ABCD matrix: a unique tool for linear two-wire transmission line modelling

Pedro L. D. Peres, Carlos R. de Souza and Ivanil S. Bonatti
School of Electrical and Computer Engineering, University of Campinas, Campinas, Brazil
E-mail: peres@dt.fee.unicamp.br

Abstract The aim of this note is to show that all the behaviour of a two-wire transmission line can be directly derived from the application of ***ABCD*** matrix mathematical concepts, avoiding the explicit use of differential equations. An important advantage of this approach is that the transmission line modelling arises naturally in the frequency domain. Therefore the consideration of frequency-dependent parameters can be carried out in a simple way compared with the time-domain. Some standard examples of transmission lines are analysed through the use of ***ABCD*** matrices and a case study of a *balun* network is presented.

Keywords ***ABCD*** matrix; electric circuit analysis; two-port networks; two-wire transmission lines

Partial differential equations for voltage and current along the line are traditionally derived from an elementary line section of length Δx using the distributed parameter model for a two-wire transmission line. Several physical interpretations can be obtained from the solution of these equations.[1,3,4,8]

The concepts of image impedance and ***ABCD*** matrix can be used as an alternative approach to the modelling of the two-wire transmission line, although the operational advantages and capabilities of the ***ABCD*** matrix model as a convenient tool for solving almost all problems related to the transmission line are not fully stressed. Indeed, the authors claim that most of the transmission line results which are usually obtained by other means can more easily be produced when nothing more than the ***ABCD*** matrix model is used. The importance of ***ABCD*** matrix modelling in transmission line theory and its indisputable advantages over other tools are presented and discussed in this paper.

ABCD matrix fundamentals

The approach presented in this section for obtaining the ***ABCD*** matrix of a two-wire transmission line is based on image parameters, i.e. image impedance and image transfer constant.[7]

Consider an electrical network having two pairs of terminals, one labelled the input (sending) terminals and the other the output (receiving) terminals. A pair of terminals at which the network can be accessed so that the currents in the two terminals are the same is called a port. This condition is assured when each port of a network is connected to a similar port of another network.[6]

A two-port network with terminal voltages and currents as specified in Fig. 1 can be described by an ***ABCD*** matrix if it is composed only of linear elements (zero initial conditions), possibly including dependent sources, but containing no independent sources.

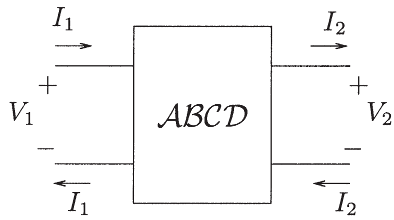

Fig. 1 *Two-port linear network.*

The ***ABCD*** matrix entries satisfy the linear relationship

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \tag{1}$$

where the voltages V_i and currents I_i represent either the Fourier (Laplace) transforms of $v_i(t)$ and $i_i(t)$ respectively, or the associated phasors ($i = 1, 2$).

Let Z_{sc} be the impedance reflected to the input terminals when the output terminals are short-circuited and Z_{oc} the corresponding input impedance when the output terminals are open-circuited. According to eqn (1), these impedances are given by

$$Z_{sc} = \frac{\boldsymbol{B}}{\boldsymbol{D}}; \qquad Z_{oc} = \frac{\boldsymbol{A}}{\boldsymbol{C}} \tag{2}$$

For reciprocal (that is, passive, linear and bilateral) two-port networks, the ***ABCD*** matrix determinant satisfies[6]

$$\boldsymbol{AD} - \boldsymbol{BC} = 1 \tag{3}$$

Furthermore, $\boldsymbol{D} = \boldsymbol{A}$ for symmetric two-port networks.

Alternatively a symmetric reciprocal two-port network can also be described by two image parameters: the characteristic impedance Z_0 and the propagation constant θ. Z_0 is the impedance at the input terminals of the two-port network when the output terminals are matched (terminated by a load impedance Z_0). The propagation constant θ is the natural logarithm of the ratio (I_1/I_2) where I_1 and I_2 are the matched condition terminal currents.

