One Hundred Years of *L'Enseignement Mathématique*

Moments of Mathematics Education in the Twentieth Century

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L'ENSEIGNEMENT MATHÉMATIQUE

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TABLE OF CONTENTS

ANALYSIS

APPLICATIONS OF MATHEMATICS: MATHEMATICS AS A SERVICE SUBJECT

PERSPECTIVES FOR MATHEMATICS EDUCATION

LIST OF PARTICIPANTS

Néstor AGUILERA (Argentina)
Michèle ARTIGUE (France)
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Jean-Pierre BOURGUIGNON (France)
Aldo BRIGAGLIA (Italy)
Daniel CORAY (Switzerland)
Ubiratan D'AMBROSIO (Brazil)
Viviane DURAND-GUERRIER (France)
Trevor FLETCHER (UK)
Ruhal FLORIS (Switzerland)
Fulvia FURINGHETTI (Italy)
Claude GAULIN (Canada)
Livia GIACARDI (Italy)
Hélène GISPERT (France)
Alexandra GOMES (Portugal)
Francine GRANDSARD (Belgium)
Pierre-Paul GRIVEL (Switzerland)
André HAEFLIGER (Switzerland)
Ernst HAIRER (Switzerland)
Gila HANNA (Canada)
Jean-Claude HAUSMANN (Switzerland)
Bernard R. HODGSON (Canada)
Geoffrey HOWSON (UK)
Jean-Pierre IMHOF (Switzerland)
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Jean-Pierre KAHANE (France)
Jeremy KILPATRICK (USA)
Colette LABORDE (France)
Thomas LINGEFJÄRD (Sweden)
Marta MENGHINI (Italy)
Alain MERCIER (France)
Philippe NABONNAND (France)
Mogens NISS (Denmark)
João PITOMBEIRA DE CARVALHO (Brazil)
Luís REIS (Portugal)
André REVUZ (France)
Nicolas ROUCHE (Belgium)
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Mansa C. SINGH (Canada)
Man-Keung SIU (China)
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Charles-Ange LAISANT (1841–1920)

INTRODUCTION

Organized jointly by the University of Geneva and the International Commission on Mathematical Instruction, a symposium on mathematics education was held in Geneva in October 2000 under the theme

"ONE HUNDRED YEARS OF *L'ENSEIGNEMENT MATHÉMATIQUE* :
MOMENTS OF MATHEMATICS EDUCATION IN THE 20TH CENTURY"

It was an occasion for celebrating the 100th anniversary of the international journal *L'Enseignement Mathématique*, founded in 1899 by Henri Fehr (Geneva) and Charles-Ange Laisant (Paris). Among periodicals devoted to mathematics education, the new journal was the first to seek explicitly an international audience; in fact, an original characteristic of the beginnings was a series of articles on the teaching of mathematics in different countries.

The fact that such a journal was launched in Geneva is not surprising if we consider the local context in which it emerged. Apart from obvious references to internationalism as seen from Henry Dunant's birth-place, it is also true that, for centuries, Geneva had been haunted by the myth of its pedagogic vocation.

To describe this strong historical background, it is useful to recall here that the city of Geneva was not a part of the Swiss Confederation until 1815. It had been an episcopal principality for several hundred years — with a relatively large degree of autonomy granted to the people — until it became an independent protestant republic, by a decree of 21st May 1536, in which the *Conseil général* promulgated the Reformation[1]). It is quite remarkable that the same decree set the principle of compulsory education, which was also declared free for the poor:

> [...] que chescung soit tenu envoyer ses enfans à l'escholle et les faire apprendre.[2]) [Encyclopédie de Genève 1986, 127]

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[1]) To avoid a common confusion, it is important to note that this decision was reached under the determining influence of Guillaume Farel, a few months *before* Jean Calvin arrived in Geneva.

[2]) [...] that everyone be required to send their children to school and have them learn.

In 1559 Jean Calvin inaugurated two new institutions: the *Collège* (with seven grades) and the *Académie.* Led by Mathurin Cordier, Théodore de Bèze, and others, both schools attracted numerous students from France, the German States, the Low Countries and the whole of Europe, right from the beginning 3) but even more after the Saint-Barthélemy massacre (1572).

Mathematics and physics were also taught. However, a formal chair of mathematics was introduced in the *Académie* only in 1724, with Jean-Louis Calandrini and Gabriel Cramer as professors.

In spite of these auspicious beginnings, many questions were raised throughout the 18th century about the quality, the content and the methods of instruction. The best known name in this connection is that of Jean-Jacques Rousseau, who was concerned with pedagogy 4). But the discussion, which involved famous scientists like Horace-Bénédict de Saussure, went on more specifically about the respective roles of the sciences and the humanities 5), or the creation of technical schools.

At the beginning of the 19th century, many educators were active on this fertile soil. One was Rodolphe Töpffer, a professor of Greek and rhetoric in Geneva's *Académie* who created his own pedagogic system and even opened a boarding school. He is also viewed by some as the inventor of the comic strip.

The cartoons selected here 6) illustrate, with a specific hint at mathematics, the kind of animated debate that was going on about the organization of primary and secondary schools (public or private) at that time. They are drawn from *Monsieur Crépin* [Töpffer 1837, 11–12], a comic strip published in Geneva in 1837. They depict a father who is trying to find a suitable preceptor for his children. In the preceding cartoon one of the first teachers he has hired explains that "all pupils now begin to proceed very well from the general to the particular". An example is provided by the cartoon about Besançon. The father is of course rather unhappy, even though the answer is mathematically correct. The next cartoon shows how the system can fail if the pupil is not particularly gifted. Considering the early date (1837), any allusion to Bourbaki and the 'new maths' movement in the third cartoon would be sheer anachronism. But the questions were already very clearly stated.

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3) John Knox, Jean de Léry, and other distinguished names, were among them.

4) *Émile, ou de l'éducation* was published in the Hague and Paris in 1762, … and burnt in Geneva in that same year!

5) Saussure published a *Projet de réforme pour le Collège* in 1774; cf. [Encyclopédie de Genève 1988, 13–14].

6) reproduced by the kind permission of Éditions du Seuil.

The teacher summons Joseph and asks him: "Where is Besançon?" Joseph instantly replies that Besançon is in the set of all things, which includes the Universe, which includes the World, which includes all four parts of the World, which include Europe, where Besançon lies.

Having called Leopold, Mr. Crépin himself asks how much will eight pounds of lard cost, at five florins a pound. Leopold instantly replies that lard is in the set of all things, which includes the Universe, which includes all three reigns, which include the animal reign, which includes the pig, which includes lard.

Mr. Crépin finds that his son Leopold is little advanced in Arithmetic. The teacher explains that, in his system, Arithmetic is the very last thing that Leopold will know. Indeed, he must first know Algebra, which he will begin to learn only after an in-depth study of quantity in general.

The *Académie* was itself undergoing a process of transformation which was giving it a status more in line with that of other universities in Europe [7]). Since Fehr began his studies and completed his doctorate in Geneva, he was certainly aware of these discussions. In connection with Furinghetti's article in this monograph, we can also mention that, as a professor in the University, he had Théodore Flournoy and Édouard Claparède as colleagues.

The first prefaces of *L'Enseignement Mathématique* show that Fehr and Laisant also wanted to associate the world of teaching to the "*great movement of scientific solidarity*" which was emerging at the end of the 19th century,

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[7]) The *Académie* evolved into the University of Geneva in 1872, with a Faculty of Medicine, independence from the clergy, etc.

notably through the organization of international meetings such as the first International Congress of Mathematicians held in Zurich in 1897. The journal immediately obtained important successes, as is testified by the gold medal at the World Fair of Brussels in 1905.

The idea of internationalism in mathematics education was crucial to the journal right from its beginning and it even led to some articles in or about Esperanto. Moreover the frequent advocacy in the journal of the importance of an international perspective played an essential role in the establishment, a few years later, of the International Commission on Mathematical Instruction.

The two editors Fehr and Laisant had proposed in 1905 to organize an international survey on the reforms needed in mathematics education, asking in particular for opinions on the following theme: "les conditions que doit remplir un enseignement complet, théorique et pratique, des mathématiques dans les établissements supérieurs" [8]) [La Rédaction 1905, 382].

In response to a question on the progress needed in the organization of the teaching of pure mathematics, the US mathematician and teacher educator, David Eugene Smith, explicitly suggested the establishment of an international commission to study the situation:

> Pour ce qui est de la première question, j'estime que la meilleure manière de renforcer l'organisation de l'enseignement des mathématiques pures, serait de créer une commission qui serait nommée par un Congrès international et qui étudierait le problème dans son ensemble. [9]) [Smith 1905, 469]

Smith repeated this suggestion in a report he presented at the Fourth International Congress of Mathematicans, held in Rome in April 1908. This Congress then adopted a resolution [10]) to the effect of appointing a committee, composed of Felix Klein (Germany) as President, George Greenhill (Great Britain) as Vice-President, and Henri Fehr (Switzerland) as Secretary-General, with the mandate to constitute an International Commission to organize a comparative Study on the methods and plans of mathematics teaching in secondary schools. This International Commission eventually developed a much wider scope of interest and became the ICMI as we know it today.

The coincidence of the spirit of internationalism between the newly established Commission on teaching and the journal *L'Enseignement Mathé-*

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[8]) The conditions to be satisfied by a complete — theoretical and practical — teaching of mathematics in higher institutions.

[9]) As regards the first question, I consider that the best way to reinforce the organization of the teaching of pure mathematics, would be the establishement of a committee appointed by an international Congress and which would study the problem in its entirety.

[10]) See in this connection *L'Enseign. Math. 21* (1920), 306.

matique, possibly combined with the presence of Fehr being active in both groups, resulted in one of the very first decisions of ICMI being to select the journal as its official organ. This was announced simply as follows in the journal as part of a report on the decision of the Rome Congress to establish an International Commission : "*L'Enseignement mathématique* servira d'organe à la Commission, dont la tâche se rattache très intimement à celle que poursuit notre revue internationale depuis dix ans." [11]) [Chronique 1908, 333]

The explicit mention of the journal as the official organ of ICMI can be seen on the front cover of the first issue following the inception of the Commission (volume 11, 1909). And it has been appearing on the cover ever since then, except for some periods when ICMI was inactive around the Second World War.

Another distinguishing feature of *L'Enseignement Mathématique* at the outset was a keen interest of the journal in the social role of mathematics and of science in general. Laisant in particular was the author of several articles on these topics. To give an idea of his fairly optimistic vision, we may quote : "Si la science pouvait devenir exclusivement utilitaire, elle perdrait sa plus grande utilité." [12]) [Laisant 1904, 342], and also : "Mesurer une science à son utilité est presque un crime intellectuel." [13]) [Laisant 1907, 121].

The aim of the symposium organized by the University of Geneva and ICMI was to look at the evolution of mathematics education over the twentieth century and to identify some guidelines and trends for the future, taking into account, among other sources, the documents, debates and related papers having appeared in *L'Enseignement Mathématique.* The emphasis was on secondary education (students in the age range of about 12 to 18 or 19 years) and also included the education of teachers.

The programme of the symposium was based on a series of invited talks. The Programme Committee [14]) had identified three main themes to be discussed — geometry, analysis, and applications of mathematics — and three different periods at which these themes were to be considered : 1900, 1950 (*i.e.* the period leading to the 'new maths'), and 2000.

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[11]) *L'Enseignement Mathématique* will serve as organ of the Commission, whose task is very intimately linked to the aim our international journal has been pursuing for ten years.

[12]) If science could become exclusively utilitarian, it would lose its greatest usefulness.

[13]) Measuring a branch of science by its usefulness is something like an intellectual crime.

[14]) The members of the Programme Committee of the Symposium were Daniel Coray (Switzerland), Fulvia Furinghetti (Italy), Hélène Gispert (France), Bernard R. Hodgson (Canada) and Gert Schubring (Germany).

It was also a gathering of some of the main actors, during the last decades, in mathematics education as considered from an international perspective, and ample time was devoted during the symposium to collective discussions on the themes presented in the talks. In this connection some participants had been invited to play the role of 'reactors'. They had the responsibility, in each session, of launching the discussion following the invited talks, partly by giving a synthesis of the presentations but more importantly by outlining the major trends and issues about the theme, both in the light of the past century and as seen from today's perspective.

This explains the subdivision adopted in this book. We must add that every contribution, in either English or French, is preceded by a fairly long abstract in the other language.

We hope these *Proceedings* will contribute to a better understanding and appreciation, among the communities of mathematicians and mathematics educators, of the evolution of mathematics education during the 20th century. The book that we are proposing to the reader aims at reflecting the spirit and the work of a symposium which, in the words of Geoffrey Howson, demonstrated

> how over the century the emphasis shifted from discussions of the mathematics to be taught to an élite, to the needs of a wider range of students and of society. […]
>
> It reminded us of the way in which two generations had tried to make enormous changes in the content of school mathematics and methods of teaching it. It gave us an opportunity to see where these earlier efforts had not been wholly successful and challenged us to determine why. With such an understanding we should be better equipped to tackle both the problems that now face us and those which will arise in the future. [Howson 2001, 183]

This endeavour would not have been possible without the generous support of the following institutions: Swiss Academy of Sciences, Swiss National Science Foundation, Commission Administrative de l'Université de Genève, Faculté des Sciences (University of Geneva), and Société Académique (Geneva). We are very grateful to all of them.

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Henri FEHR (1870–1954)

L'ENSEIGNEMENT MATHÉMATIQUE : BIRTH AND STAKES

MATHEMATICAL INSTRUCTION IN AN INTERNATIONAL PERSPECTIVE: THE CONTRIBUTION OF THE JOURNAL *L'ENSEIGNEMENT MATHÉMATIQUE*

L'éducation mathématique dans une perspective internationale : la contribution de la revue L'Enseignement Mathématique

par Fulvia FURINGHETTI

La revue *L'Enseignement Mathématique* a été fondée en 1899, dans un moment de grand ferment social et culturel qui touchait notamment les mathématiques. D'une part les États s'étaient dotés d'une organisation moderne comportant, entre autres, la mise en place de systèmes nationaux d'instruction. D'autre part, la recherche mathématique s'était développée dans différentes directions (pures ou appliquées), et ce de manière très efficace. Plusieurs revues de recherche mathématique étaient alors régulièrement publiées dans divers pays et le premier Congrès international des mathématiciens avait été organisé à Zurich en 1897. Dans le domaine de l'enseignement des mathématiques, des associations nationales d'enseignants avaient vu le jour dans de nombreux pays et des revues avaient été créées portant uniquement sur l'enseignement de cette discipline.

Les idées centrales faisant de *L'Enseignement Mathématique* un journal remarquable dans ce panorama sont l'*internationalisation* et la *communication*. Les fondateurs de la revue — le Suisse Henri Fehr (1870–1954) et le Français Charles-Ange Laisant (1841–1920) — en furent les directeurs jusqu'à leur mort. Dans la présentation du journal publiée dans le premier numéro, ils écrivaient que le monde de l'enseignement devait s'associer au grand « mouvement de solidarité scientifique ». Ils soulignaient aussi l'importance des réformes des programmes et du problème de la formation des enseignants. Pour aborder ces questions, ils préconisaient que l'on compare les systèmes d'instruction dans les différents pays et que les enseignants échangent leurs points de vue.

Le présent texte porte sur les premières années du journal, jusqu'à la première Guerre mondiale, au cours desquelles le projet des fondateurs a trouvé un terrain favorable à son développement. A cette époque, le « Comité de patronage » du journal comprenait des personnages importants du milieu mathématique (directeurs de revues mathématiques, historiens, mathématiciens), tous avec un intérêt marqué pour les problèmes liés à l'éducation mathématique. Le journal publiait

- des articles généraux,
- des nouvelles du monde académique,
- des annonces bibliographiques et des comptes rendus de livres, articles et conférences,
- des correspondances diverses,
- des enquêtes lancées par le journal, ainsi que les résultats accompagnés de commentaires.

La langue utilisée alors est quasi exclusivement le français. Les articles généraux abordent des thèmes mathématiques d'intérêt pour l'enseignement aux niveaux secondaire et tertiaire, ainsi que l'histoire, la philosophie, l'épistémologie et la psychologie. Les auteurs des articles et des lettres étaient pour la plupart français ou suisses, mais on trouvait aussi des contributions provenant d'autres pays (Algérie, Allemagne, Argentine, Autriche et les régions de son ancien empire, Belgique, Danemark, Espagne, États-Unis, Grèce, Italie, Japon, Pays-Bas, Portugal, Roumanie, Royaume-Uni, Russie, Ukraine). Déjà au cours de la première année et durant les années subséquentes, on trouve dans la revue des articles sur la situation de l'enseignement des mathématiques dans divers pays, en conformité avec les buts d'information et de communication du journal.

La correspondance permit l'établissement de contacts entre les lecteurs et fut à l'origine d'initiatives intéressantes. Par exemple, c'est une lettre d'un lecteur qui inspira une enquête lancée par le journal sur les méthodes de travail des mathématiciens. Cette enquête fut réalisée au moyen d'un questionnaire de 30 questions adressées aux lecteurs et autres mathématiciens intéressés. Le questionnaire fut préparé par Fehr avec l'aide de deux psychologues de l'Université de Genève, Édouard Claparède et Théodore Flournoy. Le journal publia le questionnaire ainsi que l'analyse des réponses et des commentaires sur les résultats, rédigés par Fehr, Claparède et Flournoy. Cette enquête contribua à attirer l'attention des mathématiciens (entre autres, Henri Poincaré) et des enseignants sur certains thèmes reliés à la psychologie et sur le problème de l'invention en mathématiques.

La discussion sur les réformes à accomplir dans l'enseignement des mathématiques, lancée en 1905 par des lettres de mathématiciens (parmi lesquels David E. Smith et Gino Loria), fut à l'origine d'un mouvement plus général qui aboutit à la fondation de la Commission internationale de l'enseignement mathématique (CIEM/ICMI) lors du Congrès international des mathématiciens tenu à Rome en 1908. A partir de 1909, le journal devint d'ailleurs l'organe officiel de la CIEM et on y publia régulièrement les annonces, actes des rencontres et enquêtes de la Commission. Fehr fut Secrétaire général de la CIEM depuis sa création jusqu'au début des années 50.

MATHEMATICAL INSTRUCTION IN AN INTERNATIONAL PERSPECTIVE: THE CONTRIBUTION OF THE JOURNAL *L'ENSEIGNEMENT MATHÉMATIQUE*

by Fulvia FURINGHETTI

INTRODUCTION

Among all the mathematical journals of the nineteenth century *L'Enseignement Mathématique* plays a particular role [1]). It was founded in 1899 by Charles-Ange Laisant and Henri Fehr and is still published today [2]). To understand why its role is so interesting I will analyse the first years of its life, in which it made a strong contribution to the birth of an international community of mathematics educators, alongside the community of mathematics researchers. We shall see that this analysis provides us with the opportunity to rethink important aspects of the history of mathematical instruction, including the birth of ICMI.

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1) Appendix 1 gives an idea of the range of the mathematical press when *L'Enseignement Mathématique* was founded. See [Friedelmeyer 1996] for further information on early mathematical journals.

2) At first the publishers were G. Carré and C. Naud in Paris (since 1902 C. Naud alone). In 1904 the administration and the printing house were transferred to Geneva, and the publishers were Gauthier-Villars (Paris) and Georg & Co. (Geneva). The present publisher is an independent foundation linked to the Department of Mathematics of the University of Geneva. The printer has been KUNDIG (Geneva) since 1903, with slight variations of the commercial name: W. Kündig & fils, Albert Kündig, Albert Kundig ('*umlaut*' removed in 1914 !), etc., now SRO-Kundig.

In the first issue of *L'Enseignement Mathématique* the editors explained the aims of the journal in the following terms:

> A l'heure où la science a tant progressé, certaines simplifications peuvent être désirables, les programmes des diverses branches de l'enseignement appellent des réformes plus ou moins complètes. Et avec cela, il y a une question fondamentale dont on ne saurait méconnaître l'importance : c'est celle de la préparation du corps enseignant.
>
> Toutes ces transformations ne sauraient s'accomplir brusquement, ni sans de sérieuses réflexions préalables. Mais, pour procéder à une telle étude d'une façon judicieuse et utile, la première des conditions n'est-elle pas de connaître ce qui se passe dans les autres pays, de savoir quel est dans chacun d'eux le mode d'organisation de l'enseignement mathématique, quels sont les programmes en vigueur, les moyens de sanction des études, etc. ? […] Malgré les relations fréquentes qui se sont établies à notre époque entre savants qui cultivent un même sujet d'étude, malgré les congrès internationaux, si brillamment inaugurés à Zurich en 1897 et dont le principe est désormais établi, le monde de l'enseignement proprement dit n'a pu s'associer jusqu'à présent à ce grand mouvement de solidarité scientifique aussi pleinement qu'il eût été désirable. […]
>
> Nous avons voulu, par la publication de notre Revue, renverser les obstacles qui s'opposent à ces communications réciproques et créer une sorte de correspondance mutuelle, continue, entre les hommes qui ont consacré leur vie à cette noble mission : l'éducation mathématique de la jeunesse.
>
> En vue de ce résultat, notre premier soin a été de donner à la publication périodique dont il s'agit un caractère franchement et hautement international. [3])
>
> [Les Directeurs 1899, 1–2]

The target readership of the journal was those who taught mathematics (at secondary and tertiary level). As stated by the directors in the introduction to the sixth volume of *L'Enseignement Mathématique* :

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[3]) At a time when science has made so much progress, some simplifications may be desirable, the programmes of the different branches of teaching require more or less complete reforms. Linked to this there is a fundamental issue whose importance should not be overlooked : namely that of the education of teachers.

All these changes ought not to be carried out in an abrupt way, nor without serious preliminary reflections. But is not the first requirement for proceeding in such a study in a judicious and fruitful way, to be aware of what happens in other countries, to know the way mathematics teaching is organised in each of them, what teaching programmes are in force, the methods by which the studies are approved, etc. ? […] Despite the strong rapport that has been established today among scientists of the same field, despite the international congresses of mathematicians, so brilliantly inaugurated in Zurich in 1897 and accepted as a principle for the future, the world of education proper has not up to now been able to join this great movement of scientific solidarity as fully as would have been desirable […].

It has been our wish, through the publication of our Journal, to overcome the obstacles to reciprocal *communications* [my italics] and to create a kind of continuous mutual correspondence between men who have devoted their lives to this noble mission : the mathematical education of young people.

In view of this aim, our first concern has been to give this periodical a clearly and openly *international* [my italics] character.

> Le mot « Enseignement » a pour nous la signification la plus large. Il veut dire enseignement des élèves, et aussi enseignement des professeurs — et d'ailleurs l'un ne va guère sans l'autre. [4]) [*L'Enseign. Math. 6* (1904), 4]

To consider those who teach as persons accomplishing a 'noble mission', rather than persons following a profession, has been a typical attitude of the past.

The social role of science and links with progress in its various forms (industry, technology, …) are among the concerns of the editors. The key words of the journal's programme are 'internationalism', 'information', 'communication' and, of course, 'teaching mathematics'. Professional mathematicians were invited to collaborate actively with the project of the journal in order to keep the teaching of mathematics in touch with the advances in the subject itself.

Behind the ideas of internationalism and solidarity expressed in the journal there lay social and political ideals, very much alive in society at the time of its foundation, but which slowly withered away in the new century. The following passage hints at the changes in the international atmosphere and their consequences for the life of the journal :

> La guerre européenne porte un coup sensible aux institutions internationales. Dans les pays belligérants et dans les pays neutres voisins tout ce que la nation compte d'hommes valides est sous les drapeaux. Il devient donc matériellement impossible de continuer les travaux faisant appel à de nombreux collaborateurs. Les œuvres de paix telles que la nôtre passent à l'arrière plan. D'ailleurs, poursuivant un idéal commun librement choisi, elles exigent une volonté d'union qu'on ne saurait demander aux savants dans une période aussi troublée que celle que nous traversons. [5]) [Fehr 1914, 477]

I consider 1914 (the year of the beginning of the First World War) as the limit of the journal's pioneering period, in which it found a propitious environment in the international social atmosphere. My analysis of the journal goes [6]) from the year of the foundation (1899) to this crucial year 1914.

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[4]) The word '*Enseignement*' [teaching] has for us the widest possible meaning. It means the teaching of pupils, as well as the teaching of teachers — and, indeed, you can hardly have the one without the other.

[5]) The European war carries with it an appreciable impact on international institutions. In the fighting countries and in the neighbouring neutral countries all those whom the nation considers to be able-bodied are in uniform. Thus it becomes practically impossible to continue with activities requiring the support of numerous collaborators. Peaceful activities like ours have to take second place. Furthermore, being in pursuit of a freely chosen common ideal, they demand a willingness for unity, which ought not to be asked of scientists in a period as troubled as the one in which we are living now.

[6]) The steps of the history of the journal until the 1970s are sketched in [de Rham 1976].

THE FOUNDERS

As is well known, early mathematical journals were strongly linked to their founders, to the extent that many of them were referred to as the journal of its founder. For example the *Journal für die reine und angewandte Mathematik* (founded by August Leopold Crelle in Berlin, first issue in 1826) was simply known as ‘Crelle’s Journal’. This was also the case for the *Journal de mathématiques pures et appliquées* (founded by Joseph Liouville in Paris, first issue in 1836) and the *Giornale di Matematiche ad uso degli studenti delle università italiane* (founded by Giuseppe Battaglini in Naples, first issue 1863). In some cases the editors were also the owners of the journal. In journals devoted to mathematics teaching the importance of the editorial line is twofold: on the one hand cultural (in connection with mathematics), on the other hand social (in connection with systems of instruction). For the period we are considering, the editorial line of *L’Enseignement Mathématique* may be seen as a real expression of the personalities of its two founders and editors.

Charles-Ange Laisant was born in Basse-Indre (France) on 1st November 1841 and died in Paris (1920). In his obituary he is described as “homme de science, éducateur, philosophe et politicien” [Buhl 1920, 73] (see also [Sauvage 1994; Pascal 1983]). Both his life and his contributions to the mathematical community are a combination of all these characteristics. He studied at the *École polytechnique* in Paris. His first mathematical works were on the application of the theory of equipollence and the method of quaternions (see [Ortiz 1999; 2001]). Afterwards he turned predominantly to social and instructional themes. His view of society, based on values such as solidarity, collaboration and communication stimulated his enterprises in the community of mathematicians. In 1894 he founded (with Émile Lemoine) the journal *L’Intermédiaire des Mathématiciens*, which was conceived as a means of providing contact between mathematicians through the exchange of questions (which were something more than mere exercises) and answers, together with bibliographical references. In the first issue of that journal [*1* (1894), question 212, 113] the idea of the organisation of international congresses of mathematicians, to be held at regular intervals, was explicitly launched. Laisant also edited the *Annuaire des Mathématiciens* (C. Naud, Paris, 1902), a publication that contained the names and addresses of living mathematicians, of scientific societies and of scientific periodicals. He was, with Poincaré, a member of the commission charged with the production of the *Répertoire bibliographique des sciences mathématiques* (a precursor of the present journals of mathematical reviews).

In the field of mathematical education and instruction Laisant's works on pedagogy of mathematics were greatly valued by his contemporaries in France and abroad. His educational and philosophical leanings are evident in the editorial line of the journal and in the contributions he made to *L'Enseignement Mathématique*. For example, Buhl [1920] writes that Laisant's ideas were imbued with the philosophy of Auguste Comte and, indeed, in the second year of *L'Enseignement Mathématique,* we find an article on the philosophy of mathematics of Comte (see [Vassilief 1900]). Interesting aspects of Laisant's personality are shown by his contributions to French political life. Ortiz [2001] reports that he was a member of the national parliament in the Third Republic, up to the end of the century. In France this was a period of social changes which had repercussions in the scientific world. It was held that advancement in science occurs according to criteria similar to those necessary for industrial development. As Ortiz put it, Laisant showed

> cierta coherencia entre las agendas científica y política […], ambas muestran una preocupación seria por introducir ideas nuevas, por abrir nuevas posibilidades, y por buscar unidad dentro de una aparente diversidad. [7]) [Ortiz 2001, 83]

Henri Fehr was born in Zurich on 2nd February 1870 and died in Geneva (1954). He studied in Switzerland and afterwards in France. His doctoral dissertation was on the method of Grassmann vectors applied to differential geometry. Jacques Hadamard gave a positive review of this work. Fehr became professor at the University of Geneva (Sciences Faculty) and later dean, vice-rector and rector. Further information on Fehr is in [Anonymous 1955; de Rham 1955; Ruffet 1955].

A prominent characteristic of Fehr's personality was his interest in the social aspects of the mathematical community and academic life. He was regarded as a "pédagogue exceptionnel" [Anonymous 1955, 7], but he was also involved in social commitments such as the committee of the fund of pensions of his colleagues. He applied his skill as an organiser in founding [8]) the Swiss Mathematical Society (of which he was president), the Foundation for the Advancement of Mathematical Sciences, and the journal *Commentarii Mathematici Helvetici*. He received national and international honours and appointments. Fehr participated, as his country's delegate, in the International Congresses of Mathematicians and was vice-president of the ICMs in Toronto

[7]) clear consistency between his political and scientific behaviours. Both in politics and in sciences he was seriously concerned with the introduction of new ideas, of opening up new possibilities, and looking for unity in things that appear different.

[8]) with R. Fueter and M. Grossmann

(1924) and Bologna (1928). He was also vice-president of IMU. He was one of the founders of ICMI (in 1908), of which he was the secretary-general until the Second World War. When ICMI was reconstituted (in Rome, 1952) Fehr was one of the members of the special committee of five appointed to draw up a plan of work in preparation for the International Congress of Mathematicians to be held in Amsterdam (in 1954). When this committee co-opted new members to form an ICMI Executive Committee, Fehr was again chosen to be the honorary president of this new committee (see [Behnke 1951–1954]). Throughout all these years Fehr was the real soul of ICMI.

The main contribution of Fehr as author of articles in the journal was in the field of mathematics instruction. He singled out some central points, such as :

- innovations in the mathematical programmes and their links with the development of science and technology, as an echo of what was happening in many countries (see [Schubring 1996]) ;
- the relationship between pure and applied mathematics and its influence on the mathematics teaching ;
- the education of mathematics teachers ;
- new trends in mathematics teaching.

From a reading of what has been written about Laisant and Fehr one is left with the impression that these two men enjoyed the esteem of all those who had contact with them. As a pair they are a good example of the integration of interests and strengths, which was essential to the enterprise of editing *L'Enseignement Mathématique*.

The format of the journal

L'Enseignement Mathématique was published every two months. Initially (see [Les Directeurs 1899]) all papers were published in French and this continued to be the predominant language, despite the fact that the editorial address of 1913 states that papers written in the official languages of the International Congresses of Mathematicians (English, French, German and Italian) were accepted, with Esperanto also allowed. One of the very few articles appearing in a non-French language (followed by a long summary in French) is "The principles of mathematics in relation to elementary teaching", the text of the talk delivered at the International Congress of Mathematicians in Cambridge (1912) by A. N. Whitehead [1913]. The admission of Esperanto was in line with the atmosphere of internationalism advocated by the editors and the strong interest in 'international languages' shown by the mathematical *milieu* (see [Roero 1999]). Actually a few contributions to the journal were written in international languages, two of them by Charles Méray [1900; 1901] illustrating the advantages of Esperanto for the internationalism of science, one by M. Frechet [1913].

From 1903, the French mathematician Adolphe Buhl (1878–1949) was added as collaborator to the editorial board and, after Laisant died, Buhl became one of the editors of the journal and held that office until his death [9]). Until 1914 the journal had a *Comité de patronage*. The international nature of the journal can be gauged from the names of the members of the first *Comité* (1899):

Paul Appell, Paris
Nicolas Bougaiev (Bougajeff), Moscow (until 1903)
Moritz Benedikt Cantor, Heidelberg
Luigi Cremona, Rome (until 1903)
Emanuel Czuber, Vienna
Zoel García de Galdeano y Yanguas, Zaragoza
Alfred George Greenhill, Woolwich, England
Felix Klein, Göttingen
Valerian Nikolajwitsch Liguine (Ligin), Warsaw (until 1900)
Paul Mansion, Gent
Gösta Magnus Mittag-Leffler, Stockholm
Gabriel Oltramare, Geneva (until 1906)
Julius Peter Christian Petersen, Copenhagen (until 1910)
Émile Charles Picard, Paris
Henri Jules Poincaré, Paris (until 1912)
Pieter Hendrik Schoute, Groningen (until 1913)
Kyparissos Stephanos, Athens
Francisco Gomes Teixeira, Porto
Alexandr Wassiljewitsch Vassilief (Wassilief), Kazan
Alexander Ziwet, Ann Arbor, Michigan

The following new members took the place of deceased members: in 1904 Vasiliy Petrovich Ermakof (Ermakoff), Kiev; Andrew Russell Forsyth, Cambridge (until 1910); Gino Loria, Genoa; David Eugene Smith, New York; in 1907 Jérôme Franel, Zurich [10]). The composition of the *Comité* covered a wide range of abilities and interests including research in mathematics or in the history of mathematics, editing journals, the writing of books, teacher education and mathematics instruction.

From 1905 the journal carried the subtitle "Méthodologie et organisation de l'enseignement. Philosophie et histoire des mathématiques. Chronique scientifique – Mélanges – Bibliographie". Following the inauguration of ICMI (in Rome, 1908) the lives of ICMI and of the journal were intertwined; since 1909 the frontispiece of the journal has described itself as the "Organe officiel

[9]) For further information on Buhl see [Fehr 1942–1950].

[10]) The *Comité de patronage* ceased to exist in 1915.

de la Commission internationale de l'Enseignement mathématique" [11]).

The normal practice (both in the past and today) is for journals to contain a set of articles and a small part devoted to selected announcements and reviews of books and articles appearing elsewhere. In *L'Enseignement Mathématique* the main articles were only a part of the journal, the other sections being just as important. In 1904, when the administration and the printing services moved from Paris to Geneva, the editors clearly explained the structures of the journal and the purposes of the different sections. They stressed again their intention to foster communication between researchers and teachers.

As an example of the format of the journal, we can look at the organisation of the material published in 1905 (the seventh year of the journal, the middle of the period under consideration). We find the following sections:

A) *Articles généraux* (General articles: Methodology and organisation of teaching, History and Philosophy)

B) *Mélanges* (Miscellanies)

C) *Correspondance* (Correspondence)

D) *Chronique* (News)

E) *Notes et documents* (Notes and documents)

F) *Bibliographie* (Bibliography: Reviews or simple announcements of treatises)

G) *Bulletin bibliographique* (Bibliographical bulletin: Content of the main mathematical journals)

The categorisation of the various contributions was not rigid and absolute; thus sometimes the same type of contribution appeared under different headings. During these years there was little change in the way the material was organised (e.g. in names of the sections, the distribution of the subjects, etc.) in order to reflect accumulated editorial experience or requests from readers, although the journal always maintained its general character of being flexible, open and multipurpose. Major changes occurred, however, when the journal became the official organ of ICMI in 1909, after which it carried reports of meetings of the commission as well as proposals for ICMI studies and their outcomes. We can obtain a better view of the nature of the journal from some detail of the contents published under the various headings.

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[11]) I will use the acronym ICMI, but in the journal the commission was termed CIEM (Commission Internationale de l'Enseignement Mathématique). In Germany the acronym was IMUK (*Internationale mathematische Unterrichtskommission*). In *The Mathematical Gazette* (1912) the commission was termed 'International Commission on Mathematical Teaching' or 'International Commission on the Teaching of Mathematics'. In the presentation of the questionnaire for the inquiry into the training of secondary teachers of mathematics [*L'Enseign. Math. 17* (1915), 129–145] we find 'International Commission on Mathematical Education' in the English translation and 'Commissione internazionale dell'insegnamento matematico' in the Italian one. Now ICMI stands for 'International Commission on Mathematical Instruction'.

GENERAL ARTICLES. Section *A*, and sometimes Section *B*, contained the sort of articles one may expect in a standard mathematical journal. The general articles were classified into various sub-sections. Over time, one of these sub-sections, 'Organisation of teaching', became of increasing importance. After the birth of ICMI this sub-section contained the various business material of the commission. Other issues concerning ICMI appeared also in the sections 'Chronique' and 'Notes et documents'.

Some of the articles, especially those by important mathematicians, had been published before elsewhere. Often papers roused discussion among the readers and reactions were published in the correspondence.

Until 1903, besides the section 'General articles' there was a section 'Études pédagogiques' (Pedagogical studies), explicit evidence of the editors' wish to stress the links with mathematics teaching.

MISCELLANIES. This section contained short articles of various types. In some years this part was attached to the correspondence section.

CORRESPONDENCE. In the 'Correspondance' section there were interesting contributions which, in some cases, could almost be considered as articles in their own right. From the point of view of interpreting the spirit of the journal this section is important in helping us to understand the readership and the kind of problems that interested them. We shall see later that it was a reader's letter that inspired a major inquiry launched by the journal (into the methods of working of mathematicians). Letters are also important in showing that countries all over the world were reached by the journal, even if letters came predominantly from Europe.

Major mathematicians figured among those who reacted to the articles published in the journal: L. E. J. Brouwer [*13* (1911), 377–380] who commented on a paper by G. Combebiac on the theory of measure, G. Peano [*8* (1906), 315–316] who reacted to the papers by E. Carvallo and V. Jamet on the convergence of series.

NEWS. This section included all kinds of news about the mathematical community: announcements of death, awards, meetings, proposals of new national programmes, monuments to be erected, the activity of societies and academies. Thanks to this section the reader was able to participate in the life of the mathematical community. This section is also important for present historians for the variety of information it provides.

NOTES AND DOCUMENTS. This section is devoted mainly to providing information on academic courses (topics developed, names of the professors). When the journal became the official organ of ICMI this section contained the Proceedings of the works of the national sub-commissions.

BIBLIOGRAPHY. This section contained short reviews of treatises. It is valuable for us today, both in giving us the reactions of contemporary readers to treatises that afterwards became famous, and also in providing us with information about forgotten works.

BIBLIOGRAPHIC BULLETIN. This section provided information about new books (authors, publishers and town of publication, format, and price) and the tables of contents of some important mathematical journals, and proceedings of academies and societies. Journals of history of mathematics and elementary mathematics are also mentioned. The list of the publications is not exhaustive, but is a very rich source of information (see Appendix 1).

THEMES AND AUTHORS

To give an idea of the themes treated in the journal, the contributions that appeared in the sections 'General articles' and 'Pedagogical articles' (excluding editorial notes published at the beginning of the volumes) have been classified in the Table below under the following categories [12]:

Alg	Algebra	**Met**	Methodology in teaching
Ana	Analysis	**Mod**	Mathematical models
App	Applications	**Nom**	Nomography
Ari	Arithmetic	**NSu**	National surveys of systems of instruction
Ast	Astronomy		
Esp	On the Esperanto language	**Org**	Organisation of mathematical instruction
FMa	Financial mathematics		
Fou	Foundational themes	**Phi**	Philosophical themes
Geo	Geometry	**PhM**	Philosophy of mathematics
His	History (including obituaries)	**Pro**	Probability
IC	ICMI	**Psy**	Psychology
Inq	Inquiries	**Soc**	Society
Log	Logarithms	**ToN**	Number Theory
Logi	Logic (including set theory)	**Tri**	Trigonometry
MaP	Mathematical Physics	**Var**	Various themes
Mec	Mechanics	**Vec**	Vectors

[12]) My classification is made from a reading of the articles, without reference to the classification in *Jahrbuch über die Fortschritte der Mathematik*, nor to the present classification in *Mathematical Reviews* or *Zentralblatt*. The journal used a classification different from mine; see the list published in volume 40 (1951–1954). On the difficulties of classifying papers, and especially papers of the past, see [Furinghetti & Somaglia 1992].

CONTRIBUTIONS THAT APPEARED IN THE JOURNAL,
CLASSIFIED BY THEMES, AND THEIR DISTRIBUTION

Year		1899	1900	1901	1902	1903	1904	1905	1906	1907	1908	1909	1910	1911	1912	1913	1914
Alg	25			3	1	2	2		4	4	2	2		1	1	1	2
Ana	60	3	3	8	1	2	6	6	5	2	3	4	1	4	4	6	2
App	2			1									1				
Ari	9		1	1			3	1		2	1						
Ast	2								1	1							
Esp	3		1	2													
FMa	1											1					
Fou	30	5	5	5		3	1	1			2	3	1	1		3	
Geo	150	7	14	8	16	12	6	9	8	9	9	9	5	15	3	8	12
His	22	1	3	2	1	1		1	3	2		3	1	1	1	2	
IC	16										1	2	4	2	3	1	3
Inq	19				1		1	3	5	7	2						
Log	2		1			1											
Logi	9						2	3	1	2							1
MaP	2											2					
Mec	22	1	1		3	1	2	2	3	1			3	1		2	2
Met	41	10	1	3	1		1	9	4		2	2	3	1	2		2
Mod	1		1														
Nom	5	1	1			2										1	
NSu	18	3		2	2			3			2	1	3	1			1
Org	19	2	2	1	1	1					1	1		1	1	1	7
Phi	6		1		3				1	1							
PhM	2				2												
Pro	4					1		1								1	1
Psy	1													1			
Soc	1						1										
ToN	16				1				1	5		2	2	1	4		
Tri	10	1	1	2			1	1	1					2	1		
Var	2							1							1		
Vec	4	1			1										2		
Totals	504	35	36	38	34	26	26	41	37	36	25	32	24	32	23	26	33

Further information may be obtained from the list of materials published in the journal itself [*40* (1951–1954), 124–174].

In the period being considered we find 212 authors of general or pedagogical articles. Some articles in the sub-section 'Organisation of teaching' are anonymous. Collaboratively written articles are unusual. Authors contributing more than five articles were: J. Andrade (France), A. Aubry (France), P. Barbarin (France), C. Berdellé (France), V. Bobynin (Russia), C. Cailler

(Switzerland), G. Combebiac (France), L. Crelier (France), H. Fehr (Switzerland), G. Fontené (France), L. Godeaux (Belgium), C. A. Laisant (France), H. Laurent (France), G. Loria (Italy), C. Méray (France), J. Richard (France), E. Turrière (France).

Contributions came from a number of countries [13]) such as Algeria, Argentina, Austria and the regions of its old Empire, Belgium, Denmark, France, Germany, Greece, Italy, Japan, The Netherlands, Portugal, Rumania, Russia, Spain, Switzerland, UK, Ukraine, USA. Usually, at the end of an article, we find the author's surname, together with a first inital and the place from which the article was submitted; some of these places are just a village so that it is not always easy to determine the country of origin. From the various data it would appear that the desire for internationalism was indeed realised, even if the core readership was in Europe. As regards contributions, France led the way, followed by Switzerland, and in total rather more than 50% of the authors wrote from these two countries.

The majority of the contributions were on geometry. That this was at that time considered to be the backbone of the mathematical instruction at secondary level in many countries is shown by the many letters from readers discussing themes related to Euclidean geometry. Many of the articles on geometry concerned descriptive geometry and, in general, those aspects which have links with applications of interest to technical institutes and faculties. Among the authors contributing to this topic, J. Andrade published attempts at new approaches to geometry suitable for technical schools. A few of the articles dealt with non-Euclidean geometries. Often one feels that behind many articles lay the problem of answering such questions as the role of rigour and axiomatic methods in the teaching of mathematics. This subject is linked to the foundational debate, very much alive in those years. Indeed, following the birth of ICMI the debate on the place of foundations in mathematical instruction became the object of specific inquiries published in the journal.

The themes of the reforms in mathematics teaching were treated in articles surveying mathematical systems of instruction and in articles that focused on specific parts of mathematics. For example, the teaching of analysis is discussed at a general level by Klein [1906], in line with the ideas expressed in the Meran Syllabus, and in articles discussing didactical problems of specific topics, such as the paper of Fehr [1905a] on the concept of function. In the following years the publication in the journal of the studies launched by ICMI contributed to raising interest in the theme of teaching calculus and analysis

[13]) I refer to the modern names of the countries.

in many countries. An echo of this interest is to be found in national journals devoted to mathematics teaching, such as the British *Mathematical Gazette* and the Italian *Bollettino della Mathesis*. The theme of analysis is typical of the relationships between curricular innovations, mathematical research and the needs of society at the time (see [Bourlet 1910]). Also the interest shown by the journal in the book *Traité de nomographie* [14]) by Maurice d'Ocagne [1899] is evidence of the interest in the applications of mathematics and the teaching of mathematics to students in technical schools.

At first glance some teaching suggestions, such as those in [Sainte Laguë 1910] on the use of squared paper, may appear naive or obvious to a modern reader, if one does not take into account the state of teaching methods of the time and the difficulty of introducing new ideas, such as the graphing of functions (see [Gibson 1904; Brock & Price 1980]).

The common mathematical interests of the two founders of the journal are illustrated by the attention paid to quaternions, the vectorial calculus and discussions about vectorial notations. In this regard we signal the article by J. S. Mackay [1905] on Peter Guthrie Tait.

The journal was open to a wide range of contributions and ideas even if, in some cases, the editors pointed out that they did not share the contributors' opinions. Among the authors there were various important characters in the world of mathematics education and of history of mathematics (Loria, Smith, etc.), famous mathematicians (E. Borel, C. Bourlet, L. E. J. Brouwer, E. Czuber, G. Darboux, F. Enriques, M. Frechet, Z. G. de Galdeano, J. Hadamard, D. Hilbert, F. Klein, H. Lebesgue, B. Levi, C. Méray, P. Painlevé, H. Poincaré, F. Gomes Teixeira, H. Weyl, etc.), philosophers such as L. Couturat, and a cohort of secondary and university teachers now forgotten, but very active at the time in contributing to the international debate.

THE INQUIRY ON THE METHOD OF WORKING OF MATHEMATICIANS

The editorial line of the journal ranged between two poles, one specifically referring to mathematics and the other concerned with pedagogical aspects linked to mathematics teaching. Psychology was a link between these two poles. In writing on mathematical topics some authors hinted at psychological

[14]) As discussed in d'Ocagne's book, 'nomography' is a theory which permits the graphical representation of mathematical laws defined by equations of any number of variables (see the review of the book in *L'Enseign. Math. 1* (1899), 368–370). A passage of d'Ocagne's book is translated in [Smith 1929].

issues. For example, the two articles [Baron 1903; Bonola 1903] show on the one hand the attention paid to strands of contemporary mathematical research (foundational studies), and on the other hand the importance of explaining mathematical facts through factors exterior to mathematics. On the reports being published on foundational studies in geometry, Bonola wrote that they

> ont non seulement élargi le domaine de la Géométrie non-euclidienne, mais ils ont encore contraint les esprits bien organisés à suivre les édifices variés et admirables qui ont été inspirés par la renaissance scientifique, caractéristique du XIXe siècle, et à se rendre compte des connexions qui existent entre les branches les plus élevées des mathématiques et d'intéressants problèmes de Psycho-physiologie. [15]) [Bonola 1903, 317]

At that time, themes of psychology, such as the necessary conditions for creativity and invention, and their links with mathematical themes such as axiomatisation, rigour and intuition, were debated hotly in the mathematical community (see [Mannheim 1909; Poincaré 1908]).

This context may explain why the inquiry into the methods of mathematicians was promoted by the journal. This was first inspired by a letter from Ed. Maillet ['Correspondance', *3* (1901), 58–59]. He wrote that it would be interesting for young researchers beginning their profession to have information on: work and research methods, general 'hygienic rules' useful to their intellectual work, how to read papers, etc.

In the same year ['Chronique', *3* (1901), 128; 219–220] the editors acknowledged the great interest roused by the idea and launched the project of a questionnaire, which had to respect confidentiality and was to be more than a mere curiosity. Many readers responded to this note and sent questions to be included in the questionnaire [16]).

The first part of the questionnaire was published in 1902 [*4*, 208–211] and the final part in 1904 [*6*, 376–378]. The questionnaire was sent to the subscribers of the journal and also distributed to the participants at the International Congress of Mathematicians at Heidelberg in 1904 and those attending the congress of Saint-Louis [*L'Enseign. Math. 6* (1904), 481]. In Heidelberg, Fehr [1905b] presented a communication on the questionnaire.

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[15]) have not only extended the domain of non-Euclidean geometry, but have also obliged well organised minds to pay regard to various and admirable constructions, inspired by the scientific renaissance characteristic of the 19th century, and to take account of the links between the highest branches of mathematics and interesting problems of psycho-physiology.

[16]) At this time Ed. Maillet was one of the editors of the journal *L'Intermédiaire des Mathématiciens.* In 1902 (t. IX) that journal published two questions [question numbers: 2446 and 2447, 263–264] on the circumstances attending mathematical creation. Answers to these questions were published in the same volume [339–343].

Copies of the questionnaire were sent to other mathematicians and to those who requested it. In 1905 [*7*, 239] the editors thanked the scientific journals which had published the questions.

The aim of the enterprise was twofold: on the one hand to collect advice useful to researchers in mathematics, and on the other hand to contribute to research in the field of psychology of professions. The questionnaire was prepared and analysed by Fehr and two psychologists at the University of Geneva, Édouard Claparède (1873–1940) and Théodore Flournoy (1854–1920). The languages for the answers were restricted to English, Esperanto, French, German, and Italian. There were 30 questions in all and the names of responders were published only with their explicit agreement. Claparède, Fehr and Flournoy published the results with comments in 1905, 1906, 1907, and 1908. There was also a report of the inquiry published in book form [Fehr *et al.* 1908].

The analysis of the results was based on the answers of more than 100 mathematicians and on a few historical notes on the lives of previous mathematicians. The journal published statistical data, comments and a few sentences from the responders. Many of the published statements carried the names of the writers, so that we know that among the responders were L. Boltzmann, M. Cantor, J. Coolidge, L. E. Dickson, H. Fehr, Z. G. de Galdeano, C. A. Laisant, C. Méray, G. Oltramare and J. W. Young. With these statements the nationality of the responder was also indicated. The answers arrived from various countries. Comments to the questions, by G. Loria [*8* (1906), 383–385, questions 6 and 9] and V. Bobynin [*9* (1907), 135–141; 389–396, questions 4 and 5] were published [17]).

This inquiry is not mere folklore; it provides material for studies in mathematics education, epistemology, psychology and sociology. Also it constitutes an early example of making explicit the feeling mathematicians have about the nature of their discipline, of their work and of their being mathematicians. This willingness of mathematicians to be more open about their feelings and how they worked was shown in later years by a number of famous works (by Hadamard, Hardy, Poincaré, etc.). As concerns the results, Fehr himself [1908] acknowledged that it was difficult to draw general conclusions. As a matter of fact, the importance of this enterprise was in identifying themes which would have important developments in the following years. Even if the intentions of the designers of the questionnaire were mainly directed towards mathematical research, the questionnaire may have had an

[17]) Appendix 2 contains the list of notes and articles dealing with this inquiry.

impact in the world of school, by introducing elements pertaining to the psychological domain in the reflection on students' mathematical performances. For example, we can cite the report of the German national commission *Psychologie und mathematischer Unterricht* by D. Katz [1913]. The review [Brandenberger 1914] reports that it was divided into three parts:

- psychology and mathematics teaching (the child's development of the representation of number and the conception of space, the methods of working of mathematicians in line with the inquiry published in *L'Enseignement Mathématique*, the psychology of great mental reckoners, teaching to low-attainers, etc.),
- psychology of technical and artistic drawing,
- psychology and pedagogy in teacher education.

This report therefore illustrates the impact which the *L'Enseignement Mathématique* inquiry had on the world of mathematics education.

In the preface to the posthumous article by Mannheim [1909], the editors reported that the inquiry had had 'unexpected side effects', for example Mannheim's and Poincaré's papers. As a matter of fact, the themes developed through this inquiry were close to the interests of a prominent member of the Committee of supporters, Henri Poincaré, who contributed four articles to the journal. The paper [Poincaré 1908], which is the text of a lecture delivered at the *Institut général psychologique*, may be seen as a bridge between the world of mathematicians and that of psychologists. It shows that this prominent mathematician, as well as Mannheim in his paper of 1909, was in agreement with the ideas of the editors of *L'Enseignement Mathématique*. At the beginning of the paper Poincaré wrote:

> La genèse de l'invention mathématique est un problème qui doit inspirer le plus vif intérêt au psychologue. C'est l'acte dans lequel l'esprit humain semble le moins emprunter au monde extérieur, où il n'agit ou ne paraît agir que par lui-même et sur lui-même, de sorte qu'en étudiant le processus de la pensée géométrique, c'est ce qu'il y a de plus essentiel dans l'esprit humain que nous pouvons espérer atteindre.
>
> On l'a compris depuis longtemps, et, il y a quelques mois, une revue intitulée *L'Enseignement Mathématique*, et dirigée par MM. Laisant et Fehr, a entrepris une enquête sur les habitudes d'esprit et les méthodes de travail des différents mathématiciens. J'avais arrêté les principaux traits de ma conférence quand les résultats de cette enquête ont été publiés; je n'ai donc guère pu les utiliser. Je me bornerai à dire que la majorité des témoignages confirment mes conclusions; je ne dis pas l'unanimité, car, quand on consulte le suffrage universel, on ne peut se flatter de réunir l'unanimité. [18]) [Poincaré 1908, 357]

[18]) The genesis of mathematical creation is a problem which ought to inspire the intense

The Swiss environment was important in encouraging this kind of interest by the journal and one may say that the roots of future developments of the Geneva school of psychology [19]) may be found in those years.

To grasp the particular atmosphere of the period it is interesting to consider the review by Claparède [*L'Enseign. Math. 14* (1912), 81–82] of the book *Henri Poincaré* (by Dr. Toulouse, Flammarion, Paris). This book presents the results of a study on Poincaré's way of reasoning and a comparison with the results obtained from similar studies on the novelist Émile Zola. Claparède reports that the conclusions of the author are that Zola's intelligence is

> consciente, logique, méthodique, paraissant faite pour la déduction mathématique […]. Au contraire l'activité mentale de M. Poincaré est spontanée, peu consciente, plus proche du rêve que de la démarche rationnelle, et semblait surtout apte aux œuvres de pure imagination […]. [20])

THE DAWN OF ICMI

The international vocation of *L'Enseignement Mathématique* was clearly established in the first volume of the journal with the articles on mathematics teaching in Spain by Z. G. de Galdeano and on the history of mathematics teaching in Russia by V. Bobynin. In successive issues similar surveys of other countries were published. The readers were gradually invited to consider the importance of comparative studies on different systems of mathematical development (intended in the broad sense of including both research and instruction).

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interest of the psychologist. It is the activity in which the human mind appears to take least from the outside world, where it acts or seems to act only of itself and on itself, so that in studying the process of geometric thinking we may hope to reach what is most essential in man's mind.

This has long been appreciated, and some months ago a journal called *L'Enseignement Mathématique*, edited by Laisant and Fehr, began an investigation of the mental habits and methods of work of different mathematicians. I had already defined the main outlines of my lecture when the results of that inquiry were published; I have not therefore been able to make much use of them. I shall confine myself to saying that the majority of witnesses confirm my conclusions; I do not say all, for when the appeal is to universal suffrage one can hardly expect to obtain unanimity.

[19]) Jean Piaget never missed an opportunity to mention the ideas of Claparède (see, for example, [Beth & Piaget 1961, 213–214]). He himself had been called by Édouard Claparède (in 1921) to teach at the J.-J. Rousseau Institute in Geneva. As a matter of fact there is a discussion of the *Inquiry* with reference to Poincaré and Hadamard's conceptions in [Beth & Piaget 1961, §26, 96–99]. We may add that Piaget's support was very useful when the second series of the journal was launched in 1955, and he even became a member of the editorial board.

[20]) conscious, logical, methodical, appearing suitable for mathematical deduction […]. On the contrary, the mental activity of Poincaré is spontaneous, hardly conscious, closer to dreams than to the rational way, and seems most of all suitable for works of pure imagination […].

These ideas became more concrete in the note [La Rédaction 1905b] published in the seventh year, where the editors call for the opinions of readers on the following questions:

1. What progress needs to be achieved in the organisation of the teaching of pure mathematics?
2. What should be the role of tertiary instruction in teacher education?
3. How should teaching be organised in such a way as to meet the requirements of pure and applied sciences better than in the past.

As indicated in the note, an important antecedent to this initiative lay in the resolution passed by the third International Congress of Mathematicians held in Heidelberg as reported on p. 53 of the Proceedings [Krazer 1905] and in a note by *La Rédaction* [1905a]. This resolution stated that the teaching of mathematics at the tertiary level had to be in accordance with the importance of technical sciences in the various countries. So it was stressed that there was a need for new professorial appointments, suitable libraries, collections of models, laboratories for drawing and practical work, etc.

J. Andrade, E. Borel, G. Loria, F. Marotte and D. E. Smith reacted to the questions with notes published in the journal [L'Enseignement Mathématique 1905], in which the key issues for reform were stressed. The discussion was enlarged to include the theme of mathematics teaching at the secondary level. Teacher education was the natural link between these two levels. Loria [1905] advocated the 'democratisation' of some theories that in those years were considered 'advanced' (e.g. analytical geometry). Smith [1905] proposed more international co-operation and the creation of a commission to be appointed in an international conference for the study of instructional problems globally. He stated that both the journal and the commission were the best means to improve mathematics teaching. Smith was relying on the community of mathematicians, but at the same time he was quite critical about it in remarking that the papers presented in the didactic section of the International Congress of Mathematicians in Heidelberg were discussions of mathematical details rather than of general problems of teaching.

The discussion on the need for educational reforms in *L'Enseignement Mathématique* was the stimulus for the idea of a commission appointed by an international congress, which had its culmination in the establishment of ICMI during the fourth International Congress of Mathematicians in Rome in 1908 (see [Howson 1984; Schubring 2003]). Fehr was appointed as the secretary-general of ICMI, with Felix Klein as president and Alfred George Greenhill as vice-president. The following year *L'Enseignement Mathématique* became the official organ of ICMI. In particular, through the journal, ICMI launched its early studies on various themes of mathematics teaching (cf. [Schubring 2003]). The reports of the discussions on these themes were published in the proceedings of important conferences organised by ICMI : 1911 in Milan, 1912 during the International Congress of Mathematicians in Cambridge, and 1914 in Paris (see Appendix 3). Also the journal published regular news about the Sub-Commissions founded in the various countries. Pamphlets and volumes on the works of these sub-commissions were also published (see [Fehr 1920]). Together they constitute important documentation on the state of mathematical instruction at that time.

CONCLUSION

The journal *L'Enseignement Mathématique* played a unique role in the mathematical community (of research and of instruction) in the period I have considered. It was a bridge between some aspects of mathematical research, the world of school and society. It was also a forum for discussing problems of mathematical instruction and education. In addition it was a bridge between mathematics and other disciplines, such as philosophy, psychology, and technology.

The journal contributed to redirecting the attention of educators away from the core of traditional mathematical instruction (Euclidean geometry) towards new themes (analytical geometry, foundational studies, and analysis) linked to technical applications or to research being developed in those years. This contributed to raising awareness in a number of countries of the need for a discussion of the nature of their mathematical programmes. For example, in Italy the scientific lyceum, whose curriculum encompasses a strong mathematical programme, was created under the influence of the debate on the teaching of calculus and the study launched by ICMI on this subject (see [Furinghetti 2001]).

Some of the ideas discussed in *L'Enseignement Mathématique* were already gaining ground in the scientific environment; the journal had the merit of being an efficient promoter of these ideas and of providing the fertile soil in which the ideas could grow.

APPENDIX 1

JOURNALS LISTED[21]) IN THE SECTION 'BULLETIN BIBLIOGRAPHIQUE' OF *L'ENSEIGNEMENT MATHÉMATIQUE* — YEARS 1899, 1900, 1901 (WITH THE TOWN OF PUBLICATION THE FIRST TIME THE JOURNAL IS QUOTED)

Acta Mathematica, Stockholm (Sweden)
American Journal of mathematics, Baltimore (USA)
Annales de la Faculté des sciences de l'Université de Toulouse, Paris, Toulouse (France)
Annali di Matematica pura ed applicata, Milan (Italy)
Annals of Mathematics, Cambridge (USA)
Archiv der Mathematik und Physik, Leipzig (Germany)
Atti della Reale Accademia dei Lincei, Rendiconti, Rome (Italy)
Bibliotheca mathematica, Zeitschrift für Geschichte der mathematischen Wissenschaften, Leipzig (Germany)
Bolletino di Bibliografia e Storia delle Scienze mathematiche, Turin (Italy)
Bulletin astronomique, Paris (France)
Bulletin de l'Enseignement technique, Paris (France)
Bulletin de la Société mathématique de France, Paris (France)
Bulletin de la Société Philomathique de Paris, Paris (France)
Bulletin de mathématiques spéciales, Paris (France)
Bulletin des Sciences mathématiques, Paris (France)
Bulletin des sciences mathématiques et physiques élémentaires, Paris (France)
Comptes rendus des séances de l'Académie des sciences, Paris (France)
Educational Review, Rochway, New York (USA)
El Progreso matematico, Zaragoza (Spain)
Giornale di matematiche di Battaglini, Naples (Italy)
Il Bollettino di matematiche e di scienze fisiche e naturali, Bologna (Italy)
Il nuovo Cimento, Pisa (Italy)
Il Pitagora, Palermo (Italy)
Jahrbuch ueber die Fortschritte der Mathematik, Berlin (Germany)
Jahresbericht der Deutschen Mathematiker Vereinigung, Leipzig (Germany)
Jornal de Sciencias mathematicas e Astronomicas, Coimbra (Portugal)
Jornal de sciencias mathematicas, physicas e naturæs, Lisboa (Portugal)
Journal de l'Ecole Polytechnique, Paris (France)

[21]) Spelling and capitalization as they appeared the first time each journal was mentioned.

Journal de mathématiques élémentaires, Paris (France)
Journal de mathématiques élémentaires; journal de mathématiques spéciales, Paris (France)
Journal de Mathématiques pures et appliquées, Paris (France)
Journal de mathématiques spéciales, Paris (France)
Journal für die reine und angewandte Mathematik, Berlin (Germany)
L'Éducation mathématique, Paris (France)
L'Enseignement chrétien, Paris (France)
L'Intermédiaire des Mathématiciens, Paris (France)
Le Matematiche pure ed applicate, Città di Castello (Italy)
Mathematische Annalen, Leipzig (Germany)
Mathesis, recueil mathématique à l'usage des écoles spéciales et des établissements d'instruction moyenne, Gand (Belgium), Paris (France)
Monatschefte für Mathematik und Physik, Vienna (Austria)
Nieuw Archief voor Wiskunde, Amsterdam (Netherlands)
Nouvelles Annales de mathématiques, Paris (France)
Nyt Tidsskrift for matematik, Copenhagen (Denmark)
Paedagogisches Archiv, Leipzig (Germany)
Periodico di Matematica (per l'insegnamento secondario), Leghorn (Italy)
Publications de la section des sciences mathématiques et naturelles de l'Université d'Upsall, Uppsala (Sweden)
Rendiconti del Circolo matematico di Palermo, Palermo (Italy)
Revista de Ciencias, Lima (Peru)
Revista trimestral de matematicas, Zaragoza (Spain)
Revue de Mathématiques (Rivista di Matematica), Turin (Italy)
Revue de mathématiques spéciales, Paris (France)
Revue de Physique expérimentale et de Mathématiques élémentaires, Odessa (Russia)
Revue générale des sciences pures et appliquées, Paris (France)
Revue scientifique, Paris (France)
Revue semestrielle des publications mathématiques, Amsterdam (Netherlands), Edinburgh (UK), Leipzig (Germany), London (UK), Paris (France)
Schweizerische Paedagogische Zeitschrift, Zurich (Switzerland)
Sciences physico-mathématiques, Moscow (Russia)
Supplemento al periodico di Matematica, Leghorn (Italy)
The american mathematical Monthly, Springfield (USA)
The Mathematical Gazette, London (UK)
Transactions of the American Mathematical Society, Lancaster, New York (USA)
Unterrichtsblätter für Mathematik und Naturwissenschaften, Berlin (Germany)
Wiadomosci Matematyczne, Warsaw (Poland)
Wiskundige Opgaven, Amsterdam (Netherlands)
Zeitschrift für das Realschulwesen, Vienna (Austria)
Zeitschrift für lateinlose hoehere Schulen, Leipzig (Germany)
Zeitschrift für Mathematik und Physik, Leipzig (Germany)
Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht, Leipzig (Germany)

APPENDIX 2

THE INQUIRY ON THE METHOD OF WORKING OF MATHEMATICIANS

QUESTIONS [22])

Text of the first questionnaire: *4* (1902), 208–211
Text of the second questionnaire: *6* (1904), 376–378

ANALYSIS OF THE RESULTS

Question 1*a*: H. Fehr, *7* (1905), 387–395
Question 1*b*: T. Flournoy, *7* (1905), 473–478
Questions 2 and 3: H. Fehr, *8* (1906), 43–48
Questions 4 and 5: H. Fehr, *8* (1906), 217–225
Questions 6–9: T. Flournoy, *8* (1906), 293–310
Questions 10–13: H. Fehr, *8* (1906), 463–475
Questions 14–17: H. Fehr, *9* (1907), 123–128
Questions 18 and 20: T. Flournoy, *9* (1907), 128–135
Question 19: T. Flournoy, *9* (1907), 204–217
Question 21: H. Fehr, *9* (1907), 306–312
Questions 22 and 23: É. Claparède, *9* (1907), 473–479
Questions 24–30: É. Claparède, *10* (1908), 152–172

NOTES ABOUT THE INQUIRY ON THE METHOD OF WORKING OF MATHEMATICIANS

- *3* (1901), 58–59 (Maillet, Ed., 'Correspondance'); 128; 219–220
- *6* (1904), 481
- *7* (1905), 63–64; 239–240
- *8* (1906), 383–385: letter of G. Loria on questions 6–9
- *9* (1907), 135–141: reflections of V. Bobynin on the answers to questions 4 and 5
- *9* (1907), 389–396: reflections of V. Bobynin on the answers to questions 11–13

[22]) For an English translation of the questions see Appendix I in Hadamard's book [Hadamard 1945, 136–141].

APPENDIX 3

INQUIRIES ON THE TEACHING OF MATHEMATICS LAUNCHED BY ICMI

– *13* (1911)

The questions discussed in the conference of the Commission in Milan (18–21 September 1911) are published on pp. 122; 443–446.

THEMES:

A) Systematic exposition of mathematics (axioms, rigour, etc.) in secondary schools; the fusion of different branches of mathematics in secondary schools.

B) The teaching of mathematics to university students of physics and of the natural sciences.

The Proceedings of the Milan conference with reports about these questions can be found in the same volume [437–511].

– *14* (1912)

The questions discussed in the conference held during the International Mathematical Congress (Cambridge, UK, 21–28 August 1912) are published on pp. 39; 132–135; 220; 299.

THEMES:

A) Intuition and experimental evidence in secondary schools.

B) Mathematics for university students of physics.

The Proceedings of the Cambridge conference with reports about these questions can be found in the same volume [441–537].

– *15* (1913)

The questions to be discussed in preparatory work for the conference to be held in Paris (1914) are reported on pp. 243; 414; 487. The translations of these questions into German, English, and Italian can be found on pp. 394–412.

THEMES:

A) Results of the introduction of calculus in secondary schools.

B) Mathematics teaching for the technical professions in higher educational institutions.

– *16* (1914)

The questions for the Paris conference are published on p. 54.

The Proceedings of the Paris conference (1–4 April 1914) with reports about these questions are published on pp. 165–226 and 245–356.

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L'ENSEIGNEMENT MATHÉMATIQUE AND THE FIRST INTERNATIONAL COMMISSION (IMUK): THE EMERGENCE OF INTERNATIONAL COMMUNICATION AND COOPERATION

L'Enseignement Mathématique et la première Commission internationale : naissance de communications et de coopérations internationales

par Gert SCHUBRING

Le centenaire de la revue *L'Enseignement Mathématique* révèle le contraste existant au début du siècle dernier entre les niveaux de développement de la communication internationale, d'une part en mathématiques et d'autre part dans leur enseignement. Depuis longtemps, en effet, il y avait chez les mathématiciens tant une communication directe que des revues mathématiques, soutenues par des auteurs et un public internationaux, ou encore des rencontres, tel le deuxième Congrès international des mathématiciens de Paris qui permit de faire un bilan des avancées du domaine tel qu'il se présentait en 1900. Mais pour ce qui est de l'éducation mathématique, un pareil niveau de communication parmi les enseignants n'a pu être établi qu'avec la fondation de la nouvelle revue, en 1899.

A l'examen des grands pays européens, on voit que la situation de l'enseignement des mathématiques y était tellement diverse pendant le XIX^{e} siècle qu'un tel manque de communication internationale est tout à fait compréhensible. Même en France, l'enseignement des mathématiques pour la grande majorité des élèves dans les lycées n'occupait qu'une place mineure, de sorte qu'une professionnalisation des enseignants de mathématiques pouvait difficilement avoir lieu. En Italie et en Angleterre, l'enseignement était caractérisé par une conception strictement classique du savoir enseigné, préconisant ainsi Euclide à la fois comme manuel et comme méthodologie pour les mathématiques.

Avec la fondation de la revue *L'Enseignement Mathématique*, la situation commença à changer : les rapports publiés dès lors sur l'état de l'enseignement des mathématiques dans divers pays firent naître le désir d'établir une coopération internationale. Le premier à proposer la création d'une commission internationale pour promouvoir une telle coopération fut l'Américain David Eugene Smith en 1905.

La Commission internationale de l'enseignement mathématique (CIEM/ICMI), créée lors du Congrès international des mathématiciens suivant, tenu en 1908 à Rome, prit très au sérieux sa mission de transformer les communications naissantes en une vraie coopération internationale. Afin de promouvoir des réformes dans l'enseignement mathématique par l'actualisation des programmes et des méthodes d'enseignement, la Commission élargit nettement le mandat restreint défini par le Congrès en invitant à participer des représentants de pays plus nombreux qu'initialement prévu (et de tous les continents). La Commission ne se contenta pas des seules écoles secondaires, mais résolut de s'intéresser à tous les types d'écoles, depuis l'école élémentaire jusqu'à l'enseignement supérieur, en passant par les écoles d'éducation générale et les écoles professionnelles ou techniques. Même si la grande majorité des membres de la Commission appartenait au secteur post-secondaire et aurait de ce fait été naturellement liée principalement aux écoles secondaires, ils soutinrent activement cette politique universaliste du Comité Central.

C'est seulement dans le choix des sujets pour des études comparatives visant à promouvoir l'esprit des réformes que l'on peut remarquer un biais résultant des orientations professionnelles des membres : parmi les huit sujets retenus, trois seulement concernaient le mandat officiel, à savoir les écoles secondaires, les cinq autres portant soit sur la transition du secondaire au supérieur, soit exclusivement sur les problèmes du secteur post-secondaire.

On présente dans ce texte un résumé des débats et des résultats concernant certaines de ces études comparatives. En conclusion, on compare le développement international de l'enseignement mathématique aujourd'hui avec la situation d'il y a cent ans.

L'ENSEIGNEMENT MATHÉMATIQUE AND THE FIRST INTERNATIONAL COMMISSION (IMUK): THE EMERGENCE OF INTERNATIONAL COMMUNICATION AND COOPERATION

by Gert SCHUBRING

This *World Mathematical Year 2000* provides us with an excellent opportunity for illustrating the specific situation of mathematics education. As we commemorate the International Congress of Mathematicians (ICM) in Paris in 1900, we can see how firmly international communication had been established among mathematicians at the time of that second international Congress. And Hilbert's famous speech set the agenda for mathematical research of a large part of the new 20$^{\text{th}}$ century. But what was the state of international communication in mathematical education? We should note that communication between different countries regarding the teaching of mathematics had been practically non-existent up to the end of the 19$^{\text{th}}$ century.

While mathematicians in Europe had been communicating for centuries and journals accepting contributions from different countries had existed since the early 19$^{\text{th}}$ century — international congresses in 1897 establishing a new, though logical development —, founding the journal *L'Enseignement Mathématique* was a bold endeavour since the necessary basis for international communication on matters of education had hardly been established at that time.

This lack of international communication with regard to mathematical education was basically due to the fact that differences were profound, even between the countries of Western Europe. These differences concerned the contents of instruction as well as the epistemology of school mathematics and the methodology of teaching. A clear indicator is the amount of teaching time allotted to mathematics, as compared to that of competing school disciplines. Thus, both the number of weekly hours and the status of mathematics teachers were indicators for the social status enjoyed by mathematics as a teaching subject in general education. I shall give a few examples to illustrate the considerable differences between European countries.

