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Options on Normal Underlyings with an Application to the Pricing of Survivor Swaptions

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Survivor derivatives are gaining considerable attention in both the academic and practitioner communities. Early trading in such products has generally been confined to products with linear payoffs, both funded (bonds) and unfunded (swaps). History suggests that successful linear payoff derivatives are frequently followed by the development of option-based products. The random variable in the survivor swap pricing methodology developed by Dowd *et al* [2006] is (approximately) normally, rather than lognormally, distributed and thus a survivor swaption calls for an option pricing model in which the former distribution is incorporated. We derive such a model here, together with the Greeks and present a discussion of its application to the pricing of survivor swaptions. © 2009 Wiley Periodicals, Inc. Jrl Fut Mark 29:1–18, 2009

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INTRODUCTION

A surge of attention in derivative products designed to manage survivor (or longevity) risk can be observed in both the academic and the practitioner communities. Examples from the former include Bauer and Russ (2006), Bauer and Kramer (2008), Blake, Cairns, and Dowd (2006), Blake, Cairns, Dowd, and MacMinn (2006), Cairns, Blake, and Dowd (2008), Dahl and Møller (2005), Sherris and Wills (2007), and Wills and Sherris (2008). In the practitioner community, JPMorgan (2007) has set up LifeMetrics, one of whose stated purposes is to "Build a liquid market for longevity derivatives." A number of banks and reinsurers, including JPMorgan, Deutsche Bank, Goldman Sachs, Société Générale, and SwissRe, have set up teams to trade longevity risk, and in April 2007, the world's first publicly announced longevity swap took place between SwissRe and the U.K. annuity provider Friends Provident. In January 2008, the world's first mortality forward contract—a q-forward contract—took place between JPMorgan and the U.K. insurer Lucida. For more details of these and other developments, see Blake, Cairns, and Dowd (2008).

Early products generally exist in linear payoff format, either funded (bonds) or unfunded (swaps). History suggests that when linear payoff derivatives succeed, there is generally a requirement for option-based products to develop as well. This study derives a risk-neutral approach to pricing and hedging options on survivor swaps, building on the survivor swap pricing model published in Dowd, Cairns, Blake, and Dawson (2006) (henceforth, Dowd et al., 2006).

Central to the option model is the observation that the random survivor premium, π, in Dowd et al. (2006) is (at least approximately) normally distributed. Normal distributions have generally been shunned in asset pricing, as they permit negative prices. In the case of survivor derivatives, however, negative prices (corresponding to a decrease in life expectancy) are just as valid as positive prices. The model presented in this study has been generalized to be applicable, not just to survivor swaptions, but to options on any asset whose price is normally distributed.[1]

Options on normally distributed underlyings were famously considered in Bachelier's (1900) model of arithmetic Brownian motion. However, as just noted, such a distribution would allow the underlying asset price to become negative, and this unattractive implication can be avoided by using a geometric Brownian motion instead. Thus, the Bachelier model came to be regarded as a dead end, albeit an instructive one, and very little has been written on it since. For example, Cox and Ross (1976) derived an option model for a normally

[1]Commentators on early drafts of this study have suggested options on spreads as another application of this model. Although spread prices frequently become negative, our investigations indicate that their distribution is rarely normal and we therefore urge caution in using our model for such applications. Carmona and Durrleman (2003) discuss spread options in great detail.

distributed underlying, but avoided the negative price problem by assuming an absorbing barrier at price zero. An option model with a normal underlying was also briefly considered, though without any analysis, by Haug (2006).[2]

Accordingly, the principal objectives of this study are two-fold: first, to set out the full analytics of option pricing with a normally distributed underlying; and, second, to show how this model can be applied to the illustrative example of valuing a survivor swaption, that is, an option on a survivor swap. This study is organized as follows. The section "Model Derivation" derives the formulae for the put and call options for a European option with a normal underlying and presents their Greeks. The section "The Distribution of Survivor Swap Premiums" shows that survivor swap premiums are likely to be approximately normally distributed. The section "A Practical Application: Pricing Survivor Swaptions" discusses how the model can be applied to price swaptions and presents results that further support the assumption that the swap premium is normally distributed. The section "Testing the Model" tests the model. The last section concludes. The derivation of the Greeks is presented in Appendix A.

MODEL DERIVATION

Consider an asset with forward price F, with $-\infty < F < \infty$. We do not consider the case of an option on a normally distributed spot price, as this is an obvious special case of an option on a forward price. We denote the value of European call and put options by c and p, respectively. The strike price and maturity of the options are denoted by X and τ, respectively. The annual risk-free interest rate is denoted by r and the annual volatility rate (or the annual standard deviation of the price of the asset) is denoted by σ.

We first observe that the put–call parity condition is independent of the price distribution and is thus applicable in our model.

$$p = c - e^{-r\tau}(F - X). \tag{1}$$

The Black & Scholes (1973)/Merton (1973) dynamic hedging strategy can be implemented if there is assumed to be a liquid market in the underlying asset. In such circumstances, a risk-free portfolio of asset and option can be constructed and the value of an option is simply the present value of its expected payoff. The values of call and put options can then be presented as

$$c = e^{-r\tau} \times P(F_\tau > X) \times (E(F_\tau | F_\tau > X) - X) \tag{2}$$

$$p = e^{-r\tau} \times P(F_\tau < X) \times (X - E(F_\tau | F_\tau < X)) \tag{3}$$

in which F_τ represents the forward price at option expiry, and $F_\tau \sim N(F, \sigma^2\tau)$.

