Lecture Notes on MA5209 Algebraic Topology-2008/2009 Semester 1

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CHAPTER 1

Simplicial Complexes

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1. Introduction

The notion of simplicial complex was introduced for making triangulations of polyhedrons. In such a way, one can cut a polyhedron into many simpler pieces called simplices. A simplex is given in usual sense that 0-simplex means a point, 1-simplex means a line segment, 2-simplex means a triangle, 3-simplex means a tetrahedron and etc. By using these simpler objects, one can gluing them together to make polyhedrons. On the other hand, by using these objects, one can obtain algebraic objects such as the free abelian groups generated by these objects, namely the direct sum of the copies of the integers $\mathbb{Z}$ labeled by these objects. The gluing rule (for making a polyhedron) gives many group homomorphisms. From these data, one then obtains so-called simplicial homology. It was then proved that the simplicial homology of a polyhedron is invariant under any deformations of the polyhedron. More precisely the simplicial homology only depends on the homotopy type of the polyhedron. For instance, one can make a lot of different triangulations on a sphere S^2. But all simplicial homologies of S^2 are the same. On the other hand, one can find that the simplicial homology of S^2 is different from the simplicial homology of a torus. From this, we can conclude that there is no way to deform a torus into a sphere. Simplicial complex then became one of powerful tools for studying polyhedrons.

The ideas in mathematics are always shared among people. Nowadays the ideas of simplicial objects have been applied to many areas in mathematics. One of the steps is to make an abstract version of simplicial complex, called *abstract simplicial complex*. For applications, one can set up a mathematical model as an abstract simplicial complex. For instance, some mathematical models from the computer science are given by abstract simplicial complexes. Whence we have an abstract simplicial complex on hand, we can make a polyhedron by taking geometric realization. Namely taking a collection of simplices and gluing them together according to the rules in an abstract simplicial complex. From this, we obtain a space and have many information such as homology, homotopy and Euler characteristic.

For understanding the deep structure in an abstract (or geometric) simplicial complex, the notion of Δ-set came out naturally. Its geometric realization then gives Δ-complexes. The notion of Δ-set can go even further for having Δ-objects on any category. This allows to have simplicial homology theory for any working category.

In this chapter, we introduce these notions step by step. So we first study geometric simplicial complexes to have some geometric intuition on these objects, and then we study abstract simplicial complexes. By ruling out the structure in abstract simplicial complexes, we introduce the notion of Δ-sets. Finally we finish this chapter by having the categorical view on Δ-objects.

2. Geometric Simplicial Complexes

2.1. Simplices. We start with the concept of (geometric) simplex. Consider the Euclidean space $\mathbb{R}^m$. Given $(n+1)$ points, we are going to make a subspace linearly spanned by these points. For instance, given three points not in a line, we can make a triangle spanned by these three points. Of course, we can also use four or more points to spanned a polyhedron. But we are interested in having minimal number of points to span a region. For instance, two points can make a line segment. Three points can make a triangle. Four points in general positions in $\mathbb{R}^3$ can make a tetrahedron.

Mathematically we need to first understand what it means in "general positions". The definition is as follows.

DEFINITION 2.1.1. A set $\{a^0, \ldots, a^n\}$ of $(n+1)$ points in $\mathbb{R}^m$ is called *geometrically independent* if the vectors $a^1 - a^0, a^2 - a^0, \ldots, a^n - a^0$ are linearly independent.

Observe that for $(n+1)$ geometrically independent points we should have the inequality $m \geq n$. We may assume that m is sufficient large for having geometrically independent points. In our setting, m is allowed to be infinite. Here the infinite dimensional Euclidean space $\mathbb{R}^\infty$ is the union of the sequence of finite dimensional space

$$\mathbb{R}^1 \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3 \subseteq \mathbb{R}^4 \subseteq \cdots \mathbb{R}^m \subseteq \mathbb{R}^{m+1} \subseteq \cdots,$$

where R^n is considered as the subspace of $\mathbb{R}^{n+1}$ in the canonical way, that is, $\mathbb{R}^n = \{(x_1, \ldots, x_n, 0) \in \mathbb{R}^{n+1}\} \subseteq \mathbb{R}^{n+1}$. In other words,

$$\mathbb{R}^\infty = \bigcup_{k=1}^{\infty} \mathbb{R}^k.$$

By writing down the coordinates, each $x \in \mathbb{R}^\infty$ has the coordinates $x = (x_1, x_2, \ldots)$ with all $x_i = 0$ except **finitely many coordinates**. Thus for $x, y \in \mathbb{R}^\infty$ the distance

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots}$$

is well-defined and defines a metric on $\mathbb{R}^\infty$. Note that for any finite points $a^0, a^1, \ldots, a^n$, there exists m such that all of a^i lie in $\mathbb{R}^m$.

PROPOSITION 2.1.2. *Let $\{a^0, \ldots, a^n\}$ be a geometrically independent set in $\mathbb{R}^m$ and let $\{a^{i_0}, \ldots, a^{i_r}\}$ be a subset of $\{a^0, \ldots, a^n\}$. Then $\{a^{i_0}, \ldots, a^{i_r}\}$ is also geometrically independent.*

PROOF. From the definition, we need to show that the vector $a^{i_1} - a^{i_0}, a^{i_2} - a^{i_0}, \ldots, a^{i_r} - a^{i_0}$ are linearly independent. Observe that

$$a^{i_s} - a^{i_0} = (a^{i_s} - a^0) - (a^{i_0} - a^0) \tag{2.1.1}$$

for $1 \leq s \leq r$. Since $a^1 - a^0, a^2 - a^0, \ldots, a^n - a^0$ are linearly independent, the vectors $a^{i_1} - a^0, a^{i_2} - a^0, \ldots, a^{i_r} - a^0$ are linearly independent. By subtracting the common vector $a^{i_0} - a^0$, the vectors in Equation 2.1.1 are linearly independent and hence the result. □

Now given a geometrically independent set $\{a^0, \ldots, a^n\}$ in $\mathbb{R}^m$, we have an n-dimensional subspace of $\mathbb{R}^m$ centered at a_0 and spanned by the vectors

$$a^1 - a^0, a^2 - a^0, \ldots, a^n - a^0.$$

By cutting a region in this subspace, we obtain a simplex. The method is as follows. Given two points a and b, the line segment between a and b is given by the points

$$\{t_0 a + t_1 b \mid 0 \leq t_0, t_1 \leq 1,\ t_0 + t_1 = 1\}.$$

It is good to write the points in the line segment in this way as the vertices a and b can be written as $a = 1 \cdot a + 0 \cdot b$ and $b = 0 \cdot a + 1 \cdot b$. Similarly given three geometrically independent points a, b and c, the triangle spanned by a, b and c are given by the points

$$\{t_0 a + t_1 b + t_2 b \mid 0 \leq t_0, t_1, t_2 \leq 1,\ t_0 + t_1 + t_2 = 1\}.$$

In general, we have the following definition.

Definition 2.1.3. A *geometric n-simplex*

$$\sigma^n = \left\{ x = \sum_{i=0}^{n} t_i a^i \mid t_i \geq 0 \text{ and } \sum_{i=0}^{n} t_i = 1 \right\} \subseteq \mathbb{R}^m$$

with subspace topology, where $\{a^0, \ldots, a^n\}$ linearly independent. Sometimes we write $\sigma = a^0 a^1 \cdots a^n$ for meaning that σ is spanned by the vertices $a^0, a^1, \ldots, a^n$. The numbers t_i are uniquely determined by the point x, which are called *barycentric coordinates* of x of σ with respect to $a^0, \ldots, a^n$. The points a^i are called the *vertices* of σ^n. If $\{a^{i_0}, a^{i_1}, \ldots, a^{i_r}\}$ is a subset of $\{a^0, \ldots, a^n\}$ with $0 \leq i_0 < \cdots < i_r \leq n$, then the subspace

$$\tau^r = \left\{ \sum_{j=0}^{r} t_j a^{i_j} \mid t_j \geq 0 \text{ and } \sum_{i=0}^{n} t_j = 1 \right\}$$

is call a *face* of σ^n. The number n is called the *dimension* of σ. A face of σ different from σ itself is called a *proper face*. The union of all proper faces of σ is called the *boundary* of σ, denoted by $\partial\sigma$. The *interior* $\mathrm{Int}(\sigma)$ of σ is defined by

$$\mathrm{Int}(\sigma) = \sigma \smallsetminus \partial\sigma,$$

sometimes called *open simplex*.

Proposition 2.1.4. *The barycentric coordinates $t_i(x)$ of x with respect to $a^0, \ldots, a^n$ are continuous on x.*

Proof. We may assume that $a^0, \ldots, a^n$ lie in $\mathbb{R}^m$ with m a sufficiently large finite number. By definition, the functions $t_i(x)$ are defined by the equation

$$x = \sum_{i=0}^{n} t_i(x) a^i.$$

Since the vectors $v^1 = a^1 - a^0, v^2 = a^2 - a^0, \ldots, v^n = a^n - a^0$ are linearly independent, we can extend it to be a basis $v^1, v^2, \ldots, v^n, v^{n+1}, \ldots, v^m$. Here $v^{n+1}, \ldots, v^m$ only depends on $v^1, \ldots, v^n$, which are independent on the points x. Consider

$$
\begin{aligned}
x - a^0 &= (t_0a^0 + t_1a^1 + \cdots + t_na^n) - (t_0a^0 + t_1a^0 + \cdots + t_na^n) \\
&\qquad \text{because } \textstyle\sum_{i=0}^n t_i = 1 \\
&= t_1(a^1 - a^0) + \cdots + t_n(a^n - a^0) \qquad (2.1.2)\\
&= t_1v^1 + t_2v^2 + \cdots + t_nv^n \\
&= t_1v^1 + t_2v^2 + \cdots + t_nv^n + 0v^{n+1} + \cdots + 0v^m.
\end{aligned}
$$

Write down this vector equation in terms of coordinates in $\mathbb{R}^m$. Let $x = (x_1, \ldots, x_m)$, $a^0 = (a^0_1, \ldots, a^0_m)$ and $v^i = (v^i_1, \ldots, v^i_m)$. Then we have the linear equation

$$(x_1 - a^0_1, \ldots, x_m - a^0_m) = (t_1, \ldots, t_n, 0, \ldots, 0)(v^i_j)_{m\times m}.$$

Since $v^1, \ldots, v^m$ form a basis for $\mathbb{R}^m$, the matrix (v^i_j) is invertible. Thus

$$(t_1, \ldots, t_n, 0, \ldots, 0) = (x_1 - a^0_1, \ldots, x_m - a^0_m)(v^i_j)^{-1}$$

and so each $t_i(x)$ is continuous on x for $1 \le i \le n$. Since

$$t_0(x) = 1 - t_1(x) - \cdots - t_n(x),$$

$t_0(x)$ is also continuous on x. This finishes the proof. □

The proof also explains why the barycentric coordinates are unique as one can solve t_i using linear equations. The general points x in the subspace spanned by the points $a^0, \ldots, a^n$ may have negative $t_i(x)$. But, from geometric observation (or just from our definition of simplex), the points x in σ must have $0 \le t_i(x) \le 1$ with $\sum_{i=0}^n t_i(x) = 1$.

PROPOSITION 2.1.5. *Let σ be an n-simplex. Then $x \in \partial\sigma \iff t_i(x) = 0$ for some $0 \le i \le n$. Thus $x \in \mathrm{Int}(\sigma) \iff t_i(x) > 0$ for all $0 \le i \le n$.*

PROOF. If $x \in \tau^r$ a proper face of σ^n spanned by $\{a^{i_0}, a^{i_1}, \ldots, a^{i_r}\}$. Then x has unique barycentric coordinate expression

$$x = \sum_{j=0}^{r} s_j a^{i_j}.$$

Since $x = \sum_{i=0}^n t_i(x)a^i$, we have $t_{i_j} = s_j$ and $t_i = 0$ for $i \notin \{i_0, i_1, \ldots, i_r\}$. Since $r < n$, there exists $t_i = 0$ for some $0 \le i \le n$. It follows that $x \in \partial\sigma \implies t_i(x) = 0$ for some $0 \le i \le n$.

Conversely if $t_i(x) = 0$ for some $0 \le i \le n$, then x lies in the face of σ spanned by $\{a^0, \ldots, a^{i-1}, a^{i+1}, \ldots, a^n\}$. The proof is finished. □

DEFINITION 2.1.6. Let $\sigma^n = a^0a^1\cdots a^n$ and $\tau^m = b^0b^1\cdots b^m$ be two simplices in $\mathbb{R}^m$. A function $f\colon \sigma \to \tau$ is called *a simplicial map* if

(1). f sends each vertex of σ to a vertex of τ, that is $f(a^i) \in \{b^0, \ldots, b^m\}$ for $0 \le i \le n$.

(2). For $x = \sum\limits_{i=0}^{n} t_i(x)a^i \in \sigma$, $f(x) = \sum\limits_{i=0}^{n} t_i f(a^i)$.

In the first condition, we allow that f sends two or more vertices to one vertex of τ. By the second condition, the simplicial map is determined by its values on vertices because $f(x)$ must be given in the form as in Condition 2. This condition just states that f must be linear.

EXAMPLE 2.1.1. Let $\sigma = a^0a^1a^2$ be a triangle and let $\tau = b^0b^1b^2b^3$. Then we have simplicial maps f like

(1). $f(a^0) = b^0$, $f(a^1) = b^0$, $f(a^2) = b^1$;
(2). $f(a^0) = b^1$, $f(a^1) = b^0$, $f(a^2) = b^1$;
(3). $f(a^0) = f(a^1) = f(a^2) = b^3$;
(4). $f(a^0) = b^1, f(a^1) = b^3, f(a^2) = b^0$.

Given two simplices σ and τ. Let $\mathrm{Hom}(\sigma, \tau)$ denote the set of all simplicial maps from σ to τ.

PROPOSITION 2.1.7. *The simplicial maps from $\sigma = a^0a^1 \cdots a^n$ to $\tau = b^0b^1 \cdots b^k$ have the following properties:*

(1). *Any simplicial map $f\colon \sigma \to \tau$ is continuous.*
(2). *Let S be the set of all functions from $\{a^0, \ldots, a^n\}$ to $\{b^0, \ldots, b^k\}$. Then there is an isomorphism of sets:* $\mathrm{Hom}(\sigma, \tau) \cong S$.

PROOF. The first assertion follows from the definition of simplicial map that $f(x) = \sum_{i=0}^{n} t_i(x)f(a^i)$ with $t_i(x)$ continuous on x by Proposition 2.1.5. The second assertion also follows from the definition of simplicial map that the simplicial map are uniquely determined by its values on vertices. □

The *standard n-simplex* is defined by

$$\Delta^n = e^0e^1 \cdots e^n = \{(t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \mid 0 \le t_i \le 1 \text{ and } \sum_{i=0}^{n} t_i = 1\},$$

where the vertices $e^0 = (1, 0, 0, \ldots, 0), e^1 = (0, 1, 0, \ldots, 0), \ldots, e^n = (0, 0, \ldots, 0, 1)$. Geometrically we can see that each n-simplex σ^n is just a copy of Δ^n. More precisely, let us define a linear embedding of Δ^n into $\mathbb{R}^m$ as follows: A function $f\colon \Delta^n \to \mathbb{R}^m$ is called a *linear embedding* if

(1). $\{f(e^0), f(e^1), \ldots, f(e^n)\}$ is geometrically independent in $\mathbb{R}^m$ and
(2). $f(x) = \sum\limits_{i=0}^{n} t_i(x)f(e^i)$ for $x = \sum\limits_{i=0}^{n} t_i(x)e^i \in \Delta^n$.