Let us begin with FRANCE, a country which was generally believed to have established a strong position of mathematics in schools since the science-minded reforms arising from the French Revolution. Actually, the influence of these reforms declined decisively after the Restoration, leaving mathematics with a rather marginal status within "general education for all" in secondary schools. For most of the 19th century, there was no systematic teaching

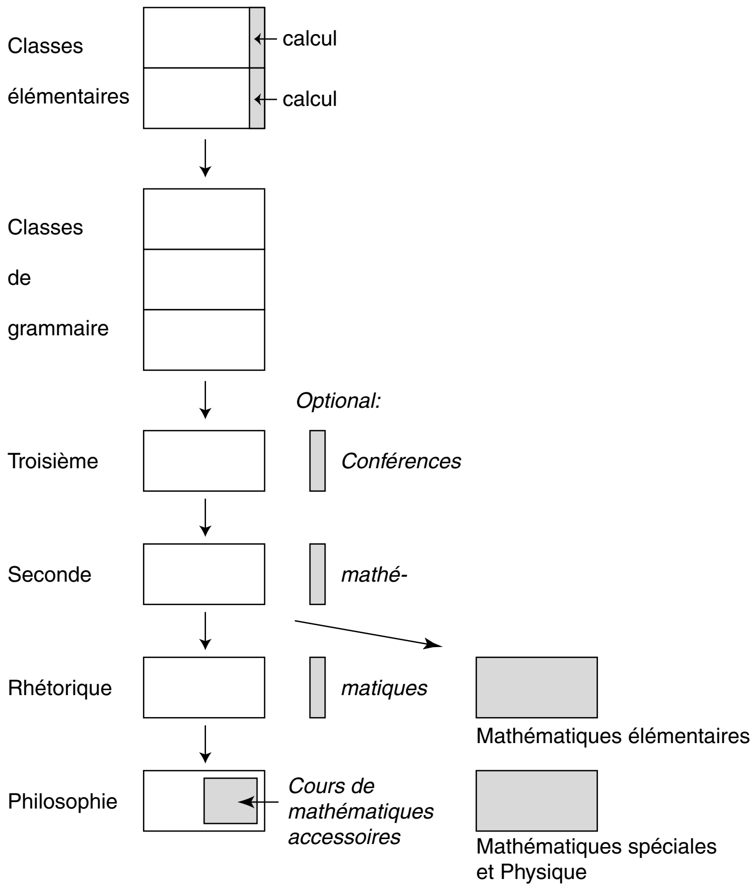

FIGURE 1
Structure of mathematics instruction in French secondary schools in 1845 [Hahn 1846]

of mathematics for developing mathematical knowledge throughout the school years — rather, it was reduced to a few basic notions in the first years and to some concentrated instruction in the final years.

As Figure 1 shows, mathematics education in French secondary schools had practically returned — as far as mathematics teaching "for all" was concerned — to the Jesuit system where mathematics was not taught before the very end of the *collèges*: in the philosophy class. For a selection of students who wanted more instruction there were optional lessons in the three senior classes. And for those who aspired to stand as candidates for the École polytechnique there were special classes during the last two years.

During the Third Republic, mathematics became more permanently established as a teaching subject "for all" — yet still at the bare minimum for most classes, gaining importance only in the final classes as the following Table shows for 1885:

1885

Sixième	1
Cinquième	1
Quatrième	1
Troisième	2
Seconde	2
Rhétorique	2
Philosophie	4

Number of weekly hours for mathematics instruction
in the classes of French secondary schools [Belhoste 1995, 500 ff.]

Actually, the lack of continuity, as well as the marginal level of instruction, in mathematics and in the sciences, had become a matter of concern of educational policy since 1847. To supplement these deficiencies in the classical *lycées*, an 'enseignement secondaire spécial' was established, although the course was entirely concerned with practice and applications and hence of a lower status. The experiment, initiated in 1852, of a 'bifurcation' between the humanities and the sciences within the *lycées* had to be abandoned a few years later [Hulin-Jung 1989]. The originally vocationally oriented *enseignement spécial* then rose in status and achieved in 1891 (now renamed '*enseignement secondaire moderne*') a status equivalent to that of classical studies [Belhoste

1995, 226 and *passim*]. Eventually, in 1902, both branches merged to form just one school, but one which was internally differentiated into two, and then four, sections:

1902

Premier Cycle

	Section A	Section B
Sixième	2	3
Cinquième	2	3
Quatrième	1 + 1 opt.	4
Troisième	2 + 1 opt.	3

Second Cycle

	Sect. A et B
Seconde	1
Première	1
Philosophie	2 optional

	Sect. C et D
Seconde	5
Première	5
Mathématiques	8

Number of weekly hours for mathematics instruction
in the classes of French secondary schools [Belhoste 1995, 577 ff.]

The reform of 1902 is famous in the history of French mathematics education for having established a decisive improvement of the status of mathematics as a teaching subject and for having introduced the notion of function as a key concept. As the Table above shows, this improvement of status affected only the science sections C and D, which constituted separate classes for the last three years. For these final years, the '*lettres*' classes suffered strong reductions in the teaching of mathematics. And a closer scrutiny of the programmes shows that the concept of function was not introduced explicitly in the mathematics syllabus for the science sections, but only in the physics syllabus of the *Seconde* — and even there only in the observations intended for teachers [1]):

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[1]) Clearly, in the programme for the final class (*Mathématiques*), the term 'function' was used.

> *Conseils généraux*. — Le professeur […] utilisera fréquemment les représentations graphiques, non seulement pour mieux montrer aux élèves l'allure des phénomènes, mais pour faire pénétrer dans leur esprit les idées si importantes de fonction et de continuité.[2]) [Belhoste 1995, 600]

Curiously enough, the only explicit introduction of the function concept in a mathematics syllabus occurs in the programme for the final class (*Philosophie*) of the '*lettres*' sections, where mathematics appeared only as an optional subject: "notion de fonction, représentation graphique de fonctions très simples"[3]) [Belhoste 1995, 595].

A decisive change, which reinforced mathematics as a permanent discipline, took place only after 1902. But mathematics still suffered from a split between general education and a better status for those students who aimed at later specialisation in mathematics. The scientific 'unity' is said to have been achieved in 1925 — by establishing a *tronc commun* for the humanities and the science sections, that means common teaching of both strands during the *premier cycle*.

In contrast to the strictly centralised French state, GERMANY was divided up into a host of separate territories: 39 since 1815, and still 25 after the establishment of the German *Reich* in 1871, with the exclusion of Austria. Actually, this German Empire was formally a mere confederation of independent states, and education was a major area where this independence was emphasised. Consequently, status, extent and organisation of mathematics instruction continued to differ considerably between these states.

One structural pattern common to all these federal states, however, was that there were two or even three parallel types of secondary schools: varying between one type featuring the values of classical Antiquity and another, decidedly modern type. Evidently, the status of mathematics depended entirely on the degree of 'modernity' represented by the respective school type. In Prussia, the dominating state of the Federation, mathematics enjoyed a relatively good position in the classical school type, with 3 to 4 hours per week, extending to 5 to 6 hours in the genuinely modern type of school, but mathematics had a rather restricted role in the other German states.

In ITALY, which had become a unified state since 1861, mathematics suffered a decline unique in all of Europe: in the only and exclusively

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[2]) *General advice*. — The teacher […] shall make frequent use of graphical representations, not only in order to let the pupils get a better understanding of the shape of the phenomena, but so as to inculcate them with the utterly important notions of function and continuity.

[3]) the concept of function, graphical representation of very simple functions.

classical type of secondary school of that time, mathematics was conceived as a classical subject, as well, with Euclid as the official text-book, together with the consequent frightening requirements of rigour. This teaching methodology resulted in ever repeated student failures, in particular in the final examination. The solution, eventually adopted in the 1880s, was to abolish the character of mathematics as a major discipline and to reduce it to the status of a minor discipline — and so remove it from the final examinations.

ENGLAND is somewhat analogous to the Italian case. It was not until the 19th century that some secondary schools, in the form of the famous 'public schools' — actually independent, private establishments —, first began to constitute a separate secondary school system, thus differentiating schools from the formerly all-embracing university colleges and arts faculties. Mathematics instruction there was dominated by closely following Euclid as a text-book and as a methodology for formalist elementary geometry.

These few examples might be sufficient to show that around 1900 the national situations regarding mathematics teaching were so profoundly different that the absence of international communication can be easily understood.

From the very moment, however, that the newly launched journal *L'Enseignement Mathématique* began to publish reports describing experiences with mathematics teaching in some countries, the demand for international communication began to rise. This demand was transformed into a concrete proposal by David Eugene Smith in 1905. Smith, professor for mathematics teaching in New York, was the first to propose an institutionalisation, not only of communication but, much more, of international cooperation. His proposal was:

> La meilleure manière de renforcer l'organisation de l'enseignement des mathématiques pures, serait de créer une commission qui serait nommée par un Congrès international et qui étudierait le problème dans son ensemble.[4])
> [Smith 1905, 469]

Thus, the journal *L'Enseignement Mathématique* has not only the seminal merit of having established international *communication* on mathematics teaching, it also encouraged taking the next step, i.e. establishing *cooperation*. Due to this excellent groundwork, Smith's proposal was adopted when he renewed it in 1908 at the next International Congress of Mathematicians. Following Smith's proposal, meanwhile complemented by some issues concerning school

[4]) The best way to reinforce the organisation of the teaching of pure mathematics, would be the establishement of a committee appointed by an international Congress and which would study the problem in its entirety.

curricula which such a committee might investigate, the Fourth International Congress of Mathematicians in Rome voted to establish an international committee on mathematics instruction, with a mandate until the next Congress to take place four years later. A skeleton team of three was nominated : Felix Klein from Germany, Henri Fehr from Geneva and George Grenhill from Great Britain. This team, named 'Comité Central', a common term at the time and without the later political connotations, elected Klein president and Fehr secretary-general to the committee. This did not only acknowledge the role of *L'Enseignement Mathématique* in launching this cooperative endeavour, the journal also had a decisive function in realising the tasks of cooperation towards an international effort in favour of mathematics education. In fact, the new committee, whose official name was used either in its German version :

– IMUK (*Internationale mathematische Unterrichtskommission*),

or in its French version :

– CIEM (*Commission internationale de l'enseignement mathématique*),

decided that *L'Enseignement Mathématique* was to become the official journal for the entire work of IMUK/CIEM.

One of the first issues facing the *Comité Central* was to decide which countries should be invited to participate in the work of IMUK. This question had in principle already been answered by the number of participating members of the ICM with voting rights. Its application to the work of IMUK, however, strikes us today as rather bizarre and the *Comité Central* itself did not appear to have been satisfied with this crude definition of cooperation. The intention was to try to tie a given country's level of activity in mathematics education to its participation in international *mathematical* communication : this level being determined by the number of participants at the four international mathematical congresses between 1897 and 1908. Countries having sent at least two mathematicians to at least two of the four congresses (Group 1) were invited to constitute a national subcommittee of IMUK and to be represented in the IMUK-body by one voting member ; for those whose number of participants had been ten or more, the number of voting members was raised to two or even three (Group 2). It is not clear to me how this arithmetic of proportion was determined.

One possible reason is that this proportionality might have served to legitimise the IMUK endeavour to the respective government : it would be unimaginable today for national subcommittees to be semi-official bodies funded by their respective countries where members had to be approved by the government of their country. For a reason yet unknown to me, the British

VOTING MEMBERS OF THE IMUK

GROUP 2	GROUP 1
Austria	Belgium
British Isles	Denmark
France	Greece
Germany	Netherlands
Hungary	Norway
Italy	Portugal
Russia	Rumania
Switzerland	Spain
USA	Sweden

government even went so far as to exercise a certain supervisory role in approving the respective delegates.

These criteria yielded a sample of eighteen countries, with nine countries in each of the two groups. As the *Comité Central* realized that the practice of mathematical activity as decided by the ICM would not embrace all countries engaged in mathematics education, it was decided to invite the following countries to participate with non-voting members. The list (in italics are those countries which demonstrated at least temporarily a certain degree of activity) shows that by now all continents were involved. As one of Klein's assistants in the IMUK work reported, the idea had been to have the "civilized countries" represented in the IMUK body [Schimmack 1911, 2].

NON-VOTING, INVITED COUNTRIES

Argentina	Canada	*Japan*
Australia	*Cape Colony*	*Mexico*
Brazil	Chile	Peru
British India	China	*Serbia*
Bulgaria	Egypt	Turkey

Besides Henri Fehr, a considerable merit in constituting this first international network is due to the efforts of D. E. Smith. Through advising F. Klein, he succeeded in establishing a great number of the personal contacts necessary for setting up the various national subcommittees.

We can have some understanding of the tasks undertaken by IMUK, and make some assessment of them, by considering the main players, namely the members of IMUK, and what constituted their relation to mathematics education.

As can be seen from the list in Appendix 1 showing the members (delegates) in 1914, the great majority of members were university professors of mathematics. Their relation to mathematics teaching was constituted by the fact that at least a considerable number of their students were intending to become secondary school teachers. Probably the only professional among them was David Eugene Smith who, from the beginning of his career, had worked in normal schools, i.e. at teacher training institutions in the USA [5]). His productivity was connected with his professorship at the Teachers College, a part of Columbia University in New York. His exclusive task there was to train mathematics teachers. He succeeded in firmly establishing mathematics education, by his own reflections on teaching methodology, by publishing school textbooks, and by extensive historical research.

The fact that the overwhelming majority of IMUK members came from higher education constituted at the same time a strength and a weakness of IMUK's work. It was strong in that these members were firmly rooted within the mathematical community. The first IMUK thus worked on educational problems as an integral part of mathematical issues — there were no conflicts, no divergences between mathematicians and educators.

On the other hand, this *enracinement* within the mathematical community was a weakness of the first IMUK, since work was not undertaken from a proper perspective of school mathematics, i.e. that of teaching mathematics in schools. This potential weakness did not become an obstacle or a problem, however: firstly, since proper communities of mathematics education had not yet emerged, and secondly since Felix Klein had successfully avoided the one-sidedness which seemed to be an inevitable outcome of the domination of IMUK by higher education and by mathematicians' concerns.

In fact, the consequence of that domination was that the ICM, when creating IMUK, had commissioned it to deal with mathematics instruction at secondary schools only. This bias in favour of secondary schools was a result of the fact that university mathematicians considered this sector as the only field where they were possibly educationally competent. Felix Klein, and the *Comité Central* with him, modified this commission and undertook to extend their task to all school levels: from primary schools up to universities and technical colleges, and even including vocational schools!

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[5]) Cf. the thesis by Donoghue [1987] for an account of Smith's growing engagement in mathematics education.

> Dans le texte même de la résolution du Congrès de Rome il n'est question que de « l'enseignement mathématique dans les écoles secondaires ». Mais, étant donné que le but de ces écoles et la durée de leurs études est très variable d'un État à l'autre, le Comité fera porter son travail sur l'ensemble du champ de l'instruction mathématique, depuis la première initiation jusqu'à l'enseignement supérieur. Il ne se bornera pas aux établissements d'instruction générale conduisant à l'Université, mais il étudiera aussi l'enseignement mathématique dans les écoles techniques ou professionnelles. […]
>
> Il s'agit donc d'une étude d'ensemble de l'enseignement mathématique dans les différents types d'écoles et à ses divers degrés, […] [6]) [CIEM 1908, 451–452]

The work of IMUK and of its national subcommittees on mathematics instruction at all these school levels was really impressive. The official list of publications established in 1920, upon the dissolution [7]) of IMUK, documents 294 contributions published in 17 countries. Evidently, the quality of these papers varies significantly. In some countries, the reports were the result of truly collective and intensive work while, for other countries, the reports were prepared by individuals. The German reports were generally noted for being the best organised.

Communication and cooperation had not been an aim in itself for the *Comité*; they were understood rather as a process with a direction, namely initiating reforms. The pivotal point for launching the first international reform movement in mathematics education was, in fact, the decision of the *Comité Central* to complement the (more or less descriptive) national reports submitted by the national sub-committees with international comparative reports on a few key topics representing the major reform concerns. These topics, perhaps unsurprisingly, reflected the fact that the viewpoints of higher education were indeed dominant in the first IMUK. Of the eight topics for written reports, only three (1, 2, 5) corresponded to the level officially designated as IMUK responsibility, i.e. the level of secondary schools, whereas the majority of five topics concerned either the transition from secondary to higher education or

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[6]) In the text adopted by the Rome Congress, only "mathematical teaching in secondary schools" is mentioned. But, given that the aims of these schools and the duration of their studies are quite different from state to state, the Committee will extend its work to the entire field of mathematical instruction, from the first initiation to superior education. It will not limit itself to institutions of general education leading to University, but it will also study mathematical teaching in technical or vocational schools. […]

We will therefore undertake a global study of mathematical teaching in the various types of schools and at its various levels, […]

[7]) For more information on the different periods of existence of IMUK, see the additonal comments made by the author during the general discussion on this theme, as reported in this volume (p. 91).

related exclusively to higher education. The titles of the reports [8]) were:

1. La rigueur dans l'enseignement mathématique dans les écoles moyennes (Milan, 1911).
2. La fusion des différentes branches mathématiques dans l'enseignement dans les écoles moyennes (Milan, 1911).
3. L'enseignement mathématique théorique et pratique destiné aux étudiants en sciences physiques et naturelles (Milan, 1911).
4. La préparation mathématique des physiciens à l'université (Cambridge, 1912).
5. L'intuition et l'expérience dans l'enseignement mathématique dans les écoles moyennes (Cambridge, 1912).
6. Les résultats obtenus dans l'introduction du calcul différentiel et intégral dans les classes supérieures des établissements secondaires (Paris, 1914).
7. La préparation mathématique des ingénieurs dans les différents pays (Paris, 1914).
8. La formation des enseignants (professeurs) des mathématiques pour les établissements secondaires. [9])

On the other hand, it is quite clear that these topics, which helped to initiate reform movements in many of the participating countries, constituted essential concerns for the relation between mathematics education and mathematics, and they have remained pivotal points for this relation even today.

I shall give an overview of the major contents of these thematically comparative reports:

1. RIGOUR

The *rapporteur* for this topic was Guido Castelnuovo from Italy. He confined himself to geometry. While efforts in the other countries were towards reducing rigour in teaching geometry, while introducing empirical methods to encourage the role of intuition, such reforms were refuted in principle in Italy [Castelnuovo 1911]. The majority of Italian mathematicians even opted, and acted, for reinforcing rigour in geometry. A presentation of geometry in a more and more axiomatic form over the school years is characteristic of this Italian approach.

[8]) In brackets we mention the year and the place of their presentation either to a general IMUK meeting or to an IMC. See also Appendix 3 in [Furinghetti 2003].

[9]) 1. Rigour in mathematics teaching in secondary schools. 2. The fusion of the various mathematical branches in secondary school teaching. 3. Pure and applied mathematics teaching for students in physics and the natural sciences. 4. The mathematical preparation of physicists at university. 5. Intuition and experimental evidence in mathematics teaching in secondary schools. 6. Results obtained from introducing calculus in the upper grades of secondary institutions. 7. The mathematical preparation of engineers in the various countries. 8. The training of mathematics teachers for secondary institutions.

The last study, decided in 1914 for the next session, was delayed due to World War I and was presented only at the 1932 Congress.

2. Fusion

The report on the second topic concentrated on the controversy whether it was preferable or not for geometry teaching to integrate planimetry and stereometry. Other possible concrete realisations of ‘fusion’, say, of integrating algebra and geometry, were not discussed. Although the *rapporteur* was a Frenchman, Bioche, the discussion was again dominated by the Italians. The controversy whether planimetry and stereometry should be taught in an integrated manner, or not, had been dividing the Italian mathematical community for decades [Bioche 1911]. The pivotal point of the controversy was the concept of the purity of the mathematical method. Since the Italian syllabus of 1906, which admitted a plurality of teaching methods, had been replaced in 1910 by an anti-fusionist one, the traditional purists in Italy had eventually won.

3. Pure and applied mathematics for students of the sciences

The *rapporteur* was this time a German, Heinrich Timerding. He presented his report along the lines of the German problematic : the hotly debated point being whether the mathematical formation necessary to study the natural sciences should be specially adapted to the future profession of the students, or whether a general education in mathematics as a common foundation for the study of the sciences was preferable. Timerding criticised the too narrow specialisation prevailing in Germany, emphasising the more general character of mathematical education in France and Italy [Timerding 1911].

Since the two topical reports to the Cambridge Congress of 1912, where the mandate of IMUK was prolonged for another four years, essentially discuss variations of the first three topics, I will now concentrate on the last two, presented at Paris in 1914.

It is quite remarkable that the culminating point of the IMUK work was reached in 1914 at its Paris session where the two topics presented corresponded exactly to the two cornerstones of Klein’s German reform programme.

In Paris, the subject that attracted the most attention and participation was “the evaluation of the introduction of calculus to secondary schools”. The topic was hotly debated, and the report on it was the most voluminous of all the international IMUK reports [Beke 1914]. This was also the topic that Klein had prepared more carefully than any other. He not only helped design the international questionnaire that dealt with this matter, but he also chose its coordinator and reporter, Emanuel Beke, a Hungarian scholar and former student of Klein, and one of the most fervent adherents of Klein's programme. Beke was responsible for transmitting the reform ideas directly to Hungary and he was highly successful in initiating an analogous movement there.

For the second topic of the Paris session, the mathematical preparation of engineers, Klein also chose a trustworthy person as chairman, namely Paul Stäckel, a close co-worker of Klein's in matters of history and of applied mathematics [Stäckel 1914]. Klein, who did not participate in the Paris session himself, was immediately informed about the course of the debates by letters from his assistants and co-workers.

Concerning the mathematical training of engineers, Lietzmann was able to report, much pleased, that "the engineers want — this was the general opinion — to get their mathematics from the mathematician, not from the engineer" [Schubring 1989, 190]. Stäckel also reported that he was quite satisfied with the international response, particularly from the French engineers, the overwhelming majority of whom had stressed "the necessity of a *culture générale* for engineers" [*ibid.*].

Although Klein and his co-workers had expected more "palpable results" with regard to Calculus, the impact of his main reform agenda — namely to introduce the concept of 'functional thinking' as a basic notion pervading the entire mathematical curriculum — proved to be enormously successful on the international level. Sooner or later, the syllabus became modernised almost everywhere, supplanting the traditional restriction to the static ideas of elementary geometry which excluded any knowledge of variables or any kind of modelling of physical processes.

When Klein had begun to understand that not only had teacher education to be improved, but also that the school curriculum needed to be profoundly modernised, he criticised the deep chasm that existed between research mathematics and school mathematics. His intention in launching the reform movement had been to bridge that gulf, but in a manner which nonetheless acknowledged the structural difference between the domains of school and research. Klein used a fitting metaphor to describe this necessary difference, adopting the term HYSTERESIS from physics, saying that there must necessarily be a certain hysteresis between the evolution of mathematical knowledge and the contents of school mathematics. This is needed to restructure the building of mathematics by integrating the new knowledge and achieving a new elementarisation of it [Klein 1933, 221]. Klein's view was that a hysteresis of about 30 years was a reasonable time span, and he criticised the fact that at the beginning of the 20th century the time lag amounted to rather more than

100 years, with the result that there was no more ‘elasticity’ (to follow the physics metaphor) and that the relation between mathematics and the school curriculum had been completely severed.

Upon reflecting on the evolution of school curricula in mathematics across the 20th century, we observe that, as an effect of the work and initiatives of *L’Enseignement Mathématique* and of IMUK, such an immense hysteresis did not occur again and that, conversely, mathematics education began to follow the great achievements of mathematics quite closely, the time lag of 30 years suggested by Klein becoming even shorter. School curricula proved able to develop in a highly dynamical manner.

I should like to recall that applied mathematics was almost non-existent in Klein’s time. It is due to efforts initiated in Göttingen, and first realised by Carl Runge, that numerical mathematics began to evolve. Probability theory itself emerged as a distinct branch only since the 1920s.

Today, however, probability theory and stochastics are not only a standard branch of secondary school mathematics, but basic probabilistic notions can even be found in the content of primary school mathematics courses.

I should also like to mention the importance of the introduction of set-theoretical notions to school mathematics. While so-called modern mathematics is seen only negatively today, we should note that the new syllabuses based on the notions of mathematical structures brought about the modernisation of the syllabuses for primary schools for the first time.

As other examples of quite recent developments in mathematical research which quickly found their way into school, I just mention non-standard analysis and fractals. Perhaps it would be well not to mention the immediate introduction of information technology to schools, since mathematics itself has problems defending its own position versus computer science.

Thus an evaluation from an historical perspective allows us to see that the objective basis for the convergence between mathematics education and mathematics itself has been decisively improved, and that the two disciplines can contribute to fruitful cooperation from very advanced levels of scientific development.

On the other hand, it seems to be a problem for many mathematicians to understand that learning processes are not determined by content alone, and that research into learning processes requires very broad conceptual and empirical work, at least where the particularly resistant subject of mathematics is concerned. The enormous progress in research on pedagogical, psychological and epistemological foundations of the learning process in mathematics is responsible for the emergence of the didactics of mathematics as a scientific discipline in its own right. For the moment, it has to be admitted, this development carries with it the danger of an estrangement (*Entfremdung*) between the mathematics community and mathematics education.

Not only is this danger visible in many countries where mathematics educators have only a precarious foothold within the mathematics departments, but it has even become acute where mathematics education is well established. As an example we can cite the so-called mathematics war in California in recent years, where mathematicians have taken up arms against mathematics educators [10]). Although it would seem that these mathematicians were not aware of their being manipulated by politicians who tried to maintain elitist concepts of education, I find it deplorable how strongly they favoured highly conservative and out-dated concepts of drill and practice and learning by rote.

Given the objective basis for a convergence between mathematics and mathematics education, and the two disciplines' mutual interest in a qualitative improvement of school mathematics and of learning mathematics, I hope that the evaluation of these One Hundred Years will contribute towards promoting and improving these relations.

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[10]) See the controversial statements by Henry L. Alder, Jerry P. Becker and Bill Jacob in: *ICMI Bulletin 44* (1998), 16–25 and *45* (1998), 9–16. The conflicts are not confined to mathematics but concern other school subjects, too (see [Stotsky 2000]).

APPENDIX 1

MEMBERS OF IMUK / CIEM IN 1914

Australia. Horatio Scott Carslaw (univ.)

Austria: Emanuel Czuber (univ); Wilhelm Wirtinger (univ.); Richard Suppantschitsch (univ.)

Belgium: J. Neuberg (univ.)

Brazil: Raja Gabaglia (sec. school)

British Isles: Sir George Greenhill (Artillery college); E.-W. Hobson (sec. school); Charles Godfrey (Naval College)

Bulgaria: A. V. Sourek

Cape Colony: S. S. Hough (Observatory)

Denmark: Poul Heegard (univ.)

Egypt: F. Boulad

France: Jacques Hadamard (Polytechn., Collège de France); Maurice d'Ocagne (Polytechn.); Charles Bioche (sec. school)

Germany: Felix Klein (univ.); Paul Staeckel (univ.); Albrecht Thaer (sec. school)

Greece: Cyp. Stéphanos (univ.)

Italy: Guido Castelnuovo (univ.); Fed. Enriques (univ.); Gaetano Scorza (univ.)

Japan: Rikitaro Fujisawa (univ.)

Mexico: Valentin Gama (Observatory)

Netherlands: J. Cardinaal (polytechn.)

Norway: Olaf Alfsen (sec. school)

Portugal: Gomes Teixeira (Polytechn.)

Rumania: G. Tzitzeica (univ.)

Russia: Nikolaj v. Sonin (univ.); Boris Kojalovic (polytechn.); Constantin Possé (univ.)

Serbia: Michel Petrovitch (univ.)

Spain: Luis-Octavio de Toledo (univ.)

Sweden: Edvard Göransson (sec. school)

Switzerland: Henri Fehr (univ.); Carl Friedrich Geiser (Polytechn.); J.-H. Graf (univ.)

USA: David Eugene Smith (Teachers College); William Osgood (univ.): John Wesley Young (univ.)

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JOURNALS OF MATHEMATICS EDUCATION, 1900–2000

Les revues d'enseignement des mathématiques, de 1900 à 2000

par Gila HANNA

On examine d'abord l'évolution du nombre des revues d'enseignement des mathématiques au siècle dernier. Puis on discute les caractéristiques de trois importantes revues internationales qui sont entièrement dédiées à la recherche en didactique des mathématiques.

Dans la première partie on présente une recherche sur les revues, effectuée au moyen des sites web de *Ulrich's International Periodicals Directory* [UIPD] et de *Zentralblatt für Didaktik der Mathematik* [ZDM] (connu aussi sous le nom d'*International Reviews on Mathematical Education*). En combinant et comparant les résultats de la recherche sur les deux sites on voit que le nombre total de revues sur l'enseignement mathématique est passé de six au commencement du siècle à 314 en 1999. Nous analysons la distribution statistique des revues selon les pays où elles ont été publiées et la décennie de leur première publication. Le caractère des revues est étudié à partir des déclarations éditoriales : on a trouvé plus de similarités que de différences. Selon ces déclarations la majorité des revues se consacrent à la compréhension et à l'amélioration de l'enseignement des mathématiques, elles publient des articles qui peuvent intéresser les enseignants et les chercheurs, et elles sont ouvertes en particulier à des contributions qui discutent des nouvelles orientations dans la recherche en didactique des mathématiques ou des potentialités des nouvelles technologies pour améliorer l'enseignement des mathématiques.

Dans la seconde partie de cette contribution, on analyse plus en détail trois importantes revues : *Educational Studies in Mathematics* (*ESM*), *Journal for Research in Mathematics Education* (*JRME*) et *For the Learning of Mathematics* (*FLM*). Les deux premières revues sont comparées à partir de leurs profils initiaux, leur déclaration de politique éditoriale et les sujets traités dans les articles. *ESM* et *JRME* commencèrent dans la même période (en 1968 et 1970 respectivement), mais elles ont pris des directions différentes dans les deux premières décennies de leur existence. *ESM* publiait des discussions et des essais, tandis que *JRME* publiait des articles basés sur la recherche

expérimentale et sur l'analyse quantitative de données (opérant surtout à partir de tests à choix multiples).

Cette situation changea par la suite d'une façon significative, bien que graduellement. A la fin des années 1970, *ESM* commença à publier une grande variété d'articles, y compris certains qui traitaient de recherche quantitative basée sur des données statistiques. Aussi le sujet des articles changea-t-il considérablement. Même si la plupart des articles étaient encore des essais conceptuels sur des aspects de la didactique des mathématiques, la revue se mit à publier des rapports de recherche sur les problèmes de la classe comme le style d'apprentissage, le travail en petits groupes, la différence de genre, l'interaction dans la classe, les attitudes des étudiants et les interprétations fausses. A partir de la fin des années 1980, *JRME* diminua le nombre d'articles basés sur les projets expérimentaux, l'inférence statistique et autres méthodes quantitatives. Dans le même temps on assista à un accroissement dans la proportion d'articles basés totalement ou en partie sur des méthodes qualitatives. L'orientation vers une convergence de contenus dans les deux revues a continué au cours de la troisième décennie.

FLM est la plus jeune des trois revues, puisqu'elle est née en 1980. La différence entre *FLM* et les deux autres revues réside dans le fait qu'elle publie souvent des articles centrés sur des disciplines qui, dans la tradition, ne sont pas associées à l'enseignement mathématique, comme la sociologie, la linguistique, l'histoire, l'art, la philosophie et l'ethnomathématique.

JOURNALS OF MATHEMATICS EDUCATION, 1900–2000

by Gila HANNA

Today, at the beginning of the 21st century, mathematics education can boast of hundreds of journals, published all over the globe and in many languages. Our historical data reveals very few journals of mathematics education on the threshold of the 20th century, when the French-language journal *L'Enseignement Mathématique* was first published in Switzerland. This certainly reflects the status of mathematics education at the time, when it was not yet recognized as a scholarly discipline. As mathematics education slowly acquired such a status, however, first in Europe and then in North America and the rest of the world, many new journals of mathematics education began to appear, and the trickle of new journals at the turn of the 20th century turned into a flood by mid century and still continues.

This growth in the number of journals over the last century reflects a significant need created by the growth in the field of mathematics education itself. In this field, as in many others, scholarly journals are the primary vehicle for the dissemination of new research findings and the primary forum for the exchange of views and methods by its practitioners. In addition, they play a crucial role in helping the professional community monitor the quality of its research and aggregate the disparate pieces of new knowledge.

It is difficult to arrive at a precise count of journals of mathematics education, because the relevant data-bases do not always use clear descriptors and unambiguous keywords. In addition, the data itself can only reflect the necessarily limited manner in which journals choose to represent themselves in their titles and keywords. Several educational journals, for example, do not have the word 'mathematics' in their title or list of descriptors, but nevertheless do publish many articles on mathematics education (and even occasional special issues entirely devoted to it). On the other hand, of course, many journals that do have the word 'mathematics' in their title never publish articles on mathematics education.

ULRICH'S INTERNATIONAL PERIODICALS DIRECTORY

A search[1]) of the electronic, on-line version of *Ulrich's International Periodicals Directory* [UIPD 2000] (which appears to provide data for up to 1999) yielded a list of 209 journals, published in 28 countries. Confirmation that this count is more or less correct came from a search of the list for 1999 of some 400 serials covered by *Zentralblatt für Didaktik der Mathematik: International Reviews on Mathematical Education* [ZDM 2000], in the areas of mathematics, mathematics education, and general education. This search yielded about 200 journals of mathematics education.

It should be pointed out that Ulrich's directory is incomplete. It does not list, for example, such well-known journals of mathematics education as *For the Learning of Mathematics* (Canada) and *La Matematica e la sua Didattica* (Italy), and it reports nothing for some countries for which it is unreasonable to assume that there are no journals of mathematics education in any local language. Its data-base contains important details on each journal reported, however, such as first year of publication, frequency of appearance, publisher, whether refereed or not, cessation of publication, and a short description of the purpose of the journal. Accordingly, the 38th edition of Ulrich's directory is the most comprehensive source available, and nearly all the information presented below is based upon it.

Refereed journals

Of the 209 journals on the list of Ulrich's directory, 61 were shown as refereed. This means only that at least 61 are refereed, as this label was obviously attached only to journals that had made a point of confirming that they were refereed. *Mathematical Thinking and Learning: an international journal* and the *American Mathematical Monthly*, for example, both of which are known to be refereed journals, were not recorded as such on this list.

Status

The status of the 209 journals listed in [UIPD 2000] was as follows:

Active: 169 Ceased: 24 Unverified: 16

The 24 ceased journals were distributed as follows: Australia (1), Canada (1), France (1), Germany (5), India (1), Papua New Guinea (2: actually it was

[1]) using the description «mathematics AND (education OR learning OR teaching)»

TABLE 1
Distribution of mathematics education journals by country
(according to Ulrich's directory)

Australia	22	Italy	4	Spain	1
Botswana	1	Japan	19	Sri Lanka	1
Canada	6	Korea	1	Sweden	1
China	9	Malaysia	5	Switzerland	4
Czech Republic	3	Netherlands	8	Taiwan	1
Finland	1	Nigeria	1	Tanzania	1
France	10	Papua New Guinea	2	United Kingdom	21
Germany	17	Poland	6	United States	55
India	4	Russia	2	*Total*	*209*
Indonesia	1	Saudi Arabia	2	*Active*	*169*

only one, appearing with different titles), Poland (1), United Kingdom (3) and United States (9). For ten of these, there was no indication of a terminal date. The other fourteen had been published for as long as 98 years (one journal, in France) and as little as four years (two journals, in the USA). The median life of the 14 journals was 11 years, with a minimum of 4 years and a maximum of 27 (excluding the journal that lasted 98 years).

Distribution of journals by decade of first publication

For 33 of the 209 journals in Ulrich's directory, there was no indication of the year in which they were first published. Therefore Figure 1, which shows the distribution of the journals by decade, covers only those 176 journals for which the year of first publication was given.

As mentioned, Ulrich's directory is clearly incomplete. Schubring [1980] lists 253 journals, and Appendix 1 of [Furinghetti 2003] indicates that over 70 journals were listed in the "Bulletin Bibliographique" of *L'Enseignement Mathématique* as appearing in the years 1899–1901. The names of these journals would seem to indicate that most of them were devoted primarily to pure and applied mathematics. It would be fair to assume, however, that a number of them did publish papers on mathematics education, and even that some of them were dedicated to that topic, so that the numbers given in Figure 1 for these years, on the basis of Ulrich's directory, can only be an underestimation.

For example, important journals such as *Periodico di matematica* (founded in 1886), *Il Bollettino di Matematica* (founded in 1902), and *Bollettino della*

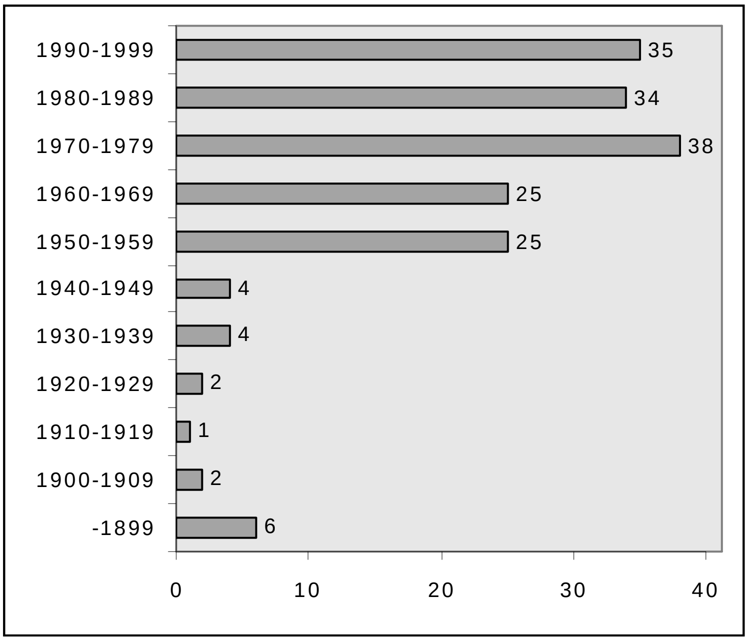

FIGURE 1
Distribution of 176 journals by first year of publication within each decade (according to Ulrich's directory)

Mathesis (founded in 1909), do not appear in Ulrich's directory even though they are written in Italian, one of the major languages used in mathematical journals and conferences of the period.

Only six journals included in Ulrich's directory began publication prior to 1900. They are :

1) *Statistics in Society* (in English), 1838. It is published three times a year by the Royal Statistical Society, Blackwell Publishers Ltd., United Kingdom. This refereed journal carries papers on economic, social and governmental issues (historical, biographical, philosophical, demographical) and on medical statistics, but appears to carry few papers related to mathematics education.

2) *American Mathematical Monthly* (in English), 1894. It is published ten times a year by the Mathematical Association of America, and is currently edited by John H. Ewing. It publishes expository articles on all facets of mathematics, pure and applied, old and new, and on teaching approaches, with regular columns devoted to reviews and to both basic and complex problems.

3) *L'Éducation Mathématique* (in French), 1898–1980 (ceased). It was published by Librairie Vuibert, Paris, France. It is described as a journal that published articles on mathematics, education, teaching methods, and curriculum.

4) *L'Enseignement Mathématique* (in French), 1899. This is an academic, scholarly journal currently published in Switzerland.

5) *Revue des Mathématiques de l'Enseignement Supérieur* (in French), 1890. Before 1998 this publication was called *Revue de Mathématiques Spéciales*. It is now published ten times a year. This publication carries information on teaching at the undergraduate and first graduate levels in France.

6) *Mathematical Gazette* (in English), 1894. This journal is currently published three times a year by the Mathematics Association, United Kingdom. It features articles about teaching and learning mathematics, with a focus on the 15–20 age range, as well as expository articles on attractive areas of mathematics.

In Figure 1, the [UIPD] data is presented by decade. In the first decade, 1900–1909, two new journals appeared : *Mathematics Teacher* in 1901 and *School Science and Mathematics* in 1908. The first, published by the National Council of Teachers of Mathematics (USA), contains papers on the improvement of mathematics instruction in junior and senior high schools, colleges, and teacher education colleges. The second, published by the School Science and Mathematics Association (USA), carries articles on assessment, curriculum, research, teacher education, learning theory, philosophy and history of science, non-traditional instruction, and the role and influence of science and technology in society.

Between 1910 and 1919 only one new journal appeared on the scene, according to this data-base : *Japan Society of Mathematical Education Journal* (in Japanese), 1919. It is published by the Japan Society of Mathematical Education and is described as being a mathematics education journal.

In the three decades from 1920 through 1949 ten new journals appeared, two in the 1920s and four in each of the next two decades. In the 1920s the two new journals were *Euclides* (in Dutch), 1925, and *Rozhledy Matematicko-fyzikalni* (in Czech and Slovak), 1921.

In the '30s the new additions were : *Dimensio* (in Finnish, with summaries in English, Finnish and Swedish), 1937, *Mathematics and Physics / Tokyo Kyokai Daigaku Rigakubu Kiyoa* (in multiple languages), 1930, *Matematika* *v Škole* (in Russian), 1934, and *Mathematics Student* (text in English) published by the Indian Mathematical Society, 1933.

The ’40s saw the publication of the *Australian Mathematics Teacher* (in English), 1945, the *Hokkaido University of Education Journal. Section 2A: Mathematics, Physics, Chemistry, Engineering* (text and summaries in both English and Japanese), 1949, *Matematyka* (text in Polish), 1948, and *Archimede*, 1949 (text in Italian, continuation of *Il Bollettino di Matematica* founded in 1902).

Over the next five decades, from 1950 to 2000, an average of 30 new journals of mathematics education arrived on the scene every ten years. In the 1950s the 25 new journals were distributed among six countries as follows: Japan (10), Germany (5), UK (5), Poland (2), USA (2), and Czechoslovakia (1). In contrast, the 25 new journals which appeared in the 1960s were distributed among as many as thirteen countries and included such prominent journals as *Educational Studies in Mathematics* and *Zentralblatt für Didaktik der Mathematik*.

The 1970s saw the publication of 38 new journals in seventeen countries; the highest concentration was in the USA (12), Australia (7) and Germany (6). Among the new journals were the well-known journals *Journal for Research in Mathematics Education* and *Journal für Mathematikdidaktik*.

Of the additional 34 new journals that appeared in the 1980s, 15 were in the USA, with the other 19 spread over 11 countries. The Australian *Mathematics Education Research Journal* and the American *Journal of Mathematical Behavior* are among these new journals. Also noteworthy in the 1980s is the appearance of the word ‘computers’ in the title of several of the new journals. Not surprisingly, the 1990s saw even more new journals with the word ‘computer’ in the title, among them the *International Journal of Computers for Mathematical Learning*. There were also a few journals devoted specifically to mathematics teacher education, such as the *Journal of Mathematics Teacher Education*.

Ulrich’s data-base does not contain information on journals that started appearing in the year 2000. For this reason it does not mention very recent additions such as the Greek journal *Themes in Education* or the Canadian *Journal of Science, Mathematics and Technology Education*, whose first issues have just appeared.

[UIPD 2000] does not mention electronic journals, but a Web search revealed that there are about a dozen new free electronic journals on mathematics education posted on the Web.

THE KARLSRUHE ZENTRUM FÜR DIDAKTIK DER MATHEMATIK DATA-BASE

The list of periodicals covered by the data-base maintained and edited by the *Zentrum für Didaktik der Mathematik* at Karlsruhe University [ZDM] was also consulted to complement the incomplete information provided by Ulrich's directory.

TABLE 2
Distribution of active mathematics education journals, in [ZDM] but not in Ulrich's, by country

Argentina	2	Hong Kong	1	Portugal	2
Australia	5	Hungary	2	Romania	2
Austria	3	India	3	Russian Fed.	1
Belgium	3	Italy	5	Singapore	3
Brazil	5	Japan	3	Slovakia	1
Canada	8	Korea	1	South Africa	2
Colombia	1	Malaysia	1	Spain	4
Croatia	2	Mexico	1	Sweden	2
Czech Republic	1	Mozambique	1	Switzerland	6
Denmark	3	Netherlands	3	Ukraine	1
Finland	1	New Zealand	1	United Kingdom	12
France	13	Norway	2	United States	20
Germany	12	Philippines	1	Yugoslavia	1
Greece	1	Poland	3	*Total*	*145*

Though [ZDM] provides up-to-date information on the presently active periodicals, it does not give details, such as first year of publication or whether journals are refereed or not.

The [ZDM] data-base lists 215 mathematics education journals, 70 of which were already part of [UIPD]. The additional 145 journals missing from Ulrich's were distributed among 41 countries as described in Table 2.

The total number of active mathematics education journals in both data-bases is 314 (169 in Ulrich's and 145 others in [ZDM]). By way of comparison, there are about 675 active mathematics journals and about 35 of these are free electronic journals [Jackson 2000].

EDITORIAL STATEMENTS

Editorial statements describe the purpose and scope of each journal and thus might be expected to help in determining differences among journals. These statements are remarkably similar to each other, however. For example, the *Journal for Research in Mathematics Education* (*JRME*) in its first issue in 1971 stated:

> [This journal] issues the invitation to researchers to submit papers which represent a coordinated approach to research on a major problem in learning, methodology, role of materials, etc. […] It is anticipated that such papers could serve as the basis for an exchange among mathematics education researchers.

This is very close to the statement in *Educational Studies in Mathematics* (*ESM*), first published in 1969:

> [*ESM*] presents new ideas and developments which are considered to be of major importance to those working in the field of mathematical education. It seeks to reflect both the variety of research concerns within this field and the range of methods to study them.

Again, this is not much different from the editorial statement of *For the Learning of Mathematics*, first published in 1980:

> The journal aims to stimulate reflection […] to generate productive discussion […] to promote criticism and evaluation of ideas and procedures current in the field […]

This is also quite close to the editorial statement of the recent journal *Mathematical Thinking and Learning*, which first appeared in 1999 and which invites high-quality articles

> that make a significant contribution to mathematical learning, reasoning, and instruction. Articles that are proactive and challenge the status quo are particularly sought. Authors who analyze future-oriented problems and offer clear, researchable questions for addressing such problems are likewise encouraged to make a contribution to the journal.

But there were journals that wanted to emphasize how different they were from existing journals. For example, the *Journal of Mathematical Behavior* (previously known as *Journal of Children's Mathematical Behavior*) which was created in 1971, stated that:

> [JMB] felt that the then-existing journals and common practices left a sizable gap which needed to be filled. It seemed that two distinct paradigms existed concerning research in mathematics education, but only one of these was represented in journals. The alternate paradigm, with which this *Journal* is concerned, is (at least roughly) defined by [the postulate that behavior is explained by information processing].

The *International Journal of Computers for Mathematical Learning*, first published in 1996, somewhat deviated from the conventional statements by saying that "the mathematics discussed is not limited by current boundaries of the discipline" and stating:

> [This journal] is dedicated to the understanding and enhancement of changes in the nature of worthwhile mathematical work that can be performed by learners, teachers and practitioners. It publishes contributions that open the way to new directions, particularly those which recognise the unique potential of new technologies for deepening our understanding. Contributions are both theoretical and empirical, practical as well as visionary.

A CLOSER LOOK AT THREE PROMINENT JOURNALS

In the following I will discuss in some detail three journals that have gained international prominence and are arguably prestigious journals. Rather than being professional journals aimed at teachers, all three are research journals devoted entirely to current research in mathematics education: *Educational Studies in Mathematics* (*ESM*), the *Journal for Research in Mathematics Education* (*JRME*) and *For the Learning of Mathematics* (*FLM*).

Educational Studies in Mathematics (*ESM*)

BEGINNINGS. *Educational Studies in Mathematics*, published in the Netherlands by Kluwer Academic Publishers, was founded by its first editor, Hans Freudenthal. The first two issues appeared together in May 1968, and consisted of the proceedings of a colloquium on mathematics education held in Utrecht in August 1967 and entitled "How to teach mathematics so as to be useful". The colloquium was an activity of ICMI. This combined issue consisted of 22 articles, of which 13 were in English, eight in French and one in German. Most of the authors were well-known mathematicians and mathematics educators, among them H. Freudenthal (The Netherlands), A. Krygovska (Poland), H. B. Griffiths (UK), T. Fletcher (UK) and A. Revuz (France).

ESM publishes papers in both English and French, though most appear in English.

POLICY STATEMENT. Oddly, *ESM* did not publish a policy statement initially. It is not until the appearance of volume 10, number 1, in February

1979, when Alan Bishop took over the editorship of the journal, that such a statement was made public:

> The journal presents new ideas and developments which are considered to be of major importance to those working in mathematics education. It seeks to reflect both the variety of research concerns within this field and the range of methods used to study it. It deals with didactical, methodological and pedagogical subjects rather than with specific programmes and organisations for teaching mathematics, and it publishes high-level articles which are of more than local or national interest.

The wording of this statement was slightly changed in 1990 when the editorship was put in the hands of a team of three editors, with Willibald Doerfler as editor-in-chief. Again the statement was slightly modified; the sentence “All papers are strictly refereed and the emphasis is on high-level articles which are of more than local or national interest” was added in 1996 when Kenneth Ruthven assumed the responsibility of editor-in-chief.

SUBJECTS OF ARTICLES. In its first issues, *ESM* published articles that can best be described as reflections on mathematics and its teaching, as well as articles on psychological and pedagogical aspects of teaching. It sought to establish strong links among scientific, educational and psychological research, mathematics, and classroom teaching. Most articles were of the essay form, presenting their arguments without relying on empirical educational data. There were almost no articles reporting on systematic quantitative evaluations of achievement, and in fact *ESM* shunned such articles. On one occasion the editor went so far as to dedicate an entire issue of the journal (vol. 6, no. 2, 1975) to a scathing critique of the North-American practice of using multiple-choice tests to measure mathematical achievement in general and of making international comparisons of achievement based on a common test in particular, as was done in the First International Mathematics Study (FIMS). Ironically, it was through this issue that *ESM* became especially well known in North America.

Through its first decade the journal continued, in the main, to publish essay articles that did not include any element of quantitative research. The table of contents for volume 4 (1973), for example, indicates that out of 33 articles, 16 discussed approaches to teaching various mathematics topics, nine were critical examinations of some psychological aspects of thinking about mathematics, two discussed linguistic variables that influence mathematics learning, two were about computers and mathematics, two addressed student behaviour, one described Soviet research, and one traced the history of reforms in mathematics education in the UK.

In the late 1970s, however, the journal began to publish a wider variety of articles, including ones that reported on quantitative research relying on statistical analyses. The subject matter became more varied as well. While it was still true that most of the articles were essays on general ideas, the journal did publish reports of research on 'real classroom problems' such as learning styles, grouping of students, gender differences, classroom interactions, and student attitudes and misconceptions. A great deal of this research, too, relied upon quantitative methods.

As the journal became somewhat more closely associated with the International Group on the Psychology of Mathematical Education (PME), in that a great number of its editorial board members were also members of PME, it tended to publish manuscripts that reflected the norms of research endorsed at the time by PME. Indeed, since the 1990s *ESM* has published two PME Special Issues, each of around 100 pages, every calendar year. These are explicitly identified as PME Special Issues, but are subject to the normal *ESM* reviewing process.

Journal for Research in Mathematics Education (*JRME*)

BEGINNINGS. The *Journal for Research in Mathematics Education* (*JRME*) is an American journal sponsored and managed by the National Council of Teachers of Mathematics (NCTM) that publishes, in English only, scholarly papers in mathematics education. Its first issue appeared in January 1970 under the editorship of David C. Johnson of the University of Minnesota. It published five research articles, two of which were "Verbal and nonverbal assessment of the conservation of illusion-distorted length" at the primary level, and "Attitude changes in a mathematics laboratory utilizing a mathematics-through-science approach" at the eighth-grade level. *JRME* editors are appointed by NCTM [Johnson, Romberg and Scandura 1994].

POLICY STATEMENT. The Editorial in the first issue invited researchers to

> […] submit papers which represent a coordinated approach to research on a major problem in learning, methodology, role of materials, etc. Such blocks of research could be published in a regular or special issue of *JRME*, or as an NCTM research monograph.

The journal was also willing to consider submissions of models and plans for large-scale research in an area, for publication in a special section called "A forum for researchers". This policy statement has undergone a few changes over the years, reflecting the changing wishes of the NCTM members as well as changes in the area of research in mathematics education. The most recent policy statement, posted on the Web, says that *JRME* is

> devoted to the interests of teachers of mathematics and mathematics education at all levels (preschool through adult). [The *JRME*] is a forum for disciplined inquiry into the teaching and learning of mathematics. The editors encourage the submission of a variety of manuscripts: reports of research, including experiments, case studies, surveys, philosophical studies, and historical studies; articles about research, including literature reviews and theoretical analyses; brief reports of research; critiques of articles and books; and brief commentaries on issues pertaining to research.

SUBJECTS OF ARTICLES. In its early years *JRME* published mainly research reports. An examination of the subject index for volume 4 (1973), for example, indicates that of the 29 articles published in that year as many as 18 are research reports on the comparative effectiveness, as measured by achievement, of various class sizes and methods of teaching. Of the remaining eleven articles, those not concerned with measuring effects, four discuss psychological theories, four review various curricula and three report on other research on mathematics education or present responses to previous contributions.

The titles of articles had a very practical sound in the first decade of the journal. They almost always included the word ‘effect’ (“The effect of class-size on…”, “The effect of organisers…”, “The effects of instruction on…”) or the word ‘comparison’ (“A comparison of three strategies…”, “A comparison of initially teaching…”), or the words ‘correlating’, ‘determining’ and ‘predicting’.

Indeed, throughout its first ten years about two-thirds of the articles in *JRME* were experimental research reports that relied on statistical techniques such as analysis of variance or regression analysis. Even after the second decade such research reports still made up about five-eighths of the articles [Lester and Lambdin 1998]. There had been some increase, however, in the proportion of articles that relied in whole or in part on qualitative methods, and that trend continued through the third decade.

RESEARCH METHODS AND THEORETICAL FRAMEWORKS IN *JRME* AND *ESM*, 1990–2000. Over the last decade, a comparison of the research paradigms reported in *JRME* articles with those reported in *ESM* reveals more similarities than differences. Although *JRME* is still perceived, at least by Europeans, as a journal that stands out in not favouring articles on qualitative research, this perception is not borne out by the data. As shown in Figure 2, if one considers all articles, including those that do not report on research at all, the proportion

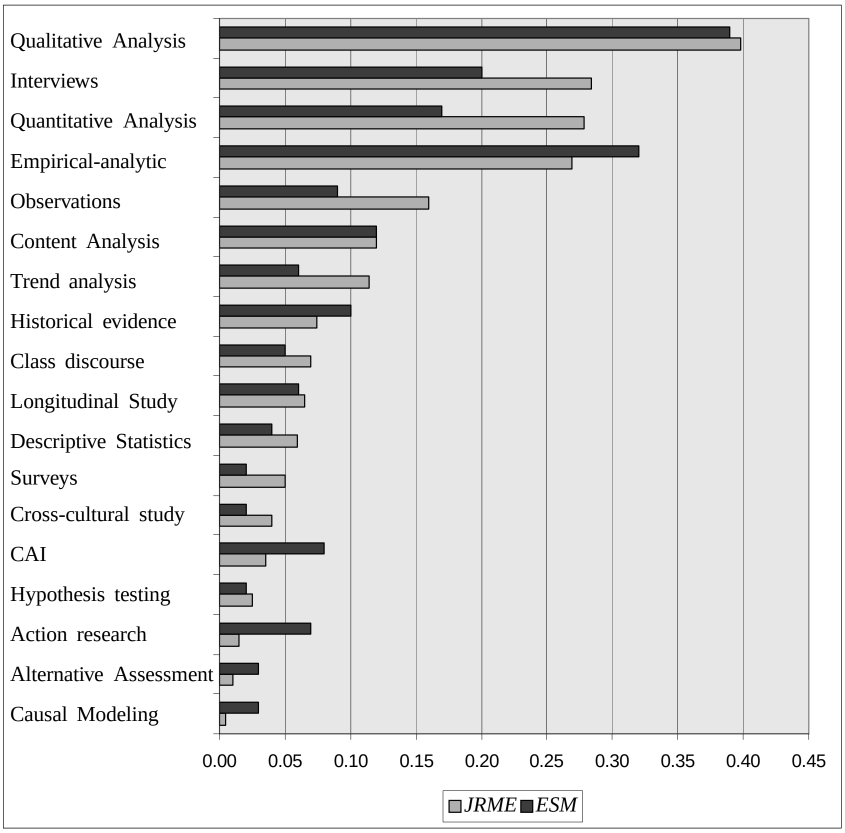

FIGURE 2
Proportion of articles in *JRME* and *ESM* using each research paradigm

of articles that make use of quantitative analysis is indeed slightly higher in *JRME* than in *ESM* (28% vs. 18%), but the proportion of articles presenting qualitative analysis is the same in both journals.

For the Learning of Mathematics (*FLM*)

BEGINNINGS. This journal owes its existence, its unique character, and its prominence in the field of mathematics education to its founder and first editor, David Wheeler. It came into being in 1980 with the help of small grants from the Canadian Social Science and Humanities Research Council and from Concordia University in Montreal. David Wheeler was the editor of

FLM for its first 50 issues, which appeared between 1980 and 1997. In 1997, after Wheeler's retirement, the Canadian Mathematics Education Study Group (CMESG) became its manager and appointed David Pimm as editor.

FLM publishes papers in both English and French, though most appear in English.

POLICY STATEMENT. From its inception, *For the Learning of Mathematics* (*FLM*) reflected the broad interests and good judgment of David Wheeler. The name of the journal was intentionally chosen to emphasize the concern for learning and to acknowledge the influence of Caleb Gattegno, known for his document *For the Teaching of Mathematics* [Gattegno 1963] and his other influential contributions to mathematics education over four decades. *FLM* quickly established itself as a respected periodical that has consistently published articles reflective of its policy statement:

> […] to stimulate reflection on and study of the practices and theories of mathematics education at all levels; to generate productive discussion; to encourage enquiry and research; to promote criticism and evaluation of ideas and procedures current in the field. It is intended for the mathematics educator who is aware that the learning and the teaching of mathematics are complex enterprises about which much remains to be revealed and understood.

In its suggestions to writers, *FLM* indicated that

> Mathematics education should be interpreted to mean the whole field of human ideas and activities that affect, or could affect, the learning of mathematics. […] The journal has space for articles which attempt to bring together ideas from several sources and show their relation to the theories or practices of mathematics education. It is a place where ideas may be tried out and presented for discussion.

A review of articles published in the early issues of *FLM* confirms that the editor had selected papers that stimulated reflection on mathematics education, in line with these suggestions and with journal policy. Some characteristic titles were "Communicating mathematics is also a human activity", "Alternative research metaphors and the social context of mathematics teaching and learning", and "When is a symbol symbolic?".

SUBJECTS OF ARTICLES. The articles published in *FLM* over the past two decades show a breadth of subject matter. They draw heavily on a number of disciplines not traditionally associated with mathematics education, such as sociology, linguistics, history, physics, arts, philosophy, and ethno-mathematics. Some indication of the place of such themes in *FLM* is given by the occurrence of certain key terms in the titles of the approximately 300 articles that appeared in the years 1980–2000. Some variation of the word ‘theory’ (‘theoretical’, ‘theories’) appeared in 23 titles, ‘history’ in as many as 20 titles, ‘ethnomathematics’ in 12, ‘language’ in 11, ‘proof’ in 9, ‘epistemology’ in 7, and ‘philosophy’ in 7.

FLM seems to have put a particular emphasis on articles that discussed the role of the history of mathematics in mathematics education (along with suggestions for its classroom use) and explored the concept of ethnomathematics and its potential application to mathematics education. In fact, *FLM* may be the first scholarly journal of mathematics education to publish papers on these topics and in so doing to contribute significantly to the development of these young areas of scholarly research. Another theme prominent in *FLM* is that of the nature of proof in mathematics and its implications for the teaching of proof.

Another way to characterize *FLM* is to look at the kind of papers it did not publish. It has very few articles, only three in fact, that follow a traditional format of research papers in mathematics education: problem statement, review of relevant literature, rationale for the method of research and research results. There are no articles reporting on purely experimental classroom research and relying solely on statistical analyses. Some of the published articles do report on empirical research, but only as part of a wider discussion of some issue. No article appeared with a title such as “The effect of X on Y”. *FLM* has published relatively few purely empirical studies, and thus it is difficult to determine whether there was an editorial preference for any particular research method.

CONCLUSION

The impressive number of active mathematics education journals is probably the most tangible indication that the discipline of mathematics education is alive and well. They have been very successful, collectively, in establishing and maintaining a wide variety of national and international forums for the publication and discussion of research on this topic. In so disseminating new information and encouraging debate, these journals also serve researchers as a valuable instrument of professional development.

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REACTION

by Jean-Pierre BOURGUIGNON

Five key points can be identified within the context of the changes that have occurred between 1900 and 2000, if we consider the key issues facing mathematics education today.

THE INTERNATIONAL DIMENSION. Internationalism was at the very heart of the genesis of *L'Enseignement Mathématique*. Of course, mathematics then was international, but there was a need to provide opportunities for communication. Today things have changed a great deal because of changes in the world we are living in rather than in mathematics itself. Travel is much easier, news is disseminated more swiftly, and altogether the way we feel about the world is very different. The internationalism then espoused by the journal is now commonplace.

LINKS BETWEEN MATHEMATICS AND SOCIETY. Several articles in the early issues of *L'Enseignement Mathématique* address the responsibility of mathematicians to educate, not only mathematicians, but also those engaged in other disciplines. That time coincided with the birth of many national societies. As far as France was concerned, a major stimulus for the creation of national scientific societies derived from the defeat of France by Germany in 1870 and the perceived view that France had not paid sufficient attention to the importance of science (in particular in its connections with engineering). Also in 1900 a big effort towards popularising science could be observed.

Today the significance of the link between mathematics and society (whose operation in a number of places is actually very dependent on sophisticated mathematics and mathematical methods) has not really percolated down to the teaching of mathematics (at least in France). It is essential to make much more explicit the link between the mathematics that is studied and the society it serves. In fact much of society is very critical of science in general and we see a decline in the number of students wishing to study science. It is very important to address this trend which comes in particular from a generally negative view of science.

A second point to make about the link between society and mathematics is a change of the vocabulary used in mathematics today compared with the early issues of *L'Enseignement Mathématique*. An example is the use of the term 'model' which now appears in almost every book and curriculum statement about mathematics and was not there in the 1900s. This is more than fashion, and shows a conceptual change in the way we view mathematics. If we look at articles about geometry for example in the early issues of *L'Enseignement Mathématique*, we have the impression it is concerned with concrete objects. The idea that mathematics represents things in an idealised way was not even present. Today this is very different and shows that this change of philosophy is now widely accepted.

AN ARENA FOR CONFRONTATION. The early issues of *L'Enseignement Mathématique* were seen by its editors as an arena for the confrontation of mathematicians and teachers of mathematics. The word 'confrontation' is not pejorative but used here in the original sense of 'coming face to face with'. The need still exists today but there are significant differences, not least the development of a separate discipline of didactics, as Gert Schubring has noted. At that time the number of research mathematicians world-wide was about 200 in all; today there are of the order of 60,000. The number of secondary school mathematics teachers today, world-wide, has become huge. Clearly a journal can no longer serve as a tool for the common interest of both groups (and, in any case, the journal is no longer playing this role). It is also noteworthy that there have been changes, at least recently, in the way teachers of mathematics and mathematicians perceive each other. To refer to the French situation, we can note that teachers of mathematics are no longer asking mathematicians to talk to them about ways of teaching mathematics (*tricks to be used in the classroom*) but are asking increasingly for information about the nature of new mathematics being done. This is a sign that a move towards both groups seeing each other as experts in their own areas is occurring.

An important new development is the increasing use of computers, which is not to be seen in any way as a threat, as implied by the comments of Gert Schubring. But in order for it not to become one, it is vital that the first experiences of students with computers in the secondary school be given by mathematicians and not by computer scientists. This means that mathematics teachers need to become familiar with computers and computer software and make substantial efforts to incorporate this tool into their own teaching. This will have consequences on the relative weights of different areas of mathematics, the usual emphasis on analysis at the end of the school curriculum being challenged by theories and objects borrowed from discrete mathematics.

L'ENSEIGNEMENT MATHÉMATIQUE AS A TOOL FOR INFORMATION FLOW. The journal saw itself as an important vehicle for the exchange of information. This is no longer a necessity. In fact, the need today is not creating a flow of information, but controlling it. This is a problem that still needs to be addressed at the right level, because we need to get the right information to the right people, and how this can be achieved is not so obvious. In any case, a paper journal is clearly no longer a suitable tool for the purpose of reaching teachers at all because of its too restricted circulation.

THE SCIENTIFIC LANDSCAPE. There has been a major change in the scientific landscape. Early issues of the journal did not really concern themselves with addressing other sciences. Today, as ever, mathematicians seem to be reluctant to talk enough and at the appropriate depth to other scientists. This attitude is extremely dangerous for mathematics, not only for opportunistic reasons, but also for the health of the discipline. One reason is that the scientific landscape itself has changed a great deal. Many new sciences have appeared (computer science has already been mentioned, but we could mention others) and the general balance between fields is now completely different, the dominating role and the now central importance achieved by the life sciences being one of the features to stress.

The life sciences are attractive to politicians (because they address, and fortunately in several cases solve, important problems connected with the health of the population) and also attractive to students (because they offer new challenges and new frontiers to explore). In fact many new ideas for mathematics come from the sciences. To give one name, which brings us back to 1900, Henri Poincaré was inspired by celestial mechanics and theoretical physics. We need to look to sciences to provide stimulation for new mathematics. Science will demand new tools, and new mathematics will be generated although we don't know what it might turn out to be. We also need to involve other sciences in order to attract students to do mathematics. This has implications for the teaching of mathematics, which must expose students to other sciences as well as mathematics. Some countries train mathematics students by exposing them only to mathematics, which is not good for the students and also potentially weakens the development of the mathematical sciences.

CONCLUDING REMARK. Initially *L'Enseignement Mathématique* provided a tool by which mathematicians and mathematics teachers from different countries could compare their experiences, which were then very different. It is still the case today that experiences are very different in different parts of the world. Even if one considers just the European Union, one finds very diverse systems of mathematics education. If a unified platform is to be introduced within the EU, a task the governments have set themselves, then the education systems in the different countries need, at least, to be compatible. The tendency of bureaucracies everywhere when asked to provide a solution to such a problem is to go for a uniform solution. While compatibility is desirable, we stress that one should guard against this leading us into a wholly undesirable uniformity.

GENERAL DISCUSSION

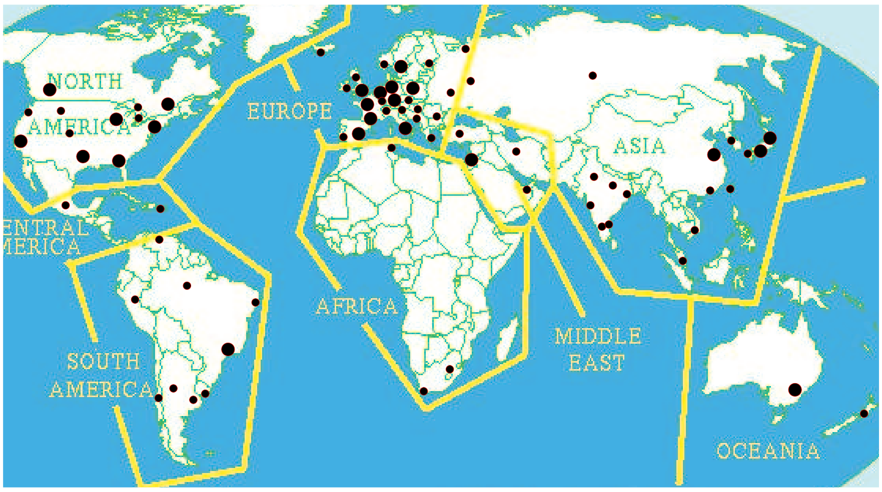
(reported by Chris WEEKS)

The discussion from the participants following J.-P. Bourguignon's reaction to the three talks divided into two areas. On the one hand there were questions and comments about the journal *L'Enseignement Mathématique* itself and on the other hand contributions to a discussion about the teaching of mathematics today, and particularly the importance of the key idea identified by J.-P. Bourguignon of 'Links with Society'.

THE JOURNAL *L'ENSEIGNEMENT MATHÉMATIQUE*. First, in answer to questions, Fulvia Furinghetti was able to provide some further information about the readership and dissemination of the journal. Although now almost entirely found in universities and other institutions, the journal in its early days was subscribed to by individual mathematicians and teachers of mathematics. Some of these appear to have been lone figures in countries like Rumania and Albania, but there was a relatively strong readership in France, Italy and Germany. Daniel Coray, on behalf of the present editorial board, reported that there are currently about 600 subscribers, although about 850 copies of each edition are printed to provide for a future time when new universities in developing nations, such as China, may wish to become subscribers and purchase previous copies. In addition to Europe and USA there is also a healthy readership in Japan.

As to the content of the journal, it was noted that, in addition to technical mathematics articles, early editions contained articles relevant to the teaching of mathematics and articles *about* mathematics. The journal today consists almost entirely of research articles (some elementary, others not so elementary) and, notwithstanding its title, *L'Enseignement Mathématique* is not a journal concerned with mathematics education. It is, however, a journal of ICMI and associated with IMU.

INTERNATIONAL DISTRIBUTION OF THE JOURNAL *L'ENSEIGNEMENT MATHÉMATIQUE*
(estimate based on shippings between 1996 and 2000)



- Switzerland : 25 ; France : 45 ; Germany : 66 ; Austria : 7 ; Italy : 27 ; Belgium : 5 ; Netherlands : 40 ; Luxembourg : 1 ; Spain : 16 ; Portugal : 4 ; UK [1]) : 42 ; Eire : 2 ;
- Denmark : 6 ; Norway : 3 ; Sweden : 7 ; Iceland : 1 ; Finland : 3 ; Croatia : 1 ; Slovenia : 1 ; Yugoslavia : 1 ; Czech Republic : 2 ; Poland : 7 ; Bulgaria : 1 ; Hungary : 5 ; Rumania : 3 ; Greece : 2 ;
- Russia : 4 ; Turkey : 1 ; Israel : 7 ; United Arab Emirates : 1 ; Tunisia : 2 ; South Africa : 2 ; Argentina : 3 ; Brazil : 8 ; Chile : 2 ; Mexico : 2 ; Uruguay : 1 ; Venezuela : 1 ; USA : 144 ; Canada : 19 ;
- Japan : 60 ; Korea : 1 ; China : 6 ; Hong Kong : 1 ; Taiwan : 1 ; Iran : 1 ; India : 9 ; Malaysia : 1 ; Singapore : 1 ; Vietnam : 2 ; Australia : 10 ; New Zealand : 2

Total [2]) *: 615*

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[1]) A number of copies sent to the UK or the Netherlands are in fact resent to other destinations (South America, Asia, Africa). The figures have therefore been corrected whenever some clear indications could be obtained from shippings in previous years, but they remain too high for these two countries.

[2]) This total includes about 100 copies which are exchanged with other periodicals from all parts of the world.

One important reason for the change in emphasis is the development of many national and international organisations now catering for the needs of teachers of mathematics, as well as researchers in the fields of psychology, history, didactics, etc. of mathematics, and these meet their client needs much better than a journal. A second reason is connected with the history of the journal, as explained further by Gert Schubring.

From the beginning, *L'Enseignement Mathématique* should not be seen as the only publication of ICMI. National reports on the state of mathematics education in different countries were solicited and these were also published. There were also ICMI Study Groups, whose reports were published. It should be noted, however, that — contrary to the present ICMI — the first ICMI was not a permanent body: It was constituted by the 1908 International Congress of Mathematicians (ICM) with a mandate of four years, until the next Congress. Since this period proved to be too short, the mandate was extended in 1912 by four more years. Due to World War I, there was no Congress in 1916. Since there was, after the War, a ban against German scientists and since the Germans had been a highly active element within the ICMI work, the first ICMI was dissolved in 1920. The first again truly international Congress in the Inter-War period, 1928 in Bologna, reconstituted ICMI with a four-year mandate, extended at each following Congress until 1936 although the work never regained its former intensity. When ICMI was re-established in 1952, it became independent of the ICMs and attained a permanent character. The first series of *L'Enseignement Mathématique*, until 1954, continued to provide a forum for discussion about mathematics education but, for reasons given above, little of this appears in the second series.

LINKS WITH SOCIETY. A number of participants were struck by "la modernité de la problématique" of the founders of *L'Enseignement Mathématique*. Certainly many of the problems facing the teaching of mathematics identified in the early issues of *L'Enseignement Mathématique* are present today.