[2]Haug presents but does not derive an option pricing formula, and does not report the Greeks or discuss possible applications of the formula.

If $N(z)$ is the standard normal cumulative density function of z, with $z \sim N(0,1)$, the corresponding probability density function, $N'(z)$, is

$$N'(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \tag{4}$$

and it follows that

$$P(F_\tau > X) = \int_X^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(X-F)^2}{2\sigma^2\tau}} dF \tag{5}$$

$$= 1 - \int_{-\infty}^X \frac{1}{\sqrt{2\pi}} e^{-\frac{(X-F)^2}{2\sigma^2\tau}} dF. \tag{6}$$

Defining $d = (F - X)/\sigma\sqrt{t}$ then gives

$$P(F > X) = 1 - N\left(\frac{X - F}{\sigma\sqrt{\tau}}\right) \tag{7}$$

$$= N\left(\frac{F - X}{\sigma\sqrt{\tau}}\right) \tag{8}$$

$$=N(d). \tag{9}$$

We next consider the conditional expected value of F_τ, i.e., the expected value of F at expiry given that the call option has expired in the money

$$E(F_\tau|F_\tau > X) = \frac{\int_X^\infty \frac{F_\tau}{\sqrt{2\pi}} e^{-\frac{(X-F)^2}{2\sigma^2\tau}} dF}{\int_X^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(X-F)^2}{2\sigma^2\tau}} dF}. \tag{10}$$

A well-known result from expected shortfall theory—see, e.g., Dowd (2005, p. 154)—shows that

$$\frac{\int_X^\infty \frac{F_\tau}{\sqrt{2\pi}} e^{-\frac{(X-F)^2}{2\sigma^2\tau}} dF}{\int_X^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(X-F)^2}{2\sigma^2\tau}} dF} = F + \sigma\sqrt{\tau} \frac{N'\left(\frac{X - F}{\sigma\sqrt{\tau}}\right)}{1 - N\left(\frac{X - F}{\sigma\sqrt{\tau}}\right)} \tag{11}$$

$$= F + \sigma\sqrt{\tau} \frac{N'(-d)}{1 - N(-d)} \tag{12}$$

$$= F + \sigma\sqrt{\tau} \frac{N'(d)}{N(d)}. \tag{13}$$

Substituting (9) and (13) into (2) gives

$$c = e^{-r\tau} \times N(d) \times \left(F + \sigma\sqrt{\tau}\frac{N'(d)}{N(d)} - X\right) \tag{14}$$

and so gives us the call option pricing formula we are seeking in (15) below.

$$c = e^{-r\tau}((F - X)N(d) + \sigma\sqrt{\tau}N'(d)). \tag{15}$$

Substituting (15) into (1) then gives

$$p = e^{-r\tau}((F - X)N(d) + \sigma\sqrt{\tau}N'(d)) - F + Xe^{-r\tau} \tag{16}$$

$$= e^{-r\tau}((F - X)(N(d) - 1) + \sigma\sqrt{\tau}N'(d)) \tag{17}$$

$$= e^{-r\tau}((X - F)(1 - N(d)) + \sigma\sqrt{\tau}N'(d)). \tag{18}$$

Thus, the corresponding put option formula is given by (19) below.

$$p = e^{-r\tau}((X - F)N(-d) + \sigma\sqrt{\tau}N'(d))s. \tag{19}$$

Table I presents the Greeks. Their derivation can be found in Appendix A.

THE DISTRIBUTION OF SURVIVOR SWAP PREMIUMS

In a survivor swap, the pay-fixed party agrees to pay defined sums at defined intervals over the life of the contract and to receive in return payments predicated on the actual survivorship of the cohort referenced in the swap contract.

TABLE I
Summary of the Model and its Greeks

	Calls	*Puts*
Option value	$e^{-r\tau}[(F - X)N(d) - \sigma\sqrt{\tau}N'(d)]$	$e^{-r\tau}[(X - F)N(-d) + \sigma\sqrt{\tau}N'(d)]$
Delta	$e^{-r\tau}N(d)$	$-e^{-r\tau}N(-d)$
Gamma	$\frac{e^{-r\tau}}{\sigma\sqrt{\tau}}N'(d)$	$\frac{e^{-r\tau}}{\sigma\sqrt{\tau}}N'(d)$
Rho (per percentage point rise in rates)	$\frac{-\tau c}{100}$	$\frac{-\tau p}{100}$
Theta (for one day passage of time)	$\frac{2\sqrt{\tau}rc - e^{-r\tau}\sigma N'(d)}{730\sqrt{\tau}}$	$\frac{2\sqrt{\tau}rp - e^{-r\tau}\sigma N'(d) + 4\sqrt{\tau}re^{-r\tau}(F - X)}{730\sqrt{\tau}}$
Vega (per percentage point rise in volatility)	$\frac{e^{-r\tau}\sqrt{\tau}N'(d)}{100}$	$\frac{e^{-r\tau}\sqrt{\tau}N'(d)}{100}$