A two-port network composed by cascading n symmetric reciprocal two-port networks, each with the same characteristic impedance Z_0 and propagation constants θ_1; θ_2; . . . ; θ_n, respectively, has an equivalent characteristic impedance Z_0 and its propagation constant θ is given by

$$\theta = \sum_{k=1}^{n} \theta_k \tag{4}$$

The relationships between the ***ABCD*** matrix parameters and the image parameters are obtained by comparing the terminal equations (voltages and currents) for both models. This procedure yields

$$\boldsymbol{A} = \boldsymbol{D} = \cosh\theta; \quad \boldsymbol{B} = Z_0 \sinh\theta; \quad \boldsymbol{C} = \frac{1}{Z_0}\sinh\theta \tag{5}$$

and also considering eqn (2)

$$Z_0 = \sqrt{\frac{\boldsymbol{B}}{\boldsymbol{C}}} = \sqrt{Z_{sc}Z_{oc}} \tag{6}$$

$$\tanh\theta = \frac{\sqrt{\boldsymbol{B}/\boldsymbol{D}}}{\sqrt{\boldsymbol{A}/\boldsymbol{C}}} = \sqrt{\frac{Z_{sc}}{Z_{oc}}} \tag{7}$$

As the following theorem shows, the computation of the parameters Z_0 and θ becomes less involved when a symmetric two-port network is split into two sections so that the right-side section is the mirror image of the left-side section.

Theorem: The image parameters Z_0 and θ of a reciprocal symmetric two-port network satisfy

$$Z_0 = \sqrt{\frac{ab}{cd}} \tag{8}$$

$$\tanh\frac{\theta}{2} = \sqrt{\frac{bc}{ad}} \tag{9}$$

where a, b, c and d are the *abcd* matrix entries of the left-side section of the bisected two-port network.

Proof: As the right-side section is the mirror image of the left-side section

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & b \\ c & a \end{bmatrix} \tag{10}$$

yielding $\boldsymbol{A} = \boldsymbol{D} = ad + bc$, $\boldsymbol{B} = 2ab$ and $\boldsymbol{C} = 2cd$. Therefore, eqn (8) for Z_0 is readily obtained from eqn (6). From eqn (7) one gets

$$\tanh\theta = 2\frac{\sqrt{\dfrac{bc}{ad}}}{1 + \dfrac{bc}{ad}} \tag{11}$$

and using the identity

$$\tanh\theta = 2\frac{\tanh\dfrac{\theta}{2}}{1 + \tanh^2\dfrac{\theta}{2}} \tag{12}$$

eqn (9) is also proved.

As a result of the theorem,

$$Z_0 = \sqrt{\frac{ab}{cd}} = \sqrt{z_{sc} z_{oc}} \tag{13}$$

$$\tanh\frac{\theta}{2} = \sqrt{\frac{bc}{ad}} = \sqrt{\frac{z_{sc}}{z_{oc}}} \tag{14}$$

where z_{sc} and z_{oc} are, respectively, the short-circuit and the open-circuit impedances of the left-side section of the bisected two-port network. The same result is usually obtained via Bartlett's bisection theorem.[10]

Transmission line fundamentals

Consider a transmission line with length d and parameters r, l, g and c, respectively the distributed resistance, inductance, conductance and capacitance per unit length. The line can be divided into n sections of equal length Δx and each one can be represented by the circuit of lumped parameters shown in Fig. 2.

The symmetrical network shown in the Fig. 2 can be bisected. The short-circuit and the open-circuit impedances of the left-side section of the bisected network are given by

$$z_{sc} = (r + sl)\Delta x/2 \tag{15}$$

$$z_{oc} = (r + sl)\Delta x/2 + \frac{1}{(g + sc)\Delta x/2} \tag{16}$$

Equations (15) and (16) can be used to obtain the characteristic impedance z_{0k} and the propagation constant θ_k of the kth generic section of length Δx

$$z_{0k} = Z_0\left(1 + \left(\gamma\frac{\Delta x}{2}\right)^2\right)^{1/2} \tag{17}$$

$$\tanh\frac{\theta_k}{2} = \gamma\frac{\Delta x}{2}\left(1 + \left(\gamma\frac{\Delta x}{2}\right)^2\right)^{-1/2} \tag{18}$$