A number of speakers commented on the importance of linking mathematics and mathematics teaching with society and this was also linked to the effectiveness and relevance of much of the teaching of mathematics today. While mathematics teaching of future mathematicians is by and large satisfactory, and is also generally satisfactory for users of mathematics such as physicists and engineers, mathematics education for the bulk of the population has serious weaknesses or worse – "quelle catastrophe" as one speaker put it. Part of the cause lay in a certain detachment of mathematics education from the world around us, as if the processes of mathematics were in themselves sufficient to justify its existence. Many speakers picked up on the point made by J.-P. Bourguignon that mathematics education must be linked to other sciences.

The sort of mathematics needed for the future non-professional user of the subject concerned a number of speakers. The day-to-day mathematical requirements for the citizen are decreasing while at the same time society is becoming more dependent on mathematics. In this sense, mathematics awareness is even more necessary but technical expertise is less necessary. Mathematics education needs to take account of this in terms of the content of the subject material and also its place in the curriculum. One danger is that mathematics could become like Latin, an optional subject for the interested scholar, and one speaker from Belgium suggested that this was already happening as a result of making it optional so that even chemistry and biology students no longer need to study mathematics.

Other points made were that the learner is a person, with a life and experience outside the mathematics lesson. He or she is not a blank sheet on which new ideas and concepts can be inscribed, but comes to the mathematics lesson with many ideas (and perhaps many false ideas). Mathematics teaching that does not engage with the person is destined to fail. Linked to this is the point made by another speaker that, certainly for the majority of the population, mathematics must have ‘a human face’. In fact it would be a worthwhile venture for ICMI to set up a study group on this very topic.

GEOMETRY

LA GÉOMÉTRIE DANS LES PREMIÈRES ANNÉES DE LA REVUE *L'ENSEIGNEMENT MATHÉMATIQUE*

Geometry in the first years of the journal L'Enseignement Mathématique

by Rudolf BKOUCHE

The journal *L'Enseignement Mathématique* was closely associated with the reform movement of scientific teaching which took place at the beginning of the 20th century: the founding texts of the reform were in fact published in this journal. We use some of these texts to discuss a few questions raised by the renewal of geometry teaching.

Understanding the global stakes of this reform implies taking into account the place that mathematics occupied in the physical and natural sciences and in technology at the turn of the 20th century. For the specific case of geometry, we must remember its dual character, at the crossroads of mathematics and physics, which conducted the reform movement through two major themes that have renewed the teaching of geometry. One is *fusion* — that is, ignoring the traditional distinction between plane and solid geometry —; the other one is *motion*, more precisely the motion of solids in three-dimensional space.

We will focus our attention on these two themes and on the way they appear in the texts published by *L'Enseignement Mathématique*. This will lead us to reconsider the empiricist conception of geometry which marks these texts so strongly and which constituted an essential aspect of the whole period under review.

LA GÉOMÉTRIE DANS LES PREMIÈRES ANNÉES DE LA REVUE *L'ENSEIGNEMENT MATHÉMATIQUE*

par Rudolf BKOUCHE

« Se demander si un enfant a des dispositions pour la Mathématique équivaut à se demander s'il en a pour l'écriture et pour la lecture. »
[Laisant 1907]

INTRODUCTION

On ne peut détacher la revue *L'Enseignement Mathématique* de la réforme de l'enseignement scientifique qui s'est développée au début du XX^{e} siècle [1]). C'est dans la revue qu'ont été publiés les textes fondateurs de la réforme et nous nous proposons dans cet exposé d'aborder, à travers quelques textes publiés dans *L'Enseignement Mathématique*, les questions posées par le renouvellement de l'enseignement de la géométrie.

Pour comprendre les enjeux généraux de la réforme, il nous faut prendre en compte la place des mathématiques dans les sciences de la nature et dans la technique au tournant des XIX^{e}–XX^{e} siècles et les conceptions positivistes et empiristes qui marquent cette époque.

En ce qui concerne la géométrie il faut alors prendre en compte son caractère mixte, au carrefour des mathématiques et de la physique [2]) ; c'est ce caractère mixte qui conduit la réforme à travers deux thèmes qui se

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[1]) Sur la réforme de l'enseignement scientifique en 1902, nous renvoyons à l'article de Bruno Belhoste [1989] ainsi qu'à notre article [Bkouche 1991a] et à l'ouvrage *Les Sciences au Lycée* [Belhoste, Gispert & Hulin 1996]

[2]) « Geometry is a physical science » écrit W.K. Clifford [1885] dans un ouvrage, *The Common Sense of the Exact Sciences*, publié après sa mort par Karl Pearson. A travers deux chapitres, "*Space*" et "*Position*", Clifford explique comment la géométrie se rattache aux sciences de la nature.

proposent de renouveler l'enseignement de la géométrie, d'une part celui de la *fusion*, c'est-à-dire l'abandon de la distinction traditionnelle entre la géométrie plane et la géométrie dans l'espace, d'autre part celui du *mouvement* et plus particulièrement du mouvement des corps solides. C'est donc sur ces deux thèmes et la façon dont ils interviennent dans les textes de la revue que nous centrerons notre exposé. Cela nous conduira à revenir sur la conception empiriste de la géométrie qui marque le tournant des XIX^e–XX^e siècles.

La place des mathématiques dans les sciences de la nature et dans la technique au début du XX^e siècle

La réforme de l'enseignement scientifique, et particulièrement de l'enseignement des mathématiques, est marquée par le succès de la mathématisation des sciences de la nature, essentiellement des sciences physiques, et le rôle des mathématiques dans le développement des techniques.

Cette mathématisation présente deux aspects, d'une part l'expression mathématique des lois de la nature issue des travaux des physiciens-mathématiciens du XVII^e siècle, d'autre part, conséquence de cette mathématisation, la possibilité accrue de prédiction des phénomènes, autant sur le plan qualitatif que sur le plan quantitatif. On pourrait citer trois grandes théories exemplaires, la mécanique rationnelle, l'électromagnétisme et l'énergétisme. Si nous ne pouvons entrer, dans le cadre de cet exposé, dans une étude de ces théories, nous insisterons cependant sur le double aspect de leur succès : théorique d'une part, technique d'autre part.

Si la science mathématisée prend cette importance dans le développement de la société, il devient nécessaire que la place des mathématiques dans l'enseignement soit réévaluée, autant dans l'enseignement général que dans l'enseignement professionnel et la formation des ingénieurs.

De nombreux textes des premières années de *L'Enseignement Mathématique* notent cette importance des mathématiques dans la connaissance de la nature, non seulement parce qu'elles mettent en forme la connaissance empirique mais aussi parce que le formalisme mathématique est lui-même créateur de connaissance.

> Les relations de la mathématique avec le monde qui nous environne ne sont donc pas accidentelles et artificielles ; et si on oubliait ces relations, on ferait perdre à cette science son caractère.

écrit Bettazzi [1900, 16] dans un article consacré à l'application des mathématiques. On pourrait citer aussi des textes sur la mécanique et son enseignement ([Laisant 1899], [Maggi 1901]), mettant en avant les liens entre la mécanique théorique, laquelle relève des mathématiques, et la mécanique pratique et la nécessité d'une théorisation.

Plus tard, l'un des principaux animateurs de la réforme de 1902 en France, Carlo Bourlet, expliquera, après avoir insisté sur la part de la connaissance de la nature dans le développement des mathématiques, combien il est juste que celles-ci, après avoir pris leur autonomie, apportent aujourd'hui leurs derniers développements aux sciences de la nature, remettant ainsi en question toute distinction entre mathématiques pures et mathématiques appliquées [Bourlet 1910].

Cette insistance sur la part des mathématiques dans la connaissance de la nature posait une double question : d'une part la nécessité d'un enseignement des mathématiques pour aborder l'étude des sciences de la nature, d'autre part la question de qui doit enseigner les mathématiques : l'enseignement des mathématiques doit-il être pris en charge pas les enseignants des disciplines qui en ont besoin ou doit-il être assuré par un corps spécifique d'enseignants de mathématiques.

C'est à cette seconde question que s'attaque Felix Klein dans un article de 1906. S'il reconnaît la nécessité d'un enseignement moins formel des mathématiques dans les Hautes Écoles Techniques (les écoles d'ingénieurs), il précise combien une étude mathématique préalable peut faciliter la résolution de certains problèmes pratiques. Cela le conduit à refuser que l'enseignement des mathématiques soit laissé aux seuls praticiens [Klein 1906].

La géométrie au carrefour des mathématiques et de la physique

La position de la géométrie est particulière dans la mesure où elle parle des objets de l'espace, plus précisément des corps solides ; en ce sens elle se rattache aux sciences de la nature. Il faut alors rappeler que la classique distinction entre géométrie rationnelle et géométrie pratique porte essentiellement sur la manière d'étudier les propriétés géométriques des corps, mais la géométrie est une, du moins jusqu'à la découverte des géométries non-euclidiennes. C'est la découverte de la multiplicité des géométries qui a conduit à distinguer une géométrie mathématique, dont l'exemple est donné par la construction hilbertienne [Hilbert 1899], et une géométrie physique qui s'intéresse à la structure de l'espace dans lequel nous vivons, ce qu'Einstein résume de la façon suivante :

> Pour autant que les propositions de la mathématique se rapportent à la réalité, elles ne sont pas certaines, et pour autant qu'elles sont certaines, elles ne se rapportent pas à la réalité [3]). [Einstein 1921 (1972), 76]

Einstein pose alors la question des relations entre cette géométrie mathématique et la géométrie physique, mais ce n'est pas encore la question de ce début de siècle qui s'interroge sur les liens entre les objets idéaux de la géométrie et les objets du monde réel.

Il semble cependant, en ce début du XX^e siècle, que la découverte des géométries non-euclidiennes ne soit pas encore comprise et acceptée par tous, comme le montre le débat récurrent sur le postulat des parallèles dans les premiers numéros de la revue. On peut alors noter que les tentatives de démonstration du postulat des parallèles ou les critiques des arguments non-euclidiens s'appuient sur des arguments d'ordre physique et l'impossibilité de penser un espace autre que l'euclidien auquel nous sommes habitués et qui est devenu un cadre de pensée [4]).

Ces résistances encore fortes devant la possibilité d'une géométrie non-euclidienne montrent que la distinction entre géométrie mathématique et géométrie physique n'est pas faite malgré les écrits des pères de la géométrie non-euclidienne, Gauss, Lobatchevski et Bolyai, puis ceux de Riemann, de Klein ou de Poincaré. Les modèles euclidiens restent loin d'être convaincants ; « les géodésiques de la pseudo-sphère de Beltrami ne sont pas des droites » expliquent certains contradicteurs. Notons qu'à la même époque (fin du XIX^e siècle et début du XX^e) des arguments contre la géométrie non-euclidienne sont développés par Cayley [1883] et Frege [1994].

En contrepoint, des auteurs vont publier des exposés de géométrie élémentaire en faisant ressortir la partie d'icelle qui ne dépend pas du postulat des parallèles, en particulier la géométrie sphérique [5]). Nous citerons ici l'ouvrage de Dassen (mathématicien argentin), *Tratado elemental de Geo-*

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[3]) Einstein distingue dans ce texte la *géométrie axiomatique pure* et la *géométrie pratique.*

[4]) En un certain sens l'espace euclidien est *devenu* une forme *a priori* de notre intuition, ce qui pourrait être une tentative d'explication des raisons qui ont conduit aux conceptions développées par Kant dans son *Esthétique transcendantale.*

[5]) L'indépendance de la géométrie sphérique par rapport au postulat des parallèles semble avoir frappé les mathématiciens d'autant que c'est à partir de considérations de géométrie sphérique que Lobatchevski développe les relations métriques et trigonométriques de sa géométrie [Lobatchevski 1840].

metría Euclídea (tome I : géométrie plane, tome II : géométrie dans l'espace), ouvrage longuement commenté dans la revue [6]).

L'aspect physique de la géométrie joue un rôle essentiel dans la mise en place de la réforme et l'on peut considérer que c'est *via* l'enseignement de la géométrie que s'affirme dans l'enseignement la conception empiriste des mathématiques.

LA QUESTION DE LA FUSION

Notons d'abord cette dissymétrie du vocabulaire scolaire français : on parle de *géométrie plane* et de *géométrie dans l'espace*, la première fait référence au plan, la seconde fait référence moins à l'espace en tant que tel qu'à ce qui a lieu dans l'espace ; si le plan est un objet géométrique que l'on peut définir, l'espace est le lieu de la géométrie, « il ne fait que fournir les lieux que les corps occupent et remplissent », comme l'explique Euler dans ses *Lettres à une Princesse d'Allemagne* [Euler 1768/1772]. Notons que l'anglais utilise les termes de *plane geometry* et de *solid geometry* [7]).

Il faut rappeler que, dans la géométrie grecque, si le plan est défini comme objet géométrique [8]), la notion d'espace n'y apparaît pas ; on y étudie des corps, essentiellement des corps solides, représentés par des figures.

La distinction entre la *planimétrie* (étude des figures planes) et la *stéréométrie* (étude des figures de l'espace) procède de multiples raisons. D'abord le rôle spécifique que joue le plan dans les formes d'expression par l'homme de son rapport au monde, comme le montrent aussi bien le dessin que l'écriture, ensuite la stéréométrie est une science difficile, comme l'explique déjà Platon au Livre VII [528 b–d] de *La République* [1966, 287], enfin, dans le cadre de la rationalité grecque, le développement logique exige que l'étude des corps solides vienne après celle des figures planes sur laquelle elle s'appuie, développement qui est celui des *Éléments* d'Euclide où la géométrie plane occupe la première partie (Livres 1–4 et 6) alors que la stéréométrie

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[6]) L'ouvrage de Dassen fait l'objet de deux comptes rendus, le premier après la publication du tome I in *L'Enseign. Math. 7* (1905), 244–246 ; le second après la publication du tome II in *L'Enseign. Math. 9* (1907), 74–75.

[7]) Cependant en allemand on dit *Geometrie der Ebene* (géométrie du plan) et *Geometrie des Raumes* (géométrie de l'espace) ; on dit aussi *ebene Geometrie* (géométrie plane) et *räumliche Geometrie* (géométrie spatiale). Je dois ces renseignements à Nicolas Rouche.

[8]) Pour une revue des diverses définitions de la notion de plan, nous renvoyons à l'ouvrage de Heath [1908, Book I] sur les *Éléments* d'Euclide.

occupe les trois derniers livres (Livres 11–13) ; ce développement implique un ordre des démonstrations, la démonstration d'une propriété de géométrie plane ne saurait ainsi s'appuyer sur des propriétés de stéréométrie. C'est cet ordre euclidien qui fonde la tradition géométrique occidentale (au sens large du terme, depuis le développement scientifique de l'époque arabo-islamique jusqu'à la révolution scientifique de l'Europe du XVII^{e} siècle). Il faut noter cependant que les principes énoncés au début du Livre I des *Éléments* (postulats, axiomes) ne concernent pas la seule géométrie plane [9]).

Nous ferons ici deux remarques. D'une part, la distinction euclidienne relève d'un ordre logique ; si les textes problématiques, tels ceux d'Archimède, prennent quelques libertés avec cet ordre, les textes d'exposition se doivent de le respecter. D'autre part, cette distinction ne se retrouve pas dans d'autres aires culturelles, comme le montrent par exemple les mathématiques chinoises qui développent directement ce qu'on pourrait appeler une *méthode des volumes* que l'on peut comparer à la *méthode des aires* des mathématiques grecques [Martzloff 1990].

Cette distinction sera remise en cause par le développement de la géométrie projective qui mettra en valeur ce que Chasles a appelé « l'alliance intime et systématique entre les figures à trois dimensions et les figures planes » [Chasles 1837 (1989), 191]. Ce point de vue conduira à utiliser l'espace pour étudier des propriétés planes.

En 1826 Gergonne, étudiant la dualité des propriétés de situation et remarquant comment celles-ci peuvent être déduites indépendamment du calcul et de la théorie des proportions à condition « pour cela de passer tour à tour de la géométrie plane à celle de l'espace et de celle-ci à la première » [Gergonne 1826], proposait déjà de revenir sur la classique division géométrie plane – géométrie dans l'espace, y compris dans l'enseignement.

A côté de ces raisons d'ordre théorique Méray, qui cite le texte de Gergonne dans la préface de l'édition de 1874 de ses *Nouveaux Éléments de Géométrie*, avance une raison d'ordre pratique, précisant que la distinction entre la géométrie plane et la géométrie dans l'espace

> est encore plus nuisible dans l'enseignement professionnel, car la pratique des Arts réclame bien plus la connaissance approfondie des principales combinaisons de droites et de plans, que celle de propositions théoriques comme les propriétés des sécantes du cercle [10]). [Méray 1874, xi–xii]

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[9]) Notons que la démonstration du premier cas d'égalité des triangles (Livre I, proposition 4) n'implique pas que les triangles soient dans un même plan. De façon générale les cas d'égalité des triangles ne sont pas des théorèmes de géométrie plane (cf. [Bkouche 2000]).

[10]) Après sa publication en 1874, l'ouvrage de Méray conduira à quelques expériences d'ensei-

On peut rapprocher ce dernier argument, qui sera repris par les partisans de la réforme, de ceux développés par Monge dans la préface de sa *Géométrie descriptive* [Monge 1798], véritable plaidoyer en faveur de cette technique nouvelle que représente, pour son inventeur, la géométrie descriptive.

Le développement de la géométrie projective conduira à mettre en place ce que l'on a appelé la *fusion* entre la planimétrie et la stéréométrie ; des ouvrages seront publiés en France et en Allemagne au milieu du XIX^e siècle sans grand succès.

L'ouvrage de Méray sera « redécouvert » par Laisant qui en exposera les principes dans *L'Enseignement Mathématique* en 1901 ; l'article de Laisant [1901] sera suivi par d'autres articles présentant les conceptions de Méray ; voir [Perrin 1903] ainsi que les articles de Méray lui-même [11]).

C'est en Italie que l'idée de la fusion se développera avec le plus de succès à la fin du XIX^e siècle. En 1873, Luigi Cremona, soucieux de développer un enseignement élémentaire de géométrie projective publiera, à l'usage des élèves des Instituts Techniques, des *Elementi di Geometria Projettiva* [Cremona 1873]. Dans la préface, Cremona insiste sur le caractère technique de l'ouvrage qui doit « conduire rapidement les élèves à appliquer les connaissances théoriques au dessin » [1873 (1875), IX] ; quant à la fusion, Cremona explique :

> Dès le commencement j'alterne sans distinction les théorèmes de géométrie plane avec ceux de la géométrie de l'espace, parce que l'expérience m'a enseigné, et d'autres l'ont remarqué avant moi, que les considérations de l'espace suggèrent bien souvent le moyen de rendre facile et intuitif ce qui serait compliqué et difficile à démontrer par la géométrie plane. [Cremona 1873 (1875), X]

Ainsi se développera en Italie un courant fusionniste, deux ouvrages seront publiés qui contribueront à populariser l'idée de la fusion entre les deux géométries, les *Elementi di geometria* de De Paolis en 1884, puis les *Elementi di geometria* de Lazzari et Bassani en 1891, correspondant à un enseignement effectivement donné à la Royale Académie de Livourne ; les idées fusionnistes, telles qu'elles se sont développées en Italie, font l'objet de deux articles de la revue, l'un de Candido [1899], l'autre de Loria [1905].

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gnement dans l'Académie de Dijon (Méray était professeur à la Faculté des Sciences de Dijon) non sans quelque réussite, mais ces expériences s'arrêteront devant l'indifférence, sinon l'hostilité, de l'Administration (!) et la méfiance de certains enseignants, comme l'explique Méray lui-même dans la préface de la seconde édition [Méray 1903] de son ouvrage (Jobard, Dijon).

[11]) [Méray 1901]; cet article est une partie d'un article plus ancien publié dans la *Revue bourguignonne de l'enseignement supérieur*, 1892. Lors de la seconde édition de ses *Nouveaux éléments de géométrie*, Charles Méray publiera un article de présentation de l'ouvrage [Méray 1904].

Ce rapprochement, dans l'enseignement, de la géométrie plane et de la géométrie dans l'espace ne sera pas sans poser problème ; ainsi Hadamard, dans la préface de la seconde édition de ses *Leçons de géométrie élémentaire*, publiée à l'époque de la réforme, refusera de « fondre la géométrie plane et la géométrie dans l'espace », et il expliquera :

> Que cette fusion soit préférable au point de vue logique, je le veux bien. Mais il me paraît que, pédagogiquement, nous devons penser tout d'abord à diviser les difficultés. Celle de « voir dans l'espace » en est une sérieuse par elle-même, que je ne considère pas comme devant être ajoutée tout d'abord aux autres. [Hadamard 1906]

La question du mouvement

Ici encore les réformateurs du début du XX^e siècle vont remettre en cause une tradition bien établie depuis la géométrie grecque.

Les raisons de l'élimination du mouvement sont à la fois d'ordre métaphysique et d'ordre scientifique ; mais peut-on distinguer science et métaphysique lorsque l'on parle de la pensée grecque ? Les paradoxes de Zénon ont montré les difficultés de parler du mouvement et l'on peut penser que devant ces difficultés, mieux valait circonscrire le domaine de la connaissance rationnelle à ce qui peut être objet de discours. C'est l'une des raisons qui conduit Platon à proclamer : « εἰ μὲν οὐσίαν ἀναγκάζει θεάσασθαι, προσήκει, εἰ δὲ γένεσιν, οὐ προσήκει » [12]) [*La République*, Livre VII, 526e].

On peut alors considérer que la force de la construction euclidienne vient de ce qu'elle élimine tout recours au mouvement dans son discours alors que la physique aristotélicienne a cédé la place lorsque le temps est devenu une grandeur géométrique et d'une certaine façon statique. Dans cette nouvelle physique géométrisée le mouvement échappe aux paradoxes des Éléates pour devenir objet de science, c'est-à-dire objet d'un discours rationnel.

Pourtant le mouvement est présent dans la géométrie grecque, d'abord avec le principe de l'égalité par superposition (l'axiome 4 ou 8 suivant les éditions des *Éléments* d'Euclide), principe que nous énonçons dans la traduction de Hoüel [1867, 13] : « les grandeurs que l'on peut faire coïncider l'une avec l'autre sont égales entre elles », principe fondateur de la géométrie dans la mesure où c'est ce principe qui permet de comparer entre eux les objets

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[12]) Si [la géométrie] oblige à contempler l'essence, elle nous convient ; si elle s'arrête au devenir, elle ne nous convient pas. [Platon 1966, 285]

géométriques et donc de les mesurer. Mais cette intervention du mouvement est vite éliminée, et c'est le rôle du principe de l'égalité par superposition que de permettre cette élimination ; en effet, ce principe une fois énoncé, le problème est de trouver des critères d'égalité *a priori* qui permettent de se dispenser de la superposition effective, ce seront les classiques cas d'égalité des triangles, lesquels sont, en ce sens, fondateurs de la rationalité géométrique euclidienne [Bkouche 1991b ; 2000]. Dans le cadre de cette rationalité, et conformément au dogme platonicien rappelé ci-dessus, l'appel à des considérations de mouvement est interdit *en droit* dans le raisonnement géomérique.

C'est la géométrisation du temps qui va conduire à la possibilité d'une étude scientifique du mouvement et en retour à son utilisation pour résoudre certains problèmes de géométrie, ainsi le problème des tangentes [13]). Cette géométrisation du mouvement conduira à définir au XIXe siècle une *géométrie du mouvement* ou *géométrie cinématique*, géométrie qu'Amédée Mannheim, dans son cours à l'École polytechnique, définit ainsi :

> La Cinématique a pour objet l'étude du mouvement indépendamment des forces ; la Géométrie cinématique a pour objet l'étude du mouvement indépendamment des forces et du temps, c'est-à-dire qu'elle a pour objet l'étude des déplacements. Nous réservons l'expression de déplacement pour un mouvement dans lequel on ne considère pas la vitesse. [Mannheim 1880, treizième leçon]

Ce n'est pas ici le lieu de développer l'étude de cette histoire mais nous citerons Jules Hoüel qui, ouvrant ainsi la voie à un renouvellement de l'enseignement de la géométrie, écrivait en 1867, dans son *Essai critique sur les principes fondamentaux de la géométrie élémentaire* :

> C'est par suite d'une confusion d'idées que plusieurs géomètres veulent bannir des éléments de géométrie la considération du mouvement.
>
> L'idée du mouvement, abstraction faite du temps employé à l'accomplir, c'est-à-dire l'idée du mouvement géométrique, n'est pas une idée plus complexe que celle de grandeur ou d'étendue. On peut même dire, en toute rigueur, que cette idée est identique avec celle de grandeur, puisque c'est précisément par le mouvement que nous parvenons à l'idée de grandeur.
>
> Ce mouvement géométrique, qu'une équivoque de langage a fait confondre avec le mouvement dans le temps, objet de la cinématique, ne peut pas dépendre d'une autre science que la géométrie pure.
>
> Il est avantageux d'introduire cette idée de mouvement géométrique le plus tôt et le plus explicitement possible. On y gagne beaucoup sous le rapport de la clarté et de la précision du langage, et l'on se trouve mieux préparé à introduire plus tard dans le mouvement les notions de temps et de vitesse.

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13) On pourrait citer les travaux de Roberval en 1636 sur la détermination des tangentes, travaux publiés en 1693 [Roberval 1693].

> C'est d'ailleurs ce que tous les auteurs font à leur insu; et il serait difficile de trouver une seule démonstration d'une proposition fondamentale de la géométrie, dans laquelle n'entre pas l'idée de mouvement géométrique, plus ou moins déguisée. [Hoüel 1867, Note II]

C'est ce courant d'idées qui permettra à Charles Méray d'utiliser le mouvement dans la construction de la géométrie élémentaire *via* les translations et les rotations; aux translations sont alors associées les notions de droite et de parallélisme, aux rotations sont associées la notion de perpendiculaire ainsi que les notions d'angle et de cercle.

Cette transgression de la tradition euclidienne sera diversement appréciée et on reprochera aux partisans de la réforme un manquement à la rigueur dans le raisonnement. Il est vrai que la construction de Méray est assez compliquée et, par cela-même, loin d'être à l'abri des critiques; sa recherche d'un enseignement plus intuitif et le souci conjoint de préserver une certaine rigueur de l'exposé explique cette complication. Mais derrière la complication du texte de Méray on voit apparaître les idées dont je viens de parler et surtout, mais c'est peut-être le point essentiel de la polémique, l'aspect expérimental de la géométrie. D'autres ouvrages paraîtront au moment de la réforme qui développeront ce point de vue tels ceux de Carlo Bourlet [1906/1908; 1912] ou celui d'Émile Borel [1905].

Il faudrait citer aussi la *géométrie de l'ajustage* développée par Jules Andrade, professeur à l'Université de Besançon, où il a proposé en 1905 un cours de chronométrie. Comme il l'explique [Andrade 1905], il destinait un tel enseignement aux fils des patrons d'horlogeries; en fait il rencontre un public d'ouvriers, ce qui l'amène à penser un cours de géométrie permettant d'accéder aux principales connaissances sans avoir suivi le cursus classique. C'est l'occasion pour lui de penser l'enseignement de la géométrie à travers les pratiques de mouvements utilisées en atelier. Il construit ainsi son cours en termes de translations et de rotations, moins comme définitions formelles mais comme descriptions de mouvements. On peut alors introduire les parallèles *via* le mouvement de translation; par ailleurs l'étude du mouvement d'un solide ayant deux points fixes permet une définition empirique de la droite comme l'ensemble des points fixes dans le mouvement d'un solide ayant deux points fixes[14]). Cela le conduit à construire une géométrie naturelle [Andrade 1908], dont il explique qu'elle est fondée cinématiquement.

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[14]) Notons que cette définition est donnée par Leibniz dans sa *Caractéristique géométrique* [Leibniz 1995, 161]. Cette remarque pose la question de la relation complexe entre le rationalisme leibnizien et la connaissance empirique.

Géométrie et dessin

Les réformateurs ont mis l'accent sur l'enseignement du dessin géométrique ; un tel enseignement a un double rôle, d'une part son importance dans l'apprentissage de la géométrie, les figures apparaissant — du moins dans la première étude de la géométrie — comme les objets de la géométrie, d'autre part sa place dans l'enseignement professionnel où la géométrie descriptive joue un rôle important comme mode de représentation des objets de l'espace. On peut noter ici le nombre important de traités de géométrie descriptive publiés dans plusieurs pays européens et recensés par la revue.

C'est en partie pour assurer la liaison entre le dessin géométrique et l'enseignement de la géométrie qu'Émile Borel proposera en 1904 la mise en place de laboratoires de mathématiques, point sur lequel nous reviendrons ci-dessous.

Une conception empiriste de la géométrie

La réforme du début du XX^{e} siècle est marquée par l'empirisme ; les mathématiques sont le moyen de décrire la nature, mais si elles ont cette propriété c'est qu'elles sont issues de connaissances naturelles comme le rappelle Bourlet [1910]. Quant à la géométrie, en tant qu'elle est science des corps solides [Poincaré 1902 (1968), 86], elle participe autant des sciences de la nature que des sciences mathématiques. On met ainsi l'accent sur la part de physique qui sous-tend la géométrie.

Nous citerons ici un article publié dans le premier volume de la revue dans la mesure où il apparaît comme un manifeste de cette conception empiriste de la connaissance, savoir, l'article de Laurent [1899], article que l'on peut situer à la fois dans la tradition des *Lumières* (de d'Alembert [1759] et Condillac [1746] à Lacroix [1804]) et dans la tradition du positivisme comtien.

« L'origine de nos connaissances se trouve dans nos sens », écrit ainsi l'auteur [Laurent 1899, 383] expliquant les trois modes de connaissance : l'observation, l'expérience (nous dirions aujourd'hui l'expérimentation), le raisonnement, et il précise : « … toute science passe par trois phases successives : 1^{o} la phase d'observation, 2^{o} la phase de raisonnement, 3^{o} la phase expérimentale » [Laurent 1899, 385].

On peut remarquer l'ordre des phases : la phase expérimentale y apparaît moins comme une phase d'élaboration de la connaissance que comme une phase de vérification [15]), la seconde place, après la phase empirique définie par l'observation, étant celle du raisonnement.

La place accordée au raisonnement montre la place de l'abstraction dans l'activité scientifique, abstraction que l'auteur définit comme « une convention que l'on fait de négliger toutes les propriétés d'un objet, pour ne tenir compte que d'une seule d'entre elles » [Laurent 1899, 393]. On peut comparer cette conception avec celle de d'Alembert qui écrit dans son *Essai sur les éléments de philosophie* :

> L'abstraction en effet n'est autre chose que l'opération par laquelle nous considérons dans un objet une propriété particulière, sans faire attention aux autres. [d'Alembert 1759 (1986), 29]

Laurent propose alors une classification des sciences qui s'appuie sur la plus ou moins grande part de la connaissance sensible qui intervient dans l'élaboration de cette science ; il place en tête la Mathématique, qu'il réduit à la science des nombres (arithmétique et algèbre) en ce que le rôle des sens y est presque nul, même s'il considère que la notion de nombre est issue des activités liées au comptage.

La Géométrie est alors considérée par Laurent comme une science physique « parce qu'elle emprunte au témoignage des sens la notion d'*espace* et la notion de *déplacement* » [Laurent 1899, 398], Laurent distinguant alors les notions de *déplacement* et de *mouvement*, cette dernière notion faisant appel à la notion de *temps*. Mais si la géométrie commence comme une science d'observation, c'est le raisonnement qui permet son développement en apportant les moyens de découvrir de nouvelles propriétés des figures. De fait l'entrée de la géométrie dans la mathématique pure est liée à la géométrie analytique considérée comme une reconstruction numérique de la géométrie ; c'est une telle reconstruction qui permet de la situer dans une science plus générale, la *Pangéométrie* [16]), généralisée elle-même avec l'invention des géométries multidimensionnelles. C'est une fois cette construction rendue possible que la géométrie devient expérimentale.

L'ordre des phases plaçant le raisonnement avant l'activité est un point important de l'article de Laurent qui d'une part permet d'éviter un réalisme naïf et d'autre part permet de distinguer l'observation empirique et l'expérimentation, cette dernière supposant une activité préalable de raisonnement.

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[15]) Cela remet en place certains jugements hâtifs sur l'empirisme et montre le rôle du raisonnement dans la conception empiriste de la connaissance.

[16]) Rappelons que *pangéométrie* est le nom que Lobatchevski donne à sa géométrie générale qui contient à la fois la géométrie euclidienne et la géométrie non-euclidienne ; cf. [Lobatchevski 1855] : ce texte a été publié simultanément en russe et en français en 1855.

Le raisonnement, quant à lui, s'appuie sur l'observation comme l'expliquera Laisant dans un article ultérieur : « Il est certain que les raisonnements ne peuvent tenir lieu de la connaissance des faits. » [Laisant 1902, 8]

C'est encore l'empirisme qui guide Méray lorsqu'il explique dans son article sur l'enseignement des mathématiques : « Entre l'exactitude d'une hypothèse physique et l'évidence d'un axiome mathématique, il n'y a donc aucune différence essentielle. » [Méray 1901, 178–179]

Il précise dans ce même article comment la géométrie s'inscrit, moins dans les mathématiques pures que dans les mathématiques appliquées. Après avoir présenté les deux grands chapitres des mathématiques, l'*Analyse mathématique*, science générale des nombres, et la *Géométrie*, qui étudie la forme et l'étendue des corps ainsi que leurs positions relatives, il écrit :

> L'usage a réuni ces deux sciences sous la dénomination générique de *Mathématiques pures*, et a groupé dans les *Mathématiques appliquées*, toutes celles résultant effectivement de l'application de l'Analyse et de la Géométrie au développement d'un très petit nombre de notions spéciales. Il semblerait plus naturel de mettre à part l'Analyse, qui prête sans cesse ses principes mais n'en emprunte aucun ailleurs, de réserver le nom de Mathématiques pures à ses diverses branches, puis de placer à sa suite, dans les Mathématiques appliquées, toutes les sciences trouvant dans ses formules un appui essentiel et continu. En les rangeant dans l'ordre où chacune est nécessaire aux suivantes mais non aux précédentes, on y rencontrerait la Géométrie, la Mécanique, la Physique mathématique, … . [Méray 1901, 173]

Plus tard, dans la préface de la seconde édition de ses *Nouveaux éléments de géométrie* il parlera de « *faits géométriques* » et de « *la vision des faits de l'espace* » [Méray 1903, vi–vii].

Mais la géométrie est marquée par le fait d'avoir été la première science rationalisée, c'est-à-dire la première à répondre aux exigences des *Seconds Analytiques* d'Aristote. De ce fait la part de connaissance empirique de la géométrie a été masquée par le discours rationnel qui la soutenait. La mécanique ou l'électromagnétisme ont été rationalisés beaucoup plus tard et leur caractère empirique reste présent.

Borel est conscient de cela lorsqu'il propose des travaux pratiques de géométrie dans l'enseignement, lors de la conférence de 1904 dont nous avons déjà parlé [Borel 1904], conférence qui résume ses conceptions pédagogiques.

L'aspect expérimental de la géométrie apparaît aussi avec les descriptions et l'usage d'instruments géométriques. Nous pourrions citer en France l'ouvrage de Fourrey [1907], qui consacre un chapitre aux instruments géométriques. Nous pourrions aussi signaler une série d'articles publiés dans *L'Enseignement Mathématique* sur la stéréoscopie.

Cela nous renvoie à une lecture empiriste des *Éléments* d'Euclide que l'on ne peut développer ici. En ce sens les propositions de réforme de l'enseignement de la géométrie restent proches du point de vue euclidien même si elles en renouvellent le style en mettant l'accent sur l'origine empirique des objets de la géométrie.

QUELQUES REMARQUES EN CONCLUSION

Nous voudrions terminer cet article en parlant de l'actualité de certains articles autour de la réforme du début du XX^{e} siècle.

Outre le fait que certains problèmes sont toujours actuels, en particulier celui de décider qui enseigne les mathématiques dans l'enseignement professionnel, la question des rapports des mathématiques avec les autres disciplines reste permanente.

Il faudrait alors revenir sur l'aspect expérimental qu'il est nécessaire de replacer, comme le fait l'article cité de Laurent, dans un cadre gobal. En particulier la part d'empirisme de la géométrie ne saurait être détachée de la part rationnelle. C'est en ce sens qu'il faut comprendre la proposition de laboratoires de mathématiques proposée par Émile Borel. Mais cela demande aussi de retrouver la part d'empirisme de la géométrie grecque.

C'est la prise en compte de cette part d'empirisme qui fait défaut dans l'enseignement d'aujourd'hui et, en ce qui concerne la géométrie, l'oubli qu'elle est aussi une science physique. Peut-être parce que — même si l'on prend en compte l'apport des mathématiques aux sciences de la nature, pour ne parler que d'elles — on a oublié que cet apport est possible uniquement parce que les mathématiques (même si elles s'en sont détournées pour mener leur vie propre) se sont construites sur des connaissances empiriques.

Le travail cité de Jules Andrade construisant ce qu'il appelle une « géométrie naturelle » n'est pas seulement un artefact pédagogique utilisé parce qu'il s'adressait à des ouvriers ne possédant pas la culture géométrique classique ; en introduisant explicitement le mouvement dans son cours de géométrie il effectuait un retour aux origines et c'est *via* ce retour aux origines qu'il pouvait espérer conduire à une meilleure compréhension de la géométrie élémentaire.

En ce sens, la lecture des textes des premiers numéros de la revue peut être un salutaire rappel pour l'enseignement d'aujourd'hui.

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GEOMETRY : 1950–70

La géométrie de 1950 à 1970

par Geoffrey HOWSON

Les deux décennies de 1950 à 1970 ont une importance particulière, non seulement parce que l'enseignement de la géométrie dans les écoles y faisait l'objet de passablement de discussions (ce qui n'a rien d'exceptionnel, puisque ce débat est toujours d'actualité), mais surtout en raison du foisonnement d'activités qui eurent lieu effectivement dans les classes. De plus, les expérimentations varièrent énormément dans leurs objectifs, les conditions de leur mise en œuvre et leurs résultats. Et il n'y avait pas de consensus sur les directions à suivre. Or, peu de traces subsistent de cette période de « fertile instabilité », et celles qui restent ne permettent pas de bien comprendre ce que les auteurs de ces innovations se proposaient d'obtenir.

Nous aborderons brièvement les questions suivantes : Qu'est-ce qui provoqua la révolution dans l'enseignement de la géométrie ? Que cherchait-on à faire ? Qu'a-t-on obtenu ? Qu'a-t-on appris ? (Ou plutôt : *Quelles leçons aurait-on pu tirer, même si cela n'a souvent pas été fait ?*)

Nous remarquons qu'au début de cette période il n'y avait aucun point de vue universellement partagé sur les buts de l'enseignement de la géométrie. Plusieurs traditions différentes et l'influence exercée par les mathématiciens des universités (qui avaient aussi une opinion sur les mathématiques qu'il fallait enseigner dans les écoles) ont toutes joué des rôles considérables pour définir la nature des réformes accomplies dans les différents pays.

En particulier, on remarque des clivages entre des présentations strictement axiomatiques et d'autres plus pragmatiques ; ou entre celles qui se raccrochaient encore à une approche "euclidienne" (mais avec de nouveaux systèmes d'axiomes) et celles qui plaçaient l'accent sur les transformations linéaires et les vecteurs en faisant de gros efforts pour réunifier l'enseignement de la géométrie avec celui des structures algébriques.

D’une manière générale, l’accent était mis presque entièrement sur la géométrie qui devait être apprise aux étudiants des établissements d’élite : gymnases, collèges, lycées, …. Cela créa des problèmes considérables lorsque, à la fin de ces décennies, les systèmes scolaires devinrent moins sélectifs, avec pour but les “mathématiques pour tous”. Mais même à l’intérieur de ces secteurs d’élite, avec des enseignants mathématiquement bien entraînés, surgirent de sérieux problèmes de formation continue.

Non seulement les enseignants devaient apprendre les mathématiques nouvellement requises, mais il devenait aussi essentiel qu’ils sachent apprécier les innovations dans un cadre mathématique plus large, afin de saisir les buts visés par les innovateurs pour se convaincre qu’ils étaient à la fois désirables et accessibles (du moins lorsque cette affirmation était justifiée !). C’est donc par rapport aux leçons à tirer dans l’art du *développement curriculaire* que l’étude de ces décennies apportera ses plus grandes satisfactions. Cependant, l’esprit d’invention, l’enthousiasme et l’ambition mathématique (quoique excessive) des innovateurs devraient servir d’aiguillon à ceux qui sont mécontents de l’enseignement mathématique qui est dispensé dans les écoles aujourd’hui.

GEOMETRY : 1950–70

by Geoffrey HOWSON

INTRODUCTION

The first thoughts of anyone studying the history of the teaching of geometry in the years 1950–70 must be of the degree of confusion concerning aims, the way in which these were implemented and also the wide range of outcomes. It was, indeed, a period of 'fertile instability' [1]). Everything seemed to be on the move — but there was no consensus concerning the direction in which moves should be made. For that reason, it would be impossible in the space allowed to discuss all those initiatives of which I am aware — and my knowledge is, of course, limited. What I shall attempt to do is to look briefly at four key questions :

- What caused the revolution in geometry teaching ?
- What was attempted ?
- What was achieved ?
- What was learned ? (Or, perhaps, "What lessons might have been learned, but often were not ?")

[1]) Freudenthal, in a very important paper [1963], pointed out that although ICMI had chosen the teaching of geometry as one of its themes for study in the years 1954–58, this aroused little interest (a report of the outcomes is to be found in *L'Enseign. Math. (2) 5* (1959)), yet once ICMI had chosen algebra and arithmetic as subjects for study in 1958–62, everyone suddenly became preoccupied with geometry. Freudenthal's paper — along with those referred to by Artin, Stone, and Lombardo-Radice [1963] — stemmed from an ICMI seminar held in 1961 in Bologna on the theme "teaching geometry at the secondary level". Other papers from that meeting can be found in *L'Enseign. Math. (2) 9* (1963). Freudenthal's own attempts to develop school geometry courses at IOWO and what is now the Freudenthal Institute in Utrecht came after 1970.

THE BACKGROUND

Many of the reasons for the reform movement in geometry were shared with other topics. In particular, these included a wish to 'update' the teaching of mathematics, and also to rewrite school curricula and textbooks so that they would not only reflect 'twentieth century' mathematics and the way in which it was used, but also ensure that there was not a gulf between school and university mathematics [2]).

However, the position of geometry was, in most countries, different from that of other subjects: the aims for geometry teaching were far from universally shared. Here, for example, it differed from algebra. No matter how much algebra was actually included in the school curriculum, there was considerable agreement (up to the late 1950s) on what school algebra might mean. After the introduction of letters to denote numbers or variables, came the construction of algebraic formulae, followed by the formation and solution of linear equations, then quadratics, then simultaneous linear, the properties of the roots of quadratic equations, of cubics, and so on.

However, the aftermath of the early 20th century school reforms had left countries following very different patterns in their geometry teaching. Moreover, the position was further complicated by the existence of various types of schools/educational institutions and the fact that these taught, and valued, different forms of geometry. The academic schools still retained a 'Euclidean' tradition, although Euclid itself had been largely swept aside. The sequence in which theorems were taught and the degree of axiomatic rigour varied from country to country, but the actual theorems — Pythagoras, what is termed either Thales' Theorem or the *Strahlensatz*, the circle theorems, and congruence and similarity properties — were still to be found in most curricula. Where there were differences, this mainly concerned the position of coordinate geometry, the attention given to 3-D work and the use made of analytical methods, particularly when, in the Senior High School, conics were introduced.

There was, though, one very significant exception. Following World War II, and during the period of US occupation, schooling in Japan had been reorganised on US comprehensive lines. Moreover, the mathematics curriculum had been recast so that it sought to emphasise the uses of mathematics

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[2]) The relationship between school and university mathematics was explored at a symposium held in Geneva in honour of Henri Fehr (see, e.g., [Behnke 1957; Freudenthal 1956; Maxwell 1956]). Concern regarding the lack of accuracy, honesty and clarity in school texts was also expressed (see, e.g., [Cockcroft 1962]) at a seminar organised by ICMI in Lausanne in 1961.

in solving social problems. What was laid down for Japanese schools (see [Nagasaki 1990]) was not an academic curriculum, but one which, in Europe, would have been associated more with 'technical' schools or *Realschulen*. That is, the mensuration of plane and solid figures, their simple properties, the ability to read and interpret maps, leading to consideration of congruence and similarity, an introduction to coordinates and the use of projection (what elsewhere might have been seen as the beginnings of 'technical drawing' or 'descriptive geometry').

"Academic, Euclidean-style" geometry formed an optional unit in the Japanese Upper Secondary School, but students "disliked" it, "[few] pupils learned geometry" and "this caused difficulties in the teaching of mathematics in the colleges and universities" [Wada *et al.* 1956, 167].

The Japanese experience raises interesting questions :

- What form should geometry take in an "education for all" ? How are the competing claims of 'academic' and 'artisan' geometry (i.e. surveying, making and reading plans and maps, technical drawing (perspective, elevations and projections)) to be reconciled ?
- Can one overcome the fact that "Euclid-style" geometry is found extremely difficult (and often uninteresting) by most students [3]) ?
- Can one ignore the benefits of an academic geometry course for those students capable of following one ? And what exactly are those benefits ?

Such questions were given little consideration at the famous Royaumont conference of 1959 [OEEC 1961]. There, attention appeared to be almost entirely directed at students in selective schools and often it appeared to be assumed that the object of secondary school mathematics was solely to prepare students for its study at university. Indeed, it might be argued that, more than this, the hidden agenda was to determine which students were capable of such study.

What is remembered most from that meeting is, of course, Dieudonné's call that "Euclid must go" [Dieudonné 1961]. The call, however, was far from novel and as we have seen there were countries in which Euclid had disappeared. (However, it should be noted that in Japan Euclid-style geometry re-entered the Lower Secondary Schools in 1958 and has remained there ever

[3]) Numerous quotes could be given to support this statement. A typical one is that by Tammadge [1987, 22] reporting on his experiences as an examiner of the top 20% or so of English students : "Euclidean geometry had been watered down to the absurd. Only a small percentage of candidates attempted questions on this topic and they normally regurgitated the theorem and collapsed when it came to the rider [i.e. a request to prove a corollary to the theorem]."

since. As two Japanese educators wrote, a curriculum "based on the needs of society was changed to one based on the needs of mathematics" [Kunimune & Nagasaki 1996].) Indeed, writing in 1911, D. E. Smith argued that

> The efforts usually made to improve the spirit of Euclid are trivial […] but there is a possibility […] that a geometry will be developed that will be as serious as Euclid's and as effective in the education of the thinking individual. If so, it seems probable that it will not be based upon the congruence of triangles, but upon certain postulates of motion […]. It will be through some effort as this, rather than through the weakening of the Euclid–Legendre style of geometry, that any improvement is likely to come. [Smith 1911]

Significantly, Dieudonné himself recommended that up to the age of 14 the teaching of geometry should be "experimental" and "part of physics, so to speak". But the emphasis should not be on "such artificial playthings as triangles" but "on basic notions such as symmetries, translations, composition of transformations, etc.". An argument that in retrospect does not look terribly revolutionary. However, from age 15 he proposed the introduction of the axiomatic method :

> The axioms should be developed from the algebraic and geometric point of view, i.e. any notion should be given with both kinds of interpretation. […] the emphasis should be on the linear transformations, their various types and the groups they form. Matrices and determinants of order 2 appear […] in a natural way in this development. [Dieudonné 1961]

Important lessons were learned by those who attempted to follow this path, for neither Dieudonné nor anyone else had experience of how to communicate 'axiomatics' to adolescents and their teachers.

WHAT HAPPENED ?

The call to reform geometry teaching was one that found a ready response. For a variety of reasons there was dissatisfaction with what was currently being offered. We shall here try to group the responses under a variety of headings. Again, it must be emphasised that all the initiatives were designed initially for students in the top 25% or so of the ability range.

APPROACHES IN THE SPIRIT OF DIEUDONNÉ — OR IN WHAT WAS THOUGHT TO BE THE SPIRIT OF DIEUDONNÉ

Here Papy's work must be mentioned. He believed that "the most fundamental and central topic of the secondary school programme is, without doubt, vector spaces" and designed an approach which cleverly combined the development of the real number system and the affine plane [4]). Many alternative approaches to Papy's, often written with different age groups of readers in mind, were provided by, for example, Choquet (in an outstanding book *L'Enseignement de la Géométrie* [5]), which was greatly to influence [6]) the school mathematics course in France), Glaymann [1969], and Levi [1971].

The emphasis on linear algebra [7]) drew strong criticism from Freudenthal who argued that the geometry to which it lends itself is restricted and by no means the type to interest students. Heaven help the child brought up on Dieudonné's approach who seeks to prove that the plane can be tessellated with regular hexagons but not with regular pentagons ! (See also [Freudenthal 1963] which criticises some of the schemes proposed for teaching geometry in the light of the work of the van Hieles, which at that time was little known

[4]) Papy's books (*Mathématique moderne*, Didier, 1963 on) aroused much interest and some appeared also, for example, in German, Italian and English translations. They were distinguished by a most imaginative use of colour and diagrams. A rationale and synopsis of their contents can be found in [Papy 1966a ; 1966b], and a summary of Papy's approach to the number systems and geometry in [Griffiths & Howson 1974, 282–286]. In these it is described how, for example, 12–13 year-olds reached the definition of a vector via the path : parallels, equipollency [(A, B) and (C, D) are *equipollent* if and only if AB is parallel to CD and AC to BD], equivalence classes, translation, vector.

This method was adopted elsewhere and, for example, Leonid Brezhnev, not usually remembered for his contributions to mathematics education, in 1981 told the Central Committee of the Soviet Communist Party that, in future, there would be no more talk of equivalence classes : a vector would revert to being something having magnitude and direction. ([Keitel 1982] gives references to a report of the speech in *Pravda*, but no details of its contents other than that they spelled the end of the Kolmogorov reforms. I am relying on my memory of reading a translation of the speech when visiting Moscow in 1982.)

It should be noted that rather than beginning with a list of axioms, Papy adopted what he described as the "progressive axiomatic approach", namely : "State clearly that which is accepted : do not say everything at one time ; state certain accepted things, little by little." This method has some pedagogical benefits, and at first glance it does tend to show how an axiom system evolves. This latter benefit is largely illusory, however, since only the author is aware of why he has selected a particular statement as an axiom (and the need for new axioms is not long in coming — e.g. in proving that equipollency is transitive). SMSG and SSMCIS (see below) both made all their axioms explicit from the start — and these were largely "statements beyond dispute". Choquet's axioms were very much a result of mathematical hindsight and at the time were called "parachute axioms", i.e. they attack problems from the rear and were not framed because they were natural assumptions.

[5]) The title of the English translation, *Geometry in a Modern Setting*, gives a better indication of the actual nature of the work.

[6]) It should be noted that many articles were written, often by distinguished mathematicians on what and how geometry should be taught in schools. *L'Enseignement Mathématique* and *Educational Studies in Mathematics* provide numerous examples of these. The vast majority, however, were never translated into classroom terms and use. Freudenthal [1963] compared this to producing theorems without proofs.

[7]) Although the 'natural' link was between the 'new' geometry and algebra, there were also attempts to link a new approach to geometry teaching with an introduction to analysis (see, e.g. [Delessert 1962]).

outside the Netherlands.) Other criticisms were to come from Artin [1963] who, in a very cogent article, reviewed what he referred to as three "extreme" views on the teaching of geometry [8]). As he pointed out, the approach using linear algebra must, at some time, call on some subtle reasoning to deal with the introduction of the notion of angle [9]).

Strong criticisms of the new French curriculum [10]) for the "junior high school years" (see [Adda 1981]), and of the influences of Choquet and Dieudonné, were also to come from, amongst others, Thom [1970; 1973], who argued for the retention of Euclid on a variety of grounds [11]), and Leray [1966] [12]). Perhaps, here, one should include an example to illustrate just how far the French reforms went (beyond anything envisaged by Dieudonné) and what was inflicted upon both students and teachers. The following definition and theorem [13]) were intended for the *classe de quatrième*, i.e. for 14 year-olds.

Étant donnée une droite graduée (Δ, g).

1° *Pour tout couple de réels* (a', b') *tel que* $a' \neq 0$, *l'application* g' *de* Δ *sur* **R** *définie pour tout élément* M *de* Δ *par :*

$$g'(M) = a' \,.\, g(M) + b'$$

est bijective.

2° *La famille de toutes les bijections ainsi définies possède la propriété :*

Pour deux bijections quelconques g' *et* g'' *de cette famille, il existe un couple* (a, b) *de nombres réels, tel que* $a \neq 0$, *et pour tout élément* M *de* Δ *:*

$$g''(M) = a \,.\, g'(M) + b\,.$$

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8) Essentially, teach Euclid and pretend it is rigorous, teach Hilbert (or a later geometrical axiom system), or use a basically algebraic axiom system.

9) A further, very interesting paper to appear at this time was by Marshall Stone [1963]. In it, Stone, takes as axiomatic that students between the ages of 15 or 16 and 18 should meet an axiomatic treatment of the Euclidean plane and space. He then goes on to lay down certain didactical rules that should characterise such a presentation. *Inter alia*, he refers to the work of Birkhoff and Beatley, Choquet, and the SMSG which are mentioned briefly in this paper. Stone was, I believe, very much concerned as a consultant to the SSMCIS Project, yet (see below) that project did not develop an axiomatic presentation of the Euclidean plane but chose to demonstrate only how an axiom system worked. By this stage, the reader of the present paper should have noticed the unprecedented (and subsequently unmatched) interest that leading mathematicians were at that time displaying in school curricula.

10) Earlier reforms of the curriculum for the higher classes in France had proved less contentious. These introduced the various types of linear transformation and demonstrated their group properties.

11) Thom argued that although algebra might be stronger in "syntax", Euclid-style geometry was stronger in "meaning".

12) A recent critical appraisal of the French reforms can be found in [CREM 2000].

13) Reprinted in "Les « maths modernes » à l'école" in "Spécial Bourbaki", *Pour la Science* (février 2000), p. 83.

> *On appelle alors* graduation *de Δ toute bijection de cette famille, et le nombre $g'(M)$ est appelé* abscisse *de M dans la graduation g'.*[14] [15]

No wonder that Leray should conclude a report to the French Academy of Sciences in 1972 by writing[16]:

> L'option ensembliste de la définition de la géométrie est une dangereuse utopie. […] Les termes scientifiques que nous avons dû employer pour l'analyser montrent combien cette réforme méconnaît les aptitudes et besoins intellectuels des adolescents qui sont élèves des […] lycées. La réforme en cours met gravement en danger l'avenir économique, technique et scientifique du Pays.[17]
> (quoted in [Adda 1981])

In other countries, vector geometry was also stressed (e.g. by Hope[18]) in England) but here the emphasis was not so much on teaching the concept of a linear space, but frequently on using vector algebra to prove those theorems about triangles that Dieudonné so despised.

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[14]) *Given a line with a graduated scale (Δ, g).*

1° *For every pair of real numbers (a', b') such that $a' \neq 0$, the map g' from Δ onto* **R** *defined for every element M in Δ by:*

$$g'(M) = a' \,.\, g(M) + b'$$

is bijective.

2° *The family of all bijections thus defined has the following property:*

For any two bijections g' and g'' from this family, there exists a pair (a, b) of real numbers, with $a \neq 0$, such that for every element M in Δ:

$$g''(M) = a \,.\, g'(M) + b \,.$$

We then call any bijection in this family a graduation *of Δ, and the number $g'(M)$ is called the* abscissa *of M in the graduation g'.*

[15]) An interesting, but not surprising, feature of the French and other reforms was the extra linguistic demands they made upon students. This, as several researchers demonstrated, led to students' mathematical success being linked even more strongly to their home and social background.

[16]) Leray had expressed his concerns about some of the "modern mathematics" in a paper published by *L'Enseignement Mathématique* [Leray 1966].

[17]) The set-theoretic option in the definition of geometry is a dangerous utopia. […] The scientific terms which we have been forced to use in order to analyse it show how much this reform misappreciates the intellectual aptitudes and needs of the adolescents who attend our […] high schools. The reform in progress seriously endangers the economic, technical, and scientific future of the Nation.

[18]) In the early 1960s, Cyril Hope led an interesting project, the *Midlands Mathematics Experiment*, which, from the beginning, sought to bring 'new' mathematics to students from a wide range of abilities. Its books had very many good points, both mathematically and pedagogically, but the project was inadequately financed and it was overshadowed by the immensely more influential SMP.

APPROACHES THROUGH TRANSFORMATION GEOMETRY

The ideas foreshadowed by D. E. Smith had gradually acquired greater currency in the 1950s and early 1960s. By that time the approach in Polish schools was already laying emphasis on the isometries and similarity transformations [19]). Elsewhere authors such as Bachmann [1959], Yaglom [1962] and Jeger [1966] (again, writing for very different types of readership) were demonstrating the possibilities of an approach to geometry based upon linear transformations [20]).

An important example of this approach was that of the School Mathematics Project in England. Strangely, and by an extremely circuitous route, this came to resemble the pre-age-15 course described by Dieudonné (although devised for somewhat older students). Originally, the course was intended as a two-year one to cover the age-range 14–16. By then students would already have met some Euclidean-style geometry including Pythagoras. The first year course comprised an experimental study of the isometries, enlargement and shearing: in Year 2 came the description of transformations in matrix terms, their composition (geometrically and algebraically) and the group of isometries. Those who attempted "Additional mathematics", an examination for high-attaining 16 year-olds, were offered the classification of frieze and plane patterns. Ideas of linearity were further developed in the 16–18 course.

Various attempts (e.g. [Maxwell 1975]) were made to provide an axiomatic basis and development for this approach to geometry teaching, but none took hold in the schools.

NEW AXIOMATIC APPROACHES TO PLANE GEOMETRY

In 1932 G. D. Birkhoff produced an axiom system which assumed the fundamental properties of the real numbers, basically permitting the use of a ruler and protractor in a natural way. As a result, using only four axioms, he was able to reach interesting theorems very quickly. Later, he and Beatley [1940] incorporated these axioms into a school text. A further attempt to

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[19]) This approach was developed in a text book series by S. Kulczycki and S. Straszewicz (see [Ehrenfeucht 1978]).

[20]) Despite the pedagogical principle that one only fully understands a definition when one meets something that does not satisfy it, I can recall no examples of non-linear transformations being exhibited except towards the end of secondary education. Then, for example, SMP's syllabus for the very high-attaining mathematics students included "inversion [and] simple conformal transformations" to be studied "amongst geometrical properties in the complex plane". ('Conformal' was dropped from the syllabus within a few years.)

proceed on these lines was made by the US *School Mathematics Study Group* (SMSG) on its establishment in 1958.

A second US project, the *Secondary School Mathematics Curriculum Improvement Study* (SSMCIS) opted for a less ambitious axiom system [21]) which, although not leading to the results of traditional Euclidean geometry, still gave students an inkling of axiomatic reasoning. Simple logical deductions were developed from these axioms and both geometric and non-geometric models of the axioms were exhibited.

In many countries, however, a completely axiomatic approach was rejected (see, e.g., [Freudenthal 1971; Griffiths & Howson 1974; Halmos & Varga 1978]) in favour of what was termed "local organisation" — small packets of deductive geometry centred on, for example, the circle theorems.

EXTENDING THE BOUNDS OF SCHOOL GEOMETRY

A marked feature of geometry teaching in the '50s and '60s was the range of geometrical topics introduced into the school curriculum. The SMP, for example, included in its original draft texts (but not in the published versions) work on non-Euclidean geometries (including the models of Beltrami and Poincaré) and also some work on finite geometries [22]). Its later texts, *SMP Books 1–5* [23]), for the age range 11–16, contained some topology (e.g., Euler's formula), graph theory (including dual graphs and applications to operational research), the "earth as a sphere" (small and great circles, etc.), as well as 'artisan' type work on surveying, perspective and technical drawing of three-dimensional objects. Elsewhere, there were similar attempts to introduce new topics, although the approach was often very much more pure and abstract. For example, the US *Contemporary School Mathematics Project* [Kaufman 1971, 284] proposed that Grade 10 geometry should include affine geometry in 2- and 3-space with considerable emphasis on characterizing the finite affine planes, together with projective planes, particularly finite projective planes. The air of unrealistic optimism (generally based upon out-of-classroom discussions, or one-off demonstration classes [24]) by enthusiasts) typical of the time is

[21]) The axiom systems of Birkhoff and SMSG are reprinted as appendices in [Tuller 1967], that for SSMCIS can be found in [Griffiths & Howson 1974].

[22]) This was, however, non-examinable, enrichment work.

[23]) These books, published by Cambridge University Press from 1965 onwards, aroused considerable international interest and were translated, in whole or parts, into many languages, including Arabic, Chinese, French, Italian, and Spanish, as well as appearing in versions adapted for use in other anglophone countries.

[24]) I am told that, under the aegis of the Institut National de la Recherche Pédagogique, there

exemplified by Kaufman's comments on these plans: "This will be an easy task, as the usual correspondence between affine and projective planes will be established." Also "duality will be stressed": thus, this should prove "an extremely interesting and meaningful topic" [Kaufman 1971, 284].

TOPSY-TURVYDOM

A remarkable trait of many reformers was their rush to try something different, usually ignoring on the way the student's existing knowledge and the type of mathematics he or she would meet outside the classroom [25]). Piaget's assertions concerning geometrical intuitions (see, e.g. [Piaget 1973, 83]) were often used to justify the early introduction of 'topology' (as this rarely amounted to much it did neither great harm nor good). More significantly, the influence of Bourbaki (whose work greatly affected Piaget, particularly in what he referred to as the three, Bourbaki-revealed, 'mother' structures of algebra, order, and topology) presumably led to the affine plane being frequently studied prior to the metric plane (see, e.g., [Kaufman 1971; Laborde 1998]). This was to cause considerably more problems. The arguments advanced for teaching "affine before metric", and those against, can be found in [UNESCO 1973, 30–31]. In the same chapter one finds what is to me an almost unbelievable statement:

> In the affine approach students are made more and more aware that in order to 'do geometry' one has only to remember the axioms and definitions pertaining to real numbers and those which give the structure of the affine vector plane.

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were some classroom trials of the French materials — but in non-representative high-attaining schools with enthusiastic and committed teachers. In England the SMP initially tested draft materials in eight or so schools most of which supplied authors to the project, and then second drafts were tested in a much wider, and more representative, range of schools before publication of a final version. Even that did not resolve all the problems, for 'successful' chapters tended to become more polished through the various versions, whereas approaches to difficult topics were often changed significantly, even in some cases up to a largely untried final version. The dangers of extrapolating from what committed, enthusiastic, well-informed teachers could achieve in the classroom to what would happen in the classrooms of typical teachers were very great and frequently ignored.

[25]) Such moves were much criticised by Freudenthal and Thom. Freudenthal [1963] was critical of what he called "didactical inversion" or "anti-didactical inversion", e.g. the use of parachute axioms. Thom [1973] invoked Haeckel's Law of recapitulation — *ontogenesis recapitulates phylogenesis* — to argue against abandoning 'historical' approaches.

I write ‘unbelievable’, not just because it would seem to me to embody both an extremely limited view of geometry and what to ‘do geometry’ might mean, but also because it was written, without any qualification, by a mathematics educator held in high regard.

The tendency to break away from old patterns is further exemplified by Halmos and Varga’s account [1978] of reforms in Hungary. There, it was argued, teaching should now proceed from the general to the particular. That is, the old-fashioned sequence of

$$\text{triangles} \to \text{quadrilaterals} \to \text{polygons} \to \begin{matrix}\text{curvilinear}\\ \text{plane figures}\end{matrix} \to \text{solid geometry}$$

should be replaced by

$$\text{sets of various objects} \to \text{sets of points} \to \text{subsets}\,.$$

I find it hard to see how such an approach could possibly succeed. Yet Varga [26]) was certainly not the well-known mathematics educator whom Freudenthal once described as having learned his psychology from Bourbaki and his mathematics from Piaget !

WHAT EFFECT DID THESE INNOVATIONS HAVE ?

In retrospect, it would be easy to dismiss the period 1950–70 as one of wasted effort and missed opportunities. So little of what was attempted still survives in schools today and geometry appears to many to be in a worse state than ever.

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[26]) Varga and the anonymous author of the quotation on ‘to do geometry’ were educators and teacher trainers of great merit who had enormous national and international influence in their time and whose influence is still felt through the work of their students. They should not be judged on these two quotations. Nevertheless the quotations have an important role in this paper in indicating the influences of Bourbaki and his followers. The brilliance of the Bourbaki exposition, which, as Blakers (see below) indicated, encouraged mathematicians to see the mathematics they had acquired in a new and restructured manner, temporarily led some educators to imagine that it might also lead to a novel and more successful pedagogical approach. Only later was it accepted that Bourbaki’s approach was pedagogically and, it could be argued, mathematically sterile.

Yet there were some successes [27]), e.g., the earlier introduction of coordinate geometry. Moreover, I believe the original *SMP Books T/T4* [28]) had considerable merits in developing students' spatial understanding, presenting modern concepts in an appropriate and achievable form, linking algebra and geometry, and laying the foundations for future mathematical learning. However, these successes were largely swept aside by changes within educational systems over which mathematicians and mathematics educators had no control and for which they were unable, or lacked the foresight, to prepare.

All countries witnessed a great widening of opportunities in those two decades. In many, the compulsory age for school attendance was raised, but in all the proportion of the age cohort remaining in school beyond that age rose sharply. Several countries, particularly in Europe had to cope with the coming of comprehensive schools. This usually had the consequence that not only were the aims for high-attainers diluted, but the mathematics teaching specialists who previously had congregated in the selective schools were now more widely and thinly distributed. It is significant that in England the original *SMP Books 1–5* continued to be used (from the early 1980s in a revised, but essentially unchanged form) in some of the remaining independent (non-state) and selective schools until the advent of the National Curriculum in the early 1990s. However, in the vast majority of state schools the lower secondary school geometry curriculum became so diluted that by the mid-1980s matrices and the geometry/algebra links had entirely vanished. As in other countries, the transformation geometry that was by now commonly taught lacked well-defined aims.

Other significant societal influences were linked to economic changes — for example, the way in which countries such as the Netherlands changed from agricultural to essentially industrial societies — and the effect these had on the role of mathematics and mathematicians within society. Euclidean geometry no longer seemed so important, and there was not a steady stream of well-qualified mathematics graduates returning to schools to teach the subject — now they were offered a much wider range of career opportunities.

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[27]) Defining 'success' is far from easy: does it mean "sold a lot of copies" or "mathematically and pedagogically sound — and appropriate for the target age group"? Certainly I should class the series of texts published in the 1970s by Lombardo-Radice (an Italian mathematician greatly interested in improving school mathematics, see, e.g. [Lombardo-Radice 1963]) and Lina Mancini Proia (a pioneering schoolteacher) as a success using the second criterion — but I suspect it was not a great financial success for the publishers. Another influential Italian schoolteacher contemporary who wrote much on geometry was Emma Castelnuovo (see, e.g. [Castelnuovo 1966]), but whose work is perhaps better represented by, say, [Castelnuovo & Barra 1976].

[28]) Published by the Cambridge University Press, 1964–65. Here I should declare a personal interest and possible bias, for I edited Books T and T4 and also Books 1–3 of the later series.

One result of these pressures was that geometry's place in the upper secondary school became seriously weakened. Statistics and probability, and other 'utilitarian' topics, began to take preference over Menelaus and Ceva and a synthetic approach to conic sections. Moreover, first-year university courses no longer always included a unit on geometry. What university mathematicians came to deplore was not the students' absence of geometrical knowledge concerning triangles (as Dieudonné made clear), but that of 'proof', which it was assumed came from the study of Euclidean geometry. For certainly, "local organisation" promised more in this direction than it ever achieved. (Normally because the 'packets' were so small that the work on them had little effect on examination performance, with the result that teachers and students often neglected them.)

WHAT LESSONS WERE TO BE LEARNED ?

Perhaps the most obvious lesson to be learned from the '50s and '60s was just how difficult it is to bring about serious curriculum changes. First there is the problem of designing sensible new curricula. Then, one is faced with the enormous task of explaining the new curriculum to teachers [29]). This means, not only describing the new content, suggesting how it might best be taught and examined, and providing pupils' texts, but also conveying the purpose of the changes, and convincing teachers that new goals are attainable and will prove of benefit to their students.

Certainly, the role played by university mathematicians came to be greatly questioned as the two decades came to an end. Did they see school mathematics as simply the first step in an assembly line that would eventually turn out a mathematics graduate ? Had they realised the problems of translating abstract concepts into school terms and of providing motivation for their study ? For one fact became increasingly apparent : the idea that explaining mathematics clearly and logically would automatically yield success was ill-founded. One can only agree with Rosenbloom who wrote :

[29]) Only Alexander Wittenberg of the major critics of 'modern mathematics' seemed fully to spell out the importance of the role of teachers : "The most crucial single factor for sound teaching of mathematics is and remains the *teacher.*" [Wittenberg 1965, 307] This paper, representative of his other writings, shows a great grasp not only of the mathematical problems and those of the learner, but also of effecting change within educational systems. Regrettably he died in 1965 and his warnings went largely unheeded.

> During the last years a number of expositions of vectors in mathematics have appeared. […] We find many of these […] unsatisfactory. Often the authors give no natural and non-trivial problems which lead the student to feel a need for vector concepts. A number of these presentations seem overly abstract. Some approaches overemphasise the affine aspects, which lend themselves to heavily algebraic treatment and neglect metric geometry. There is often a neglect of geometric intuition. [Rosenbloom 1969]

Here, I should like to draw specific attention to one of Rosenbloom's points that might otherwise pass unnoticed : the manner in which many of the innovations were not accompanied by a bank of problems that could not only be given to students, but would also catch their attention and interest. Until we arrive at that ideal world, there will also be the need for a variety of suitable, sensible problems that can be set in tests and examinations. Several topics which had sound mathematical and pedagogic merits suffered in the 60s simply because they were brought into ill-repute by examiners lacking in vision and thought.

Even when an approach combined both mathematical and psychological know-how this was no guarantee of success. I recall a textbook series, produced by a well-known cognitive psychologist who had a good command of mathematics, that failed simply because it was so dull : there was no flair, nothing that would grasp the interest and attention of pupils and little to provide motivation.

Yet, to ignore the mathematicians was to run the risk of providing texts that lacked obvious mathematical purpose, or even correctness (and as a reviewer of books I saw a number of horrors). It was clear that the skills and knowledge of mathematicians, psychologists and teachers who have day-to-day contact with students were all required. Moreover, their work had to be guided by knowledge of the way that the educational system in which they worked was moving and by the goals that society — the government, parents and employers — held for it.

However, even the best designed curricula cannot be implemented overnight and the problems of actually effecting successful change in the classroom were greatly underestimated. A typical reaction was described by Blakers concerning Australia :

> Many mathematics teachers gained new insights into the subject and hoped [often incorrectly] that these […] could be shared with their students [30]). Another group of teachers was hostile to the changes, sometimes out of conservatism and sometimes out of genuine concern for their inappropriateness ; but the

[30]) Although Blakers is here writing of schoolteachers, his remarks must also surely hold good for many of the university mathematicians involved in the reforms.

> majority [of teachers] were rather indifferent, being quite willing to teach different content and approaches […] provided that someone else would […] tell them how to go about it. [Blakers 1978, 153]

The words of Beeby, the New Zealand educator, describe the results of such a situation:

> There is one thing that distinguishes teaching from all other professions, except perhaps the Church — no change in practice, no change in the curriculum has any meaning unless the teacher understands it and accepts it. If a young doctor gives an injection under instruction, or if an architect as a member of a team designs a roof truss, the efficiency of the injection or the strength of the truss does not depend upon his faith in the formula he has used. With the teacher it does. If he does not understand the new method, or if he refuses to accept it other than superficially, instructions are to no avail. At the best he will go on doing in effect what he has always done, and at the worst he will produce some travesty of modern teaching. [Beeby 1970, 46]

These are hard lessons to have to digest[31]). Moreover the problem of recruiting sufficient high-quality teachers has increased everywhere and will add to the difficulty of effecting curriculum change. Yet improvements must be made[32]). It is important then that this is done bearing in mind all the constraints and that we do not fall into the trap of unthinking and unrealistic optimism that blighted so many of the innovations of the period 1950–70.

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[31]) Hard facts concerning the difficulties of major curricular innovation are most effectively spelled out in [Dalin 1978]. The author had directed a major project of the *Centre for Educational Research and Innovation* (CERI – an offshoot of OECD) which considered the effectiveness of curriculum reform in a number of countries. Several volumes of case studies were published by OECD, but it is Dalin's summary that is recommended reading to all interested and concerned with curriculum development.