Lack of market completeness means that survivor swap contracts cannot be priced with the zero-arbitrage methodology observed in interest rate swaps. Instead, a survivor premium, π, is factored into the payments of the pay-fixed party. In the example cited later in this study, the pay-fixed party pays $\$K \times (1 + \pi)$ on each anniversary for the number of members of a pre-defined cohort expected at the date of trading the swap to survive until that anniversary and receives in return $\$K$ for every actual cohort survivor at that anniversary. Thus, a pension provider can turn an unknown survivorship liability into a series of fixed payments. π is effectively the price of the survivor swap. π can be positive or negative, depending on which party is observed to be at greater risk; π is also volatile.

Now let $l(s, t, u)$ be the probability-based information available at s that an individual who is alive at t survives to u. It follows that for each $s = 1, \ldots, t$, we get

$$l(s, t-1, t) = l(s-1, t-1, t)^{1-\varepsilon(s)} \tag{20}$$

where $\varepsilon(s)$ is the longevity shock in year s (Dowd et al., 2006, p. 5). This means that the probability of survival to t is affected by each of the one-year longevity shocks $\varepsilon(1), \varepsilon(2), \ldots, \varepsilon(t)$. Dowd (2005, p. 5) then suggests that the longevity shocks $\varepsilon(1), \varepsilon(2), \ldots, \varepsilon(t)$. can be modeled by the following transformed β distribution

$$\varepsilon(s) = 2y(s) - 1 \tag{21}$$

where $y(s)$ is β-distributed. As the β is defined over the domain $[0, 1]$, the transformed β $\varepsilon(s)$ is distributed over domain $[-1, +1]$, where $\varepsilon(s) > 0$ indicates that longevity unexpectedly improved, and $\varepsilon(s) < 0$ indicates the opposite.

The premium π is set so that the initial value of the swap is zero, and, in the case of a the simple vanilla swaps considered in Dowd et al. (2006), this implies that

$$\pi = \frac{E\prod_{i=t}^{n-1} l_i^{1-\varepsilon(i)}}{\prod_{i=t}^{n-1} l_i} - 1 \tag{22}$$

where $E\prod_{i=t}^{n-1} l_i^{1-\varepsilon(i)}$ is the value of the fixed leg and $\prod_{i=t}^{n-1} l_i$ is the value of the floating leg, and l_i is the probability of survival to i. Equation (22) tells us that, in general, the swap premium is related to a weighted average of the expectation of $n - 1$ independent $\varepsilon(i)$ shocks, each of which follows a transformed β distribution. So what is the distribution of π?

The first point to note is that the average of $n - 1$ independent and identically distributed shocks falls under the domain of the central limit theorem: this immediately tells us that the distribution of π tends to normality as n gets large. In the present case, n is the number of years ahead over which the relevant survivor rate is specified, so n realistically might vary from 1 to the time that the cohort concerned has completely died out (and this might be around 50 years for a cohort of current age 65). Hence, we can say at this point that π tends to approach normality as n gets large, but (especially for low value of n) the approximation to normality may be limited, depending on the distribution of $\varepsilon(i)$.

This takes us to the distribution of $\varepsilon(i)$ itself. In their original study, Dowd et al. (2006) suggested that the size of π reflected projected longevity improvements since the time that the pre-set leg of the swap was initially set. In the past, the pre-set leg would typically have been based on projections from a mortality table that was prepared years before and projected fairly small mortality improvements. By contrast, the floating leg would be set by, say, the output of a recently calibrated stochastic mortality model that might have projected stronger mortality improvements. However, as time goes by, we would expect all "views" of future mortality to be generated by reasonably up-to-date models, and differences of views would become fairly small. Hence, we would expect π to decline over time as swap counterparties become more sophisticated.

If we accept this line of reasoning, then the sets of β distribution parameters presented in Table 1 of Dowd et al. (2006) lead us to believe that the most plausible representation of mortality given in that Table is their case 1: this is where $\varepsilon(i)$ has a zero mean and a standard deviation of around 2.2%, and this occurs where the β distribution has parameters $a = \text{b} \approx 1{,}000$. Thus, if we accept this example as plausible,[3] we might expect the parameters of the β distribution to be both high and approximately equal to each other. This is convenient, because statistical theory tells us that a β distribution with $a = b$ tends to normality as $a = b$ gets large (see, e.g., Evans, Hastings, & Peacock, 2000, p. 40). Thus, if we choose a and b to be large and equal (e.g., 1,000), then $\varepsilon(i)$ will be approximately normal.

We therefore have two mutually reinforcing reasons to believe that the distribution of premia might be approximately normal.