where

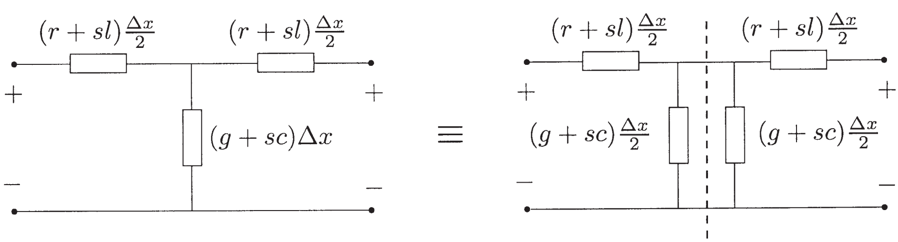

Fig. 2 *A transmission line section of length* Δx = d/n.

$$Z_0 = \sqrt{\frac{r+sl}{g+sc}}; \quad \gamma = \sqrt{(r+sl)(g+sc)} \tag{19}$$

As Δx approaches zero, eqn (17) shows that z_{ok} approaches Z_0 and eqn (18) shows that θ_k approaches zero. Nevertheless, the number n of sections approaches infinity so that the product $n\Delta x$ remains constant (the line length d). As the first-order approximation for θ_k is $\gamma\Delta x$, eqn (4) produces $\theta = \gamma d$ for the line propagation constant.

Equations (19)–(20) summarise the application of the ***ABCD*** matrix modelling to the two-wire transmission line of length d

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \cosh(\gamma d) & Z_0 \sinh(\gamma d) \\ \frac{1}{Z_0}\sinh(\gamma d) & \cosh(\gamma d) \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \tag{20}$$

It is worthwhile mentioning that eqn (20) is well known and that, traditionally, it is obtained from the transmission line differential equations, as for instance in Ref. [2]. Another alternative and interesting way to obtain eqn (20) from the Caley-Hamilton theorem has been presented in Ref. [9].

Transmission line computations

In order to stress the capabilities of the ***ABCD*** matrix, the input impedance of a two-wire transmission line terminated by an impedance Z_2 is computed. Equations (1) and (20) provide the relation

$$Z_{in} = \frac{\boldsymbol{A}Z_2 + \boldsymbol{B}}{\boldsymbol{C}Z_2 + \boldsymbol{D}} = \frac{Z_2 + Z_0 \tanh\theta}{Z_0 + Z_2 \tanh\theta} Z_0 \tag{21}$$

Note that $Z_{in} = Z_0$ for $Z_2 = Z_0$ (matched output) and $Z_{in} = Z_0 \tanh\theta$ for $Z_2 = 0$ (short-circuited output).

When the line is fed by a voltage E the voltage and the current at any distance x from the source can also be easily computed using the ***ABCD*** matrix:

$$Z_x = \frac{Z_2 + Z_0 \tanh(\gamma(d-x))}{Z_0 + Z_2 \tanh(\gamma(d-x))} Z_0 \tag{22}$$

$$I_x = \frac{E}{Z_x \cosh(\gamma x) + Z_0 \sinh(\gamma x)} \tag{23}$$

$$V_x = Z_x I_x \tag{24}$$

As a numerical example, consider a two-wire transmission line with the following parameters: $d = 1$ km; $l = 0.55$ mH/km; $c = 24.44$ nF/km; $g = 10$ nS/km; $r = 0.2254$ mΩ/km. These parameter values have been chosen in order to produce a distortionless media, thus providing an easy interpretation of the results. Note, however, that the computation method proposed here also applies to any other parameter values. For this line the surge impedance and the propagation velocity are respectively given by

$$\sqrt{\frac{l}{c}} = 150\Omega; \quad \frac{1}{\sqrt{lc}} = 272750\,\text{km/s} \tag{25}$$

The variation of the impedance Z_{in} with the frequency is shown on the right-hand side of Fig. 3 for a load impedance of $50\,\Omega$. Although the load impedance is constant, the reflected impedance magnitude varies significantly as is typical for a mismatched line. Its maximum is attained at a frequency which corresponds to the quarter-wavelength condition, i.e. $d = 1\,\text{km}$ implies $\lambda = 4\,\text{km}$ and consequently this frequency is $f = 272750/4 = 68.19\,\text{kHz}$. As expected, the phase curve points out that the input impedance presents either an inductive or a capacitive characteristic when the frequency varies. Figure 3 (left) shows the impedance Z_{in} for a load of $450\,\Omega$. The magnitude of the reflected impedance is $450\,\Omega$ for zero frequency and attains a minimum of $50\,\Omega$ at $68.19\,\text{kHz}$. Again, the reflected impedance can be capacitive or inductive, being capacitive for low frequency in this case.