[32]) Data from the *Third International Mathematics and Science Study* (TIMSS) relating to the pure geometrical knowledge of 18+ specialist mathematics students in their last year of secondary school make dismal reading. For example, an open response item asking students to prove a triangle was isosceles (given angle facts relating to two of its altitudes) was answered correctly by fewer than 10% of US specialist students; the corresponding percentages for two groups of high-scoring students, from Greece and France, were 65 and 53 respectively. ('Specialist students' represented approximately 14%, 10% and 20% respectively of the three countries age cohorts.) However, before one reads too much into these results and "the students' abilities to think logically", it should be noted that the percentages of students able to identify which of four statements could be logically deduced from the fact that (to paraphrase) 21 letters had been placed in 4 pigeon holes were: US, 70.0; France, 61.8 and Greece, 33.8.

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GÉOMÉTRIE – PÉRIODE 2000 ET APRÈS

Geometry — Period 2000 and after

by Colette LABORDE

The teaching of geometry lost importance in the second half of the 20^{th} century, but there is now a widespread agreement on the necessity of maintaining and renewing it. Four main issues for geometry learning and teaching in the 21^{st} century are discussed: the development of spatial awareness; the dual nature of geometry and the relationship between theoretic and intuitive geometrical knowledge; the learning process to formal reasoning; the use of *Information and Communication Technology*.

THE DEVELOPMENT OF SPATIAL AWARENESS. Geometry is rooted in actions and motions in the material environment. However, spatial awareness does not arise spontaneously and school should contribute to its development. Evidence has been gathered showing that children principally develop controls and spatial knowledge of a small-size space (*microspace*, the space of objects that they can take in their hands and manipulate) and have inadequate representations of both *mesospace* (from 0.5 to 50 times the size of an individual) and *macrospace* (in which objects cannot be grasped by hand or eye, e.g. the space of the city). The teaching of geometry deals mainly with small objects and diagrams which are drawn on a sheet of paper. But it is also essential to deal with modelling relations in meso- and macrospace with concepts like direction and angle. Research has also demonstrated that elementary-school children are unable to identify geometric relations even on diagrams of microspace. They are attracted by special features of the diagrams which are not relevant from a geometric point of view: segments are not considered to be parallel if they are not of the same length. Students should therefore learn to detach the geometric invariants from their visual appearance.

THE DUAL NATURE OF GEOMETRY: TWO REGISTERS. Geometry is based on the use of two registers, the register of diagrams and the register of language. Language allows one to describe geometric objects and relations using specific terminology, while diagrams in 2D geometry play an ambiguous role. On the one hand they refer to theoretic objects; on the other hand they offer *graphical-spatial* properties, which can generate a perceptual activity from the individual. In the traditional way of teaching geometry the theoretic properties are assimilated to graphical ones, with the illusory aim to abstract from a diagram the properties of the theoretic object it represents. As a consequence, pupils often draw the conclusion that one can construct a geometric diagram using only visual cues; or deduce a property empirically by checking on the diagram. When pupils are asked to construct a diagram, the teacher would like them to work at the geometry level, using theoretical knowledge. In reality they very often reason at the graphic level and try to satisfy visual constraints only.

Teaching the distinction between spatial graphic relations and theoretical geometric relations should certainly devolve on geometry classes, but pupils should also integrate the ability to move between theoretical objects and their spatial representations, as well as the ability to recognize geometric relations in a diagram and to imagine all possible diagrams attached to a geometric object.

This is particularly time-consuming for 3D geometry and little emphasis has been laid on its teaching, except in some countries such as the Netherlands in which 3D and 2D geometry are taught in interaction. The use of ICT is also of great help to compensate the lack of familiarity with 3D geometry.

REASONING IN GEOMETRY. As research evidence suggests, numerous conceptual difficulties arise in the learning of formal proof. Students fail to distinguish between empirical and deductive arguments; in general they rather turn to empirical arguments. Two approaches have been developed worldwide:

- one in which writing a proof is detached from the heuristic search leading to it: the text of a proof is viewed as a specific text based on the distinction between the epistemic value of a proposition ('true', 'false', 'plausible') and its theoretic status (hypothesis, conclusion, theorem). What matters is only the theoretic status of propositions, since all propositions involved have the epistemic value 'true';
- an integrative approach in which proof as a product is the result of a long process of exploring, conjecturing, and arguing. Long-term projects of such innovative teaching have been especially developed in Italy.

The fact that a proof performs a diversity of functions is now accepted by the research community, as well as the necessity of developing this diversity in teaching. In particular, proof as a means of explanation of visual phenomena seems to play a critical role in the coming era which will give more room to dynamic geometry environments offering enhanced visualization.

THE USE OF ICT FOR TEACHING AND LEARNING GEOMETRY. Spatio-graphical and geometric aspects are very much interrelated in the new kind of diagrams provided by dynamic geometry computer environments. In such environments the diagrams result from a sequence of primitives which are expressed in geometric terms chosen by the user. When an element of such a diagram is dragged by means of the mouse, the diagram is modified but all geometric relations used in its construction are preserved. These artificial realities could be compared to entities of the real world: they appear to react to the user's manipulations by following the laws of geometry in the same way as material objects obey the laws of physics. A crucial feature of these realities is their quasi-independence from the user as soon as they have been created. When the user drags one of its elements, the diagram is modified according to the geometric way defined by its construction and not as the user wishes. Computer diagrams are external objects, whose behaviour requires the construction of an interpretation by the students. Geometry is a means, among others, of achieving this interpretation.

GÉOMÉTRIE – PÉRIODE 2000 ET APRÈS

par Colette LABORDE

1. UN SUJET DE PRÉOCCUPATION DANS LE MONDE ENTIER

La fin de l'année 2000 a clos une deuxième moitié de siècle mouvementée relativement à l'enseignement de la géométrie, qui d'une grande uniformité sur le plan international avant les années 60, est passé à une forte hétérogénéité à l'heure actuelle [Hoyles *et al.* 2001]. Il est cependant loisible de se persuader à la lecture de l'étude ICMI « *Perspectives on the Teaching of Geometry for the 21^st^ Century* » [Mammana & Villani 1998] [1]) que des interrogations communes et des régularités transcendent cette grande variabilité de la place et du contenu de la géométrie dans les curricula des différents pays du monde, à savoir :

- le constat banal selon lequel la géométrie enseignée n'est plus ce qu'elle était avant les années 60 ;
- la nécessité de maintenir, voire de renforcer l'enseignement de géométrie de l'école élémentaire à la fin de l'école secondaire (de 6 à 18 ou 19 ans) ;
- le souci de préserver les liens de l'enseignement de la géométrie et de la démonstration, tout en en rénovant les modalités ;
- la nécessaire prise en compte de l'apparition de logiciels de construction géométrique et/ou de géométrie dynamique.

Il n'est pas anodin qu'en France, la commission de réflexion sur l'enseignement des mathématiques, présidée par Jean-Pierre Kahane, ait produit un rapport d'étape sur la géométrie et son enseignement [CREM 2000] [2]), alors que simultanément en Angleterre, pays de tradition scolaire fort différente, un groupe de travail réfléchissait sur l'enseignement et l'apprentissage de la

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[1]) Voir en particulier le chapitre 7 : "Changes and trends in geometry curricula". Voir aussi le compte rendu fait de cette étude par V. Villani dans *L'Enseign. Math. (2) 46* (2000), 411–415.

[2]) http://smf.emath.fr/Enseignement/CommissionKahane/

géométrie à l'initiative de la *Royal Society* et du *Joint Mathematical Council* et publiait en juillet 2001 les conclusions de sa réflexion et ses recommandations [3]). A la suite de l'ébranlement qu'a subi l'enseignement de la géométrie un peu partout dans le monde à l'occasion de la réforme des mathématiques modernes, se sont faites jour, ici et là, après les années 80, des adaptations diverses, quelquefois sans véritable cohérence interne, qui ont conduit de façon internationale à s'interroger sur l'enseignement de la géométrie et à soutenir son enseignement. Les raisons pour enseigner la géométrie sont aussi assez universellement partagées. Elles peuvent être résumées par les quelques points suivants :

- le développement pour tout individu de la maîtrise de l'environnement spatial ;
- les liens entre géométrie et vision ou intuition spatiales ;
- l'apprentissage du raisonnement ;
- l'utilité de connaissances géométriques dans la vie courante et l'importance de la géométrie dans de nombreuses applications contemporaines (courbes de Bézier, imagerie médicale, infographie, images de synthèse, …) et dans d'autres sciences comme la physique, la chimie, la mécanique ou la biologie.

Ces raisons ont une incidence certaine, tant sur les objectifs que peut poursuivre un curriculum de géométrie dans les années à venir que sur le choix des contenus et des formes d'enseignement. Notre exposé reprendra certains des points listés ci-dessus et y ajoutera le rôle possible joué par les nouvelles technologies. Il prendra évidemment en compte les recherches des dix dernières années sur l'enseignement et l'apprentissage de la géométrie dans différents pays du monde.

2. L'apprentissage de la maîtrise de l'espace

La géométrie, en tant que théorie, s'est développée sous des exigences multiples, dont à un extrême celle de maîtriser l'espace dans lequel vit l'humanité, et à l'autre de résoudre des problèmes issus de la théorie même. Un des premiers objectifs de l'enseignement de la géométrie est bien celui d'apprendre à l'enfant à saisir les situations spatiales dans toute leur complexité [Rouche 2000]. Les connaissances spatiales sont celles qui permettent à

[3]) *Teaching and Learning Geometry 11–19*, Policy document 15 (01 July 2001). The Royal Society, London.

l'individu d'agir et de communiquer dans son environnement spatial : prendre un objet, se déplacer en tenant compte des obstacles, repérer la position relative d'objets, décrire et représenter des scènes spatiales. Il est clair que la géométrie est un outil de modélisation puissant pour contrôler, agir, anticiper sur l'espace puisque, comme dit plus haut, elle tire en partie son origine de problèmes spatiaux à résoudre. On doit à Freudenthal [1973] d'avoir insisté sur cette fonction essentielle de la géométrie qui a pu être oubliée lors de la réforme des mathématiques modernes.

Des connaissances spatiales à construire

Mais si les « racines familières » (selon l'expression de Rouche [2000]) de la géométrie se trouvent bien dans les actions sur l'espace matériel, le déplacement et le repérage, il importe de reconnaître que ces connaissances spatiales ne sont pas construites de façon spontanée par les enfants et sont souvent inexistantes chez eux, dès que la taille de l'espace devient grande [Berthelot & Salin 1998, 72–73]. L'espace qui nous entoure exige des connaissances et des contrôles de nature spatiale fort différents suivant sa taille.

Brousseau [1983] propose ainsi de distinguer trois espaces : le *micro-espace*, celui dont la taille permet de prendre des objets, les bouger ; le *méso-espace*, celui dans lequel on vit, se déplace, et que la vision peut appréhender (les objets mesurent entre 0,5 et 50 fois la taille de l'individu), par exemple celui de la salle de classe ; le *macro-espace*, celui dans lequel on ne peut atteindre les objets ni par le geste ni par la vue, par exemple celui de la ville. Si des élèves de fin d'école élémentaire ont appris à reconnaître sur une feuille de papier que le dessin d'un quadrilatère est un carré en mesurant ses côtés, ils échouent largement à la tâche de reproduire un rectangle de taille celle de la base d'un banc, se fondant toujours sur des mesures de longueur et ne sachant pas contrôler les angles [Berthelot & Salin 1998].

La géométrie enseignée a en effet trop tendance à s'appuyer sans contrôle sur un rapport privilégié à l'espace réservé au traitement de petits objets ou de tracés tenant sur une feuille de papier, sur l'évidence perceptive : « on voit bien que… ». Il importe donc d'élargir les moyens d'action et de contrôle des enfants sur l'espace, en introduisant à l'école élémentaire des connaissances géométriques comme celle de visée ou d'angle. Les outils de modélisation qu'apporte la géométrie seront alors non seulement susceptibles de contribuer à la maîtrise de l'espace mais ils seront aussi la source de phénomènes qui posent question grâce aux connaissances géométriques déjà disponibles et qui sinon passeraient inaperçus. L'enseignement de la géométrie se doit d'instaurer une dialectique entre spatial et géométrique qui permette à l'enfant de progresser sur les deux plans.

D'autres recherches [Argaud 1998] montrent que même les connaissances du micro-espace de la feuille de papier sont très limitées chez les élèves de fin d'école élémentaire. Argaud [1998, 290–297], qui a travaillé de façon très approfondie à l'école élémentaire, a pu constater à plusieurs reprises chez des élèves de 9 à 11 ans l'échec important à des tâches de reconnaissance spatiale d'un quadrilatère, de deux segments parallèles, de deux segments perpendiculaires. Voici un exemple de réponses obtenues dans une classe de CM2 (10–11 ans) :

a) D est–il un quadrilatère (Fig. 1) ?

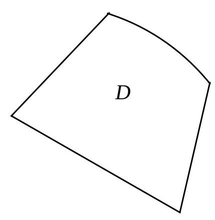

FIGURE 1
14 élèves sur 26 répondent oui

b) J est-il un quadrilatère (Fig. 2) ?

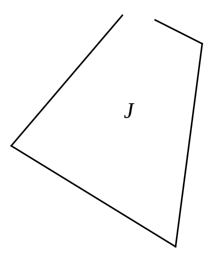

FIGURE 2
10 élèves sur 13 répondent oui. Les arguments sont : « il a 4 côtés » (8 élèves), « même s'il n'est pas fini, il a 4 côtés » (1 élève), « il a 4 arêtes » (1 élève).

c) Les segments sont-ils parallèles (Fig. 3) ?

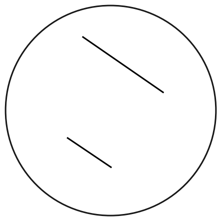

FIGURE 3

Seize élèves sur vingt-quatre disent qu'ils ne sont pas parallèles. Après discussion en binômes, cinq paires affirment que les segments ne sont pas parallèles, avec des arguments comme : « ils ne sont pas en face, ils sont décalés, ils ne sont pas de la même longueur ». Cinq autres paires déclarent que les segments sont parallèles (avec des arguments comme : « si on les prolonge, ils ne se touchent pas »). Deux paires ne se prononcent pas.

Comme on peut le constater, la difficulté réside pour les élèves dans l'identification des propriétés spatiales pertinentes qui caractérisent la propriété géométrique concernée. De la même façon, les élèves d'école moyenne (11–15 ans) ne reconnaissent pas aisément un losange, voire un carré s'ils ne sont pas dans une position prototypique, ou un axe de symétrie s'il n'est pas vertical. Or, si la géométrie contribue au développement de connaissances spatiales, elle s'appuie également sur ces dernières car le raisonnement géométrique est fondé sur une interaction entre le voir et le savoir [Parzysz 1988], ou encore selon les termes de Fishbein [1993] entre *figural* et *conceptuel*. La géométrie est de nature duale et c'est une de ses caractéristiques qui la rend si précieuse et qui motive son enseignement.

3. LA NATURE DUALE DE LA GÉOMÉTRIE : LE SENSIBLE ET L'INTELLIGIBLE

On sait que la Grèce antique avait déjà débattu de la place de la géométrie — relevait-elle du sensible ou de l'intelligible ? — et que la décision avait été prise en faveur des fameuses idéalités. La nature duale de la géométrie a depuis été maintes fois exprimée, comme le font encore Hilbert et Cohn-Vossen dans la préface de l'ouvrage *Anschauliche Geometrie* :

> In der Mathematik wie in aller wissenschaftlichen Forschung treffen wir zweierlei Tendenzen an : die Tendenz zur Abstraktion — sie sucht die *logischen* Gesichtspunkte aus dem vielfältigen Material herauszuarbeiten und dieses in

systematischen Zusammenhang zu bringen — und die andere Tendenz, die der Anschaulichkeit, die vielmehr auf ein lebendiges Erfassen der Gegenstände und ihre *inhaltlichen* Beziehungen ausgeht. [4]) [Hilbert & Cohn-Vossen 1932, v]

La géométrie, en tant que domaine des mathématiques, met en jeu des objets et des relations théoriques mais elle est issue d'une modélisation de l'espace matériel qui nous environne. "Toucher", "couper", "avoir une forme ronde", "être près de", sont autant de locutions désignant des phénomènes spatiaux que la géométrie exprime en termes de relations entre des objets définis par la théorie : « les droites sont sécantes », « la droite est tangente au cercle », … Dans certains cas, le langage de l'espace et celui de la géométrie ont recours aux mêmes termes, lorsqu'il s'agit de rendre compte de propriétés courantes : être parallèle, se couper à angle droit. Il importe cependant de distinguer deux référents différents, si l'on cherche à mieux comprendre les difficultés d'apprentissage des élèves en géométrie : d'un côté, un domaine théorique [5]) issu d'une modélisation de l'espace mais se développant ensuite en grande partie à partir de problèmes internes à la théorie et ayant son propre mode de validation (démonstration), de l'autre une réalité spatiale liée à notre environnement que certains appellent géométrie expérimentale [Chevallard 2001] ou naturelle [Houdement & Kuzniak 1999], dans laquelle la validation se fait par l'expérience sensible.

Existence en géométrie de deux registres sémiotiques

Les objets et relations de la géométrie sont extériorisés soit dans un registre de type discursif, par des énoncés dans des langages plus ou moins formels, soit dans un registre de type figural, par des dessins ou encore des entités que nous appellerons spatio-graphiques, retournant ainsi en quelque sorte à l'origine spatiale de la géométrie. Ces deux types d'expression des objets géométriques sollicitent une appréhension ainsi que des connaissances et des contrôles de nature différente de la part de l'individu. Les dessins appellent une appréhension globale en donnant à voir des relations spatiales, tandis que les énoncés sollicitent une appréhension linéaire et analytique d'un discours

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[4]) En mathématiques, comme dans toute recherche scientifique, on trouve deux types de tendances : la tendance à l'*abstraction* (elle cherche à dégager les aspects *logiques* de la diversité du matériel étudié et à corréler celui-ci de façon systématique et cohérente) et, d'autre part, la tendance à la *conceptualisation intuitive*, qui aspire plutôt à une compréhension vivante des objets en se concentrant sur le *fondement concret* de leurs rapports réciproques.

[5]) Certes, on peut distinguer des niveaux d'affinement de la théorie au sein même de la géométrie théorique. En particulier, nombre de démonstrations effectuées à l'école secondaire comportent des imperfections dont on sait qu'elles seraient susceptibles d'être éliminées dans une géométrie que Chevallard [2001] qualifie de théorique mathématique.

qui renvoie plutôt aux objets théoriques géométriques. Les contrôles mis en jeu par l'appréhension du dessin sont en un premier temps de type perceptif, tandis que les contrôles sur les énoncés se font par des connaissances géométriques. C'est bien d'abord la reconnaissance visuelle du contact d'une droite avec un cercle qui nous conduit à identifier une tangente sur un dessin et c'est en un deuxième temps que grâce à nos connaissances géométriques, nous nous assurons du bien-fondé géométrique de cette impression visuelle, par exemple en vérifiant que la droite est bien perpendiculaire au rayon.

La distinction entre spatio-graphique et géométrique

Il est évident que de façon spontanée les élèves de fin d'école primaire et de début de collège ne peuvent distinguer le spatio-graphique du théorique et que c'est à l'enseignement qu'échoit la mise en place de cette distinction, qui est une entreprise de longue haleine. De nombreux constats de cette confusion chez les élèves ont pu être faits. Là où l'enseignement attend une mise en œuvre d'une propriété géométrique, les élèves ont recours à un constat visuel, comme dans le cas de la construction d'une tangente à un cercle, issue d'un point P extérieur au cercle : les élèves la tracent par tâtonnement sur le dessin en mettant en œuvre un contrôle spatial, la bonne droite est obtenue à l'œil quand la règle touche le cercle. Les élèves travaillent spontanément sur les propriétés spatiales du dessin dans un contrat graphique alors que l'enseignement cherche à mettre en place un contrat géométrique [Arsac 1993].

Le jeu entre géométrique et spatio-graphique

Il semble donc que non seulement le théorique soit à construire par les élèves, mais aussi les rapports entre des propriétés géométriques théoriques et les représentations spatio-graphiques. L'interprétation d'un dessin comme celui d'un objet géométrique, la prise en considération de ce dessin comme une représentation parmi d'autres de l'objet géométrique en question consiste en ce que nous appelons la *figure* [Laborde & Capponi 1994]. En géométrie plane euclidienne, on a peine à imaginer qu'il soit difficile d'aller au-delà du dessin en tant qu'entité matérielle et de considérer la (ou les) figure(s) attachée(s) au dessin, tant est fort le sentiment d'évidence que l'on « voit » la figure. Le passage en géométrie non-euclidienne, par la décentration qu'il occasionne, permet de mieux prendre conscience que l'association entre le dessin et l'objet géométrique qu'il représente ne va pas de soi. En effet, la géométrie non-euclidienne a recours aux dessins de la géométrie euclidienne qui ne sont alors plus analogiques comme ils l'étaient en géométrie euclidienne : une droite n'est ainsi plus représentée par un trait rectiligne dans le modèle de Poincaré. Le rôle symbolique du dessin est plus apparent, comme l'écrit Turc [1914] :

> Ne pouvant construire les figures lobatschewskiennes, c'est d'une manière symbolique que l'on trace les figures planes sur un plan et que l'on y dessine les figures de l'espace. Cette représentation symbolique suffit pour fixer les idées et suivre le raisonnement.

Certes le sentiment d'évidence au vu du dessin peut aussi être obtenu, mais au prix d'une longue pratique. Thibault et Labarre [1996, 221] indiquent combien des étudiants canadiens d'un cours de géométrie de premier cycle universitaire en mathématiques et en éducation mathématique ont des difficultés à reconnaître une droite dans le dessin d'une ligne courbe dans un modèle de géométrie hyperbolique et combien le travail dans plusieurs modèles de géométries non-euclidiennes peut être fructueux pour dépasser ce type de difficulté. Arsac [1998] tire les mêmes conclusions sur l'intérêt d'un tel travail en formation des maîtres, qui permet de sensibiliser les formés aux nombreux postulats implicites en cours dans la géométrie euclidienne enseignée et à la prise de conscience que l'évidence est toute relative.

Or, il nous paraît essentiel que les élèves sachent utiliser un jeu d'interactions entre contrôles spatiaux et contrôles théoriques. En effet, la complémentarité des deux registres et des appréhensions et contrôles joue un rôle important dans la résolution de problèmes en géométrie, comme l'ont souligné différents travaux [Fishbein 1993 ; Mariotti 1995 ; Duval 1998]. C'est parce que l'expert sait mettre en place une interaction entre ces deux registres, qu'il avance dans la résolution d'un problème de géométrie. Il sait distinguer les phénomènes spatiaux pertinents pour le problème, des propriétés spatiales non intéressantes, il sait interpréter en termes de géométrie ce qu'il voit, ou même il sait grâce à la géométrie voir plus dans le dessin que ce que l'on peut y voir de façon immédiate, et enfin il sait tirer des conséquences géométriques de ce qu'il constate sur le dessin. Rouche [2001, 51] considère que la pensée géométrique commençante met en jeu trois processus — percevoir, concevoir et inférer — qui s'intègrent dans l'activité. Mener à bien un raisonnement en géométrie repose sur une interaction entre le registre discursif et celui des figures, entre l'appréhension de propriétés spatiales et l'usage de propriétés théoriques.

Un des objectifs de l'enseignement de la géométrie réside donc dans l'organisation de situations d'apprentissage de traitements des dessins en géométrie, où il s'agit pour les élèves de dépasser ce que Duval [1998] appelle « l'appréhension perceptive » immédiate qui peut entraver l'analyse géométrique pour aboutir à une « appréhension opératoire » du dessin : cette dernière consiste à savoir reconnaître des (sous-)configurations clés, à travailler le dessin, le modifier pour rendre plus visible une telle configuration. Des tâches de construction de figures géométriques, de reproduction de figures, ou en sens inverse de description de figures en langage naturel contribuent à cet apprentissage. Pour construire ou décrire une figure, on doit sélectionner les propriétés qui la caractérisent ; de telles activités préparent ainsi la démonstration dans laquelle des propriétés données en impliquent d'autres.

LA GÉOMÉTRIE DANS L'ESPACE

Si la géométrie s'appuie sur des dessins plans, elle est cependant destinée à modéliser l'espace à trois dimensions. La lecture géométrique d'un dessin plan d'un objet à trois dimensions pose des difficultés supplémentaires aux élèves, puisque des évidences spatiales sur le dessin (comme la concourance de droites) ne traduisent pas nécessairement des propriétés géométriques correspondantes. Il est unanimement recommandé dans le monde entier de ne pas négliger la géométrie dans l'espace et les problèmes de représentation d'objets de l'espace, les règles de représentation prenant justement appui sur des propriétés géométriques. Des recherches sur l'apprentissage de la géométrie 3D ont d'abord porté sur les capacités de représentation dans le plan d'objets solides, ou de lecture de dessins 2D d'objets 3D [Gaulin 1985 ; Bessot & Eberhard 1986 ; Parzysz 1991], puis sur des processus d'enseignement [Bessot & Eberhard 1987 ; Osta 1998 ; Douady & Parzysz 1998]. Il semble bien que les procédés de représentation en perspective posent des problèmes durables aux élèves, un peu partout dans le monde. Parzysz [1988] propose de les interpréter en termes de compromis « voir/savoir ». En absence de connaissances géométriques, les élèves cherchent à rendre compte dans le dessin à la fois de ce qu'ils savent sur l'objet solide et de ce qu'ils voient de ce dernier. De plus, leurs dessins peuvent chercher à reproduire des stéréotypes de dessins familiers comme celui du cube en perspective cavalière. Seul un enseignement fondé sur des connaissances géométriques permet de dépasser ces difficultés. Seuls dix pour cent d'élèves de fin d'école secondaire (15–17 ans) affirment que l'ombre projetée sur un plan horizontal, par une lumière quasi-ponctuelle, d'un carré fait de tiges rigides et parallèle à ce plan est toujours un carré ; les autres élèves hésitent lorsque la source de lumière n'est pas sur l'axe vertical du carré. C'est le raisonnement à l'aide de théorèmes sur les intersections de plans et droites, puis l'introduction de l'homothétie, qui permet de dépasser les hésitations [Douady & Parzysz 1998].

Or, dans les faits, trop souvent les chapitres de géométrie dans l'espace sont les premiers laissés de côté par les enseignants en cas de manque de temps. On peut soupçonner que les enseignants se sentent moins familiers avec cette géométrie souvent mal connue. Des questions considérées comme triviales en géométrie plane ne recueillent pas de réponses immédiates, telle la suivante : « dans un tétraèdre, les hauteurs sont-elles concourantes ? ». Le recours plus difficile à l'expérimentation en 3D explique cette absence de familiarité. Les nouvelles technologies facilitent l'expérimentation, comme le montrent les figures 4 et 5 d'un tétraèdre orthocentrique, modifié ensuite par manipulation directe d'un de ses sommets dans un prototype de Cabri-géomètre 3D.

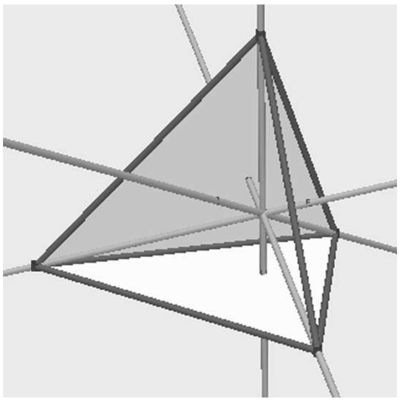

FIGURE 4
Un tétraèdre orthocentrique et ses hauteurs

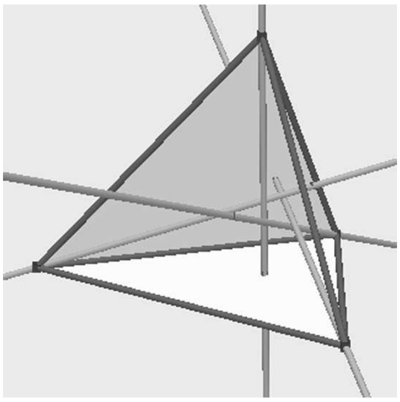

FIGURE 5
Après modification d'un des sommets

La commission française de réflexion sur l'enseignement des mathématiques suggère deux thèmes pour l'enseignement de la géométrie dans l'espace : les polyèdres et la géométrie sphérique. Comme on le voit, ces deux thèmes sont d'importance pour une meilleure connaissance du monde dans lequel nous vivons et pour les applications à d'autres disciplines qui utilisent les modèles fournis par la géométrie, ainsi que le souligne le rapport anglais mentionné au début de cet exposé. Un autre apport de l'enseignement de la géométrie dans l'espace réside dans l'usage de connaissances de géométrie plane auxquelles toute activité de géométrie dans l'espace fait appel. C'est justement le recours au jeu entre 2D et 3D qui fait le plus souvent cruellement défaut aux élèves de l'école secondaire. Des curricula comme ceux des Pays Bas, qui instaurent une véritable interaction entre plan et espace tout au long de la scolarité, sont de ce point de vue particulièrement bienvenus.

4. Raisonnement géométrique et démonstration

Dans la droite ligne de la tradition des *Éléments* d'Euclide, la géométrie a été considérée, jusqu'il y a encore récemment, comme le lieu d'exercice privilégié de la démonstration. On a pu s'interroger sur le bien-fondé de cette tradition pour initier les élèves à la déduction. En effet, il s'agit de faire comprendre aux élèves qu'en géométrie, ils doivent s'aider de ce qu'ils voient pour avoir des idées dans la résolution d'un problème de géométrie, mais qu'ils n'ont plus le droit d'utiliser ce qu'ils voient quand il s'agit de raisonner « rigoureusement » et doivent alors se cantonner au niveau théorique. Un contrat didactique d'un type nouveau doit être installé en classe et cela n'est pas simple.

De nombreuses recherches au plan international ont montré combien l'entrée dans la démonstration est difficile pour les élèves de l'enseignement secondaire voire universitaire, qui préfèrent donner des arguments empiriques plutôt que développer un raisonnement déductif [Balacheff 1991; Harel & Sowder 1998; Chazan 1993]. Ces arguments empiriques reposent sur des constats perceptifs de relations spatio-graphiques sur le dessin. La démonstration n'est pas perçue comme un outil de résolution de problème ou d'explication et reste souvent une activité formelle dépourvue de sens [Hanna & Jahnke 1993; De Villiers 1994].

Les interprétations de ces difficultés et les propositions d'enseignement ont pu soit envisager l'activité globale d'élaboration d'une démonstration allant de sa recherche à la production d'un texte, soit se centrer particulièrement sur les spécificités du texte de démonstration.

Dans la seconde catégorie de recherches, Duval [1991] a montré que le fonctionnement du texte de démonstration diffère complètement du texte argumentatif. Dans un texte de démonstration, ce qui importe n'est pas la valeur épistémique de vérité ou de vraisemblance d'une proposition mais son statut opératoire (hypothèse du problème, conclusion d'un pas de démonstration), et ce qui permet le déroulement est la substitution d'une ou de propositions par une autre jusqu'à ce qu'on aboutisse à la proposition cible. Une argumentation en revanche repose sur l'accumulation des arguments et leur force. Une des implications pour l'enseignement est l'accent à mettre sur l'apprentissage de la rédaction de la démonstration. La démonstration est un texte qui a un fonctionnement discursif spécifique qui ne peut être appris par simple imitation.

Alors que les travaux de Duval séparent la partie heuristique et la rédaction de la démonstration, d'autres recherches ont au contraire vu le texte de démonstration comme indissociable de son élaboration et du problème à résoudre. Ainsi des chercheurs italiens [Bartolini Bussi & Boero 1998 ; Bartolini Bussi *et al.* 1999 ; Douek 1999] voient-ils une continuité ou encore ce qu'ils appellent une *unité cognitive* entre la phase de production de conjectures et celle de leur justification, le raisonnement qui conduit aux conjectures fournissant les arguments qui servent dans la construction de la preuve. Ces travaux mettent justement en évidence l'évolution des argumentations des élèves, obtenues grâce aux interventions de l'enseignant et aux échanges entre élèves organisés par l'enseignant. Mais, comme ces chercheurs le soulignent, le choix des problèmes, et plus largement du thème mathématique sur lequel porte l'étude qui débouche sur la construction de preuve, est fondamental. Ils considèrent que le choix de *faire produire des théorèmes* (selon leur expression) par des élèves est particulièrement pertinent lorsque la géométrie apparaît comme un outil de modélisation de phénomènes fondamentaux de l'expérience quotidienne des élèves, telles les ombres produites par le soleil ou la représentation en perspective. Cette approche confère une valeur culturelle à la géométrie qui déborde le strict plan mathématique. Les conceptions premières des élèves élaborées dans le cadre de la vie quotidienne sont erronées dès qu'il s'agit d'aller au-delà des premières évidences. La dimension théorique de la géométrie est ainsi construite comme permettant d'avoir un contrôle plus étendu sur ces phénomènes mal maîtrisés. Deux caractéristiques de ces projets d'enseignement autour de la preuve en géométrie sont essentielles :

- la gestion de la classe par l'enseignant et ses interventions, en particulier dans les discussions collectives organisées dans la classe, qui contribuent à l'avancée dans la « théorisation » par les élèves,
- le temps long des projets (plusieurs mois voire un an).

La conception de situations d'enseignement contribuant à l'apprentissage de la preuve est perçue par tous, enseignants, formateurs d'enseignants et chercheurs, comme fondamentale. Les projets italiens fournissent une approche dans laquelle la preuve contribue à expliquer ou prévoir des phénomènes externes aux mathématiques. Dans une autre approche, la preuve est un moyen de dépasser l'incertitude relative à une assertion mathématique, comme dans les situations de validation de [Brousseau 1997]. Cette pluralité d'approches traduit la diversité des fonctions de la preuve en mathématiques et très certainement il importe que l'enseignement reflète cette diversité et ne se restreigne pas à une vue étroite de la preuve dans laquelle il s'agit de démontrer la validité d'énoncés proposés par l'enseignant.

De Villiers [1994] a souligné la diversité des fonctions d'une preuve en géométrie et en particulier défendu l'idée que l'enseignement pouvait prendre davantage appui sur la fonction d'explication. La discussion sur ces fonctions est redevenue particulièrement d'actualité alors que les environnements de géométrie dynamique pénétraient l'enseignement. Certains ont pu en effet craindre l'anéantissement de la nécessité de l'enseignement de la démonstration provoqué par ces environnements qui donnent à voir « trop facilement » nombre de propositions très probablement valides puisque vérifiées dans la déformation continue de la figure lors d'un déplacement quelconque d'un de ses éléments. L'existence de ces nouvelles possibilités de visualisation change profondément le rapport à l'évidence sensible en géométrie et donc l'enseignement de la géométrie. Le paragraphe qui suit est consacré à ces environnements.

5. Des logiciels de géométrie dynamique à manipulation directe

L'idée de géométrie dynamique n'est pas neuve. La propriété de la somme des angles d'un triangle est ainsi introduite par Clairaut par l'idée que lors du déplacement de C sur le côté AC fixe, la variation de l'angle B est compensée par celle de C :

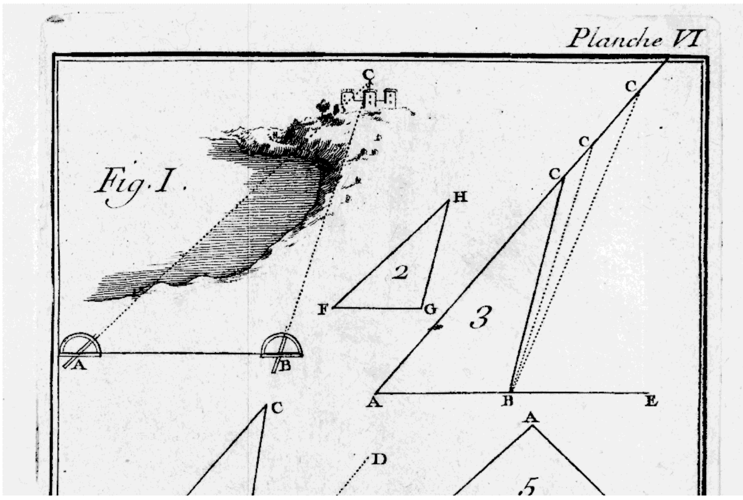

FIGURE 6

Supposons, par exemple, que BC tournant autour du point B, s'écarte de AB, pour s'approcher de BE, il est clair que pendant que BC tournerait, l'angle B s'ouvrirait continuellement ; & qu'au contraire l'angle C se resserrerait de plus en plus ; ce qui d'abord pourrait faire présumer que, dans ce cas, la diminution de l'angle C égalerait l'augmentation de l'angle B, & qu'ainsi la somme des trois angles A, B, C, serait toujours la même, quelle que fût l'inclinaison des lignes AC, BC, sur la ligne AE.

[Clairaut 1741, première partie *LXII*]

Deux siècles plus tard, a lieu une proposition faite en France par Méray [1874] d'enseigner la géométrie à partir du mouvement : le mouvement de translation permet d'introduire le parallélisme, le mouvement de rotation la perpendicularité. Si Méray s'est intéressé au mouvement des « figures solides » indéformables [1906 (3e édition), p. 3], il évoque cependant [p. 7] la notion de figure variable :

> Une déformation lente et sans rupture, d'un corps mou comme une pâte ferme, ou flexible comme un ressort, […] nous montre une succession de plusieurs figures solides inégales mais l'origine commune qui caractérise toutes ces figures, les ressemblances plus ou moins accentuées qui sont observables entre elles, permettent de voir dans le phénomène une même figure, que le déplacement de ses divers points dans l'espace fait varier dans sa forme.
> [Méray 1906, 7]

Depuis une quinzaine d'années sont apparus dans différents pays du monde des logiciels de « géométrie dynamique » [6]) qui réalisent cette idée de figure variable, en offrant des entités graphiques sur l'écran de l'ordinateur ou de la calculatrice

- qui sont le résultat d'une suite d'opérations exprimées en termes de géométrie ;
- que l'on peut déplacer en manipulation directe avec la souris ;
- et dont le comportement au cours du déplacement est contrôlé par une théorie géométrique (celle sous-jacente au logiciel) ; les relations géométriques ayant servi dans le programme de construction du dessin sont conservées au cours du déplacement [Laborde 1995].

En un mot, les dessins à l'écran, une fois construits, ont un comportement qui ne suit pas forcément les désirs de leur auteur. Dans les modifications continues qu'entraîne le déplacement d'un élément du dessin, sont offertes à la perception des propriétés spatiales dont on sait que l'évolution est régulée par la géométrie. On pourrait comparer ces réalités spatio-graphiques d'un nouveau type aux objets du monde réel, en disant qu'ils résistent aux manipulations de l'individu en suivant les lois de la géométrie. L'observation de leurs comportements peut être à l'origine de spéculations sur des relations géométriques satisfaites par le référent géométrique correspondant. En particulier, des propriétés spatio-graphiques invariantes au cours du déplacement sont de très bonnes candidates à être des relations géométriques. La caractéristique de tels logiciels est d'établir un lien entre les deux registres de la

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[6]) Citons ceux qui ont donné lieu à des recherches internationales : *Cabri-géomètre* (Laborde & Straesser, 1990), *Geometer's Sketchpad* (Key Curriculum Press, 1993), *Geometry Inventor* (Arcavi & Hadas, 2000)

géométrie. On peut cependant craindre que les apprenants débutants ne disposant pas d'une mobilité contrôlée entre spatio-graphique et géométrique ne sachent utiliser pleinement l'amplification que permettent ces logiciels dans la visualisation de propriétés spatiales.

Plusieurs recherches et projets d'enseignement au niveau de l'enseignement secondaire ou fin d'école primaire se sont donné comme objectif de mettre au point des situations d'apprentissage de cette mobilité, en tirant parti des possibilités offertes par ces environnements. Ces derniers permettent de concevoir de nouveaux types de tâches dans lesquelles les connaissances géométriques servent à (re)produire, expliquer, prédire des phénomènes spatio-graphiques constatés visuellement.

La (re)production de phénomènes spatio-graphiques avec le logiciel correspond à des problèmes de construction géométrique dans lesquels les spécifications d'un objet sont données soit discursivement, soit sous forme d'un dessin dynamique. L'objet reproduit doit satisfaire aux spécifications et le déplacement est un outil fort d'invalidation de procédés de construction au jugé, fréquents chez les élèves comme dit plus haut. Le contrat selon lequel ce n'est pas le dessin produit qui intéresse l'enseignant mais le procédé d'obtention de ce dessin, ne semble plus tirer son origine d'une demande de l'enseignant mais de règles d'usage du logiciel. Cela ne signifie pas qu'il soit immédiatement approprié par les élèves [Bellemain & Capponi 1991; Strässer 1992]. Diverses recherches internationales ont mis en évidence nombre de procédés de construction chez les élèves mélangeant le jugé et l'usage de propriétés géométriques [Noss *et al.* 1994; Jones 1998; Hölzl 1996; Healy 2000]. La distinction à établir entre spatio-graphique et géométrique est longue à construire par les élèves pour qu'elle soit véritablement appropriée. Les environnements de géométrie dynamique apparaissent comme un moyen de médier la distinction spatio-graphique géométrique par la résistance au déplacement et offrent des rétroactions aux constructions des élèves leur montrant l'inadéquation de leurs constructions au jugé. En cela ils contribuent à l'apprentissage de cette distinction.

Les tâches de prédiction sont particulièrement adaptées aux logiciels de géométrie dynamique. Il s'agit pour les élèves de prédire le comportement d'éléments de la figure dans le déplacement. Pour être correct, cette prédiction prend appui sur des savoirs géométriques. Par exemple, comment se déplace le cercle image d'un cercle donné dans une homothétie lorsque le centre d'homothétie se rapproche du cercle donné ? (La plupart des élèves donnent une réponse erronée.) La confrontation entre les prédictions des élèves et le comportement observé peut être la source d'un conflit cognitif qui s'avère moteur dans l'apprentissage de la géométrie.

L'explication de phénomènes visuels peut être sollicitée lorsque ces derniers sont surprenants pour les élèves. Donnons un exemple avec des élèves français de 15–16 ans utilisant le logiciel Cabri-géomètre [Clarou *et al.* 2001]. Deux vecteurs $\overrightarrow{MA}$ et $\overrightarrow{MB}$ d'origine commune M sont représentés à l'écran. On construit, avec l'outil "somme de vecteurs", la somme $\overrightarrow{MC}$ de $\overrightarrow{MA}$ et $\overrightarrow{MB}$ et l'on déplace le point M. Si l'enseignant demande aux élèves, de dire ce qu'ils constatent de remarquable, ils ne savent que répondre : la plupart du temps, ils n'ont rien « vu ». L'existence d'un point invariant par lequel passent tous les vecteurs sommes d'origine M n'est pas remarquée, en absence de matérialisation de ce point. La situation devient tout autre dès que l'on déplace M en laissant la trace du vecteur somme (outil "Trace"). Le point invariant apparaît comme un étranglement de la forme à deux dimensions créée par la trace (Fig. 7). Tous les élèves le remarquent et leur surprise est grande. C'est le moment adéquat pour l'enseignant pour demander aux élèves d'expliquer ce phénomène étonnant. Les connaissances théoriques servent alors d'outil d'explication d'un phénomène visuel.

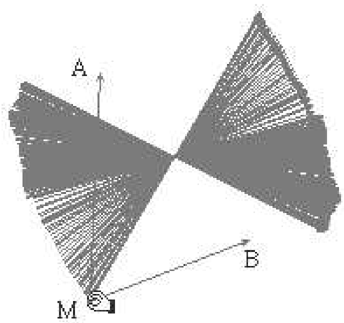

FIGURE 7

L'outil "Trace" a permis la visualisation du point invariant car il l'a placé au sein d'une région de dimension 2, alors qu'auparavant il n'était que sur un objet de dimension 1, le vecteur somme. Le logiciel a servi d'amplificateur visuel de phénomènes mathématiques.

La démonstration peut ainsi prendre sens dans de tels environnements si elle remplit une nouvelle fonction, celle d'expliquer des phénomènes surprenants pour les élèves. Dans le même esprit, des tâches de construction dans ces logiciels d'objets impossibles (par exemple, triangle avec deux bissectrices intérieures perpendiculaires, quadrilatère avec trois angles droits) sont en général des moments mathématiques forts pour les élèves, par le sentiment de conviction que crée l'environnement de l'impossibilité de ces objets, en opposition à l'attente des élèves. Le recours à la démonstration peut ainsi être introduit par l'enseignant dans la première tâche pour répondre à la question du pourquoi spontanément posée par la majorité des élèves. En effet, seul le recours au théorique peut expliquer une non-existence puisqu'il est impossible d'exhiber un objet satisfaisant aux conditions. Il est à noter que ce peut être l'occasion d'introduire un raisonnement par l'absurde. La seconde tâche utilisée à l'école élémentaire favorise la prise de conscience du caractère de nécessité que possèdent certaines propriétés géométriques dès lors que d'autres sont vérifiées. Le caractère apodictique des mathématiques n'est pas perçu par les élèves d'école primaire ou de début d'enseignement secondaire qui voient bien davantage les propriétés d'un objet géométrique comme des qualités indépendantes (un carré a quatre côtés égaux, il a quatre angles droits, etc.). Or la prise de conscience de ces liens de nécessité est un constituant indispensable de l'apprentissage de la démonstration. Les environnements de géométrie dynamique, par le sentiment de certitude qu'ils offrent, peuvent être utilisés à cette fin.

Le problème pour l'enseignant est d'identifier les situations géométriques qui conduisent à ce sentiment de surprise. La conception de telles situations qui nécessitent une grande connaissance des élèves n'est donc pas facile et requiert du temps et de l'expérience. Il a fallu quelques siècles et beaucoup de géométrie pour que le silicium du sable sur lequel étaient tracés les dessins éphémères d'Archimède concrétise les expériences de pensée de Clairaut et de Méray. Le défi pour l'enseignement est qu'elles ne deviennent pas des réalités virtuelles banales pour les élèves de la fin du 21^{e} siècle.

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REACTION

by Nicolas ROUCHE

Rudolf Bkouche has shown how, due to the evolution of the sciences and techniques, due also to the development of projective and non-Euclidean geometries, there emerged around 1900 an empiricist view of geometry, and how this view affected teaching.

Now, if the relationship between geometry and experience appears to be so strong, it seems natural to investigate what geometry owes to our perceptions, which are the primary sources of our experience. This question is important as far as teaching is concerned. As evidence of this importance, we need only consider, with Colette Laborde, how students see so many disputable things in geometrical drawings. On this point of perceptions — and in the scope of this note —, let me mention only the contribution of Ernst Mach (see especially [Mach 1900]), already at the beginning of the twentieth century. This author pointed out the narrow scope of the human being's sense organs. In the case of sight, for instance, we do not see things clearly if they are either too large or too small, too close or too distant, or simply badly oriented with respect to us. Mach explains the particular conditions under which two congruent (isometric) figures are recognized as such : this happens when both figures are in a frontal plane, at a reasonable distance from the eyes, and providing that they are mutual images of each other, either by a reflection whose axis belongs to the symmetry plane of the observer or by a horizontal translation. In all other cases, the congruence is recognized with difficulty or not at all.

Now, recognizing the congruence of two figures is no negligible operation. It is probably the first step of the human being towards geometry.

What do we have at our disposal to ascertain congruence, when our sight fails ? Mach again gives the answer. We can bring the figures in a position of correct perception, in the sense explained above. We can also superimpose one figure over the other, and Rudolf Bkouche reminded us that this simple operation is a starting point for Euclidean geometry. Mach calls these actions 'mechanical operations'. But the objects cannot always be transported that way. And in case they cannot, our only resort lies in some 'operations of the intellect' : *we begin to elaborate and apply theorems.*

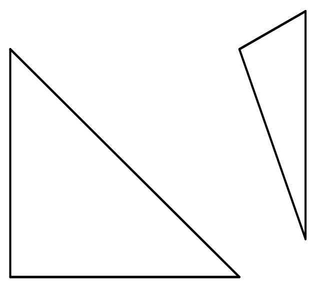

FIGURE 1

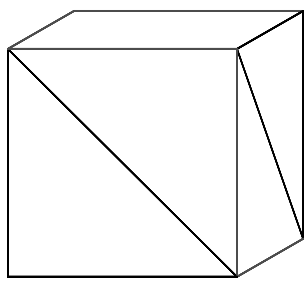

FIGURE 2

By way of example, let us consider two triangles as in Fig. 1. Suppose we are told that it is a photographic picture of two triangles situated in space in an unknown way. Are they congruent ? We do not know ! But if, as in Fig. 2, we are given some indications as to their dimensions, we succeed in concluding that they are congruent. This we infer by an intellectual operation, whatever way we proceed : for example by noticing that the faces of a cube are congruent and that each triangle is half such a face, or by observing that our triangles have equal sides in pairs.

This example suffices to show that plane geometry, before being the geometry of the plane, is first of all *the geometry of plane objects whatever their disposition in space*. This throws some light on the issue of fusion between plane and solid geometry as commented by R. Bkouche.

But let us come back to this basic fact : geometry has its roots in the visual, tactile and kinaesthetic perceptions. This is something we should remember when we want to revise the way it is taught. And this shows, in passing, that revising the teaching of geometry is not a matter for mathematicians only.

Let us now consider the fifties and sixties. Referring in general to the so-called 'new maths', Geoffrey Howson emphasizes a most relevant social fact, namely that the inspiration for this reform came from university mathematicians. These were concerned essentially with gifted students in senior high schools at a time, moreover, when these schools were attended by a minority of the population. G. Howson asks the frightening question :

"Did they see school mathematics as simply the first step in an assembly line that would eventually turn out a mathematics graduate ?" This circumstance resulted, in several countries, in an axiomatic form of teaching, using an excess of technical terms and an unrealistically large and varied subject matter. It is true that certain countries have been preserved from such exaggerations.

On the other hand, this curriculum planned for the élite inspired subsequently, but only subsequently, the teaching dispensed to everyone, including the pupils of kindergartens and primary schools.

However, already before the fifties, Piaget had been active in the field of elementary school (see especially [Piaget & Inhelder 1947]). As G. Howson reminded us, his assertions were taken to justify the teaching of geometry in the direction from the poorest to the richest structures, in particular by putting topology first (see also [Piaget 1966]). It is commonly admitted nowadays that the work of Piaget on the psychological genesis of geometry was biased by too narrow a conception of the essence of mathematics. As demonstrated by Hans Freudenthal, the result was that Piaget, in his investigations on geometry, proposed to the children only such situations — Freudenthal would have said *phenomena* —, lacking in variety, as suited his structuralist thesis.

One might conclude that, to revise the teaching of geometry, one should first revise Piaget (if one dare say that in Geneva), starting from another basis, and of course relying on all relevant contributions from psychologists and psychomotricians, but also with the collaboration of mathematicians.

These are too rarely interested in kindergarten and primary teaching. But if we accept the idea that each stage of geometry learning is rooted in all the preceding ones, then the problem in no way reduces itself, as many mathematicians believe, to producing the best rational discourse, appropriate for 12 or 13 year-old children. And mathematicians could add much relevance to the studies of the most elementary learning activities, as observed by Freudenthal. They did not collaborate enough at the time of Piaget.

Now, arriving at the third contribution in this panel and having no sufficient knowledge of the geometric software packages — a subject of major importance —, I would like to say a few words about the important analysis by Colette Laborde of all the misunderstandings originating in the figures in geometry courses. This analysis will certainly open the eyes of many teachers.

On the other hand I found it interesting that this analysis relies on a clear distinction between perceived and conceived spaces, the visual and the geometric, between the drawing (usually called *figure*), which is perceptive and global, and the discourse, which is theoretical and analytic (linear). Colette

Laborde asserts that “the teaching of geometry relies heavily on what one usually calls figures, the common use of which in geometry is to help in the heuristic phase of problem solving”. She adds :

> Il s’agit de faire comprendre aux élèves qu’en géométrie, ils doivent s’aider de ce qu’ils voient pour avoir des idées dans la résolution d’un problème de géométrie, mais qu’ils n’ont plus le droit d’utiliser ce qu’ils voient quand il s’agit de raisonner « rigoureusement » et doivent alors se cantonner au niveau théorique.[1]) [Laborde 2003, 145]

One may wonder whether the role of geometric figures (drawings) is limited to the heuristic activity. Do there not exist particular figures which, because they represent a situation with few degrees of freedom, offer clearly to the sight, as well as to the mind, the variants, infinite in number, of this situation ? Should the well known Indian proof of the Pythagoras theorem, as sketched in Fig. 3, be reformulated away from the figure ? Or is it not a quality of this proof that it demands no such formal expression ?

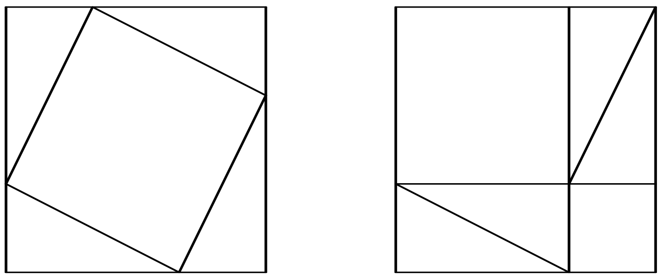

FIGURE 3

This type of observation is not proper to geometry. To mention just another example, an arithmetic property is sometimes given a proof which is expressed by considering particular numbers. Such a proof is said to be ‘paradigmatic’. This procedure works only when the particular situation refers unambiguously to all the other situations which have to be encompassed, but if this is the case, the procedure works and satisfies the most demanding mind.

[1]) The duty of teachers is to let the students understand that, although in geometry they have to rely on what they see to get ideas on how to solve a problem, they no longer have the right to utilise what they see as soon as the question is to reason “rigorously”; and then they must confine themselves to the theoretical level.

Vygotski once wrote : "The analysis of reality with the help of concepts occurs much sooner than the analysis of the concepts themselves" [1934 (1997)].

Should we not admit that the analysis of reality with the help of concepts is a fundamental activity to be exercised without end, in particular during all the school years, and that the analysis of concepts, which is much more difficult, comes along when the analysis of reality with the help of concepts encounters difficulties ?

I would like to conclude by emphasizing the inventory proposed by Geoffrey Howson of the conditions of success of a curriculum reform. There are conditions of various orders, and each of them is probably necessary but certainly insufficient, and often also beyond our competence. In this perspective, I recall a quotation of L. Henkin which could be read about ten years ago on the front page of an important Belgian report on mathematical teaching :

> Changing the mathematical teaching system is a long run. Those who will engage in that and try to complete it in a decennium will not succeed and will be entirely disappointed. Changing the education system, I know for sure that mathematicians cannot do it, teachers cannot do it, politicians cannot do it, it demands a global cooperation effort. [Ministère de l'Éducation 1990]

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GENERAL DISCUSSION

(reported by Marta MENGHINI)

The discussion about geometry following Nicolas Rouche's reaction divided into two areas, namely :

- the historical motivations that determined the reform movements in Europe and the United States during the 20th century, particularly in the second half of the century, leading to the reduction or elimination of classical Euclidean geometry from the curriculum ;
- the didactic and cognitive links between the perception of geometric properties and their proof.

These two areas are closely related, since the latter was a major motivation for changes in the school mathematics curriculum, particularly in the second half of the 20th century. The comments of the participants in the discussion, however, are summarised here separately.

HISTORICAL AND CULTURAL PERSPECTIVES. Geoffrey Howson and André Revuz both addressed the first point, reflecting the situations in Great Britain and France respectively. The rationale for eliminating classical Euclidean geometry was apparently the same in both countries. To begin with, there was a general feeling that things were not quite right with education in general, and the same impression prevailed also in other countries. Then there was an awareness of the difficulties inherent in presenting proofs at school level, particularly the difficulty of deciding where a proof begins and how much is to be assumed. This is sometimes referred to as “the starting-point problem”. Commonly, axioms were perceived by students as “absolute truths” rather than as reasonable starting points. Yet, the rules of procedure were often not stated clearly enough to allow “playing the game” of proving.

However, the reactions to the situation were different in the two countries. England and Wales chose a path which, by chance, was partly similar to that recommended by Dieudonné. The approach to the reform of school mathematics was not axiomatic in nature and, in fact, has sometimes been considered to be in the style of physics rather than mathematics.

In France, initially, the problems of teaching mathematics were answered institutionally by setting up the IREM [1]) groups, charged with the responsibility of both in-service teacher training and research into mathematics teaching practice. Only later did there arise a demand for major changes to the curriculum, which then adopted a Bourbakist approach, following the changes that were already under way in Belgium. The answer to the problem of axioms was to abandon the Euclidean postulates entirely and replace them by new axioms together with a different way of introducing them.

During the same period in Italy there were no official changes in the mathematics curriculum at all. In fact, the debate about geometry teaching had been initiated at the beginning of the century. At that time it focused on the interaction between empiricism and axiomatics. Aldo Brigaglia commented that this did not represent a contraposition of views. Peano, in fact, had been a supporter of empiricism, although he was rigidly axiomatic in his expositions. On the other hand, Cremona was not particularly keen about axiomatics in research, but his geometry was abstract, not empirical. Finally, Enriques' axioms for teaching had a psychological genesis. During the '60s and '70s certain aspects of the European mathematics teaching reforms were adopted in Italy, by means of integrating Choquet's axiomatics (which represented a way of introducing measurement and geometrical transformations) into the classical structure of geometry. The Italian cultural tradition did not allow for a disappearence of congruence and similarity criteria for triangles, as had occurred during a period of thirty years in France, where they were replaced for ten years by linear algebra and then by the more empirical geometric transformations.

DIDACTIC AND COGNITIVE ASPECTS OF GEOMETRIC PROPERTIES AND THEIR PROOF. Several speakers pointed out that today the attitude about the teaching of geometry has changed, particularly regarding the relationship between intuition and proof. Euclidean geometry without tools is not the same thing as Euclidean geometry with tools, and undoubtedly DGS (*Dynamic Geometry*

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[1]) *Institut de recherche sur l'enseignement des mathématiques*

Software) has a role in the revival of this debate. The topic generated a lively discussion, often in the form of open questions, such as :

- how is it possible to obtain a proof after visualizing a property with DGS ?
- is there a dichotomy between the figure and the formal explanation ?
- are axioms the bridges between visualization and proof ?

But there were also questions of a more fundamental nature, such as :

- why, more generally, is it necessary to provide a proof for a property that looks evident ?
- what leads us to abstraction ?
- what is perceptive understanding with respect to proof ?

Underlying the discussion was the recognition of the deductive nature of proof and its presence in a number of disciplines. Perception without (deductive) proof is unsatisfactory and some speakers held that geometry was the logical analysis of our perception of space. The question remains, however, as to what draws us to this analysis ? What is fundamental to the problem is not the existence of the two aspects of understanding — intuition and proof —, but the link between them. Without such a link, there would be two different disciplines : on the one hand the figure, reasoning on the other ; but then mathematics — which is based entirely on reasoning — would seem to shun empiricism.

Colette Laborde expressed the opinion that a dichotomy between these two aspects should not be made, and that there is, on the contrary, a dialectic. Geometry cannot be done without both aspects. But for the purpose of analysing the problems that students encounter when doing geometry, it is interesting, and fruitful, for the observer to make a distinction between diagrams and theory. On the other hand, the nature of the dialectic between reasoning and empiricism needs to be explored. Gila Hanna, for example, pointed out that it is not always the case that in visualization we see that something is true, and in abstraction we explain why it is true. Abstraction can also be used to show that something is true, without ever showing why it is true : there are proofs that do not mention anything about explanations.

On the other hand, the example of a "proof without words" given by Nicolas Rouche cannot be brought as an example of uselessness of a formal explanation. Such diagrams, inherently implying a proof, do not need a formal proof for a mathematician, because mathematicians are experts and are able to extract the relevant properties from the drawing. However, it cannot be assumed that 12–13 year-old children would be able to understand. Also other types of proof, such as proofs by contradiction, question the role of visualization, since it is necessary to use theory in these cases. As against that, visualization is certainly a powerful tool. In using DGS, for example, students are convinced, in less time than with paper and pencil, of the non-existence of an object (for example of a triangle with two perpendicular bisectors).

Many speakers were of the opinion that new developments in dynamic geometry would lead more and more towards aiding the proof of geometrical results. Nonetheless, we still need to bear in mind that tools are only tools, and that other questions about mathematics may arise (such as the interplay between the continuous and the discrete). In this sense, DGS has more mathematics embedded in it than ruler and compasses alone. Finally, tools should be evaluated not for their perfection, but mainly for their appropriateness for teaching.

ANALYSIS

L'ENSEIGNEMENT DU CALCUL DIFFÉRENTIEL ET INTÉGRAL AU DÉBUT DU VINGTIÈME SIÈCLE

The teaching of calculus at the beginning of the twentieth century

by Jean-Pierre KAHANE

An international conference of ICMI was held in Paris in April 1914. A detailed account was published in *L'Enseignement Mathématique*. One item in the conference agenda was the inquiry that ICMI had conducted all over Europe on the results obtained from introducing calculus in the upper grades of secondary institutions.

In the first section, we discuss the rather optimistic report written by E. Beke [1914], which talks about mathematics as "an instrument of progress". In spite of its eloquent conclusion on the usefulness of calculus in the perspective of promoting the study of "scientific humanities", the situation in the various countries was quite diverse, and the divisions were even amplified in the aftermath of the First World War.

The second section is devoted to the context of calculus teaching, i.e. exercices, text-books, and teacher training. We look more specifically at the situation in France and in Hungary.

In two appendices we reproduce two specimens of examinations taken by students at the Sorbonne, resp. in the upper grade of the *lycée*, around 1900.

L'ENSEIGNEMENT DU CALCUL DIFFÉRENTIEL ET INTÉGRAL AU DÉBUT DU VINGTIÈME SIÈCLE

par Jean-Pierre KAHANE

Cette communication est divisée en deux parties. La première expose la situation et les thèses en présence d'après le compte rendu de la conférence internationale d'avril 1914. La source principale est le rapport d'E. Beke [1914]. J'évoquerai aussi les contributions d'E. Borel et de Ch. Bioche, et la discussion. La seconde partie est consacrée au contexte (exercices, livres, formation des enseignants), en insistant sur la France et la Hongrie. En complément, on trouvera des exemples de ce que l'on pouvait attendre des élèves à l'époque.

1. La conférence internationale d'avril 1914

Cette conférence internationale de l'enseignement mathématique, soigneusement préparée, se tint à Paris du 1er au 4 avril 1914. Un compte rendu détaillé en est donné dans *L'Enseignement Mathématique* [Beke 1914]. La conférence rassemblait plus de 160 participants venant de 17 pays ; les pays les plus représentés sont la France (82), la Hongrie (15), l'Allemagne (14), la Suisse (12), la Russie (10). L'ordre du jour comprenait deux points :

A) les résultats obtenus dans l'introduction du calcul différentiel et intégral dans les classes supérieures de l'enseignement moyen, et

B) la place et le rôle des mathématiques dans l'enseignement technique supérieur.

C'est du point A) qu'il sera question ici.

Le discours inaugural de G. Castelnuovo, parlant au nom de Felix Klein, président de la CIEM, empêché, indique les raisons du choix de Paris comme siège de la conférence :

> De la France est partie en 1902 l'initiative d'une réforme organique de l'enseignement traditionnel des écoles moyennes. Le plan d'études de cette époque a introduit d'une manière systématique, avant les autres pays, les notions de dérivées et de fonctions primitives dans les programmes des lycées. Il est donc naturel de constater ici ce que l'expérience de dix ans a pu enseigner à ce sujet. [Castelnuovo 1914, 190]

Émile Borel, l'un des artisans de cette réforme de 1902, fait la conférence inaugurale sur le thème de « l'adaptation de l'enseignement secondaire aux progrès de la science » [Borel 1914]. Il constate d'abord que, pour toutes sortes de raisons, « l'enseignement secondaire ne peut évoluer que très lentement », et il insiste :

> Toute modification trop brusque ou trop considérable risque d'être fâcheuse pendant un temps assez long ; on peut même affirmer d'une manière presque absolue que toute modification est tout d'abord nuisible et, pendant la période d'adaptation, entraîne plus d'inconvénients que d'avantages. [Borel 1914, 201]

Cependant l'enseignement secondaire n'est pas immuable, il évolue, dans toutes les matières, et les mathématiques ne sauraient être à part :

> […] les changements doivent être lents ; mais peut-être n'est-il pas excessif de penser qu'il est aussi absurde pour le professeur de mathématiques de l'enseignement secondaire de paraître ignorer Galilée, Descartes, Newton et Leibniz qu'il le serait pour le professeur de chimie d'ignorer Lavoisier, ou pour le professeur d'histoire de négliger la Révolution française. [Borel 1914, 207]

Au surplus, la géométrie analytique et le calcul différentiel et intégral sont, au moins dans leurs éléments, plus proches de l'expérience commune que beaucoup de mathématiques dites élémentaires. Les lecteurs de journaux, lorsqu'ils regardent un graphique,

> font de la géométrie analytique sans le savoir ; parfois même, en discutant sur la rapidité plus ou moins grande des oscillations de ces graphiques et sur les conséquences qu'on peut en tirer, ils font, sans le savoir, du calcul différentiel et du calcul intégral. [Borel 1914, 206]

Enfin, dit Émile Borel, la valeur éducative des matières nouvelles ne le cède en rien aux matières anciennes, même si « la masse des professeurs ne peut arriver du premier coup à une technique pédagogique aussi bonne pour les matières nouvelles que la technique traditionnelle l'était pour les anciennes ».

En résumé, « tout changement est mauvais pendant qu'on le réalise et […] un changement, s'il n'est pas absurde, devient bon une fois qu'il est réalisé depuis un certain temps » [Borel 1914, 210].

C'est le rapport d'E. Beke, professeur à l'université de Budapest, qui constitue la charpente du point A) de l'ordre du jour. Il avait été préparé par un questionnaire et par des rapports partiels venant d'Allemagne, d'Australie, d'Autriche, du Brésil, du Danemark, des États-Unis, de France, de Hollande, de Hongrie, des Iles Britanniques, d'Italie, de Norvège, de Russie, de Serbie et de Suisse. *L'Enseignement Mathématique* en publie un résumé [Beke 1914, 222–225], le texte complet [Beke, 245–282] et le questionnaire qui l'a préparé [Beke, 283–284]. J'en dégagerai seulement quelques-uns des points saillants.

L'introduction du rapport est vibrante d'optimisme. Les mathématiques sont « la langue naturelle de la pensée » et « l'instrument du progrès » ; « on attend des sciences plus d'effet que par le passé, pour la formation des esprits ». Et surtout ceci, qu'on ne peut lire sans émotion :

> Je crois pouvoir affirmer, sans crainte de me tromper, que c'est un haut idéal d'internationalisme qui nous a réunis ici. Nous avons senti que l'éducation de la jeunesse n'a pas seulement pour but de former, d'accroître et de maintenir les forces vives d'une nation et l'esprit national, de doter du patrimoine commun les ouvriers actifs de la civilisation nationale ; elle a aussi la tâche encore plus noble de créer et de faire vivre un idéal commun à toute l'Humanité. [Beke 1914, 246]

Nous sommes le 2 avril 1914. Dans quelques mois, la guerre.

La conclusion du rapport est également éloquente, sur la nécessité de promouvoir des « humanités scientifiques » et sur la place dans ce cadre de l'objet du débat, le calcul différentiel et intégral : il faut

> faire répandre dans le cercle le plus large possible, parmi tous les hommes qui cultivent la Science, la connaissance du Calcul infinitésimal qui est la Science du changement, principe éternel du monde, qui est l'instrument indispensable de tout raisonnement scientifique et qui, enfin, représente une création magnifique de l'esprit humain. [Beke 1914, 282]

Allons directement au questionnaire. Il concerne les lycées, les gymnases classiques ou *réaux*, les établissements similaires des divers pays et aussi, en principe, les écoles normales d'instituteurs (mais il n'y a pas de réponse de ce côté).

1) Qu'a t-on introduit ? Le calcul différentiel est-il limité aux fonctions d'une variable ? Quelles fonctions étudie-t-on ? *Quid* du calcul intégral ? de la formule de Taylor ? des équations différentielles ?

2) Quel est le degré de rigueur ? Se contente-t-on d'une introduction géométrique ? Utilise-t-on la notion de limite ? Démontre-t-on des théorèmes sur les limites, tels que $\lim(1/a) = 1/\lim a$? Fait-on usage des différentielles, et comment ? Fait-on intervenir le reste dans la formule de Taylor ? Parle-t-on de fonctions non-dérivables ? Introduit-on rigoureusement les nombres irrationnels ?

3) Quelle est la méthode ? A-t-on déjà vu des fonctions simples et leur représentation graphique ? Emploie-t-on la notation de Leibniz ? Commence-t-on par le calcul différentiel ou par le calcul intégral ? Comment introduit-on l'intégrale : intégrale définie (limite d'une somme) ou intégrale indéfinie (primitive) ? Quels manuels, quels ouvrages ?

4) Quelles applications ? Par exemple, *quid* des maxima et minima ? des développements en séries entières ? de l'interpolation, de l'extrapolation, du calcul des erreurs ? du calcul des aires et des volumes ? de la mécanique (vitesse, accélération, travail, moments d'inertie, etc.) ? de la physique, en particulier pour l'optique et l'électrodynamique ?

5) Quels allègements par ailleurs ?

6) Quel accueil pour la réforme ?

Le rapport de Beke donne une analyse détaillée des réponses. Ce qui frappe est la diversité des méthodes, en particulier dans l'introduction de l'intégrale. En général cependant, on se borne aux fonctions d'une variable, on préfère la notation de Lagrange à celle de Leibniz, on ne parle pas de fonctions non-dérivables, et les nombres irrationnels ne sont introduits que sur des exemples (extraction de racines). La série de Taylor figure dans peu de programmes. Les applications principales sont la recherche des extrema et les calculs d'aire et de volume.

Mais ce résumé de résumé doit être complété, immédiatement, par un aperçu sur la diversité des situations. En Allemagne, les réponses sont différentes selon qu'elles proviennent de Bavière, Wurtemberg, Bade et Hambourg, où les éléments du Calcul infinitésimal sont au programme des écoles, ou de Prusse et de Saxe où ils se trouvent enseignés sans être au programme. Un cas extrême est celui de Hambourg. Les dérivées y sont enseignées depuis 1874, et aussi, à tire optionnel, les intégrales. En 1897, les intégrales font partie du programme également. En Hongrie, depuis 1899 la plupart des lycées enseignent les dérivées et les intégrales ; depuis longtemps étaient dans les programmes la géométrie analytique, les représentations graphiques de fonctions simples, les maxima et minima. Les Anglais insistent sur la qualité de leurs livres. En Italie, rien ; pas d'analyse dans les lycées (cependant qu'avec Dini, Peano, Volterra et d'autres, l'Italie joue un rôle phare en analyse au plan de la recherche). En Roumanie, plusieurs variables. En Suisse, une particularité mérite attention. Voici une résolution de l'Association suisse des professeurs de mathématiques, en 1904 :

> La notion de fonction et les problèmes fondamentaux qui s'y rattachent *appartiennent* au programme de l'enseignement mathématique des écoles moyennes. [Beke 1914, 256]

C'est moi qui ai signalé "appartiennent". Ils ne disent pas "devraient appartenir", en s'adressant à une instance supérieure. Ils *décident* [1]).

Je n'ai pas encore parlé de la situation en France. Beke en parle abondamment, et c'est l'objet d'un rapport spécial de Charles Bioche [1914], professeur au lycée Louis le Grand.

Ce qui est le plus spécial en France, c'est l'existence des classes de Mathématiques spéciales. Le calcul intégral est renvoyé à ce niveau. Au lycée, on se borne donc, avant le baccalauréat, au calcul différentiel : c'est plus qu'en Italie, mais beaucoup moins qu'à Hambourg ou Budapest. La réforme de 1902 a introduit les dérivées en classe de seconde (14–15 ans), mais la récente réforme de 1912 reporte cette introduction à la classe de première (15–16 ans). Les dérivées des fonctions trigonométriques $\sin x$, $\cos x$, etc. sont renvoyées en classe de Mathématiques (16–17 ans).

Bioche donne des exemples d'épreuves de mathématiques proposées dans les classes de première et de Mathématiques. J'y reviendrai à la fin de cet article.

La discussion apporte des indications complémentaires où éclatent les différences entre les différents pays. Dans l'ensemble, les participants approuvent la réforme. Une forte voix discordante est celle du mathématicien autrichien R. Suppantschitsch, qui écrit [2]) : « Die Frage, ob die Einführung der Infinitesimalrechnung einstimmig als ein entscheidender Fortschritt zu betrachten sei, ist mit NEIN zu beantworten. » (Est-ce un progrès ? NON !)

Et il ajoute : « Ich halte daher schon jetzt die Frage für sehr diskutierbar, ob die Infinitesimalrechnung AUS der Schule nicht wieder verschwinden soll. » (Il est temps de discuter de renvoyer le calcul infinitésimal DEHORS.)