Let $\mathrm{LEmb}(\Delta^n, \mathbb{R}^m)$ be the set of all linear embeddings from Δ^n into $\mathbb{R}^m$. Given a linear embedding $f\colon \Delta^n \to \mathbb{R}^m$. Then we have an n-simplex $\sigma^n = f(e^0)f(e^1) \cdots f(e^n)$ given by the image of Δ^n under f. Let $g\colon \Delta^n \to \mathbb{R}^m$ be another linear embedding determined by

$$g(e^i) = f(e^{\sigma(i)})$$

for $\sigma \in \Sigma_{n+1}$, the symmetric group, permuting the indices $0, 1, \ldots, n$. Then

$$g(x) = \sum_{i=0}^{n} t_i(x)f(e^{\sigma(i)})$$

for $x = \sum\limits_{i=0}^{n} t_i(x)e^i$. The image $g(\Delta^n) = g(e^0)g(e^1) \cdots g(e^n) = e^{\sigma(0)}e^{\sigma(1)} \cdots e^{\sigma(n)}$ with the same n-simplex $\sigma = f(\Delta^n)$ with vertices relabeled. Let $\mathrm{Simx}_n(\mathbb{R}^m)$ denote the set of all n-simplices in $\mathbb{R}^m$, where two n-simplices are regarded as the same if their underline spaces are the same. From this setting, we obtain the following proposition:

Proposition 2.1.8. *Let the symmetric group Σ_{n+1} act on* $\mathrm{LEmb}(\Delta^n, \mathbb{R}^m)$ *induced by permuting the vertices of Δ^n. Then for any $n < m$, there is an isomorphism*

$$\mathrm{LEmb}(\Delta^n, \mathbb{R}^m)/\Sigma_{n+1} \cong \mathrm{Simx}_n(\mathbb{R}^m)$$

given by $f \mapsto f(e^0)f(e^1)\dots f(e^n)$. □

This proposition gives a view for considering an n-simplex as a linear embedding from Δ^n into $\mathbb{R}^m$. Intuitively this is more difficult to image what an n-simplex looks like. However this new treatment will help us to construct *simplicial complexes with labels in a topological space* which we will discuss in details later.

2.2. Geometric Simplicial Complex. Roughly speaking, a geometric simplicial complex is a geometric object consisting of a collection of simplices. The rule for being a simplicial complex requires (1) every face of a simplex in the complex should still lie in the complex; and (2) the intersection of any two simplices is either empty or a common face. For having (geometric) simplicial complexes with arbitrary many simplices and arbitrary dimension of its simplices, we need to extend our m-dimensional Euclidean space to an infinite dimensional vector space as follows:

Let J be an arbitrary index set, and let $\mathrm{Map}(J, \mathbb{R})$ be the J-fold Cartesian product of $\mathbb{R}$ with itself. An element in $\mathrm{Map}(J, \mathbb{R})$ is a function from J to $\mathbb{R}$ denoted by $x = (x_\alpha)_{\alpha \in J}$. The set $\mathrm{Map}(J, \mathbb{R})$ is a vector space with addition given by the usual component-wise addition and multiplication by scalars, that is, $(x + y)_\alpha = x_\alpha + y_\alpha$ and $(cx)_\alpha = cx_\alpha$ for $\alpha \in J$.

Let $\mathbb{R}^J$ be the subset of $\mathrm{Map}(J, \mathbb{R})$ consisting of all points $(x_\alpha)_{\alpha \in J}$ such that $x_\alpha = 0$ for **all but finitely many values** of α. Then $\mathbb{R}^J$ is a vector subspace of $\mathrm{Map}(J, \mathbb{R})$ with a basis given by e^α for $\alpha \in J$, where e^α is the map from J to $\mathbb{R}$ with $e^\alpha(\alpha) = 1$ and $e^\alpha(\beta) = 0$ for $\beta \neq \alpha$. If $J = \mathbb{N}$ the set of natural numbers, then $\mathbb{R}^N = \mathbb{R}^\infty$ defined in the previous subsection.

Note. $\{e^\alpha\}_{\alpha \in J}$ does NOT form a basis for $\mathrm{Map}(J, \mathbb{R})$ if J is an infinite set. Since $\{e^\alpha_{\alpha \in J}\}$ is a basis for $\mathbb{R}^J$, we can also consider that $\mathbb{R}^J$ is the vector subspace of $\mathbb{R}^J$ generated by e^α for $\alpha \in J$. In other words, $\mathbb{R}^J$ is the smallest vector subspace of $\mathrm{Map}(J, \mathbb{R})$ containing e^α for each $\alpha \in J$. The space $\mathbb{R}^J$ has a *metric* defined by setting

$$|x - y| = \sqrt{|x_\alpha - y_\alpha|^2_{\alpha \in J}}.$$

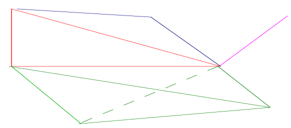
Here the above formula is well-defined for vector $x, y \in \mathbb{R}^J$ because $x_\alpha = y_\alpha = 0$ except finitely many indices α.

Now we give a mathematical definition of geometric simplicial complex.

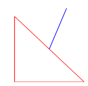
Definition 2.2.1. A *geometric simplicial complex* K is a collection of simplices, all contained in some Euclidean space $\mathbb{R}^J$ for some index set J such that

(1). if σ^n is a simplex in K and τ^p is a face of σ^n, then τ^p is in K; and
(2). if σ^n and τ^p are simplices of K, then $\sigma^n \cap \tau^p$ is either empty, or a common face of σ^n and τ^p.

In the following pictures, the left hand-side one is a simplicial complex but the right hand-side one is not a simplicial complex.



Simplicial complex



Not a simplicaial complex

Proposition 2.2.2. *Let K be a simplicial complex and let $x \in \bigcup_{\sigma \in K} \sigma$. Then there exists a unique simplex σ of K such that $x \in \mathrm{Int}(\sigma)$.*

Proof. Let σ be a simplex such that

$$\dim \sigma = \min\{\dim \tau \mid x \in \tau \text{ and } \tau \in K\}.$$

Namely σ is a minimal dimensional simplex of K that contains x. Then x does not lie in any face of σ because otherwise there would a simplex (as a face of σ) containing x with dimension strictly less than $\dim \sigma$. It follows that $x \in \mathrm{Int}(\sigma)$.

For proving the uniqueness, let τ be any simplex of K such that $x \in \mathrm{Int}(\tau)$. Then $\sigma \cap \tau \neq$ as both of them contains x and so $\sigma \cap \tau$ is a common face μ of σ and τ by definition. Since

$$x \in \mathrm{Int}(\sigma) \cap \mathrm{Int}(\tau) \subseteq \sigma \cap \tau = \mu,$$

μ can not be a proper face of σ because any proper face of σ disjoins with the interior of σ. Thus $\mu = \sigma$. Similarly $\mu = \tau$ and so $\tau = \sigma$. The proof is finished. □

Definition 2.2.3. Let K be a simplicial complex.

(1). The *dimension* of a simplicial complex K is defined to be

$$\dim K = \sup\{\dim \sigma \mid \sigma \text{ is a simplex } of K\}.$$

So an n-dimensional simplicial complex means a simplicial complex without simplices of dimension higher than n. Note that it is possible to have $\dim K = \infty$. In such a case, for any given n, K has simplices which dimensions greater than or equal to n. If $\dim K < \infty$, we call K a *finite dimensional simplicial complex*.

(2). If L is a sub-collection of K that contains all faces of its elements, then L is a simplicial complex in its own right, called a *subcomplex* of K.

(3). One special subcomplex of K is the collection of all simplices of K dimension at most n, called the *n-skeleton* of K denoted by $\mathrm{sk}_n K$.

(4). The points of the collection $\mathrm{sk}_0 K$ are called *vertices* of K.

According to the definition, all simplices of K must be located in $\mathbb{R}^J$. Let

$$|K| = \bigcup_{\sigma^n \in K} \sigma^n \subseteq \mathbb{R}^J$$

be the union of all simplices of K. This gives a subset of $\mathbb{R}^J$ and so one has the subspace topology on $|K|$. For our purpose for having continuity property of simplicial maps which will be discussed later, we define a new topology on $|K|$:

DEFINITION 2.2.4. Let K be a simplicial complex and let $|K|$ be the union of its simplices. Each simplex σ has its natural subspace topology in $\mathbb{R}^J$. (**Note.** If J is an infinite set, for each simplex σ, there exists a large finite number $m >> 0$ such that $\sigma \subseteq \mathbb{R}^m \subseteq \mathbb{R}^J$. Clearly the subspace topology of σ in $\mathbb{R}^m$ is the same as subspace topology in $\mathbb{R}^J$ as both topologies are induced by the standard metric given by distance.) Then the *weak topology* on $|K|$ is defined by requiring that

a subset A of K is closed if and only if $A \cap \sigma$ is closed in σ for each $\sigma \in K$.

PROPOSITION 2.2.5. *Let K be a simplicial complex and let $|K|$ have the weak topology. Then a subset U is open in $|K|$ if and only if $U \cap \sigma$ is open in σ for each $\sigma \in K$.*

PROOF. Suppose that U is open. Then $A = |K| \smallsetminus U$ is closed. Let $\sigma \in K$. Then

$$\sigma \cap U = \sigma \smallsetminus \sigma \cap A$$

is open as $\sigma \cap A$ is closed.

Conversely let U be a subset of $|K|$ such that $U \cap \sigma$ is open in σ for each $\sigma \in K$. Let $A = |K| \smallsetminus U$. Then

$$\sigma \cap A = \sigma \smallsetminus \sigma \cap U$$

is closed for each $\sigma \in K$. Thus A is closed and so $U = |K| \smallsetminus A$ is open. □

The weak topology on $|K|$ is different from the subspace topology on $|K|$ in general.

EXAMPLE 2.2.1. Let $K = \{t \mid 0 \leq t \leq 1\}$ be the simplicial complex with 0-simplices $\{t\} \subseteq \mathbb{R}$ labeled by $0 \leq t \leq 1$. Then

$$|K| = [0, 1]$$

as the subsets of $\mathbb{R}$. But the topology on $|K|$ is different from the interval $[0, 1]$. We claim that the topology of $|K|$ is actually discrete. Let A be any subset of $|K| = [0, 1]$. For each simplex $\{t\}$, Then $A \cap \{t\} = \{t\}$ or $\emptyset$. In each case, $A \cap \{t\}$ is closed. From the definition of the topology on $|K|$, A is closed. It follows that $|K|$ has discrete topology. □

EXAMPLE 2.2.2. Let σ_n^2 be a 2-simplex spanned by the points

$$(0, 1), (1/(n + 1), 0) \text{ and } (1/n, 0)$$

in $\mathbb{R}^2$ and let τ be the 1-simplex spanned by the points $(0, 0)$ and $(0, 1)$. Let K be the collection of τ and its faces, and σ_n^2 and its faces for $n = 1, 2, 3, \ldots$. Then K is a simplicial complex with $|K|$ is the triangle spanned by $(0, 0), (0, 1)$ and $(1, 0)$.

Let $A = \{(1/n, 0) \mid n = 1, 2, 3, \cdots\}$. Then A is closed in $|K|$ under weak topology because A intersects with each simplex is closed.

But, for subspace topology, A is not closed because the closure of A is given by $\{(0, 0), (1/n, 0) \mid n = 1, 2, 3, \ldots\}$. □

However under certain hypothesis, the weak topology on $|K|$ coincides with the subspace topology. A *finite simplicial complex* means a simplicial complex which has only finitely many simplices.

PROPOSITION 2.2.6. *Let K be a finite simplicial complex. Then the weak topology on $|K|$ coincides with the subspace topology on $|K|$.*

PROOF. Suppose that U is open under subspace topology. Then $U \cap \sigma$ is open in σ for each simplex σ because $\sigma \subseteq |K|$ is a subspace. Thus U is open under the weak topology.

Conversely suppose that U is open under the weak topology. Let $x \in U$. Since K has only finitely many simplices, we may assume that $K = \{\sigma_1, \ldots, \sigma_s, \tau_1, \ldots, \tau_t\}$ with $x \in \sigma_i$ and $x \notin \tau_j$ for $1 \leq i \leq s$ and $1 \leq j \leq t$. For each $1 \leq i \leq s$, since $x \in U \cap \sigma_i$ with the property that $U \cap \sigma_i$ is open in σ_i, there exists $\epsilon_i > 0$ such that the open ball

$$\operatorname{Int}(D_{\epsilon_i}(x)) \cap \sigma_i \subseteq U \cap \sigma_i,$$

where $\operatorname{Int}(D_\epsilon(x)) = \{y \in \mathbb{R}^J \mid d(x,y) < \epsilon\}$. Since $x \notin \bigcup\limits_{j=1}^{t} \tau_j$ with the property that $\bigcup_{j=1}^{t} \tau_j$ is closed, there exists $\epsilon_0 > 0$ such that

$$\operatorname{Int}(D_{\epsilon_0}(x)) \cap \left(\bigcup_{j=1}^{t} \tau_j \right) = \emptyset.$$

Let $\epsilon = \min\{\epsilon_0, \epsilon_1, \ldots, \epsilon_s\}$. Then

$$\begin{array}{rcl} \operatorname{Int}(D_\epsilon(x)) \cap |K| & = & \operatorname{Int}(D_\epsilon(x)) \cap \left(\bigcup_{i=1}^{s} \sigma_i \cup \bigcup_{j=1}^{t} \tau_j \right) \\ & = & \operatorname{Int}(D_\epsilon(x)) \cap \left(\bigcup_{i=1}^{s} \sigma_i \right) \\ & & \text{ because } \epsilon \leq \epsilon_0 \\ & = & \bigcup_{i=1}^{s} \operatorname{Int}(D_\epsilon(x)) \cap \sigma_i \\ & \subseteq & \bigcup_{i=1}^{s} U \cap \sigma_i \\ & \subseteq & U. \end{array}$$

Thus U is open under subspace topology. The proof is finished. □

In the theory of simplicial complexes, we always assume that $|K|$ is a topological space with weak topology. As we have seen above, this topology coincides with subspace topology when K is a finite simplicial complex. But the weak topology is pretty different from the subspace topology in general. An advantage of weak topology is as follows:

PROPOSITION 2.2.7. *Let K be a simplicial complex and let X be a topological space. Then a function $f\colon |K| \to X$ is continuous (under the weak topology of $|K|$) if and only if f restricted to each simplex of K is continuous.*

PROOF. If $f\colon |K| \to X$, then clearly $f|_\sigma\colon \sigma \to X$ is continuous for any simplex σ of K.

Conversely, suppose that $f|_\sigma\colon \sigma \to X$ is continuous for any simplex σ of K. Let U be an open subset of X. Then, for any simplex σ of K,

$$f^{-1}(U) \cap \sigma = f|_\sigma^{-1}(U)$$

is open in σ because $f|_\sigma$ is continuous. From the definition of weak topology, $f^{-1}(U)$ is open. Thus f is continuous. □

DEFINITION 2.2.8. A topological space X is called a *polyhedron* if there exists a simplicial complex K such that X is homeomorphic to $|K|$. In this case, the simplicial complex K is called a *triangulation* of X.

EXAMPLE 2.2.3. The unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ is homeomorphic to the boundary of Δ^3. Let K be the simplicial complex consisting of all proper faces of Δ^3. Then $S^2 \cong |K|$. Thus S^2 is a polyhedron.

2.3. Simplicial Maps. We have known simplicial maps between simplices. This concept can be extended for simplicial complexes as follows:

DEFINITION 2.3.1. Given simplicial complexes K and L, a function

$$f\colon |K| \to |L|$$

is called a *simplicial map* if it satisfies the following conditions:

(1). If a is a vertex of K, then $f(a)$ is a vertex of L.
(2). If $a^0a^1 \cdots a^n$ is a simplex of K, then $f(a^0), f(a^1), \ldots, f(a^n)$ span a simplex of L (possibly with repeats).
(3). If $x = \sum_{i=0}^n t_i a^i$ is a point in a simplex $a^0a^1 \cdots a^n$ of K, then

$$f(x) = \sum_{i=0}^n t_i f(a^i).$$

That is f is *linear* on each simplex.

From the definition, for having a simplicial map, one needs:

(1). a function f which sends the vertices of K to vertices of L such that
(2). Whenever $a^0, a^1, \ldots, a^n$ span a simplex of K, $f(a^0), f(a^1), \ldots, f(a^n)$ spans a simplex of L.

PROPOSITION 2.3.2. *A simplicial map $f\colon |K| \to |L|$ is continuous.*

PROOF. The assertion follows from that f restricted to each simplex is continuous. □

EXAMPLE 2.3.1. Let $K = \{t \mid t \in [0, 1]\}$ be the simplicial complex of vertices labeled by $0 \le t \le 1$ and let $L = \{0, 1\}$ be the simplicial complex consisting of two vertices. Let $f\colon K \to L$ be the function with $f(t) = 0$ for $0 \le t \le 1/2$ and $f(t) = 1$ for $1/2 < t \le 1$. Then f is a simplicial map because f sends every simplex (only vertex) of K to a simplex of L. By using the weak topology, $f\colon |K| \to |L|$ is continuous. On the other hand, $f\colon |K| \to |L|$ is not continuous under subspace topology.

PROPOSITION 2.3.3. *Suppose that $f\colon \mathrm{sk}_0 K \to \mathrm{sk}_0 L$ is a bijective correspondence such that the vertices $a^0, a^1, \ldots, a^n$ spanned a simplex in K if and only*

$$f(a^0), f(a^1), \ldots, f(a^n)$$

spanned a simplex in L. Then the induced simplicial map $f\colon |K| \to |L|$ is a homeomorphism, called linear isomorphism or simplicial homeomorphism of K with L.

