Robin Hudson's Pathless Path to Quantum Stochastic Calculus

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Abstract

Robin Hudson's work on quantum central limit theorems, quantum Brownian motion, quantum stopping times and "formal" quantum stochastic calculus is reviewed and reappraised.

1 Introduction

Quantum stochastic calculus was created by Robin Hudson and K.R. Parthasarathy. The key paper which contained almost all the basic results was published in 1984 in "Communications in Mathematical Physics" [19]. Here we find, in particular, the construction of quantum stochastic integrals, quantum Itôs formula, the existence and uniqueness of linear quantum stochastic differential equations (QSDEs), necessary and sufficient conditions for unitarity of solutions and the dilation of quantum dynamical semigroups (at least for one degree of freedom). In this article I will focus on the "prehistory" of quantum stochastic calculus with a particular emphasis on Robin Hudson's contribution. This will cover work that was published in the period 1971-84. As can be seen by a quick journey to MathSciNet, this was a highly productive period for Robin. I am not going to survey all his papers from this period in this article or even all of those that he wrote in the area of quantum probability. What I will present is four key milestones - the quantum central limit theorem, quantum Brownian motion, quantum stopping times and the heuristic version of quantum stochastic calculus that preceded its rigorous development in [19].

A brief comment on the title. In the ancient Chinese philosophy of Taoism, the mysterious tao is often described (at least in contemporary English translation) as a "pathless path". Of course in quantum theory, a particle does not have a "path" in the usual sense and consequently path space techniques are inappropriate tools for studying quantum processes.

Notation. If $\mathcal{H}$ is a complex Hilbert space then $B(\mathcal{H})$ is the algebra of all bounded linear operators on $\mathcal{H}$. If T is a densely defined closeable linear operator defined on $\mathcal{H}$ with adjoint $T^\dagger$, then any proposition involving $T^\#$ should be read twice, once for T and once for $T^\dagger$.

2 Quantum Central Limit Theorem

Robin's first paper on quantum probability was joint work with his PhD student Clive Cushen [9]. The opening words of the introduction to this paper are almost a clarion call for quantum probability: "In recent years there has been an increasing awareness that the foundations of quantum mechanics lie in a non-commutative analogue of axiomatic probability theory." In order to formulate a quantum central limit theorem (CLT), Cushen and Hudson first needed to decide what should a "quantum random variable" be and how could a sequence of these be "identically distributed and independent"? The basic (bosonic) quantum random variable is a canonical pair (q, p) of linear self-adjoint operators acting in a complex Hilbert space $\mathcal{H}$ and satisfying the Heisenberg commutation relation

$$[p, q] := pq - qp = iI,$$

on a suitable dense domain. The distribution of the pair (q, p) is determined by a mixed state which is identified with its density operator ρ. Now consider an infinite sequence $((q_n, p_n), n \in \mathbb{N})$ of such canonical pairs satisfying

$$[q_i, q_j] = [p_i, p_j] = 0; [p_i, q_j] = i\delta_{ij}I,$$

for each $i, j \in \mathbb{N}$ and equipped with the state ρ. They are said to be *independent* if all finite subsets are independent in the sense that if $A \subset \mathbb{N}$ is finite with $A = \{i_1, \ldots, i_N\}$ then ρ_A is unitarily equivalent to $\rho_{i_1} \otimes \cdots \otimes \rho_{i_N}$. Here ρ_A is the reduced density operator defined on $L^2(\mathbb{R}^N)$ by

$$\mathrm{tr}(\rho_A T) = \mathrm{tr}(\rho\iota(T)),$$

for all $T \in B(L^2(\mathbb{R}^N))$ and ι is the canonical embedding of $B(L^2(\mathbb{R}^N))$ into $B(\mathcal{H})$ that is determined by von Neumann's uniqueness theorem. The reduced states $\rho_{i_1}, \ldots, \rho_{i_N}$ are similarly obtained by taking $A = \{i_1\}, \ldots, \{i_N\}$ (respectively). The sequence $((q_n, p_n), n \in \mathbb{N})$ comprises *identically distributed* quantum random variables if $\rho_{i_1} = \cdots = \rho_{i_N}$ for every finite set $A \subset \mathbb{N}$. Now we average. For each $n \in \mathbb{N}$, define

$$\overline{p}_n = \frac{1}{n}(p_1 + \cdots + p_n), \overline{q}_n = \frac{1}{n}(q_1 + \cdots + q_n).$$