L'opinion de Suppantschitsch apparaissait alors comme extrêmement minoritaire. C'est elle, cependant, qui triompha en France après la guerre de 1914–1918.

La fracture de la guerre a été non seulement la ruine des belles espérances exprimées par Beke, mais un sévère coup d'arrêt à la réflexion sur l'enseignement des mathématiques. La CIEM déchirée, divisée, ne s'en est jamais remise.

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[1]) On trouvait une situation analogue en Allemagne, comme décrit dans [Schubring 2003, 53].

[2]) *L'Enseign. Math. 16* (1914), 305.

2. Contexte, commentaires et compléments

Le début du vingtième siècle est une période faste pour la recherche mathématique. De jeunes mathématiciens apparaissent sur le devant de la scène : Borel, Baire, Lebesgue, Fejér, Riesz, pour m'en tenir à l'analyse et à la France ou la Hongrie.

L'enseignement supérieur suit très inégalement. C'est en France qu'il manifeste le plus de retard. Les épreuves de calcul différentiel et intégral à la Sorbonne, comme celles d'analyse supérieure, sont intéressantes et difficiles, mais ne se distinguent guère, jusque vers 1950, de celles qui étaient proposées à la fin du 19e siècle. Par exemple, l'intégrale de Lebesgue en est absente. C'est en Russie, en Pologne, en Hongrie, que l'analyse telle qu'elle est issue des travaux de Borel, Baire et Lebesgue sera enseignée et popularisée.

Cependant l'édition mathématique est florissante, à tous les niveaux. Le *Jahrbuch über die Fortschritte der Mathematik* en donne une image au plan mondial. En France existent d'excellents ouvrages d'enseignement, et, parmi eux, des livres d'exercices à l'intention des élèves et des professeurs.

L'analyse y apparaît souvent à l'ombre de l'algèbre. Carlo Bourlet a publié en 1896 ses *Leçons d'algèbre élémentaire* [Bourlet 1896]. Il y introduit les dérivées et les applique à l'étude de la variation des fonctions, avant de passer aux logarithmes et aux intérêts composés (façon de concevoir l'exponentielle), puis, en appendice, d'introduire les nombres complexes et l'étude des fonctions circulaires. Plus de 250 exercices accompagnent ces *Leçons*.

Bien plus tard, en 1915, un *Précis d'algèbre pour les classes de première C et D*, de J. Girod [1915], comprend « les formules générales pour le calcul des dérivées » et une foule d'applications. Par exemple [chap. 6, théorème 3] : « si $x > 0$, $y > 0$ ont une somme constante, le produit $x^p y^q$ est maximum quand x et y sont proportionnels à leurs exposants ». Les 200 exercices que contient l'ouvrage sont en majorité des sujets d'examen donnés au baccalauréat : beaucoup de questions d'analyse présentées comme problèmes de géométrie.

L'analyse apparaît sous son nom dans un livre de Baire de 1907, *Leçons sur les théories générales de l'analyse* [Baire 1907]. C'est un excellent ouvrage, très rigoureux, mais qui s'arrête aux fondements.

L'analyse classique, celle qui s'enseigne dans les universités, fait l'objet d'un recueil, par E. Fabry, en 1913, de « problèmes d'analyse mathématique » [Fabry 1913]. Le livre comprend 279 énoncés et solutions. Les rubriques sont : quadratures, intégrales multiples, fonctions analytiques, équations différentielles, courbes planes, surfaces et leurs lignes remarquables, équations différentielles et leurs applications géométriques, différentielles totales. Cela donne une idée de la culture mathématique des professeurs de lycée les plus solidement formés.

Il est intéressant de comparer la contribution des mathématiciens les plus éminents à la formation des professeurs de lycée dans trois pays bien représentés à la conférence de Paris : la France, l'Allemagne et la Hongrie.

En France, l'exemple est Jacques Hadamard, avec ses excellentes *Leçons de géométrie élémentaire* [Hadamard 1898/1901]. Il s'agit bien de leçons adressées aux professeurs, mais en principe elles s'adressent aussi aux élèves. Le jeu est bien de faire un exposé élémentaire de la géométrie. Cela sera le parti pris également de Lebesgue, bien plus tard, dans ses cours sur les coniques. Ouvrages séduisants, ingénieux, et délibérément éloignés du mouvement des mathématiques contemporaines.

En Allemagne, Felix Klein s'adresse aux professeurs à partir d'une conception très différente. Il s'agit d'aborder les mathématiques élémentaires d'un point de vue moderne : *Elementarmathematik vom höheren Standpunkte aus* [Klein 1908/1909]. Par exemple, pourquoi y a-t-il des nombres transcendants ? Parce que l'ensemble des nombres algébriques est dénombrable, et que l'ensemble des nombres réels ne l'est pas ; c'est la démonstration de Cantor. Felix Klein s'attache à éclairer le champ du point de vue le plus élevé possible. Son influence internationale est considérable — il préside la CIEM, il a été l'initiateur de l'Union mathématique internationale. Mais surtout, son audience en Allemagne est immense. Son jubilé à Göttingen, en décembre 1918, sera la première manifestation scientifique en Allemagne au sortir de la guerre.

Revenons à Hadamard. Il est plus jeune que Klein, mais c'est un mathématicien de la même envergure. Lui aussi sera, plus tard, président de la CIEM. C'est un passionné d'enseignement. Mais, pour lui, il y a rupture complète entre sa recherche et l'enseignement. Klein, lorsqu'il s'adresse aux enseignants, s'inspire le cas échéant de son *programme d'Erlangen*. Hadamard, qui préside la séance où parle Beke, est le spécialiste mondial de la série de Taylor et de son prolongement analytique. Y a-t-il quelque chose à en tirer pour l'enseignement ? « Non, dit-il, la série de Taylor n'a pas sa place au lycée », et cela clôt le débat.

En Hongrie, le lien entre recherche et enseignement a pris des formes originales. Il existe une compétition à la fois très sélective et très populaire dans le milieu mathématique, ouverte aux étudiants qui entrent à l'Université. C'est la compétition Schweitzer, qui existe toujours, et dont les organisateurs comme les lauréats sont parmi les meilleurs mathématiciens du pays. Il existe une revue destinée aux professeurs de l'enseignement secondaire et aux étudiants, rédigée par les mathématiciens les plus créatifs. C'est *Matematikai Lapok*. Leopold Fejér (Fejér Lipot), qui a passé sa thèse en 1902, introduit une pratique à laquelle il sera fidèle sa vie durant : tout ce qu'il publie en français ou en allemand est également publié en hongrois. Tout cela crée une relation très forte entre chercheurs, professeurs et étudiants, et le hongrois est une langue en état de marche au plan scientifique autant que les grandes langues de communication. Le poids qu'a E. Beke dans la conférence, ce n'est pas seulement d'avoir dressé un état des lieux clair et complet ; c'est aussi de représenter à la conférence une tradition remarquable de *liaison entre recherche, enseignement, et culture nationale.*

Pour conclure, il ne me semble pas mauvais de montrer ce qui était demandé aux étudiants d'université et aux élèves de lycée en France au cours des années 1900. On trouvera donc ci-après les sujets d'examen en calcul différentiel et calcul intégral à Paris en 1901. Comme je l'ai déjà dit, le type de sujets n'a pas beaucoup varié pendant cinquante ans. Mais il n'est pas évident que les étudiants de licence d'aujourd'hui sachent les traiter (annexe 1). L'annexe 2 est empruntée à l'exposé de Ch. Bioche. C'est le sujet d'une composition de mathématiques, à traiter en 2 heures et demie, pour des élèves de la classe de Mathématiques (16–17 ans). La forme de l'énoncé montre que les élèves sont habitués à représenter certaines figures de révolution par leur section méridienne. Peu de connaissances sont nécessaires ; les candidats bacheliers d'aujourd'hui sauraient-ils les mobiliser ? Mais n'est-il pas vrai qu'à chaque époque les performances des aînés apparaissent à la fois comme un peu étranges et indépassables ?

ANNEXE 1

Examens de la Sorbonne, juillet 1901

Certificat de Calcul différentiel et Calcul intégral

Épreuve écrite (2 juillet, de 8h. à midi)

1° Une famille de courbes gauches (T) est représentée, dans un système d'axes rectangulaires, par les équations

$$x^2 + 2y^2 = \alpha z^2$$
$$x^2 + y^2 + z^2 = \beta z$$

où α et β représentent deux constantes arbitraires. On demande 1) de démontrer que ces courbes sont les trajectoires orthogonales d'une famille de surfaces (S), dépendant d'un paramètre arbitraire, et de trouver l'équation générale de ces surfaces ; 2) de prouver que les sections des surfaces (S) par des plans passant par Oz sont des lignes de courbure de ces surfaces et de trouver la seconde famille de lignes de courbure.

2° Soit $R(x, y)$ un polynôme entier à deux variables x, y, de degré $p+q-1$ au plus par rapport à l'ensemble des deux variables. Démontrer que l'intégrale double

$$I = \iint R(x,y)\,\frac{\partial^p((x^2-a^2)^p)}{\partial x^p}\,\frac{\partial^q((y^2-b^2)^q)}{\partial x^q}\,dx\,dy\,,$$

étendue à l'intérieur du rectanle limité par les droites $x = +a$, $x = -a$, $y = +b$, $y = -b$ est nulle, quels que soient les coefficients du polynôme $R(x, y)$.

Épreuve pratique (2 juillet, de 2h$\frac{1}{2}$ à 3h$\frac{1}{2}$)

Calculer l'intégrale définie

$$\int_0^1 \frac{\sqrt[3]{x^2(1-x)}}{(1+x)^3}\,dx$$

soit par des moyens élémentaires, soit en se servant de la théorie des résidus.

Examinateurs : MM. Goursat, Picard, Hadamard.

Sur 37 candidats, 13 ont subi les épreuves orales et 11 ont été reçus.

ANNEXE 2

Composition de mathématiques
(classe de *Mathématiques*, 2h.30)

On considère le solide formé par un cône SAA' et un cylindre $ABB'A'$ ayant la même longueur de génératrice : $SA = AB = a$. Soit x la hauteur SH du solide.

1º Exprimer le volume V du solide au moyen de a et de x.

2º Trouver pour quelles valeurs de x le volume V est maximum. Calculer ce maximum en hectolitres dans le cas où $a = 1\,\mathrm{m}$.

3º Construire la courbe qui représente les variations de la fonction

$$y = \frac{3V}{\pi a^3}$$

en représentant par a l'unité de longueur graphique.

4º Calculer l'aire comprise entre la courbe et la corde joignant le point d'abscisse 1 et le point d'abscisse 2.

5º Déduire de la considération de la courbe combien il y a de valeurs de x pour lesquelles y prend une valeur donnée. Calculer les valeurs de x qui correspondent à $y = 3$.

[Bioche 1914, 287–288]

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LEARNING AND TEACHING OF ANALYSIS IN THE MID TWENTIETH CENTURY : A SEMI-PERSONAL OBSERVATION

L'étude et l'enseignement de l'analyse au milieu du vingtième siècle : une observation semi-personnelle

par Man-Keung SIU

Cet article présente une observation semi-personnelle sur l'enseignement de l'analyse au milieu du 20e siècle. Au lieu de compter uniquement sur les expériences glânées pendant ses années scolaires, l'auteur rassemble aussi des informations à partir d'articles parus dans *L'Enseignement Mathématique* durant la décennie 1955–65, de rapports pertinents sur les programmes scolaires et universitaires, de quelques manuels d'un usage commun au milieu du 20e siècle, et de correspondances et interviews effectuées par l'auteur avec des mathématiciens de différents pays, qui étaient soit étudiants soit jeunes professeurs dans les années 50 ou au début des années 60.

Cet article commence par décrire les mathématiques à l'école pendant les années 50 et le cadre de connaissances des lycéens qui se lançaient dans l'étude de l'analyse, matière habituellement dénommée *calcul infinitésimal* dans la plupart des pays. Le calcul différentiel et intégral a été introduit au lycée au début du 20e siècle. Son programme s'est plus ou moins stabilisé avant les années 50 en un enseignement généralement intuitif et dénué de rigueur mathématique, mettant l'accent sur les calculs mécaniques plutôt que sur la compréhension conceptuelle. En se basant sur sa propre expérience, l'auteur décrit le plaisir qu'il avait à résoudre des problèmes de calcul infinitésimal sans comprendre vraiment les théories sous-jacentes. Ainsi, bien que le *Théorème Fondamental de l'Analyse* fût expliqué dans le manuel, l'auteur n'était pas conscient de l'importance de ce beau résultat à cette époque-là. Sans avoir saisi le concept d'une intégrale définie comme limite d'une certaine somme, il lui était même difficile d'utiliser le calcul intégral pour résoudre autre chose que des problèmes stéréotypés. Malgré cette compréhension imprécise (mais avec une compétence technique raisonnablement suffisante), l'auteur était fasciné par le calcul différentiel et intégral à l'école secondaire.

La discussion se poursuit sur la situation de l'analyse dans le mouvement des "mathématiques modernes". L'introduction de sujets non traditionnels et l'insistance sur l'abstraction et les structures dans ce courant de pensée ne semblent pas avoir affecté fondamentalement l'enseignement de l'analyse en tant que discipline scolaire, ce qui signifie qu'un difficile problème pédagogique restait difficile pendant ce mouvement, … et aussi après. Il est vrai que, durant cete période, l'analyse était enseignée à des élèves de plus en plus jeunes en classes de niveau de plus en plus bas, que l'emploi d'ensembles et d'applications était adopté pour la poursuite de l'exactitude mathématique, et qu'il existait une tendance à l'abstraction. Mais, même sans l'emploi d'ensembles ni d'applications, même sans la forte dose d'abstraction, les élèves trouvaient l'analyse difficile parce qu'ils ne comprenaient pas les concepts mis en jeu et donc ne pouvaient pas saisir la *signification* de cette matière.

La transition de l'école à l'université posait (et pose encore aujourd'hui) des problèmes même plus graves. Par exemple :

a) Dans quelle mesure une transition en douceur est-elle en rapport avec une formation "classique" en géométrie ? Non seulement la connaissance de la géométrie à l'école d'autrefois permettait à l'élève de s'habituer avant ses études universitaires à la notion de démonstration et de logique, mais la géométrie en elle-même est aussi une matière avec laquelle on peut s'exercer à la discipline logique et *en même temps* à l'imagination libre. Développer une sympathie pour la géométrie permet aussi à l'élève de considérer des problèmes dans d'autres domaines avec un point de vue géométrique.

b) Dans quelle mesure le mouvement bourbakiste a-t-il influencé l'enseignement universitaire de l'analyse, bien qu'il semble n'avoir eu que peu d'influence sur les mathématiques à l'école ?

En ce qui concerne l'enseignement de l'analyse à l'école ou à l'université du 21e siècle, que peut-on apprendre en faisant référence au passé ? Les questions contemporaines relatives à l'étude de l'analyse, comme l'emploi de la technologie moderne, la pertinence de cette matière dans la vie quotidienne, ou son rapport avec la technologie informatique, n'étaient pas à l'ordre du jour au milieu du 20e siècle. Cependant, les difficultés rencontrées par l'élève à cette époque-là sont toujours les difficultés rencontrées par l'élève d'aujourd'hui. Ce qui est à l'origine de ces difficultés, c'est la controverse perpétuelle entre les techniques et la compréhension, entre le concret et l'abstraction. Cette controverse, qui malheureusement se développe parfois en une fausse dichotomie stérile, était un thème retentissant au siècle dernier à partir de l'introduction du calcul infinitésimal dans les programmes scolaires jusqu'à la nouvelle réforme qui a eu lieu dans les années 80 et 90.

L'auteur conclut par une remarque sur l'emploi du terme "analyse" au cours de l'histoire : on peut y voir comme un rappel de l'unité fondamentale des quatre matières principales : analyse, arithmétique, algèbre et géométrie, unité que, nous l'espérons, nos élèves peuvent comprendre et apprécier.

LEARNING AND TEACHING OF ANALYSIS IN THE MID TWENTIETH CENTURY: A SEMI-PERSONAL OBSERVATION

by Man-Keung SIU

1. A (SOMEWHAT PERSONAL) PROLOGUE

When I was approached by the organizers of the Symposium as a prospective speaker, I certainly felt honoured but at the same time surprised and timorous. As students in mathematics it is true that we all study analysis; and as teachers in mathematics we have all taught some courses in calculus or analysis. But I tend to view myself just as a plain mathematics teacher who believes in the value of the cultural and historical dimensions of this discipline in general education.

So, this paper presents a semi-personal observation of the learning and teaching of analysis in the mid 20th century. Besides relying on the learning experience of my school days in the 1960s, I also gathered information from papers published in *L'Enseignement Mathématique* during the decade 1955–65, from relevant reports[1]) on the school and university mathematics curriculum, from some textbooks in common use in the mid 20th century, and from correspondence or interviews with some mathematicians of different nationalities who were either students or young teachers in the 1950s and early 1960s.

It must be admitted that this part of the project is not carried out systematically nor scientifically and tends to be anecdotal. But it has a certain interest because it gives a non-Western perspective. Papers in *L'Enseignement Mathématique* usually recount learning and teaching in European countries,

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[1]) The following are particularly pertinent: [ICMI/UNESCO 1967–79], [UNESCO 1985], [Artigue 1991; 1996], *Proceedings* of International Congresses of Mathematicians (ICMs) and International Congresses on Mathematical Education (ICMEs) held between 1950 and 1980.

sometimes also in North American countries, but very seldom in Asia. When I was a school pupil, the tidal wave of the ‘new math’ movement had not yet swept over Hong Kong, so I was brought up in a more traditional curriculum, modelled after the British system, sometimes with a time lag, as Hong Kong was in those days a British colony. I began undergraduate study in 1963. It was in the university that I first felt the impact of modern mathematics.

2. Learning and teaching of analysis in the 1950s and 1960s

What was school mathematics like in the 1950s ? There is no dearth of reports on this question even if we confine our search to *L'Enseignement Mathématique*. Instead of going over them one by one, let me just quote a passage from a report by Howard F. Fehr :

> This, then is the picture of what the pupil has been taught. What does he really know ? This is hard to tell, but it can be said that the 15 year old in all countries, who has continued his study of mathematics through the first 9 or 10 years of school can compute in a mature manner with the positive rational numbers, in a decimal system of notation, even though he cannot rationalize what he does ; he has a fairly useful and practical knowledge of geometry with respect to mensuration and common relationships ; and he can manipulate algebraic expressions and solve equations and problems in a structureless system of algebra. He can make simple deductions, but his entire concept of proof, if any, is limited to that of theorems in geometry. He really does not know what mathematics is, or how it is applied, but he has a large body of information, upon which, if he is inclined or interested, a study of mathematics can be built in the ages 16 years to 21 years. [H. F. Fehr 1959, 66–67]

(As an aside, from what I observe of my own students today, most of them still do not know what mathematics is, but they have a far more meagre body of information mastered ; so, they experience difficulty in continuing their study of mathematics.) Fehr continues with the aims of mathematical instruction labelled as : 1) mathematics for the better life, i.e. for its intrinsic value, or for its own sake ; and 2) mathematics for a better living, i.e. for its application to science, technology, and social problems that will result in more efficient practical day by day living.

This should ring equally true forty years after it was written, but today people seem to pay more and more attention to 2), rather than 1), at most paying some lip-service to the latter.

It was with the academic background depicted above that students in the upper secondary school embarked on the study of analysis, which was usually just called ‘calculus’ in most places (as it still is today). Calculus had been introduced into the upper secondary school curriculum in the early part of the 20th century. It met with some success, for instance in the reform of 1902 in France [Artigue 1996]. By the 1950s the syllabus was more or less stabilized to include: differentiation and integration; simple applications such as rate of change, maxima and minima, area and volume, centre of mass, moment of inertia; the trigonometric, logarithmic and exponential functions; Taylor series expansions of functions. The instruction was in large part intuitive and informal, emphasizing calculation rather than conceptual understanding. For instance, the idea of *limit* was explained through an intuitive sense, that the dependent variable approaches a certain value as the independent variable approaches a given one. The derivative of a function $f(x)$ at $x = a$ is described geometrically as the slope of the tangent line to the curve $y - f(x) = 0$ at the point $(a, f(a))$ of the Euclidean plane $\mathbf{R}^2$. From this description students were led to compute the limit of a Newton quotient $[f(a+h) - f(a)]/h$ as h approaches 0, from which point on they were drilled in the calculation of the derivatives of many different functions. The subject was meant to be preparatory for those who would go on to university and need it as a tool or prerequisite for further mathematical pursuits.

Let me illustrate with my own learning experience. I came into contact with calculus before formal upper secondary schooling when in the preceding summer vacation my mathematics teacher kindly offered to give me extra lessons out of the popular American textbook *Elements of the Differential and Integral Calculus* by W. A. Granville [1904 (1929)]. One would read definitions like:

> the variable v is said to approach the constant ℓ as a limit when the successive values of v are such that the numerical value of the difference $v - \ell$ ultimately becomes and remains less than any preassigned positive number, however small,

and recipes like:

> GENERAL RULE FOR DIFFERENTIATION:
>
> FIRST STEP. In the function replace x by $x + \Delta x$, and calculate the new value of the function, $y + \Delta y$.
>
> SECOND STEP. Subtract the given value of the function from the new value and thus find Δy (the increment of the function).
>
> THIRD STEP. Divide the remainder Δy (the increment of the function) by Δx (the increment of the independent variable).
>
> FOURTH STEP. Find the limit of this quotient when Δx (the increment of the independent variable) varies and approaches zero as a limit. This is the derivative required.

Integration is introduced as the inverse operation to differentiation, viz. "to find a function $f(x)$ whose derivative $f'(x) = \phi(x)$ is given". The definite integral follows next, again with a recipe:

FIRST STEP. Find the indefinite integral of the given differential expression.

SECOND STEP. Substitute in this indefinite integral first the upper limit and then the lower limit for the variable, and subtract the last result from the first.

In my school days I enjoyed doing all these, but I would not claim that I really understood what was going on. I still did not really understand what was going on even after studying later upper secondary school textbooks such as *Techniques of Mathematical Analysis* by C. J. Tranter [1957]. I solved quite a number of exercises from that book, most of them of a rather technical nature, such as:

- If a, b are positive and $a + b = 1$, show that $ab \leq \frac{1}{4}$ and deduce that
$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2}\,.$$
- Prove that, if $y = x^2 \cos x$, then
$$x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + (x^2 + 6)\, y = 0\,.$$
Deduce that, when $x = 0$,
$$(n-2)(n-3) \frac{d^ny}{dx^n} + n(n-1) \frac{d^{n-2}y}{dx^{n-2}} = 0\,.$$
- If $\pi y = \int_0^\pi \cos(x \sin\theta)\, d\theta$, show that
$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0\,.$$

Occasionally I ran into some theoretical discussion, such as:

- If $f(a) = f(b)$, there is at least one point x in the open interval (a, b) at which $f'(x) = 0$.
- If $f'(x) > 0$ for every x in the open interval (a, b) then $f(x)$ is strictly increasing throughout the interval.

Somewhere in the book there is a passage on the Fundamental Theorem of Calculus. But I was not aware of the significance of this beautiful result at the time. In fact, without grasping the concept of a definite integral as the limit of a certain summation, I would have had difficulty in applying integration at will to solve problems other than the stereotyped ones. If I had studied with all my might the book *A Course of Pure Mathematics* by G. H. Hardy [1908

(1952)], I could have understood better. But the book was clearly beyond my comprehension at the time — I bought the book because of its misleading title, since the subject I registered for the university entrance examination was called 'Pure Mathematics' (to be distinguished from another subject called 'Applied Mathematics')! However, one thing I remember vividly about reading that book is its appendix on two proofs of the Fundamental Theorem of Algebra with the beginning remark that "it belongs more properly to analysis" — in the mind of a 16 year-old, it is strange to learn that the root of an algebraic equation has to do with infinitesimal analysis.

Despite this shadowy understanding (but with reasonably adequate technical competence in computation) I was thrilled with the subject of calculus. During the summer vacation when my teacher gave me the extra tutoring, I could solve a problem, albeit in a rather formal symbol-pushing manner without knowing what a differential equation is, that asked for the *escape speed* of a rocket. A rocket (of mass m) is to be launched straight up from the surface of Earth (of mass M and radius R). The student is asked to show that the minimum speed v_0 at which the rocket must be launched to reach a distance r_0 from the centre of Earth is given by $v_0 = \sqrt{2GM\left(\frac{1}{R} - \frac{1}{r_0}\right)}$ and to deduce that the escape speed is $\sqrt{\dfrac{2GM}{R}}$. I started the calculation with the formula $-m\dfrac{dv}{dt} = \dfrac{GMm}{r^2}$ (this much is physics) followed by a formal manipulation

$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = \frac{dv}{dr}v$$

that yielded $-mv\dfrac{dv}{dr} = \dfrac{GMm}{r^2}$, or $-vdv = \dfrac{GM}{r^2}dr$, so that

$$-\int_{v_0}^{0} vdv = \int_{R}^{r_0} \frac{GM}{r^2}dr\,;$$

hence $-\dfrac{v^2}{2}\Big|_{v_0}^{0} = -\dfrac{GM}{r}\Big|_{R}^{r_0}$, or $\frac{1}{2}v_0^2 = -GM\left(\frac{1}{r_0} - \frac{1}{R}\right)$, i.e. $v_0 = \sqrt{2GM\left(\frac{1}{R} - \frac{1}{r_0}\right)}$. (Just to indicate how shadowy my understanding was at that stage, I should confess that I added the minus sign in the starting formula *after* I found out that I arrived at an expression $\frac{1}{2}v_0^2 = GM\left(\frac{1}{r_0} - \frac{1}{R}\right) < 0$, which is blatantly wrong!) Finally, by putting $r_0 = \infty$, I obtained the escape speed $v_0 = \sqrt{\frac{2GM}{R}}$. This little feat on my part was particularly thrilling at the time when — a little over three years before — the first Soviet sputnik had been launched into orbit, heralding the age of space travel!

3. THE ‘NEW MATH’ MOVEMENT AND ANALYSIS

The word ‘*sputnik*’, mentioned in a paper on mathematics education, is evocative of the ‘new math’ movement. There is no need to go into the history of this movement, nor into the dispute on the pros and cons during and after it, although this is eventful, instructive and worth the discussion. There is a large body of literature on that. For an overview in the UK and the USA one can consult [Thwaites 1972] and [CBMS 1975]. Two thought-provoking papers published in *L’Enseignement Mathématique* discussed the issues in the early phase of the movement: [Freudenthal 1963] and [Wittenberg 1965]. The introduction of untraditional subject matter and the emphasis on abstraction and structure in the new math movement did not seem to have a profound influence on the learning and teaching of analysis as a school subject, in the sense that what was a difficult pedagogical problem before the movement remained a difficult pedagogical problem during and after it.

It is true that in the new math movement, the subject of analysis, or at least notions and topics closely related to it (such as function, absolute value, error estimation, sequence and series, manipulation of inequalities, metrics, etc.), was taught in even lower grades for even younger pupils. Also, the language of sets and mappings was adopted for the pursuit of preciseness and there was a trend towards abstraction. But even without the language of sets and mappings, even without the strengthened dose of abstraction, students found analysis (calculus) difficult, not really because they could not do the (routine) calculations but because they did not understand and could not make good *sense* of the subject.

The transition from school to university posed (as it still does today) even more serious problems. Several papers which appeared in *L’Enseignement Mathématique* in the mid 1950s were devoted to that issue [Freudenthal 1956; Maxwell 1956; Behnke 1957; Delessert 1965]. Kay Piene also discussed this issue in an address in the ICM54 at Amsterdam [Piene 1956]. Again, let me illustrate with my own learning experience. Besides studying textbooks such as [Courant 1937], [Apostol 1957], or [Rudin 1953 (1964)], I had also access to Chinese textbooks which were modelled after the Soviet tradition of textbooks, written by authors like A. I. Khinchin [2]) or G. M. Fichtenholz, that were popular in the USSR and in China in the mid 20$^{\text{th}}$ century. The course would start with a detailed discussion of the completeness of the real number system in its various formulations and the basic theorems about a continuous

[2]) A particularly lucid account, still worth studying today, is the little book *Eight Lectures on Mathematical Analysis* [Khinchin 1943 (1965)].

function defined on a closed interval. Thus I had the opportunity to sample the very rigorous treatment of analysis by the Soviet school, which was hard work but good solid training. In Europe, budding mathematicians in their first year at university went through a similar rigorous diet of textbooks by authors like Édouard Goursat, Georges Valiron, or Jean Bass. The consequence of going through such a rigorous diet is that one either makes it or one gets irretrievably lost. In the mid 20th century, when tertiary education in most places still catered for élitism, this state of affair was allowed to go on. With the opening up of tertiary education in later decades, the problem is getting more and more noticeable and has to be faced and resolved.

4. Two questions on transition from school to University

There are two questions I wish to raise concerning the transition from school to university pertaining to the learning and teaching of analysis.

a) By observing students I have taught from the mid 1970s till now, I find that I was quite fortunate to have experienced a smoother transition. To what extent has this to do with my ‘classical’ education in geometry ? As Richard Courant puts it in an introductory remark to his famous textbook on calculus [1937], “its intimate association with geometrical ideas and its stress on individual niceties give the older mathematics a charm of its own”.

I was brought up with a large dose of synthetic geometry replete with lots of proofs and construction problems. Not only was I accustomed to the notion of proof and logic before starting undergraduate study, but in school geometry I tasted the joy of discovery and the joy of succeeding in understanding something which was tangible (you can at least draw some pictures even if you do not know why it has to be like that at first) but not obvious (you do not know why it is like that at first). Geometry is a subject in which one can exercise logical discipline and free imagination *at the same time*. Developing a liking for geometry also enabled me to look at problems in other subjects from a geometric viewpoint. This helped in particular in the study of analysis. After all, calculus is the process of linear approximation, and linear problems fall within the purview of linear algebra, which is in itself akin to geometry. Of course, it does not work all the time and different persons are accustomed to different ways of thinking ; nevertheless it offers an alternative. Many students today are not accustomed to this flexibility in framework in their study of mathematics.

b) How extensive is the influence of ‘Bourbakism’ felt on the teaching of analysis in the university ? In school mathematics, in my experience, there was relatively little influence. I first learnt about the work of Bourbaki from my teachers at university. I cannot say much more further on this issue as I have not found out enough, and besides it would be somewhat remote from school mathematics, which is what we are more focused on in our discussion.

5. What can we learn from looking at the past ?

What can we learn from looking at those bygone days as far as the learning and teaching of analysis in the school or university classroom of the 21st century are concerned ? Back in those days, the all-purpose electronic computer was just making its début ; the hand-held calculator was still a luxury in the classroom ; the Internet was not thought of even in works of science fiction ; the mathematics curriculum was not as broad and as diversified ; the application of analysis was mainly confined to physics and engineering. As a result, contemporary issues in the learning and teaching of analysis, such as the use of modern technology in the classroom, the relevance of the subject in daily life or the relationship of the subject to information technology, were not yet on the agenda.

However, the difficulties a student encountered in those days are still the difficulties a student encounters today. In this sense, many of these contemporary issues, though they may play a significant role in improving the learning and teaching of the subject, are secondary since they may well breathe new life into the subject but they are not at the root of the difficulty. What is at the root is the perennial controversy between computational skill and understanding, between concreteness and abstraction. This controversy, which can sometimes unfortunately develop into an unnecessary false dichotomy, is a theme which reverberated throughout the last century from the introduction of calculus into the school curriculum to the calculus reform of the 1980s and 1990s. In this connection let me quote two relevant passages from two great teachers :

> The point of view of school mathematics tempts one to linger over details and to lose one’s grasp of general relationships and systematic methods. On the other hand, in the ‘higher’ point of view there lurks the opposite danger of getting out of touch with concrete details, so that one is left helpless when faced with the simplest cases of individual difficulty, because in the world of general ideas one has forgotten how to come to grips with the concrete. The reader must find his own way of meeting this dilemma. In this he can only

> succeed by repeatedly thinking out particular cases for himself and acquiring a firm grasp of the application of general principles in particular cases; herein lies the chief task of anyone who wishes to pursue the study of Science.
> [Courant 1937, 2–3]

> Abstraction was not a hormone which can be imposed from outside, but one that the patient must generate for himself in response to appropriate stimulation.
> [Quadling 1985, 94]

Finally, let me inject a historical remark on the term 'analysis'. In the ancient Greek usage of this word it means, in contrast to 'synthesis', the process of working backward from what is sought until something already known is arrived at. Towards the end of the 16th century François Viète used the term 'analysis' to denote algebra, since he did not favour the word 'algebra' (coming from the Arabic word *al-jabr*) which has no meaning in any European language. In his book *In artem analyticen isagoge*, much of the algebra developed is motivated by the intention to solve geometric problems. In this sense it is related to the ancient usage of the word in describing the process. This was brought even more into focus by the work of René Descartes, as illustrated by the famous appendix "La géométrie".

When calculus came on stage in the era of Isaac Newton and Gottfried Wilhelm Leibniz, they regarded the subject as an extension of the algebra of the infinite, in which lots of functions were expressed as power series that behave like polynomials, just longer! This was again emphasized in the work of Joseph Louis Lagrange with the title *Théorie des fonctions analytiques*. Leonhard Euler titled his book on calculus *Introductio in analysin infinitorum*. By and by the term 'analysis' was used to denote the study of calculus and its extension. In the preface to an unpublished book on analysis, Henri Lebesgue discusses the relationship and distinction between arithmetic, algebra and analysis (published posthumously as [Lebesgue 1956]). Gustave Choquet [1962] emphasizes the inseparable relationship between algebra, geometry and analysis throughout his paper. Thus, the subject of analysis is closely tied in with the subjects of arithmetic, algebra and geometry. Maybe this historical episode is a reminder of the integrated unity of the four basic subjects in the mathematics curriculum, which we wish our students to realize and to appreciate.

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ANALYSIS 2000 : CHALLENGES AND OPPORTUNITIES

L'analyse en l'an 2000 : défis et opportunités

par Lynn Arthur STEEN

Dans cette contribution on essaie de tracer les lignes de développements possibles de l'enseignement de l'analyse. Il importe de souligner que dans le monde de l'éducation le mot "analyse" est plutôt utilisé au sens de l'américain *calculus*. Il s'agit de calcul différentiel et intégral conçu pour les masses ; il se concentre sur des procédés et des recettes, alors que l'analyse s'intéresse aux preuves et aux définitions fondamentales.

En tout cas, l'analyse n'occupe plus à présent la position dominante qui était la sienne il y a cinquante ans. Quels sont les facteurs contemporains de nature à influencer le futur de l'enseignement de l'analyse ?

D'abord on énumère des facteurs liés à l'environnement : facteurs sociaux (responsabilité publique, mutations technologiques, enseignement à distance, économie globale) ; facteurs liés à l'éducation (méthodes d'évaluation, nombre élevé d'étudiants, dont la préparation n'est pas homogène), facteurs mathématiques (diversification des mathématiques, "mathématiques pour tous").

A propos des réformes de l'enseignement du calcul différentiel et intégral, on mentionne deux mouvements qui se sont développés aux États-Unis. Il y a vingt ans l'intêret pour l'informatique a remis en question la suprématie de l'analyse comme porte d'entrée aux mathématiques universitaires. Ce mouvement a eu le soutien de scientifiques, d'administrateurs, de politiciens et de mathématiciens. Dans le même temps, mais indépendamment, le cours intitulé "*Calculus*" devenait un symbole de haute qualité de l'instruction : les parents comme les élèves revendiquaient pour ce cours un rôle central dans le programme du secondaire. Le but n'était nullement de *perfectionner* ce cours, mais de permettre au plus grand nombre de *passer l'examen*, si possible avec des notes élevées.

La réforme a changé les cours d'analyse plus dans la pédagogie et le contexte que dans les contenus. On met l'accent surtout sur les ordinateurs, les calculatrices de poche, le travail en groupes, les projets d'étudiants, l'écriture, la modélisation, l'établissement de liens et les applications. On demande aux étudiants d'avoir une plus

profonde compréhension des relations entre les formules, les graphes, les nombres et les descriptions verbales. Mais l'impact le plus important de la réforme de l'enseignement de l'analyse a été l'engagement des professeurs d'université dans la discussion sur l'apprentissage des mathématiques élémentaires destinées au plus nombre.

L'avenir de l'enseignement du calcul différentiel et intégral met en jeu plusieurs questions. D'abord il y a la manière dont les étudiants apprennent le sujet et les obstacles cognitifs. Par exemple, il faut évoquer le passage de l'algèbre à l'analyse, l'influence du contexte et des expériences quotidiennes sur l'apprentissage, le rapport entre l'éducation secondaire et tertiaire, le problème de la rigueur. Mais il est également important de se demander comment les mutations technologiques ont transformé l'approche de l'analyse. Comme les problèmes d'apprentissage sont difficiles à résoudre, peut-être les mathématiciens devraient-ils prendre connaissance des recherches qui se font dans les neurosciences sur les constructions mentales des mathématiques.

ANALYSIS 2000: CHALLENGES AND OPPORTUNITIES

by Lynn Arthur STEEN

Talking about challenges facing analysis education at the turn of the new century is a daunting task, well beyond the wisdom of any one person. However, it does provide a rare opportunity to think about what might be rather than what is, to imagine inquiries unencumbered by the constraints of current literature or prevailing orthodoxy. Thus my role is not that of a mathematician who proves theorems that last forever, but that of a prophet who imagines a future that is yet to emerge. I take some comfort in the knowledge that similar predictions on other anniversary occasions have set a very low standard for this genre.

Instead of beginning as a mathematician would, with definitions and notation, I shall begin, as an amateur prophet might, with clarifications and distinctions. Although my assigned subject is *analysis*, in the world of education analysis tends to mean *calculus*. Indeed, the vast majority of analysis enrollments are in elementary calculus. The distinctions between what mathematicians call analysis and what students call calculus are clear to anyone who has taught both courses. *Calculus focuses on procedures and templates, analysis on definitions and proofs.* Calculus is for the masses, analysis is for the 'mathes' — those who plan to specialize in mathematics. In this paper, my focus will be on calculus.

A second distinction is about mathematics vs. education. During the last half-century mathematics has expanded enormously, both in the diversity of its specialties and in the pervasiveness of their roles in society. These so-called *mathematical sciences* encompass a diverse and rapidly expanding part of human intellectual accomplishment. Although still vigorous both in its own right and as a supporting tool for other parts of mathematics, analysis now represents a much smaller fraction of mathematical practice than it did fifty years ago, before linear programming or digital computers, before combinatorics or bioinformatics, before data mining and string theory.

Paradoxically, during this same half-century the role played by calculus in education has expanded enormously, often irrationally. Today the mathematical focus of secondary school is calculus, not mathematics. Even as the mathematical sciences have built avenues of intellectual exchange extending in many different directions, the signposts of society still direct everyone to enter mathematics on the traditional highway of calculus. The question we must confront at the beginning of this new century is whether this traditional route still serves mathematics well or whether it might not be prudent to explore other options through which students can enter the world of mathematics.

1. Environmental scan

As is fashionable these days among those whom we in the United States suspiciously call ‘policy wonks’, I begin with an environmental scan to survey briefly some powerful contemporary factors that are likely to have significant bearing on the future of calculus. First are four societal factors:

- *Public accountability.* All over the world, government leaders are raising their expectation that universities serve broad public purposes, not merely elite professional interests. This demand arises both from increasing democratization of governments — who must now be at least somewhat responsive to the people they govern — and from rapid increases in secondary and tertiary enrollments that are supported to a great degree by public funds. Accountability pressures on mathematics departments are now external, not just internal; they arise as often from parents and politicians as from professionals and peers.
- *Technology.* As computers become standard tools of employment and research, proficiency in professional use of technology becomes an obligation of education. Mathematics educators now must deal not only with the question of how technology can enhance (or impede) mathematical learning and how technology changes priorities for mathematics content, but also how technology changes the way mathematics is expected to be performed. Technology’s challenge to mathematics education is now about ends as well as means.
- *Telecommuting.* Mathematical skills, being purely cerebral, can be bought and sold anywhere on a worldwide market that is increasingly linked with high speed Internet connections. Thus the market for mathematical skills is no longer local or parochial but international. As graduates can now

sell technical skills to employers anywhere in the world, so students and teachers can also join a community of learners without borders. Thus despite lingering national differences in mathematics curricula, calculus as a subject is increasingly shaped by international contexts.

- *Global economy.* As nations compete in a technologically sophisticated international economy, the demand for technically trained employees who can manage complex industries is increasing far more rapidly than is the demand for researchers and scientists. Thus the economic demand for mathematical skills is now tilted in the direction of breadth rather than depth, for the practical skills of calculus rather than the theoretical skills of analysis, and even more for the data-based skills of statistics rather than the function-based skills of algebra.

Now three educational factors:

- *Diversification of education.* With the growth of a "parallel universe" [Adelman 2000] of higher education — on-line courses, for-profit colleges, certificate programs, and virtual universities — students now have many more options for their education. For better or worse, school and college mathematics departments will no longer have hegemony over calculus.
- *Assessment.* More and more, educational quality is being measured by outputs (student learning) rather than inputs (faculty teaching). Increasingly, calculus will be judged not by the syllabus, textbook, or instruction, but by the degree to which students can demonstrate that they meet the objectives of the course — many of which are totally absent from traditional tests.
- *Enrollment pressure.* In virtually every country the number of students completing secondary school and entering post-secondary education is rising, bringing to upper secondary and university levels students of very different skills, backgrounds, and motivations. These changes, worldwide in scope, increase enormously the pedagogical challenges of teaching calculus.

Finally, two mathematical factors:

- *Diversification of mathematics.* The expansion of mathematical methods into such diverse areas as genetics, finance, and even cinema has significantly changed the balance of mathematical practice. New tools developed for these new applications, especially in combinatorics, geometry, and data analysis, have displaced analysis from the leading role it has played for nearly three hundred years during which it has been at the center of the mathematical universe. As analysis no longer plays the lead role in mathematics, so calculus is gradually losing its claim to be the centerpiece of mathematics education.

- *Math for all.* Worldwide, educational and governmental leaders now include mathematics as part of the common core of learning expected of all students. However, especially in light of the changes in the mathematical sciences, one must now ask whether ‘math for all’ should continue to mean, as it has in recent decades, ‘calculus for all’. There is some evidence that accelerated and excessively narrow mathematical requirements have created a backlash, leading some countries to scale back mathematical requirements in the schools.

2. Calculus reform

In the United States, and to varying degrees in other countries, one of the major factors influencing calculus in recent years has been a movement marching under the banner of ‘calculus reform’. The progress and setbacks of this campaign provide a valuable case study in change, illustrating both how calculus responds to external pressures as well as how it manages to retain surprising equilibrium despite these pressures.

I apologize for focusing in this international forum on a movement that arose and remains anchored in the United States. I do this for both weak and strong reasons. The weak reason is that it is what I know best. The strong reason is that students and teachers in the United States come literally from all over the world. Thus the challenges we face are in some ways representative of the challenges faced by students and teachers in other countries. Indeed, our mixture of cultures in a single nation, a single city, a single school or university, even a single classroom, creates extraordinary instructional challenges that may be as internationally representative as anywhere on earth.

Twenty years ago, rising interest in computer science began to challenge the primacy of calculus as the gateway to university mathematics. Computing both pulled students away from calculus and challenged the importance of topics taught in calculus. One response was a campaign led by US college and university mathematicians to make calculus more "lean and lively", to make it "a pump not a filter" in the educational pipeline [Douglas 1986; Steen 1988]. The *reform movement* drew support from very different sources — from scientists who were frustrated by the inability of students to use calculus intelligently in real applications, from administrators who were angered by high failure or withdrawal rates from calculus courses, from politicians who saw in the global economy an increased need for technicians rather than theoreticians, and from mathematicians who recognized that calculus instruction had become, in Bernard Hodgson's memorable image, "*l'enseignement sclérosé*" [Hodgson 2000].

Simultaneously but independently, calculus in the United States also became a political totem, a supposedly objective and unassailable surrogate for high standards to which politicians could appeal in supporting or attacking various education proposals. Backed by political rhetoric that proclaimed calculus as the epitome of academic accomplishment, parents and students began pushing calculus into the secondary curriculum, primarily through the vehicle of the *Advanced Placement (AP) program.* This political campaign had nothing to do with reform but everything to do with status and access. Its goal was not to *improve* calculus but to enable more secondary school students to *pass* calculus, preferably with high grades (see, e.g., [Stewart 1997] and [Spence 2000]).

Needless to say, these two movements had rather different aims and objectives. The goal of the AP course soon became numbers: more offerings, more participants, more passing grades. The course itself is anchored by a traditional syllabus set by an external committee that also establishes the national exam for the course. Recently the syllabus was changed slightly to reflect some aspects of the calculus reform movement, but even these small changes produced great anxiety among AP teachers who depend on the course's stability and predictability for their success in getting students through the exam.

Even as AP calculus became the gold standard of secondary school mathematics, the tertiary level calculus reform movement tried with increasing energy to destabilize those entrenched aspects of postsecondary calculus that, in reformers' eyes, were not serving students well. For better or for worse, reform calculus became a magnet for unfulfilled goals of every prior progressive movement in mathematics education. Ten years after calculus reform had begun, a federally sponsored assessment of the movement found over a dozen different goals, only a few of which had anything to do with the content of calculus [Tucker & Leitzel 1995].

Three content goals, not among the first on the list, did imply fluency in traditional skills :

- Develop improved in-depth understanding of specific mathematical concepts.
- Employ calculus techniques in other disciplines and in novel situations.
- Reason analytically, qualitatively, and quantitatively.

Several cognitive goals stressed robust problem solving skills :

- Represent problems algorithmically, graphically, verbally, numerically, and symbolically.
- Use technology in solving problems.
- Deal with complex, often ill-defined problems.
- Translate problems from one form to another.

Some behavioral goals concerned how students work and interact with others :

- Communicate mathematical ideas both in writing and orally.
- Become independent mathematical learners.
- Work effectively in groups.

Finally, attitudinal goals focused on students' feelings about mathematics :

- Understand the role of experimentation, conjecture, verification, and abstraction.
- Develop positive attitudes about one's ability to do mathematics successfully.
- Appreciate mathematics' elegance and structure.
- Pursue further work in quantitative fields.

3. Evaluation and assessment

It is now fifteen years since the movement to reform calculus began, time enough to assess its short-term impact. It turns out that reform courses and textbooks — in the United States the text largely defines the course — do not differ much from traditional courses in content, but they do differ significantly in pedagogy and context. Reform calculus typically places much greater emphasis on calculators, computers, cooperative learning, students projects, writing, modeling, connections, and applications ; it makes only modest changes in the way limits, derivatives, integrals, and series are approached or developed. Although evidence is scarce and unreliable, the consensus of experts and the conclusions of several studies is that the majority of students in the United States still study calculus from texts and in classrooms that have changed very little during the fifteen years of the calculus reform movement. However, a sizeable minority — typical estimates range from 25% to 40% — encounter calculus in courses that attempt to implement at least some of the strategies common to reform courses [Tucker & Leitzel 1995, 23–27; Roberts 1996, 157–8].

These students do face different expectations. As evidenced by final examinations, students in fully reformed courses, compared with their peers in traditional courses, are expected to have much deeper understanding of the relations among formulas, graphs, numbers, and verbal descriptions. For example, in a traditional US course, a typical exam problem at the end of the first term would give the formula for a function and ask the student to find the derivative and then sketch both the function and its derivative. In a reform course, a typical problem might give students six or eight unlabeled graphs and ask them to decide which are derivatives of which others and to explain how they figured it out [Roberts 1996, 74–132].

Convincing evidence of changes in students' mathematical performance, cognitive habits, work behavior, or attitude towards mathematics is both scarce and decidedly mixed. Since the goals of a reform course are so different from those of a traditional course, the lack of clear-cut evidence is not surprising [Schoenfeld 1997; Gold, Keith & Marion 1999, 229–256]. Different students learn different things well, and different things poorly; some thrive under the reform regimen, others chafe. Even good students often encounter difficulty when they shift from one style of course to another because it upsets their own expectations of what is required. Direct comparison of outcomes is difficult, if not nonsensical — a bit like comparing the outcomes of a course in Shakespeare with a course in James Joyce.

By far the greatest impact of the calculus reform movement is the engagement of university mathematicians, sometimes for the first time, in serious, productive discussions about teaching and learning elementary mass-marketed mathematics [Tucker & Leitzel 1995, 36–42]. Ideas for improved pedagogy spin off from these discussions and take root in other courses across the curriculum. Many university mathematicians have discovered that they share a common agenda with secondary school mathematics teachers who are working to implement new standards for school mathematics. Perhaps even more surprising, many university administrators find in the reform movement meritorious evidence of scholarship that both advances the art of teaching and resonates with institutional evaluation for tenure and promotion [Joint Policy Board 1994].

4. Inquiry and scholarship

Calculus reform is one manifestation of efforts to improve the teaching and learning of analysis — and more broadly, of higher mathematics. Another is the growing interest in research in mathematics education at the upper secondary and undergraduate levels (where analysis education begins). These are not unrelated, of course, since most advocates of calculus reform claim to ground their recommendations in the results of research. (Not surprisingly, most critics of the reform movement (e.g., [Wu 1997; Askey 1999]) dispute the relevance of that same research.)

I tend to adopt a broad view of research, one aptly expressed by the phrase "disciplined inquiry" that Mogens Niss employed in his plenary address at ICME-9 [Niss, forthcoming]. This phrase nicely subsumes four distinct yet interconnected aspects of scholarship (of discovery, of synthesis, of application, and of teaching) that Ernest Boyer identified in his well-known monograph *Scholarship Reconsidered* [Boyer 1990]. The literature is filled with evidence of more focused or more idiosyncratic definitions, especially of educational research. I am not an expert in this area, but neither are the tens of thousands who teach calculus. So in this respect, I stand in their shoes as one who is more a practitioner than a theoretician of education. It is not natural for me, or for them, to ask about belief systems or metacognition, much less about APOS (action–process–object–schema) stages or statistically significant p-values (e.g., [Kaput & Dubinsky 1994; Dubinsky, Kaput & Schoenfeld 1994–1998]). Valuable as these may be to experts, they do not communicate in the language spoken by practitioners.

So instead of attempting to outline a research agenda cast in the traditional language of educational research, I want to suggest a variety of issues concerning the future of calculus that I believe could benefit in coming years from disciplined inquiry and reconsidered scholarship. Some of these issues are about cognition and understanding, some about teaching and learning, and some about policy and practice. Professionals may consider some of these issues out of the normal bounds of educational research. But I believe, and I hope you will too, that not only calculus but also analysis and, indeed, all of mathematics would nonetheless benefit from these kinds of disciplined inquiry.

Before delving into the issues, I need to make one more distinction — that between descriptive and normative questions, between inquiry into 'what is' vs. questions about 'what should be'. For example, in the history of calculus reform, one of the key results from the 'what is' inquiry was evidence of astonishingly high rates of dropout, failure, and repetition in calculus as taught in US colleges and universities. No other subject was even close. Another indicator, perhaps a bit more subjective, was the widespread sense among both mathematicians and users of mathematics that even students who completed calculus with good grades were, in too many cases, incapable of using it intelligently and expeditiously.

These failures of traditional calculus were documented by descriptive investigations. In contrast, reform calculus — the movement — was the result of disciplined inquiry into the normative question of what calculus *should* look like if we want to improve retention and improve performance in subsequent courses. As we have seen, this inquiry also led to additional goals, to cognitive, behavioral, and attitudinal expectations that were not central to the reform effort in its early, descriptive, hand-wringing stage. For some mathematics teachers, these newer goals — for example, writing, using technology, mathematical modeling, and group projects — are strategic objectives aimed at bringing about more traditional goals of improved retention and performance. For others these goals are ends in themselves, dimensions of what it means to learn calculus in the twenty-first century.

In most instances, descriptive and normative inquiries occur simultaneously on parallel paths. Ideas and evidence from one influence the other, generally for the betterment of both. Nonetheless, insofar as possible, I believe that it is important to keep the distinction clear, much as journalists, ideally, attempt to maintain a relatively clear line between reporting and advocacy. That said, I begin with some issues that I believe merit disciplined *descriptive* inquiry.

5. Descriptive issues

How many students study calculus ? What do they learn ? This is basic. In the United States, we gather some data addressing this issue, but it is relatively infrequent and not easy to interpret because much calculus in secondary schools is embedded in other courses, or taught superficially as a warm-up for later formal study [Loftsgaarden, Rung & Watkins 1997]. Other nations may do their own studies, but there seems to be no source of accurate international enrollment information on courses such as calculus that are optional and taught in many different kinds of educational settings. (And now we have the added complication of on-line Internet courses.) Calculus is one of relatively few optional courses of broad significance that is taught everywhere in the world, so it may merit special attention as a surrogate for the quality and extent of advanced education in different nations.

In what ways does calculus depend on its setting? This inquiry probes the universality of calculus by seeking explicit information about variability that may be due to external contexts: different nations, different educational settings (secondary school, technical college, university), different student abilities, different settings for diverse applications (physics, biology, business). What is common and what is variable? Are there any universals in terms of definitions, problems, theorems, applications, sequencing? Are there particulars that are either widely adopted or widely ignored depending on context? An international survey of calculus instruction could help define the subject by identifying what is central and what is peripheral.

How well do calculus tests assess the goals of calculus courses? This is an important empirical question because tests, not syllabi, determine what most students actually learn. In traditional courses in the United States, tests ask primarily for routine calculations plus one or two predictable applications and perhaps some simple proofs [Steen 1988, 177–211]. Exams for reform courses include a wider variety of conceptually interesting problems, often expecting fluency not just with formulas and functions, but also with graphs, numerical tables, and verbal descriptions [Roberts 1996, 74–132]. But in neither case does one find many questions that reveal the broader goals of reform calculus nor the deeper goals of traditional courses.

How important is calculus now that graphing and symbol manipulating software is widely available? Is calculus still as important as it was during the three hundred years between Isaac Newton and Bill Gates when it was the only tool available for most scientific models? Put another way, if digital computers had been invented before calculus, would calculus ever have gained the central position it now occupies in mathematics education? Now that virtually any set of differential equations can be solved digitally, closed-form analytical solutions are no longer the gold standard of mathematical modeling: visual representations of changes in behavior under different values of parameters offer far more insight than do symbolic solutions. This question calls for empirical investigation into the practice of mathematics to determine the degree to which the new digital empire has replaced the analysis empire that has been in power for most of the last three centuries. Results of this investigation will almost surely lead to a revised map of mathematics.

Which aspects of calculus require human expertise and which are best performed by computers? This inquiry into the content of calculus in the computer age naturally splits into two parts — calculus as it is practiced and calculus as it is taught. Long experience has provided mathematics teachers with a pretty good mental map of the logical and instructional sequencing appropriate for calculus in its traditional form when most student work is done with paper and pencil. But now that computer software can do much of what students have historically been asked to do, and now that computer systems are widely used in every profession where calculus plays a significant role, we need to envision calculus differently. Technology is doing for calculus what a new subway does for a city and what international air transportation is doing for the globe: it is realigning relationships and changing psychological distances. A new map of calculus would reflect these changed relationships.

What mathematical uses of calculators and computers are inappropriate? Twenty years' experience shows rather convincingly that calculators can, if wisely used, enhance students' experiences in mathematics class. In these masterful classes, students learn mathematics well, gain facility with a handheld mathematical machine, and emerge enjoying mathematics more than those who study the same material in more traditional contexts. However, university mathematicians universally complain that large numbers of students use calculators inappropriately — either to perform calculations that they really should do in their heads or on paper, or to give approximate numerical answers to problems where an exact answer (e.g. $\pi\sqrt{2}$) would reveal much greater understanding than the numerical approximation (4.4428829). Mathematicians are not alone in this concern; older adults often complain when they see young store clerks rely on calculators to perform simple calculations. So this inquiry is not about whether calculators enhance or diminish learning, but whether it is possible to distinguish between appropriate and inappropriate uses of calculators in a way that will be useful for teaching and learning. (If this could be done, then perhaps mathematicians might be able to reach consensus, now regrettably lacking, on the appropriate role of calculators in calculus.)

How does the way calculus is taught influence students' ability to use calculus in further study and professional practice? Most research into pedagogy is concerned about its relation to learning. This proposed inquiry points in a different direction by asking not about the effect of teaching practice on student learning but about its effect on students' subsequent ability to use what they have learned. Students' notorious inability to transfer learning from one context to another — from mathematics class to economics class, for example, or from classroom to work place — is widely recognized as a generic problem of learning, not a special disability of mathematics. Student resistance to employing good writing outside of language classes is equally well known. The issue for inquiry is whether and how the circumstances of learning can facilitate transfer of what has been learned.

How much of calculus is learned outside of calculus class? We know from many studies that young children pick up considerable mathematics outside of school: at home, in the playground, in stores, on television, and now at the computer. Children's mathematical minds are shaped substantially both by what happens in and outside of mathematics class. Historically, this has also been true for calculus students. Typically, students learned calculus deeply primarily by using it for physics. (And they learned algebra fluently primarily by using it for calculus.) Today relatively few mathematics students study physics, but many study business and economics where calculus-based models abound. We need to learn systematically (rather than anecdotally) just how important these out-of-math-class experiences are for full mastery of calculus. Does it matter much for mastery of derivatives and integrals if the primary application is to physics — the historic taproot of calculus — or to newer sciences such as economics or biology? More interestingly, can average students really learn calculus well if they study only calculus? Might it be that the subtle nature of infinite processes requires the anchor of a real model (rather than only mathematical definitions) for the mind to construct an appropriate and usable mental model of calculus?

What special cognitive hurdles are involved in the transition from algebra to analysis? All mathematics teachers recognize an enormous gap between the knowledge and skills developed in secondary school courses in geometry and algebra — knowledge that is primarily static and procedural — and the dynamic subtlety of limits and limit-based concepts such as the real number line, the derivative, and the integral. This gap is partly due to conflicts between ordinary and mathematical language (e.g., *limit* as a barrier; *continuous* as incessant; *infinite* as unfathomable), but even more to the increasing level of abstraction that is inherent in nested quantification (e.g., "for every epsilon there exists a delta such that …"). It took mathematicians two centuries to figure out how to express the fundamental definitions of calculus in such a way as to avoid contradictory inferences when reasoning about infinite processes. It is not enough for calculus teachers to just understand the logical resolution of these paradoxes of the infinite — the so-called 'arithmetization' of analysis introduced in the nineteenth century. The issue requiring inquiry is how these logical resolutions may in fact exacerbate (rather than resolve) the psychological impediments students face in dealing with infinite processes.

What characteristics of students, teachers, schools, and policy account for the significant differences between secondary and tertiary education? Calculus, the introduction to analysis, straddles the active fault line where secondary education pushes up against the plate of tertiary education. Stress is evident everywhere — on students, teachers, schools — and discontinuities abound. Students face the daunting challenge of leaving the security of algebra for the uncertainties of analysis at the same time as they traverse the steep terrain leading from secondary to higher education. Although anticipation of educational and intellectual hazards will not eliminate them, foreknowledge can help both students and teachers develop strategies to minimize their negative effects. The purpose of this inquiry is to identify and describe special impediments to students' intellectual growth in these years due to factors other than the special character of analysis.

6. Normative issues

Shifting gears, I conclude with some normative questions, with suggestions for more speculative and subjective inquiry into possible futures for calculus, and *ipso facto*, for analysis education.

What is the best way to introduce students to university mathematics? In particular, is calculus the right course for the majority of students? No one disputes that calculus is both significant and sublime. But are these sufficient reasons to require that all students enter higher mathematics through this single gateway? Students in the social, behavioral, and life sciences — now the majority of clients of mathematics — need a much broader portfolio of mathematical methods, notably statistical, combinatorial, and computational. Other themes, such as optimization or modeling, that represent more widely applicable areas of mathematics can provide intellectual challenges equal to those of calculus as well as a synthesis of methods from algebra, analysis, geometry, and combinatorics. Might multiple gateways to mathematics help reverse the worldwide decline in students who specialize in university-level mathematics?

Should mathematicians welcome increasing numbers of students to calculus? Many of the complaints one hears from calculus teachers about students' lack of preparation or motivation are consequences of society's pressure, wisely or not, to push more and more students into higher levels of mathematics. Many mathematicians talk as if they would much prefer that mathematics had remained an elite subject for a select few rather than become a basic subject for mass education. Analysis (that is, calculus) is where this argument is joined, since everyone acknowledges that all students must study the other foci of school mathematics — arithmetic, algebra, and geometry. Might it be better for mathematics, and for students, if calculus were delayed until fewer (and more mature) students were enrolled ?

Is calculus the right course to introduce the rigor of analysis ? Calculus not only represents the crowning achievement of the age of science, but it also serves as a phase transition in students' mathematical passage from algebra to analysis. It is where the clear crystals of arithmetic, algebra, and geometry liquefy into the fluid ideas of limits, derivatives, and real numbers. Earlier, when calculus served a very limited population of motivated and screened students who were preparing for careers in mathematically based fields, it functioned with three main goals : to teach students the mechanics of calculus, to introduce some of its myriad applications, and to introduce epsilon-delta arguments. But as calculus has come to serve a much more diverse clientele, this latter goal has gradually disappeared : syllabi now focus on tools and applications, assessments stress procedures. Thus most students who study calculus experience only results and applications, not methodology or foundations. Determining the appropriate balance between the pragmatic and intuitive on the one hand and the formal and rigorous on the other hand is largely a matter of values, a reflection of the goals one wants to achieve in the course.

Should mathematics stake its future on calculus ? Mathematics continues to hold a place of privilege in an increasingly crowded curriculum. Unlike most subjects, mathematics is universally compulsory for school children and widely recommended long after it becomes optional. Increasingly, its special place is being challenged, especially by information technology. Historically, mathematicians (and others) could easily make a strong case for retaining a focus on calculus not just for instrumental reasons of convenience but also on intellectual, cultural, pragmatic, social, and scientific grounds. But now calculus faces two significant challenges — from technology that can best even experts on most calculus tasks, and from significant shifts in the balance of power among major fields of mathematics. Might there now be better exemplars to make the case for mathematics to an increasingly skeptical public ?

Should calculus look the same everywhere ? How much local context should be reflected in the content, applications, context, and pedagogy of a calculus course ? Can one expect the same textbook or syllabus to work as well in South Dakota as in South Africa ? This inquiry touches on one of the deepest epistemological issues of mathematics education — the degree to which mathematics should be taught as a pristine objective discipline independent of local culture and customs, or as one of many subjective aspects of culture that is tightly bound to problems of local interest. The former emphasis tends in the direction of pure mathematics, the latter in the direction of quantitative literacy. At risk of oversimplification, one might say that the former reflects an elitist perspective, the latter a populist agenda. Which is better for the future of mathematics ?

How much technology should be taught in calculus class ? As bank clerks use calculators (rather than traditional algebra) to compute interest and loan payments, so engineers use professional computer software (rather than traditional calculus) to solve analysis problems. These tools include symbol manipulating software, interactive visualization, mathematically active notebooks, applets, and modern web-based communication tools. In this age, calculus students can rightly expect not only to learn traditional procedures, concepts, and applications, but also to acquire modest fluency in using the professional tools by which calculus is now practiced. Notwithstanding these legitimate expectations, there remains a fundamental question of purpose and value that needs to be thoughtfully addressed : Should calculus in school prepare people technologically for calculus at work, or should it instead focus primarily on understanding fundamentals as preparation for further study ? One cannot rightly assess the success of a course or program, much less design appropriate strategies, without first reaching agreement on its goals.

Should computer notation become part of standard calculus instruction ? As silicon computers began taking on the task formerly carried out by human computers, programmers ran right into one of the major roadblocks that has historically created so much difficulty for mathematics students : ambiguous and two-dimensional notation. To make things work, programmers introduced new notation that could be typed on a standard keyboard and interpreted unambiguously by software that lacked human intelligence for guessing meaning from context. This new notation is ubiquitous : $*$ for multiplication, $\wedge$ for exponents, : for ranges of summation, $:=$ for assignment, etc. Yet mathematics texts continue, for the most part, to use only traditional notation that evolved to meet nineteenth century needs when computers were nowhere in sight. Should mathematics adjust its standards so that students learn a single unambiguous system that can be communicated readily by e-mail ? Might the clarity of notation that helped computers also help students ?

What is the appropriate balance of content and context ? Discussions of mathematics curricula often take place along two relatively independent dimensions : more or less (abstract) mathematical content, and more or less (authentic) worldly context. Different approaches to calculus can readily be plotted on a content-context plane — some are high in one dimension and low in another, some are low in both (because they focus mostly on mechanics at the expense of both content and context). Few, perhaps none, are high in both content and context because that would require far more time and effort than students ordinarily have available. Choices must be made to achieve a suitable balance. Like so many other questions about calculus, the answer to this question reveals more about the values of the person who provides the answer than it does about the nature of the subject.

Should theories of cognition influence the way analysis is taught? For at least a decade, if not longer, those who actively pursue research in mathematics education have been exploring cognitive issues related to learning calculus. Much attention has been given, for example, to the 'APOS' stages in students' developing grasp of the concept of function: these are first seen as actions (calculating values), then as processes (reflecting on actions collectively), then as objects (encapsulated processes), and finally as schemata (classes of objects defined by shared properties). Functions have also been studied as representations of correspondences and of covariation among related variables, ideas that continually evolve in students' minds [Harel & Dubinsky 1992]. Not only functions but also other familiar mathematical objects — the number line, the meaning of equality, the idea of area, even the idea of number itself — undergo successive reconstruction as students pass through the phase transition between algebra and calculus [Tall 1991]. The issue I suggest for investigation is not which of these theories can be confirmed, but whether and to what degree any of these theories have significant impact on the effectiveness of teaching. Should it matter what theory of cognition, if any, a calculus teacher believes?

Should mathematicians pay attention to what neuroscientists are learning about the mental constructs of mathematics? Heretofore, all theories of mathematical cognition have been based exclusively on behavioral evidence — rather like theories of illness before the germ theory of disease. Now, however, physical and biological evidence from neuroscience is beginning to appear, evidence about electrical activity in the brain when it engages in mathematical thought [Butterworth 1999; Dehaene 1997; Lakoff & Núñez 2000]. Although this evidence is still primitive, it may help suggest or clarify possible explanations for what we observe about the struggles students have in learning mathematics. For example, it now seems clear that the well-known difficulty of mastering the multiplication table has physiological roots — since the words for numbers are handled in a different part of the brain than other words. Might we someday learn something about how the brain deals with limits or other abstractions of analysis? Might such insight improve the teaching and learning of calculus?

7. Summing up

The challenges facing analysis education at the beginning of the twenty-first century, and the second century of *L'Enseignement Mathématique*, are quite different from those one might have listed fifty or one hundred years ago. Calculus is now relatively more important in more students' lives even while it represents a relatively smaller fraction of the expanding mathematical universe. Calculus is no longer, as it once was, the unchallenged tool of applied mathematics; statistics, linear algebra, and combinatorial methods all have strong claims to the crown of mathematical utility. Moreover, calculus is not even the only option for accomplishing its etymological purpose, namely calculating with what used to be quaintly called 'infinitesimals'. Computers, thank you, can now do all those things perfectly well.

As we enter the new century, I suggest that we must be forthright in challenging all historic assumptions about the role of calculus in the mathematics curriculum. Take nothing for granted; put all options on the table. If calculus were not now entrenched, would we choose to give it such high priority in such a crucial place in the curriculum? Are there better choices to help attract students to the advanced study of mathematics? What benefits might ensue if secondary school mathematics were freed of the burden of preparing all students for calculus?

Answers we give to these questions may well determine the fate of higher mathematics education, whether it will be seen as the relic of a glorious past that has now been overtaken by new subjects such as information technology, or whether it will be seen, as it has been for most of the last century, as a conveyor of skills and understandings that are crucially important for all educated people. We dare not take the future for granted.

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REACTION

Enseignement et apprentissage de l'analyse : que retenir du passé pour penser le futur ?

par Michèle ARTIGUE

En réaction aux trois contributions sur l'enseignement et l'apprentissage de l'analyse au 20e siècle, présentées respectivement par Jean-Pierre Kahane, Man-Keung Siu et Lynn Arthur Steen, j'ai été particulièrement sensible à plusieurs phénomènes.

J'ai été d'abord frappée par le fort contraste existant entre la situation au début du 20e siècle et la situation actuelle. Le début du siècle voit l'entrée modeste mais efficace de l'enseignement du calcul différentiel et intégral dans les lycées, dans de nombreux pays, l'enthousiasme des pionniers, le sentiment qu'ils expriment avec force d'avoir réussi dans leur entreprise. La situation actuelle semble au contraire caractérisée par un sentiment croissant de crise que les différentes réformes entreprises dans le monde entier ne parviennent ni à enrayer, ni même à réduire.

Le second point qui m'a frappée, c'est la permanence tout au long du siècle des mêmes questions, en particulier les suivantes : Comment répercuter dans le choix des contenus d'enseignement l'évolution scientifique de ce domaine, celle des rapports entre mathématiques et autres disciplines ? Comment prendre en compte la diversité des besoins des élèves et étudiants ? Comment trouver un juste équilibre entre intuition et rigueur dans les premiers contacts avec le monde de l'analyse ?

Mais la présentation de Lynn Steen a par ailleurs bien montré qu'au delà de ces permanences, émergent des questions nouvelles, comme les suivantes : Comment penser l'enseignement de l'analyse aujourd'hui compte tenu de la massification de l'enseignement secondaire, et maintenant universitaire, et de l'hétérogénéité croissante des étudiants qui en résulte ? Comment prendre en compte dans cet enseignement l'évolution technologique ? Qu'offre, pour penser l'enseignement et l'apprentissage de l'analyse, la recherche didactique qui s'est développée maintenant depuis plus de vingt ans dans ce domaine et que pourrait-on en attendre de plus ?

Ces questions nous montrent clairement que les contextes changent et que les solutions aux problèmes d'enseignement et d'apprentissage de l'analyse qui se posent aujourd'hui ne peuvent être simplement empruntées au passé, quelles qu'aient été ses réussites.

Dans ce texte, je souhaite contribuer à la discussion sur ces questions, en respectant l'approche historique qui a été celle du Symposium et me semble être particulièrement productive. Quand on réfléchit sur les questions d'enseignement et d'apprentissage, se pencher sur l'histoire est quelque chose d'essentiel. Cela aide à percevoir l'organisation actuelle d'un enseignement, ses choix et ses valeurs, comme le résultat d'une longue histoire partiellement dépendante des cultures mathématiques et surtout éducatives dans lesquelles cette histoire s'est inscrite. Cela aide à mieux comprendre la nature et la force des contraintes avec lesquelles les systèmes éducatifs doivent compter, les résistances qui souvent s'opposent aux changements curriculaires, les effets à moyen ou long terme souvent inattendus de telle ou telle décision. Cela aide enfin à intégrer à la réflexion la vision dynamique nécessaire pour penser l'avenir.

Je reviendrai donc dans un premier temps sur le contraste souligné précédemment et essaierai de tirer quelques leçons de son analyse. Dans un deuxième temps, j'essaierai de contribuer à la réflexion sur les questions que pose aujourd'hui l'enseignement des débuts de l'analyse au lycée, en identifiant notamment les ressources offertes par les recherches didactiques menées dans ce domaine.

REACTION

Learning and teaching analysis :
What can we learn from the past in order to think about the future ?

by Michèle ARTIGUE

AN EVIDENT HISTORICAL CONTRAST

What we observe at the beginning of the 20th century is the timid introduction of differential and integral calculus in curricula which are still dominated by geometry. For instance, in the new 1902 French syllabus for scientific sections at grade 12, analysis — which is included in the algebra part — takes up only half a page from a total of nine. The teaching of analysis had limited ambitions : developing an efficient elementary knowledge of calculus. These ambitions met both rigour and usefulness requirements as they were expressed at that time : this calculus was free from the metaphysics of infinitesimals, it offered unified methods for computing geometric, mechanical, and physical magnitudes, for studying variation and motion. Teachers were provided with efficient mathematics resources which introduced them to the new contents to be taught, beyond the sole student-textbooks. We must add that high-school students were part of the social elite and constituted a rather homogeneous population : for instance, in France less than 10% of the population entered the high-school system at that time [Belhoste 1996].

With the turn of the mid-century came a substantial evolution. Man-Keung Siu evoked his personal experience as a secondary-school student in the early sixties, when calculus teaching was a stabilized corpus, taught in an intuitive and informal way which emphasized calculation rather than conceptual understanding. As he said, he enjoyed doing calculus problems, applying the taught recipes — for calculating the escape speed of a rocket for instance —, but he did not really understand what was going on.

