The material of Chapter IV, on the structure of complete local rings, is due to the reviewer. Let A be a complete local ring, M its maximal ideal, ϕ the canonical homomorphism of A onto A/M. If A and A/M have equal characteristics p, then A contains a field K such that $\phi(K)=A/M$. If $p=0$, then this can be readily derived from "Hensel's Lemma," the statement of which for complete local rings is just what one expects; in the more difficult case $p>0$, the proof follows from the lifting theorem stated below. (A particularly simple proof in this equal-characteristic case has been recently given by A. Geddes [J. London Math. Soc. vol. 29 (1954) pp. 334–341].) Suppose now that A/M has characteristic $p>0$ (which is certainly the case if this characteristic is different from that of A, on which we make no assumption), let B be a complete discrete valuation ring of characteristic zero whose maximal ideal is generated by p, and assume τ a homomorphism of B onto A/M (existence of B and τ is well known). The lifting theorem then asserts the existence of a homomorphism σ of B into A such that $\phi\sigma=\tau$. From this can be deduced that every complete local ring is the homomorphic image of a power series ring over K or B, and also is (in the absence of zero divisors) a finite module over such a power series ring. Some further theorems on structure are proved and some applications made to the ideal theory in complete local rings.

In Chapter V, it is proved that if a ring with nucleus lacks zero divisors and is integrally closed then its completion also possesses these properties. This is a generalization of Zariski's theorem on the analytical irreducibility and analytical normality of normal varieties. A form of the Weierstrass preparation theorem is given and with its aid unique prime factorization is proved in the power series rings described in the preceding paragraph.

In Chapter VI there is defined a Kronecker product for two M-adic rings. Some other questions are briefly considered.

I. S. Cohen

Editorial note: The preceding review was found among Professor Cohen's papers after his death. Attached notes indicate that a final paragraph was to have mentioned the historical notes at the end of the chapters; the extensive bibliography (98 items); and the existence of a number of misprints, some merely typographical, others actual mistakes (pp. 22, 35, 37, 51). The mistake which Cohen considered most serious appears to be on p. 35, fourth line from the bottom.

Funzioni ipergeometriche confluenti. By F. G. Tricomi. Rome, Cremonese, 1954. 16+312 pp. 3500 Lire.

This book on confluent hypergeometric functions differs quite considerably from the same author's *Lezioni sulle funzioni ipergeometriche confluenti* reviewed recently in this *Bulletin* (vol. 60 (1954) pp. 185–189), and is to all intents and purposes a new book. Almost the only common points with the *Lezioni* are: the notation for confluent hypergeometric functions (which differs somewhat from that most commonly adopted), some of the applications discussed in the book, and, of course, the general point of view. In every other respect the differences are much more important than the similarities. The earlier work gave an introduction to confluent hypergeometric functions and was self-contained to a remarkable extent; the present work is a treatise on the subject, demanding from the reader both more knowledge, and more willingness to look up references. The author did not aim at encyclopedic completeness, yet he endeavored to give, and fully succeeded in giving, a comprehensive picture of the theory of confluent hypergeometric functions and of their applications in physics, engineering, and probability theory. Readers of earlier books by the same author need not be told that the presentation is excellent, the arrangement well planned, and the style both lucid and readable.

The so-called special functions present a somewhat peculiar problem of exposition in that books on them are read alike by physicists, engineers, and professional mathematicians; by beginners and experts; by specialists and casual users; shortly by a wide circle of readers with vastly differing mathematical training and interests. A monograph giving detailed proofs is apt to become repetitious and boring as well as exceedingly bulky, and one giving outlines of proofs only is likely to become so sketchy that it will amount to little more than a collection of results. The first kind of monograph will be an invaluable work of reference, but it is unlikely to be readable; the second kind may, with consummate skill on the part of the author, be made into a serviceable work of reference, but it will be singularly unenlightening as to methods and trends. Both kinds will be useful tools in the hands of an expert, and neither of them will be a favorite with the general reader or the beginner. It takes all the author's well-known skill to steer a reasonable middle course, and he must be complimented on having produced one of the best monographs of this kind in recent years. All basic results, and many other results, are proved in full detail, some other results are stated with references to the literature where proofs can be found, yet others are merely mentioned, again with references to existing literature. The book is well documented without overwhelming the reader with a multitude of references. Papers are referred to in footnotes throughout the book, and a list of 25 books is given at the end of the volume, where there is also an impressive list of 25 papers by the author on confluent hypergeometric functions and related topics.

In Chapter I, the author introduces

$$xy'' + (c - x)y' - ay = 0 \tag{1}$$

as the canonical form of the confluent hypergeometric equation, reduces the general linear differential equation of the second order with linear coefficients to this canonical form, and gives some other differential equations which can be reduced to (1). If c is not an integer, Kummer's series

$$\Phi(a, c; x) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)}{c(c+1)\cdots(c+n-1)} \frac{x^n}{n!}, \tag{2}$$

and $x^{1-c}\Phi(a-c+1, 2-c; x)$ form a fundamental system of solutions of (1). Instead of Φ, the author often considers

$$\Phi^*(a, c; x) = \frac{1}{\Gamma(c)} \Phi(a, c; x), \tag{3}$$

the latter function being an entire function of all three of its arguments. An investigation of the elementary properties of Φ follows, including Kummer's transformation, differentiation and recurrence relations, Laplace transform pairs and infinite series involving Φ, special cases of Φ including Bessel functions, Laguerre polynomials, incomplete gamma functions, etc. The chapter concludes with the author's well known expansion

$$\Phi^*(a, c; x) = e^{x/2}(kx)^{(1-c)/2}\sum_{n=0}^{\infty} A_n \left(\frac{x}{4k}\right)^{n/2} J_{c+n-1}(2(kx)^{1/2}), \tag{4}$$

where $k = c/2 - a$, and the A_n are certain coefficients determined by a generating function.