PROOF. From the assumption, the inverse $f^{-1}\colon L \to K$ is also a simplicial map and hence the result. □

PROPOSITION 2.3.4. *Let $f\colon K \to L$ be a simplicial map. Then*

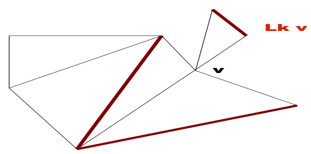
(1). *The image $f(K)$ is a simplicial subcomplex of L.*
(2). *The preimage $f^{-1}(L_0)$ is a simplicial subcomplex of K for any simplicial subcomplex L_0 of L.*

PROOF. Because both $f(K)$ and $f^{-1}(L_0)$ are closed under face operations. □

2.4. Stars and Links. An important concept is the star of a vertex in a simplicial complex.

DEFINITION 2.4.1. Let K be a simplicial complex and let v be a vertex of K. The *star* of v in K, denoted by $\mathrm{St}(v)$ or $\mathrm{St}(v, K)$, is the union of the interior of those simplices of K that have v as a vertex. Its closure, denoted by $\bar{\mathrm{St}}(v)$, is called the *closed star* of v in K. The set $\bar{\mathrm{St}}(v) \smallsetminus \mathrm{St}(v)$ is called the *link* of v in K, denoted by $\mathrm{Lk}(v)$.

A picture for the link of v is as follows:



PROPOSITION 2.4.2. *Let v be a vertex of K. Then the closed star $\bar{\mathrm{St}}(v)$ is the union of all simplices of K having v as a vertex.*

PROOF. Let A be the union of all simplices of K having v as a vertex. Write $A = \bigcup\limits_{\alpha} \sigma_\alpha$, where σ_α has v as a vertex. From the definition of star, we have $\mathrm{St}(v) \subseteq A$.

We first check that A is closed. Let τ be any simplex of K. Then

$$\begin{aligned} \tau \cap A &= \tau \cap (\textstyle\bigcup_\alpha \sigma_\alpha) \\ &= \textstyle\bigcup_\alpha \tau \cap \sigma_\alpha. \end{aligned}$$

From the definition of simplicial complex, $\tau \cap \sigma_\alpha$ is either empty or a common face of τ and σ_α. Thus $\tau \cap A$ is a union of some faces of τ. Since τ only has finite faces, $\tau \cap A$ is a finite union of closed subsets of τ. Hence $\tau \cap A$ is closed. By the definition of weak topology, A is closed.

Now let B be any closed set of $|K|$ such that $\mathrm{St}(v) \subseteq B$. Let σ be a simplex of K that has v as a vertex. Then from the definition of star

$$\mathrm{Int}(\sigma) \subseteq \mathrm{St}(v) \subseteq B.$$

It follows that the closure

$$\sigma = \overline{\mathrm{Int}(\sigma)} \subseteq \overline{B} = B$$

and so $A \subseteq B$.

This proves that A is the closure of $\mathrm{St}(v)$ and so $A = \bar{\mathrm{St}}(v)$. $\square$

The star can be measured by continuous functions defined as follows: Let v be a vertex in K and let x be a point in $|K|$. Then x is interior to precisely one simplex of K, whose vertices are (say) $a^0, a^1, \dots, a^n$. Then

$$x = \sum_{i=0}^{n} t_i a^i$$

with $t_i > 0$ and $\sum\limits_{i=0}^{n} t_i = 1$. We define the *barycentric coordinate* $t_v(x)$ of x with respect to v by setting

$$t_v(x) = \begin{cases} t_i & \text{if} \quad v = a^i \text{ for some } 0 \leq i \leq n \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 2.4.3. *Let K be a simplicial complex and let v be a vertex. Then $t_v \colon |K| \to \mathbb{R}$ is continuous.*

PROOF. Given any simplex σ of K, $t_v(x)|_\sigma$ is either identically 0 or the barycentric coordinate of x with respect to the vertex v of σ. Thus $t_v(x)|_\sigma$ is continuous. By the definition of the (weak) topology on $|K|$, $t_v(x)$ is continuous on $|K|$. □

PROPOSITION 2.4.4. *Let K be a simplicial complex and let v be a vertex. Then*

$$\mathrm{St}(v) = \{x \in |K| \mid t_v(x) > 0\}.$$

Thus $\mathrm{St}(v)$ is an open neighborhood of v in $|K|$.

PROOF. If $x \in \mathrm{St}(v)$, then there exists a simplex σ with $x \in \mathrm{Int}(\sigma)$ and σ having v as one of its vertices by definition. From the definition of t_v, we have $t_v(x) > 0$.

Conversely let $x \in |K|$ with $t_v(x) > 0$. Let σ be the simplex such that $x \in \mathrm{Int}(\sigma)$. From the definition of t_v, σ has v as one of its vertices. Thus $x \in \mathrm{St}(v)$. □

PROPOSITION 2.4.5 (Star Covering). *Let K be a simplicial complex. Then*

$$|K| = \bigcup_{v \in \mathrm{sk}_0 K} \mathrm{St}(v).$$

PROOF. For any $x \in |K|$, there exists a unique simplicial σ such that $x \in \mathrm{Int}(\sigma)$. Let v be a vertex of σ. Then $x \in \mathrm{St}(v)$ and hence the result. □

PROPOSITION 2.4.6. *Let $x \in \bar{\mathrm{St}}(v)$. Then the line segment xv lies in $\bar{\mathrm{St}}(v)$. Moreover the line segment starting from v meets $\mathrm{Lk}(v)$ exactly one point.*

PROOF. Let $x \in \bar{\mathrm{St}}(v)$. By Proposition 2.4.2, there exists a simplex σ of K such that $x \in \sigma$ and σ has v as a vertex. Since both x and v lie in σ, the line segment

$$xv \subseteq \sigma \subseteq \bar{\mathrm{St}}(v)$$

and hence the result. □

2.5. Subdivisions.

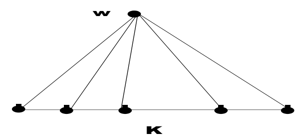
DEFINITION 2.5.1. Let K be a geometric simplicial complex in $\mathbb{R}^J$. A complex K' is called to be a *subdivision* of K if

(1). Each simplex of K' is contained in a simplex of K.
(2). Each simplex of K equals to the union of **finitely many** simplices of K'.

These conditions imply that the union of the simplices of K' is equal to the union of the simplices of K, that is, $|K| = |K'|$ as sets. The finiteness part of condition (2) guarantee that $|K'|$ and $|K|$ are equal as topological spaces.

DEFINITION 2.5.2. Suppose that K is a simplicial complex in $\mathbb{R}^J$, and w is a point in $\mathbb{R}^J$ such that each ray emanating from w intersects $|K|$ in at most one point. We define the *cone on K with vertex w* to be the collection of all simplices of the form $a^0a^1 \cdots a^p w$, where $a^0a^1 \cdots a^p$ is a simplex in K, along all faces of such simplices. Denote the cone by $K * w$.

From the definition, the vertices of $K * w$ consist of w and the vertices of K. The simplices of $K * w$ consists of the simplices of K together with the simplices of the form $a^0a^1 \cdots a^p w$ with $a^0a^1 \cdots a^p$ a simplex of K. The cone $K * w$ is pictured as follows:



DEFINITION 2.5.3. Let K be a simplicial complex. Suppose that L_p is a subdivision of $\operatorname{sk}_p K$. Let σ be a $(p+1)$-simplex of K. Note that $|\partial\sigma|$ is a polyhedron of a subcomplex of $\operatorname{sk}_p K$ and so it is a polyhedron of a subcomplex, denoted by L_σ, of L_p. If w_σ is an interior point of σ, then the cone $L_\sigma * w_\sigma$ is a simplicial complex whose underlying space is σ. We define L_{p+1} to be the union of L_p and the simplicial complexes $L_\sigma * w_\sigma$ as σ runs over all $(p+1)$-simplices of K. Then L_{p+1} is a simplicial complex, called *subdivision of $\operatorname{sk}_{p+1} K$ obtained by starring L_p from the points w_σ.*

DEFINITION 2.5.4. Let $\sigma = v^0v^1 \cdots v^n$ be an n-simplex. The *barycenter* of σ is the point

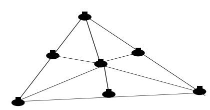
$$\hat{\sigma} = \sum_{i=0}^{n} \frac{1}{n+1} v^i,$$

that is $\hat{\sigma}$ is the point of $\operatorname{Int}(\sigma)$ all of those barycentric coordinates with respect to the vertices are equal.

If σ is 1-simplex, then $\hat{\sigma}$ is the midpoint. If σ is a 0-simplex, then $\hat{\sigma} = \sigma$. In general, $\hat{\sigma}$ is the centroid of σ.

DEFINITION 2.5.5. Let K be a simplicial complex. We define a sequence of subdivisions of the skeletons of K as follows: Let $L_0 = \operatorname{sk}_0 K$. Assume that L_p is defined as a subdivision of $\operatorname{sk}_p K$. Let L_{p+1} be the subdivision of $\operatorname{sk}_{p+1} K$ obtained by starring L_p from the barycenter of the $(p+1)$-simplices of K. The union $\bigcup_{p=0}^{\infty} L_p$ is a subdivision of K, called *barycentric subdivision* of K, denoted by $\operatorname{sd} K$. Define *the iterated barycentric subdivision* recursively by $\operatorname{sd}^n K = \operatorname{sd}^{n-1}(\operatorname{sd} K)$ for $n > 1$.

Let σ be a 2-simplex. Then $\operatorname{sd}\sigma$ is shown in the picture below:



The simplices in $\operatorname{sd} K$ can be described as follows. Define a partial order on the simplices of K by setting $\sigma_1 < \sigma_2$ if σ_1 is a proper face of σ_2.

PROPOSITION 2.5.6. *The simplicial complex* $\operatorname{sd} K$ *equals to the collection of all simplices of the form*

$$\hat{\sigma}_1\hat{\sigma}_2\cdots\hat{\sigma}_n,$$

where $\sigma_1 < \sigma_2 < \cdots < \sigma_n$ *in* K.

PROOF. The proof is given by induction on p that the assertion holds for $\operatorname{sk}_p K$ for each $p \geq 0$. The assertion holds for $\operatorname{sk}_0 K$ as $\operatorname{sd}\operatorname{sk}_0 K = \operatorname{sk}_0 K$. Suppose that the assertion holds for $\operatorname{sk}_p K$. By definition of barycentric subdivision, the assertion holds for $\operatorname{sk}_{p+1} K$. Since $\operatorname{sd} K = \bigcup\limits_p \operatorname{sd}\operatorname{sk}_p K$, the assertion follows. □

An important property of barycentric subdivision is as follows, which plays a key role for proving the simplicial approximation theorem in the next section. The *diameter* $\operatorname{diam}(\sigma)$ of a simplex σ means the length $l = \max\{|x-y| \mid x, y \in \sigma\}$. For any simplicial complex K, let

$$\operatorname{mesh}(K) = \sup\{\operatorname{diam}(\sigma) \mid \sigma \text{ is a simplex of } K\}.$$

If K is a finite complex, then $\operatorname{mesh}(K) < \infty$ which is the maximum diameter of the simplices of K. If K has infinite simplices, it is possible that $\operatorname{mesh}(K) = \infty$.

THEOREM 2.5.7. *Let K be a finite dimensional simplicial complex. Suppose that* $\operatorname{mesh}(K) < \infty$. *Then for any $\epsilon > 0$, there is a positive integer N such that* $\operatorname{mesh}(\operatorname{sd}^N K) < \epsilon$.

PROOF. **1.** *If $\sigma = a^0a^1\cdots a^n$ is a simplex, then the diameter*

$$\operatorname{diam}(\sigma) = \max\{|a^i - a^j| \mid 0 \leq i \leq j \leq n\},$$

the maximum distance between vertices.

Let $l = \max\{|a^i - a^j| \mid 0 \leq i < j \leq n\}$. For each vertex a^i, let

$$D(a^i, l) = \{x \mid |x - a^i| \leq l\}$$

be the ball of radius l centered at a^i. Then $D(a^i, l)$ is convex. Since $D(a^i, l)$ contains all vertices of σ, $\sigma \subseteq D(a^i, l)$. Thus

$$|x - a^i| \leq l$$

for any $x \in \sigma$ and each $0 \leq i \leq n$.

Now given any $x \in \sigma$, consider the ball $D(x, l) = \{y \mid |y - x| \leq l\}$. Then $\sigma \subseteq D(x, l)$ because each $a^i \in D(x, l)$ as $|a^i - x| \leq l$. Thus, for any $z \in \sigma$, $|x - z| \leq l$. Hence $\operatorname{diam} \sigma = l$.

2. *If $\sigma = a^0 a^1 \cdots a^n$ is a simplex, then for any $x \in \sigma$*

$$|\hat{\sigma} - x| \leq \frac{n}{n+1} \operatorname{diam}(\sigma).$$

Note that for each a^t

$$\begin{aligned}
|a^t - \hat{\sigma}| &= \left| a^t - \sum_{i=0}^{n} \tfrac{1}{n+1} a^i \right| \\
&= \left| \sum_{\substack{0 \leq i \leq n \\ i \neq t}} \tfrac{1}{n+1} (a^t - a^i) \right| \\
&\leq \sum_{\substack{0 \leq i \leq n \\ i \neq t}} \tfrac{1}{n+1} |a^t - a^i| \\
&\leq \tfrac{n}{n+1} \operatorname{diam}(\sigma).
\end{aligned}$$

Let $l' = \max\{|a^t - \hat{\sigma}| \mid 0 \leq t \leq n\}$. Then $l' \leq \frac{n}{n+1} \operatorname{diam}(\sigma)$. The ball

$$D(\hat{\sigma}, l') = \{x \mid |x - \hat{\sigma}| \leq l'\}$$

contains all vertices of σ and so $\sigma \subseteq D(\hat{\sigma}, l')$. It follows that

$$|x - \hat{\sigma}| \leq l' \leq \frac{n}{n+1} \operatorname{diam}(\sigma)$$

for any $x \in \sigma$.

3. *Let K be an n-dimensional finite simplicial complex. Then*

$$\operatorname{mesh}(\operatorname{sd} K) \leq \frac{n}{n+1} \operatorname{mesh}(K).$$

The proof is given by induction on the skeleton $\operatorname{sk}_p K$. For $\operatorname{sk}_0 K$, we have $\operatorname{mesh}(\operatorname{sd} \operatorname{sk}_0 K) = \operatorname{mesh}(\operatorname{sk}_0 K) = 0$. Suppose that

$$\operatorname{mesh}(\operatorname{sd} \operatorname{sk}_p K) \leq \frac{p}{p+1} \operatorname{mesh}(\operatorname{sk}_p K).$$

Consider $\operatorname{sk}_{p+1} K$. By the definition, $\operatorname{sd} \operatorname{sk}_{p+1} K$ is the union of $\operatorname{sd} \operatorname{sk}_p K$ and the simplices of the form $\tau * \hat{\sigma}$ for $(p+1)$-simplices σ of K, where τ is a simplex of $\operatorname{sd} \partial \sigma$. If σ' is a simplex of $\operatorname{sd} \operatorname{sk}_p K$, then

$$\operatorname{diam}(\sigma') \leq \frac{p}{p+1} \operatorname{mesh}(\operatorname{sk}_p K) \leq \frac{p}{p+1} \operatorname{mesh}(\operatorname{sk}_{p+1} K) \leq \frac{p+1}{p+2} \operatorname{mesh}(\operatorname{sk}_{p+1} K).$$

If $\sigma' = \tau * \hat{\sigma}$ with τ a simplex of $\operatorname{sd} \partial \sigma$, then, by Step 2,

$$|v - \hat{\sigma}| \leq \frac{p+1}{p+2} \operatorname{diam}(\sigma) \leq \frac{p+1}{p+2} \operatorname{mesh}(\operatorname{sk}_{p+1} K)$$

for any vertex v of τ and, by induction,

$$|v-w| \leq \frac{p}{p+1}\operatorname{mesh}(\operatorname{sk}_{p+1} K) \leq \frac{p+1}{p+2}\operatorname{mesh}(\operatorname{sk}_{p+1} K)$$

for any vertices v and w of τ. Thus

$$\operatorname{diam}(\sigma') \leq \frac{p+1}{p+2}\operatorname{mesh}(\operatorname{sk}_{p+1} K)$$

by Step 1. The induction is finished and hence the statement.
4. Let $\dim K = n$ and let $d = \operatorname{mesh}(K)$. Then

$$\operatorname{mesh}(\operatorname{sd}^N K) \leq \left(\frac{n}{n+1}\right)^N d.$$

Thus $\operatorname{mesh}(\operatorname{sd}^N K) \to 0$ as $N \to \infty$ and hence the result. □

Corollary 2.5.8. *Let K be a finite simplicial complex. Then for any $\epsilon > 0$, there is a positive integer N such that the diameters of any simplices of $\operatorname{sd}^N K$ are less than ϵ.* □

In practice it may be necessary to subdivide only part of a simplicial complex K, so as to leave alone a given subcomplex A. For doing subdivision in this case, we have the relative skeleton filtration defined as follows: Let $\operatorname{sk}^A_{-1} K = A$ and let $\operatorname{sk}^A_n K$ be the union of A and the simplices σ of K with $\dim \sigma \leq n$.