With the 'new maths' movement, new ambitions developed for the teaching of mathematics and, as a specific case, for the teaching of analysis. Beyond efficient calculus, what was now being pursued, in many countries including France, was an approach to analysis as a structured theoretical field where concepts were defined formally and where approximation played a fundamental role [Artigue 1996]. Analysis became independent of algebra and the balance between analysis and geometry progressively changed in the high-school curriculum. The rejection of the structural values of the 'new maths' reform, when it occurred (as it did in France in the early eighties), did not alter this tendency. Reducing the role given to algebraic structures and to the set-theoretic dimensions which had been emblematic of that reform, cleared some curricular space which analysis tended to occupy. It is important to remark that this increasing role of analysis took place in a mathematical environment which was becoming less extensive (for instance, in France, astronomy, mechanics, and kinematics had progressively disappeared from the high-school curriculum) and where the connections with other scientific disciplines, which were deemed so important at the beginning of the century, were gradually fading out.

The present situation, as stressed by L. Steen, is characterized both by an over-representation of analysis in many high-school curricula and by a general feeling of crisis which seems to transcend cultural differences. In his text L. Steen has described the US situation: the teaching of analysis is optional at the secondary level; what is taught is still mainly calculus, but the teaching content is ever more challenged by the advances of computer technology, which seems able to take charge of the full range of abilities and competence expected from the students. In France, the context is not quite the same. Analysis teaching is compulsory, its aims go beyond mere calculus, even if theorization and formalization have been progressively reduced during the last twenty years. In fact, our educational system, facing the constraints of mass teaching, is much less able to foster the epistemological values it claims to develop. These difficulties generate strong frustrations. Moreover, the predominant role of analysis in the syllabus, as is proved by its key position in the *baccalauréat*[1]), is challenged by other mathematical topics: mainly, statistics and probability theory or discrete mathematics. Statistics and probabilities are more and more considered to be necessary parts of the mathematical culture of educated citizens living in a democratic country

[1]) the national examination ending high-school studies, which allows entrance into university.

like France. Discrete mathematics is strongly favoured by the technological evolution and the connections between mathematics and computing.

Thus, the crisis does not look everywhere the same, but it is a perception which transcends cultural differences. There is an obvious necessity of rethinking the teaching of analysis at the secondary level. How relevant will it be at the turn of the new century if we take into account the current institutional constraints (especially mass education and the resulting students' heterogeneity), technological advances or the evolution of social and scientific needs ? These questions have been extensively addressed by L. Steen in his contribution. He emphasized that there is not a unique solution. In fact, each of the answers we can imagine necessarily includes some ideological choices which we have to integrate. Even though, as pointed out by M.-K. Siu, the fundamental difficulties of the learning of analysis have not substantially changed, what has been stressed above shows also that radical changes in contexts and values prevent us from simply borrowing solutions from the past.

SO, WHAT CAN WE LEARN FROM THE PAST ?

What was perceived as an adequate solution at the beginning of the 20th century or in the eighties cannot be seen as a possible solution now. But the past certainly helps to question the present tendencies, values, and choices. We will use history within this perspective in what follows, when evoking three separate points : the respective role of geometry and graphical representations in the meaning of analytic concepts, the transition between algebra and analysis, and technological issues.

FROM GEOMETRY TO GRAPHIC VISUALIZATIONS : HOW DO THE CONCEPTS OF ANALYSIS BECOME MEANINGFUL FOR STUDENTS ? Up to the seventies, as was stressed by J.-P. Kahane and M.-K. Siu, the geometrical and kinematic culture of students certainly played an essential role by allowing students to give some meaning to computational practices which were far from being clearly understandable. Today after the theoretical attempts of the 'new maths' reform and the reduction of the mathematical students' landscape, graphical visualizations offered by calculators and computer software are apparently supposed to play a similar role. All over the world, the curricula emphasize the necessity (in the first contacts with functions and analysis) of interconnecting the algebraic approaches, which have been predominant for a long time, with numeric and graphical ones [Robert & Speer 2001]. The triad "numeric–graphic–algebraic" has become emblematic of different projects such as the *Harvard Calculus Project* in the USA. Didactic research, more globally, tends to offer arguments in support of such a strategy, by showing the role played in conceptualization processes by a certain amount of flexibility between different semiotic registers of representation, as is for instance extensively discussed in [Duval 1995]. Beyond that, the dynamic character of the graphical visualizations produced by computer technology, and the possibility of connecting such visualizations with mechanical devices in order to simulate motion and also to analyse and predict the effects of changes in speed and acceleration, offer us, today, new ways of exploiting the potential provided in the past by classical kinematic perspectives. Research conducted by J. Kaput and his colleagues [Kaput 1992], in connection with the software MathCars, is a good example : this pioneering work showed how a first qualitative contact with variational issues and concepts can be introduced very early, at a time when students do not yet master the algebraic techniques that are generally considered a prerequisite to learning in this area.

Up to what point are graphic visualizations efficient tools for supporting the development of mathematical knowledge in analysis ? What kind of tasks, of problematic situations can allow the students to maximize the expected benefit of these techniques ? What are their real potential and limits ? Are they adequately exploited in current teaching with graphic calculators ? Even if research allows us to approach these questions better today, they remain widely open. Moreover, there is no clear evidence that graphic calculators, as they are normally used in high-school educational contexts, offer the possibility of a deep conceptual understanding in analysis. Graphical visualization is not meaningful per se. Most research results show that the students' spontaneous interpretation of graphical phenomena may be quite different from what is expected by the teacher, who reads these phenomena through his own mathematical background (see for instance [Trouche 1994]). Loading graphical representations with mathematical substance requires specific apprenticeship which is often underestimated. It is far from easy to connect graphical explorations and symbolic representations in order to go beyond the evident limitations of graphical descriptions (for instance, they do not allow the control of approximation orders, which is essential in analysis). Research certainly provides some useful resources for this work (see [Tall 1996] for a synthetic view), but, as is often the case in education, what is discovered in an experimental setup does not seem easy to share and generalize. The fact that most calculus reform projects in the US are not closely linked to research, as evidenced by N. Speer in a recent survey [Robert & Speer 2001], is not anecdotal from this point of view. We will come back to this point further on.

THE TRANSITION BETWEEN ALGEBRA AND ANALYSIS. In his contribution L. Steen has evoked this transition and stressed its logical demands. In this paragraph, I would like to evoke briefly the diversity of the reconstructions at play in this transition and the long-term evolution they require. I will then discuss the curricular implications that such considerations can have.

Entering a new field of mathematical knowledge generally requires some cognitive reconstructions that are not easy to achieve. This cognitive cost is very often underestimated by educational systems; generally, most of these reconstructions are left to the personal responsibility of students and this is the origin of persistent difficulties and failures. Such difficulties have been extensively investigated in the case of the transition between arithmetic and algebra [Bednarz, Kieran & Lee 1996]. Surely, analysis also obeys the general situation.

Not all the cognitive reconstructions that are required for entering the field of analysis are of the same nature [Artigue 2001a]. Some of them deal with mathematical objects or practices which are familiar to students before they get acquainted with the official teaching of analysis. Some others are more internal to the field of analysis. They can result from the polysemy of mathematical concepts (like the integral or the derivative) and from the fact that the different facets of such notions cannot all be introduced at the same time. Others still are linked to the fact that, as H. Poincaré underlined at the beginning of the century in his famous lecture on mathematical definitions [Poincaré 1904], a mathematical concept cannot, as a rule, be introduced to students from the beginning in its most achieved form. Different levels of conceptualization have to be progressively reached, each one corresponding to a kind of balance which is partially broken when a more complex level is striven for. In order to analyse the transition between algebra and analysis, we shall need to go into these different categories, as the relationships between algebra and analysis depend on the levels of conceptualization that one adopts.

For instance, when the teaching of analysis begins at high-school level, students have already met irrational numbers and functions. These objects are implied in their mathematical practices, and the conceptions they generate emerge from these practices. Mathematical practices can be described, as Chevallard does in his anthropological approach to mathematics education [Chevallard 1992; Bosch & Chevallard 1999], in terms of praxeologies structured around tasks, techniques developed for solving these tasks, and the technological and theoretical discourses which are used to motivate, explain, and justify these techniques. Thus the real numbers that emerge from students' practices are algebraic objects with a dense order, a geometric representation (the real line), and numerical representations or approximations, among which the finite decimal expansions given by calculators play an essential role. As we said before, entering the field of analysis requires some complex reconstructions. Research data show that, even if students declare that the real order is a dense order, they are tempted to think that some numbers can, in some sense, be successors. For instance 0.999… is very often considered to be the last number before 1. In fact, the symbolic notation evokes an infinite process whose terms are all less than 1, and it looks as though students could not detach the *result* of the process from the process itself. Tests at the entrance of French universities have regularly shown that the majority of science students answered the question : "what can you say about a and b if, for every positive integer n, $|a - b| < 1/n$?", by saying that a and b are infinitely close or successors. Such conceptions obviously make it difficult to produce or even understand proofs in analysis. Research also shows that the conceptions the students have of the real line are at variance with what is expected in analysis, at a certain level at least. A spontaneous conception of continuity develops on the basis of our physical experience with space and motion [Lakoff & Nuñez 2000 ; Nuñez, Edwards & Matos 1999]. The real line is thus at first linked to the idea of continuous motion, of trajectory ; real points are not constitutive of it, they appear more as points on a pre-existing line, as milestones along a highway. Understanding the essential role played by completeness, perceiving continuous functions as functions that preserve the continuity of subsets, requires subsequent changes.

When they begin to be taught analysis, students have already developed some experience with functions. Once more, their relations to functional objects emerge from the practices these objects have been involved in. Today, these practices generally favour an introduction to functions through modelling situations, combining numerical, graphical and algebraic work. Variational issues are also considered mainly by relying on numerical and graphical evidence, or on the properties of prototypical functions. These practices are of a point-wise or of a global nature. Entering the field of analysis forces one to reconsider them by taking into account the localization of the perspective that is proper to analysis and the dialectic interplay that can be developed between local and global perspectives. Linearity, which was a global concept, has for instance also to be thought of as a local property shared by a large class of functions, and students need also to understand how such local conditions can fully characterize mathematical objects, as is the case with differential equations.

Even without entering formal analysis and ε-η proofs, algebraic practices have to be reconsidered. In the previous algebraic work, all the different components of an algebraic expression were given the same weight; solving an equation meant finding all the numbers satisfying that equation. In analysis, the management of algebraic expressions has to take into account the different orders of magnitude of the terms and to look for what is predominant and what can be neglected. Working with inequations is no longer playing the same game; it means combining this differentiated treatment of expressions with the play on intervals or neighbourhoods induced by the local perspective. The technical work thus deeply changes and becomes more complex, as well as the heuristics and the control processes.

The transition towards higher levels of conceptualization and what is generally called 'formal analysis' — where objects are dealt with on the basis of their formal definitions — needs reconstructions of a different nature. As stressed by many researchers [Tall 1996; Robert 1998] and evidenced by empirical results, there is a deep gap between what I have evoked above and formal analysis, even if what I have evoked goes beyond mere calculus in the classical sense. Formalized concepts, such as the formalized concept of limit, cannot be built in continuity with the intuitive sources linked to social and physical experience. In essence they are proof-generated concepts [Lakatos 1976] answering foundational needs. As pointed out by Robert [1998], understanding such mathematical needs is far removed from the mathematical culture of secondary and even university students, and some specific strategies have to be designed in order to allow them to penetrate this new culture.

Taking all these facts into consideration obliges us to think in new terms the problem of the secondary teaching of analysis. Even if we are convinced that, for cultural reasons, students cannot leave secondary school without developing some contact with the ways infinity and infinite processes are approached by mathematicians, analysis is not necessarily the most appropriate domain for all categories of students. Entering the epistemological field of analysis as we perceive it today, whatever choice we make, is costly and involves long-term processes. This field can certainly be approached in different ways, as we have tried to show. High-school teaching, without reducing to mere algebraic calculus, nor looking for inaccessible aims such as formal levels of conceptualization, can offer a coherent way that initiates the necessary reconstructions. Nevertheless, the cost of such an initiation should not be minimized and, for students who do not require such expertise for their professional future, the compulsory character of analysis at high-school level should certainly not be taken for granted.

TECHNOLOGICAL ISSUES. As regards this essential point, I wish to emphasize that we still live under a vision which, in my opinion, prevents us from addressing technological issues efficiently. The values of mathematics teaching and learning are today defined essentially by referring to paper and pencil learning environments and practices, as though mathematical values could ideally be thought of in an absolute way, without taking into account the current instruments of mathematical work, the way these shape our mathematical practices and change our mathematical needs. Technology is expected to improve learning processes, students' motivation, etc., and its educational legitimacy seems to depend on its ability to fulfil these requirements.

Research work in analysis has considered technology within different complementary perspectives:

- through a *programming* perspective, programming activities being seen as a way to solve the process/object gap (the development and use of the language ISETL in the framework of APOS theory [Dubinsky & Mac Donald 2001] can be considered a paradigmatic example of this perspective);
- through a *semiotic* perspective, using technology as a means of visualizing mathematical objects and connecting semiotic representations, as is the case for instance in [Borba & Confrey 1996];
- through an *experimental* perspective, technology being seen as a means of developing more experimental approaches involving motion, simulation, and problems coming from different scientific contexts.

This has produced interesting results and engineering designs but, on the whole, it did not seriously challenge the traditional vision, which is therefore poorly sensitive to crucial issues such as:

- the computer transposition of mathematical knowledge [Balacheff 1994] and the ways in which such a transposition can affect our relationship to mathematical objects and our mathematical needs;
- the complexity of the instrumental genesis through which an artefact such as a calculator or a piece of software becomes an efficient mathematical tool with its specific mathematical needs.

As evidenced by recent research on computer algebra systems (CAS) [Guin & Trouche 1999; Lagrange 1999; Defouad 2000], it seems especially important to develop this kind of awareness in a domain such as analysis. In fact, CAS were not designed for educational purposes but for professional use; however, they are likely, in the near future, to become tools as ordinary as graphic calculators are today in secondary schools.

Thus, as was stressed by Lynn Steen, conceiving the future of analysis in secondary schools today requires an important change of perspective :

> Mathematics educators now must deal not only with the question of how technology can enhance (or impede) mathematical learning and how technology changes priorities for mathematics content, but also how technology changes the way mathematics is expected to be performed. Technology's challenge to mathematics education is now about ends as well as means. [Steen 2003, 194]

CONCLUDING REMARKS

As I have tried to demonstrate, educational research has a lot to offer to our reflection through the results it has obtained :

- about students' conceptions and difficulties, with evident coherence in the results all over the world, in spite of natural differences in cultural and institutional contexts,
- about the dysfunctionings of educational systems in that area, the nature and strength of the constraints that oppose to evolution, and about ecological and viability issues,

but also through :

- the theoretical frames and concepts it has developed for thinking about these questions, such as the APOS theory quoted by L. Steen, which put to the fore the qualitative gap linked to the transition between the process and object conceptions of mathematical objects, which plays an essential role in learning processes in analysis,
- the engineering designs it has developed and tested, such as for instance the design elaborated by the group AHA (*Heuristic Approach to Analysis*) in Louvain-la-Neuve [Groupe A.H.A. 1999], which offers a rather ambitious programme for high-school level relying on a strong epistemological analysis and trying to progress from mental objects, with the meaning given to this term by H. Freudenthal, to constructed objects.

But research has also evident limits and certainly does not allow one today to answer many of the issues raised by L. Steen. Some of its limitations can be regarded as contingent, like those related to the poor development of research in specific areas such as the teaching of analysis in service courses. Others are not so contingent but are linked to the complexity of educational processes. Research can inform reflection and choices, it cannot guide them in a mechanical way, neither can it be asked to take in charge, in a scientific way, questions which mainly depend on ideological or political values. Moreover, its results, contrary to mathematical theorems, are time and space dependent and, even if coherent, they are not established within one unified paradigm and do not lead to identical didactic choices [Artigue 2001b]. These characteristics certainly make their transmission and use difficult, beyond local communities of research, and contribute to explaining the limited impact they have had up to now on innovation and reform projects.

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GENERAL DISCUSSION

(reported by Marta MENGHINI)

Following Michèle Artigue's reaction, there was a lively debate about analysis and the problems facing the teacher and the learner. The two different aspects of analysis, namely its usefulness in applications and as a service subject, and its conceptual role, were illuminated by the discussion about how, and to whom, the subject should be taught.

The tension between analysis as a tool and analysis as a branch of pure mathematics was illustrated in an extended contribution by Jean-Pierre Kahane, devoted to the historical development of Fourier analysis. Fourier first adapted trigonometric series to study the propagation of heat. Analysis was applied to physical investigations in order to find general procedures aimed at calculating the sums of series. The usefulness of series is not their convergence, but their power to compute. During the 19^{th} century Fourier series became something else : they provided the opportunity for Dirichlet to state the first important theorem about the convergence of functions and also for Riemann to produce his idea of integrals. Finally, in the spirit of the end of the 19^{th} century, what was taught as Analysis became Foundations of Analysis. Later Bourbaki considered fundamental structures to be paramount in mathematics, and so analysis was then taught as General Topology. And now ? We are turning again to discrete Fourier transforms, now adapted to new tools and applied to many new fields.

The major part of the discussion revolved around the teaching of analysis and the difficulties this entails, which are necessarily linked to the conceptual problems already indicated. Certainly analysis is a mathematical tool which is useful to non-mathematicians. Hence it is important to note that if mathematicians offer only traditional (pure) analysis then the subject will be taught by scientists and engineers to their students in the way that suits them best. Furthermore, motivation for learning the subject can be found in its possible applications and this should be used by mathematicians.

Most of the participants who spoke in favour of a more applied view of analysis were referring to university level courses, but one speaker made the point that, at school level too, it is exactly the awareness of the applications of analysis, particularly its physical interpretations (which are too often absent), that helps to clarify the general meaning of analysis. And Ubiratan D'Ambrosio suggested that the idea of 'mathematics in a lab' should now be renewed, with the addition of computer facilities, as a step towards more theoretical reflections.

New technology offers new opportunities in teaching and learning analysis and many participants were keen to see the development of more conceptual and structured tasks, in order to develop thinking in analysis; the issues regarding conceptual and procedural knowledge should not be neglected, and this is particularly important now, in the era of CAS (*Computer Algebra Systems*) and DGS (*Dynamic Geometry Software*), when using technology and computer transposition.

Even the 'practical' problem of determining areas takes on a conceptual aspect if the historical approach of teaching integral calculus before derivatives is followed. Trevor Fletcher maintained that integration seems to be easier than differentiation, because people studied it successfully 2000 years before. He remembered that some decades ago there were people who seriously considered developing elementary analysis by dealing with integration before differentiation, assuming that would be a better motivation for the learning of analysis. His question whether this is an issue today produced many reactions.

Jean-Pierre Kahane recalled that at the beginning of the 20^{th} century the general feeling among Italian mathematicians had been to begin with integration and there had been discussions about how this could be done. But he pointed out that for computating integrals one needs derivatives, regardless of the fact that integration was first to appear in the work of Archimedes. A balance is therefore necessary. A possibility could be to introduce integration first, presenting areas and volumes as motivation and then, soon after, to introduce derivatives in order to perform computations.

Similarly, another speaker recalled that in the '60s Tom Apostol published textbooks for teaching calculus, starting with the integral and later making the connection with derivatives. Conceptually it was very nice, but there were problems with students studying physics, and the order was reversed. Many of these approaches, which sound attractive to mathematicians, seem to have been abandoned because of the difficulty of not having developed computational tools early enough for the student, in other words because of the need to preserve analysis as a 'service subject'.

The question whether the derivative or the integral is the more fundamental concept of analysis was present in a number of contributions to the discussion. Certainly the idea of area seems more fundamental but this raises the conceptual problem of area itself. Given the concept of integration and the concept of real number we could revisit concepts which have been accepted before, such as the 'fact' that the area of a rectangle is given by the product of its two sides. The question is whether this is a theorem to prove, or a well-stated definition.

It is also important not to lose sight of the difficulties inherent in the teaching of analysis. Geoffrey Howson recalled that analysis was never easy to teach even at university level. In the '60s those people who were developing the SMP (*School Mathematics Project*) were faced with the problem of how to approach the concept of differential, and what was the key idea they wanted to be left in the student's mind. On this there was no agreement. There were two different approaches : one went through developing functions and mappings, the other aimed to emphasize the role of approximation. Unfortunately, whatever the approach, the concept of a limit was soon needed, after which the derivative was to be introduced, and finally algebraic techniques in analysis followed. Which approach to use still seems to be the crux of the problem, together with the problem of notation.

Another participant reminded the audience that the fact that analysis is introduced so late in the curriculum seems to render unnecessary the use of a simpler language, which raises the question of the advisability of looking at it in a more elementary way.

But the problem of what sort of analysis, which approach to use and, fundamentally, whether to begin with the derivative or the integral, still remains and is by no means new. Gert Schubring concluded the discussion by noting that the question of which parts of calculus should be introduced in schools, remains a problem today just as it had been at the beginning of Klein's reform movement. By common agreement, analysis should not be reduced to a set of formal techniques, so how can calculus be introduced in a way that provides conceptual understanding ?

A final remark, endorsed by the participants, was that the question of identifying conceptual tasks in analysis would be an appropriate topic for a special issue of *L'Enseignement Mathématique*.

APPLICATIONS OF MATHEMATICS:
MATHEMATICS AS A SERVICE SUBJECT

LES DÉBATS AUTOUR DES APPLICATIONS DES MATHÉMATIQUES DANS LES RÉFORMES DE L'ENSEIGNEMENT SECONDAIRE AU DÉBUT DU VINGTIÈME SIÈCLE

The debates around mathematics applications in the reforms of secondary teaching at the beginning of the twentieth century

by Philippe NABONNAND

One of the primary aims for establishing the *International Commission on Mathematical Instruction* (ICMI) was to publish contributions on the reforms of mathematical instruction. The journal *L'Enseignement Mathématique* soon became ICMI's official organ and a forum for debating questions pertaining to pedagogy, didactics, curriculum planning, and the organization of mathematical instruction.

At the close of the 19th century, the advances of the sciences and the growth of industrial society showed how necessary it had become to reform mathematics education. The demand for well-trained technicians and engineers led to the introduction of technological and vocational education at the secondary level and above. The question as to the position of mathematical training within an increasingly practical educational system thus arose. Applied mathematics, or more precisely applications of mathematics, was the main focus for reformers. Most of the authors who published articles in *L'Enseignement Mathématique* were in favour of this spirit of reforms.

The chief objective of the present paper is to use the articles published in the periodical to provide an understanding of the position which applications of mathematics were to have in the new curricula emerging at the beginning of the 20th century. Reformers argued in favour of introducing new topics of applied mathematics, advocating less theoretically oriented and more application-minded mathematical training.

Surprisingly, there were no significant developments in curricula along the lines of the intended changes, except for graphical and numerical methods. While reformers wanted mathematical training to meet the requirements of users and of the other sciences, it was, in their view, still far more important to introduce calculus into the curricula of secondary schools. Arguing in favour of introducing applications into mathematical instruction helped merely to convince the community of mathematics teachers that this evolution was inevitable and to induce them to combat reforms intended to *bowdlerize* mathematics into a set of 'useful' practical tricks.

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INTRODUCTION

Publier des contributions traitant des questions de réforme de l'enseignement des mathématiques et présenter des panoramas décrivant l'histoire et l'organisation de cet enseignement dans différents pays figurent parmi les principaux objectifs de la revue *L'Enseignement Mathématique* au moment de sa création en 1899. De nombreuses contributions à cette revue évoquent donc le contexte international de réforme de tous les ordres d'enseignement au début du 20e siècle et se font plus particulièrement l'écho des débats et des interrogations sur les questions de l'enseignement des mathématiques. En devenant à partir de 1908 l'organe officiel de la Commission internationale de l'enseignement mathématique (CIEM), la revue organisera et suscitera, au moins jusqu'en 1914, la discussion sur les questions de pédagogie, de didactique, de définition des *curricula* et d'organisation de l'enseignement des mathématiques.

La plupart des auteurs écrivant dans *L'Enseignement Mathématique* sont des partisans et même, pour beaucoup, des propagandistes de l'esprit de réforme. Bien entendu, la situation reste très variée d'un pays à l'autre et les volontés réformatrices s'expriment différemment en fonction des conjonctures nationales. Néanmoins, les nations se considérant alors [1]) comme « civilisées » sont toutes traversées au début du 20e siècle par un puissant courant de réforme de l'enseignement, en particulier de celui des mathématiques.

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[1]) En 1914, la Conférence internationale de l'enseignement mathématique de Paris réunit des délégués d'Allemagne, Autriche, Belgique, Danemark, Égypte, Espagne, États-Unis, France, Hollande, Hongrie, Îles Britanniques, Italie, Roumanie, Russie, Serbie, Suède et Suisse.

> Partout les mêmes problèmes se posent, et presque dans les mêmes termes, partout aussi l'on envisage à peu près les mêmes solutions. S'il y a entre les solutions adoptées ici et là des différences, qui tiennent évidemment au génie propre de chaque nation, il y a plus de ressemblances, plus de points communs entre elles qu'on ne serait tenté de le supposer au premier abord. [2])
> [Darboux 1914, 192–193]

Une des raisons le plus souvent invoquées pour justifier la nécessité de ces réformes est l'exigence de former des techniciens et des ingénieurs pour les besoins de l'industrie alors en pleine expansion. Alors que l'on assiste depuis la seconde moitié du 19e siècle à l'émergence d'un enseignement technique et professionnel au niveau secondaire et supérieur, la question de la formation mathématique dispensée par ces établissements (et en particulier celle de la place à accorder aux mathématiques appliquées et aux applications des mathématiques) devient alors cruciale. L'enseignement général n'est pas épargné par la volonté de réforme, ni par la question des applications. Comme le souligne Carlo Bourlet [3]), le développement des sciences et la croissance des connaissances scientifiques sont une autre des raisons majeures avancées par les promoteurs des réformes.

> L'enseignement des mathématiques, dans nos lycées, collèges et gymnases de tous pays, passe actuellement par ce que d'aucuns nomment une crise et qui n'est, en somme, qu'une fièvre de croissance, un malaise né de la rapidité même de l'évolution du savoir humain. [Bourlet 1910, 373]

Pour la plupart des réformateurs, les nouveaux programmes doivent insister sur l'unité de la science. En conséquence, on ne peut plus dans tous les ordres d'enseignement se contenter d'un enseignement abstrait mais on doit au contraire accorder un rôle essentiel à l'expérience pour la présentation et la compréhension des théories mathématiques ainsi qu'aux applications de celles-ci aux autres disciplines mathématiques.

Le but essentiel de cet article est de tenter de comprendre, à partir des contributions parues dans *L'Enseignement Mathématique*, quelle place occupent les applications des mathématiques dans les nouveaux plans d'étude qui apparaissent au début du 20e siècle ; les réformateurs plaident-ils pour l'apparition de nouveaux chapitres de mathématiques appliquées dans les

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[2]) Gaston Darboux poursuivait son discours avec peut-être un peu d'optimisme en 1914 : « Malgré les apparences, qui sont quelquefois contraires, les nations se rapprochent de plus en plus les unes des autres, elles tendent de plus en plus à former une humanité civilisée, un concert des peuples dans lequel chacun doit s'attacher à exécuter sa partie de manière à concourir à l'harmonie de l'ensemble et au bien de tous. »

[3]) Professeur au Conservatoire national des arts et métiers (Paris), délégué français à la CIEM.

curricula de mathématiques ou prônent-ils un enseignement des mathématiques moins abstrait et plus ouvert sur les applications des mathématiques ?

1. Quel contexte ?

Un contexte, diversement analysé, de développement de l'enseignement scientifique dans l'enseignement secondaire

La plupart des intervenants s'accordent à souligner l'importance des évolutions subies par l'enseignement secondaire. Chacun reconnaît que les progrès de la science et de l'industrie au 19e siècle doivent se traduire dans les plans de formation et les programmes. Pour certains, il ne s'agit que d'adapter l'enseignement secondaire à une nouvelle définition de la notion de culture générale. La mission de l'enseignement secondaire reste, pour ces derniers, essentiellement éducative. Par contre, pour d'autres, les réformes doivent intégrer une nouvelle visée de l'enseignement : former des usagers des sciences au service de l'industrie. De plus, les missions peuvent se décliner différemment selon les ordres d'enseignement. Il faut rappeler que si l'objectif de l'enseignement secondaire, classique ou moderne, est de dispenser aux élèves une culture, celui de l'enseignement professionnel et technique est avant tout de former à un métier. La question de la formation scientifique se pose donc radicalement de manière différente selon les ordres d'enseignement : les débats autour de l'évolution de la notion de culture générale et des rapports entre culture classique et scientifique concernent essentiellement l'enseignement secondaire moderne. Les *curricula* scientifiques des établissements techniques sont la plupart du temps subordonnés aux exigences de la formation pratique et technique.

Ainsi, au congrès de Bruxelles de la CIEM (1910), A. Matthias [4]) souligne le bouleversement que sont en train de connaître les *curricula* dans son pays. Les sciences, depuis le début du 20e siècle, ne sont plus cantonnées à un rôle marginal dans le plan de formation. Au contraire, dispenser une véritable culture scientifique devient une des missions de l'enseignement (en particulier secondaire) :

> Le véritable rôle des sciences dans l'enseignement moyen a été longtemps méconnu sous l'influence prépondérante des études classiques. Aujourd'hui on reconnaît leur valeur éducative. [CIEM 1910, 387]

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[4]) Représentant du ministère prussien de l'instruction publique.

De même, E. Beke [5]), dans son rapport général [Beke 1914], considère que les causes profondes du mouvement de réforme de l'enseignement sont à chercher dans « la transformation, survenue au 20e siècle, des idées sur la culture générale et dans les efforts de l'enseignement secondaire tendant à suivre la transformation de ces idées », et non pas dans les lacunes de l'enseignement ou l'insuffisance de ses résultats.

Pour d'autres, le rééquilibrage des programmes en faveur de l'enseignement des sciences ne peut qu'entraîner une évolution radicale des objectifs de l'enseignement secondaire. Ainsi, Carlo Bourlet, dans sa conférence de 1910 sur la pénétration réciproque des mathématiques pures et des mathématiques appliquées dans l'enseignement secondaire, prône la priorité du rôle social de l'école sur celui plus traditionnel de formation des individus.

> Notre rôle [celui des enseignants] est terriblement lourd, il est capital, puisqu'il s'agit de rendre possible et d'accélérer les progrès de l'Humanité tout entière. Ainsi conçu, de ce point de vue général, notre devoir nous apparaît sous un nouvel aspect. Il ne s'agit plus de l'individu, mais de la société ; et, lorsque nous recherchons la solution d'un problème d'enseignement, nous devons choisir une méthode non pas suivant sa valeur éducative pour l'élève isolé, mais uniquement suivant sa puissance vulgarisatrice pour la masse. [Bourlet 1910, 374]

Pour Bourlet, la valeur d'un enseignement ne se mesure plus seulement à l'aune d'une quelconque valeur formatrice mais aussi à son utilité.

> Un enseignement moderne ne saurait se contenter de cultiver les facultés de l'esprit, il doit savoir le meubler de faits, nombreux et précis. Nous n'avons pas à former des philosophes qui vivront en savants ermites, mais des hommes d'action qui devront contribuer, pour leur part, au progrès humain. [Bourlet 1910, 374]

Dans sa conférence sur l'adaptation de l'enseignement secondaire aux progrès de la science, Émile Borel, pourtant partisan des réformes [6]), est nettement moins enthousiaste devant les évolutions utilitaristes de l'enseignement, qu'il considère cependant comme inévitables. Selon Borel, la conception de l'enseignement secondaire selon laquelle « il s'agit avant tout de former des hommes cultivés, possédant cette "culture générale" si difficile à définir dogmatiquement, mais dont l'idée est cependant fort claire » [Borel 1914, 199] est remise en cause par les partisans d'un enseignement strictement utilitaire. Borel, tout en défendant « une évolution lente, sage et prudente » souligne

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5) Professeur à l'université de Budapest, délégué hongrois à la CIEM.

6) Borel prononce le 3 mars 1904 une conférence dans laquelle il prône, entre autres, une orientation plus pratique des exercices de mathématiques et défend une conception à la fois théorique et pratique de l'éducation mathématique [Borel 1904].

d'ailleurs les dangers qu'il y aurait à suivre une tendance trop utilitariste et à suivre de trop près les modes passagères.

Une crise de l'enseignement des mathématiques qui appelle des réformes profondes

Les mathématiques étant reconnues comme l'instrument indispensable à l'étude des phénomènes naturels et économiques, le rôle formateur des mathématiques est réévalué. Les mathématiciens et les enseignants de mathématiques de tous pays accomplissent en quelques années une véritable révolution culturelle. Les nouveaux programmes et les nouvelles instructions insistent sur la nécessité d'un enseignement qui s'inscrive dans la vie pratique et offre de nombreuses applications. L'élève doit se rendre compte des liens nombreux qui existent entre les sciences mathématiques et la vie pratique. Il faut agir sur le développement de la pensée non pas par des connaissances isolées mais par des connaissances qui soient en relation étroite avec l'activité journalière et les idées usuelles. Pour réaliser cet objectif les réformateurs, en général, insistent sur quatre points :

1) tenir compte des domaines de la vie pratique, en particulier développer l'intérêt pour les questions économiques ;
2) développer la conception de l'espace en présentant la géométrie de manière plus intuitive [7]) et en centrant l'étude de la géométrie sur celle de transformation géométrique [8]) ;
3) utiliser les représentations graphiques ;
4) introduire les éléments du calcul infinitésimal avec les applications.

Il faut aussi noter que le souci d'assurer une formation mathématique plus appliquée est le signe de la crainte de voir l'enseignement des mathématiques assuré directement par les utilisateurs (professeurs de physique, de technologie, …). Cette inquiétude est particulièrement sensible en Allemagne. Ainsi, H. E. Timerding [9]) fait état au congrès de Milan de la CIEM (1911), d'une « grande tendance *a*mathématique ou *anti*mathématique » en Allemagne. Il donne ainsi l'exemple d'une école forestière « où l'on a supprimé le poste

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[7]) Certains réformateurs proposent de présenter et même de démontrer expérimentalement les théorèmes de géométrie.

[8]) Pour plus de précisions, on peut consulter l'article de R. Bkouche dans ce volume ou [Bkouche 1991].

[9]) Professeur à l'École technique supérieure de Braunschweig, membre de la Sous-commission allemande de la CIEM.

de professeur de mathématiques et transmis l'enseignement mathématique à des professeurs sortant de la carrière forestière elle-même » [CIEM 1911, 485].

De même, C. Godfrey [10]), au congrès international des mathématiciens de Rome, dans sa description du processus de réformes en Angleterre, insiste sur l'importance des utilisateurs des mathématiques, en l'occurrence les ingénieurs. En effet, selon lui [Godfrey 1908], malgré la prise de conscience de certains enseignants, l'institution ne put se réformer elle-même et « l'impulsion nécessaire vint des ingénieurs ». Ces derniers se sont rendu compte de la nécessité d'une meilleure formation scientifique pour leur corps et affirment « qu'on ne peut pas savoir trop de mathématiques pourvu que ce soit de bonnes mathématiques ». Avec la création au sein de l'université de Cambridge d'une section d'ingénieurs, l'enseignement des mathématiques « sans base pratique » est en question.

> Ce mouvement amena la formation de divers comités qui comparèrent les opinions des hommes du métier et des maîtres d'école et trouvèrent que l'accord était possible sur la plupart des points. Les professeurs reconnurent que des sujets utiles pouvaient être aussi éducatifs que les futilités conventionnelles qui avaient fini par s'identifier avec les mathématiques enseignées dans les écoles. De même que les mathématiques supérieures pures gagnent en valeur et en intérêt par un contact plus intime avec les problèmes posés par les physiciens et deviennent en revanche irréelles et sans but quand elles sont séparées de leurs applications, de même les mathématiques élémentaires ont trouvé leur salut dans l'introduction des applications sans nombre fournies par la vie industrielle moderne. [Godfrey 1908, 462]

Les nouveaux programmes d'arithmétique intègrent ainsi l'usage des tables de logarithmes à 4 décimales plus pratiques et « pouvant être enseignées dès l'âge de 14 ans ». Quelques écoles ont même prévu un cours de travaux expérimentaux simples dans un laboratoire (pour les élèves de 13 à 15 ans) et faisant explicitement partie du cours de mathématiques [Godfrey 1908, 464]. De même, l'enseignement de la géométrie comporte une part expérimentale, l'enseignement de la trigonométrie est essentiellement numérique (usage de tables, résolution de problèmes pratiques basés sur des observations faites par les élèves avec un théodolite simplifié) [Godfrey 1908, 472]. Quant à la mécanique, l'enseignement de la statique est fondé sur un cours expérimental. Par contre, l'enseignement de la dynamique reste théorique car « il n'est pas facile d'organiser des travaux expérimentaux sur ce sujet ». Il en résulte des difficultés pour une grande part des élèves concernant cette partie. La conclusion de Godfrey est que la trigonométrie et la statique formeront peut-

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10) Directeur du R. N. College, Osborne.

être pendant un certain temps la limite des études mathématiques de la plupart des jeunes gens d'une école publique.

Dans une note de 1909 du *Board of Education* [11]) concernant l'enseignement de la géométrie dans l'enseignement secondaire, il est indiqué qu'« il est maintenant usuel d'adjoindre plus ou moins de travail graphique à l'algèbre » et qu'ainsi « on fait entrer un élément de réalité dans un sujet très abstrait et irréel, ce qui ne peut avoir que de bons résultats » [Bruce 1910, 248]. Des exemples de résolution graphique d'équations du second degré, et de représentation de fonctions du second degré, sont décrits. Un premier intérêt (pratique) d'une telle méthode est d'entraîner au calcul numérique ainsi qu'à « l'exactitude dans les mesures et le dessin ». D'autre part, des élèves plus avancés pourront étendre ces techniques aux équations de degré 3 et pourront « se rendre compte de la puissance de la méthode qu'ils ont entre les mains ». Le second intérêt (pédagogique) est qu'avec ces méthodes, « les élèves se rendront ainsi évidemment maîtres des notions essentielles de l'algèbre ». Cependant, il ne faut pas sacrifier la rigueur de l'exposé de l'algèbre, et le travail graphique « ne doit pas être un but, mais un moyen ».

En Allemagne, le courant réformateur, particulièrement actif, est organisé autour de la personnalité de Félix Klein qui depuis fort longtemps prônait une modernisation des enseignements de mathématiques et une prise en compte plus importante des mathématiques appliquées [12]). Les propositions de Méran (1905) et de Stuttgart (1906) définissent le renforcement de l'intuition de l'espace en fondant l'enseignement de la géométrie sur la notion de transformation géométrique et le développement de l'idée de fonction comme objectifs principaux de l'enseignement des mathématiques dans les écoles supérieures.

Pour parvenir à ce but, il faut

1) ordonner l'enseignement de façon à mieux l'adapter au développement naturel de l'esprit;
2) développer autant que possible cette faculté d'observation mathématique des phénomènes qui nous entourent par un choix approprié d'applications;
3) arriver peu à peu à la conception de l'unité de la science en concentrant tout l'enseignement autour de la notion de fonction, aussi bien au point de vue algébrique qu'au point de vue géométrique. [EM 1911a, 67]

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[11]) Bureau du gouvernement britannique qui s'occupe des questions d'enseignement public (au sens français du terme). Le *Board of Education* avait pris en charge l'organisation et les travaux de la Sous-commission britannique de la CIEM.

[12]) Sur le mouvement de réforme de l'enseignement des mathématiques en Allemagne, on peut consulter en particulier [Schubring 1989], [Tobies 1989] ou l'article de G. Schubring dans ce volume.

Ces propositions sont appliquées plus ou moins officiellement dans la plupart des États allemands à partir de 1908. De plus, en Prusse, il est accordé aux trois genres d'écoles (*Gymnasien*, *Realgymnasien* et *Oberrealschulen*) l'égalité « en ce qui concerne les droits qu'elles accordent », traduisant ainsi institutionnellement un certain rééquilibrage entre les formations classique et moderne ainsi qu'entre enseignement général et enseignement à vocation plus technique.

La réforme de 1902 en France accorde aux sciences une place beaucoup plus importante dans les filières classique et moderne de l'enseignement secondaire. Les nouveaux programmes accordent aux formations scientifique et littéraire la même importance ; la volonté affichée par le ministère est de promouvoir une culture scientifique au même titre qu'une culture littéraire [13]).

> Les lettres sont et resteront comme par le passé, des institutrices éprouvées qu'il serait impossible de suppléer dans leur domaine. Mais dans le domaine qui est celui des sciences positives, on attend des sciences plus d'effets que par le passé, pour la formation des esprits. (L. Liard (1904), cité par Beke [1914, 245–246])

Pour Gaston Darboux, à qui fut confiée la présidence de la commission de révision des programmes scientifiques, les principaux acquis en mathématiques de la réforme de 1902 se résument aux quatre points suivants :

> 1° l'introduction dans l'enseignement élémentaire du Calcul des dérivées et même de notions de Calcul intégral ;
> 2° l'emploi systématique dans la géométrie des méthodes de transformation qui simplifient l'étude et apportent un principe de classification ;
> 3° le développement donné aux applications qui sont posées par la pratique, à l'exclusion de ces problèmes qui n'ont aucune racine dans la réalité ;
> 4° le développement aussi complet que possible de l'initiative personnelle chez tous les élèves qui prennent part à l'enseignement et une préoccupation incessante d'une bonne formation de l'esprit.
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> [Darboux 1914, 197]

Partout en Europe [14]) se pose la question de réformer l'enseignement des mathématiques dans un sens plus pratique, ce qui se traduit le plus souvent par la volonté de fonder à partir de l'expérience quotidienne l'enseignement de la géométrie et celle d'introduire dans les cursus le calcul différentiel et intégral, reconnu comme l'outil essentiel des applications des mathématiques dans les autres sciences.

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13) Pour plus de précisions sur la réforme de 1902 en France, on peut consulter les ouvrages de B. Belhoste [1995], N. Hulin [2000], ainsi que [Belhoste, Gispert & Hulin 1996].

14) Le mouvement de développement des enseignements de mathématiques à la fin du 19e siècle n'affecte pas l'Italie où il subit un recul, devenant même optionnel dans certaines filières [EM 1912c, 253].

2. Les applications dans les discussions autour de l'enseignement des mathématiques

Dans sa déclaration d'intention [CIEM 1908, 445–458], la CIEM se donne comme objectif de « faire une enquête et publier un rapport général sur les tendances actuelles de l'enseignement mathématique dans les divers pays » [CIEM 1908, 450]. Cet état des lieux concerne l'enseignement de la première initiation à l'enseignement supérieur, et aussi bien l'enseignement général que l'enseignement technique ou professionnel. Même si elle n'est pas cantonnée à cet ordre d'enseignement, la question des mathématiques appliquées est ressentie par les auteurs de *L'Enseignement Mathématique* comme beaucoup plus cruciale lorsqu'il s'agit des formations techniques, en raison en particulier de leur relative nouveauté et donc d'une moindre pesanteur des traditions.

> En raison de l'importance croissante que prennent ces écoles [les écoles techniques ou professionnelles] et des exigences nouvelles qu'on ne cesse de montrer vis-à-vis de l'enseignement mathématique, il y aura lieu d'accorder dans cette enquête une large place aux mathématiques appliquées. [CIEM 1908, 452]

A quoi servent les mathématiques dans une formation ?

Les partisans les plus résolus des réformes, tout en insistant sur la valeur utilitaire des mathématiques, ne cèdent en rien aux opposants ou aux réticents [15]) quant à la valeur éducative et disciplinaire de celles-ci. Certes, les positions sont plus ou moins nuancées selon les types de formation et le statut de “branche principale” ou “branche secondaire” que les mathématiques y occupent. De plus, les discours sur la mission culturelle de l'enseignement concernent essentiellement l'enseignement secondaire classique et moderne. Néanmoins, les réformateurs vont s'attacher à montrer que l'ouverture des *curricula* aux applications et la disparition des programmes de mathématiques d'un certain nombre de notions et de méthodes, comme certaines virtuosités purement calculatoires de géométrie analytique ou d'arithmétique, au profit de l'introduction du calcul différentiel et intégral ne constituent en rien un appauvrissement du contenu du cours de mathématiques et encore moins un affaiblissement de sa valeur formatrice.

Bourlet, qui défend la prééminence du rôle social de l'enseignement en général, prône bien entendu un enseignement essentiellement utilitariste :

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[15]) Les contributeurs de *L'Enseignement Mathématique* et les participants aux diverses commissions de la CIEM sont pour l'essentiel des partisans des réformes. Le discours des opposants n'apparaît donc, dans le corpus que nous nous sommes fixé, qu'indirectement, dans les réponses des partisans des réformes.

> […] il ne nous est plus permis maintenant de présenter à nos élèves la science mathématique sous un aspect purement spéculatif et […] il nous faut, coûte que coûte, plus encore pour rendre service à la société dans son ensemble, qu'à chacun de nos étudiants en particulier, nous efforcer de faire plier les abstractions mathématiques aux nécessités de la réalité. [Bourlet 1910, 374]

Selon Bourlet, le fait majeur au début du 20e siècle est la prédominance de l'industrie parmi les activités humaines. La nécessité se fait jour de préparer « les jeunes gens […] à connaître, à pratiquer et à faire progresser les sciences expérimentales où cette industrie puise ses forces ». Il faut donc écarter de l'enseignement des mathématiques tout ce qui n'aura pas une utilité plus ou moins directe dans les applications. Cependant, un tel programme ne conduit pas nécessairement à une baisse de contenu du cours de mathématiques, car si l'« on fait un tableau complet des connaissances strictement indispensables à un ingénieur ordinaire, on s'aperçoit aussitôt que le champ ainsi borné est encore immense ». Parmi les évolutions nécessaires, l'enseignement de l'analyse en France a dû s'adapter aux sciences appliquées en introduisant par exemple la notion de fonction, « base de toute étude des phénomènes naturels, et de sa représentation graphique » dans des manuels de préparation au baccalauréat. Cette évolution fondamentale du programme d'analyse que constitue l'introduction du calcul différentiel dans les programmes des dernières années du lycée est due, selon Bourlet, à l'apparition d'enseignements de physique et de mécanique dans l'enseignement secondaire. Il conclut en affirmant que « la limite entre les mathématiques pures et les mathématiques appliquées n'existe pas, car ces deux sciences, loin d'être séparées, doivent sans cesse s'entraider et se compléter » [Bourlet 1910, 386].

De même, la réforme qui s'opère en Allemagne dans les années 1908–1909 passe par une orientation plus pratique et plus appliquée des mathématiques et accorde à l'intuition et aux applications un rôle prépondérant [EM 1910a, 63]. En 1912, dans un rapport sur les problèmes commerciaux et l'enseignement des mathématiques dans les écoles secondaires, Timerding [EM 1912a, 60–63] reprend la discussion de l'objectif et de l'utilité de l'enseignement des mathématiques.

> Tout l'enseignement dépend du but que l'on assigne à l'école. Les uns veulent que, par une gymnastique intellectuelle intense, elle habitue l'esprit à bien penser et craignent toutes les questions pratiques que compliquent trop les contingences de la vie pour qu'elles soient un bon aliment de la pensée pure. Les autres, se défiant des esprits trop logiques, désirent, au contraire, que l'école inculque des connaissances précises à ses élèves et les mette en contact avec la complexité des choses. [EM 1912a, 60]

Il ne choisit pas entre ces deux points de vue dont on a vu qu'ils dépendent essentiellement du public auquel on s'adresse ; par contre, il développe une idée intéressante : un enseignement délibérément appliqué comme celui de l'arithmétique politique (un mélange de théorie des probabilités appliquées, de mathématiques financières, de statistiques, de mathématiques appliquées aux assurances, …) peut néanmoins satisfaire les exigences culturelles de l'enseignement secondaire général et est l'occasion de montrer que « les notions mathématiques ne sont pas arbitraires, mais qu'on y a été amené par la force des choses ».

On retrouve le même type de préoccupation dans un rapport sur l'enseignement des mathématiques dans les *Realschulen* suédoises. Le but de l'enseignement des mathématiques dans ces établissements à vocation intermédiaire et de caractère technique doit être essentiellement pratique, ce qui nécessite des méthodes d'enseignement pratique. Mais, « la résolution d'un problème par une équation ne doit pas exclure systématiquement le raisonnement lorsqu'il peut être utile » [EM 1911e, 171].

En lisant ces déclarations, on pourrait penser que l'opposition entre les points de vue des tenants et des opposants des réformes est, en fait, des plus ténues. Pourtant, le fossé reste beaucoup plus profond qu'il n'apparaît dans le discours des réformateurs. En effet les opposants aux réformes, en exprimant leurs inquiétudes relatives au contenu et à l'affaiblissement de la valeur éducative du cours de mathématiques, s'alarment surtout de la perte d'autonomie de cet enseignement. Il est vrai qu'en insistant sur les applications des mathématiques et les liens de celles-ci avec les autres sciences, les réformes remettent en cause l'architecture de la formation mathématique dans l'enseignement secondaire (et supérieur) et soumettent l'enseignement purement disciplinaire des mathématiques à celui plus général d'une méthode scientifique [Perry 1909, 137].

Ainsi, les nouveaux programmes des mathématiques dans les établissements secondaires supérieurs en Allemagne visent explicitement à donner aux élèves :

> un coup d'œil scientifique sur la parenté des sujets mathématiques traités à l'école ; une certaine aptitude de la conception mathématique et son emploi à la résolution de problèmes particuliers ; enfin et surtout la pénétration de l'importance des mathématiques pour la connaissance exacte de la nature. [EM 1906, 58]

Avec de tels objectifs, l'élève est donc censé acquérir une formation en mathématique « non seulement précieuse en elle-même » mais qui sera utile dans l'exercice de sa profession, au moins pour ceux qui se destinent à une carrière scientifique ou technique. On retrouve le même type de soucis pédagogiques dans les nouveaux programmes autrichiens publiés en 1909 :

> Dans les conditions actuelles des écoles réales, l'enseignement mathématique a pour but la pratique des mathématiques élémentaires, y compris la notion de fonction, comme préparation aux écoles supérieures ; il ne doit pas avoir en vue une culture spéciale, mais contribuer au développement général de l'esprit par la science. [EM 1910b, 338]

Pour ce qui intéresse les mathématiques appliquées et plus généralement la notion d'application des mathématiques, ce programme se traduit par :

1) une simplification du champ d'étude par la liaison des branches ayant des relations les unes avec les autres, et
2) une adaptation du programme de mathématiques aux branches correspondantes et aux applications de la vie réelle.

Quelle rigueur pour un enseignement pratique ?

Une des principales inquiétudes quant aux réformes est qu'en s'ouvrant aux applications, l'enseignement des mathématiques perde toute rigueur. Traditionnellement, le rôle formateur essentiel des mathématiques est justement de faire acquérir aux élèves la méthode logique et l'esprit de rigueur. Ceux qui défendent un enseignement fondé sur les applications considèrent que le rôle formateur des mathématiques réside aussi dans l'acquisition d'une méthode scientifique dans laquelle les mathématiques ont toute leur place.

A. N. Whitehead [1913, 105–113], qui évoque cette crainte d'une perte de la rigueur dans l'enseignement mathématique, distingue deux catégories de public dans l'enseignement élémentaire : ceux qui désirent limiter leur formation mathématique et ceux qui au contraire ont besoin d'une éducation mathématique plus conséquente pour leur carrière professionnelle future. Pour les premiers, selon Whitehead, l'enseignement mathématique, même s'il doit rester à un niveau élémentaire, doit viser deux objectifs : développer la faculté d'abstraction et développer la faculté de raisonnement logique. Aussi l'enseignement des mathématiques doit-il être d'une rigueur logique sans concession. Cependant, il ne faut pas oublier que la précision logique est un but et non le point de départ de l'enseignement, et donc celle-ci doit être obtenue « par approximations successives ». Concernant la seconde catégorie d'élèves, il considère comme une erreur profonde l'opinion (largement majoritaire à ses yeux) qu'il soit possible d'enseigner les mathématiques avancées du seul point de vue de l'utilité pour les physiciens ou ingénieurs sans s'intéresser à la logique et la théorie. Il est important pour les physiciens et ingénieurs d'avoir un esprit entraîné mathématiquement (*mathematically trained mind*) et on ne peut pas se contenter d'une connaissance quasiment mécanique en vue des applications. Whitehead concède que l'on peut introduire les notations et les premières notions du seul point de vue des applications mais l'éducation mathématique des scientifiques appliqués doit consister à rendre ces notions précises et à donner des démonstrations précises. Veronese défend une position analogue au congrès de Milan de la CIEM (1911). Une des questions soumises à la discussion du congrès était : Dans quelle mesure peut-on tenir compte, dans les écoles moyennes (lycées, collèges, gymnase, écoles réales, etc.), de l'exposé systématique des mathématiques ? La discussion tourne autour d'une opposition entre un enseignement déductif, supposé rigoureux, et un enseignement intuitif expérimental, plus laxiste. Pour Veronese en raison du rôle éducatif des mathématiques, le rôle d'un enseignement de type intuitif et expérimental doit être réduit à préparer un enseignement axé sur la seule déduction, au moins pour les écoles qui préparent à l'enseignement supérieur. En particulier,

> si l'industrialisme ou l'utilitarisme matériel avait […] des influences prépondérantes dans l'enseignement des écoles moyennes, les mathématiciens devraient les combattre. [16]) [CIEM 1911, 465]

Même Bourlet, dont on a vu l'engagement militant pour un enseignement plus pratique et plus appliqué des mathématiques, affirme qu'il ne faut « rien sacrifier des qualités de rigueur, de logique et de précision qui sont l'apanage des mathématiques ». De même, Timerding, qui ne veut surtout pas aller à l'encontre de la tendance expérimentale qui se développe dans l'enseignement de la physique en Allemagne et qui prône un enseignement des mathématiques qui, sans renoncer à ses buts propres, doit « tenir compte de la réalité et des applications », a le souci non seulement de la rigueur de cet enseignement mais aussi que ces exigences ne soient pas contredites lors des applications :

> On ne doit pas, ici, rappeler l'élève à la rigueur et à l'exactitude, alors que là un laisser-aller commode dans l'expression et le raisonnement est non seulement permis, mais encore donné en exemple. [EM 1911b, 69–70]

Pourtant, d'autres arguments se font entendre ; selon le même Timerding, les objectifs que l'on assigne à l'enseignement mathématique interrogent aussi la méthode de cet enseignement. En particulier, il se demande s'il faut « observer toute la rigueur même dans un enseignement élémentaire où l'on ne veut traiter que les premiers principes dans un but pratique » ou au contraire si l'on ne peut pas se servir d'une induction partielle au lieu de la déduction pure « pour

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[16]) Veronese nuance légèrement sa position en soulignant qu'il ne faut « pas faire de la rigueur excessive dans l'enseignement moyen ».

faciliter les études » ou même avoir recours à des méthodes expérimentales [CIEM 1911, 495].

De la même manière, G. Scorza [17]), auteur d'un rapport sur l'enseignement des mathématiques dans les écoles et instituts techniques italiens (enseignement secondaire), regrette les « préoccupations excessives de prudence rigoriste » qui compliquent inutilement l'apprentissage [EM 1912e, 420]. Au congrès de Milan, J. W. A. Young [18]) accorde, lui aussi, une importance capitale aux approches de type pratique et considère que

> les débuts devraient se faire d'une façon concrète, aussi bien dans l'enseignement secondaire en général que dans le travail d'une année particulière ou dans l'exposition d'un sujet spécial quelconque; les procédés abstraits (abstraits relativement à la maturité et au degré d'avancement de l'élève) n'apparaissant que pour éviter de trop nombreuses répétitions concernant des exemples concrets essentiellement pareils. [Young 1911, 474]

Selon Young, le but de l'enseignement de la classe n'est pas de faire des mathématiques abstraites, mais plutôt des mathématiques présentant par-ci par-là des procédés abstraits.

Quelle place pour les applications dans un cours de mathématiques ?

La question du contenu d'un cours de mathématiques appliquées dans l'enseignement secondaire n'est jamais abordée explicitement dans les débats de *L'Enseignement Mathématique*. Une des raisons est qu'il n'y a pas d'accord sur la formation des utilisateurs de mathématiques : doit-on dispenser un cours de mathématiques générales centré autour d'un corpus de notions considérées comme utiles ou, au contraire, un cours centré sur les seules applications pratiques ? Une solution est de tenter de concilier les deux objectifs :

> Il faut que ce programme renferme des questions d'ordre réellement pratique et ne soit pas réduit à une pure gymnastique cérébrale, ce qui ne veut pas dire toutefois que le cours de mathématiques appliquées soit transformé en un cours de physique expérimentale.
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> Un programme bien compris, qui initierait les auditeurs aux méthodes fondamentales de la physique et leur fournirait en même temps des résultats de nature mathématique en évitant cependant de trop grandes difficultés analytiques, constituerait une excellente base d'action commune pour le mathématicien et le physicien. [EM 1912b, 73]

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[17]) Professeur à l'Institut technique de Palerme, délégué italien à la CIEM.

[18]) Professeur à l'Université de Chicago, délégué des États-Unis à la CIEM.

De même, Fehr soutient une position médiane et affirme qu'à côté des cours destinés aux mathématiciens et physiciens, il est nécessaire « qu'il y ait un cours dit de mathématiques générales portant sur les notions les plus utiles ». Il souligne que d'ailleurs les problèmes commencent avec la définition de ce que sont ces notions. Pédagogiquement, il insiste sur la nécessité de travaux pratiques qui « doivent montrer à l'étudiant, mieux qu'on ne peut le faire par des exemples dans un cours général, comment les mathématiques interviennent réellement dans les applications ». Il ajoute :

> Il est désirable que les écoles supérieures apportent une attention toute spéciale au développement de cet enseignement pratique pour en faire un véritable *laboratoire mathématique.* [CIEM 1911, 494]

De plus, il faut distinguer les formations pour lesquelles les mathématiques sont une matière principale et celles où elles sont considérées plutôt comme une « branche accessoire, destinée à abréger certains raisonnements et à formuler d'une façon particulièrement brève tout un ensemble de résultats ». Ainsi, P. Rollet [19]), dans son rapport sur l'enseignement technique secondaire en France, présente l'enseignement mathématique dans ce type de formation comme n'étant ni une fin, ni un but (au contraire des formations généralistes dispensées dans les collèges et lycées). Il y a donc lieu d'écarter toutes méthodes et démonstrations « qui ne concourent pas à la fin cherchée ou au but poursuivi », à savoir former des ouvriers, des contremaîtres ou des techniciens. Il est par contre vivement conseillé d'insister sur les liens avec les cours techniques et les applications.

> Acceptant l'influence du milieu technique dans lequel ils vivent, les professeurs de mathématiques ont su caractériser nettement leur enseignement et lui donner son adaptation pratique, tout en ne perdant pas de vue le rôle éducatif qui reste le propre des mathématiques. [EM 1912d, 325]

Selon H. Grünbaum [20]), dans un rapport sur l'enseignement mathématique en Allemagne dans les écoles techniques moyennes pour l'industrie mécanique, ces dernières assignent aux mathématiques le rôle de science accessoire (par opposition à celui de science éducative dans les lycées), destinée à résoudre des problèmes techniques. Dans ce type de formation, les applications sont le but suprême à poursuivre, et l'enseignement des mathématiques, des sciences naturelles et des branches techniques s'y fait simultanément. Les résultats mathématiques principaux doivent être énoncés et démontrés, en écartant

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[19]) Directeur de l'École municipale professionnelle Diderot à Paris, membre de la Sous-commission française de la CIEM.

[20]) Collaborateur de la Sous-commission allemande de la CIEM.

systématiquement tous les sujets qui n'ont pas d'applications techniques, tels que la trigonométrie sphérique, la géométrie synthétique, etc. Le choix des sujets traités dans le cours doit se conformer au critère de l'application pratique :

> A ce titre, les calculs les plus simples, les constructions géométriques les plus élémentaires doivent être exercés aussi bien que les parties soi-disant supérieures des mathématiques, tels que les éléments du calcul infinitésimal, dont l'emploi est courant dans les publications techniques. [EM 1911c, 156]

La géométrie descriptive, la mécanique et les méthodes graphiques constituent le programme de mathématiques appliquées.

De même, les nouveaux programmes autrichiens des écoles techniques insistent sur l'importance du calcul numérique et de la notion d'approximation, sur la possibilité de présenter graphiquement la fonction logarithmique et sur l'utilisation des tables numériques. Les instructions insistent sur l'importance des exercices et des problèmes qui doivent « toucher aux différentes branches de l'enseignement » et présenter des rapports avec la vie courante [EM 1910b, 330–332].

Enfin, il faut tenir compte des traditions souvent nationales qui influent sur la conception générale du cours de mathématiques. Dans la conclusion de sa conférence déjà citée, Bourlet [1910] précise que l'enseignement en France est traditionnellement généraliste et « oblige les élèves à recevoir une instruction générale très étendue ». Il poursuit en soulignant qu'en France, à la différence de l'Allemagne, « on ne concevrait pas un cours de mathématiques uniquement pour des chimistes fait dans l'esprit de la spécialisation étroite ».

Il ne faut pas négliger non plus que la question d'un enseignement de mathématiques appliquées est à la fois beaucoup plus cruciale et plus simple à résoudre pour les formations supérieures d'ingénieurs ou de physiciens que dans l'enseignement secondaire même technique. En effet, personne ne remet en cause la nécessité d'un enseignement de mathématiques pour ces formations et tout le monde s'accorde avec les idées exposées par E. Possé dans son rapport sur l'enseignement technique supérieur en Russie :

> L'étude des sciences mathématiques n'est pas le but principal des ingénieurs, mais elle leur est indispensable comme étude auxiliaire, les Mathématiques étant la base de toutes les sciences techniques précises. [EM 1911f, 337]

Dans les formations techniques supérieures, le cours de mathématiques appliquées apparaît souvent comme un complément d'une formation générale antécédente. La formation des techniciens et ingénieurs, selon Jouglet[21]) dans une conférence sur l'organisation de l'enseignement technique dans

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[21]) Ingénieur des arts et métiers.

les écoles d'arts et métiers [CIEM 1910, 401], doit comporter trois volets : a) la culture générale, b) la culture scientifique et c) la culture technique [22]). Dans cette perspective, l'enseignement mathématique est défini comme l'étude de compléments de mathématiques nécessaires à la poursuite des questions de mécanique et d'électricité. De la même manière, E. Czuber [23]), dans sa description de l'enseignement des mathématiques pures à l'école technique supérieure de Vienne [EM 1911d, 164–166], assigne à cet enseignement (calcul infinitésimal et géométrie) de servir de base aux autres branches, de permettre à l'étudiant la lecture des livres techniques et de lui donner une indépendance mathématique suffisante. Cet enseignement est complété par un cours de mathématiques appliquées ; à cet égard, il faut signaler un exemple intéressant concernant cette école, celui du cours de technique d'assurance qui comprend, outre des conférences de mathématiques pures, le calcul des probabilités, la statistique mathématique et les mathématiques des assurances.

Concernant l'enseignement secondaire, les débats autour de l'idée d'une formation mathématique appliquée, ou au moins tenant compte des applications, se polariseront souvent autour de l'introduction de la notion de fonction et de l'initiation au calcul différentiel.

Bourlet précise que l'enseignement des lycées suffit pour donner aux jeunes gens les connaissances mathématiques (géométrie pure, analytique et descriptive, trigonométrie, algèbre, calcul différentiel et intégral) dont on peut avoir besoin dans le commerce, l'économie politique et même les constructions civiles et l'architecture.

Poske [24]), dans une conférence sur l'enseignement de la physique [CIEM 1910, 392], pose très nettement la question de la formation mathématique des physiciens et affirme que les notions fondamentales du calcul infinitésimal doivent être fournies par l'enseignement mathématique. De même, la conclusion la plus importante du rapport de Timerding sur les mathématiques dans les traités de Physique est

> qu'il est urgent d'introduire les notions de dérivée et d'intégrale dans le programme de mathématiques des collèges, et cela assez tôt pour qu'elles puissent être utilisées et appliquées concrètement dans les leçons de physique des classes supérieures. [EM 1911b, 71]

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[22]) Jouglet présente l'organisation des enseignements dans les écoles des arts et métiers. Ces écoles constituent l'aristocratie des écoles techniques en France et revendiquent pour leurs meilleurs diplômés le titre d'ingénieur. La présentation des *curricula* en termes de culture est caractéristique d'une volonté de se démarquer des autres établissements d'enseignement moins prestigieux, qui ont moins de revendications culturelles.

[23]) Professeur à l'École technique supérieure de Vienne, délégué autrichien à la CIEM.

[24]) Professeur à Berlin, membre de la Sous-commission allemande de la CIEM.

Les mêmes arguments sont utilisés par le rapporteur sur l'enseignement mathématique en Suède, qui souligne que « le plan d'études tend à donner à la notion de fonction la place de notion centrale fondamentale » et qu'« il semble que l'expérience ait démontré que la notion d'intégrale elle-même peut être enseignée à des élèves de capacités moyennes et qu'elle peut être pour eux d'un grand intérêt et d'une réelle utilité » [EM 1911e, 341]. En particulier, l'introduction des notions fondamentales du calcul infinitésimal permet aux élèves qui se destinent aux études techniques supérieures, d'aborder celles-ci avec plus de facilité.

CONCLUSION

Malgré la diversité de leurs discours, les réformateurs expriment de manière quasi unanime l'exigence de prendre en compte, dans l'enseignement et en particulier dans celui des mathématiques, les évolutions de la société et des demandes sociales vis-à-vis de l'enseignement. Cette exigence devrait, selon eux, se traduire dans les programmes par une prise en compte plus importante des applications des mathématiques, à la fois parce qu'il faut former professionnellement des ouvriers spécialisés, des techniciens ou des ingénieurs, mais aussi parce qu'il faut former des individus qui vont vivre en contact avec un monde de plus en plus industrialisé. De manière surprenante, à l'exception de quelques méthodes graphiques ou numériques, l'esprit de réforme ne suscite pas d'évolution radicale des programmes vers les applications, ni même d'accord ou de réflexion pour en dégager. La volonté de faire évoluer les *curricula* de mathématiques dans un sens plus pratique s'exprime certes par la prise en compte des besoins mathématiques des autres disciplines scientifiques, mais l'introduction du calcul différentiel et intégral dès l'enseignement secondaire supérieur est la marque essentielle de l'évolution des esprits au sujet du rôle formateur des mathématiques.

Si l'on ne peut nier la volonté chez les réformateurs de promouvoir un enseignement plus pratique des mathématiques, on peut néanmoins penser qu'à leurs yeux il est beaucoup plus important d'introduire le calcul différentiel et intégral dans les programmes. L'argument de la prise en compte des applications des mathématiques sert à la fois à convaincre les communautés des enseignants de mathématiques de l'inéluctabilité de ces évolutions, à défendre la place des mathématiques dans les *curricula* et à combattre les tentations de faire enseigner les mathématiques sous forme de recettes par les utilisateurs que peuvent être les enseignants de physique ou de disciplines techniques.

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APPLICATIONS : LES MATHÉMATIQUES COMME DISCIPLINE DE SERVICE DANS LES ANNÉES 1950–1960

Applications : mathematics as a service subject in the fifties and sixties

by Hélène GISPERT

The fifties and sixties saw the rise, all over the world, of the 'new maths' reform movement. Considering the harsh criticisms of these different reforms which were expressed as early as the seventies, we might wonder whether the reformers had even thought of mathematical applications. An examination of the journal *L'Enseignement Mathématique* has in fact been surprisingly rich in this respect. From the beginning of the fifties till the end of the sixties, ICMI — and the reformers — paid particular attention to the applications of mathematics, some of the ICMI activities and numerous initiatives being reported in *L'Enseignement Mathématique*.

The first part of this paper deals with two studies carried out by ICMI. The first one, launched in 1952, had its origin in the idea that "mathematical instruction at a given time is intimately linked to the function that mathematics and mathematicians fulfill at that time". The reports stressed the fact that the post-war period, which appeared dramatically new and obviously marked by the development of applications, entailed the necessity of a reform of teacher-training taking applied mathematics into account. The second study, ten years later, underlined the little attention paid to the teaching of mathematics applications in all countries except Scandinavia. We shall see what were, then, the positions and priorities of ICMI : at the end of this period (1967) a colloquium was held to answer the question "how to teach mathematics so as to be useful", which is not the same thing as "how to teach useful mathematics".

In the second part of the paper, I study the reflections which were made by other organizations engaged in renewing mathematical instruction, in particular within the framework of OECD. Several leading figures, fundamentally committed and representative of the new maths reform movement (especially in France and in Belgium), insisted on the necessary integration of applied mathematics in new mathematical curricula, both for its topics and its methods.

Lastly, in a third part, I consider the apparent contradiction between the inescapable reference to Bourbaki, which stuck to the new maths movement, and the strong concern of reformers about the applications of mathematics. We may also ask : what were the fields of application concerned when the reformers stressed the universal mathematical language and structures to the prejudice of traditional calculus or geometry ? In fact, what was involved there, was a weakening of the bonds between previously associated disciplines, which had not been replaced by new connections.

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par Hélène GISPERT

Cet exposé, comme c'est la règle pour ce colloque, se doit de combiner un thème et une époque en se pliant à une contrainte, celle de s'appuyer sur la revue *L'Enseignement Mathématique* et ce qu'elle a publié des réflexions et des débats en partie impulsés par la Commission internationale de l'enseignement mathématique (CIEM). Cette règle, dans mon cas, pourrait paraître paradoxale.

En premier lieu, la période des années 1950–1960 a été celle de la montée, partout dans le monde, du mouvement pour la réforme des mathématiques modernes. Au vu des critiques et bilans des différentes réformes, assénés dès les années 1970, on pourrait effectivement se demander s'il y avait une place pour les applications des mathématiques dans les réflexions des réformateurs.

En second lieu, toujours dans ces années, la revue *L'Enseignement Mathématique* était devenue avant tout une revue de mathématiques. Elle avait entamé une seconde série en 1955, après la parution de deux tomes pour l'ensemble des années 1942–1954 (1942–1950 pour le premier et 1951–1954 pour le second) qui terminaient la première série inaugurée en 1899. Que pouvait-on bien trouver concernant l'enseignement des mathématiques, et plus encore des mathématiques comme discipline de service, dans les numéros de ces deux décennies ?

En fait, et c'est en soi un premier résultat intéressant, le dépouillement fut étonnamment fructueux ; la CIEM, dont la vie et les nombreuses initiatives de cette période ont été à quelques exceptions près rapportées dans la revue, fut dès le début des années 1950 et jusqu'en 1970 préoccupée par les applications des mathématiques comme l'étaient d'ailleurs les réformateurs pour l'enseignement des mathématiques modernes, dont ce fut un souci premier. Cet intérêt de la revue m'a conduit à organiser mon propos comme suit.

Une première partie sera consacrée à deux enquêtes de la CIEM. La première, intitulée « Le rôle des mathématiques et du mathématicien dans la vie contemporaine », est une des deux premières enquêtes lancées par la nouvelle CIEM en 1952 ; ses résultats sont publiés dans la revue en 1955. L'autre enquête, qui date du début des années 1960, est présentée dans la revue en 1964 avec un titre en anglais : « Which subjects in modern mathematics and which applications in modern mathematics can find a place in programs of secondary school instruction ? ».

Dans une seconde partie, j'étudierai les réflexions menées dans le cadre d'autres organismes que la CIEM — mais d'une certaine façon en coopération avec elle — impliqués dans la rénovation de l'enseignement des mathématiques et en particulier dans le cadre de l'Organisation de coopération et de développement économiques (OCDE). Enfin, j'examinerai dans la troisième partie l'apparente contradiction entre l'incontournable référence à Bourbaki, qui accompagne le mouvement de réforme des mathématiques modernes, et ce souci appuyé des applications des mathématiques qu'ont manifesté les réformateurs.

1. Les mathématiques modernes comme discipline de service dans *L'Enseignement Mathématique* : d'une enquête à l'autre

Amsterdam, 1954 : le rôle des mathématiques et du mathématicien dans la vie contemporaine [1])

A l'origine de cette première enquête on trouve une idée affirmée dès les premières lignes du rapport que G. Kurepa présente en 1954 à Amsterdam au Congrès international des mathématiciens [2]) et qui, je pense, garde toute son actualité aujourd'hui. « L'enseignement des mathématiques à une époque donnée, écrit-il, est intimement lié avec le rôle que les mathématiques et les mathématiciens jouent à l'époque en question. » Il s'agit ici d'une question clé pour la CIEM qui devrait permettre « d'examiner le problème de l'éducation mathématique prise comme un tout ».

