n(C)\, d\mu_\zeta. \tag{19}$$

This theorem, which follows immediately from B), shows that for subsets of the set M_n, the measure μ is completely determined by the functions m_n.

D) *If $\tau(m_{\zeta'}) = \tau(m_{\zeta''})$, then $\tau(\zeta') = \tau(\zeta'')$.* (Theorem (II), No. 1, §4).

We construct the decompositions (9) corresponding to the decompositions ζ' and ζ'':

$$M' = M'_0 + \bigcup_{n=1}^{\infty} M'_n, \quad M'' = M''_0 + \bigcup_{n=1}^{\infty} M''_n,$$

and denote by ζ'_0 and ζ''_0 the decompositions induced by the decompositions ζ' and ζ'' in M'_0 and M''_0. Let V be an isomorphic mod 0 mapping of the space M'/ζ' onto M''/ζ'' carrying $m_{\zeta'}$ into $m_{\zeta''}$. In view of the results of the preceding No., there exists an isomorphism mod 0, U, of the set M'_0 onto the $U(C'M'_0) = VC' \cdot M''_0$ for an arbitrary element C' of the decomposition ζ'. In view of the relations (17), (18), and (19), U can automatically be extended to be an isomorphism mod 0 of the space M' onto the space M'', carrying ζ' into ζ''.

Bibliography

1. P. R. Halmos, and J. v. Neumann, *Operator methods in classical mechanics*. II, Ann. of Math. (2) **43**, 332-350 (1942).
2. J. v. Neumann, *Einige Sätze über messbare Abbildungen*, Ann. of Math. (2) **33**, 574-586 (1932); p. 582.
3. P. R. Halmos, *The decomposition of measures*, Duke Math. J. 8, 386-392 (1941).
4. W. Ambrose, P. R. Halmos, S. Kakutani, *The decomposition of measures*. II, Duke Math. J. **9**, 43-47 (1942).
5. V. Rohlin, *On the classification of measurable decompositions*, Doklady Akad. Nauk SSSR (N.S.) **58**, 29-32 (1947).
6. F. Hausdorff, *Mengenlehre*, 3rd.ed., de Gruyter, Berlin and Leipzig, 1935.
7. O. Nikodym, *Sur une généralisation des intégrales de M. J. Radon*, Fund. Math. **15**, 131-179 (1930).
8. A. N. Kolmogorov, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Springer, Berlin, 1933.
9. P. R. Halmos, *On a theorem of Dieudonné*, Proc. Nat. Acad. Sci. U.S.A. **35**, 38-42 (1949).

On partial derivatives

[Izvestiya Akademii Nauk SSSR. Seriya Matematičeskaya 13, 425–446 (1949)]

G. P. Tolstov

Translated from

Г. П. Толстов

О частных производных

[Известия Академии Наук СССР.
Серия Математическая 13, 425-446 (1949)]

by Edwin Hewitt

First published by the American Mathematical Society in 1952
as Translation Number 69. Reprinted without revision.