The voltage and current profiles along the line (terminated with $450\,\Omega$) are shown on the left-hand side of Fig. 4 for $f = 136.38\,\text{kHz}$, which corresponds to the $\lambda/2$ condition. Note that the midline voltage is a minimum because the distance from the load is $\lambda/4$ and, accordingly, the current is a maximum. The voltage and current profiles along the line (terminated with $50\,\Omega$) are shown on the right-hand side of Fig. 4 for $f = 136.38\,\text{kHz}$. Note that the midline voltage is a maximum and the current is a minimum in this case.

The frequency responses of the voltage at the load terminals for the line terminated with $50\,\Omega$, $150\,\Omega$ and $450\,\Omega$ are as shown in Fig. 5. Note that the frequency response varies significantly when the load is not matched with the line. This condition should be avoided to prevent distortions when the signal travels down the line.

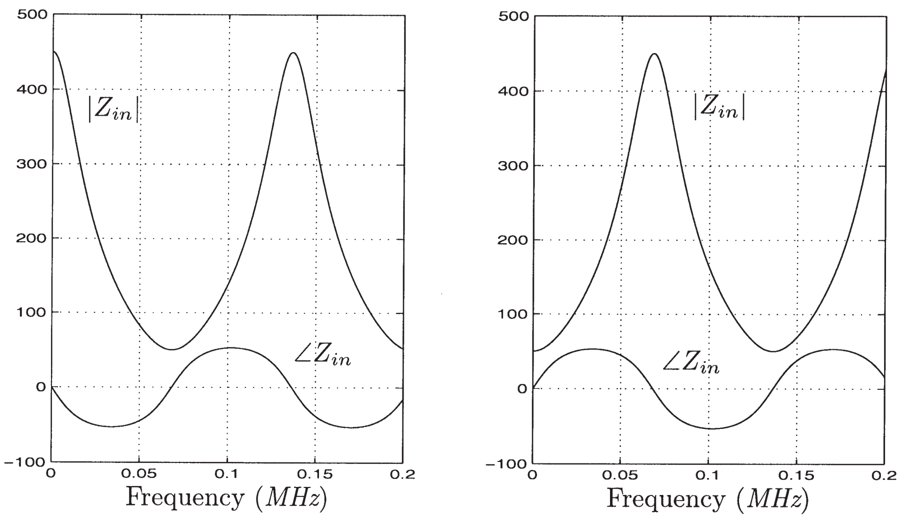

Fig. 3 *Input impedances of the line terminated with 450 Ω (left) and 50 Ω (right) as a function of the frequency (magnitude in ohms, phase in degrees).*

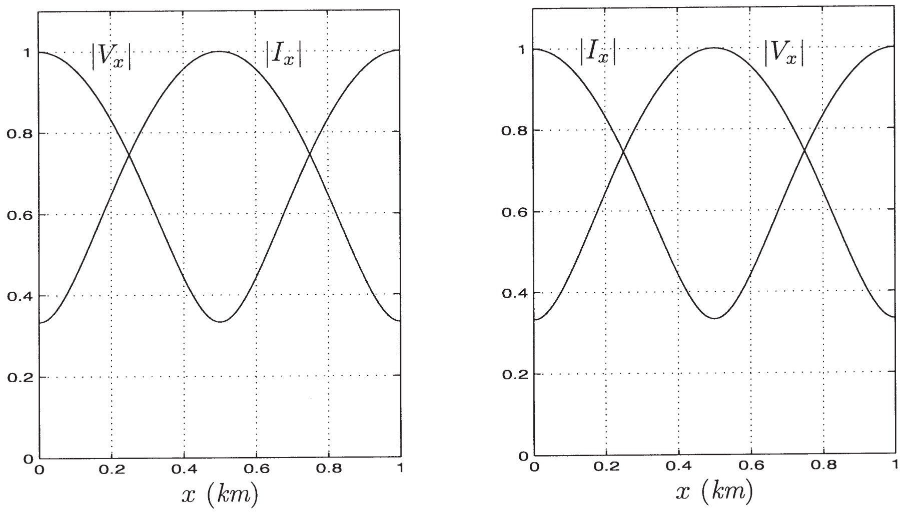

Fig. 4 *Normalized voltage and current profiles along the line terminated with 450Ω (left) and 50Ω (right) for f = 136.38 kHz.*