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[1]) Les documents relatifs à cette enquête sont publiés dans le tome de 1955 de *L'Enseignement Mathématique*, premier tome de la deuxième série. Dans le dernier tome de la première série, celui correspondant aux années 1951–1954, on trouve le compte rendu des décisions prises par le Comité exécutif de la CIEM pour la préparation du Congrès international des mathématiciens d'Amsterdam où figure l'annonce et la présentation de cette enquête [pages 72 à 75].

[2]) « Rapport général » [Kurepa 1955], suivi en annexe du texte de lancement de l'enquête et du questionnaire associé.

Six sous-comités nationaux de la CIEM avaient pris une part active à l'enquête au moment du congrès. Les délégués des sous-commissions nationales avaient en effet été invités à consulter des personnes qualifiées dans tous les domaines de la vie contemporaine et à ne négliger aucun des aspects du problème, « dans l'ordre social, dans l'ordre intellectuel, dans l'ordre scientifique, dans l'ordre des applications pratiques ». L'ensemble de leurs rapports ont été publiés dans la revue à la suite du rapport de synthèse que Kurepa présente au congrès [3]). Sans revenir sur l'ensemble des contenus de ces rapports, qui débordent bien évidemment du seul thème des applications, je m'attacherai ici à dégager quelques-unes des idées fortes ayant effectivement trait à notre thème.

Tout d'abord, la réflexion qu'a souhaité lancer la CIEM est présentée comme nécessaire dans la mesure où cette période de l'après-guerre est définie comme radicalement nouvelle et ce pour deux raisons. On assiste en effet à la fois à des « acquisitions vraiment révolutionnaires » dans le savoir mathématique et à la création de laboratoires de mathématiques dans des entreprises économiques, industrielles, commerciales, etc., « fait sans précédent dans l'histoire de l'humanité » [Kurepa 1955, 93 et 98].

Si le premier des deux arguments avait déjà été avancé cinquante ans plus tôt par Carlo Bourlet [1910], le second est nouveau et spécifique de cette période. Le rapport de synthèse et différents rapports nationaux (dont ceux des U. S. A. et des Pays-Bas) soulignent en effet l'influence des années de guerre dans cette évolution. L'après-guerre est manifestement une période nouvelle, marquée par le développement des applications. Cette nouveauté touche à la fois le savoir mathématique lui-même, avec le développement de nouvelles branches issues des recherches de guerre, et la place des mathématiques dans la société [4]).

Ainsi, la création de chaires de mathématiques appliquées dans les universités, le développement du nombre de mathématiciens travaillant hors du champ académique comme aux U. S. A. où leur proportion parmi les mathématiciens augmente très rapidement dans ces années : plus du quart des deux cents étudiants ayant obtenu leur doctorat en 1951 travaillent au moment de l'enquête dans le monde industriel ou pour le gouvernement. Toujours aux États-Unis, on emploie en 1952 cinquante fois plus de statisticiens qu'en 1945 dans le domaine du contrôle de qualité. Le rapport de Kurepa note la création

[3]) Il s'agit des rapports de E. Kamke (Allemagne), O. Weinberger (Autriche), G. Darmois (France), D. van Dantzig (Pays-Bas), G. Ascoli (Italie) et A. M. Gleason (U. S. A.).

[4]) Voir à ce propos l'étude d'Amy Dahan [1996] sur l'essor des mathématiques appliquées aux U. S. A. et l'impact de la deuxième guerre mondiale.

de nombreuses nouvelles sociétés savantes qui ont un lien étroit avec les mathématiques, comme les sociétés de statistique, les sociétés biométriques, les sociétés de mathématiques industrielles ou encore, aux U. S. A., la *Society for Quality Control* qui compte déjà six mille membres en 1953.

Dans ce contexte Kurepa s'interroge sur la définition du mathématicien et, considérant l'évolution des emplois des « gens s'occupant de mathématiques », il considère ce qu'il appelle des « mathématiciens au sens large » dont font partie les « mathématiciens pratiques que sont les ingénieurs ». Cela le conduit à questionner, sinon à rejeter, tout clivage entre mathématiques pures et mathématiques appliquées.

La rédaction même du questionnaire diffusé par la CIEM traduit un axe fort de cette enquête, à savoir la relation nouvelle que les mathématiques entretiennent avec les autres disciplines. Il est intéressant, tout d'abord, de relever le vocabulaire employé dans les rapports à ce propos; ils parlent d'« enchevêtrement », d'« interconnexions » des mathématiques aux autres disciplines. Ces autres disciplines appartiennent tout autant aux sciences dites dures qu'aux sciences humaines ou aux sciences économiques. Les rapports en multiplient les exemples qui, en particulier aux U. S. A., apparaissent également marqués par les domaines issus de recherches menées durant la guerre. Ils notent, de plus, que cet appel de mathématiques dans tant de disciplines n'est pas la conséquence d'une propagande ou d'une campagne de publicité des mathématiciens, mais correspond à une véritable demande autonome de ces différents champs.

Kurepa et les autres rapporteurs insistent alors sur le fait que c'est à partir de ces mutations de l'activité mathématique, des évolutions du contenu même de la notion de mathématiques et de mathématicien, qu'il s'agit de penser l'enseignement des mathématiques. Les applications, les besoins d'un « modern consumer of mathematics » [5]), impliquent la nécessité d'une réforme de la formation intégrant de nouveaux contenus théoriques fondamentaux; ce dont il y a besoin c'est d'applications des mathématiques modernes ou de mathématiques appliquées modernes. Sont citées, par exemple, l'idée de dépendance stochastique, l'introduction de l'aléatoire, la recherche opérationnelle, l'analyse numérique à cause de nouvelles machines à calculer, la statistique, la théorie de l'information…

Mais il y a plus; il est nécessaire, affirment-ils, de dépasser l'idée dominante selon laquelle l'enseignement secondaire devrait se réduire aux mathématiques pures et au seul raisonnement purement déductif. Ce dernier, en effet, ne saurait

[5]) Cette expression est employée par van Dantzig dans son rapport.

être transféré dans les différents domaines des mathématiques appliquées et on laisse alors de côté, par exemple, la logique inductive ou la notion de plausibilité.

Au-delà de ces constats sur la place des mathématiques et des mathématiciens dans la société, sur les conséquences attendues et nécessaires sur l'enseignement des mathématiques, un seul rapport rend compte de changements effectifs dans l'enseignement secondaire. Il s'agit du rapport de la sous-commission des Pays-Bas, seul pays où, par exemple, des éléments de statistique ont été introduits dans l'enseignement secondaire ; pour d'autres pays, les exemples de changement qui sont avancés dans les rapports concernent l'enseignement supérieur, comme en France avec l'Institut de statistique de Paris. Le rapport de Weinberger mentionne, par contre, les efforts, a priori non couronnés de succès, des congrès internationaux de statistique qui se sont tenus depuis la guerre pour l'introduction d'un enseignement de statistique dans les lycées.

STOCKHOLM, 1962 : WHICH SUBJECTS IN MODERN MATHEMATICS AND WHICH APPLICATIONS IN MODERN MATHEMATICS CAN FIND A PLACE IN PROGRAMS OF SECONDARY SCHOOL INSTRUCTION ? [6])

Cette deuxième enquête, lancée en 1958 par la CIEM, se positionne différemment par rapport à notre sujet. Son titre même détache les applications de ce qui serait le corpus de mathématiques modernes. Il y aurait d'une part les sujets des mathématiques, et d'autre part les applications de ces sujets. Cette mise à l'écart, qui tranche par rapport au ton de l'enquête précédente, est confirmée par le contenu de l'enquête.

L'enquête s'appuie sur 21 rapports nationaux dont seuls les noms des rapporteurs, avec leurs pays, figurent dans la revue [7]). Le rapport de synthèse présenté par J. G. Kemeny [1964] au Congrès international de 1962 souligne le degré de similarité de ces contributions nationales à la fois quant aux propositions pour introduire de nouveaux sujets mathématiques dans l'enseignement et

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[6]) Cette enquête de la CIEM est annoncée avec un titre en français dans la lettre circulaire du 28 juin 1958 du bureau de la CIEM aux dirigeants des sous-commissions nationales, publiée dans le tome 5 (1959) de *L'Enseignement Mathématique*. Le rapport de Stockholm sur l'enquête est paru dans le tome 10 (1964) de la revue [Kemeny 1964].

[7]) Les pays qui ont participé à l'enquête sont l'Allemagne, l'Angleterre, l'Argentine, l'Australie, le Danemark, la Finlande, la France, la Grèce, la Hongrie, l'Inde, Israël, l'Italie, le Luxembourg, la Norvège, les Pays-Bas, la Pologne, le Portugal, la Sierra Leone, la Suède, la Suisse et les U. S. A.

pour le peu d'attention accordé à l'enseignement des applications des mathématiques [8]).

Dans tous ces rapports, il n'est en effet envisagé qu'un seul sujet "appliqué", la statistique. Ils proposent l'introduction des probabilités et de la statistique qui apparaissent toujours couplées, les probabilités étant considérées comme sujet de mathématiques pures et la statistique comme sujet de mathématiques appliquées. On est loin de la diversité des réflexions de 1954.

Certains des rapporteurs nationaux s'émeuvent de la situation qu'ils ont pu constater dans leur pays où, alors qu'un énorme effort a été fait pour améliorer l'enseignement des mathématiques pures, le sujet des mathématiques appliquées a apparemment été oublié. Les Scandinaves ont, semble-t-il, une place d'avant-garde dans ce souci de promotion des applications : leurs rapports indiquent en effet que le *Scandinavian Committee for the Modernizing of School Mathematics* a adopté en 1960 un projet en cinq points dont le premier est ici remarquable. Il s'agit d'étudier les besoins en mathématiques à la fois pour le monde industriel et pour l'université et de proposer à partir de cela de nouveaux curricula.

Dans son rapport de synthèse, Kemeny souligne les insuffisances notées dans les différents rapports nationaux, relaie les inquiétudes des rapporteurs et présente des recommandations particulières. Il propose que l'enseignement des applications des mathématiques soit une des priorités dans les études de la CIEM pour les quatre années à venir ; il suggère à cette fin, mis à part la statistique, trois types d'applications : les applications des mathématiques à la physique, la programmation linéaire — peut-être le seul exemple, écrit-il, où l'étudiant pourrait constater un lien véritable avec les sciences sociales — et le libre usage des machines à calculer dans l'enseignement. Reprenant des points de l'enquête précédente de 1954, Kemeny insiste finalement sur le fait que l'enseignement de mathématiques appliquées suppose de développer dans l'enseignement de nouvelles habitudes de pensée qui diffèrent souvent de celles à l'œuvre dans les mathématiques abstraites.

Le rapport se termine enfin par la recommandation de lancer plusieurs études, dont une sur un problème notoirement négligé dans le passé et qui concerne les mathématiques appliquées : « How can the teaching of applied mathematics in our high schools be modernized ? »

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8) L'expression adoptée par Kemeny au début de la quatrième partie de son rapport consacrée aux applications des mathématiques est particulièrement forte : « It is painfully clear, in reading the 21 national reports, that ... » [Kemeny 1964, 166].

UNE DIFFÉRENCE DE TON MANIFESTE : ÉVOLUTION DE LA CIEM OU DIFFICULTÉS OBJECTIVES ?

Les applications des mathématiques et leur enseignement n'ont ainsi ni la même place ni le même rôle dans ces deux études de la CIEM. Cette différence de ton entre l'étude de 1954 et celle de 1962 est-elle due à une évolution dans les positions ou les priorités de la CIEM ? Ou bien est-elle due à des difficultés objectives à la prise en compte de ces applications dans les milieux mathématiques traditionnels responsables de l'enseignement ?

Il faut noter tout d'abord que l'enquête d'Amsterdam de 1954 aurait dû être poursuivie et élargie [9]). Ce ne fut pas le cas, comme en témoigne le compte rendu d'une réunion du comité exécutif de la CIEM en 1955 paru dans *L'Enseignement Mathématique* [CIEM 1955, 198–201]. Elle fut mise de côté, puis abandonnée, à la suite d'autres propositions d'enquêtes de Hans Freudenthal sur des points beaucoup plus techniques ou didactiques et sur des sujets mathématiques plus ciblés dont la géométrie.

En même temps, dans cette étude de 1954, le rapport de la sous-commission des U. S. A. soulignait les limites de la nouvelle situation créée par le développement d'après-guerre des applications des mathématiques. Le nouvel intérêt pour les applications les plus modernes des mathématiques dans un si grand nombre de disciplines, l'implantation de laboratoires de mathématiques dans l'industrie, l'économie, etc., n'avaient pas encore provoqué de changement majeur dans le milieu mathématique américain, la plupart des mathématiciens ayant un poste dans le monde universitaire. De même, Jeremy Kilpatrick mentionne que les réformateurs, aux U. S. A., étaient majoritairement des mathématiciens universitaires impliqués plutôt dans des recherches de mathématiques pures que dans des recherches de mathématiques appliquées [Kilpatrick 1996].

Cependant, malgré l'abandon — à mon avis significatif — de l'enquête de 1954, la CIEM ne va pas déserter le champ des applications des mathématiques. En premier lieu, en 1960, elle organise à Belgrade un colloque sur la coordination des enseignements de mathématiques et de physique et engage une enquête sur l'enseignement des mathématiques pour les physiciens. Cette enquête est présentée par Charles Pisot au Congrès international des mathématiciens à Moscou en 1966. Dans son rapport, publié par *L'Enseignement Mathématique*, il souligne que la préoccupation fondamentale de

[9]) Voir le compte rendu d'activité de la CIEM après le congrès d'Amsterdam : « Le rapport de la seconde enquête tentait de dresser un inventaire de l'application des mathématiques aux activités et métiers les plus variés. L'enquête, qui n'en est qu'à ses débuts, sera poursuivie et un rapport sera présenté au prochain congrès, à Edimbourg. » [Behnke 1951–54, 90]

l'enseignement au niveau 16–18 ans doit être la coordination aussi étroite que possible entre les mathématiciens et la physique. Là, écrit-il, est le nœud de l'enseignement secondaire [Pisot 1966, 203].

Il semble ici que la CIEM privilégie le lien à la physique, le lien avec les physiciens. Or, les recommandations que André Lichnérowicz, alors président de la CIEM, avance dans son rapport sur la période 1963–1966 insistent pourtant sur l'importance qu'il y aurait à prendre en compte les problèmes propres de liaison qui se posent à l'enseignement des mathématiques « non seulement pour les sciences de la nature et pour la technique, mais aussi pour les sciences économiques et sociales » [Lichnérowicz 1966]. Mais peut-être y a-t-il une réelle difficulté à dépasser le stade des intentions et à trouver au sein de la CIEM, ou des différents milieux mathématiques nationaux, des hommes pour mener à bien de telles initiatives. En effet, en 1967, la CIEM cherche à lancer en collaboration avec l'UNESCO un projet d'ouvrage qui illustre l'application des mathématiques à diverses sciences, mais il y a quelque difficulté à trouver un mathématicien qui se charge de sa réalisation. De plus, toujours en 1967, des propositions faites par la Grande-Bretagne pour d'éventuelles études de la CIEM sur les applications des mathématiques à la biologie et aux recherches sociales, ou sur l'importance du calcul automatique à tous les niveaux scolaires, ne sont pas retenues [CIEM 1967].

C'est sur un tout autre plan qu'est débattue, au colloque d'Utrecht organisé par la CIEM en 1967, la question de l'utilité des mathématiques pour les autres sciences et des conséquences à en tirer pour son enseignement. Son titre même, « How to teach mathematics so as to be useful », indique un glissement d'intérêt ; les préoccupations de ce colloque sont essentiellement d'ordre didactique et du registre de la classe, et les références aux contextes social, économique et même mathématique notées précédemment semblent avoir disparu. Ce colloque ne correspond à aucune enquête préalable de la CIEM et, il me semble important de le souligner, n'est ni annoncé, ni commenté dans *L'Enseignement Mathématique*. C'est dans le premier numéro de la toute nouvelle revue *Educational Studies in Mathematics* fondée par Hans Freudenthal, alors président de la CIEM, que sont publiés les Actes de ce colloque. Il semble que nous ayons là un tournant [10]) dans la nature des liens que la CIEM a entretenus avec la revue *L'Enseignement Mathématique*.

Dans son intervention à ce colloque, Henry Pollak (des *Bell Telephone Laboratories*), un des acteurs les plus présents des initiatives de ces années

[10]) Daniel Coray fait remarquer que 1967 est également l'année du décès de Jovan Karamata, qui dirigeait la revue depuis le début de la 2e série et qui avait aussi créé la collection des *Monographies de l'Enseignement Mathématique.*

sur l'enseignement des mathématiques, souligne le nouveau problème auquel ont à faire face les éducateurs : considérant qu'à la fin des années 1960 « it is just not possible to object to applications of mathematics as a part of the educational process. Why therefore do we have so much difficulty ? Why was it necessary to hold this meeting ? » [Pollak 1968] Une première difficulté tient à la contradiction suivante que Hans Freudenthal met en avant dans son intervention d'ouverture. En effet, indique-t-il, après avoir précisé qu'enseigner des mathématiques de manière à ce qu'elles soient utiles n'est pas la même chose qu'enseigner des mathématiques utiles :

> Useful mathematics may prove useful as long as the context does not change, and not a bit longer, and this is just the contrary of what true mathematics should be. Indeed it is the marvellous power of mathematics to eliminate the context. […] In an objective sense the most abstract mathematics is without doubt the most flexible. In an objective sense, but not subjectively […]. [11])
> [Freudenthal 1968, 5]

D'où la réponse que Willy Servais apporte à la question-titre du colloque, qui montre toute la diversité des biais possibles dans les références à cette notion d'utilité pour ce qui est des priorités à dégager pour les curricula : « *assurer les conditions d'apprentissage qui font pratiquer la mathématique pour ce qu'elle est : une activité créatrice de structures qui permettent de saisir la réalité* » [Servais 1968, 53].

Le colloque met en avant un deuxième type de difficulté, cette fois d'ordre pédagogique ou didactique, qui concerne tant les élèves ou les étudiants que les enseignants face à ce que l'on ne nomme pas encore *modélisation* mais "mathématisation moderne". Face à la portée et au style des applications contemporaines des mathématiques et des procédés de cette mathématisation moderne, le professeur ressent un sentiment désagréable d'incertitude, d'incompétence, de dilettantisme et cherche à éviter ces situations à la limite. De plus, obstacle supplémentaire, il n'est pas certain — comme le fait remarquer A. Z. Krygovska [1968] — qu'il soit aisé de trouver des types de situation que l'on puisse mathématiser avec profit en classe. Enfin, une troisième difficulté débattue lors du colloque tient cette fois aux seuls étudiants dont l'attitude dans les classes montre qu'ils n'attendent de leurs enseignants de mathématiques que des procédés qui leur donnent rapidement accès aux résultats.

[11]) Les mathématiques utiles peuvent se montrer utiles tant que le contexte ne change pas, et pas un moment de plus, ce qui est juste le contraire de ce que les vraies mathématiques devraient être. En effet, n'est-ce pas le merveilleux pouvoir des mathématiques que d'éliminer le contexte ? […] En un sens objectif les mathématiques les plus abstraites sont indubitablement les plus flexibles. En un sens objectif, mais pas subjectivement […].

Quelles ques soient les difficultés pédagogiques, il semble nécessaire d'arriver à les surmonter. C'est ce dont témoigne la conclusion de l'intervention déjà mentionnée de Freudenthal intitulée « Why to teach mathematics so as to be useful ? ». Reprenant des accents déjà entendus soixante ans plus tôt dans des conférences reproduites dans *L'Enseignement Mathématique*, Hans Freudenthal s'inquiète de ce que les utilisateurs de mathématiques pourraient dessaisir les mathématiciens de l'enseignement de leur discipline si ceux-ci n'arrivaient pas à enseigner les mathématiques de sorte qu'elles soient utilisables. Ce serait alors la fin de toute éducation mathématique.

Cela dit, un an plus tard, au premier Congrès international sur l'enseignement des mathématiques (ICME) qu'organise la CIEM à Lyon, cette question ne fait pas recette. Seuls deux intervenants, sur la quarantaine qui prirent la parole, traitèrent d'applications ou de mathématisation. L'un d'eux était à nouveau Pollak qui analysait dans son intervention [Pollak 1969] un certain nombre de problèmes renvoyant à des situations de mathématisation.

2. Les colloques de l'OECE et de l'OCDE : « le recours aux applications dans l'enseignement des mathématiques » [12])

Dans son rapport d'activité pour la période 1959–1962, la direction sortante de la CIEM insiste sur la participation de plusieurs de ses membres aux initiatives de l'OECE, puis [13]) de l'OCDE, pour la modernisation de l'enseignement mathématique [ICMI 1963]. Ces initiatives commencent avec le colloque de Royaumont en 1959 en France, puis celui de Dubrovnik et celui d'Athènes en 1963. Les membres de la CIEM engagés dans ces colloques sont des responsables importants de la commission. Marshall H. Stone (vice-président puis président de la CIEM de 1955 à 1962) est l'organisateur du colloque de Royaumont ; quant à Athènes, André Revuz (secrétaire de la CIEM de 1959 à 1962) est responsable du texte français de l'ouvrage issu du colloque édité par Howard F. Fehr, directeur du Department of Mathematical Education de l'Université de Columbia, impliqué dans les activités de la CIEM et consultant pour l'OCDE.

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[12]) Il s'agit du titre d'une des sessions de la conférence d'Athènes de l'OCDE en 1963.

[13]) L'*Organisation de coopération et de développement économiques* a pris en 1961 la relève de l'*Organisation européenne de coopération économique* (qui avait été créée au lendemain de la seconde guerre mondiale pour administrer l'aide des États-Unis et du Canada dans le cadre du Plan Marshall de reconstruction de l'Europe).

L'implication d'un organisme tel que l'OCDE dans un mouvement de modernisation de l'enseignement des mathématiques, en particulier dans celui qui aboutit aux différentes réformes dites *des mathématiques modernes* dans le monde, pourrait avoir de quoi surprendre. Le colloque d'Athènes s'en explique, une intervention soulignant que la réforme de l'enseignement s'est faite sous l'impulsion de trois forces différentes : celle des mathématiciens qui voulaient une présentation des mathématiques en accord avec les théories modernes, celle des psychologues de l'adolescence qui ont montré la possibilité d'un meilleur apprentissage, et celle des dirigeants de l'industrie — nous retrouvons ici l'OCDE — qui demandaient une meilleure présentation des applications des mathématiques [14]).

J'examinerai ici plus particulièrement ce colloque d'Athènes et l'ouvrage [OCDE 1963] qui en est issu. Quatrième ouvrage de la direction des affaires scientifiques de l'OCDE dans la série « Pour un enseignement rénové des sciences », consacré à « l'enseignement moderne des mathématiques », ce livre — dont le titre traduit l'ambition : *Mathématiques modernes; guide pour enseignants* — contient en effet un chapitre consacré aux applications. La présence d'un tel chapitre, parmi les six du livre, intitulé "Applications dans la modernisation des mathématiques", revêt un intérêt tout particulier. La prise en compte des applications est en fait affirmée plusieurs fois dans l'ouvrage, et pas seulement dans le chapitre qui y est spécifiquement dévolu. Le chapitre 5, où sont exposées les conclusions et les recommandations du colloque, y consacre, par exemple, plusieurs items dont je donne ici les idées directrices.

Il s'agit tout d'abord d'inclure dans les programmes les théories et matières qui ont acquis une grande importance, soit par leur rôle unificateur dans l'exposé des mathématiques, soit par l'intérêt particulier qu'ils présentent pour les applications dans d'autres sciences, en particulier les sciences physiques. Une liste est donnée dans laquelle les deux registres envisagés ne sont pas séparés ; citons : les ensembles, les relations et les fonctions, les principes et les structures algébriques fondamentales, les espaces vectoriels et l'algèbre linéaire, le calcul différentiel, le calcul des probabilités et la statistique, les théories mathématiques qui ont un lien avec les calculatrices électroniques. On y souligne l'importance, dans les programmes, de traiter dans une seule et même étude les structures logiques et les problèmes.

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[14]) Voir dans [OCDE 1963, 194] le résumé de la discussion qui a suivi l'intervention de H. Pollak.

En second lieu, il faut améliorer les méthodes d'enseignement et développer l'aptitude des élèves à rechercher la solution de problèmes concrets dans un cadre mathématique. Ce dont il est question, en fait, c'est de développer une approche de la modélisation mathématique, appelée ici « chercher le modèle », le mot "modèle" étant d'ailleurs à cette occasion utilisé pour la première fois, du moins dans la littérature sur l'enseignement des mathématiques que j'ai pu consulter. Dans cette perspective la conférence invite l'ensemble des pays à constituer des séries systématiques d'exemples d'applications des mathématiques adaptées à l'enseignement secondaire, ce qui suppose une recherche permanente dans laquelle il est urgent de s'engager.

Enfin, la résolution finale insiste énormément sur l'importance des applications des mathématiques modernes dans la formation des professeurs, sur la valeur de l'enseignement des mathématiques par les applications, sur la nécessité de justifier les notions mathématiques par leurs applications et de faire comprendre que les mathématiques sont utiles à la société. Quant au champ des applications, l'importance des liens avec les physiciens est à nouveau mise en avant.

Cette insistance sur les applications, sur leur nécessaire intégration à l'enseignement mathématique, tant dans son esprit et ses contenus que dans ses méthodes, est d'autant plus remarquable qu'elle apparaît comme collectivement portée par le colloque. Et cela, dans un colloque où parmi les orateurs de première importance figurent certes Henry Pollak — omniprésent, nous l'avons dit, à tous ces congrès ou conférences —, mais aussi André Revuz, Willy Servais, secrétaire de la *Commission internationale pour l'étude et l'amélioration de l'enseignement mathématique* (CIEAEM), créée après-guerre, ou Georges Papy, toutes personnalités très engagées et très représentatives du mouvement de réforme des mathématiques modernes, en France et en Belgique notamment.

Cependant, cette insistance dans les conclusions et recommandations de la conférence semble quelque peu volontariste. Elle ne reflète pas, en particulier, la part que les intervenants, tant dans les conférences que dans les discussions qui suivirent, ont réellement réservée à ces questions d'applications des mathématiques [15]).

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[15]) Ainsi, si une journée a été consacrée à Athènes au thème « recours aux applications dans l'enseignement des mathématiques », le programme de cette journée apparaît bien mince et ces questions ne resurgissent que pour les conclusions.

3. Et qu'en a-t-il été alors de l'impact de Bourbaki sur les réformes des mathématiques modernes ?

Le nom commun le plus souvent associé aux mathématiques dites modernes est celui de *structures*, le nom propre, en France mais aussi dans le monde, est celui de *Nicolas Bourbaki* qui lui est si étroitement lié. Or, Bourbaki ne s'est pas intéressé aux mathématiques appliquées et a ignoré des théories mathématiques si essentielles et si présentes dans les applications, à commencer par les probabilités et la statistique. N'y a-t-il pas alors une contradiction entre ce souci appuyé des applications des mathématiques et les images qui collent au mouvement de réforme des mathématiques modernes ?

Il faut noter, tout d'abord, que malgré les conditions qui ont provoqué sa naissance, Bourbaki, comme groupe de mathématiciens, ne s'est jamais vraiment intéressé aux questions d'enseignement, de curriculum ou de pédagogie, ni dans le supérieur, ni encore moins dans le secondaire. De plus, il serait faux, y compris en France et en Belgique, de limiter la modernisation de l'enseignement des mathématiques à la prise en compte de la nouvelle architecture des mathématiques introduites par la notion de structures.

Cette modernisation est par exemple définie en trois points dans le guide pour les enseignants [OCDE 1963]. Si le premier point est bien : « unifier l'ensemble du sujet en insistant sur les structures fondamentales », les deux autres points relèvent des applications et consistent à inclure dans les programmes les théories qui ont acquis une grande importance du fait de leurs applications dans d'autres sciences (entre autres la statistique) et à améliorer les méthodes d'enseignement par le recours à la résolution de problèmes concrets dans un cadre mathématique.

Il ne semble pas qu'il y ait eu, entre les différents acteurs de la promotion des mathématiques modernes dans l'enseignement, de désaccords fondamentaux sur cette nécessaire prise en compte des applications. Tous en soulignent l'importance, soit dans des prises de position individuelles, soit en s'associant à des manifestations qui insistent tout particulièrement sur ce point. Même la CIEAEM, qui dans ses propres publications [16]) ne semble pas concernée par cette dimension, s'y trouve de fait associée par la présence et le rôle de son président Servais au sein de la CIEM et au colloque de l'OCDE de 1963.

Il y a par contre débat sur la façon de décliner cette ouverture aux applications. Quelles mathématiques sont utiles pour quelles applications ?

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[16]) Voir par exemple [Piaget & al. 1955].

Enseigner les mathématiques afin qu'elles soient utiles et applicables peut signifier prioritairement — et quelquefois uniquement, dans la réalisation effective des principes — "apprendre à abstraire" et non "viser les applications directes". Cette conception conduit, de fait, à faire l'impasse sur l'introduction de nouvelles théories (statistique par exemple), sur l'apprentissage de la modélisation par le traitement de cas concrets, sur l'initiation à d'autres formes de raisonnement que la stricte logique déductive, etc.

La façon de concevoir l'utilité et l'applicabilité des mathématiques, et donc leur enseignement, dépend en fait des champs d'applications visés, des disciplines concernées. Si la CIEM et l'OCDE insistent dans les années 1960 sur la physique, il n'en est pas de même, par exemple, de la Commission ministérielle française de réforme de l'enseignement des mathématiques dirigée par André Lichnérowicz. Dans sa première déclaration officielle, la Commission relativise les liens de tout temps privilégiés entre les mathématiques et la physique et insiste, au contraire, sur la diversification des disciplines et des champs de l'activité humaine qui ont besoin des mathématiques.

> La mathématique joue un rôle privilégié pour l'intelligence de ce que nous nommons le réel, le réel physique comme le réel social. Notre mathématique sécrète par nature l'économie de pensée et, par là, permet seule de classer, de dominer, de synthétiser […]. La mathématique a été, depuis toujours, discipline auxiliaire des sciences physiques et de l'art de l'ingénieur. Elle est devenue désormais, au même titre, discipline auxiliaire, aussi bien d'une grande partie des sciences biologiques et médicales que de l'économie et des sciences humaines. […] Elle porte partout témoignage du fonctionnement même de notre esprit. [Commission ministérielle 1967]

De là l'importance donnée à l'universalité du langage mathématique et des structures, qui rendent « les mathématiques contemporaines infiniment plus applicables », au détriment des attributs plus traditionnels des « mathématiques dites classiques » que sont le calcul infinitésimal et la géométrie. Le choix des théories, des outils à privilégier dans l'enseignement des mathématiques est en effet lié aux questions précédentes. Mais ce choix fait débat, y compris lorsqu'on ne considère qu'un seul champ d'application.

Un bon exemple en est le lien à la physique qui a été en France, durant le mouvement de réforme des mathématiques modernes, un point très litigieux. Les scientifiques français, par le biais de leurs associations de spécialistes, protestèrent contre les nouveaux programmes de mathématiques des sections scientifiques des lycées issus de la réforme des mathématiques modernes. Ils s'inquiétaient du nouveau visage de l'enseignement scientifique en France à la suite des effets de ce qu'ils appellent en 1970 la *rénovation pédagogique* et qu'ils caractérisaient comme une reconversion aux mathématiques modernes et une réduction accrue de la place accordée aux sciences expérimentales. Ils dénoncèrent « l'envahissement par les mathématiques délibérément les plus abstraites » ainsi que

> cette école de dogmatisme [qui] a pour dernier souci, et de motiver ses abstractions par référence initiale à quelque problème concret, et de veiller à fournir aux autres disciplines les outils mathématiques (ou, si l'on préfère, de "calcul") qui leur sont nécessaires. [17])

Ces débats et ces tensions provoquèrent une rupture dans la commission Lichnérowicz. Charles Pisot, le rapporteur de l'enquête de 1966 de la CIEM sur la formation des ingénieurs, démissionna de la Commission (dont il était membre) et contribua à la création d'une nouvelle association, l'Union des professeurs et utilisateurs de mathématiques (UPUM) qui milita pour préserver une formation mathématique traditionnelle des scientifiques et ingénieurs.

En fait, malgré tous les discours et principes favorables aux applications, à des applications modernes des mathématiques dites *modernes*, discours tenus, nous l'avons vu, au colloque d'Utrecht de la CIEM, à la réunion d'Athènes, dans la Commission Lichnérowicz, le mouvement des mathématiques modernes se solde effectivement, dans la réalité des programmes, par un affaiblissement des anciennes solidarités disciplinaires. Et cela, il est important de le souligner, sans qu'elles aient été réellement remplacées par de nouvelles.

Au-delà de ce constat, quelles conclusions peut-on tirer de l'étude de cette période ? Tout d'abord, Geoffroy Howson l'a dit, il faut savoir lire les textes du passé — même d'un passé récent — pour saisir les enjeux d'une période. Les réformateurs des années 1950–1960 ont été face à un enjeu nouveau : la scolarisation secondaire de tous.

Au début du vingtième siècle, en France par exemple, les seules solidarités disciplinaires pour la grande masse des enfants, qui ne connaissent alors que l'enseignement primaire élémentaire [18]), sont l'arpentage et le dessin linéaire. Pour les autres, ceux qui poursuivent leur scolarité au-delà de 12 ou 13 ans, s'il y a effectivement dans l'enseignement mathématique secondaire un lien revendiqué à la physique, à la mécanique, à l'astronomie, cela ne concerne que

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[17]) Communiqué de 1970 de l'Union des physiciens, de la Société française de physique et de la Société chimique de France, publié dans [Hulin 1992, 41].

[18]) La situation française est d'une certaine façon particulière ; les États-Unis, par exemple, connaissent dès le début du siècle une école unique, gratuite et obligatoire jusqu'à 14 ans. Mais la différence de registre dans les solidarités disciplinaires de l'enseignement des mathématiques, que j'explicite dans le cas français, demeure à mon avis pertinente pour tous les pays. Tous connaissent en effet, à un niveau scolaire ou un autre, une séparation entre des enseignements à finalités avant tout pratiques et d'autres dont la finalité est plus libérale ; voir [Nabonnand 2003].

les classes de second cycle de l'enseignement des lycées (qui à cette époque était payant); dans l'enseignement primaire supérieur, où le calcul infinitésimal n'est pas développé, ce sont les liens avec le monde du commerce, de la banque, de la mécanique des machines et non pas de la mécanique rationnelle qui sont affirmés. Si l'on poursuit au-delà de la première guerre mondiale l'étude du cas français, on constate que l'évolution de l'enseignement mathématique secondaire dans l'entre-deux-guerres prend de plus en plus de distance avec les applications revendiquées au début du siècle. La vocation libérale et désintéressée de cet enseignement destiné à une élite n'a pas fait bon ménage avec les ambitions des réformateurs du début du siècle.

Au lendemain de la seconde guerre mondiale, la question des applications dans l'enseignement mathématique se pose, pour tous les pays, avec une nouvelle dimension. Comme le souligne M. H. Stone en 1952 à Rome, à l'occasion de la reprise des activités de la CIEM comme commission de la toute nouvelle Union mathématique internationale (UMI), cette question se pose dans le cadre d'une instruction populaire obligatoire que vont devoir assumer tous les pays [Stone 1953]. Les débats sur les applications et les mises en œuvre éventuelles des projets en deviennent singulièrement plus complexes.

Face à ces défis, quels pouvaient être les appuis des réformateurs, de la CIEM ou d'ailleurs, conscients de la nécessité du développement des mathématiques comme discipline de service et de l'introduction de cette dimension dans l'enseignement ? Ils ne viendront pas du monde mathématique lui-même. Comme l'avait laissé penser le rapport de la sous-commission des U. S. A. présenté par Kemeny, les changements dans le monde mathématique après guerre ne furent pas aussi radicaux [19]) que le laissait supposer la première enquête de la CIEM. Tant du point de vue des intérêts mathématiques que des débouchés professionnels, les "mathématiques académiques" dominent toujours le monde mathématique et plus particulièrement la partie intéressée par les questions d'enseignement. Les appuis ne sont pas venus non plus du monde enseignant qui s'est avéré très conservateur. Les difficultés liées à la formation des maîtres dans le domaine des applications des mathématiques sont d'ailleurs sans cesse soulignées dans les divers colloques ou enquêtes.

Ce constat appelle alors plusieurs questions qui ont déjà été posées dans ce Symposium. Quel biais la CIEM apporte-t-elle dans l'étude que l'on peut faire de l'évolution et de l'histoire de l'enseignement des mathématiques dans

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[19]) On peut lire à ce sujet, dans l'article d'Amy Dahan [1996] déjà cité, la partie qu'elle consacre à la *Conference on training in applied mathematics* tenue à l'université de Columbia (New York) en 1953 (voir en particulier p. 196).

le vingtième siècle ? De quoi, de qui la CIEM est-elle représentative ? Voilà, me semble-t-il, des questions tout à fait pertinentes qu'il serait intéressant de traiter dans la perspective du centenaire à venir de la CIEM.

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APPLICATIONS OF MATHEMATICS '2000'

Les applications des mathématiques "an 2000"

par Mogens NISS

Aussi loin que remontent les idées d'enseignement et d'apprentissage des mathématiques, certains aspects de leurs applications se sont retrouvés, explicitement ou implicitement, dans les programmes scolaires de la plupart des pays. S'il est vrai qu'avant le 20ᵉ siècle les applications des mathématiques concernaient principalement l'enseignement primaire, elles ont par la suite fait leur chemin dans d'autres niveaux, en parallèle avec le phénomène de la massification de l'enseignement secondaire et post-secondaire. Dans le cas de l'enseignement secondaire, ce développement s'est produit au cours de la période 1900–1960. Il s'est poursuivi au niveau post-secondaire depuis les années 1960, de sorte que les applications des mathématiques sont maintenant présentes au premier cycle universitaire un peu partout dans le monde.

Ce texte porte sur le développement des applications des mathématiques durant le dernier tiers du 20ᵉ siècle. Au cours de cette période, les tendances générales que nous venons de mentionner étaient présentes, bien que sous des formes nouvelles et dans un contexte nouveau. Un aspect caractéristique en est que l'accent sur certaines branches des mathématiques, habituellement regroupées sous le vocable de "mathématiques appliquées", fut remplacé par la notion générale d'*application des mathématiques* (de quelque sorte que ce fût). De plus, la démarche ne fut pas restreinte à l'étude de quelques exemples triés sur le volet d'applications "toutes cuites" — de façon typique, sous la forme de "modèles" standard — mais fut élargie de manière à englober *tout le processus* de mise en œuvre des mathématiques dans des contextes extra-mathématiques. Le processus fut bientôt connu sous le nom de "modélisation" ou de "construction de modèles" (ou encore, dans le jargon scolaire, de "résolution de problèmes réels").

On peut sommairement identifier cinq raisons, somme toute assez différentes, pour accorder un rôle explicite aux applications et à la modélisation dans le curriculum mathématique. Ces raisons se retrouvent dans les arguments invoqués en faveur des applications et de la modélisation. A savoir : *l'argument "utilité"* (les applications et la modélisation doivent faire partie des programmes afin de rendre les étudiants capables d'utiliser les mathématiques dans d'autres domaines, à l'école ou en dehors); *l'argument "compétence critique*" (il s'agit de fournir aux étudiants les compétences leur permettant de comprendre et d'apprécier les emplois des mathématiques significatifs sur le plan social); *l'argument "formation*" (c'est un outil efficace pour favoriser le développement général, chez l'individu, d'attitudes, d'habiletés et de comportements en résolution de problèmes); *l'argument "perception des mathématiques*" (l'application des mathématiques dans des domaines extra-mathématiques représente une caractéristique importante des mathématiques, et la modélisation constitue justement le véhicule par lequel passe l'application); *l'argument "apprentissage des mathématiques*" (les applications et la modélisation renforcent à la fois l'apprentissage des mathématiques et la motivation pour leur étude).

Ces raisons ont été avancées par différents défenseurs, à différentes époques et avec des accents différents. Le mouvement des "mathématiques modernes" et ses retombées furent l'objet d'attaques par des groupes influents dans la société, car les programmes révisés ne fournissaient pas des diplômés aptes à utiliser ce qu'ils avaient appris. Cette critique, en grande partie fondée sur l'argument "utilité", n'était pas le propre des employeurs et des praticiens hors du monde de l'éducation, puisqu'elle était aussi partagée par les mathématiciens œuvrant dans le domaine de l'industrie ainsi que par certains enseignants et didacticiens des mathématiques. La philosophie sous-jacente était que pour permettre aux gens de développer des compétences en rapport avec les applications et, surtout, avec la modélisation, il fallait les leur enseigner. Au cours des années 70 et au début de la décennie suivante, la portée de l'argument utilitaire fut élargie aux qualifications concernant la vie de tous les jours et même la citoyenneté. Pour certains éducateurs, cela déboucha sur l'argument "compétence critique". Vers 1970, un autre groupe d'enseignants et de didacticiens des mathématiques conjugua ses efforts avec certains "utilitaristes". Non seulement les "maths modernes" ne satisfaisaient-elles pas les attentes des défenseurs de l'utilitarisme, mais une majorité d'étudiants éprouvaient même des difficultés à affronter les mathématiques qu'on leur enseignait. En mettant l'accent sur le rôle-clé de la mathématisation de la réalité dans la compréhension de ce que sont les mathématiques, ces enseignants et didacticiens privilégiaient l'argument "apprentissage des mathématiques". L'argument "perception des mathématiques", qui visait surtout les niveaux secondaire et post-secondaire, commença à émerger au cours des années 70 et 80, tandis que l'argument "formation" connut son apogée au cours de la décennie 70.

Il est également possible de distinguer diverses phases en ce qui concerne le développement des programmes et leur réalisation dans la pratique scolaire. Au cours de la première phase, durant les années 70, l'accent était mis sur la défense même des applications et de la modélisation, soutenues par la conception et la présentation de banques de modèles et d'applications ainsi que par le compte rendu d'expériences d'enseignement, à petite et à grande échelle. Au cours de la décennie suivante, l'attention porta sur les besoins des étudiants engagés dans de véritables activités de modélisation, toutes les étapes du processus de modélisation devant être prises en compte de façon sérieuse. La décennie 90, quant à elle, n'a pas vu de développements fondamentalement nouveaux, les applications et la modélisation ne représentant plus un "sujet chaud". Ce n'est cependant qu'au cours de cette décennie que la recherche didactique sur les applications et la modélisation a vraiment pris son envol. Mais pour ce qui est de la réalité de la salle de classe normale, c'est une tout autre histoire : il semble en effet que ne s'y soit jamais vraiment établie une solide tradition d'activités ayant trait aux applications et à la modélisation.

APPLICATIONS OF MATHEMATICS '2000'

by Mogens NISS

1. INTRODUCTION

As long as there has been mathematics education, some aspects of the applications of mathematics have been present in the curriculum, explicitly or implicitly, in almost all countries. In former times, i.e. before the 20th century, this pertained particularly to primary education. When higher educational levels became widely accessible to the general populace, the application of mathematics made its way into the agenda of mathematics education for these higher levels. This happened first to secondary education, during the period 1900–1960. Then this development gradually gained momentum at the tertiary level since 1960 and is now rather manifest at the undergraduate level in most places of the world.

This should come as no surprise. Basically, when mathematics education is being supplied to recipients who are not studying mathematics in order to become involved in its *production* (as researchers) or *reproduction* (as teachers), mathematics education will necessarily have to be justified by reference to matters that are extraneous to mathematics itself. Traditionally, such justification has consisted in referring either to the value mathematics has for the formation of the character and personality of the individual, or to the usefulness of mathematics to extra-mathematical life in everyday practice, on the job, in the community, or in society.

The expansion of the educational system since the '60s, so as to make it cater for new and larger groups of students. has led to exactly this. Mathematics has been given, at still higher levels, to more and more students who will not primarily be preoccupied with the production or reproduction of mathematics as a discipline.

While this development has put the application of mathematics on the agenda of new (higher) educational levels, this does not imply that the application of mathematics has been given constantly increasing weight in the actual teaching and learning of mathematics. Throughout the 20th century one can actually discern oscillations between 'utilitarian' periods, during which the relevance and usefulness of mathematics education to extra-mathematical life have been emphasised, and 'puristic' periods during which the intrinsic values of mathematics as part of human culture, or its significance to the formation of the personality, are in the centre [Niss 1981; 1996]. However, these oscillations have been superimposed on the general trend already mentioned. So, although the role and importance of applications and modelling are not increasing in a literal sense they are, in fact, so *on average*.

This paper concentrates on the development with respect to the application of mathematics in the last third of the 20th century. Within that period, too, the general trends outlined above are present, even though they have new backgrounds and take new forms during the period. It is a characteristic feature of the development during these three decades that the focus on certain branches of mathematics which were granted the status of 'applied mathematics' was soon superseded by the general notion of 'applying mathematics' (of whatever sort). This, in turn, was not confined to studying (samples of) particular, ready-made applications of mathematics — typically in the form of standard 'models' — but became extended so as to encompass *the entire process* of putting mathematics to use towards matters extra-mathematical. This process became known under the label of 'modelling' or 'model building'; in school contexts also sometimes as 'real-world problem solving'.

In order to generate a working terminology, let us use the shorthand 'applications and modelling' for the whole package, including 'applied mathematics', 'applications of mathematics', 'mathematical models', and 'mathematical modelling', although there are important differences between these elements to keep in mind. The shift of emphasis from the former to the latter during the period we are considering will be further discussed below.

2. JUSTIFICATION OF APPLICATIONS AND MODELLING IN THE CURRICULUM

In essence one can identify five quite different reasons for assigning a manifest role to applications and modelling in a mathematics curriculum. These reasons are reflected in the arguments put forward in the mathematics education literature in favour of applications and modelling [Blum & Niss

1991]. To avoid misunderstandings, these arguments are not (necessarily) my arguments, but arguments detected in the literature (cf. also [UNESCO 1973, 81–82] and [Kaiser-Messmer 1986]). The order of presentation is neither chronological nor a manifestation of the popularity or priority assigned to each argument.

(a) *The 'utility' argument.* Ultimately, the majority of students of mathematics will be users rather than producers or reproducers of mathematics. Since experience shows that the ability to use mathematics is not an automatic consequence of knowing or even mastering pure mathematics, this ability has to be taught. So, applications and modelling should be included in the mathematics curriculum (at least for the majority of students) in order to enable them to use mathematics in other fields inside or outside of school.

(b) *The 'critical competency' argument.* It is a fact that mathematics is being widely used in society, actually to such an extent that it contributes to the shaping of society, for better and for worse. Multitudes of decisions and actions of great societal and social significance are being taken under the involvement of mathematics. Whether such use is justified or not, it requires both mathematical and application and modelling competencies to understand and judge it. So, applications and modelling should be included in the mathematics curriculum in order to equip its recipients with a critical competency that allows them to understand and judge socially significant uses of mathematics.

(c) *The 'formative' argument.* It is a major goal of mathematics education to help the formation of active, autonomous, creative, and flexible individuals. One aspect of this is the ability to identify, pose, formulate, handle, and ultimately solve problems that one encounters on one's way. Sometimes such problems may involve the use of mathematics, not infrequently in a non-routine setting. Sometimes they don't. However, in either case applications and modelling are an efficient tool for fostering general problem solving attitudes, abilitites and behaviours in individuals. So, this is a reason for including applications and modelling in the mathematics curriculum.

These three arguments have one thing in common: they all focus on "what applications and modelling can do for preparing students to master life outside of mathematics". Since, of course, mathematics is a necessary (but not sufficient!) prerequisite to applications and modelling, if one or more of these arguments prevail they also lead to considering the issue “what mathematics (education) can do for applications and modelling”, and by transitivity, “what mathematics (education) can do for life outside of mathematics”. The following two arguments, in contrast, have the opposite orientation, in that they bring answers to the question : “What can applications and modelling do for mathematics, and its teaching and learning ?”

(d) *The ‘picture of mathematics’ argument*. Let us assume that mathematics education is meant to provide its recipients with a comprehensive, balanced, and representative picture of mathematics as a discipline and of its position and role in the world. As the application of mathematics to extra-mathematical areas is indeed a characteristic feature of mathematics, and as modelling is the vehicle through which the appplication of mathematics is brought about, applications and modelling should occupy a manifest position in the mathematics curriculum.

(e) *The ‘learning of mathematics’ argument*. Numerous mathematical entities (including objects, phenomena, relations, and concepts) are to a large extent linked to corresponding entities in the real world. Oftentimes mathematical entities are abstractions derived from attempts to model aspects of this world. So, in order really to understand these entities it is necessary or at least very helpful to consider their origin in modelling. Moreover, since most mathematical concepts, methods and results are actually applicable to dealing with matters outside of mathematics, working with applications and modelling serves to underpin and scaffold the understanding and appreciation of mathematics. So, both the learning of mathematics and the motivation for studying it are strengthened by giving applications and modelling a manifest position in the mathematics curriculum.

3. TRENDS 1967–2000

The reasons outlined above are given different emphases by different quarters at different times.

The era of the ‘*new* (or *modern*) *maths*’ has been dealt with, in this symposium, by Hélène Gispert. Suffice it here to emphasise that society’s overarching reason for supporting curriculum reform during the ’60s was clearly utilitarian. Society was convinced that mathematical knowledge was crucial to a rapidly growing portion of occupations which required independent, flexible, creative, and efficient employees able to activate their mathematical knowledge in non-routine situations originating in their work [Cooper 1985, chap. 5–8]. As a matter of fact, the reformers behind the ‘new maths’ movement agreed completely [OECD 1961, 107]. They just found that nothing mathematical was so useful and applicable as good, pure, general mathematical structures (e.g. logic, sets, relations, groups). By virtue of their generality these structures were not tied to specific contexts but were, on the contrary, of potential universal relevance to all sorts of fields from the world outside of mathematics. It also belongs to this part of the story that it was the ‘new maths’ movement that was instrumental in including probability and statistics in the secondary school curriculum because of the significance of these topics to the application of mathematics.

Employers in general, and industrialists in particular, have always insisted on the usefulness and applicability of mathematics as the ultimate justification for its prominent position in the educational system. How the educational system will achieve that the people it ‘produces’ possess the knowledge and skills needed to use mathematics in the work place is, in principle, of secondary importance as long as it happens.

However, it soon became clear that the ‘new maths’ reform did not ‘keep its promises’ by producing the sorts of people that were in demand by employers. Graduates from the reformed curricula turned out not to be any better at using mathematics for practical purposes than was the case before. One could even say that they were worse in this respect because now many of them could not even do the computations, say in arithmetic, they used to be able to do. It is true that the reformed curricula did produce graduates from secondary and tertiary education who had benefited greatly from the new approach, in terms of pure mathematical knowledge and insight. But this did not automatically enable them to make use of what they had learnt.

So, the results of the ‘new maths’ came under harsh attack by large and influential quarters in society (for an example, see [Hammersley 1968]). This provided one essential stimulus for upgrading the role of applications and modelling in mathematics curricula at virtually all levels. Needless to say, this line of reasoning was largely based on the ‘utility’ argument. It was shared not only by employers and practitioners outside of the mathematics education community but also by increasingly strong and vocal groups of mathematicians [Pollak 1968; 1976; Klamkin 1968], some of whom worked in industry, and mathematics educators [Ford & Hall 1970], primarily in the Anglo-Saxon world, but also in Germany, the Netherlands, and Scandinavia.

The underlying philosophy, based on experience, was (and is) that if you want people to obtain competence in applications and modelling you have to teach it to them. This goes for all the aspects of the applications and modelling package, but above all for modelling. To become able to carry through the entire modelling process from problem identification and specification, through within-model problem solving, to model validation, and including all the steps in between, requires both substantial training and multiple experiences, which in turn have to be obtained by means of special activities. It is — according to those who put forward the utility argument — the responsibility of mathematics education to take care of such training and to provide such multiple experiences, in order ultimately to ensure that the applications and modelling competency thus formed has a sufficiently wide and general scope. If circumstances allow for it, collaboration with other subjects, such as physics, biology, economy, technology, is certainly most valuable too, but only in mathematics is it possible to generate and display a broad spectrum of applications and modelling experiences taken from different fields.

Along the road, during the 1970s and early '80s, the scope of the utility argument was expanded from focusing mainly on job qualifications to emphasising also people's everyday life and citizenship. This happened concurrently with the political and societal development that took place in that decade, which led to a strong emphasis in several quarters on the socio-political relevance of education. The well-known 'Cockcroft report' [Cockcroft 1982] in the UK gave particular emphasis to the need of preparing people in general for everyday life and citizenship. For some mathematics educators, especially but certainly not exclusively in Germany and Scandinavia (e.g. Damerow *et al.* [1974], Mellin-Olsen [1987], Skovsmose [1994], and Volk [1979], amongst many others), this gave rise to the 'critical competency' argument which at first sight may be perceived as a variant or at least an offspring of the 'utility' argument, although its focus is in fact rather different.

Around 1970 a different group of mathematics educators joined forces with some of the 'utilitarians'. Not only did the 'new maths' fail to fulfil the expectations or demands of utilitarian quarters; it also turned out that a major proportion of students had severe difficulties in coming to grips with the mathematics taught to them as such, whether or not an applications and modelling perspective was present. To many students, mathematics appeared to be a completely abstract game played according to some random set of rules coming out of the blue. Perhaps the game could sometimes be good fun, to some students, but it was played in splendid isolation from the rest of the world.

This state of affairs was observed with great concern by a number of mathematics educators, amongst whom Hans Freudenthal was a protagonist. Already in 1967 an international colloquium, sponsored by ICMI, the IMU, and the Dutch government, was held in Utrecht under the title “How to teach mathematics so as to be useful”. It gathered a varied group of mathematicians and mathematics educators who insisted on the need to take the application of mathematics seriously. The proceedings of that colloquium were published in the first volume of *Educational Studies in Mathematics* in 1968. Freudenthal’s opening address deserves to be quoted in part:

> Since mathematics has proved indispensable for the understanding and the technological control not only of the physical world but also of the social structure, we can no longer keep silent about teaching mathematics so as to be useful. [...] The large majority of students are not able to apply their mathematical classroom experiences, neither in the physics or chemistry school laboratory nor in the most trivial situations of daily life. [...] In its first principles mathematics means mathematizing reality, and for most of its users this is the final aspect of mathematics, too. [...] If we do not succeed in teaching mathematics so as to be useful, users of mathematics will decide that mathematics is too important a teaching matter to be taught by the mathematics teacher. Of course this would be the end of all mathematics education. [Freudenthal 1968, 4–8]

While this quotation primarily emphasises the utility of mathematics, Freudenthal in his address also touched upon the key role of mathematisation of reality in understanding what mathematics is all about, and in coming to grips with its concepts and methods. Thus he and other participants in the colloquium (e.g. Krygowska [1968]) also invoked and emphasised the ‘learning of mathematics’ argument. This is an argument which has always been strong with mathematics educators and (some) university mathematicians, whereas it has never attracted much attention in quarters outside of the educational system.

The same is true with the ‘picture of mathematics’ argument which began to emerge amongst some (but rather few) mathematics educators and mathematicians in the ’70s and ’80s, primarily addressing secondary and tertiary levels.

As finally regards the ‘formative’ argument, mainly put forward by mathematics educators, this had its heydays in the ’70s, but then lost some of its momentum along with the growth of a general distrust of transfer in the ’80s and ’90s.

4. Applications and modelling in the curriculum

So much about the arguments for applications and modelling in mathematical education. How about the actual development of curricula and implementation in classroom practice ? Here it is possible to discern different phases.

In the first phase, in the '70s, the main emphasis was on the very advocacy of applications and modelling, which was supported by the generation and presentation of source collections of models and applications and by reporting teaching experiments in small or large scale, all of which were meant to serve two purposes. Firstly, to demonstrate that it is actually possible to find examples or cases that are viable and accessible to the teaching of mathematics at various levels. Secondly, to serve as a bank of resources for teachers and textbook authors who wanted to include some element of applications and modelling in their teaching. In addition to establishing such resource banks, much work was put into recruiting particularly interested teachers to take up applications and modelling work in their own teaching practice. The establishment in 1983 in the USA of COMAP, the *Consortium for Mathematics and its Applications*, as an organisation to further the creation of curricula and teaching materials was a reflection of this development [Garfunkel 1998]. However, in spite of the danger of oversimplifying the situation in the '70s, it seems fair to state that the majority of applications and modelling teaching round the world focused on ready-made models that were presented to students for further study.

It was also in the '70s that applications and modelling began to figure in prominent places in the ICMEs. At ICME-3 in Karlsruhe in 1976, a full section was devoted to applications and modelling, and an often quoted report by Henry Pollak (with the slightly misleading title "The interaction between mathematics and other school subjects") was published by UNESCO [1979, 232–248]. Also at ICME-4 in Berkeley, 1980, applications formed a major theme, reflected in a chapter of 70 pages in the Proceedings [Zweng *et al.* 1983, no. 8].

In the '80s attention was drawn to the need for students to engage in real modelling activities, where all the stages of modelling were to be taken seriously, also those that cannot be labelled as predominantly mathematical. Since this is time consuming, such work was found primarily in experimental or innovative settings characterised by a great deal of enthusiasm on the part of those involved. Quite a few large scale projects in applications and modelling were established in the '80s.

At the level of countries, there is little doubt that the UK had a leading role in emphasising modelling [Ford & Hall 1970; Burghes 1980]. Various groups such as the Spode Group were formed to promote and develop modelling activities in school and university. In many of the new polytechnics in Britain a lot of work was done in this direction. It is also on the initiative of British mathematics educators that the series of *International Conferences on the Teaching of Mathematical Modelling and Applications* (the ICTMAs) was instigated, the first one being held in Exeter in 1983 [Berry *et al.* 1985]. Since then an ICTMA has been held every odd year [Berry *et al.* 1986; Blum *et al.* 1989; Niss *et al.* 1991; de Lange *et al.* 1993; Sloyer *et al.* 1995; Houston *et al.* 1997; Galbraith *et al.* 1998] with ICTMA-11 going to be held in Milwaukee in 2003. As the scope of these conferences became widened from tertiary applications and modelling, lower educational levels were taken on board as of the late ’80s. A few years ago, the informal community behind the ICTMAs decided to formalise itself into the *International Community for the Teaching of Mathematical Modelling and Applications* [www.infj.ulst.ac.uk/ictma/]. At ICTMA-10 in Beijing 2001, this community further decided to apply for the status of Affiliated Study Group of ICMI.

The way in which applications and modelling were included in the ICMEs of the 1980s (1980 not included) was changed to give more emphasis to the modelling aspect. This happened along with the new format adopted for the ICMEs initiated at ICME-5 in Adelaide (1984). At that congress, Theme Group 6 was devoted to “applications and modelling” [Carss 1986, 197–211]. At ICME-6 in Budapest (1988), two theme groups dealt with matters pertaining to applications and modelling, no. 3 “problem solving, modelling and applications”, and no. 6 “mathematics and other subjects” [Hirst & Hirst 1988]. Based on the work of these two theme groups a joint book was subsequently published [Blum *et al.* 1989].

In the ’90s one can hardly say that fundamentally new developments occurred. The trends of the ’70s and ’80s were consolidated in some places (for an overview, see [de Lange 1996]), whereas they lost steam in others. In general it seems fair to claim that applications and modelling was no longer such a ‘hot topic’ in the ’90s. Indeed, it continued to be cultivated in ICTMA circles, which were also expanded in the ’90s. Applications and modelling were given a *status quo* position in ICME-7 in Québec, except that modelling was emphasised even more than in the past. Working Group 14 dealt with “modelling in the classroom” [Breiteig *et al.* 1993]. In ICME-8 in Seville, one could discern a weakening of the focus, in that applications and modelling tended to be spread over a larger set of groups. Thus, one of the Working Groups had the title "linking mathematics with other school subjects", while one of the Topic Groups was called "problem solving throughout the curriculum". No group specifically addressed applications and modelling as such. At ICME-9 in Tokyo/Makuhari 2000, applications and modelling re-entered the congress programme in the Topic Study Group (no. 9) on "mathematical modelling and links between mathematics and other subjects". It should be added that it was only in the '90s that didactical research into applications and modelling gained some momentum [Niss 2001]. As a reflection of this momentum, ICMI has decided to mount an ICMI Study on the teaching and learning of mathematical applications and modelling.

One thing is what happens at the level of the didactics of mathematics in terms of discussion, research and development. Quite a different thing is what really happens in the mainstream classroom. Although I have no solid data to offer, it does seem from reports of protagonists in the field that substantial activity in applications and modelling was never secured an unchallenged foothold in such classrooms. Some attention appears to be paid in most places, but serious emphasis is confined to special places and institutions.

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REACTION

by Gerhard WANNER

Even though I have coauthored a textbook on analysis [Hairer & Wanner 1995], I see myself as a specialist in numerical analysis, not as an expert in mathematics education. Like others, I was a pupil and a student myself some 40 years ago. Since then I have taught mathematics at various levels to students of computer science, physics, and mathematics. But, after the three brilliant lectures we have heard, I have the awkward feeling of re-experiencing Galileo's dialogue between two intelligent participants (Salviati, Sagredo) and a stupid one (Simplicio) — with the only improvement that here we have *three* intelligent ones against one stupid.

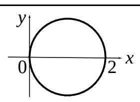
All the texts mentioned, throughout the century, stress the importance of applied mathematics in teaching. My overall impression is that they differ mainly in what is meant, in each period, by 'applied mathematics' :

- notion of function from algebraic and geometric viewpoints for computations of 'modern' industrial applications, use of 4-decimal logarithms in the early century;
- probability and stochastics, operational research, numerical analysis in the years 1950–1970; much of this increased interest in applications came from industrial research labs in the sequel of World War II and after the 'sputnik shock';
- applications and modelling of large "real world problems" in the third period.

Again and again the importance of teaching applied mathematics has been emphasized, not only for preparing students to master their future life *outside* of mathematics, but also for improving the *learning of mathematics* itself.

TABLE 1

Level of acquaintance with high-school mathematics
(beginning first-year students in Geneva, October 1985)

Faculty:	Bio	PhM	Info	Geo	Chm	Eco	GE	For
1. A cylinder of revolution with radius r and altitude h has surface area $A =$ ☐	70%	86%	63%	81%	82%	37%	56%	43%
2. Compute, without using a calculating machine: $\mathrm{Log}\,5 + \mathrm{Log}\,20 = \mathrm{Log}$ $=$ ☐ (Log denotes logarithm to base 10)	32%	60%	57%	6%	60%	17%	29%	29%
3. One throws two dice simultaneously. Give the probability that the sum of numbers on their sides is equal to 3 ☐	49%	50%	33%	50%	55%	35%	41%	35%
4.  Give the equation of the drawn circle. ☐	27%	47%	31%	18%	49%	11%	22%	21%
5. To what value does the function $f(x) = \frac{1}{1+x^2}$ tend as $x \to \infty$? ☐	90%	93%	92%	75%	95%	84%	87%	85%
6. The set $\{\vec{a}, \vec{b}, \vec{c}\}$ contains a pair of orthogonal vectors. Which one? $\vec{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} -3 \\ -2 \\ 5 \end{pmatrix}$ ☐ *and* ☐	38%	64%	49%	37%	75%	26%	36%	28%
7. Compute the derivative of the function $f(x) = \sqrt{x}$ $f'(x) =$ ☐	81%	88%	84%	93%	86%	57%	71%	67%
8. Compute $\int_0^1 x^2\,dx =$ $=$ ☐	74%	88%	72%	62%	80%	44%	60%	53%
TOTAL	58%	72%	60%	53%	73%	39%	50%	45%
Students with 7 or 8 correct answers:	14%	43%	29%	12%	35%	5%	14%	10%
Students with 0 or 1 correct answer:	4%	3%	2%	0%	2%	18%	10%	16%

Bio = Medicine, Biology, Pharmacy; PhM = Physics, Mathematics; Info = Computer Science;
Geo = Geology; Chm = Chemistry; Eco = Economics and Social Sciences;
GE = Global percentage for Geneva;
For = Foreign students and students with incomplete secondary schooling

ACTUAL FACTS IN SCHOOLS. Despite these admirable resolutions and discussions about the didactics of mathematics, research and development, it is a striking reality that in mainstream classrooms one can hardly find any serious activity in applied mathematics. This concluding remark of M. Niss' contribution ("Although I have no solid data to offer …" [Niss 2003, 282]) is also supported by my own personal experience. Indeed, I took part in a commission set up by the Rectors of all Swiss universities in 1985. The object was to study the acquaintance with high-school mathematics of students entering the universities. All first-year students in science, medicine, and economics were given an anonymous questionnaire with eight questions of an extremely simple nature ('simple' as compared to the usual curriculum). The results for Geneva are displayed in Table 1. Among the various questions, the more elementary and the more applied they were (like the probability quiz), the worse were the results. At the other extreme, abstract limits, derivatives, and integrals were known best.

Another episode in the same direction was David Mumford's astonishment when he

> tried (unsuccessfully) to get each high school in which my children were enrolled to go outside during geometry and find out how tall the oak in the yard really is. Instead they buckled under to the educational establishment …
> [Mumford 1999, 5]

WHAT COULD BE THE REASONS FOR THIS DISCREPANCY ?

In the next paragraphs we suggest three principal reasons which can be invoked to analyse this phenomenon.

PURE MATHEMATICS HAS HIGHER REPUTE. There are numerous statements by pure mathematicians (the 'true' mathematicians of Godement) against applications :

> Les sectateurs des mathématiques appliquées, de l'analyse numérique et de l'informatique font preuve dans toutes les universités du monde de tendances impérialistes beaucoup trop manifestes pour que les vrais mathématiciens … [1])
> [Godement 1998, IX]

This tradition, which is absent from the work of Newton, Euler or Gauss, goes back, as Felix Klein explains, to the times of Abel and Jacobi :

[1]) The partisans of applied mathematics, numerical analysis, and computer science display, in all the world's universities, imperialist tendencies which are much too manifest for true mathematicians to …

> … durch die Gründung des *Journals für die reine und angewandte Mathematik* (1826). Nimmt man heute einen Band dieser Zeitschrift zur Hand, so mag der Titel vielleicht Verwunderung erregen. […] Das neuhumanistische Ideal der reinen Wissenschaft als Selbstzweck, das die Verachtung aller Nützlichkeit im gemeinen Sinne in sich barg, führte bald zu einer geflissentlichen Abkehr von allen der Praxis zugewandten Bestrebungen […] und stempelte es zu einem Organ abstrakter Spezialmathematik von strengster Ausprägung, die ihm den Scherznamen "Journal für reine, unangewandte Mathematik" eingetragen hat. [2])
> [Klein 1926, 94–95]

Future high-school teachers learn this frame of mind during their studies at university.

> Nous n'avons *pas* besoin d'ordinateur. [3]) [G. de Rham, to the Dean of the Faculty, Geneva, 1970]

> Les mathématiques pures, c'est comme un tableau de Rembrandt ; les mathématiques appliquées, c'est comme un peintre en bâtiment. [4]) [A member of the Maths Department in Geneva, around 1975]

Not surprisingly, they are badly prepared for applications.

> Ich habe immer wieder beobachtet, daß Mathematiker und Physiker mit abgeschlossenem Examen über theoretische Ergebnisse sehr gut, aber über die einfachsten Näherungsverfahren nicht Bescheid wußten. [5]) [Collatz 1951, V]

APPLIED MATHEMATICS IS SEEN ONLY AS UTILITARIAN. Even the title of this chapter "Applications of mathematics : mathematics as a service subject" makes us believe that the *only* role of applied mathematics is to be useful : something *served* by noble servants in dress coat to physicists, chemists, biologists sitting there as the *clientele*, after being prepared in *la haute cuisine* by pure mathematicians.

It appears widely forgotten that — *the other way round* — applications often had an enormous influence on the development of pure mathematics itself.

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[2]) … by founding the *Journal for pure and applied mathematics* (1826). If today we browse through a volume of this publication, we may well find the title astonishing. […] The neo-humanistic ideal of pure science for its own sake, which implied the contempt of every form of utility in the ordinary sense, soon led to deliberately turning away from all aspirations oriented towards practice. […] This hallmarked it as an organ of abstract special mathematics of the strongest mintage and earned it the humorous name of 'Journal for pure, unapplied mathematics'.

[3]) We have *no* need for computers.

[4]) Pure mathematics is like a painting by Rembrandt ; applied mathematics is more like a house-painter's job.

[5]) I have repeatedly observed that mathematicians and physicists, upon completion of their exams, knew a lot about theoretical results but nothing about the simplest approximation procedures.

History is rich in such examples, beginning with Kepler's astronomy used for the discovery of Newton's gravitation law and of differential calculus; the brachystochrone problem leading to variational calculus (this will be further developed in the next paragraph); astronomical calculations by Euler and Gauss:

> In 1735 the solving of an astronomical problem, proposed by the Academy, for which several eminent mathematicians had demanded some months' time, was achieved in three days by Euler with aid of improved methods of his own. [...] With still superior methods this same problem was solved later by K. F. Gauss in one hour! [Cajori 1893 (1980), 232]

Another example is Fourier's theory of heat, which led to Dirichlet's concept of function, Riemann's integral, Cantor's set theory and Sturm-Liouville's spectral theory in Hilbert spaces:

> J'ajouterai que le livre de Fourier a une importance capitale dans l'histoire des mathématiques et que l'analyse pure lui doit peut-être plus encore que l'analyse appliquée. [6]) [Poincaré 1895, 1]

Further, Poincaré's *Mécanique céleste* led to symplectic geometry and KAM-theory; Curtiss and Hirschfelder's chemical calculations led to the theory of stiff differential equations, and so forth.

We may also mention *logarithms*, which were invented in the early 17th century for the sake of practical calculations and have been used for several centuries. After pocket calculators came into use, logarithms were thought to be superfluous for school teaching, and a whole generation of students never saw them in high school. The enormous importance they have for all sciences was thereby totally overlooked. The same fate is happening now to *descriptive geometry.*

APPLIED MATHEMATICS IS MORE DIFFICULT TO TEACH. The teaching (and understanding) of applied mathematics gives rise to additional difficulties for the teacher and for the students, such as: managing computer tools, learning programming languages, tracking wrong answers due to rounding errors, knowing the application field like physics or chemistry, working with physical constants and conversion factors, and also the difficulty of translating the applied problem conveniently into a mathematical one. For example, it is easier to solve

$$5 \cdot 8 = 3 \cdot x$$

[6]) I shall add that Fourier's book is of utmost importance in the history of mathematics and that it has been even more profitable, perhaps, to pure than to applied analysis.

than to answer the question

5 workers accomplish a certain task in 8 hours;
how many hours will 3 workers need?

A famous example in the same vein (though more serious) concerns the *brachystochrone problem.* Its history is outlined in the following three quotations.

> PROBLEMA NOVUM *Ad cujus solutionem Mathematici invitantur.* Datis in plano verticali duobus punctis A & B, assignare Mobili M viam AMB, per quam gravitate sua descendens, & moveri incipiens a puncto A, brevissimo tempore perveniat ad alterum punctum B.[7]) [Bernoulli 1696]

Initially it was a problem of applied sciences, asking for the curve of fastest descent, which de l'Hôpital could not solve:

> Ce probleme [me] paroist des plus curieux et des plus jolis [que] l'on ait encore proposé et je serois bien aise de m'y appliquer; mais pour cela il seroit necessaire que vous me l'envoyassiez reduit à la mathematique pure, car le phisique m'embarasse ...[8]) [letter of de l'Hôpital to Joh. Bernoulli, 15 June 1696: NFG *63*, 319]

Only with the knowledge of a certain amount of physics did this become a problem of pure mathematics:

> Voycy donc maintenant comme le probleme se reduit à la pure mathematique: D'entre toutes les lignes AMB qui joignent deux points donnés A et B on cherche la nature de celle dont
>
> $$\int \frac{\sqrt{dx^2 + dy^2}}{\sqrt{y}}$$
>
> soit la plus petite.[9]) [letter of Joh. Bernoulli to de l'Hôpital, 30 June 1696: NFG *64*, 321]

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[7]) Let two points A and B be given in a vertical plane. Determine the path AMB that a point M must follow so that, starting from A, it reaches B in the shortest time under its own gravity.

[8]) This problem looks to me one of the most intriguing and one of the nicest that has ever been proposed, and I would much like to dedicate myself to it; but it would be necessary for this that you send it to me reduced to pure mathematics, since the physical [aspect] disturbs me ...

[9]) Here is now how the problem reduces to pure mathematics: From among all paths AMB which connect two given points A and B, one is looking for the nature of the one for which

$$\int \frac{\sqrt{dx^2 + dy^2}}{\sqrt{y}}$$

is smallest.

Our last quotation, from Daniel Bernoulli, shows what difficulties can arise — *the other way round* — for applied scientists.