It is easy to see that $(\overline{q}_n, \overline{p}_n)$ form a canonical pair and the main result of the paper is to prove the quantum central limit theorem:

$$\lim_{n \to \infty} \operatorname{tr}(\overline{\rho}_n T) = \operatorname{tr}(\rho_\sigma T), \tag{2.1}$$

for all $T \in B(L^2(\mathbb{R}))$. Here $\overline{\rho}_n$ is the reduced density operator corresponding to the canonical pair $(\overline{q}_n, \overline{p}_n)$ and ρ_σ is a quantum Gaussian state on $L^2(\mathbb{R})$, i.e. a thermal state of the quantum harmonic oscillator having variance $\sigma \geq 1$ (see Example 2 in [3] for insight into the sense in which this state is "Gaussian" and [28] for an expository account of quantum Gaussian states.)

A key ingredient in the proof is the use of what are here called *quasi-characteristic functions* which are defined for $x, y \in \mathbb{R}$ by

$$f_{p,q}(x, y) = \operatorname{tr}(\rho U_{x,y}),$$

where $U(x, y) = e^{i(xp+yq)}$ is the Weyl operator. Indeed the authors establish a Glivenko-type convergence theorem to the effect that for (q_n, p_n) to converge "in distribution" (i.e. in the sense of 2.1) it is sufficient for the associated sequence of quasi-characteristic functions to converge pointwise to a function on $\mathbb{R}^2$ that is continuous at the origin.

This paper was followed by the fermionic version [13]. In this case, the appropriate analogue of the canonical pairs are fermionic annihilation and creation operators $(a_n, a_n^\dagger), n \in \mathbb{N})$ which satisfy the canonical anticommutation relations (CARs):

$$\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0; \{a_i, a_j^\dagger\} = \delta_{ij} I,$$

for $i, j \in \mathbb{N}$, where $\{A, B\} := AB + BA$ is the "anticommutator". Once again we get a new representation of the CARs by averaging:

$$\overline{a}_n^{\#} := \frac{1}{n}(a_1^{\#} + \cdots + a_n^{\#}),$$

and the fermionic central limit theorem yields convergence of corresponding reduced states to a fermionic quasi-free state (i.e. a fermionic Gaussian).

In this work, the key tool was cumulants rather than quasi-characteristic functions. The development of the appropriate fermionic version of the latter required the use of Grassman algebra techniques [2].

The Cushen-Hudson work is without doubt a landmark paper, and not just for its influence on quantum probability. The study of quantum central limit theorems is a flourishing enterprise in its own right which has attracted the attention of a large number of authors since the early 1970s, see e.g. [11], [10], [12] and [21]. Indeed the recent article [21] lists seven distinct areas in which quantum central limit theorems have been developed and applied including quantum information theory, graph theory and combinatorics. K.R.Parthasarathy told the author[1] that "reading this paper for the first time deeply influenced his subsequent mathematical life" (sic.)

3 Quantum Brownian Motion

The first paper to study quantum Brownian motion was published by Robin together with another of his PhD students[2] Anne Cockcroft in 1977 [7]. In this paper, the passage of time is modelled by the closed interval $[0, 1]$, but the authors could just as easily used $\mathbb{R}^+ := [0, \infty)$ and this became the standard choice in later work. A *quantum Brownian motion*[3] is a pair $(P(t), Q(t), t \in [0, 1])$ of self-adjoint operator-valued functions acting in a complex Hilbert space $\mathcal{H}$ together with a distinguished vector ψ to determine expectations such that:

(i) $[P(s), P(t)] = [Q(s), Q(t)] = 0; [P(s), Q(t)] = is \wedge t$
for all $s, t \in [0, 1]$.

(ii) $P(0) = Q(0) = 0$.