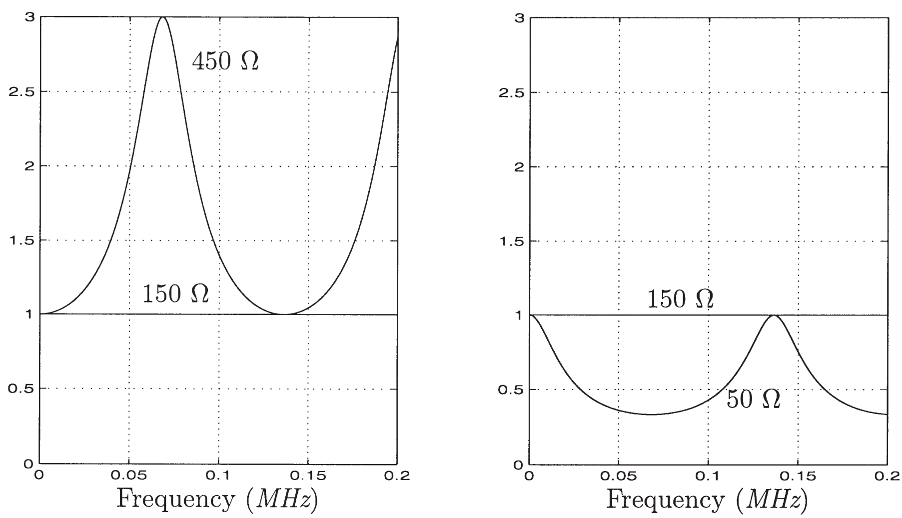

Fig. 5 *Load voltage frequency response for the line terminated with 150Ω and 450Ω (left); 150Ω and 50Ω (right).*

Study case: The balun

The usefulness of the ***ABCD*** matrix as a tool for line computation is highlighted once more in this section by analyzing the *balun* circuit.[5,7] The need for baluns (a contraction for *balanced to unbalanced*) arises in coupling a transmitter to a balanced transmission line as the output circuits of most transmitters have one side grounded. A non-symmetrical balun is also used for matching two circuits with different terminal impedances as when a remote source is connected to a 300 Ω antenna through a 75 Ω coaxial cable. Figure 6 shows a balun implemented via two identical line sections with the same characteristic impedance Z_0, propagation constant λ and length d. As one side is connected to the ground, the two lines must have a length such that the balanced end is effectively decoupled from the parallel-connected end. This requires a length that is an odd multiple of a quarter wavelengths.

Now the ***ABCD*** matrix is used to demonstrate that the source is matched with the load when $Z_1 = Z_0/2$ and $Z_2 = 2Z_0$ and to compute the transfer function of the balun for different line parameters.

From Fig. 6 one gets

$$V_1 = V_1' = V_1''; \quad I_1 = I_1' + I_1'' \tag{26}$$

$$V_2 = V_2' + V_2''; \quad I_2 = I_2' \tag{27}$$

$$I_2' = I_2'' \tag{28}$$

Equation (28) applies when the balun is balanced and parallel-connected ends are decoupled.

For this case the $\mathbb{ABCD}$ matrix for the balun circuit is given by

$$\begin{bmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}/2 & \boldsymbol{B} \\ \boldsymbol{C} & 2\boldsymbol{D} \end{bmatrix} \tag{29}$$

Note that the $\mathbb{ABCD}$ matrix determinant is equal to one.