> Den Hrn. D'Alembert halte ich für einen grossen mathematicum in abstractis; aber wenn er einen incursum macht in mathesin applicatam, so höret alle estime bey mir auf: seine Hydrodynamica ist viel zu kindisch, dass ich einige estime für ihn in dergleichen Sachen haben könnte. Seine pièce sur les vents will nichts sagen und wenn Einer alles gelesen, so weiss er so viel von den ventis, als vorhero. Ich vermeinte, man verlange physische Determinationen und nicht abstracte integrationes. Es fängt sich ein verderblicher goût an einzuschleichen, durch welchen die wahren Wissenschaften viel mehr leiden, als sie avancirt werden, und wäre es oft besser für die realem physicam, wenn keine Mathematik auf der Welt wäre. [10]) [letter of Daniel Bernoulli to Euler, 26 January 1750: Fuss *56,* 649–650]

Another weak point with applied mathematics is that the subjects easily become boring, outdated, politically incorrect or ludicrous. For example, the above problem of the five workers can have the flavour of a big manager exploiting underpaid workers, while the equation $5 \cdot 8 = 3 \cdot x$ is perfectly neutral. The same impression is conveyed by the many 'applied' examples in Euler's *Algebra* (*Opera Omnia*, vol. 1), where there is always someone who lends money or buys clothes in long forgotten currencies. It is also interesting to compare the 'applied' articles in the first issues of the *Crelle Journal*, which discuss some uninteresting motions of steam engine parts or water-works, with the 'pure' contributions of Abel and Jacobi, which still today glitter in eternal youth and beauty.

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[10]) I hold Mr. D'Alembert for a great mathematicum in abstractis; but when he makes an incursum into mathesin applicatam, then I no longer have any estime: his Hydrodynamica is much too childish for me to have any estime for him in such matters. His pièce sur les vents is meaningless and when one has read it all, one knows as much about the ventis as before. I imagined that one required physical determinations and not abstract integrationes. One begins to see a pernicious goût creeping in, from which the true sciences suffer much more than they advance. Would it often not be better for the realem physicam if there were no mathematics on earth?

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GENERAL DISCUSSION

(reported by Chris WEEKS)

The discussion from the participants, following Gerhard Wanner's reaction to the three talks, fell not surprisingly into the two broad topics of:

- what meaning should be attached to the term 'applied mathematics'?

and

- what does this imply for the mathematics curriculum?

An aspect of the second proved to be of major importance, namely:

- what is the role of assessment in the teaching of applied mathematics?

The discussion did not fall neatly into just one of these categories as each had implications for the other. In fact, there was the view that the term 'applied mathematics' was itself a construct of the teaching of mathematics. As one French participant put it:

> The distinction between real and applied mathematics is not very clear. At the beginning of my career I was a pure mathematician; now I am an applied mathematician, but personally I cannot see the difference. Certainly there is a difference, but it is made by those people who write 'pure' or 'applied' on their hats and maintain their independence; we even see some who will not talk to one another. In my view it is not a matter of epistemology or didactics but "une affaire humaine".

THE MEANING OF APPLIED MATHEMATICS. The meaning of 'applied mathematics', or what meaning should be given to the term, dominated the beginning of the discussion. Mathematics, where it is not undertaken as it were for its own sake, takes place within some exterior context and the mathematics then becomes 'applicable'. Another term, now in constant use, is 'modelling'. These ideas imply mathematics being used in some way to deal with some problem not present in the mathematics itself. But the question that then arises is whether or not the activity of applied mathematics is in reality physics or engineering or economics, for which mathematics is just one of the tools being used, or whether it is actually pure mathematics being carried out according to some given rules. Several speakers were of the view that the essential of applied mathematics lies in the translation of the real-world problem into a mathematical form. Once this mathematisation has been accomplished, the problem then becomes (simply) a problem of pure mathematics — though the solution would need to be interpreted within the originating context. The example of L'Hôpital's reaction to Bernoulli's brachystochrone problem, given by Gerhard Wanner, was found to be a useful reference example for the discussion.

The real essence of applied mathematics is to simplify reality in order to carry out some mathematics on it. Even then, when the problem is solved, it is only rarely a reasonable approximation to the reality from which the problem derived. Of course there are occasionally some good mathematical models of the real world, but then the equations are often too difficult to solve (a good example is the differential equations describing fluid dynamics).

Maybe it would be better to think of modelling, or applied mathematics, as an enrichment (of context and ideas) and not as a separate subject. 'Mixed' mathematics would be a better term. For example, for the study of abstract algebra earlier in the 20th century there was no mathematics book available, so it was necessary to turn to the physicists (who knew more about groups early in the 20th century than did professional mathematicians). Yet the theory of groups is now considered to be a topic belonging exclusively to pure mathematics. "It is a constant fact in mathematics that mathematics comes from elsewhere", added one speaker.

Another speaker pointed out that every person learning mathematics starts out from a situation where there is no distinction between pure and applied mathematics. Children just have questions puzzling them, what Freudenthal called 'phenomena'. In this sense the distinction between pure and applied mathematics is quite artificial.

There was also the question of whether 'applied mathematics' in some way showed that mathematics was 'useful' (a theme that was picked up again in the discussion of the place of applied mathematics in the mathematics curriculum). But the question of the utility of mathematics begs a question. Useful for what? One speaker proposed that there were two ways in which mathematics was useful. There was the traditional role of mathematics as a 'service subject' to other disciplines, where its utility is evident. But there was also the idea that mathematics was useful to society, and that mathematics education had the additional sociological benefit of preparing students to be effective members of society, and of being able to contribute to the development, management, and efficacy of society. For mathematics to be useful to society, added another speaker, it is not a matter of teaching mathematics so as to be useful, but of teaching useful mathematics, which is a different matter.

Mogens Niss added that when making a distinction between pure and applied mathematics, we should make a distinction between the epistemological level and the sociological level. One problem is that, even if there is no difference at the epistemological level, there is a distinction at the sociological level. And the problem is that at school (and also at university) this distinction is being perpetuated. In his view, the distinction is that you are an applied mathematician when the question comes from outside and you want an answer; you are a pure mathematician if, when you are inspired by problems that come from outside, you seek to provide some sort of generality of solutions.

APPLIED MATHEMATICS IN THE MATHEMATICS CURRICULUM. There was a general view that the applications of mathematics deserved a place within the school curriculum. Many have argued for this for a long time and had they been successful in bringing applicable mathematics into the school curriculum we would not be having this discussion at all and the intended ICMI Study into applied mathematics would not be necessary. Certainly we need a lobby group when we see that the distance between what we have and what we want is so great. This in itself would warrant an ICMI Affiliated Study Group on modelling. The curriculum needs to be penetrated to the extent that the difference between 'pure' and 'applied' mathematics simply disappears.

There are projects which seek to cross the borders between mathematics and other disciplines. André Revuz referred to an initiative at Paris VII university where the teaching of a course was shared by mathematicians and physicists in the first year of DEUG (*Diplôme d'études universitaires générales*). Each week, in a one hour session the same question was presented by both teachers (who did not always agree). All were happy with this (and the students were delighted). To talk to people of different disciplines is to enrich both disciplines.

If you ask the general public what it is in mathematics that is useful, you find references to elementary mathematics, perhaps with sophisticated applications (elementary ratio, probability, statistics, etc.), or else undergraduate level mathematics. It is very rare to find evidence of applications of upper secondary school mathematics. So a reason why applications do not penetrate school mathematics at that level is simply because there are none: very few people make extensive use of, say, trigonometric identities or quadratic equations.

Of course, the greatest difficulty in doing applied mathematics is the translation of the problem from the ‘real’ context to a mathematical task (and this is true at all levels). The reason why applied mathematics is so difficult is because there are no givens and no correct answers. There is also the problem that validation of the solution of a problem lies outside the mathematical context (see the discussion on assessment later).

The challenge for mathematics is that mathematics for all is hard to achieve. However there is a great deal of mathematics being taught outside mathematics lessons — in other disciplines — and if we wish to determine the extent of penetration of mathematics into other disciplines we need to look at what is being taught there. One consequence of the penetration of mathematics into other disciplines is that the mathematics there is being taught by practitioners of these other disciplines; economists teach the maths needed for doing economics, physicists for physics, etc. If mathematicians are not willing to cross subject boundaries and engage with the teachers of these other subjects, what then happens to the teaching of mathematics? If all the ‘useful’ mathematics is no longer taught in mathematics classrooms by mathematics teachers, what happens to the mathematics left for the mathematics classroom? The mathematician becomes a teacher of mathematics that leads nowhere. And the real danger then is that mathematics attains the status that Latin had in the 20^{th} century — part of our cultural heritage, but gradually abandoned as a useful tool for living in today’s society.

Trevor Fletcher, a retired inspector of schools in England and Wales, saw that the distinction between cultural mathematics and utilitarian mathematics was important but deplorable. Certainly at school level, mathematics should see itself as having the task of explaining something, rather than as contemplating itself. In fact, he claimed, the only justification for mathematics being compulsory as a school subject was that it had the power to explain.

A final point made was that motivation is an important factor in learning mathematics, and applications of mathematics can be important in encouraging curiosity.

ASSESSMENT. The role played by assessment in the mathematics curriculum has important implications for the teaching of applications of mathematics. One speaker pointed out that teaching does not take place in an abstract context but in a reality which is the current school reality. Here assessment plays a significant role and the reason why more applications of mathematics are not taught can be explained simply by the fact of examinations at all levels. When examinations assess applications of mathematics, it is only sterilised and formalised presentations of the applications of mathematics that are examined. We can only penetrate school syllabuses if we take up the challenge : how can it be examined ? Against that, Djordje Kadijević reminded the conference that there were examples of experiments in assessment of applications of mathematics, at least as regards the use of *computer assisted systems* (CAS), which have been used in Austria for about 10 years.

Mogens Niss was asked to say some more about applied mathematics and the problems of assessment. The problem for schools and universities, he said, was that they do not want the students to go all the way with tackling a problem. One problem is that validation of the solution requires expertise from outside the mathematics classroom. Another is that assessment carries important implications of costs (in time and resources).

At his own university (Roskilde University Center), the students tackle applications of mathematics through topic-based work. Typically, though not exclusively, the mathematics application would be in a science subject and there would be joint supervision. Otherwise there would be a primary supervisor, but then the student is always free to go to others for advice, which is freely given. Certainly the overall costs in time and resources are expensive compared with traditional examining, but this is a cost they are willing to bear. This can be done because the university is new, and so there are no entrenched assumptions, and also because it is small. Perhaps really radical things can only be done on a small scale. Mogens Niss asked us to think of his situation as a laboratory, where practices are tried out which others can observe and then adapt the practices to their own institutions. It is not his view that the model should be copied as it stands.

PERSPECTIVES
FOR MATHEMATICS EDUCATION

STAKES IN MATHEMATICS EDUCATION FOR THE SOCIETIES OF TODAY AND TOMORROW

Les enjeux de l'éducation mathématique pour les sociétés d'aujourd'hui et de demain

par Ubiratan D'AMBROSIO

On examine dans cette contribution l'émergence des mathématiques comme une discipline dans les systèmes scolaires à partir du XIX^{e} siècle. La préoccupation des éducateurs de nombreux pays visant à offrir un enseignement mathématique actualisé dominait les réflexions sur les objectifs, les contenus et les méthodes de cet enseignement.

Ce mouvement s'étendit à tous les pays et devint ouvertement international. C'est dans ce cadre que la revue *L'Enseignement Mathématique* fut créée à Genève en 1899 et que la *Commission internationale de l'enseignement mathématique* fut établie par le Congrès international des mathématiciens à Rome en 1908. Des sociétés nationales surgirent dans plusieurs pays.

Après la Seconde Guerre mondiale, les processus de décolonisation et la nécessité d'instruire les jeunes pour d'autres genres de travail en les préparant à une société de consommateurs où ils soient conscients de l'état du monde, tant du point de vue de l'environnement que de la société, de l'économie et de la géopolitique, requièrent une nouvelle réflexion sur une éducation mathématique regroupant de nouvelles matières résultant d'une pensée interdisciplinaire et même transdisciplinaire, suite à l'effondrement de barrières culturelles et à l'établissement de nouvelles techniques de communication et d'information. Les objectifs, les contenus et les méthodes de l'instruction mathématique subissent des transformations profondes, dont cette contribution tente de tenir compte pour un enseignement mathématique renouvelé.

STAKES IN MATHEMATICS EDUCATION FOR THE SOCIETIES OF TODAY AND TOMORROW

by Ubiratan D'AMBROSIO

THE SCENARIO IN THE TRANSITION FROM THE 19TH TO THE 20TH CENTURY

The transition from the 19th to the 20th century was marked by the effects of the three major revolutions of the Modern World : the *Industrial Revolution,* responsible for new models of production and labor relations ; the *American Revolution*, establishing a new model of governance and political leadership ; and the *French Revolution*, focusing on new social relations, advancing the modern concept of citizenship and generating new demands for bureaucracy and administration.

In geopolitics, the 19th century established the great empires which emerged from the colonial order. At the same time, new intellectual and material instruments, developed on the basis of Newtonian science, were the bases for the establishment of the new sciences of sociology and economics. There was also the noticeable emergence of new directions in Christianity, to a great extent proposing new perceptions of man and society. History was seen as a determinant of the state of the world.

The sciences, particularly mathematics, were consolidating the directions proposed since the 17th and 18th centuries. Rigor, precision, correctness were seen as major goals to be attained, both for the advancement of knowledge and as attributes of a valuable personality. Technology was an evidence of the righteousness of these pursuits. The Eiffel Tower, the Solvay Institute, the Nobel Prizes, are symbols of the achievements of modern technology.

The world order was apparently tranquil. The results of the Raj revolt and of the Zulu War, the successes of the Creoles (Americans born of European descent) in establishing independent mirror-nations in the Americas, were all indicators of this tranquility.

This World order implied the recognition of some universals. Mathematics was firmly established as a symbol of universal knowledge. And mathematicians from the entire world decided to assemble. An International Congress of Mathematicians was held in Zurich in 1897. Its membership reveals the dominating concept of universality: 197 members from 15 European countries plus 7 members from the USA. Another symbol of universality would be Esperanto, a universal language. Its structure, grammar, and vocabulary also reveal the prevailing concept of universality. Some mathematicians adhered to Esperanto.

The second International Congress of Mathematicians was convened in Paris, in 1900. The 232 participants came from 26 countries. Non-Europeans [1]) came from Algiers (1), Argentina (1), Brazil (1), Canada (1), Mexico (1), Peru (2), USA (8). The congress was marked by the conference of David Hilbert delivered in the section on *History and Pedagogy*, entitled simply 'Mathematical Problems', in which he formulated 23 problems which would mark the development of mathematics and, of course, Mathematics Education, throughout the 20th century.

In the second paragraph of his paper, Hilbert says:

> History teaches the continuity of the development of science. We know that every age has its own problems, which the following age either solves or casts aside as profitless and replaces by new ones. […] the close of a great epoch not only invites us to look back into the past but also directs our thoughts to the unknown future. [Browder 1976, 1]

Almost one hundred years later, Stephen Smale would also refer to the interest of dealing with mathematical problems:

> … mathematics is more like art than other sciences. But there is one special difference, I find, which is that mathematics tends to be correct. Mistakes in mathematics are rarely significant for very long. But also mathematics tends to be more irrelevant. There is so much of mathematics that tends to go off in directions which are appreciated only by a very few, irrelevant even to the rest of mathematics. So I think there is a bigger danger in mathematics than there is in other sciences of tendencies to go off into irrelevancies, i.e., into things that are correct but not important. [Casacuberta & Castellet 1992, 88–89]

Not only correctness, interest, relevance, appreciation, have always been present in reflections about mathematics and mathematics education, but the utility of mathematics has also been emphasized. In the meeting of the British

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[1]) In the next congress, held in Heidelberg in 1904, there was a larger number of participants (336), but only 18 came from outside Europe: 15 from the United States, 1 from Argentina, and 2 from Japan. In that same year, Japan would surprise the world by defeating the powerful Russian navy.

Association, called to discuss the reform proposed by John Perry, his concept of usefulness is, in itself, a program [Perry 1901, 4–5]. Its eight points focus on mental pleasure, on logical styles of thinking, on preparing for further studies and professions. Although Perry's focus was received with criticism and hostility [2]) by many mathematicians, his concept of usefulness has since then been recurrent in discussions about "why teach mathematics". Perry is but an example of the movement towards a renewed mathematics education on the eve of the 20th century. Even at the risk of being accused of banalizing the discussions about Education, I claim that the discussions about curriculum focus on *why*, *what* and *how* to teach, or *objectives*, *contents* and *methods* [D'Ambrosio 1983]. Much emphasis has been given to contents and methods. Objectives have essentially been taken for granted.

MATHEMATICS EDUCATION REFORM

In the second part of the 19th century, the social order was showing signs of fragility. The call for emancipation by the popular classes and by women had obvious reflections in education [3]). The success of the United States in building up a new nationality showed the potential of schools for social change. The claim for social change was strong. The developments of a new production system, as a result of the Industrial Revolution, called for a new kind of worker. The home, so important when artisanal production was prevailing, did not respond to the needs of the future worker.

The relation education–society was profoundly affected by two factors: the need for a new kind of worker; the great progress of psychology, with the recognition of specific behavior and special needs of children. These two factors, the development of productive social capabilities and the modern "discovery of the child" are, according to Manacorda [1989], characteristic of the new school emerging in the second half of the 19th century. The new educational thinking called for updated mathematical contents, as well as for new methods of teaching, reflecting findings in learning. An active school was proposed: hands-on projects, learning by doing. Motivation should have priority.

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[2]) 'hostility' is the tone of the reaction of F. A. Forsyth, as recognized by himself [Perry 1901, 39].

[3]) Quite interesting are the remarks of Mrs. W. N. Shaw on supporting mathematics teaching for girls: "… those powers in which the feminine mind is said to be peculiarly deficient — the powers of accurate observation and logical reasoning…" [Perry 1901, 50].

Mathematics Education was central for the new needs of labor. This is seen in Perry's and in several other proposals, where we recognize an appeal to psychology. In 1905 the Young couple wrote :

> We have heard it asserted that it would be harmful to a young child to be 'plagued with Geometry'. Geometry, as we have in view, is no plague. It requires no school-room, no blackboard, no special apparatus. It does not even require a trained teacher, nor does it demand the child's attention for long periods at a time. It is essentially a subject for home, and for an early age. [Young & Young 1905, vi]

And, resorting to paper folding, there follows good traditional Euclidean geometry. A few years later, Klein wrote :

> The child cannot possibly understand if numbers are explained axiomatically as abstract things devoid of content, with which one can operate according to formal rules. […] While this goes without saying, one should — *mutatis mutandis* — take it to the heart, that in all instruction, even in the university, mathematics should be associated with everything that is seriously interesting to the pupil at that particular stage of his development and that can in any way be brought into relation with mathematics. [Klein 1908, 4]

A conservative posture, encouraged by the recent theoretical advances of classical mathematics and its applications, pointed to a formal and structured organization of curricula. Objectives, aimed at keeping the established social and world order, were taken for granted. Contents were almost entirely agreed upon, methods would differ widely. Particularly interesting is Klein's argument in favor of calculating machines [Klein 1908, 17–22].

L'Enseignement Mathématique was founded in 1899 for contact, exchange of information and comparison of educational systems. I will not comment on the first series of the journal. This has been done in a very thoughtful and complete way by others. I will just add a few remarks, which may be taken as curiosities or may suggest reflections of a different nature :

1) all the so-called 'general' papers are written in French, except one in Esperanto, by M. Fréchet (1913), and a tiny few [4]) in English, by C. Runge (1912), D.E. Smith (1912), A.N. Whitehead (1913), and G.A. Miller (1915);

2) there are only two papers by Felix Klein, one in 1906 and another, joint with A. Gutzmer, in 1908, both about reforms in Germany;

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[4]) Of course English, German, Italian, and even Esperanto, do occur in other texts, in particular in reports of discussions.

3) very few articles came from South America, though there is one by Nicolas Besio Moreno [5]), from Argentina (1920);

4) in H. Fehr's report [1953] on the *Commission internationale de l'enseignement mathématique*, the USSR is not mentioned a single time, but there is a reference to a paper on mathematics education in Russie (*sic*), by D. Sintsof [6]). According to Manacorda [1989], many educational reforms in the Soviet Union, affecting particularly the objectives of education, and mathematics education in particular, were taking place.

In the International Congress of Mathematicians, held in Rome in 1908, an International Commission on the Teaching of Mathematics was created by initiative of David Eugene Smith, from the USA, with the main objective of comparing methods and plans of studies of different countries. A retrospective of this commission was given by the outgoing chairman, H. Fehr, in 1952 [Fehr 1953]. In the same year, 1952, the General assembly of the International Mathematical Union (IMU), convened in Rome, established the International Commission on Mathematical Instruction (ICMI), as a commission of the IMU.

AFTEREFFECTS OF WORLD WAR II

The Second World War represents a landmark in recent world history. It paved the way for a new world and social order. The emergence of a new technology, called *technoscience*, strongly grounded in new areas of mathematics, asked for profound reforms of mathematics education. This was generally called Modern Mathematics Movement. New priorities for economic reconstruction and defense prompted the adoption of new ideas in Mathematics Education, anchored on advances in the theory of cognition. The research conducted by Jean Piaget was a strong support in the Western world, a role played by A. R. Luria and Lev Vygotsky in the Soviet Union.

A major consequence of World War II was the establishment of the United Nations, in 1945. The charter was signed by 51 states. Soon it was enlarged by former colonies that achieved or were granted independence. The ideal of education for all was explicit in the Universal Declaration of Human Rights, issued by the United Nations in 1948. International cooperation was channeled through UNESCO, the Organization of American States, and similar organizations, as well as bilateral cooperation, establishing a new professional

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5) a renowned engineer who was well informed on educational matters.

6) *L'Enseign. Math. 32* (1933), 81–87.

international figure: the expert, or consultant, or *coopérant*. The rhetoric of development implied the priority of mathematics over other disciplines in the curriculum. The Modern Mathematics Movement was seen as a valid option also for the nations belonging to the group called "Third World" [7]). I will not engage in pro and con arguments about the Modern Mathematics Movement. Instead, I will address social and cultural issues, which were largely disregarded in the movement.

The Cold War, immediately following the end of World War II, was marked by reconstruction of the economies, together with enormous spending in defense, which also provided funds for scientific research. Mathematics particularly benefited from generous military funding. At the same time, peace movements were active and the appeal of a new social and political order was intense. In the Third World, this appeal provided a fertile ground for civil wars and dictatorships, and a superb arena for the Cold War.

The postwar period was also marked by intense demographic change. Granting independence was, in most cases, accompanied by very loose immigration procedures by most colonial metropoli. Emigration became a goal. The more developed nations were subjected to a large number of legal and illegal immigrants. This was added to the demand for civil rights, which was particularly strong in the USA. Social exclusion, intimately associated with cultural differences, emerged as a new focus of attention. Multicultural education, particularly language education, became a new issue. Indeed, children, and even adults, may acquire, in schooling, different kinds of cultural identity, thus perceiving different orders of understanding themselves and the world. The social implications of language education, recognized as a major factor responsible for exclusion, drew the attention of educators [Freire 1970; Bernstein 1971].

The reflections focusing on language led scholars to examine the nature of languages and the cognitive implications of different languages. It was accepted that language was a cultural manifestation. Different cultures have different languages. The social implications of not respecting language differences were easily recognized. *But few would dare similar thinking about mathematics.*

In the International Congress of Mathematicians held in Oslo in 1936, it was decided that the next congress would take place in the USA in 1940. As a consequence of the war, the congress was convened only in 1950, in Cambridge, Massachusetts. The International Mathematical Union (IMU) was

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[7]) A general view of the movement, particularly focused on Brazil, can be seen in [B. S. D'Ambrosio 1987].

founded[8]) in 1951 and its first General Assembly took place in Rome in 1952. In this meeting, it was approved that an International Commission on Mathematical Instruction (ICMI), a reorganization of the pre-war International Commission on the Teaching of Mathematics, be formalized as a commission of IMU.

After World War II, mathematics was firmly established as universal knowledge. The political atmosphere just after World War II conduced, with respect to mathematics, to what has been called the '*American Declaration of Universality*', which proposed that mathematicians should convene independently of national allegiance [Lehto 1998, 74]. The universality implied objectives and goals for mathematics education defined irrespective of social and cultural parameters. In 1952 it was decided that ICMI should perform a study on the role of mathematics and mathematicians in the contemporary world. Duro Kurepa conducted the study, which was reported to the International Congress of Mathematicians in Amsterdam, 1954. The report gives special attention to several new directions of mathematical activity. Particularly interesting is the recognition of the importance of calculators :

> Les mathématiques se sont rapprochées de la physique et de la psychologie et on est en train d'examiner le mécanisme du « penser », du « calculer » et d'autres fonctions psychiques et intellectuelles.[9]) [Kurepa 1955, 100–101]

Recognizing the profound changes in the world, Kurepa gives a very comprehensive view of mathematics ; he sees its role, because of the imminence of the presence of robots, as essential for the mutual understanding between individuals and collectivities in order to have a global apprehension of the world. The role of mathematics in *Weltanschauung* is clearly stated as a moral duty of the mathematician.

The questionnaire continued to produce interesting responses and reactions. The most intriguing, opening new perspectives for mathematics education, came from the renowned Japanese mathematician Yasuo Akizuki, a member of the Executive Committee of ICMI. In line with Kurepa, Akizuki proposed an emphasis on the reflective side of mathematics, looking into the world as a whole. He made a strong point for introducing the history of science and mathematics in all levels of teaching. The most interesting point in his argument is the recognition that mathematics is a *cultural product*. He recognizes that Western mathematics is present in Asia, and says :

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[8]) For a full story of the I.M.U., see [Lehto 1998].

[9]) Mathematics gets closer to physics and to psychology and we are in the course of looking into the mechanisms of 'thinking', of 'calculating' and other psychic and intellectual functions.

> Oriental philosophies and religions are of a very different kind from those of the West. I can therefore imagine that there might also exist different modes of thinking even in mathematics. Thus I think we should not limit ourselves to applying directly the methods which are currently considered in Europe and America to be the best, but should study mathematical instruction in Asia properly. Such a study might prove to be of interest and value for the West as well as for the East. [Akizuki 1959, 289]

Although anthropologists and psychologists had been showing interest in different kinds of mathematics, better saying, different ways of mathematizing in different cultures, Akizuki's proposal did not attract the attention of the mathematical community until Claudia Zaslavsky published *Africa Counts*, in 1973 [Zaslavsky 1973].

New directions of mathematics education reforms

The Royaumont Seminar, in 1959, convened by initiative of the European Economic Cooperation Administration, which later became the Organization for Economic Co-operation and Development (OECD), established the major guidelines for the movement known as *Modern Mathematics*. Policies for universal implementation of the recommendations called for regional cooperation.

Marshall Stone, as the President of ICMI (1959–1962), marshaled cooperation in Mathematics Education in the Americas. The USA had been active in cooperation for development in the region, of vital importance in Cold War strategy, mainly through the Organization of the American States [10]). In 1961 the First Inter-American Conference on Mathematics Education was held in Bogotá, Colombia, largely financed by the NSF. Prestigious mathematicians from the USA and from Europe attended the conference, as well as representatives from almost every country of the Americas. As a result, the Inter-American Committee on Mathematics Education was founded, with Marshall Stone as its first President. In 1966 a Second Inter-American Conference on Mathematics Education was held and since then, in different countries of the Americas, conferences were convened every four years [Barrantes & Ruíz 1998]. The last one, the 10th IACME, took place in Maldonado, Uruguay, in November 1999.

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[10]) Indeed, national research councils had been created in almost every country in the Americas, modeled on the National Science Foundation.

The series of International Congresses on Mathematical Education (ICME) gave the possibility of IMU focusing more on contents regarding Mathematics Education. In 1971 the IMU created the "Union Lectures", designed as an expository series of four to six lectures surveying current research themes. It was decided that the lectures would be published in *L'Enseignement Mathématique*. This was a prompt response to acts that may be interpreted as weakening the relations of *L'Enseignement Mathématique* with ICMI. Hans Freudenthal, while President of ICMI (1967–1970), sponsored the creation of *Educational Studies in Mathematics*, which obtained financial support from UNESCO. The following President of ICMI, James Lighthill (1971–1974), created an *ICMI Bulletin* in 1972, as a means of spreading information about ICMI activities. It was considered to play a complementary role to that of *L'Enseignement Mathématique*, which remains the official organ of ICMI. The *ICMI Bulletin*'s appeal to mathematics educators all over the world increased steadily, as it attained regularity of publication and a large worldwide coverage of events and issues. Indeed, looking retrospectively on the issues covered by the *ICMI Bulletin*, as seen in issue n° 47 (December 1999), we notice a very different character from *L'Enseignement Mathématique*. It is natural to question the necessity and the reason for this dual approach.

Scenarios for the transition from the 20th to the 21st century

Before World War II, the objectives of mathematics education, aimed at keeping the established social and world order, were taken for granted. Contents were almost entirely agreed upon and methods would not differ substantially. Anchored in advances in the cognitive sciences and the new possibilities of calculation and information retrieval, the Modern Mathematics movement brought new contents and new methods of mathematics education. However, the objectives of mathematics education were unclear. And they remain so. The rhetoric of personal and social advances is not clearly focused, and exclusion seems to be the most noticeable effect of mathematics education. The question "Why teach mathematics ?" seems to be the crux [11]). But together with this question come other questions about the nature of mathematics and how to handle mathematics teaching. Mathematics in the making ? Mathematics of everyday life ? Mathematics grounded in cultural traditions ? Mathematics as fun ? Good old classical mathematics ? Although not dichotomic, these strands do represent didactical options.

[11]) I dealt with this question in ICME 3, Karlsruhe, 1976 [D'Ambrosio 1979].

To face these recurrent questions, I see as fundamental a new, broader understanding of the socio-cultural history of mathematics and of its education. This calls for attention to non-conventional sources, rather than to purely academic ones. The Program *Ethnomathematics* [D'Ambrosio 1992] provides the instruments to deal with these questions. The growing interest in ethnomathematics prompted the creation of an International Study Group in Ethnomathematics (ISGEm) in 1985. Since then, the publication, twice a year, of a *Newsletter* with international circulation, the realization of the First International Congress of Ethnomathematics, and the establishment of a web site [12]) have givem worldwide visibility to ethnomathematics. An electronic journal, *Mathematical Anthropology and Cultural Theory* [13]), has been recently created. As expected, criticism of ethnomathematics is mounting. Ethnomathematics was drawn into the Math Wars ! Recently *The Chronicle of Higher Education* started a very lively discussion on ethnomathematics. The discussions, which are going on, are an echo to the completely mistaken title of the article "*Good-bye, Pythagoras !*" [Greene 2000].

Good old Pythagoras will ever be present to enable us to fly and to communicate via the internet and many other important achievements of modern civilization. The "male and female triangles" of the Xingu culture will not do that. And, of course, there is no point in teaching a boy or girl in Chicago the way the Xingu culture classifies triangles, except if shown as a piece of folklore, which indeed harms the proposal of ethnomathematics. Critics of ethnomathematics and, regrettably, even some supporters of ethnomathematics, are missing the point. Ethnomathematics does not propose to replace academic mathematics by folkloric mathematics, or Mickey Mouse mathematics, as it used to be called in the sixties. I am afraid the ethnomathematics proposal risks being seen — and practiced ! — in the same distorted way as the Modern Mathematics movement. Soon a modern Tom Lehrer will sell thousands of CDs ridiculing ethnomathematics, as Buffalo Bill's *Wild West Show* did with Native American culture.

The key issue is much deeper. It asks for a discussion of the major objectives of education and of schooling in the future. And, of course, how does mathematics fit in this future. The difficulty resides in a simple question : what do we know about the future ? Clearly, the way we see the future — and the way we want the future — guide our actions in the present.

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[12]) http://www.rpi.edu/~eglash/isgem.htm

[13]) http://www.SBBay.org/MACT

THE FUTURE

I am not alone in dreaming of a future without hatred and bigotry, with peace among nations, peace in society, peace with the environment, individuals in peace with themselves. IMU is sponsoring *World Mathematical Year 2000*. The Assembly General of the United Nations has proclaimed 2001 to 2010 as the "*International Decade for a Culture of Peace and Non-violence for the Children of the World*". The Assembly called on relevant United Nations bodies, non-governmental organizations (NGOs), religious bodies and groups, educational institutions, artists and the media to support the Decade actively for the benefit of every child in the world. How do we respond to this call ? How does the IMU resolution and the UN resolution relate ? Shouldn't they be intrinsic to each other ? After all, mathematics is the dorsal spine of modern civilization.

To give priority to peace may be a dream, maybe utopia, which justifies my efforts as an educator and my joy as a grandfather, who hopes to survive to become a great-grandfather ! The future generations must live in a better world than the one which our and the previous generations before us have constructed. What can we offer to the future generations ? Not our model. But a *critical* view of this model and of the knowledge system in which it was built. Mathematics is recognized as central to this knowledge system. Hence, mathematics, and its history, are subjected to this critical view [14]).

It is important to understand some characteristics of the so-called echo-boom generation (those born between 1977 and 1997). The boom generation, responsible for much of the expansion of the educational systems in the sixties and for the important events of 1968, is now reaching retirement age. The effects of this retirement play an important role in public finances, hence on politics. Much of my data refers to the USA but can easily be extrapolated to the developed world, where educational decisions set the standard. The echo-boom generation is the largest ever, about 80 million young people, spanning from 3 to 23 years old, with an enormous purchasing power. Cinema and television — currently the object of political disputes over who controls them — are of lesser importance for the echo-boomers. Instead, they access the Internet and they feel control over it. Effectively, they have control, as hackers demonstrate. They benefit from the technology gap between generations, which include parents, teachers, politicians, executives and decision makers in general.

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[14]) This was a motivation for my proposal of editing a section for the *Zentralblatt für Didaktik der Mathematik* on "Mathematics, peace and ethics". It appeared in *Zbl. Didaktik Math. 30* (June 1998), Heft 3.

The economy, work and personal relations will be the most affected. They will be in control in a couple of decades. Most probably they will not be co-opted by the system, as it happened with the 'rebels' of the sixties. The echo-boomers are creating a new culture [15]).

How will mathematics fit in this new culture ? We need to understand how, in other epochs, mathematics was affected by changing scenarios. This justifies my insistence on history of mathematics, not as understanding the development of a corpus of knowledge, but as a response to societal changes. Does society influence [16]) the development of mathematics ?

Throughout history, in every culture, we recognize the efforts to develop instruments : i) to communicate ; ii) to cope with reality ; iii) to understand and to explain reality, providing the tools of critical thinking ; iv) to define strategies for action. These goals can be identified in every ethnomathematics [17]). Greek mathematics focused mainly on iii) and iv). The propædeutical character of mathematical education, which has been intrinsic to curriculum development in the 20th century, emphasized ii).

In face of the new facilities of computation and information retrieval, there is no place for the propædeutical character of mathematical education. Numbers, figures, signs are *communicative instruments*, enriching the capability of discourse and conversation, of description. Critical familiarity with them, embedded in diversified cultural environments, is part of dealing with communicative instruments. Cultural environments mean calculators and computers if they are around, beads and threads if they are around, paper, pencil, chalk and blackboard if they are around. To create a school environment detached from the socio-cultural environment is justified only if projecting into the future, like the use of fiction as a pedagogical tool. Discontinuities between school and out-of-school environments must be accompanied by a critical reflection, not as a teaching device. Teaching goes on more and more in out-of-school environments. Both mathematics and ethnomathematics provide instruments to socialize the quantitative and qualitative ways of dealing with the surrounding reality.

It is a responsibility of schools to prepare students to generate new realities, prospective or imaginary, that is, to be creative ; to be able to explain and understand reality with the capability of moving into the future equipped with

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15) See an interesting article in the journal *The Futurist 34 (5)* (Sept.–Oct. 2000).

16) Quite provocative is the paper by Loren R. Graham 'Do Mathematical Equations Display Social Attributes ?' [Graham 2000].

17) See [Urton 1997].

strategies of action. This requires abstraction, conceptualization, in essence the domain of *analytic instruments*. Both mathematics and ethnomathematics provide such instruments, as history shows us.

Technology is part of our world, and will be even more so in the future. From birth through death, we improve the capabilities of our body, the distribution of biological or sky time, our reachable and productive space, as well as our capability of communicating with living and dead individuals and cultures, through technology. When, why and how artifacts and instruments can be used, combined, improved and invented, that is, critical familiarity with *material instruments* that are part of modern civilization, already or potentially accessible, is an important objective of education. Both mathematics and ethnomathematics are intrinsic in such instruments, as history shows us.

In the educational systems of the future I see mathematics, as well as ethnomathematics, inbuilt in the effort of critically providing communicative, analytic and material instruments [18]).

Production and labor will be present in every model of society of the future. The forms they will take surely will reveal the presence of high technology in everyday life. Jobs, as we understand them today, will most probably disappear [19]). The social access of minorities is directly related to their acquiring communicative, analytical and material instruments, in other ways, to the implementation of the *trivium* LITERACY–MATHERACY–TECHNORACY.

IN GUISE OF CONCLUSION

Is mathematics, as we today understand it in our curricula, prone to disappear? I believe so. Curricular organization and assessment, the pillars of current school mathematics, reveal their fragility. The propædeutical character of mathematics, responsible for curricular organization, is challenged. Signs of this are the proliferation of remedial courses and of self-contained specific training courses, such as those provided by enterprise universities. World War II revealed the efficiency of task-training, in line with Skinner's proposals. Good achievement in mathematics, as assessed by standardized and similar varieties of tests, such as SIMS, TIMSS and others, seems to be unrelated to society

[18]) Elsewhere [D'Ambrosio 1999] I called these respectively *literacy*, *matheracy* and *technoracy*.

[19]) For a discussion of labor in the future, see [Reich 1992]. Harsh views of the future of employment, revealing the inadequacy of current educational systems, can be read in [Forrester 1999].

development. The downfall of the Soviet Union and the critical view of education in the Asian countries are signs of this.

What will be the role of a mathematics teacher? I believe mathematics teachers, understood as those who simply teach mathematics, will disappear. But there will be need for teachers who do mathematics. Even doing mathematics using only addition of three-digit numbers!

The teacher of the future will be a resource-companion of students in the search for new knowledge [20]). The more mathematics a teacher knows, a better resource he/she will be; the more curiosity about the new, a better companion he/she will be. The characteristics of the new teacher cannot be only the result of special training. Teachers of old generations will retire and echo-boomers will become the new teachers. I trust they will be able to help build a better world.

How does this *L'Enseignement Mathématique* fit in this scenario? Surely, there is always room for a good, respected, mathematical journal in this future. But does it fulfill the needs of a new *enseignement mathématique*? Not in the current format.

Maybe, following the vocation of its founders — who clearly understood what was going on in the transition from the 19^{th} to the 20^{th} century and were able to fulfill the need of a journal with the characteristics of *L'Enseignement Mathématique* —, a second rebirth might give rise to a new transdisciplinary and transcultural journal of mathematics and mathematics education, opening its pages to historical, anthropological, and cultural issues.

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SCIENTIFIC SOLIDARITY TODAY AND TOMORROW

La solidarité scientifique aujourd'hui et demain

par Jeremy KILPATRICK

Les sociétés modernes intègrent les mathématiques dans les programmes scolaires pour des raisons à la fois pratiques et intellectuelles. Historiquement, les mathématiques ont été enseignées au niveau de l'école primaire pour fournir des outils de résolution des problèmes pratiques, tandis qu'elles sont entrées dans le système éducatif secondaire et supérieur à travers les humanités (*liberal arts*). Depuis le milieu du 20e siècle, des enjeux pratiques et intellectuels pour l'apprentissage des mathématiques ont été visés dans la plupart des pays et à tous les niveaux de la scolarité. A la fin de ce siècle, les enjeux pratiques ont pris une telle importance par rapport aux enjeux intellectuels que ces derniers risquent d'être éclipsés.

A l'image du 20e siècle, qui a vu une expansion mondiale de l'éducation secondaire universelle, il est probable qu'au 21e siècle on assistera à une expansion similaire du secteur universitaire avec une demande importante de mathématiques scolaires. Nos sociétés ont accru la quantité de mathématiques qu'elles attendent en vue de compétences numériques générales et, de ce fait, retardent le moment où beaucoup d'étudiants commencent leurs études spécialisées conduisant aux mathématiques avancées. La transition entre l'éducation secondaire et l'éducation supérieure en mathématiques va requérir une attention particulière.

La deuxième moitié du 20e siècle témoigne de l'émergence et de la disparition aussi bien de projets curriculaires concernant les mathématiques que de nombreux centres nationaux d'éducation mathématique. La CIEM devrait étudier comment de tels centres peuvent être renforcés, et favoriser non seulement le développement et la recherche sur les programmes mais aussi la formation professionnelle continue dont les enseignants ont besoin. Un défi majeur pour le siècle à venir est d'augmenter la solidarité entre chercheurs et enseignants de mathématiques à tous les niveaux (de l'école primaire à l'université).

SCIENTIFIC SOLIDARITY TODAY AND TOMORROW

by Jeremy KILPATRICK

I should like to discuss questions of the stakes in mathematics education for the societies of today and tomorrow: For whom is mathematics taught, and why? What 'political tools' (in their broadest sense) should we mathematics educators develop for reflecting on how to teach? These are not minor questions, and with limited space to discuss them, I clearly can address only some aspects of each one.

I have decided to concentrate on the question of why societies include mathematics in the school curriculum and for whom it is taught. Regarding tools for reflection, I have only a few possibly provocative comments to make at the end.

1. WHY TEACH MATHEMATICS?

Over the years, a variety of justifications have been offered for teaching mathematics [1]). These justifications can be classified in a variety of ways. For example, some are concerned with why every modern society has made mathematics a compulsory part of schooling, whereas others concern the reasons individual students might have for studying mathematics. In this discussion, I address only the former, the reasons that societies provide mathematics instruction to their members. I make a further simplification: there are essentially two broad categories of reasons for people to learn mathematics — the *practical* and the *intellectual.*

Practical justifications for teaching mathematics range from a society's need to have a numerate citizenry, one that can cope with the quantitative

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[1]) For reviews and more elaborated discussions, see [D'Ambrosio 1979; Niss 1996; Stanic 1984].

demands of everyday life in the home, marketplace, and workplace, to its need for people who can build its buildings and bridges, develop its seeds and medicines, and design its airplanes and computers. Societies invest in school mathematics because they want to improve their technological and socio-economic standing, as Niss points out, “either as such or in competition with other societies/countries” [1996, 13]. They have political, economic, and even military reasons for wanting some of their members to know and be able to use advanced mathematics in solving a variety of practical problems. They also want as many people as possible to be numerate at some basic level.

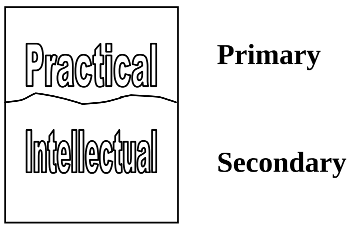
Intellectual justifications for teaching mathematics range from a society’s need for an educated citizenry that understands and appreciates the role mathematics has played in building that society and developing its culture to its need for people who can extend mathematics into new realms. Deductive reasoning, proof, and axiomatic structure, for example, are features of mathematics that ought to be understood and appreciated by educated people in a modern society. Mathematics has historically been taught at least in part because people have believed that its study develops habits and attitudes that are important for their intellectual development. Even today, a cogent argument can still be made that all students, and not just those who go on to become mathematicians and create new mathematical ideas, can profit intellectually and aesthetically from studying the mathematics developed in their society and elsewhere.

These two types of justification are connected with the way in which mathematics has been institutionalized in schools across the centuries. All societies, as they have developed methods for symbolizing ideas, have also developed their own mathematical systems, which have then had to be taught to the next generation. Today’s school mathematics curricula can be seen as resulting in large part from the collision of two tectonic plates.

The first plate developed within primary education. Societies have traditionally established some form of primary education as a means of giving children a rudimentary education in reading, writing, and elementary arithmetic. Since the mid-nineteenth or early twentieth century, free, universal, and compulsory primary education has been available in the industrialized countries of the world, and the second half of the twentieth century saw a tremendous expansion of primary education in developing countries. Mathematics is virtually universal in the curriculum of primary education, with the traditional role of mathematics teaching in primary schools being to prepare children for their future societal roles. In principle, they were learning mathematics as a tool for solving practical problems.

In contrast, the second tectonic plate developed within Western secondary and tertiary education. When the seven liberal arts were first formulated around 100 BC, mathematics — cutting across the four liberal arts of arithmetic, geometry, music, and astronomy — had already acquired a prestigious place in the curriculum of the academy. For Plato, mathematics was the test for the best minds. Secondary and higher education should aim not at the accumulation of knowledge but rather at the development of the intellect. Because mathematics was included among the liberal arts, it remained alive in the monastic and cathedral schools of medieval Europe and in the universities that were their successors even when little original research was being done in mathematics and the level of mathematical teaching was not very high. As new universities were founded during the nineteenth century around the world and the older universities expanded and developed in Europe, mathematics in secondary education was directed toward university preparation. It retained its liberal arts character.

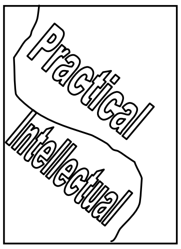
In a simplified view, therefore, the tectonic plates of justification for the traditional school mathematics curriculum looked roughly as follows:



Primary school mathematics aimed at teaching the practical side of mathematics, and, for those who continued their education, secondary school mathematics aimed at the intellectual side. This rough picture of tectonic plates characterized many curricula at the beginning of the twentieth century, and in some countries it is still a reasonable portrayal.

As the figure might suggest, the middle grades — marking the transition between primary and secondary education — became one of the most contentious parts of the curriculum as societies struggled to organize school mathematics at those grades to allow a smooth transition from one set of goals to another. It is no accident that the middle grades were characterized in the mid-twentieth century as “the doldrums of school mathematics”, nor that many of the curriculum reform efforts that became known as the ‘new math’ began in the middle grades.

As part of those reform efforts, although it had begun earlier, the tectonic plates began to shift in many societies. Mathematics in the primary grades acquired some of the intellectual character of secondary mathematics, and, as more secondary students and teachers embraced practical reasons for the study of mathematics, it became a more practical subject in the secondary grades. The line separating the practical and intellectual justifications for studying mathematics began to cut across grades rather than to separate levels of education, as below:



Primary

Secondary

The shift in the line of demarcation became even greater in the last three decades of the twentieth century as technology made the study of applications of mathematics and mathematical modeling not only more feasible but also more desirable for those students who would go on to use mathematics in their careers to solve practical problems.

Despite its many oversimplifications, this analysis allows us to see the following:

1. For centuries, societies have had both practical and intellectual justifications for teaching mathematics, and these remain intertwined in today's school mathematics.
2. The twentieth century witnessed a broadening of the mathematics curriculum at all levels, which allowed practical and intellectual aims to cross more grades.
3. In the last three decades, more arguments have been put forward for practicality in school mathematics, and students now have more opportunities to use in realistic situations the mathematics they are learning.

What is at stake for the societies of tomorrow are the intellectual justifications for school mathematics: Can they be maintained in the face of what appears to be an overwhelming push for practicality? In 1904, Poincaré was bemoaning the extraordinary fact that so many people find mathematical definitions and proofs obscure and emphasizing the importance to mathematicians of making their subject comprehensible to engineers. In today's societies, politicians, parents, students, and many mathematics teachers clamor for mathematics that students can use to solve practical problems, whether they become engineers or not. The old claims for the intellectual and aesthetic value of learning mathematics are seldom heard today except among mathematicians and some mathematics teachers who continue to argue for the disciplinary value of the subject. If one sees 'pure mathematics' as the consequence of intellectual justifications and 'applied mathematics' as the consequence of practical justifications, it appears as though the pure side of school mathematics is gradually being eclipsed by the applied side. Will the future see a total eclipse or a reappearance from somewhere of justifications for developing the intellect ? Perhaps we will again one day hear the argument that there is nothing so practical as a good mathematical abstraction.

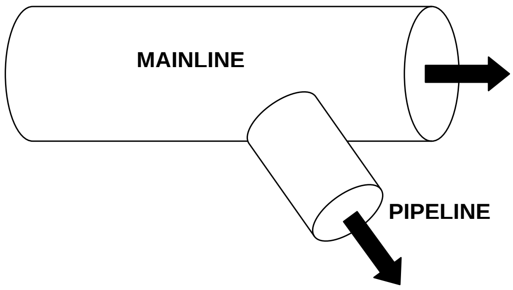
2. For whom is mathematics taught ?

The twentieth century saw the expansion of education at all levels in virtually every country of the world. In many respects, it might be called the century of the secondary school. Before the Second World War, "secondary education, especially at the upper level, and to an even greater extent higher education, was only open to limited numbers of the corresponding age-group, and usually only to boys" [UNESCO 2000, 14]. By the end of the century in many countries of the world, the opportunity for some form of secondary education had in effect been extended to all students. A recent report from UNESCO summarized the situation for developing countries :

> [In the past half-century,] school enrolment has doubled or even tripled in the developing countries, according to the level of education. By the middle of the 1950s, [the] groundswell had reached secondary education, where structures had been unified (creation of comprehensive middle-schools), arriving in the 1970s at post-secondary and higher education, the generalization of which — i.e. extension to over half of a generation — is a reality today in many countries. [UNESCO 2000, 15]

The previous figures showing justifications for the curriculum, with school mathematics sitting within a rectangle, clearly do not portray who takes mathematics. First, one should remember that even as access to secondary education has improved dramatically, there are still many countries of the world in which the number of students completing secondary school is well less than half of their age cohort. If we were to draw a picture of the fraction of the cohort in school at each grade, and therefore available to be taught school mathematics, we would have, for each country, a figure roughly approaching a trapezoid or triangle and not a rectangle.

Second, every society has to deal with built-in tensions that arise from the disparate goals of, first, educating all students to be numerate and, second, educating some students to study mathematics at advanced levels. Modern societies want essentially everyone to learn basic mathematics as part of the preparation they need for adult life. Not everyone, however, needs or wants to study mathematics at the advanced levels that society must demand of some if that society is to develop and maintain its scientific and technical prowess. Society needs what has been termed a *mainline* to provide numerate citizens, and it needs a *pipeline* to provide specialists in mathematics. Of course, this picture is highly oversimplified, since societies have many ways of setting up mainlines and pipelines for mathematics, and they have often developed different pipelines for students headed for different careers that require a knowledge of advanced mathematics.



Regardless of how society has structured its educational system, however, at some point in each pupil's education, a decision needs to be made as to whether, and if so when, the pupil will leave the school mathematics mainline to enter a school mathematics pipeline. When that decision is made, who makes it, and what the consequences will be for the pupil are questions that every society has had to face. One reason mathematics has come in for so much scrutiny over the past century has been its role in most societies as the filter or gatekeeper that determines who enters which pipelines since they lead not only to advanced study of mathematics but also to differentially desirable careers and social advantages.

In some cases, the decision point to enter a school mathematics pipeline comes at the transition from elementary to secondary education, and the kind of secondary school the pupil enters will determine whether he or she is in a pipeline. In other cases, the decision is made within a secondary school as the student is encouraged or not to pursue studies leading to advanced mathematics. And in still other cases, no decision is made until the student enters tertiary education and chooses or is chosen for a career demanding further study of mathematics. The general tendency during the twentieth century, as I see it, was to delay the decision point and sometimes to allow the decision to be reversed later if necessary. The decision has often been delayed more than in the past because of changes in what it means to be numerate. The sheer amount of mathematics that students need to know for numeracy today has been redefined upward in many countries.

Furthermore, the criteria used in making the decision have changed somewhat in many societies. Traditionally, performance on an examination in mathematics has been the main way to decide who stays in the mainline and who enters a pipeline. The quality of the examination has not always been high, the criterion for passing the examination has usually been set quite arbitrarily, and at times the connection between the examination and the school curriculum has been questionable. But schools have almost always found themselves needing to devote substantial time to preparing students for such examinations. In some educational systems during the twentieth century, the judgment of the mathematics teacher began to play a more important role, as did, in some cases, the desires of the pupil and his or her parents.

As we consider challenges for the societies of the future, we can expect to see access to the study of mathematics continuing to expand around the world. A reasonably safe prediction is that if the twentieth century was the century of secondary education, the coming century will be the century of tertiary education. Note that I say tertiary education and not university education or university-level education. In many societies, the university is being supplemented by a variety of postsecondary institutions that lead to careers for which university studies would be neither necessary nor suitable.

Universities as we know them today are only a couple of centuries old, although of course their roots go back several millennia. Change seems inevitable for universities over the next century, but the direction and magnitude of that change are far from clear. A number of commentators [Brown & Duguid 1996; Duderstadt 1997; Noam 1995; Reid 1998] view the twenty-first century university as likely to retain many if not all of its traditional functions but also to change in sometimes drastic ways under the pressure of electronic technology. In particular, Eli Noam, an economist, sees the economic foundation of universities collapsing because face-to-face teaching is becoming too expensive. In contrast, John Seely Brown and Paul Duguid see a hybrid university emerging, in which institutional arrangements are changed and the university devolves into components — a degree-granting body, the academic staff, the campus facilities, and the students — that are no longer tied tightly together. Brown and Duguid envision a middle way between the natural centralizing tendencies of the monolithic university and the vision advanced by so many futurists of a completely dispersed distance education mediated by technology.

In all of these scenarios, many more people enter universities and related institutions as tertiary education evolves into a more democratic and open "global knowledge industry" [Duderstadt 1997]. The challenge to school mathematics, regardless of which scenario comes about, will be to prepare increasing numbers of students to enter tertiary education equipped to continue learning not only mathematics but also the other subjects they will study. That challenge is nontrivial in view of the experience of the past century, which suggests that a rapid expansion of enrollments in advanced courses in school mathematics is almost inevitably accompanied by a perceived decline in the content and rigor of those courses[2]). Creating a smooth transition from secondary to tertiary education so that society's intellectual and practical goals for learning mathematics are kept in some sort of balance is part of the challenge.

L'Enseignement Mathématique was founded in the spirit of scientific solidarity that brought together mathematicians and teachers of mathematics in a mutual effort to improve the teaching and learning of mathematics. From the beginning, the editors saw both the intellectual and the practical demands that would be placed on the subject of mathematics. As they said in the editorial in the inaugural issue:

> L'avenir de la civilisation dépend en grande partie de la direction d'esprit que recevront les jeunes générations en matière scientifique; et dans cette éducation scientifique l'élément mathématique occupe une place prépondérante. Soit au point de vue de la science pure, soit à celui des applications, le XX^e siècle, qui va s'ouvrir, manifestera des exigences auxquelles personne ne doit ni ne peut se dérober.[3]) [Fehr & Laisant 1899, 5]

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[2]) For further discussion of the challenge of "ensuring mathematics for all", as well as other challenges facing mathematics educators in the United States and Canada, see [Kilpatrick & Silver 2000].

[3]) The future of civilization depends mainly on the direction that the younger generations will receive in understanding scientific subject matter, and in this scientific education the mathematical element occupies a dominant place. Whether from the point of view of pure science or from that of applications, the approaching twentieth century will impose requirements that must and cannot be concealed from anyone.

Their words are as pertinent for the twenty-first century as they were for the twentieth.

3. Tools for reflecting on how to teach

Scientific solidarity has been demonstrated in a different way in the invention of curriculum development projects in mathematics. Although they may seem to have been around forever, curriculum development projects are actually a phenomenon of the last half of the twentieth century. Before that time, schools had a mathematics curriculum, of course, but it had not been developed through a project. All through the twentieth century, groups were organized to look critically at school mathematics, study various curriculum problems, and make recommendations. The International Commission on Mathematical Teaching was a pioneer in that process, stimulating a series of national reports early in the century. Most of the groups working on curriculum, however, confined their activities to the production of reports and recommendations that were seldom accompanied by teaching materials or efforts to try out the recommendations [Howson, Keitel & Kilpatrick 1981, 68]. The idea that the curriculum could be studied and purposefully developed, rather than just being allowed to evolve, led ultimately to the first project in 1951 with the establishment of the University of Illinois Committee on School Mathematics at the dawn of the new math era. The curriculum development project, which was modeled after military and public health projects aimed at a specific result — a weapon, a vaccine, a therapy —, brought together mathematicians and teachers of school mathematics in a new kind of partnership and solidarity.

One of the great lessons taught by the new math era, however, was that curriculum development and solidarity within a project are not enough. Every teacher is involved in curriculum development at some level, and if ideas and materials developed by projects are to take root in classrooms, teachers need to be educated and supported in becoming more competent, autonomous developers and users of curricula [Howson, Keitel & Kilpatrick 1981, 260]. That is why I see, as an essential item on the agenda for improving the practice of mathematics teaching in the coming century, the creation of new forms of continuous professional development for teachers of mathematics.

It is far from clear what these forms might be or how they might be organized institutionally. The past half-century saw the establishment in many countries of centers and institutes where curriculum development could take place. In some cases, these centers were able to undertake, in addition to curriculum development, programs of research in mathematics education, but most of them — with the striking exception of the *Instituts de recherche sur l'enseignement des mathématiques* (IREMs) in France — did not have the resources, or even the mission, to do very much in the way of professional development. Furthermore, many of these centers have had a precarious existence. Governments have changed, bringing into office administrators who were not convinced that such centers warranted their continued support. Foundations have turned their funding elsewhere, figuring that they had launched an effort that needed to continue on its own. Officials of universities that housed a center have wanted it brought under their control or wanted it terminated as an unnecessary drain on their resources. I cannot tell you how many letters I have been asked to write and have written over the past several decades to administrators threatening drastic cuts of one sort or another in which I testified to the good work a center had done and the need for its continued existence. These centers have sometimes been able to find new sources of funding or have been able to reinvent themselves, but often they have continued on in a very diminished form, and sometimes they have essentially disappeared from view. Last January, I was present in Gothenburg for the inauguration of the Swedish National Center for Mathematics Education (NCM). I was delighted to see this enterprise underway — with careful planning, talented people, and what appeared to be a firm commitment from the national government for sustained support — but I also could not help worrying, in view of the history of similar ventures elsewhere, what the center might look like a decade from now.

Any enterprise that is set up to address the continuous professional development of mathematics teachers, in my view, cannot avoid also addressing matters of curriculum development and of research. That is an ambitious program for anyone to undertake, and it is not clear that the type of center structure available in the past will continue to work. In particular, pressures to move such centers out from under the umbrella of the university are likely to increase in many countries. Somehow, ways need to be found to insulate such enterprises from the winds of politics both academic and civic.

A useful activity for the International Commission on Mathematical Instruction (ICMI) or some other organization to take on would be to survey systematically the history and fate of institutes and centers for mathematics education around the world over the past three decades or so to see what might be learned from the various ways in which work has been organized, funding secured, and programs conducted. Such a survey might suggest how new institutional structures might be built to house the sort of programs I envision for integrating professional development with curriculum development and research.

In 1992, ICMI began a study entitled "What is research in mathematics education, and what are its results ?" The purpose, as expressed in the letter of invitation to conduct the study, was "to review the state of the field and to begin a dialogue with other scientific communities" [Sierpinska & Kilpatrick 1998, x]. Those communities included, specifically, the community of mathematicians. The ICMI Executive Committee felt that mathematicians did not know enough about what was happening in research in mathematics education and were questioning whether it had yielded any results at all worth considering. Anna Sierpinska and I edited the report of the study, which appeared in 1998 and was entitled *Mathematics Education as a Research Domain*. Because we wanted to convey the point that the study had not yielded a single, definitive answer to the questions it addressed, we made what turned out to be a tactical error and added the subtitle *A Search for Identity*. This resulted in reviewers making wisecracks about adolescents seeking identity and asking what other field would be questioning whether or not it is a research discipline. Mathematicians, of course, would never engage in such questioning.

Perhaps the questioning comes from the history of the field, and in particular, from the way mathematics education has developed internationally, as illustrated in the ICMI. One can also ask, what other field would have the officers of its premier international organization appointed by a group outside the field ? The ICMI is a commission of the International Mathematical Union (IMU) and is therefore subject to IMU's oversight. Could it be that the insecurity and apparent disarray of the field, despite its growth and accomplishments through the twentieth century, might stem in part from the way it has been treated by mathematicians ?

In 1984, in an address at the Fifth International Congress on Mathematical Education, I made the following modest proposal :

> Perhaps the field [of mathematics education] has reached a point in its development where it needs to set up a permanent executive — a secretariat that would facilitate communication among mathematics educators around the world. […] The International Commission might even wish to sponsor some sort of individual membership organization, possibly with a newsletter, so that interested persons might maintain contact with one another in the four years between congresses. [Kilpatrick 1985, 20]

I felt then, as I feel now, that the ICMI needed more autonomy as well as more involvement by its constituency. Since that time, I have served on the ICMI Executive Committee, and I have a better understanding of the reasons, not only historical but also economic, that have kept the ICMI from becoming or initiating a membership organization despite the growth of the field. Nonetheless, I believe that the ICMI needs to become an independent group of some sort.

I don't want to be misunderstood. I am certainly not suggesting that mathematics educators should sever all ties with mathematicians. Both groups have much to learn from each other. I have recently been very encouraged, despite the so-called math wars (cf. [Schubring 2003, 63]), to encounter mathematicians who are beginning to understand better than in the past and truly to appreciate the complexity and difficulty of the issues that mathematics educators are wrestling with in their research and their practice. For too long, mathematics education has been seen as a field open to anyone knowledgeable about mathematics in which having an interest and perhaps some experience in teaching is sufficient for entry. That attitude is, I think, beginning to change.

I used to think that it would be helpful to mathematics education if it could be seen as included among what are called 'the mathematical sciences'. I have changed my mind. The tactic is a little like the one used during the so-called 'back-to-basics' movement of the 1970s in North America : define problem solving as a basic skill and then embrace the slogan "back to basics". Such rhetorical tricks never work. Mathematics education is not a branch of mathematics, nor does it belong among the arts and sciences. It is a separate field with very different traditions, foundations, problems, methods, and results. It is much more contingent on history and culture than mathematics could ever be, and that is part of the reason for what outsiders perceive as a field in disarray.

A major challenge for the twenty-first century is for mathematicians and mathematics educators to modify their mutual relationship. Moreover, both need to modify their relationship with teachers. In a recent book, Ellen Lagemann [2000] documents the widening gap during the twentieth century between US education researchers and school-teachers. The condescension researchers in mathematics education have shown teachers of school mathematics has sometimes mirrored the condescension they have received themselves from mathematicians. All three groups need to learn to work more productively together with greater understanding of their differences and greater appreciation for the strengths the others are bringing to the enterprise. The relationships need to shift from outmoded paternalism to true fraternalism. Only then can we have scientific solidarity in learning and teaching mathematics as well as with mathematics itself.

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REACTION

The growing importance and challenges of mathematics education

by Hyman BASS

Both Jeremy Kilpatrick and Ubiratan D'Ambrosio offer us a grand historical perspective on mathematics education, reaching back into the 19th century. Perhaps appropriately, they dwell more on yesterday and today than on tomorrow.

D'Ambrosio emphasizes the social and cultural context of mathematics. While both authors treat the question "Why teach mathematics ?", they offer different kinds of analyses. Discussing the transition from the twentieth to the twenty-first centuries, D'Ambrosio writes :

> Anchored in advances in the cognitive sciences and the new possibilities of calculation and information retrieval, the Modern Mathematics movement brought new contents and new methods of mathematics education. However, the objectives of mathematics education were unclear. And they remain so. […] The question "Why teach mathematics ?" seems to be the crux. But together with this question come other questions about the nature of mathematics and how to handle mathematics teaching. Mathematics in the making ? Mathematics of everyday life ? Mathematics grounded in cultural traditions ? Mathematics as fun ? Good old classical mathematics ? [D'Ambrosio 2003, 310]

The Ethnomathematics Program, of which D'Ambrosio is a founder, addresses these questions in a socio-cultural and historical context.

> Throughout history, in every culture, we recognize the efforts to develop instruments : i) to communicate ; ii) to cope with reality ; iii) to understand and to explain reality, providing the tools of critical thinking ; iv) to define strategies for action. [D'Ambrosio 2003, 313]

He sees the role of mathematics education in different historical periods as emphasizing different subsets of these four characteristics.

> In the educational systems of the future I see mathematics, as well as ethnomathematics, inbuilt in the effort of critically providing communicative, analytic and material instruments. [D'Ambrosio 2003, 314]

> The teacher of the future will be a resource-companion of students in the search for new knowledge. The more mathematics a teacher knows, a better resource he/she will be; the more curiosity about the new, a better companion he/she will be. The characteristics of the new teacher cannot be only the result of special training. Teachers of old generations will retire and echo-boomers will become the new teachers. I trust they will be able to help build a better world. [D'Ambrosio 2003, 315]

Kilpatrick's perspective is more internal to the culture of mathematics as a discipline, and to the infrastructure of the mathematics research community and of the mathematics educational enterprise.

Broadly characterizing the genres of missions of mathematics education as either *practical* or *intellectual*, he suggests that, in the early twentieth century, practical aims dominated primary education, while intellectual aims prevailed at the secondary level. As the century progressed, both practical and intellectual aims emerged as important at both levels. Gradually, stimulated in part by technology, the practical assumed increasing dominance over the intellectual, a development that he notes with some lament. He further observes that, in the twentieth to twenty-first century transition, tertiary (i.e. post-secondary, but not necessarily university) level education would become more widespread and important.

Another motif concerns the bifurcation in mathematics education between general students (*mainline*), and those needing advanced mathematical study for mathematically intensive professions (*pipeline*). Kilpatrick notes that the bifurcation point, where one course of study or the other is decided, is evolving to a later point in time. Moreover, the exam-based criterion for these two pathways is being progressively broadened.

Kilpatrick offers some important observations and views concerning curriculum development, professional development of teachers, and education research. He calls attention to a remarkable historical fact:

> The idea that the curriculum could be studied and purposefully developed, rather than just being allowed to evolve, led ultimately to the first project in 1951 with the establishment of the University of Illinois Committee on School Mathematics at the dawn of the 'new math' era. The curriculum development project, which was modeled after military and public health projects aimed at a specific result — a weapon, a vaccine, a therapy —, brought together mathematicians and teachers of school mathematics in a new kind of partnership and solidarity. [Kilpatrick 2003, 326]

Most mathematics education reform interventions have been founded on the development of new curricular materials. However, their implementation has often proved disappointing, an outcome judged by many to have been a consequence of inadequate concern for the capacity-building that would have been needed to make these curricula usable and effective in school. Kilpatrick refers to [Howson *et al.* 1981]:

> Every teacher is involved in curriculum development at some level, and if ideas and materials developed by projects are to take root in classrooms, teachers need to be educated and supported in becoming more competent, autonomous developers and users of curricula. [Kilpatrick 2003, 326]

He draws a conclusion that has become now almost universally acknowledged:

> That is why I see, as an essential item on the agenda for improving the practice of mathematics teaching in the coming century, the creation of new forms of continuous professional development for teachers of mathematics. [Kilpatrick 2003, 326]

And he further argues that

> Any enterprise that is set up to address the continuous professional development of mathematics teachers, in my view, cannot avoid also addressing matters of curriculum development and of research. [Kilpatrick 2003, 327]

and that

> … ways need to be found to insulate such enterprises from the winds of politics, both academic and civic. [Kilpatrick 2003, 327]

As co-chair of an ICMI Study on Mathematics Education Research, Kilpatrick has reflected at length on the character and status of this field. With regard to its relation to mathematics, he insists that

> Mathematics education is not a branch of mathematics […] It is a separate field with very different traditions, foundations, problems, methods, and results. It is much more contingent on history and culture than mathematics could ever be, and that is part of the reason for what outsiders perceive as a field in disarray. [Kilpatrick 2003, 329]

For reasons such as these he has advocated a fundamental revision of what are seen to be the anachronistic and patriarchal organization of the governance and of ICMI within the International Mathematical Union (IMU), and its attendant lack of autonomy. Kilpatrick notes for example that the officers and EC of ICMI are elected by the IMU, without voting participation by ICMI. This is an issue of continuing concern and debate, to which I am now a direct witness. It seems likely that progress will be evolutionary, not revolutionary, and this is perhaps to be desired.

In a concluding comment, Kilpatrick notes that mathematicians and mathematics education researchers are joined by a third professional community of practice, school mathematics teachers, and that the whole triangle of mutual relationships is in need of improvement.

> The condescension researchers in mathematics education have shown teachers of school mathematics has sometimes mirrored the condescension they have received themselves from mathematicians. All three groups need to learn to work more productively together with greater understanding of their differences and greater appreciation for the strengths the others are bringing to the enterprise. [Kilpatrick 2003, 329]

Some afterthoughts

Both Ubi D'Ambrosio and Jeremy Kilpatrick have provided us with invaluable historical canvases of the sweep of educational issues and the institutional forms of addressing them over the past century, and more. I am most familiar with the setting of Kilpatrick's presentation, and I am in broad agreement with his analyses and recommendations.

In these brief remarks, let me bring into sharper focus a few points that I feel are now particularly pressing.

First of all, I cannot overemphasize the importance of *continuing professional development of teachers*. Developing frameworks and curriculum materials is crucial and challenging work, but it is far easier than building human capacity in a profession of human improvement like teaching. Any stable educational improvement at systemic scale depends in large measure on this. This requires growth in our knowledge base (research), and implementation takes time and sustained support from public institutions. Other countries, such as Japan, provide examples of a professional culture of teaching that supports continuous professional development.

In every aspect of educational improvement, it must (tacitly if not explicitly) be understood that every recommendation must pay a tax to the need for concomitant professional development. This need is now widely acknowledged and appreciated. It is the subject of a recently launched ICMI Study, co-chaired by Deborah Ball and Ruhama Even. This applies in particular to the matters discussed below.

Finally, let me open a bit wider the discussion of "Why study mathematics?" As D'Ambrosio emphasizes, the answers to this change with history and culture, and they are now conditioned also by the presence of powerful technology. Our unquestioned premise of the central place of mathematics in the school curriculum (often 12 years of instruction) is now under widespread assault (in Europe, in Japan, and in some quarters in almost every country). This is producing policies that can have profound impact on the cultural, intellectual, and material standing of mathematics, and its professional communities, in the educational enterprise.

The traditional rationale for mathematics study has been (to give it a finer grained classification than Kilpatrick's *practical* and *intellectual*) a mixture of pragmatic, economic, social, intellectual, and cultural reasons. *Pragmatic* reasons include the need to learn the basic skills of arithmetic and of measurement, and the rudimentary geometric concepts and figures. *Economic* arguments are based on the quantitative literacy demanded by the rapidly evolving technological workplace, and the desire to remain competitive in the world economy. Mathematics can be justified for *social* reasons, because it provides the resources for responsible citizenship in a modern industrial democracy. The *intellectual* justification is that mathematics is the enabling discipline for all of science, and that it offers fundamental tools of analysis, quantitative expression, and disciplined reasoning. Finally, its *cultural* warrant is that mathematics exposes students to some of the most subtle and sublime achievements of the human spirit.

Yet people can still question whether these arguments suffice to justify so many years of mathematical study. Who today will be doing significant arithmetic without a calculator ? How many people will ever have occasion as adults to solve a quadratic equation ? Even engineers have no need to *prove* that their mathematical methods work. Why do our cultural arguments for the study of mathematics merit such different responses than those advocating the study of ancient languages and history ?

Most of my colleagues, mathematicians and teachers, have strong personal convictions about the importance of a solid mathematics education for all citizens, but these are increasingly seen to be expressions of belief systems rather than persuasive, empirically-based, arguments for public policy.

So the first step in convincing policy makers is to convince ourselves. The answers must be framed in new ways, and grounded in new conceptions of what a contemporary mathematics curriculum, and learning goals for all students, must look like.

For example, one argument is that mathematics study cultivates analytical reasoning skills that are a powerful general intellectual resource. To this one might first observe that the teaching of mathematical reasoning is an endangered species in much of current mathematical instruction, with widely observed disabling effects. Secondly, the implicit argument for knowledge-transfer is vividly challenged within the very domain of concern to us here —

mathematics education. Highly competent mathematicians have been frequently guilty of making strong general claims and judgments about mathematics education that are empirical in nature, yet based on limited and often anecdotal evidence. They exhibit a kind of cavalier undisciplined reasoning in mathematics education that they would never countenance in mathematics itself.

Questions about the place and nature of mathematics education comprise a web of complex and subtle relationships, but they are crucial to the evolution of mathematics education, and so they demand our serious attention and reflection. Here is not the place to enter into the important issues that these questions bring to the fore. Some aspects of this set of questions are being engaged by a project on quantitative literacy, led in part by Lynn Steen, who is chief author of a discussion document, *Mathematics and Democracy : The Case for Quantitative Literacy*, published in 2001 by the Mathematical Association of America. There are few contemporary questions about mathematics education more important than those concerned with the kind, scope, and nature of the mathematics needed by all responsible citizens in a modern technological democracy, and how such education can be realized.

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