(iii) For $\Delta := (a, b] \subset [0, 1]$, define the canonical pair (p_Δ, q_Δ) by
$$p_\Delta := \frac{p(b) - p(a)}{\sqrt{b - a}}, q_\Delta := \frac{q(b) - q(a)}{\sqrt{b - a}}.$$
For arbitrary pairwise disjoint $(\Delta_n, n \in \mathbb{N})$, the sequence $((p_{\Delta_n}), q_{\Delta_n})), n \in \mathbb{N})$ consists of independent and identically distributed (i.i.d.) canonical pairs having quantum Gaussian distributions with mean 0 and variance $\sigma^2 \geq 1$ in the state determined by ψ.

[1] E-mail communication in April 2010

[2] Robin has always been extremely generous in sharing his ideas with others and many PhD students, including the author, have been beneficiaries of this largesse.

[3] In [7] the authors used the terminology "quantum Wiener process", but "quantum Brownian motion" became the preferred usage amongst practitioners.

Note that each of $(Q(t), t \geq 0)$ and $(P(t), t \geq 0)$ are separately equivalent to the probabilist's Brownian motion but the non-trivial commutation relation in (i) ensures that these are not simultaneously diagonisable. The key probabilistic input of the definition is in (iii) which extends to a non-commutative framework the fact that a "classical" Brownian motion $(B(t), t \geq 0)$ has stationary and independent increments with each random variable $\frac{B(t)-B(s)}{\sqrt{t-s}} \sim N(0, \sigma^2)$.

The main result of this paper is to show that any quantum Brownian motion is unitarily equivalent to the pair of "co-ordinate" and "momentum" field operators which are indexed by the indicator function $1_{[0,t)}$ and are associated to a cyclic representation of the extremal universally invariant quasi-free state ω_σ on the Weyl CCR algebra over $\mathcal{H}$ defined on Weyl operators $W(f)$ by

$$\omega_\sigma(W(f)) = e^{-\frac{1}{2}\sigma^2 ||f||^2}$$

for each $f \in \mathcal{H}$. The case $\sigma = 1$ is the Fock state.

In later years, the development of quantum stochastic calculus, made it more convenient to identify quantum Brownian motion with the annihilation/creation operator valued process $(A(t), A^\dagger(t), t \geq 0)$ defined by

$$A(t) = \frac{1}{\sqrt{2}}(Q(t) + iP(t)), A^\dagger(t) = \frac{1}{\sqrt{2}}(Q(t) - iP(t)),$$

which satisfy the commutation relations (CCRs):

$$[A(s), A(t) == [A(s)^\dagger, A(t)^\dagger] = 0, [A(s), A^\dagger(t)] = s \wedge tI,$$

for all $s, t \geq 0$. The Cockroft-Hudson theory then tells us that every quantum Brownian motion for which $\sigma = 1$, is unitarily equivalent to $(a(1_{[0,t)}), a^\dagger(1_{[0,t)}), t \geq 0)$ where $a(\cdot), a^\dagger(\cdot)$ are the Fock annihilation and creation operators acting in boson Fock space $\Gamma(L^2(\mathbb{R}^+))$ with distinguished vector the Fock vacuum Ω. When $\sigma > 1$, quantum Brownian motion is often said to be "non-Fock". In this case, the reference Hilbert space is $\Gamma(L^2(\mathbb{R}^+)) \otimes \Gamma(\overline{L^2(\mathbb{R}^+)})$ and we have

$$A(t) = \lambda a(1_{[0,t)}) \otimes I + \mu I \otimes a^\dagger(1_{[0,t)}), A^\dagger(t) = \lambda a^\dagger(1_{[0,t)}) \otimes I + \mu I \otimes a(1_{[0,t)}),$$

where $\lambda^2 - \mu^2 = 1, \lambda^2 + \mu^2 = \sigma^2$. The distinguished vector is $\Omega \otimes \overline{\Omega}$. The non-Fock quantum Brownian motions have a deep and beautiful mathematical structure (see e.g. [16]) and it may well be that they haven't yet been exploited to their full potential.