A PRACTICAL APPLICATION: PRICING SURVIVOR SWAPTIONS

As noted earlier, a practical illustration of the usefulness of this option pricing model can be found in the pricing of survivor swaptions. A survivor swaption

[3]We emphasize that this model is limited insofar as it assumes that mortality shocks are drawn from a single, age-independent, distribution. This is a convenient assumption for our illustrative purposes, and is comparable to the flat term structure often assumed in option pricing models. For a more general model, which allows for age-dependent shocks, see Cairns (2007) and Dawson et al (2008).

gives the right, but not the obligation, to enter into a survivor swap contract at a specified rate of π. The term π is also a risk premium reflecting the potential errors in the expectation of mortality evolution and π can be positive or negative, depending on whether greater longevity ($\pi > 0$) or lower longevity ($\pi < 0$) is perceived to be the greater risk. It will also be zero in the case where the risks of greater and lower longevity exactly balance. Survivor swaptions can take one of two forms: a payer swaption, equivalent to our earlier call, in which the holder has the right but not the obligation to enter into a pay-fixed swap at the specified future time; and a receiver swaption, equivalent to our earlier put, in which the holder has the right but not the obligation to enter into a receive-fixed swap at the specified future time.

In order to price the swaption using the usual dynamic hedging strategies assumed for pricing purposes, we are also implicitly assuming that there is a liquid market in the underlying asset, the forward survivor swap. Although we recognize that this assumption is not currently valid in practice, we would defend it as a natural starting point, not least because survivor swaptions cannot exist without survivor swaps.

We now consider an example calibrated on swaptions that mature in five years' time and are based on a cohort of U.S. males who will be 70 when the swaptions mature. The strike price of the swaption is a specified value of π and for this example, we shall use an at-the-money forward option, i.e., X is set at the prevailing level of π for the forward contract used to hedge the swaption. Setting the option to be at-the-money forward means that the payer swaption premium and the receiver swaption premium are identical.

Using the same mortality table as Dowd et al. (2006), and assuming, as they did, longevity shocks, $\varepsilon(i)$, drawn from a transformed β distribution with parameters 1,000 and 1,000 and a yield curve flat at 6%, Monte Carlo analysis with 10,000 trials shows the moments of the distribution of π for the forward swap to have the following values:

Mean	0.001156
Annual variance	0.000119
Skewness	0.008458
Kurtosis	3.029241

The distribution is shown in Figure 1 below.

A Jarque–Bera test on these data gives a test statistic value of 0.476. Given that the test statistic has a χ^2 distribution with two degrees of freedom, this test result is consistent with the longevity shocks following a normal distribution.

Figure 2 below shows the options premia for both payer and receiver swaptions across π values spanning ± 3 standard deviations from the mean. The γ, or convexity, familiar in more conventional option pricing models is also seen here.

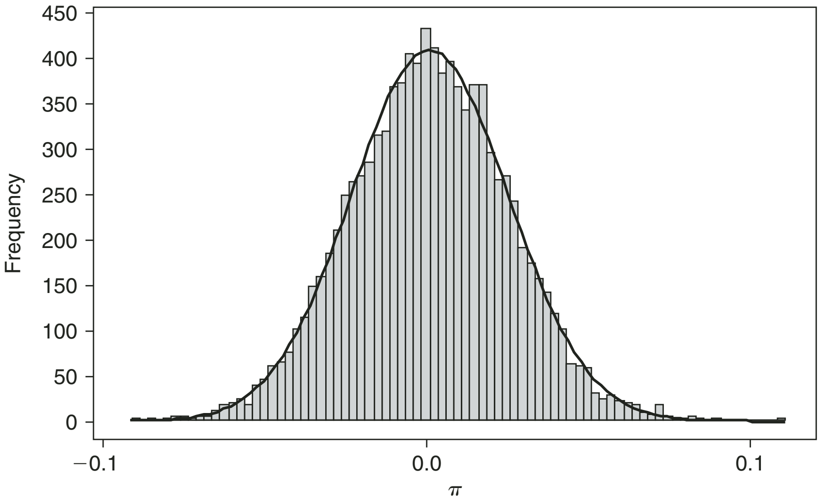

FIGURE 1
Distribution of the values of the p for a 45 year survivor swap starting in five years' time, from a Monte Carlo simulation of 10,000 trials with $\varepsilon(i)$ values drawn from a transformed β distribution with parameters (1,000; 1,000). A normal distribution plot is superimposed.

As with conventional interest rate swaptions, the premia are expressed in percentage terms. However, whereas with interest rate swaptions, the premia are converted into currency amounts by multiplying by the notional principal, with survivor swaptions, the currency amount is determined by multiplying the percentage premium by $N\sum_{t=1}^{50}A_{expiry,t}K(t)S(t)$ in which N is the cohort size, $A_{expiry,t}$ is the discount factor applying from option expiry until time t, $K(t)$ is the payment per survivor due at time t, $S(t)$ is the proportion of the original cohort expected to survive until time t, such expectation being observed at the time of the option contract, and where all members of the cohort are assumed to be dead after 50 years.[4] $N\sum_{t=1}^{50}A_{expiry,t}K(t)S(t)$ is known with certainty at the time of option pricing.

Figure 3 below shows the changing value of the at-the-money forward payer and receiver swaptions as time passes. The rapid price decay as expiry approaches, again familiar in more conventional options, is also seen here.