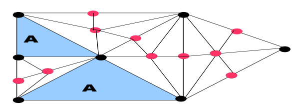
Definition 2.5.9. Let K be a geometric simplicial complex in $\mathbb{R}^J$ and let A be a subcomplex of K. A complex K' is called to be a *subdivision of K relative A* if

(1). Each simplex of K' is contained in a simplex of K.
(2). Each simplex of K equals to the union of **finitely many** simplices of K'.
(3). A is a subcomplex of K'.

The last condition requires that the simplices of the subcomplex A do not get subdivision.

Definition 2.5.10. Let K be a simplicial complex and let A be a subcomplex of K. We define a sequence of subdivisions of the relative skeletons $\operatorname{sk}^A_n K$ of K as follows: Let $L_{-1} = A$ and let $L_0 = \operatorname{sk}^A_0 K$. Assume that L_p is defined as a subdivision of $\operatorname{sk}^A_p K$. Let L_{p+1} be the subdivision of $\operatorname{sk}_{p+1} K$ obtained by starring L_p from the barycenter of the $(p+1)$-simplices of K NOT in A. The union $\bigcup_{p=0}^{\infty} L_p$ is a subdivision of K, called *barycentric subdivision of K relative to A*, denoted by $\operatorname{sd}(K, A)$. Define *the iterated barycentric subdivision* recursively by $\operatorname{sd}^n(K, A) = \operatorname{sd}^{n-1}(\operatorname{sd}(K, A))$ for $n > 1$.

The barycentric subdivision relative to a subcomplex is shown by the picture:



Recall that $\tau < \sigma$ if and only if τ is a proper face of σ.

Proposition 2.5.11. *Let K be a simplicial complex and let A be a subcomplex. Then the vertices of* $\mathrm{sd}(K,A)$ *are the barycenters of the simplices of $K \smallsetminus A$, together with the vertices of A. Distinct points*

$$a^1, a^2, \ldots, a^q, \hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_m$$

(with $\dim \sigma_i \leq \dim \sigma_{i+1}$ for each i) span a simplex of $\mathrm{sd}(K,A)$ if and only if $a^1, a^2, \ldots, a^q$ span a simplex σ of A, σ_j is a simplex of $K \smallsetminus A$ for $1 \leq j \leq m$ with $\sigma < \sigma_1 < \sigma_2 < \cdots < \sigma_m$.

Note. If $m = 0$, then it just requires $a^1, \ldots, a^q$ that span a simplex of A. If $q = 0$, it just requires $\sigma_m > \sigma_{m-1} > \cdots > \sigma_1$ and in this case $\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_m$ forms a simplex disjoint from A.

Proof. The proof follows by induction on the relative skeleton $\mathrm{sk}_n^A K$. $\square$

2.6. Regular Neighborhoods. Let K be a simplicial complex. For any vertex v of K, we have an open neighborhood $\mathrm{St}(v)$ of v. For any simplicial subcomplex L of K, we have the following concept.

Definition 2.6.1. Let L be a simplicial subcomplex of K. The open set

$$N(L) = \bigcup \{\mathrm{St}(v) \mid v \text{ is a vertex of } L\}$$

is called the *regular neighborhood* of L in K.

Let L be a simplicial subcomplex of K. Define the function $t_L \colon |K| \to [0, \infty)$ by

$$t_L(x) = \sum_{v \in \mathrm{sk}_0(L)} t_v(x).$$

Given $x \in |K|$, there is a unique simplex $\sigma = a^0 \cdots a^q$ such that $x \in \mathrm{Int}(\sigma)$. From the definition, $t_v(x) > 0$ if and only if $v = a^i$ for some $0 \leq i \leq q$. Thus the above summation is a finite summation and so it is well-defined and t_L is continuous. Note that

$$\sum_{i=0}^{q} t_{a^i}(x) = 1.$$

The value t_L is part of $\sum\limits_{i=0}^{q} t_{a^i}(x)$, that is, the summation of $t_{a^i}(x)$ for $a^i \in L$. Thus

$$0 \leq t_L(x) \leq 1$$

for any $x \in |K|$.

PROPOSITION 2.6.2. *Let L be a simplicial subcomplex of K. Then $x \in N(L)$ if and only if $t_L(x) > 0$.*

PROOF. $t_L(x) > 0$ if and only if $t_v(x) > 0$ for some $v \in \mathrm{sk}_0(L)$, if and only if $x \in \mathrm{St}(v)$ for some $v \in \mathrm{sk}_0(L)$. $\square$

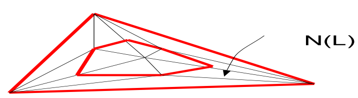
Let X be a space. A subspace A is called a *strong deformation retract* of X if there is a homotopy $F\colon X \times I \to X$ such that $F(x,0) = x$, $F(x,1) \in A$ for $x \in X$ and $F(a,t) = a$ for $0 \le t \le 1$. Equivalently A is a strong deformation retract of X if and only if the identity map id_X is homotopic to a self map $r\colon X \to X$ relative to A with $r(X) \subseteq A$.

We will discuss when $|L|$ is a strong deformation retract of $N(L)$.

EXAMPLE 2.6.1. Let $\sigma = a^0a^1a^2$ be a 2-simplex and let $L = \partial\sigma$ be the boundary of σ. Then $N(L) = \sigma$. In this case, $|L|$ is not a strong deformation retract of $N(L)$.

Let $K = \mathrm{sd}(\sigma, L)$ be the barycentric subdivision of σ. Then $N(L) = |K| \smallsetminus \{\hat{\sigma}\}$. In this case, $|L|$ is a strong deformation retract of $N(L)$. If we take the closure $\overline{N(L)}$ of $N(L)$ in $|K|$, then $\overline{N(L)} = |K| = \sigma$ and $|L|$ is not a strong deformation retract of $\overline{N(L)}$.

Let $K = \mathrm{sd}^2(\sigma, L)$. Then $|L|$ is a strong neighborhood retract of $\overline{N(L)}$ by the following picture.



$\square$

DEFINITION 2.6.3. A simplicial subcomplex L of K is called a *full subcomplex* if L contains all simplices $\sigma \in K$ whose vertices lie in L. Namely if all of the vertices of σ lie in L, then σ is a simplex of L.

It is possible that a simplicial subcomplex is not a full subcomplex. For instance, if K is an n-simplex σ with $n > 0$ and $L = \partial\sigma$, then L is not a full subcomplex of K.

PROPOSITION 2.6.4. *Let L be a full subcomplex and let $N(L)$ be the regular neighborhood of L in K. Then $|L|$ is a strong deformation retract of $N(L)$.*

PROOF. Define a map $r_L\colon N(L) \to |L|$ by setting

$$r_L(x) = \sum_{v \in \mathrm{sk}_0(L)} \frac{t_v(x)}{t_L(x)} v. \tag{2.6.1}$$

We check that r_L is a well-defined map. Since $x \in N(L)$, $t_L(x) > 0$ and so coefficients $t_v(x)/t_L(x)$ is well-defined. Given $x \in N(L) \subseteq |K|$, there is a unique simplex $\sigma = a^0a^1 \cdots a^q$ of K such that $x \in \mathrm{Int}(\sigma)$. We may assume that $a^0, a^1, \ldots, a^s \in L$ with $s \geq 0$ and $a^{s+1}, \ldots, a^q \notin L$. Then

$$r_L(x) = \sum_{i=0}^{s} \frac{t_{a^i}(x)}{t_L(x)} a^i$$

is a point that lies the simplex spanned by a^0, a^1,..., a^s. By the definition of full subcomplex, $a^0a^1 \cdots a^s$ is a simplex of L. Thus $r_L(x) \in |L|$. Clearly r_L is continuous.

If $x \in |L|$, then $t_L(x) = 1$ because $x \in \mathrm{Int}(\sigma)$ with $\sigma \in L$. In this case $r_L(x) = \sum\limits_{v \in \mathrm{sk}_0(L)} t_v(x)v = x$ by barycentric coordinates of σ. Thus $r_L|_{|L|} = \mathrm{id}_{|L|}$.

Define the linear homotopy $H\colon N(L) \times I \to N(L)$ by setting

$$H(x,t) = (1-t)x + tr_L(x) \tag{2.6.2}$$

for $x \in N(L)$ and $0 \leq t \leq 1$. We check that $H(x,t) \in N(L)$. Given $x \in N(L) \subseteq |K|$, there is a unique simplex $\sigma = a^0a^1 \cdots a^q$ of K such that $x \in \mathrm{Int}(\sigma)$. As in the previous paragraph, we may assume that $a^0, \ldots, a^s \in L$ with $s \geq 0$. From the definition, $r_L(x)$ is a point in the face $a^0a^1 \cdots a^s$ of the simplex $a^0a^1 \cdots a^q$. The line segment joining x and $r_L(x)$ also lies in $N(L)$. Thus $H(x,t) \in N(L)$. Now $H(x,0) = x$ for $x \in N(L)$, $H(x,1) = r_L(x) \in |L|$ for $x \in N(L)$. If $x \in |L|$, then $r_L(x) = x$ and so $H(x,t) = (1-t)x + tx = x$ for $0 \leq t \leq 1$. The proof is finished. $\square$

PROPOSITION 2.6.5. *Let L be a simplicial subcomplex of K. Then L is a full subcomplex of* $\mathrm{sd}(K,L)$.

PROOF. By Proposition 2.5.11, the simplices in $\mathrm{sd}(K,L)$ are given in the form

$$\tau = a^1a^2 \cdots a^q \hat{\sigma}_1 \hat{\sigma}_2 \cdots \hat{\sigma}_m,$$

where $a^1, a^2, \ldots, a^q$ span a simplex σ of L, σ_j is a simplex of $K \smallsetminus L$ for $1 \leq j \leq m$ with $\sigma < \sigma_1 < \sigma_2 < \cdots < \sigma_m$. If all of the vertices of τ lie in L, then $m = 0$ and so τ is a simplex of L. $\square$

By the above two propositions, we obtain the following theorem.

THEOREM 2.6.6. *Let L be a simplicial subcomplex of K. Then $|L|$ is a strong deformation retract of its regular neighborhood in* $\mathrm{sd}(K,L)$. $\square$.

Now we consider the closure $\overline{N(L)}$ of the regular neighborhood $N(L)$.

PROPOSITION 2.6.7. *Let L be a simplicial subcomplex of K. Then*

$$\overline{N(L)} = \bigcup_{v \in \mathrm{sk}_0(L)} \overline{\mathrm{St}}(v) = \bigcup \{\tau \mid \tau \textit{ has at least one vertex in } L\}.$$

Thus $\overline{N(L)}$ is the polyhedron of the simplicial subcomplex of K consisting of all of those simplices of K that are faces of the simplices with at least one of its vertices in L.

PROOF. Let v be a vertex of L. From $\mathrm{St}(v) \subseteq N(L)$, we have

$$\overline{\mathrm{St}}(v) \subseteq \overline{N(L)}$$

and so

$$N(L) \subseteq \bigcup_{v \in \mathrm{sk}_0(L)} \overline{\mathrm{St}}(v) \subseteq \overline{N(L)}.$$

We check that $\bigcup_{v \in \mathrm{sk}_0(L)} \overline{\mathrm{St}}(v)$ is closed. By Proposition [**?**], $\overline{\mathrm{St}}(v)$ is the union of the simplices τ with v as one of its vertices. Thus

$$\bigcup_{v \in \mathrm{sk}_0(L)} \overline{\mathrm{St}}(v) = \bigcup \{\tau \mid \tau \text{ has at least one vertex in } L\}.$$

Let σ be any simplex of K. Recall from the definition of simplicial complex that $\sigma \cap \tau = \emptyset$ or a common face of σ and τ for any simplex τ. Thus

$$\sigma \cap \bigcup_{v \in \mathrm{sk}_0(L)} \overline{\mathrm{St}}(v)$$

is a union of some faces of σ and so a closed subspace of σ. It follows that $\bigcup_{v \in \mathrm{sk}_0(L)} \overline{\mathrm{St}}(v)$ is closed under the weak topology of $|K|$ and hence the result. $\square$

Proposition 2.6.8. *Let L be a simplicial subcomplex of K. Let $N(L)$ be the regular neighborhood of L in K and let $N'(L)$ be the regular neighborhood of L in* $\mathrm{sd}(K, L)$. *Then* $\overline{N'(L)} \subseteq N(L)$.

Proof. By Proposition 2.5.11, the simplices in $\mathrm{sd}(K, L)$ are given in the form

$$\tau = a^1 a^2 \cdots a^q \hat{\sigma}_1 \hat{\sigma}_2 \cdots \hat{\sigma}_m,$$

where $a^1, a^2, \ldots, a^q$ span a simplex σ of L, σ_j is a simplex of $K \smallsetminus L$ for $1 \leq j \leq m$ with $\sigma < \sigma_1 < \sigma_2 < \cdots < \sigma_m$. By Proposition 2.6.7, $\overline{N'(L)}$ is the union of the simplices

$$\tau = a^1 a^2 \cdots a^q \hat{\sigma}_1 \hat{\sigma}_2 \cdots \hat{\sigma}_m$$

with $q \geq 1$. For $x \in \overline{N'(L)}$, there exists τ given in the above form such that $x \in \tau$. Then x lies in the interior of a face of τ. We may assume that $x \in \mathrm{Int}(a^1 a^2 \cdots a^s \hat{\sigma}^1 \cdots \hat{\sigma}^t)$ with $s \geq 0$, where $\sigma < \sigma_1 < \sigma_2 < \cdots < \sigma_t$. If $t = 0$, then $x \in |L| \subseteq N(L)$. If $t > 0$, then $x \in \mathrm{Int}(\sigma_t)$ because x has positive barycentric coordinate on $\hat{\sigma}^t$ which induces that x has positive coordinates on all vertices of σ_t. Since σ_t has vertices $a^0, \ldots, a^q$ in L, $x \in N(L)$ and hence the result. $\square$

Proposition 2.6.9. *Let L be a full subcomplex of K and let $N'(L)$ be the regular neighborhood of L in* $\mathrm{sd}(K, L)$. *Then $|L|$ is a strong deformation retract of* $\overline{N'(L)}$.

Proof. Consider the maps $r_L \colon N(L) \to |L|$ and $H \colon N(L) \times I \to N(L)$ defined in Equations (2.6.1) and (2.6.2), respectively. Let $x \in \overline{N'(L)} \subseteq N(L)$. There exists a unique simplex τ of $\mathrm{sd}(K, L)$ such that $x \in \mathrm{Int}(\tau)$, where τ is given in the form $\tau = a^1 a^2 \cdots a^s \hat{\sigma}^1 \cdots \hat{\sigma}^m$ with $a^1 a^2 \cdots a^s$ a face of a simplex $\sigma = a^1 a^2 \cdots a^q$ of L and $\sigma < \sigma_1 < \cdots < \sigma_m$. If $m = 0$, then $x \in |L|$ and $r_L(x) = H(x, t) = x$ for $0 \leq t \leq 1$. If $m > 0$, then $x \in \mathrm{Int}(\sigma_m)$ as in the proof of the previous proposition. Since x has positive barycentric coordinates on all vertices of σ_m, $t_{a^i}(x) > 0$ for $i = 1, 2, \ldots, q$. It forces that $s = q$. Thus $\tau = a^1 a^2 \cdots a^q \hat{\sigma}^1 \cdots \hat{\sigma}^m$ and so $\sigma = a^1 \cdots a^q$ is a face of τ. From the definition of r_L, $r_L(x)$ is a point in the simplex $a^1 a^2 \cdots a^q$. Note the line segment joining x and $r_L(x)$ lies in τ. Thus

$$H(x, t) \in \overline{N'(L)}$$

for $x \in \overline{N'(L)}$ and $0 \leq t \leq 1$.