The Cockcroft-Hudson paper is justly celebrated as marking the birth of quantum Brownian motion. Perhaps less well-known is the follow-up paper

[8] by the same authors (but now augmented with S.Gudder) which directly followed it in the same volume of the "Journal of Multivariate Analysis". Here the authors establish a functional central limit theorem for quantum Brownian motion. The set-up is as follows. Suppose that we have a sequence of canonical pairs $((p_n, q_n), n \in \mathbb{N})$ that are i.i.d. with respect to a state ω for which

$$\omega(q_n) = \omega(p_n) = \omega(\{p_n, q_n\}) = 0; \omega(q_n^2) = \omega(p_n^2) = \sigma^2,$$

for all $n \in \mathbb{N}$, where $\sigma^2 \geq 1$. For each $t \in [0, 1]$ consider the operators defined by

$$P_n(t) = \frac{1}{\sqrt{n}}(p_1 + \cdots + p_{[nt]} + (nt - [nt])p_{[nt]+1}),$$

$$Q_n(t) = \frac{1}{\sqrt{n}}(q_1 + \cdots + q_{[nt]} + (nt - [nt])q_{[nt]+1}).$$

It follows that for each $s, t \in [0, 1]$

$$[P_n(s), P_n(t)] = [Q_n(s), Q_n(t)] = 0, [P_n(s), Q_n(t)] = i(s \wedge t + r_n),$$

where $\lim_{n \to \infty} r_n = 0$. The authors demonstrate that the sequence $((P_n, Q_n), n \in \mathbb{N})$ converges "weakly" to quantum Brownian motion of variance σ^2. A large part of the paper grapples with the question of what weak convergence might mean in this context. The authors build an elaborate technical apparatus which inter alia requires the "compact uniform closure" of a C*-algebra with respect to a sequence of states. Readers who want to know more about this are referred to the original paper.

The importance of the Cockroft-Hudson paper [7] lies firstly in its identification of what would become two of the key fundamental noise processes of quantum stochastic calculus and secondly in providing a model for latter versions of non-commutative Brownian motions appearing in different contexts such as the fermionic [2], free [32], "twisted" [6] and monotone [25]. There have not been many developments in the literature on quantum probabilistic properties of quantum Brownian motion outside its use as noise in quantum stochastic calculus, although a recent paper by the author [3] has established a Lévy-Cielsielski type series expansion in terms of a Schauder system. On the other hand, classical Brownian motion remains a topic of intense study for classical probabilists as it continues to yield deep and fascinating secrets (see e.g. [26] for an account of recent progress.) Is quantum probability missing an opportunity here?

4 Quantum Stop Times

Stopping times play a very important role in classical probability, probabilistic potential theory and many applications (e.g. consider the problem of pricing an American option in mathematical finance.) In the classic book by Chris Rogers and David Williams there is a wonderful quote from Sid Port that I can't resist including here (see [30], p.9 [4]): "The one thing probabilists can do which analysts can't is stop - and they can never forgive us for it."

In [14] Robin introduced the concept of a quantum stopping time and proved the strong Markov property for quantum Brownian motion. Before going on to describe this it may be worth recalling the classical result. Let $(B(t), t \geq 0)$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$ and adapted to a filtration $(\mathcal{F}_t, t \geq 0)$. Let T be a stopping time, i.e. a random variable defined on Ω that takes values in $[0, \infty]$ such that the event $(T \leq t) \in \mathcal{F}_t$ for all $0 \leq t < \infty$. The strong Markov property asserts that $(B(T + t) - B(t), t \geq 0)$ is a Brownian motion adapted to the filtration $(\mathcal{F}_{T+t}, t \geq 0)$ and independent of $\mathcal{F}_T := \{A \in \mathcal{F}, A \cap (T \leq t) \in \mathcal{F}_t \text{ for all } t \geq 0\}$.