TESTING THE MODEL

The derivation of the model is predicated on the assumption that implementation of a dynamic hedging strategy will eliminate the risk of holding long or short positions in such options. We test the effectiveness of this strategy by

[4]This approach is equivalent to that used in the pricing of an amortizing interest rate swap, in which the notional principal is reduced by pre-specified amounts over the life of the swap contract.

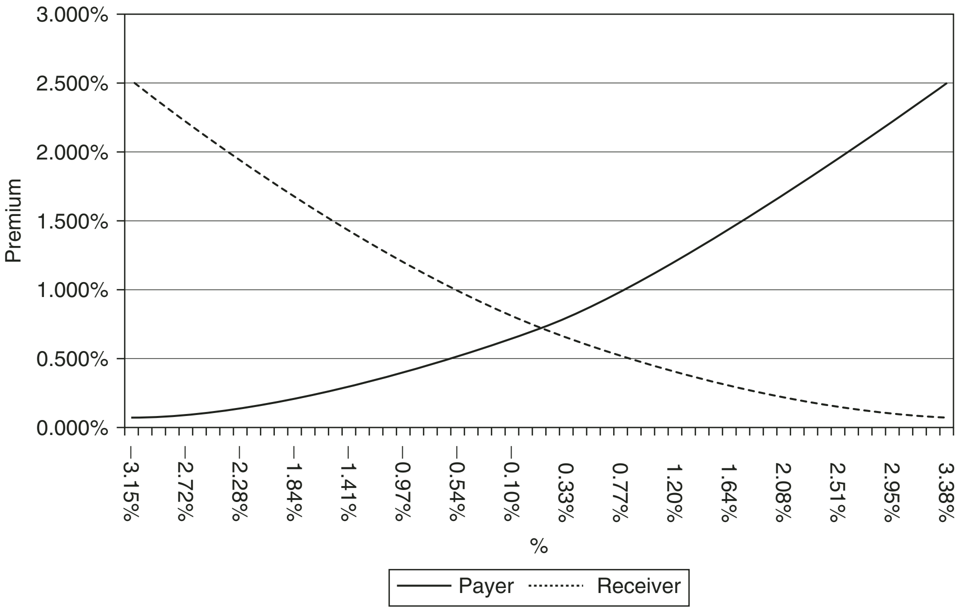

FIGURE 2
Premia for the specified survivor swaptions.

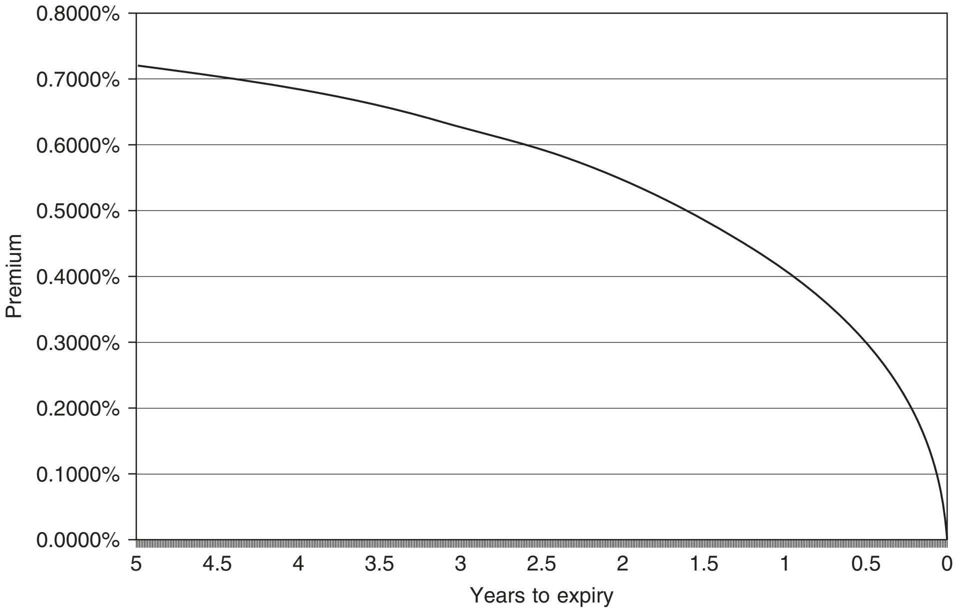

FIGURE 3
Options premia against time.

simulating the returns to dealers with (separate) short[5] positions in payer and receiver swaptions, and who undertake daily rehedging over the five year (1,250 trading days) life of the swaptions. We use Monte Carlo simulation to model the evolution of the underlying forward swap price, assuming a normal distribution. We assume a dealer starting off with zero cash and borrowing or depositing at the risk-free rate in response to the cash flows generated by the dynamic hedging strategy. As Merton (1973, p. 165) states, "Since the portfolio requires zero investment, it must be that to avoid 'arbitrage' profits, the expected (and realized) return on the portfolio with this strategy is zero." Merton's model was predicated on rehedging in continuous time, which would lead to expected and realized returns being identical. In practice, traders are forced to use discrete time rehedging, which is modeled here. One consequence of this is that on any individual simulation, the realized return may differ from zero, but that over a large number of simulations, the expected return will be zero. This is actually a joint test of three conditions:

1. The option pricing model is correctly specified—Equations (15) and (19) above;
2. The hedging strategy is correctly formulated—Equations (A1) and (A2) below, and
3. The realized volatility of the underlying asset price matches the volatility implied in the price of the option. We can isolate this condition by forecasting results when this condition does not hold and comparing observation with forecast. The dealer who has sold an option at too low an implied volatility will expect a loss, whereas the dealer sells at too high an implied volatility can expect a profit. This expected profit or loss of the dealer's portfolio, $E[V_p]$, at option expiry is

$$E[V_p] = se^{r\tau}\frac{\partial c}{\partial \sigma_{implied}}(\sigma_{implied} - \sigma_{actual}) \tag{23}$$

$$= \sqrt{r}N'(d)(\sigma_{implied} - \sigma_{actual}) \tag{24}$$

in which $\sigma_{implied}$ and σ_{actual} represent, respectively, the volatilities implied in the option price and actually realized over the life of the option.

We have conducted simulations across a wide set of scenarios, using different values of π, $\sigma_{implied}$ and σ_{actual} and different degrees of moneyness. In all cases, we ran 250,000 trials and in all cases, the results were as forecast. By way of example,[6] we illustrate in Table II the results of the trials of the option

[5]The returns to long positions will be the negative of returns to short positions.

[6]Results of the full range of Monte Carlo simulations are available on request from the corresponding author.

TABLE II
Results of Monte Carlo Simulations of Delta Hedging Strategy

		Payer				*Receiver*			
$\sigma_{implied}$ (%)	σ_{actual} (%)	*Expected Value* (%)	*Mean* (%)	*Standard Deviation* (%)	*t-Stat*	*Expected Value* (%)	*Mean* (%)	*Standard Deviation* (%)	*t-Stat*
1.088998	1.088998	0.0000	0.0002	0.2092	0.40	0.0000	0.0002	0.2036	0.42
1.088998	0.988998	0.0892	0.0894	0.1897	0.45	0.0892	−0.0894	0.1853	0.46
1.088998	1.188998	−0.0892	−0.0891	0.2287	0.24	−0.0892	−0.0891	0.2221	0.24

Note. Simulations carried out using @Risk, with 250,000 trials.

illustrated in Figures 1 and 2. The *t*-statistics relate to the differences between the observed and the forecast mean outcomes.

The reader will note that the differences between the observed and expected means are very low and statistically insignificant. This reinforces our assertion that the model provides accurate swaption prices.

CONCLUSION

Interest in survivor derivatives from both the academic and practitioner communities has grown rapidly in recent times, partly because of the economic importance involved in the risk being managed and partly because of the significant intellectual challenges of developing such products. Trading in such derivatives is in its early stages and has largely been confined to linear payoff products. Option-based products seem inevitable and, given the normal distribution of the random variable, π, used in the Dowd et al. (2006) survivor swap pricing methodology, a model for pricing options on normally distributed assets is required. The model derived in this study is a straightforward adaptation of the Black–Scholes–Merton approach and provides a robust solution to this problem.

APPENDIX A: DERIVATION OF THE GREEKs

Delta (Δ_c, Δ_p)

The option's Δs follow immediately from (17) and (21)

$$\Delta_c = \frac{\partial c}{\partial F} = e^{-r\tau}N(d). \tag{A1}$$

$$\Delta_p = \frac{\partial p}{\partial F} = -e^{-r\tau}N(-d). \tag{A2}$$

Gamma(Γ_c, Γ_p)

$$\Gamma_c = \frac{\partial^2 c}{\partial F^2} = \frac{\partial \Delta_c}{\partial F} = \frac{\partial e^{-r\tau} N(d)}{\partial F} \tag{A3}$$

$$= e^{-r\tau} \frac{\partial d}{\partial F} \times \frac{\partial N(d)}{\partial d} \tag{A4}$$

$$= \frac{e^{-r\tau}}{\sigma \sqrt{\tau}} N'(d). \tag{A5}$$

$$\Gamma_p = \frac{\partial^2 p}{\partial F^2} = \frac{\partial \Delta_p}{\partial F} = -\frac{\partial e^{-r\tau} N(-d)}{\partial F} \tag{A6}$$

$$-e^{-r\tau} \frac{\partial(-d)}{\partial F} \times \frac{\partial N(-d)}{\partial d} \tag{A7}$$

$$= \frac{e^{-r\tau}}{\sigma \sqrt{\tau}} N'(-d) \tag{A8}$$

$$= \frac{e^{-r\tau}}{\sigma \sqrt{\tau}} N'(d) \tag{A9}$$

Rho $(\mathrm{P}_c, \mathrm{P}_p)$

$$\mathrm{P}_c = \frac{\partial c}{\partial r} = -\tau c. \tag{A10}$$

$$\mathrm{P}_p = \frac{\partial p}{\partial r} = -\tau p. \tag{A11}$$

Theta (Θ_c, Θ_p)

$$\frac{\partial c}{\partial \tau} = \frac{\partial}{\partial \tau} e^{-r\tau} [(F - X)N(d) + \sigma \sqrt{\tau} N'(d)]. \tag{A12}$$