Now the homotopy $H|_{\overline{N'(L)} \times I} \colon \overline{N'(L)} \times I \to \overline{N'(L)}$ gives a strong deformation retraction of $\overline{N'(L)}$ into $|L|$. □

COROLLARY 2.6.10. *Let L be any simplicial subcomplex of K and let $N''(L)$ be the regular neighborhood of L in $\mathrm{sd}^2(K, L)$. Then $|L|$ is a strong deformation retract of $\overline{N''(L)}$.*

PROOF. Since L is a full subcomplex of $\mathrm{sd}(K, L)$, the assertion follows from Proposition 2.6.9. □

Exercises

EXERCISE 2.1. Prove the following statements:

(1). If L is a subcomplex of K, then $|L|$ is a closed subspace of $|K|$. In particular, if $\sigma \in K$, then σ is a closed subspace of $|K|$.
(2). $|K|$ is Hausdorff.
(3). If K is finite, then $|K|$ is compact. Conversely if a subset A of $|K|$ is compact, then $A \subseteq |L|$ for some finite subcomplex L of K.

EXERCISE 2.2. Prove the following statements:

(1). If K is a simplicial complex, then the intersection of any collection of subcomplex of K is a subcomplex of K.
(2). If $\{K_\alpha\}$ is a collection of simplicial complexes in $\mathbb{E}^J$, and if the intersection of every pair $|K_\alpha| \cap |K_\beta|$ is the polyhedron of a simplicial complex which is a subcomplex of both K_α and K_β, then the union $\bigcup\limits_\alpha K_\alpha$ is a simplicial complex.

Project

The following topic may be used as research projects of undergraduate/master students.

Topology on Polyhedrons. The beginners might be worried about the weak topology on polyhedrons $|K|$. For having better understanding on the weak topology, one may look at a general notion of compactly generated topology. A topological space X is called *compactly generated* if it satisfies the following property:

> A subset A of X is closed if and only if $A \cap C$ is closed in X for any compact subspace C of X.

An advantage of compactly generated topology is that: Suppose that X has compactly generated topology. Then a function $f \colon X \to Y$ is continuous if and only if f restricted to every compact subspace is continuous.

Proposition 1. Let K be a simplicial complex. Then the weak topology on $|K|$ is compactly generated.

PROOF. First $|K|$ is Hausdorff because the identity map of $|K|$ is continuous from the weak topology to the subspace topology, and $|K|$ is Hausdorff under the subspace topology as it is a metric space. (**Note.** If $f \colon X \to Y$ is an injective continuous function and Y is Hausdorff, then X is Hausdorff.) It follows that every compact subset of $|K|$ is closed.

Let A be a closed subset of $|K|$ and let C be any compact subset of $|K|$. Then $A \cap C$ is closed. Thus A is a closed subset of $|K|$ under compactly generated topology.

Conversely let A be a subset of $|K|$ such that $A \cap C$ is closed in $|K|$ for every compact subset C. Let σ be any simplex of $|K|$. Since σ is compact, $A \cap \sigma$ is closed in $|K|$ by the assumption and so $A \cap \sigma$ is closed in σ. It follows A is closed by the definition of weak topology. $\square$

Given a topological space X, one can have a new topology on X defined by: A is closed if and only if $A \cap C$ is closed in X for every compact subset C of X. This gives a new topology on X, called *compactly generated topology induced by the topology of X*.

Suppose that X is Hausdorff. Then every compact subset of X is closed and so every closed subset of X must be closed under the induced compactly generated topology. In other words,

$$\begin{array}{cl} & \{\text{closed subsets under the induced compactly generated topology }\} \\ \supseteq & \{\text{ closed subsets under the topology of } X\}. \end{array}$$

Now for a polyhedron $|K|$ there are three canonical topologies now:

$$\begin{array}{cl} & \text{weak topology} \\ \supseteq & \text{compactly generated topology induced by subspace topology} \\ \supseteq & \text{subspace topology .} \end{array} \tag{2.6.3}$$

A **proposed project** could be given by exploring the relation between the above three topologies such as giving examples for the inequalities in general; and making the statements that the equalities hold under certain hypothesis. One of the statements could be as follows:

proposition 2. Let K be a simplicial complex with $|K| \subseteq \mathbb{R}^J$. Then the weak topology of $|K|$ coincides with the subspace topology if and only if for every simplex σ there exists an open neighborhood $V(\sigma)$ of σ in $\mathbb{R}^J$ such that $V(\sigma)$ intersects with finitely many simplices of K.

Proof. $\Longleftarrow$ Let U be an open subset of $|K|$ under weak topology. It suffices to show that U is open under subspace topology. Let $x \in U$. Then there exists a unique simplex σ of K such that $x \in \operatorname{Int}(\sigma)$. From the assumption, there is an open neighborhood $V(\sigma)$ such that $V(\sigma)$ only intersects with finitely many simplices of K. Let $\sigma_1, \ldots, \sigma_q$ be all of the simplices of K with $\sigma_i \cap V(\sigma) \neq \emptyset$. Let K_1 be the simplicial subcomplex of K consisting of σ_i and their faces for $1 \leq i \leq q$. Then K_1 is a finite simplicial complex with

$$V(\sigma) \cap |K| \subseteq |K_1|.$$

Under the weak topology of $|K_1|$,

$$U \cap V(\sigma) \cap |K_1|$$

is open. Since K_1 is a finite simplicial complex, $U \cap V(\sigma) \cap |K_1|$ is open in $|K_1|$ under subspace topology with $x \in U \cap V(\sigma) \cap |K_1|$. Thus there exists a small open ball $\text{Int}(D_\epsilon(x))$ centered at x such that

$$\text{Int}(D_\epsilon(x)) \cap |K_1| \subseteq U \cap V(\sigma) \cap |K_1| \text{ and } \text{Int}(D_\epsilon(x)) \subseteq V(\sigma)$$

because $V(\sigma)$ is open in R^J containing σ and $x \in \sigma$. Now

$$\text{Int}(D_\epsilon(x)) \cap |K| \subseteq V(\sigma) \cap |K| \subseteq |K_1|$$

and so

$$\text{Int}(D_\epsilon(x)) \cap |K| = \text{Int}(D_\epsilon(x)) \cap |K_1| \subseteq U \cap V(\sigma) \cap |K_1| \subseteq U.$$

Thus U is open under the subspace topology.

$\Longrightarrow$ Let $O_r(\sigma) = \{y \in \mathbb{R}^J \mid d(y, \sigma) < r\}$. Here $d(y,\sigma) = \inf\{d(x,y) \mid x \in \sigma\}$ is the distance between σ and y. Then $O_r(\sigma)$ is an open neighborhood of σ. We prove the following statement first.

> *Let $V(\sigma)$ be any open neighborhood of σ. Let τ be a simplex such that $\tau \cap V(\sigma) \neq \emptyset$. Then* $\text{Int}(\tau) \cap V(\sigma) \neq \emptyset$.

Since $\tau \cap V(\sigma)$ is a non-empty open subset of τ, $\text{Int}(\tau) \cap V(\sigma) \neq \emptyset$ and hence the statement.

Now suppose that the conclusion is not true. Then there exists a simplex σ of K such that for every open neighborhood of σ intersects with infinitely many distinct simplices of K. From the above statement, every open neighborhood of σ intersects with the interior of infinitely many distinct simplices of K. Consider special open neighborhoods $O_{1/n}(\sigma)$. Construct a sequence of τ_n of K recursively as follows:

(1). τ_1 is a simplex of K such that τ_1 is not a face of σ with $\text{Int}(\tau_1) \cap O_1(\sigma) \neq \emptyset$.
(2). τ_n is a simplex of K such that τ_n is not a face of $\sigma, \tau_1, \ldots, \tau_{n-1}$ with $\text{Int}(\tau_n) \cap O_{1/n}(\sigma) \neq \emptyset$.

Let $y_n \in \text{Int}(\tau_n) \cap O_{1/n}(\sigma)$ and let $A = \{y_n \mid n = 1, 2, \ldots, \}$. For each simplex τ of K, consider $A \cap \tau$. If $y_j \in \tau$, since $y_j \in \text{Int}(\tau_j)$, $\tau \cap \tau_j \neq \emptyset$ and so $\tau \cap \tau_j$ is a common faces of τ_j and τ. Since the face $\tau \cap \tau_j$ contains $y_j \in \text{Int}(\tau_j)$, we have $\tau \cap \tau_j = \tau_j$. It follows that τ_j is a face of τ. Since τ has only finitely many faces, $A \cap \tau$ is a finite set and so $A \cap \tau$ is closed in τ for each simplex τ of K. Thus A is a closed set under the weak topology. From the assumption that the weak topology coincides with the subspace topology, A is closed under the subspace topology.

Now since $d(y_n, \sigma) < 1/n$, there exists $x_n \in \sigma$ such that $d(y_n, x_n) < 1/n$. Since σ is compact with $\sigma \subseteq \mathbb{R}^m$ for some $m < \infty$, there is a convergent subsequence x_{n_k} of x_n. Let $x_0 = \lim\limits_{k\to\infty} x_{n_k} \in \sigma$. Then the subsequence y_{n_k} converges to x_0 because

$$d(y_{n_k}, x_0) \leq d(y_{n_k}, x_{n_k}) + d(x_{n_k}, x_0) \leq \frac{1}{n_k} + d(x_{n_k}, x_0) \to 0.$$

Since $x_0 \notin A$ because each $y_j \notin \sigma$ and $x_0 \in \sigma$, A is not closed under the subspace topology which contradicts to that A is closed. The proof is finished now.

□

There are a lot of properties of compactly generated topology. The classical reference on this topic is Steenrod's paper [**22**].

3. Homotopy Classes, Homotopy Groups and the Fundamental Groups

3.1. Homotopy. A *pair of spaces* (X, A) means a space X with a subspace A of X. If A is empty, the pair $(X, \emptyset)$ is simply denoted by X. The *product* $(X, A) \times (Y, B)$ means the pair of spaces $(X \times Y, (A \times Y) \cup (X \times B))$. In particular, $(X, A) \times Y = (X \times Y, A \times Y)$. A *map* $f \colon (X, A) \to (Y, B)$ means a continuous map $f \colon X \to Y$ such that $f(A) \subseteq B$.

Let $f, g \colon (X, A) \to (Y, B)$ be maps. We call f *homotopic* g relative to A, denoted by $f \simeq g \mathrm{rel}\, A$, if $f|_A = g|_A$ and there exists a map

$$F \colon (X, A) \times [0, 1] \to (Y, B)$$

such that

(1). $F(x, 0) = f(x)$ for $x \in X$,
(2). $F(x, 1) = g(x)$ for $x \in X$ and
(3). $F(a, t) = f(a)$ for $a \in A$ and $0 \leq t \leq 1$.

PROPOSITION 3.1.1. *The homotopy relation* $\sim$ rel A *is an equivalence relation on the set of all maps from* (X, A) *to* (Y, B)*.*

PROOF. For any map $f \colon (X, A) \to (Y, B)$, then the constant homotopy $F(x, t) = f(x)$ is a homotopy from f to itself. Thus $f \simeq f$ rel A.

Suppose that $f \simeq g$ rel A under a homotopy F. Let $F'(x, t) = F(x, 1 - t)$ for $x \in X$ and $0 \leq t \leq 1$. Then $g \simeq f$ rel A under F'.

Suppose that $f \simeq g$ rel$A$ under F and $g \simeq h$ rel A under G. Define the homotopy $F' \colon (X, A) \times I \to (Y, B)$ by

$$F'(x, t) = \begin{cases} F(x, 2t) & \text{for} \quad x \in X \text{ and } 0 \leq t \leq 1/2, \\ G(x, 2t - 1) & \text{for} \quad x \in X \text{ and } 1/2 \leq t \leq 1. \end{cases}$$

Then $f \simeq h$ rel A under F'. □

Let $[X, A; Y, B]$ denote the quotient of the set of all maps from (X, A) to (Y, B) by the homotopy relation $\sim$ rel A, called the set of *homotopy classes* from (X, A) to (Y, B). For a map $f \colon (X, A) \to (Y, B)$, the homotopy class represented by f is denoted by $[f]$, that is $[f] = \{g \colon (X, A) \to (Y, B) \mid g \simeq f \text{ rel } A\}$.

In homotopy theory, the most interesting spaces are pointed spaces, where a pointed space means a space with a choice of basepoint $*$. Let X and Y be pointed space. The set of homotopy classes $[(X, *_X), (Y, *_Y)]$ is simply denoted by $[X, Y]$. From the definition, the set $[X, Y]$ is quotient of the set of all *pointed maps* from X to Y (that is the maps $f \colon X \to Y$ such that $f(*_X) = *_Y$) by the *pointed homotopy relation*, where a pointed homotopy means a map $F \colon X \times I \to Y$ such that $F(*_X, t) = *_Y$ for $0 \leq t \leq 1$.

3.2. Path Homotopy Classes and the Fundamental Groups. Let X be a space. A *path* means a continuous map $\lambda \colon [0, 1] \to X$. Two paths $\lambda, \lambda' \colon [0, 1] \to X$ are called *homotopic*, denoted by $\lambda \simeq \lambda'$, if $\lambda(0) = \lambda'(0)$, $\lambda(1) = \lambda'(1)$ and $\lambda \simeq \lambda'$ rel $\{0, 1\}$. The path homotopy class of λ is denoted by $[\lambda]$, that is,

$$[\lambda] = \{\lambda' \colon [0, 1] \to X \mid \lambda' \simeq \lambda\}.$$

Let $\lambda, \mu \colon [0, 1] \to X$ be two paths such that $\lambda(1) = \mu(0)$. The *path product* $\lambda * \mu \colon [0, 1] \to X$ is defined by

$$\lambda * \mu(t) = \begin{cases} \lambda(2t) & \text{for} \quad 0 \leq t \leq 1/2, \\ \mu(2t - 1) & \text{for} \quad 1/2 \leq t \leq 1. \end{cases}$$

LEMMA 3.2.1. *Let* $\lambda, \mu\colon [0,1] \to X$ *be two paths such that* $\lambda(1) = \mu(0)$. *Suppose that* $\lambda \simeq \lambda'$ *and* $\mu \simeq \mu'$. *Then*

$$\lambda * \mu \simeq \lambda' * \mu'.$$

PROOF. Let $F\colon \lambda \simeq \lambda'$ and $G\colon \mu \simeq \mu'$ be the path homotopies with $F(t,0) = \lambda(t)$, $F(t,1) = \lambda'(t)$, $G(t,0) = \mu(t)$ and $G(t,1) = \mu'(t)$. Define

$$F'(t,s) = \begin{cases} F(2t,s) & \text{if} \quad 0 \le t \le 1/2, \\ G(2t-1,s) & \text{if} \quad 1/2 \le t \le 1. \end{cases}$$

Then F' is a path homotopy between $\lambda * \mu$ and $\lambda' * \mu'$. □

For each point $b \in X$, let c_b denote the constant path at b, that is $c_b(t) = b$ for $0 \le t \le 1$. For any path λ, the inverse path λ^{-1} is defined by $\lambda^{-1}(t) = \lambda(1-t)$ for $0 \le t \le 1$.

PROPOSITION 3.2.2. *The path product satisfies the following properties:*

(1). *The path product is associative up to path homotopy. More precisely, if* $\lambda_1(1) = \lambda_2(0)$ *and* $\lambda_2(1) = \lambda_3(0)$, *then* $(\lambda_1 * \lambda_2) * \lambda_3 \simeq \lambda_1 * (\lambda_2 * \lambda_3)$.

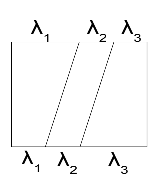
(2). *The constant paths play as the units for path product up to path homotopy. More precisely* $c_{\lambda(0)} * \lambda \simeq \lambda * c_{\lambda(1)} \simeq \lambda$.

(3). *The inverse path is the inverse of path product up to path homotopy. More precisely* $\lambda * \lambda^{-1} \simeq c_{\lambda(0)}$ *and* $\lambda^{-1} * \lambda \simeq c_{\lambda(1)}$.