The second chapter is devoted to integral representations of confluent hypergeometric functions. Integration of (1) by means of Laplace integrals leads to (2), and to a second solution which may be defined as

$$\Psi(a, c; x) = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-xt}t^{a-1}(t+1)^{c-a-1}dt \tag{5}$$

when Re $a > 0$, Re $x > 0$, and by analytic continuation for other values of its arguments. Differentiation and recurrence formulas, the relation

$$\Psi(a, c; x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\Phi(a, c; x) + \frac{\Gamma(c-1)}{\Gamma(a)}x^{1-c}\Phi(a-c+1, 2-c; x), \tag{6}$$

the limiting form of (6) for integer c follow, and the transformation $\Psi(a, c; x) = x^{1-c}\Psi(a-c+1, 2-c; x)$ is deduced from (6). The connection with Whittaker's functions $M_{k,m}(x)$ and $W_{k,m}(x)$ and with Bessel functions is also discussed. This chapter contains further integral representations involving Bessel functions, hypergeometric functions, Legendre functions, integral representations of the Mellin-Barnes type, and some miscellaneous results including results on the derivatives of confluent hypergeometric functions with respect to the parameters a, c.

In Chapter III the author discusses the asymptotic behavior of confluent hypergeometric functions, the zeros of these functions, and gives a description of their graphs for real a, c, x. This chapter is unusually well organized. Although full proofs could not be given everywhere, the reader obtains a very complete picture, including some recent results by the author and others.

In Chapter IV the author turns to special confluent hypergeometric functions. He starts with the incomplete gamma functions

$$\gamma(\alpha, x) = \int_0^x e^{-t}t^{\alpha-1}dt, \qquad \Gamma(\alpha, x) = \int_x^\infty e^{-t}t^{\alpha-1}dt$$

whose investigation is based on the function

$$\gamma^*(\alpha, x) = \frac{\gamma(\alpha, x)}{x^\alpha\Gamma(\alpha)} = \Phi^*(\alpha, \alpha+1; -x).$$

The properties of these functions are largely derived from properties of Φ, Ψ, but here again the complete description of the asymptotic behavior, zeros, and graphs in the real domain, deserves special praise. Error functions and related functions, and the exponential integral and related functions are special incomplete gamma functions and presented in this context. Parabolic cylinder functions, which arise when $c = 1/2$ or $3/2$, are also included in this chapter.

The fifth and last chapter contains some examples of applications of confluent hypergeometric functions in various fields. The examples selected by the author, the two body problem of wave mechanics, bending of elastic plates, resultant of a large number of random vectors, water waves, the distribution of integers which can be represented as the sums of two kth powers, and the reflection of electromagnetic waves on a parabolic cylinder, show the wide range of applications of these functions.

A table of Laplace transform pairs involving confluent hypergeometric functions, the list of references mentioned above, and an index conclude this excellent volume.

The printing is excellent, and very few misprints were noted by this reviewer. Tricomi's book is the first volume in a new series of mathematical monographs sponsored by the Italian *Consiglio Nazionale della Ricerche.* It is a promising beginning, and in wishing the new venture every success, one can hardly do better than express the hope that future volumes of the series will be as useful and as readable as Tricomi's book.

A. ERDÉLYI

The language of taxonomy—An application of symbolic logic to the study of classificatory systems. By J. R. Gregg. New York, Columbia University Press, 1954. 9+70 pp. $2.50.

The subtitle and almost every page of this little book hold forth promise to mathematicians on the alert for radical new applications of mathematics, but I fear that they will be rather disappointed. The specific subject matter seems dry and relatively unimportant for mathematicians and biologists alike; the method chosen by the author seems ineffective and tedious; there are scarcely any interesting arguments or deductions presented or alluded to; and the final results are meager.

The general problem of classifying a vast set of objects has many aspects of intellectual and perhaps of mathematical interest, especially when the objects are organisms, with their philogenetic connections. This book is confined to discovering a description of the formal, set theoretic, structure of the biological taxonomic systems in actual use, completely abstracted from how these systems depend on the structure and philogeny of organisms. For example, it is within the scope of the book to say that a species is a set of organisms and a genus a set theoretic union of species but not to say when two organisms should be assigned to a common species or genus.

As the author says, biologists have over the centuries evolved a remarkably accurate and effective language for describing the forms of organisms, but they have as yet no special language for talking about taxonomic systems, as opposed to talking taxonomy. The author seeks to supply the missing language with the aid of symbolic logic. I believe he would have done far better to look to a powerful,