In [14] Robin works with the quantum Brownian motion $(P(t), Q(t)), t \geq 0)$ of variance $\sigma^2 \geq 1$ with distinguished state vector ψ. The role of the σ-algebra is played by the von Neumann algebra $\mathcal{N} := \{P(t), Q(t), t \geq 0\}''$, i.e. the smallest von Neumann algebra containing all the spectral projections of the Ps and Qs and a filtration in this context is the family of increasing sub-algebras $(\mathcal{N}_\lambda, \lambda \geq 0)$ where $\mathcal{N}_\lambda := \{P(t), Q(t), 0 \leq t \leq \lambda\}''$. A stopping time T is then a positive self-adjoint operator having spectral decomposition $T = \int_0^\infty \lambda dE(\lambda)$ which is such that $E(\lambda) \in \mathcal{N}_\lambda$ for all $\lambda \geq 0$.[5] At least formally the random time-shifted quantum Brownian motion should be

$$P_T(t) := \int_0^\infty (P(t + \lambda) - P(\lambda))dE(\lambda), Q_T(t) := \int_0^\infty (Q(t + \lambda) - Q(\lambda))dE(\lambda)$$

for each $t \geq 0$. In order to give these formal expressions a rigorous meaning, Robin defines them indirectly as infinitesimal generators of the unitary operator-valued spectral integrals defined for each $x \in \mathbb{R}$ by

$$U_{P(t)}(x) := \int_0^\infty e^{ixP(t+\lambda)} e^{-ixP(\lambda)} dE(\lambda), V_{Q(t)}(x) := \int_0^\infty e^{ixQ(t+\lambda)} e^{-ixQ(\lambda)} dE(\lambda), \tag{4.2}$$

[4] The page reference is to the Cambridge University Press edition.

[5] Later literature on quantum stop times usually defined these directly in terms of projection-valued measures - see e.g. [29].

so $(U_{P(t)}(x), x \in \mathbb{R})$ and $(V_{Q(t)}(x), x \in \mathbb{R})$ are each strongly continuous one-parameter unitary groups and we have

$$U_{P(t)}(x) = e^{ixP(t)}, V_{Q(t)}(x) = e^{ixQ(t)},$$

for each $t \geq 0, x \in \mathbb{R}$. Much of the technical work in the paper involves the construction of integrals of the type considered in (4.2) as limits of Riemann sums in the strong operator topology.

Before we can state the quantum strong Markov property, we need the concepts of pre-T and post-T von Neumann algebras which we'll denote as $\mathcal{N}_{T]}$ and $\mathcal{N}_{[T}$ respectively. These are defined by

$$\mathcal{N}_{T]} := \{A \in \mathcal{N}, AE(\lambda) = E(\lambda)A \in \mathcal{N}_\lambda \text{ for all } \lambda \geq 0\},$$

$$\mathcal{N}_{[T} := \{P_T(t), Q_T(t), t \geq 0\}''.$$

The strong Markov property states that $\mathcal{N}_{T]}$ and $\mathcal{N}_{[T}$ are independent in the state ψ [6] and that $((P_T(t), Q_T(t)), t \geq 0)$ is a quantum Brownian motion of variance σ^2.

A corresponding strong Markov property was established by the author for fermion Brownian motion in [2]. A key later development of quantum stopping times was the paper [29] by Parthasarathy and Sinha in which the tensorial factorisation of Fock space over $L^2(\mathbb{R}^+)$ corresponding to the splitting $f \to f1_{[0,t)} + f1_{[t,\infty)}$ was extended to the case where t is replaced by a quantum stopping time. Another paper worth mentioning (which sadly has never been followed up in the literature) is a very interesting study of first exit times by J.-L.Sauvageot [31]. Although there has continued to be sporadic work on quantum stopping times (see e.g. [15] for a recent survey article by Robin) it seems that a breakthrough is still needed to forge it into a tool that is of similar value in quantum probability to its commutative counter-part.

5 Formal Quantum Stochastic Calculus

In the last part of this survey I will focus on work carried out during the early 1980s. A great deal of the standard conceptual structure of quantum stochastic calculus was developed by Robin and co-workers (principally K.R.Parthasarathy and R.F.Streater) from a heuristic viewpoint. The rigorous development came a lot later. In many ways these formal calculations (which are quite satisfactory to most physicists) constitute the essence of the subject.

[6] i.e. $\langle\psi, AB\psi\rangle = \langle\psi, A\psi\rangle\langle\psi, B\psi\rangle$ for all $A \in \mathcal{N}_{T]}, B \in \mathcal{N}_{[T}$.