By the product rule

$$\frac{\partial c}{\partial \tau} = [(F - X)N(d) + \sigma \sqrt{\tau} N'(d)] \frac{\partial}{\partial \tau} e^{-r\tau}$$

$$+ e^{-r\tau} \frac{\partial}{\partial \tau} [(F - X)N(d) + \sigma \sqrt{\tau} N'(d)]. \tag{A13}$$

$$\text{Let } A = [(F - X)N(d) + \sigma \sqrt{\tau} N'(d)] \frac{\partial}{\partial \tau} e^{-r\tau}. \tag{A14}$$

$$\text{Let } B = e^{-r\tau} \frac{\partial}{\partial \tau} [(F - X)N(d) + \sigma \sqrt{\tau} N'(d)] \tag{A15}$$

$$\therefore \frac{\partial c}{\partial \tau} = A + B. \tag{A16}$$

$$A = [(F - X)N(d) + \sigma \sqrt{\tau} N'(d)] \frac{\partial}{\partial \tau} e^{-r\tau}$$

$$= -re^{-r\tau} [(F - X)N(d) + \sigma \sqrt{\tau} N'(d)] \tag{A17}$$

$$= -rc \tag{A18}$$

$$\therefore \frac{\partial c}{\partial \tau} = -rc + B. \tag{A19}$$

Applying the sum rule

$$B = e^{-r\tau}\frac{\partial}{\partial \tau}[(F - X)N(d) + \sigma\sqrt{\tau}N'(d)]$$

$$= e^{-r\tau}(F - X)\frac{\partial}{\partial \tau}N(d) + e^{-r\tau}\frac{\partial}{\partial \tau}\sigma\sqrt{\tau}N'(d). \tag{A20}$$

$$\text{Let } C = e^{-r\tau}(F - X)\frac{\partial}{\partial \tau}N(d). \tag{A21}$$

$$\text{Let } D = e^{-r\tau}\frac{\partial}{\partial \tau}\sigma\sqrt{\tau}N'(d) \tag{A22}$$

$$\therefore \frac{\partial c}{\partial \tau} = -rc + C + D. \tag{A23}$$

The chain rule then implies

$$C = e^{-r\tau}(F - X)\frac{\partial}{\partial \tau}N(d) = e^{-r\tau}(F - X)\frac{\partial}{\partial d}N(d)\frac{\partial d}{\partial \tau} \tag{A24}$$

$$= -\frac{e^{-r\tau}\,d(F - X)N'(d)}{2\tau} \tag{A25}$$

$$\therefore \frac{\partial c}{\partial \tau} = -rc - \frac{e^{-r\tau}d(F - X)N'(d)}{2\tau} + D. \tag{A26}$$

By the product rule

$$D = e^{-r\tau}\frac{\partial}{\partial \tau}\sigma\sqrt{\tau}N'(d) = e^{-r\tau}N'(d)\frac{\partial}{\partial \tau}\sigma\sqrt{\tau} + e^{-r\tau}\sigma\sqrt{\tau}\frac{\partial}{\partial \tau}N'(d). \tag{A27}$$

$$\text{Let } E = e^{-r\tau}N'(d)\frac{\partial}{\partial \tau}\sigma\sqrt{\tau}. \tag{A28}$$

$$\text{Let } F = e^{-r\tau}\sigma\sqrt{\tau}\frac{\partial}{\partial \tau}N'(d). \tag{A29}$$

$$E = e^{-r\tau}N'(d)\frac{\partial}{\partial \tau}\sigma\sqrt{\tau} = \frac{e^{-r\tau}\sigma N'(d)}{2\sqrt{\tau}} \tag{A30}$$

$$\therefore \frac{\partial c}{\partial \tau} = -rc - \frac{e^{-r\tau}d(F - X)N'(d)}{2\tau} + \frac{e^{-r\tau}\sigma N'(d)}{2\sqrt{\tau}} + F. \tag{A31}$$

By the chain rule

$$F = e^{-r\tau}\sigma\sqrt{\tau}\frac{\partial}{\partial \tau}N'(d) = e^{-r\tau}\sigma\sqrt{\tau}\frac{\partial}{\partial d}N'(d)\frac{\partial d}{\partial \tau} \tag{A32}$$

$$= \frac{e^{-r\tau}\sigma\sqrt{\tau}d^2N'(d)}{2\tau} \tag{A33}$$

$$\therefore \frac{\partial c}{\partial \tau} = -rc - \frac{e^{-r\tau}d(F-X)N'(d)}{2\tau} + \frac{e^{-r\tau}\sigma N'(d)}{2\sqrt{\tau}} + \frac{e^{-r\tau}\sigma\sqrt{\tau}d^2N'(d)}{2\tau}. \tag{A34}$$