PROOF. (1). A homotopy is given by

$$F(t,s) = \begin{cases} \lambda_1\left(\frac{4}{s+1}t\right) & \text{if} \quad 0 \le t \le \frac{s+1}{4}, \\ \lambda_2(4t-s+1) & \text{if} \quad \frac{s+1}{4} \le t \le \frac{s+2}{4}, \\ \lambda_3\left(\frac{4}{2-s}\left(t - \frac{s+2}{4}\right)\right) & \text{if} \quad \frac{s+2}{4} \le t \le 1. \end{cases}$$

A picture of this homotopy is as follows:



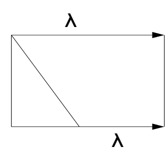
(2) The homotopies between $c_{\lambda(0)} * \lambda \simeq \lambda$ and $\lambda * c_{\lambda(1)} \simeq \lambda$ are given by

$$F(t,s) = \begin{cases} \lambda(0) & \text{if} \quad 0 \le t \le \frac{1-s}{2}, \\ \lambda\left(\frac{2}{1+s}\left(t - \frac{1-s}{2}\right)\right) & \text{if} \quad \frac{1-s}{2} \le t \le 1, \end{cases}$$

and

$$G(t,s) = \begin{cases} \lambda\left(\frac{2}{s+1}t\right) & \text{if} \quad 0 \le t \le \frac{s+1}{2}, \\ \lambda(1) & \text{if} \quad \frac{s+1}{2} \le t \le 1, \end{cases}$$

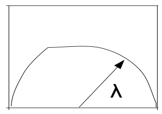
respectively. A picture of the homotopy F is as follows:



(3) A homotopy between $\lambda * \lambda^{-1} \simeq c_{\lambda(0)}$ is given by

$$F(t,s) = \begin{cases} \lambda\left(1 - 2\sqrt{(s-1/2)^2 + t^2}\right) & \text{if} \quad 0 \le \sqrt{(s-1/2)^2 + t^2} \le 1/2, \\ \lambda(0) & \text{if} \quad \sqrt{(s-1/2)^2 + t^2} \ge 1/2. \end{cases}$$

By replacing λ to be λ^{-1}, one gets the homotopy between $\lambda^{-1} * \lambda$ and $c_{\lambda(1)}$. A picture of the homotopy F is as follows:



□

A *loop* means a path λ with $\lambda(0) = \lambda(1)$. Let $*$ be the basepoint of X. The set of the path homotopy classes of all loops with $\lambda(0) = \lambda(1) = *$ is called the *fundamental group* of X, denoted by $\pi_1(X)$. According to the above proposition, $\pi_1(X)$ is a group with the multiplication induced by the path product.

An important tool for computing fundamental groups is the Seifert-Van Kampen Theorem. A proof can be found in Hatcher's book [**7**, page 43-46].

THEOREM 3.2.3 (Seifert-van Kampen Theorem). *Let X be a space and let U and V be open subsets of X such that $X = U \cup V$. Let $j_1\colon U \cap V \to U$ and $j_2\colon U \cap V \to V$ be the inclusions. Suppose that $U \cap V$ is path-connected. Then*

$$\pi_1(X) = \pi_1(U) \coprod_{\pi_1(U\cap V)} \pi_1(V)$$

the free product$\pi_1(U)$ and $\pi_1(V)$ with amalgamation through group homomorphisms $j_{1*}\colon \pi_1(U \cap V) \to \pi_1(U)$ and $j_{2*}\colon \pi_1(U \cap V) \to \pi_1(V)$. □

3.3. Homotopy Groups. Let X be a pointed space and let S^n be the n-sphere with the North pole $(1, 0, \ldots, 0)$ as the basepoint. The *n-th homotopy group* $\pi_n(X)$ is defined by the set of pointed homotopy classes $\pi_n(X) = [S^n, X]$. From the definition, $\pi_n(X)$ is the quotient of all pointed continuous maps from S^n to X by the pointed homotopy. If $n = 0$, then $\pi_0(X)$ is only a set with a bijection to the set of path-connected components of X. $\pi_1(X)$ is the fundamental group. The multiplication on $\pi_n(X)$, $n \geq 1$, is defined as follows:

Let $I = [0, 1]$ and let $q\colon I^n \to S^n$ be the map given by the composite

$$I^n \xrightarrow{\text{pinch}} I^n/\partial I^n \cong S^n,$$

where ∂I^n is the boundary of the n-cube I^n. Then q induces a bijection of the sets of homotopy classes

$$q^*\colon [I^n, \partial I^n; X, *_X] \cong [S^n, *_{S^n}; X, *_X].$$

Now let $f, g\colon I^n \to X$ be maps such that $f(\partial I^n) = g(\partial I^n) = \{*_X\}$. Define the product

$$f * g(t_1, t_2, \ldots, t_n) = \begin{cases} f(2t_1, t_2, t_3, \ldots, t_n) & \text{if} \quad 0 \le t_1 \le 1/2, \\ g(2t_1 - 1, t_2, t_3, \ldots, t_n) & \text{if} \quad 1/2 \le t_1 \le 1. \end{cases}$$

Then $f * g(\partial I^n) = \{*_X\}$. By the proof of Proposition 3.2.2, the set

$$\pi_n(X) \cong [I^n, \partial I^n; X, *_X]$$

is a group with the multiplication induced by the above product.

If $n \ge 2$, we can define another product by setting

$$f \star g(t_1, t_2, \ldots, t_n) = \begin{cases} f(t_1, 2t_2, t_3, \ldots, t_n) & \text{if} \quad 0 \le t_1 \le 1/2, \\ g(t_1, 2t_2 - 1, t_3, \ldots, t_n) & \text{if} \quad 1/2 \le t_1 \le 1. \end{cases}$$

In the set $\pi_n(X)$, let $[f] * [g] = [f * g]$ and $[f] \star [g] = [f \star g]$.

Proposition 3.3.1. *Let $n \ge 2$. Then $\pi_n(X)$ is an abelian group under the multiplication $*$ or $\star$, and $[f] * [g] = [f] \star [g]$ for any $[f], [g] \in \pi_n(X)$.*

Proof. From the definition we have the equation

$$(f_1 * f_2) \star (g_1 * g_2) = (f_1 \star g_1) * (f_2 \star g_2)$$

for any maps $f_1, f_2, g_1, g_2\colon I^n \to X$ with $f_i(\partial I^n) = g_i(\partial I^n) = \{*_X\}$ for $i = 1, 2$. It follows that there is an equation

$$([f_1] * [f_2]) \star ([g_1] * [g_2]) = ([f_1] \star [g_1]) * ([f_2] \star [g_2])$$

for any $[f_1], [f_2], [g_1], [g_2] \in \pi_n(X)$.

Note that both multiplications $*$ and $\star$ have the identity 1 represented by the constant map $c(t_1, \ldots, t_n) = *_X$.

$$[f] \star [g] = ([f] * 1) \star (1 * [g]) = ([f] \star 1) * (1 \star [g]) = [f] * [g]$$

for $[f], [g] \in \pi_n(X)$. Thus the multiplication $*$ coincides with $\star$.

Now

$$[f] \star [g] = (1 * [f]) \star ([g] * [1]) = (1 \star [g]) * ([f] \star [1]) = [g] * [f] = [g] \star [f]$$

and so $\pi_n(X)$ is abelian. □

Since $\pi_n(X)$ is abelian for $n \ge 2$, we write $[f] + [g]$ for $[f] * [g] = [f] \star [g]$.

4. Simplicial Approximation Theorem

4.1. Simplicial Approximation.

Definition 4.1.1. Let K and L be simplicial complexes. Let $f\colon |K| \to |L|$ be a continuous map. A simplicial map $g\colon K \to L$ is called a *simplicial approximation* to f if, for each vertex v of K,

$$f(\mathrm{St}(v, K)) \subseteq \mathrm{St}(g(v), L).$$

If f is a simplicial map, then f is a simplicial approximation to itself because f sends each simplex of K to a simplex of L and hence $f(\mathrm{St}(v, K)) \subseteq \mathrm{St}(f(v), L)$.

Proposition 4.1.2. *Let $h\colon |K| \to |L|$ and $k\colon |L| \to |M|$ have simplicial approximation $f\colon K \to L$ and $g\colon L \to M$, respectively. Then $g \circ f$ is a simplicial approximation to $k \circ h$.*

Proof. We know that $g \circ f$ is a simplicial map. If v is a vertex of K, then

$$h(\mathrm{St}(v,K)) \subseteq \mathrm{St}(f(v),L)$$

because f is a simplicial approximation to h. Thus

$$k(h(\mathrm{St}(v,K))) \subseteq k(\mathrm{St}(f(v),L)) \subseteq \mathrm{St}(g(f(v)),M)$$

because g is a simplicial approximation to k. □

Proposition 4.1.3. *Let K and L be simplicial complexes, and let $f\colon |K| \to |L|$ be a continuous map. Suppose that, for each vertex a of K, there exists a vertex b of L such that $f(\mathrm{St}(a,K)) \subseteq \mathrm{St}(b,L)$. Then there exists a simplicial approximation g to f, such that $g(a) = b$ for each vertex a of K.*

Proof. It suffices to check that $g(a^0), \dots, g(a^n)$ span a simplex of L whenever $a^0, a^1, \dots, a^n$ span a simplex of K.

Let $\sigma = a^0 a^1 \cdots a^n$ be the simplex of K spanned by $a^0, a^1, \dots, a^n$. Let $x \in \mathrm{Int}(\sigma)$ be a point in the interior. Then

$$x \in \bigcap_{i=0}^{n} \mathrm{St}(a^i)$$

It follows that

$$f(x) \in \bigcap_{i=0}^{n} f(\mathrm{St}(a^i)) \subseteq \bigcap_{i=0}^{n} \mathrm{St}(g(a^i)),$$

that is $t_{g(a^i)}(f(x)) > 0$ for each $0 \le i \le n$. By the definition of the function t_v, the unique simplex that contains $f(x)$ in its interior must have each $g(a^i)$ as a vertex, and hence has a face spanned by $g(a^0), g(a^1), \dots, g(a^n)$. □

Let A be a subspace of X and let $f, g\colon X \to Y$ be maps such that $f(a) = g(a)$ for $a \in A$. Recall that f is called homotopic to g relative to A if there is a homotopy $F\colon X \times I \to Y$ such that $F(x,0) = f(x)$, $F(x,1) = g(x)$ and $F(a,t) = f(a)$ for $x \in X$, $a \in A$ and $0 \le t \le 1$.

Theorem 4.1.4. *Let K and L be simplicial complexes, and let $f\colon |K| \to |L|$ be a continuous map. Then any simplicial approximation g to f is homotopic to f relative to the subspace of K of those points x such that $f(x) = g(x)$.*

Proof. Let x be a point of K. Then there is a unique simplex $\sigma = a^0 a^1 \cdots a^n$ such that $x \in \mathrm{Int}(\sigma)$. From the proof of the above proposition, $f(x)$ lies in the interior of a simplex τ of L that contains a face spanned by $g(a^0), \dots, g(a^n)$. Thus τ contains the point $g(x)$, and so the line segment $f(x)g(x)$ lies in τ. Since the linear homotopy

$$F(x,s) = (1-s)f(x) + sg(x)$$

can be defined, the assertion follows. □

4.2. Simplicial Approximation Theorem in Absolute Case.

Theorem 4.2.1 (Finite Simplicial Approximation Theorem). *Let K and L be simplicial complexes. Suppose that K is finite. Let $f\colon |K| \to |L|$ be a continuous map. Then there exists N such that the map f has a simplicial approximation $g\colon \mathrm{sd}^N K \to L$.*

PROOF. Since $\{\mathrm{St}(w) \mid w \text{ is a vertex of } L\}$ is an open cover of $|L|$, the space $|K|$ is covered by the open sets

$$\mathcal{A} = \{f^{-1}(\mathrm{St}(w)) \mid w \text{ is a vertex of } L\}.$$

Since $|K|$ is a compact metric space, there exists a **Lebesgue number** λ such that any subset of $|K|$ with diameter less than λ lies in an element of $\mathcal{A}$. (If there were no such λ, there is a sequence C_n of subsets of $|K|$ such that $\operatorname{diam} C_n < 1/n$ but C_n does not lie in any element of $\mathcal{A}$. Choose $x_n \in C_n$. By compactness, there is a subsequence x_{n_i} convergent to a point $x \in |K|$. Then x lies in an element $f^{-1}(\mathrm{St}(w))$ of $\mathcal{A}$, and so $C_{n_i} \subseteq f^{-1}(\mathrm{St}(w))$ when i is sufficiently large. This gives a contradiction.)

Choose N such that each simplex of $\mathrm{sd}^N K$ has diameter less than $\lambda/2$. Let v be a vertex of $\mathrm{sd}^N K$. Then the diameter of $\mathrm{St}(v)$ is less than λ because, for $x, y \in \mathrm{St}(v)$, there exist simplices σ and τ of $\mathrm{sd}^N K$ such that both σ and τ have v as a vertex with $x \in \sigma$ and $y \in \tau$ and so

$$|x-y| \leq |x-a| + |y-a| < \frac{\lambda}{2} + \frac{\lambda}{2} = \lambda.$$

Thus $\mathrm{St}(v) \subseteq f^{-1}(\mathrm{St}(w))$ for some vertex w of L. The assertion follows from Proposition 4.1.3 now. $\square$

4.3. Simplicial Approximation to the Identity Map. Now we start to consider relative case. We will use barycentric subdivisions relative to a simplicial subcomplex. Observe that the identity map $|\mathrm{sd}(K,A)| \to |K|$ is not a simplicial map if $K \smallsetminus A \neq$ because the vertices $\hat{\sigma}$ of $\mathrm{sd}(K,A)$ are not the vertices of K. The following proposition gives a simplicial approximation to the identity map $|\mathrm{sd}(K,A)| \to |K|$.

PROPOSITION 4.3.1 (Simplicial Approximation to the Identity Map). *Let K be a simplicial complex and let A be a simplicial subcomplex of K. Let f be any function assigning simplices σ of $K \smallsetminus A$ to one of its vertices. Then there exists a simplicial approximation h to the identity map $|\mathrm{sd}(K,A)| \to |K|$ such that the restriction $h|_A = \mathrm{id}_A$ and $h(\hat{\sigma}) = f(\sigma)$ for each simplex σ of $K \smallsetminus A$.*

PROOF. First we need to check that h is a simplicial map. Let τ be a simplex of $\mathrm{sd}(K,A)$. Then $\tau = a^1a^2\cdots a^p\hat{\sigma}_1\hat{\sigma}_2\cdots\hat{\sigma}_q$, where $a^1, a^2, \ldots, a^q$ span a simplex σ_0 of A, σ_j is a simplex of $K \smallsetminus A$ and $\sigma_0 < \sigma_1 < \sigma_2 < \cdots < \sigma_q$. We check that the vertices

$$\{a^1, a^2, \ldots, a^p, f(\hat{\sigma}_1), f(\hat{\sigma}_2), \ldots, f(\hat{\sigma}_q)\} \tag{4.3.1}$$

span a simplex in K. Since $\sigma_0 < \sigma_1 < \sigma_2 < \cdots < \sigma_q$, each σ_j is a face of σ_q and so any vertex of σ_j is a vertex of σ_q. Thus the elements in Equation (4.3.1) are vertices of σ_q and therefore they span a simplex in K.

Next we check that h is a simplicial approximation to $\mathrm{id}\colon |\mathrm{sd}(K,A)) \to |K|$. Let v be any vertex of $\mathrm{sd}(K,A)$. We need to show that

$$\mathrm{St}(v, \mathrm{sd}(K,A)) \subseteq \mathrm{St}(h(v), K).$$

Let τ be a simplex of $\mathrm{sd}(K,A)$ having v as a vertex. We are going to show that:

> *There exists a simplex μ of K such that μ has $h(v)$ as a vertex and the interior* $\mathrm{Int}(\tau) \subseteq \mathrm{Int}(\mu)$.

If so, then $\mathrm{Int}(\tau) \subseteq \mathrm{St}(h(v), K)$ for any simplex τ of $\mathrm{sd}(K, A)$ having v as a vertex. From the definition of star, we then conclude that $\mathrm{St}(v, \mathrm{sd}(K, A)) \subseteq \mathrm{St}(h(v), K)$.

Now we prove the above statement. By Proposition 2.5.11,

$$\tau = a^1 a^2 \cdots a^p \hat{\sigma}_1 \hat{\sigma}_2 \cdots \hat{\sigma}_q,$$

where $a^1, a^2, \ldots, a^p$ span a simplex σ_0 of A, σ_j is a simplex of $K \smallsetminus A$ for $1 \leq j \leq q$ and

$$\sigma_0 < \sigma_1 < \sigma_2 < \cdots < \sigma_q.$$

If $q = 0$, then v is a vertex of A and so $h(v) = v$. In this case, we choose $\mu = \tau$.