The image impedances Z_{ti} and Z_{ri} are given by

$$Z_{ti} = \sqrt{\frac{\mathbb{AB}}{\mathbb{DC}}} = \frac{Z_0}{2}; \quad Z_{ri} = \sqrt{\frac{\mathbb{DB}}{\mathbb{AC}}} = 2Z_0 \tag{30}$$

The balun output voltage V_2 is given by

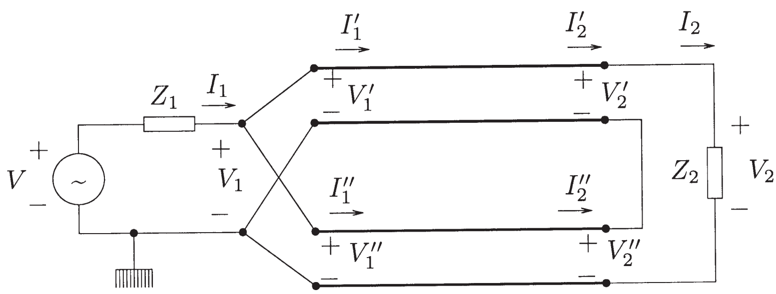

Fig. 6 *Schematic representation of a* balun *(balanced-unbalanced) composed by two parallel transmission lines with characteristic impedance* Z_0. *The* balun *matches a load* $Z_2 = 2Z_0$ *with a source impedance* $Z_1 = Z_0/2$.

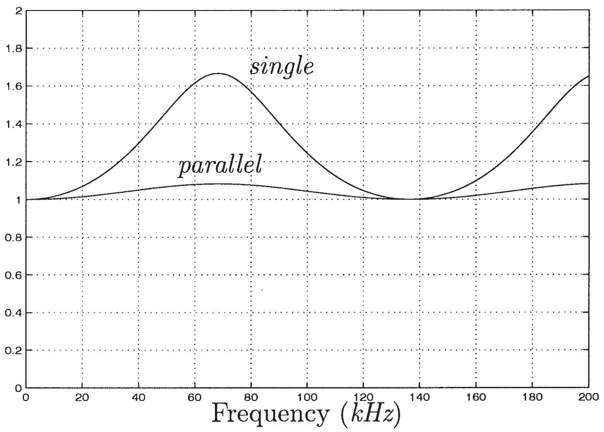

Fig. 7 *Voltage frequency responses (normalized at 0 Hz for comparison). The source and the load impedances are 50 Ω and 450 Ω, respectively, and the characteristic impedance of the line is 150 Ω.*

$$V_2 = V \frac{Z_{in}}{Z_1 + Z_{in}} \frac{Z_2}{\mathbb{A}Z_2 + \mathbb{B}} \tag{31}$$

where Z_{in} is given by

$$Z_{in} = \frac{\mathbb{A}Z_2 + \mathbb{B}}{\mathbb{C}Z_2 + \mathbb{D}} \tag{32}$$

The matching condition $Z_2 = 2Z_0$ implies $Z_{in} = Z_0/2$ and considering eqn (29), $V_2 = V\exp(\theta)$. Therefore, for a lossless balun, the gain is one (absolute value). However, the matching condition is not always present. As a final example, suppose that a remote load of 450 Ω (1 km away from the source, for example) is to be driven by a voltage source (output impedance of 50 Ω) through a transmission line with characteristic impedance of 150 Ω. A simple way to decrease the mismatch effects is obtained by arranging two parallel 150 Ω transmission lines as shown in the balun scheme (Fig. 6), i.e. using series connection at the load and parallel connection at the source. Considering the two situations, namely, *one single line* or *two parallel lines* between the source and the load, Fig. 7 shows the voltage responses at the load terminals as a function of the frequency. As expected, when some arrangement similar to the balun scheme of Fig. 6 is used the response is less sensitive to mismatched conditions.

Conclusion

This paper is mainly aimed at applying the ***ABCD*** matrix as sole tool for the simulation of two-wire transmission line characteristics. The authors' purpose was to draw special attention to the fact that this approach is very convenient for carrying out typical transmission line calculations. Moreover, as shown in this paper, the ***ABCD*** matrix is very easily obtained for a two-wire transmission line from its distributed parameters per unit length. Some typical transmission line behaviours related to matched and mismatched load conditions were shown to be quite conveniently calculated by using the ***ABCD*** matrix as an analysis tool. Its application for the modelling of a line balance converter (*balun*) is presented in this paper as a particular example.

Acknowledgement

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