At this time, the basic quantum processes were understood to be the annihilation/creation pair $(A(t), A^\dagger(t), t \geq 0)$ (where $A^\#(t) = a^\#(1_{[0,t)})$ acting in boson Fock space $\Gamma(L^2(\mathbb{R}^+))$ and equipped with the vacuum vector to determine expectations. The filtration was induced by the canonical isomorphism between $\Gamma(L^2(\mathbb{R}^+))$ and $\Gamma(L^2([0,t))) \otimes \Gamma(L^2([t,\infty))$ which maps each exponential vector $e(f)$ to $e(f1_{[0,t)}) \otimes e(f1_{[t,\infty)})$. In order to define formal quantum stochastic integrals we write "$dA^\#(t) := a^\#(1_{[t,t+dt)})$". Everything follows from the eigenrelation:

$$A(t)e(f) = \left(\int_0^t f(s)ds\right) e(f),$$

for each $f \in L^2(\mathbb{R}^+)$. Taking a deep breath, we then find that formal differentiation yields:

$$dA(t)e(f) = f(t)e(f)dt, \tag{5.3}$$

and so for suitable operator-valued processes $(F(t), t \geq 0)$ we can define the quantum stochastic annihilation integral $\int_0^t F(s)dA(s)$ by its action on exponential vectors:

$$\left(\int_0^t F(s)dA(s)\right) e(f) = \left(\int_0^t F(s)f(s)ds\right) e(f).$$

The creation integral is obtained by formal adjunction:

$$\begin{aligned}\left\langle e(f), \left(\int_0^t G(s)dA^\dagger(s)\right) e(g)\right\rangle &= \left\langle \left(\int_0^t G^\dagger(s)dA(s)\right) e(f), e(g)\right\rangle \\ &= \int_0^t \overline{f(s)}\langle e(f), G(s)e(g)\rangle,\end{aligned}$$

for each $f, g \in L^2(\mathbb{R}^+)$. The celebrated quantum Itô formula is summarised in the following table:

	$dA^\dagger(t)$	$dA(t)$	dt
$dA(t)$	dt	0	0
$dA^\dagger(t)$	0	0	0
dt	0	0	0

These formal relations are suggested by the following type of calculation using the CCRs and (5.3):

$$\begin{aligned}\langle e(f), dA(t)dA^\dagger(t)e(g) &= dt\langle e(f), e(g)\rangle + \langle e(f), dA^\dagger(t)dA(t)e(g)\rangle \\ &= dt\langle e(f), e(g)\rangle + \langle dA(t)e(f), dA(t)e(g)\rangle \\ &= (dt + o(dt))\langle e(f), e(g)\rangle\end{aligned}$$

The preceeding heuristic calculations were all given a precise rigorous meaning in the seminal paper [19].

The non-trivial Itô correction term $dA(t)dA^\dagger(t) = dt$ will only contribute to formal differentiation of terms that violate Wick ordering. This insight was the basis of a short note by Hudson and Streater [20] whose title says it all - "Itô's formula is the chain rule with Wick ordering." They consider processes that take the Wick-ordered form

$$M(t) := \sum_j c_j f_j(A(t)^\dagger, t) g_j(A(t), t),$$

where each $c_j \in \mathbb{C}$ and f_j and g_j are smooth. Define formal partial derivatives by

$$\frac{\partial M(t)}{\partial A(t)^\dagger} := \sum_j c_j \partial_1 f_j(A(t)^\dagger, t) g_j(A(t), t), \quad \frac{\partial M(t)}{\partial A(t)} := \sum_j c_j f_j(A(t)^\dagger, t) \partial_1 g_j(A(t), t)$$

$$\text{and} \quad \frac{\partial M(t)}{\partial t} := \sum_j c_j [\partial_2 f_j(A(t)^\dagger, t) g_j(A(t), t) + f_j(A(t)^\dagger, t) \partial_2 g_j(A(t), t)],$$

where for $i = 1, 2$, ∂_i denotes partial differentiation with respect to the ith variable. The authors then show that

$$dM(t) = \frac{\partial M(t)}{\partial A(t)^\dagger} dA^\dagger(t) + \frac{\partial M(t)}{\partial A(t)} dA(t) + \frac{\partial M(t)}{\partial t} dt.$$