Suppose that $q > 0$. We choose $\mu = \sigma_q$. From the previous paragraph, $h(v)$ is a vertex of μ and so we only need to check that $\mathrm{Int}(\tau) \subseteq \mathrm{Int}(\mu)$. Let $x \in \mathrm{Int}(\tau)$. Then

$$x = \sum_{i=1}^{p} s_i a^i + \sum_{j=1}^{q} t_j \hat{\sigma}_i$$

with $s_i, t_j > 0$ and $\sum_{i=1}^{p} s_i + \sum_{j=1}^{q} t_j = 1$. Let $b^0, b^1, \ldots, b^m$ be the vertices of σ_q. Let

$$a^i = \sum_{k=0}^{m} s_{i,k} b^k$$

$$\hat{\sigma}_j = \sum_{k=0}^{m} t_{i,k} b^k$$

with $s_{i,k}, t_{i,k} \geq 0$, $\sum_{k=0}^{m} s_{i,k} = 1$ and $\sum_{k=0}^{m} t_{j,k} = 1$. Since $\hat{\sigma}_q$ is the barycenter of σ_q, $t_{q,k} = \frac{1}{m+1}$ for $0 \leq k \leq m$. Now

$$x = \sum_{k=0}^{m} (\sum_{i=1}^{p} s_i s_{i,k} + \sum_{j=1}^{q} t_j t_{j,k}) b^k.$$

with

$$\sum_{k=0}^{m} \left(\sum_{i=1}^{p} s_i s_{i,k} + \sum_{j=1}^{q} t_j t_{j,k} \right) = 1.$$

For each k, we have

$$\sum_{i=1}^{p} s_i s_{i,k} + \sum_{j=1}^{q} t_j t_{j,k} \geq t_q t_{q,k} = \frac{t_q}{m+1} > 0.$$

Thus $x \in \mathrm{Int}(\sigma_q)$ and hence the result. $\square$

COROLLARY 4.3.2. *Let K be a simplicial complex and let A be a simplicial subcomplex of K. Let a be any vertex of* $\mathrm{sd}(K, A)$. *Then there exists a vertex b of K such that*

$$\mathrm{St}(a, \mathrm{sd}(K, A)) \subseteq \mathrm{St}(b, K)$$

such that if $a \in A$, then we can choose $b = a$. Moreover if B is a full subcomplex of K such that $B \cap A = \emptyset$, then for a vertex a of $\mathrm{sd}(K, A)$ *not in $|B|$, there exists a vertex b of K such that $b \notin |B|$ and*

$$\mathrm{St}(a, \mathrm{sd}(K, A)) \subseteq \mathrm{St}(b, K).$$

PROOF. Let h be a simplicial approximation to the identity. Let a be any vertex of $\mathrm{sd}(K, A)$. By the definition of simplicial approximation,

$$\mathrm{St}(a, \mathrm{sd}(K, A)) = \mathrm{id}(st(a, \mathrm{sd}(K, A))) \subseteq \mathrm{St}(h(a), K).$$

Since $h|_A = \mathrm{id}_A$, $h(a) = a$ if $a \in A$.

Now let B be a full subcomplex of K such that $B \cap A = \emptyset$ and let a be a vertex of $\mathrm{sd}(K, A)$ not in $|B|$. If $a \in A$, then

$$\mathrm{St}(a, \mathrm{sd}(K, A)) \subseteq \mathrm{St}(a, K)$$

by the previous step. If $a \notin A$, then $a = \hat{\sigma}$ for a simplex σ of $K \smallsetminus A$. Since $a = \hat{\sigma} \notin |B|$, the simplex σ has at least one vertex not in B because $\sigma \in B$ by the assumption that B is a full subcomplex. By Proposition 4.3.1, we can make a choice of the simplicial approximation h to the identity by requiring that if a is a vertex of $\mathrm{sd}(K, A)$ not in $|B|$, then $h(a)$ is a vertex not in $|B|$ by the above proposition. The assertion follows. □

4.4. Simplicial Approximation Theorem in Relative Case.

DEFINITION 4.4.1. Let K be a simplicial complex and let A be a subcomplex of K. The *supplement* of A in K, denoted by $\bar{A}$, is the set of simplices of $\mathrm{sd}(K, A)$ that have NO vertices in A. Clearly $\bar{A}$ is a subcomplex of $\mathrm{sd}(K, A)$, which is the same as the subcomplex of $\mathrm{sd}\, K$ of simplices having no vertices in $\mathrm{sd}\, A$.

The next lemma states that the maximum diameter of the stars of $\mathrm{sd}^N(K, A)$ of vertices in the supplement $|\bar{A}|$ tends to 0.

LEMMA 4.4.2. *Let K be a simplicial complex with a subcomplex A. Suppose that there are finitely many simplices of $K \smallsetminus A$. Given any $\epsilon > 0$, there exists N such that*

$$\sup\{\mathrm{diam}\ \mathrm{St}(v, \mathrm{sd}^N(K, A)) \mid v \in |\bar{A}|\} < \epsilon.$$

Before to give the proof, let's make some observations. Let τ be a 1-simplex of $\mathrm{sd}^2(K, A)$. Suppose that τ has a vertex v in A. Then another vertex of τ is either in A or the barycenter $\hat{\sigma}$ for a simplex σ of $\mathrm{sd}(K, A)$ having v as a vertex. Thus $\sigma \notin \bar{A}$ by the definition of $\bar{A}$, and so $\hat{\sigma} \notin |\bar{A}|$. In other words, NO 1-simplices of $\mathrm{sd}^2(K, A)$ that can have vertices in both A and $|\bar{A}|$. It follows that NO n-simplices of $\mathrm{sd}^2(K, A)$ that can have vertices in both A and $|\bar{A}|$. (If an n-simplex had vertices v in A and w in $|\bar{A}|$, then its face from v to w is a 1-simplex having its vertex v in A and w in $|\bar{A}|$ which contradicts to that no 1-simplex can have vertices in both A and $|\bar{A}|$.)

PROOF. Let $\hat{A}$ denote the supplement of A in $\mathrm{sd}(K, A)$, that is, $\hat{A}$ is the set of simplices of $\mathrm{sd}^2(K, A)$ that have no vertices in A. If τ is a simplex of $\mathrm{sd}^2(K, A)$ with a vertex in $|\bar{A}|$, then τ has no vertices in A from the above observation. Thus $\tau \in \hat{A}$. Note that the subdivision $\mathrm{sd}^N(K, A)$ includes the **non-relative** subdivision $\mathrm{sd}^{N-2}\, \hat{A}$ of $\hat{A}$. Thus $\mathrm{sd}^{N-2}\, \hat{A}$ is a subcomplex of $\mathrm{sd}^N(K, A)$.

Now we show by induction that:

> *For each $N \geq 2$ if τ is a simplex of $\mathrm{sd}^N(K, A)$ having a vertex in $|\bar{A}|$, then $\tau \in \mathrm{sd}^{N-2}\, \hat{A}$.*

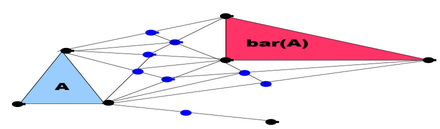
This statement has been proved when $N = 2$. Suppose that it holds for $N-1$ with $N \geq 3$. Let τ be a simplex of $\operatorname{sd}^N(K, A)$ with a vertex in $|\bar{A}|$. Then τ has the expression $a^1a^2 \cdots a^p \hat{\sigma}_1 \cdots \hat{\sigma}_q$, where $a^1, \ldots, a^p$ span a simplex σ_0 of A, each σ_j is a simplex of $\operatorname{sd}^{N-1}(K, A) \smallsetminus A$ for $1 \leq j \leq q$ with $\sigma_0 < \sigma_1 < \cdots < \sigma_q$. Since each a^i is not in $|\bar{A}|$, we may assume that $\hat{\sigma}_j \in |\bar{A}|$ for some $1 \leq j \leq q$. It follows that σ_j has a vertex in $|\bar{A}|$. Since σ_j is a face of σ_q as $\sigma_j < \sigma_q$, σ_q has a vertex in $|\bar{A}|$. By induction, $\sigma_q \in \operatorname{sd}^{N-3} \hat{A}$ and σ_0 must be empty as it is a face of the simplex σ_q of $\operatorname{sd}^{N-3} \hat{A}$. It follows that $\tau \in \operatorname{sd}^{N-2} \hat{A}$. The induction is finished and hence the statement.

Thus for any vertex v of $\operatorname{sd}^N(K, A)$ such that $v \in |\bar{A}|$

$$\begin{array}{rcl} \operatorname{St}(v, \operatorname{sd}^N(K, A)) & = & \bigcup\{\operatorname{Int}(\tau) \mid \tau \text{ is a simplex of } \operatorname{sd}^N(K, A) \text{ with } v < \tau\} \\ & \subseteq & \bigcup\{\operatorname{Int}(\tau) \mid \tau \text{ is a simplex of } \operatorname{sd}^{N-2} \hat{A} \text{ with } v < \tau\} \\ & = & \operatorname{St}(v, \operatorname{sd}^{N-2} \hat{A}). \end{array}$$

By Theorem **??**, the diameters of simplices in $\operatorname{sd}^{N-2} \hat{A}$ tends to 0 as $N \to \infty$. The assertion follows. $\square$

Now we construct a new simplicial complex $K^+ = \operatorname{sd}(\operatorname{sd}(K, A), A \cup \bar{A})$. In other words, K^+ is obtained by doing barycentric subdivision on those simplices of $\operatorname{sd}(K, A)$ that are not in A and $\bar{A}$. Thus K^+ is a subdivision between $\operatorname{sd}(K, A)$ and $\operatorname{sd}^2(K, A)$. A picture is as follows:



The vertices v of K^+ are one of the following three cases:

Type I. v is a vertex of A;
Type II. v is a vertex of $\bar{A}$;
Type III. v is the barycenter $\hat{\sigma}$ for a simplex σ of $\operatorname{sd}(K, A)$ that has vertices in both A and $\bar{A}$.

By Proposition 4.3.1, there exists a simplicial approximation h to the identity of $|K^+| = |\operatorname{sd}(\operatorname{sd}(K, A), A \cup \bar{A})| \to |\operatorname{sd}(K, A)|$ such that h on the vertices of K^+ is given by the following rule:

(1). If v is a vertex of A, then $h(v) = v$;
(2). If v is a vertex of $\bar{A}$, then $h(v) = v$;
(3). If $v = \hat{\sigma}$ for a simplex σ of $\operatorname{sd}(K, A)$ that has vertices in both A and $\bar{A}$, then $h(v)$ is a vertex of σ in A.

The first two rules requires that $h|_{A \cup \bar{A}} = \operatorname{id}_{A \cup \bar{A}}$. The last rule forces h to push down the vertices of the form $\hat{\sigma}$ into A.

Lemma 4.4.3. *If v is a vertex of A, then $h(\mathrm{St}(v, K^+)) \subseteq \mathrm{St}(v, A)$.*

Proof. Let τ be a simplex of K^+ having v as a vertex. Since

$$K^+ = \mathrm{sd}(\mathrm{sd}(K, A), A \cup \bar{A}),$$

τ has the expression

$$\tau = a^1 a^2 \cdots a^p \hat{\sigma}_1 \cdots \hat{\sigma}_q,$$

where $a^1, a^2, \cdots, a^p$ span a simplex σ_0 in $A \cup \bar{A}$, $\sigma_1, \ldots, \sigma_q$ are simplices of $\mathrm{sd}(K, A)$ that have vertices in both A and $\bar{A}$, and $\sigma_0 < \sigma_1 < \cdots < \sigma_q$. Since $v \in A$ is a vertex of τ, $p \geq 1$ and $v \in \{a^1, \ldots, a^p\}$. It follows that $a^i \notin \bar{A}$ for each $1 \leq i \leq q$ because if not, then σ_0 is a simplex having vertices in both A and $\bar{A}$ which contradicts to that $\sigma \in A \cup \bar{A}$. Thus τ only have Type I and Type III vertices, namely τ has NO vertices in $\bar{A}$. By the definition of the map h, we have

$$h(a^i), h(\hat{\sigma}_j) \in A$$

for $1 \leq i \leq p$ and $1 \leq j \leq q$. Hence $h(\tau)$ is a simplex of A having v as vertex because $h|_A = \mathrm{id}_A$. The assertion follows by taking the union of the interior of τ having v as a vertex. □

Proposition 4.4.4. *Let K and L be a simplicial complexes and let A be a subcomplex of K. Let*

$$h\colon |K^+| = |\,\mathrm{sd}(\mathrm{sd}(K, A), A \cup \bar{A})| = |K| \to |\,\mathrm{sd}(K, A)| = |K|$$

be the simplicial map defined above. Let $f\colon |K| \to |L|$ be any continuous map such that $f|_{|A|}$ is a simplicial map. Suppose that there are finitely many simplices of $K \smallsetminus A$. Then there exists N such that the composite $f \circ h$ has a simplicial approximation

$$g\colon \mathrm{sd}^N(K, A) \to L$$

with the property that $g|_{|A|} = f|_{|A|}$.

Proof. Let

$$\mathcal{A} = \{(f \circ h)^{-1}(\mathrm{St}(w, L)) \mid w \text{ is a vertex of } L\}$$

be an open covering of $|K|$. Let $\hat{A}$ be the supplement of A in $\mathrm{sd}(K, A)$ as discussed in the proof of Lemma 4.4.2. By the assumption, $K \smallsetminus A$ has only finitely many simplices. Thus $\hat{A}$ is a finite complex and so $|\hat{A}|$ is compact. Hence the covering $\mathcal{A} \cap |\hat{A}|$ of $|\hat{A}|$ has a Lebesgue number λ. By Lemma 4.4.2, there exists $N \geq 2$ such that

$$\sup\{\mathrm{diam}\ \mathrm{St}(v, \mathrm{sd}^N(K, A)) \mid v \in |\hat{A}|\} < \lambda.$$

(**Note.** Here we replace $\hat{A}$ as $\bar{A}$ in Lemma 4.4.2 by considering $\mathrm{sd}(K, A)$ as K.)

From Proposition 4.1.3, it suffices to show that, for every vertex v of $\mathrm{sd}^N(K, A)$, there exists a vertex w of L such that

$$\mathrm{St}(v, \mathrm{sd}^N(K, A)) \subseteq (f \circ h)^{-1}(w).$$

Case I. $v \in |\hat{A}|$.

Then

$$\mathrm{diam}\ \mathrm{St}(v, \mathrm{sd}^N(K, A)) < \lambda$$

and so there exists a w such that $\mathrm{St}(v, \mathrm{sd}^N(K, A)) \subseteq (f \circ h)^{-1}(w)$.

Case II. $v \notin |\hat{A}|$.

By Corollary 4.3.2,

$$\mathrm{St}(v, \mathrm{sd}^N(K,A)) \subseteq \mathrm{St}(b, \mathrm{sd}^2(K,A))$$

for some vertex b of $\mathrm{sd}^2(K,A)$ NOT in $|\hat{A}|$. Then $b \in A$. (This is because, by definition, $\hat{A}$ is the subcomplex of $\mathrm{sd}^2(K,A)$ of the simplices having no vertices in A. In particular, $\hat{A}$ contains all of vertices, as 0-simplices, that are not in A.) We claim that

$$\mathrm{St}(b, \mathrm{sd}^2(K,A)) \subseteq \mathrm{St}(b, K^+).$$

Let τ be a simplex of $\mathrm{sd}^2(K,A)$ having a vertex $b \in A$. Then τ has the expression

$$\tau = a^1 a^2 \cdots a^p \hat{\sigma}_1 \hat{\sigma}_2 \cdots \hat{\sigma}_2,$$

where $a^1, \ldots, a^p$ span a simplex σ_0 of A, σ_j is a simplex of $\mathrm{sd}(K,A) \smallsetminus A$ for $1 \le j \le q$, with $\sigma_0 < \sigma_1 < \cdots < \sigma_q$. Since b is a vertex of τ, $b \in \{a^1, \ldots, a^p\}$ with $p \ge 1$. Since σ_0 is a face of each σ_j for $1 \le j \le q$, b is a vertex of σ_j for $1 \le j \le q$. It follows that $\tau \in K^+ = \mathrm{sd}(\mathrm{sd}(K,A), A \cup \bar{A})$. This proves that

$$\mathrm{St}(b, \mathrm{sd}^2(K,A)) \subseteq \mathrm{St}(b, K^+).$$

Now by Lemma 4.4.3, we have

$$h(\mathrm{St}(b, K^+)) \subseteq \mathrm{St}(b, A).$$

Since $f|_{|A|}$ is a simplicial map, we have

$$f(\mathrm{St}(b, A)) \subseteq \mathrm{St}(f(b), L).$$

By combining the previous equations, we have

$$\begin{array}{rcl} \mathrm{St}(v, \mathrm{sd}^N(K,A)) & \subseteq & \mathrm{St}(b, \mathrm{sd}^2(K,A)) \\ & = & \mathrm{St}(b, K^+) \\ & \subseteq & h^{-1}(\mathrm{St}(b, A)) \\ & \subseteq & (f \circ h)^{-1}(\mathrm{St}(f(b), L)). \end{array}$$

Hence $f \circ h$ has a simplicial approximation $g\colon \mathrm{sd}^N(K,A) \to L$.