Tidying up

$$\frac{\partial c}{\partial \tau} = -rc + e^{-r\tau}N'(d)\left(-\frac{d(F-X)}{2\tau} + \frac{\sigma}{2\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}d^2}{2\tau}\right) \tag{A35}$$

$$= -rc + \frac{e^{-r\tau}N'(d)}{2\tau}(-d(F-X) + \sigma\sqrt{\tau} + \sigma\sqrt{\tau}d^2) \tag{A36}$$

$$= -rc + \frac{e^{-r\tau}N'(d)}{2\tau}(-\sigma\sqrt{\tau}d^2 + \sigma\sqrt{\tau} + \sigma\sqrt{\tau}d^2) \tag{A37}$$

$$= -rc + \frac{e^{-r\tau}\sigma\sqrt{\tau}N'(d)}{2\tau} \tag{A38}$$

$$= -rc + \frac{e^{-r\tau}\sigma N'(d)}{2\sqrt{\tau}}. \tag{A39}$$

As it is conventional for practitioners to quote Θ as the change in an option's value as one day passes

$$\Theta_c = \frac{2\sqrt{\tau}rc - e^{-r\tau}\sigma N'(d)}{730\sqrt{\tau}}. \tag{A40}$$

The equivalent value for a put option can be obtained quite easily from put–call parity and Equation (A40).

$$p = c - e^{-r\tau}(F-X) \tag{A41}$$

$$\therefore \frac{\partial p}{\partial \tau} = \frac{\partial c}{\partial \tau} - \frac{\partial}{\partial \tau}e^{-r\tau}(F-X) \tag{A42}$$

$$= -rc + \frac{e^{-r\tau}\sigma N'(d)}{2\sqrt{\tau}} - re^{-r\tau}(F-X) \tag{A43}$$

$$= -r(p + e^{-r\tau}(F-X)) + \frac{e^{-r\tau}\sigma N'(d)}{2\sqrt{\tau}} - re^{-r\tau}(F-X) \tag{A44}$$

$$= -rp + \frac{e^{-r\tau}\sigma N'(d)}{2\sqrt{\tau}} - 2re^{-r\tau}(F-X) \tag{A45}$$

$$\therefore \Theta_p = \frac{2\sqrt{\tau}rp - e^{-r\tau}\sigma N'(d) + 4\sqrt{\tau}re^{-r\tau}(F-X)}{730\sqrt{\tau}}. \tag{A46}$$

Vega $\left(\frac{\partial c}{\partial \sigma}, \frac{\partial p}{\partial \sigma}\right)$

$$\frac{\partial c}{\partial \sigma} = \frac{\partial}{\partial \sigma} e^{-r\tau}[(F - X)N(d) + \sigma\sqrt{\tau}N'(d)] \tag{A47}$$

$$= e^{-r\tau}(F - X)\frac{\partial}{\partial \sigma}N(d) + e^{-r\tau}\frac{\partial}{\partial \sigma}\sigma\sqrt{\tau}N'(d). \tag{A48}$$

$$\text{Let } G = e^{-r\tau}(F - X)\frac{\partial}{\partial \sigma}N(d). \tag{A49}$$

$$\text{Let } H = e^{-r\tau}\frac{\partial}{\partial \sigma}\sigma\sqrt{\tau}N'(d). \tag{A50}$$

$$\frac{\partial c}{\partial \sigma} = G + H. \tag{A51}$$

By the chain rule

$$G = e^{-r\tau}(F - X)\frac{\partial}{\partial \sigma}N(d) = e^{-r\tau}(F - X)\frac{\partial}{\partial d}N(d)\frac{\partial d}{\partial \sigma} \tag{A52}$$

$$= \frac{-e^{-r\tau}d(F - X)N'(d)}{\sigma} \tag{A53}$$

$$= -e^{-r\tau}\sqrt{\tau}d^2N'(d). \tag{A54}$$

By the product and chain rules

$$H = e^{-r\tau}\frac{\partial}{\partial \sigma}\sigma\sqrt{\tau}N'(d) = e^{-r\tau}N'(d)\frac{\partial}{\partial \sigma}\sigma\sqrt{\tau} + e^{-r\tau}\sigma\sqrt{\tau}\frac{\partial}{\partial d}N'(d)\frac{\partial d}{\partial \sigma} \tag{A55}$$

$$= e^{-r\tau}N'(d)\sqrt{\tau} + e^{-r\tau}\sigma\sqrt{\tau}N'(d)\frac{d^2}{\sigma} \tag{A56}$$

$$= e^{-r\tau}\sqrt{\tau}(d^2 + 1)N'(d) \tag{A57}$$

$$\therefore \frac{\partial c}{\partial \sigma} = e^{-r\tau}\sqrt{\tau}(d^2 + 1)N'(d) - e^{-r\tau}\sqrt{\tau}d^2N'(d) \tag{A58}$$

$$= e^{-r\tau}\sqrt{\tau}N'(d). \tag{A59}$$

As practitioners generally present vega in terms of a one percentage point change in volatility, we present vega here as

$$\frac{\partial c}{\partial \sigma} = \frac{e^{-r\tau}\sqrt{\tau}N'(d)}{100}. \tag{A60}$$

Put–call parity shows that the vega of a put option equals the vega of a call option

$$p = c - e^{-r\tau}(F - X) \tag{A61}$$

$$\therefore \frac{\partial p}{\partial \sigma} = \frac{\partial c}{\partial \sigma} - \frac{\partial}{\partial \sigma} e^{-r\tau}(F - X) = \frac{\partial c}{\partial \sigma}. \tag{A62}$$

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