In the two papers [17] and [18], Hudson and Parthasarathy investigate *quantum diffusions*. These are prototypes for the quantum stochastic processes (in the sense of [1]) that eventually became know as *quantum stochastic flows* or *Evans-Hudson flows* (see e.g. [24] and [27]). The authors work in the space $\mathfrak{h} := \mathfrak{h}_0 \otimes \Gamma(L^2(\mathbb{R}^+))$ where $\mathfrak{h}_0$ is a complex, separable Hilbert space which carries a representation of the CCRs. So we have a pair $(a, a^\dagger)$ of mutually adjoint linear operators acting in $\mathfrak{h}_0$ and satisfying $[a, a^\dagger] = 1$. This bosonic system is then perturbed by quantum noise under the constraint that the commutation relation is preserved in time. So we obtain mutually adjoint processes $(a_t^\#, t \geq 0)$ that satisfy $[a_t, a_t^\dagger] = 1$ for all $t \geq 0$. These are required to be *adapted* to the Fock filtration in that each $a_t^\# = a_1^\#(t) \otimes I$ where $a_1^\#(t)$ operates non-trivially on $\mathfrak{h}_0 \otimes \Gamma(L^2([0, t)))$. The form of the perturbation is given by

$$da_t = F(t)dA(t) + G(t)dA^\dagger(t) + H(t)dt,$$

and applying quantum Itô's formula to the CCRs yields the *restraint equations* (which are to be read pointwise in t):

$$[F, a^\dagger] = [a, G^\dagger] = 0,$$
$$[H, a^\dagger] + [a, H^\dagger] = F^\dagger F - GG^\dagger.$$

Furthermore they obtain formal conditions for the dynamics to be induced by a unitary operator-valued process $(U(t), t \geq 0)$. Indeed the unitarity requirement implies the form:

$$dU(t) = U(t)\left(LdA^\dagger(t) - L^\dagger dA(t) + \left(i\mathcal{H} - \frac{1}{2}L^\dagger L\right)dt\right),$$

with $U(0) = I$, where L and $\mathcal{H}$ are (ampliations of) linear operators acting on $\mathfrak{h}_0$ with $\mathcal{H}$ being formally self-adjoint. In order to obtain

$$a^\#(t) = U(t)(a^\# \otimes I)U(t)^\dagger$$

for all $t \geq 0$, it is shown that we must have

$$F = [L, a], G = [a, L^\dagger], H = i[\mathcal{H}, a] - \frac{1}{2}(L^\dagger La - 2L^\dagger aL + aL^\dagger L),$$

so $H = \mathcal{L}(a)$ where $\mathcal{L}$ is the Lindblad generator. Indeed it is precisely the generator of the quantum dynamical semigroup $(T_t, t \geq 0)$ defined by

$$T_t(X) = \mathbb{E}_0(U(t)(X \otimes I)U(t)^\dagger),$$

for each $X \in B(\mathfrak{h}_0)$, where $\mathbb{E}_0$ denotes the vacuum conditional expectation. A fermionic version of some of these ideas was developed in [4].

As was pointed out above, these ideas were made fully rigorous in [19] which also introduced the conservation process[7] into quantum stochastic calculus and thus completed the trio of basic integrators. The theory developed therein has been described and extended in a number of monographs and surveys (see e.g. [27], [24], [5], [23]) so the reader will surely forgive me if I stop at this point.

In conclusion, the period 1971-1984 saw a remarkable period of activity from Robin and his collaborators which led from the quantum central limit theorem to quantum Brownian motion and then to the development of quantum stochastic calculus. It is perhaps a little unfair to compare this to the gap between the first use of the central limit theorem by Abraham de Moivre in 1733 and the discovery of stochastic calculus by Kiyosi Itô in the 1940s (see [22] for a concise historical account of developments leading to the birth of the latter), nonetheless it is certainly a considerable achievement.

[7] Sometimes called the “gauge” or “number” process

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