For checking that, we can make a choice of g such that $g|_{|A|} = f|_{|A|}$. Let v be a vertex of A. In the argument of Case II above, we can choose $b = v$ by Corollary 4.3.2. Thus we have

$$\mathrm{St}(v, \mathrm{St}^N(K,A)) \subseteq (f \circ h)^{-1}(\mathrm{St}(f(v), L)).$$

Thus the simplicial map g sends v to $f(v)$ for each vertex $v \in A$. It follows that $g|_{|A|} = f|_{|A|}$. The proof is finished now. □

THEOREM 4.4.5 (Relative Simplicial Approximation Theorem). *Let K and L be a simplicial complexes and let A be a subcomplex of K. Let $f\colon |K| \to |L|$ be any continuous map such that $f|_{|A|}$ is a simplicial map. Suppose that there are finitely many simplices of $K \smallsetminus A$. Then there exists N and a simplicial map $g\colon \mathrm{sd}^N(K,A) \to L$ such that $g|_{|A|} = f|_{|A|}$ and g is homotopic to f relative to $|A|$.*

PROOF. Let g be the simplicial map in Proposition 4.4.4. Then $g|_{|A|} = f|_{|A|}$ and g is homotopic to $f \circ h$ relative to A by Theorem 4.1.4. Since h is a simplicial approximation to the identity map with $h|_A = \mathrm{id}_A$, $h \simeq \mathrm{id}_{|K|}$ relative to A and so $f \circ h \simeq f$ relative to A. It follows that $g \simeq f$ relative to A. □

4.5. Some Applications. Recall that the homotopy groups $\pi_n(X)$ is defined by

$$\pi_n(X) = [S^n, X]$$

the set of homotopy classes of the pointed continuous maps from S^n to X up to pointed homotopy (namely homotopy relative to the basepoint). A direct consequence of simplicial approximation theorem is as follows:

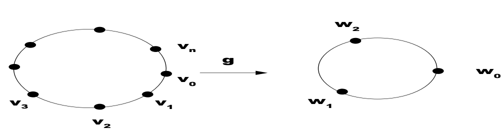
THEOREM 4.5.1. *$\pi_r(S^n) = 0$ for $r < n$.*

PROOF. Let $f\colon S^r \to S^n$ be a pointed continuous map. Let K be the simplicial complex such that $|K| \cong S^r$ and let L be the simplicial complex such that $|L| = S^n$. (We can choose K and L as the boundary of an $(r+1)$-simplex and an $(n+1)$-simplex, respectively.) Consider S^r and S^n as polyhedron of simplicial complexes. By the simplicial approximation theorem, there exists a subdivision K' of K and a simplicial map $g\colon |K'| = |K| \cong S^r \to |L| \cong S^n$ such that g is homotopic to f relative to the basepoint. Since g is a simplicial map, the map g must send K' into the skeleton $\mathrm{sk}_r(L)$ because K' is an r-dimensional simplicial complex. Since $r < n$, $\mathrm{sk}_r(L) \neq L$ and so g is not onto. Thus there is a point $x \in S^n$ such that $g(S^r) \subseteq S^n \smallsetminus \{x\} \cong \mathbb{R}^n$. Hence g is homotopic to the constant map relative to the basepoint by a linear homotopy. It follows that f is homotopic to the constant map relative to the basepoint. □

Another direct consequence is to give a computation of the fundamental group.

THEOREM 4.5.2. *$\pi_1(S^1) = \mathbb{Z}$.*

PROOF. By simplicial approximation theorem, we only need to consider the simplicial maps from a subdivision K' of the circle K to the circle K, where K is the boundary of a 2-simplex with vertices w_0, w_1, w_2 in the order of counter-clockwise along the circle, where w_0 is regarded as the basepoint. Let K' has vertices $v_0, v_1, \ldots, v_n$ in the order of counter-clockwise along the circle, where v_0 is regarded as the basepoint. Let $g\colon K' \to K$ be a pointed simplicial map. See the picture:



Then g maps the circle $(v_0, v_1, \ldots, v_n, v_0)$ (in order) into the sequence

$$(g(v_0), g(v_1), \ldots, g(v_n), g(v_0)),$$

where $g(v_i)$ is one of w_j with $g(v_0) = w_0$. If $g(v_i) = g(v_{i+1})$, then g is constant on the 1-simplex $[v_i, v_{i+1}]$ and so, up to homotopy, we can remove $g(v_{i+1})$. For the subsequence $(g(v_i), g(v_{i+1}), g(v_{i+2}))$, if $g(v_i) = g(v_{i+2})$, then $(g(v_i), g(v_{i+1}), g(v_{i+2}))$ is a path from $g(v_i)$ to $g(v_{i+1})$ and backwards to $g(v_i)$ and so we can collapse this subsequence. Assume that $v_0, \ldots, v_n$ have minimal number of vertices in the homotopy class of g. Then the only sequence of $(g(v_0), g(v_1), \ldots, g(v_n), g(v_0))$ is given by one of the following:

(1). constant map (w_0),

(2). around the circle n-times positively:

$$(w_0, w_1, w_2, w_0, w_1, w_2, w_0, \ldots, w_0, w_1, w_2, w_0),$$

or

(3). around the circle m-times negatively:

$$(w_0, w_2, w_1, w_0, w_2, w_1, w_0, \ldots, w_0, w_2, w_1, w_0).$$

This shows that any pointed continuous map $f\colon S^1 \to S^1$ is homotopic to one of the maps

$$g_n\colon S^1 \to S^1 \quad z \mapsto z^n$$

for $n \in \mathbb{Z}$, relative to the basepoint. In particular, $\pi_1(S^1)$ is generated by $[g_1]$. On the other hand, one can show that g_n is not homotopic to g_m relative to the basepoint if $n \neq m$, as g_n is the continuous mapping that goes around S^1 for n times. This gives that $\pi_1(S^1) = S^1$. $\square$

Example 4.5.1 (Cohomotopy Sets). Let X be a pointed space. Then the n-th cohomotopy set is defined to be

$$\pi^n(X) = [X, S^n],$$

the set of homotopy classes of pointed continuous maps from X to the n-sphere up to pointed homotopy. For instance, $\pi_r(S^n) = [S^r, S^n] = \pi^n(S^r)$. The fundamental problem in algebraic topology is to determine $\pi_r(S^n)$ for general r and n. When $n = 1$, it was known that $\pi_r(S^1) = 0$ for $r \neq 1$ and $\pi_1(S^1) = \mathbb{Z}$. When $n > 1$, we know from the above theorem that $\pi_r(S^n) = 0$ for $r < n$. When $r = n$, $\pi_n(S^n) = \mathbb{Z}$. (This is a direct consequence of Hurewicz Theorem which gives that, for simply connected spaces, the first non-trivial homotopy group is the same as the corresponding homology group. Or one can try to directly compute $\pi_n(S^n)$ using simplicial methods or other methods.) For $r > n$, $\pi_r(S^n)$ is known up to certain range by very nontrivial works contributed by many topologists, but far unknown for general r even if $n = 2$. By using simplicial methods, one might try a different approach to study the higher homotopy groups of spheres.

Consider S^n as the boundary of an $(n+1)$-simplex σ^{n+1}. So, as a simplicial complex, S^n has $(n+2)$ vertices labeled in order by $\{0, 1, 2, \ldots, n+1\}$. Observe that, in σ^{n+1}, any nonempty subset of the vertices $\{0, 1, 2, \ldots, n+1\}$ spans a simplex of σ^{n+1}. Since S^n is the boundary of σ^{n+1}, any *proper* subset of $\{0, 1, 2, \ldots, n+1\}$ spans a simplex of S^n.

Let K be a simplicial complex and let $\mathrm{sk}_0\, K$ be the set of vertices of K. Suppose that $f\colon K \to S^n$ is a simplicial map. Then f sends vertices of K to the vertices of S^n. Thus, for each vertex v of K, there is a color $f(v) = i$ for some $0 \leq i \leq n+1$.

This colorizes the vertices of K by using $(n+2)$ different colors. Suppose that $v^0, v^1, \dots, v^r$ span a simplex of K. Then

$$\{f(v^0), f(v^1), \dots, f(v^r)\}$$

must span a simplex of S^n, equivalently, $\{f(v^0), f(v^1), \dots, f(v^r)\}$ is a proper subset of $\{0, 1, \dots, n+1\}$. Namely at least at one color is missing in $\{f(v^0), f(v^1), \dots, f(v^r)\}$.

Conversely suppose that there is a coloration on the vertices of K using $(n+1)$ different colors, that is there is a function from $\mathrm{sk}_0 K$ to $\{0, 1, \dots, n\}$, such that if $v^0, v^1, \dots, v^r$ span a simplex of K, then at least one color is missing among the colors of $v^0, v^1, \dots, v^r$. Then the coloration induces a unique simplicial map from K to S^n.

This establishes the one-to-one correspondence between the set of simplicial maps from K to S^n and the set of colorations on vertices of K that satisfying the above rule. Thus, for studying simplicial maps from K to S^n, one can study how to make those colorations on the vertices of K satisfying the above rule. For studying $\pi_r(S^n)$, one needs to understand:

(1). The colorations on the vertices of K satisfying the above rule, where $|K| \cong S^r$, namely K is a triangulation of S^r. This will give simplicial maps $K \to S^n$. In general, one may study the colorations on the vertices of K, where K is a triangulation of a manifold.

(2). The colorations on the vertices of K satisfying the above rule, where $|K| \cong S^r \times [0,1]$. This will control the homotopy.

The above two problems are not well-understood so far. But it will be very interesting if one could make progress on this as it attacks the fundamental open problem in algebraic topology. □

4.6. Fundamental Groupoids and Fundamental Groups of Simplicial Complexes. By using simplicial approximation theorem, we can compute the fundamental group of a simplicial complex K using the simplicial structure in K.

DEFINITION 4.6.1. Let K be a simplicial complex. An *edge path* in K, from a vertex v^0 to a vertex v^n, is a sequence of vertices $\alpha = (v^0, v^1, \dots, v^n)$ such that for each $0 \le i \le n-1$, v^i, v^{i+1} spans a simplex of K (that is $v^i = v^{i+1}$ or $v^i v^{i+1}$ is a 1-simplex). Let i_α denote the initial vertex v^0 of α and let e_α denote the ending vertex v^n of α. An *edge loop* at v means an edge path α with $i_\alpha = e_\alpha = v$.

Let $\alpha = (v^0, v^1, \dots, v^n)$ and $\beta = (w^0, w^1, \dots, w^m)$ be two edge paths in K with $v^n = w^0$. Then the *product* $\alpha \star \beta$ is the edge path given by $(v^0, v^1, \dots, v^n, w^1, \dots, w^m)$. In this definition, the product $\alpha \star \beta$ is well-defined if and only if $e_\alpha = i_\beta$. The inverse path of α is defined by

$$\alpha^{-1} = (v^n, v^{n-1}, \dots, v^0).$$

Clearly the associativity of the product

$$(\alpha \star \beta) \star \gamma = \alpha \star (\beta \star \gamma) \tag{4.6.1}$$

holds if $e_\alpha = i_\beta$ and $e_\beta = i_\gamma$. So we can write $\alpha \star \beta \star \gamma$ for $(\alpha \star \beta) \star \gamma$ or $\alpha \star (\beta \star \gamma)$ whence it is well-defined. Moreover the product has the left and right identities:

$$(i_\alpha) \star \alpha = \alpha \quad \text{and} \quad \alpha \star (e_\alpha) = \alpha. \tag{4.6.2}$$

From Equations (4.6.1) and (4.6.2), let v be a fixed vertex of K, the set

$$\Omega^{\mathrm{edge}}(K; v) = \{\alpha \mid \alpha \text{ is an edge path in } K \text{ with } i_\alpha = e_\alpha = v\} \tag{4.6.3}$$

is a monoid under the product $\star$.

DEFINITION 4.6.2. Let K be a simplicial complex. Two edge paths α and β are called *equivalent*, denoted by $\alpha \sim \beta$, if one can be obtained from another by a finite sequence of the following elementary equivalences or their inverses

(a) $(\ldots, v^i, v^i, \ldots) \sim (\ldots, v^i, \ldots)$;
(b) $(\ldots, v^{i-1}, v^i, v^{i+1}, \ldots) \sim (\ldots, v^{i-1}, v^{i+1}, \ldots)$ if v^{i-1}, v^i, v^{i+1} spans a simplex of K (not necessarily 2-dimensional).

In other words, $\sim$ is the equivalence relation generated by the above two elementary equivalences.

PROPOSITION 4.6.3. *Let $\alpha_1, \beta_1, \alpha_2, \beta_2$ be edge paths in K such that $\alpha_1 \sim \beta_1$ and $\alpha_2 \sim \beta_2$. Then*

(1). *If $e_{\alpha_1} = i_{\alpha_2}$, then $\alpha_1 \star \alpha_2 \sim \beta_1 \star \beta_2$.*
(2). $\alpha_1^{-1} \sim \beta_1^{-1}$.
(3). $\alpha_1 \star \alpha_1^{-1} \sim (i_{\alpha_1})$ *and* $\alpha_1^{-1} \star \alpha_1 \sim (e_{\alpha_1})$.

PROOF. The proof follows from the definition of the equivalence relation $\sim$. □

DEFINITION 4.6.4. Let K be a simplicial complex. The *simplicial fundamental groupoid* of K is the category $\mathcal{C}^\pi(K)$ whose objects are the vertices of K and whose morphisms from a vertex v to a vertex w are the equivalence classes of edge paths from v to w. The composition operation in the category $\mathcal{C}^\pi(K)$ is induced by the $\star$ product. Fixing a vertex v of K, the *simplicial fundamental group* $\pi(K, v)$ is defined to the set of the equivalence classes of edge paths from v to v. Note that $\pi(K, v)$ is the set of the $\mathcal{C}^\pi(K)$-morphisms from v to v, which is a group with the multiplication induced by the $\star$ product.

PROPOSITION 4.6.5. *Let K and L be simplicial complexes and let $f\colon K \to L$ be a simplicial map. Let α and β be two edge paths in K with $\alpha \sim \beta$. Then*

$$f(\alpha) \sim f(\beta).$$

Thus f induces a functor $f_*\colon \mathcal{C}^\pi(K) \to \mathcal{C}^\pi(L)$ and a group homomorphism

$$f_*\colon \pi(K, v) \to \pi(L, f(v)).$$

PROOF. The proof follows immediately from the definition of the equivalence relation. □

Let X be a space. Recall that the *fundamental groupoid* $\mathcal{C}^{\pi_1}(X)$ of X is the category whose objects are the points in X and whose morphisms from a point a to b are the path homotopy classes of paths from a to b. Intuitively the fundamental groupoid is given by draw morphisms as path homotopy classes between any two points in X. The fundamental group is then given by the path homotopy classes of the loops in X.

Let $\mathcal{C}$ be category. A *full subcategory* of $\mathcal{C}$ means a category $\mathcal{C}_0$ whose objects are contained in $\mathcal{C}$ and the morphisms between two objects in $\mathcal{C}_0$ are all of the $\mathcal{C}$-morphism between these two objects. For instance, let **Set** be the category of sets, **Group** be the category of group and **Ab** be the category of abelian groups. Then **Group** is not a full subcategory of **Set** because the morphisms between two groups in **Group** are homomorphisms. But **Ab** is a full subcategory of **Group**. We are going to show that $\mathcal{C}^\pi(K)$ is a full subcategory of $\mathcal{C}^{\pi_1}(|K|)$. As a consequence, we obtain that $\pi(K, v) = \pi_1(|K|, v)$, namely the simplicial fundamental group coincides with the fundamental group.

We can consider an edge path in K as a path in the polyhedron $|K|$. More precisely, given an edge path $\alpha = (v^0, v^1, \ldots, v^n)$ in K, define a path

$$\lambda_\alpha \colon [0,1] \to |K|$$

such that $\lambda_\alpha\left(\frac{i}{n}\right) = v^i$ and λ_α is linear between i/n and $(i+1)/n$ for $0 \leq i \leq n-1$.

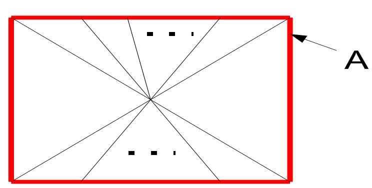
Lemma 4.6.6. *Let K be a simplicial complex and let α and β be two edge paths. Suppose that $i_\alpha = i_\beta$ and $e_\alpha = e_\beta$. Then $\alpha \sim \beta$ if and only if $\lambda_\alpha \simeq \lambda_\beta$.*

Proof. $\Longrightarrow$ If $\alpha \sim \beta$ is given by an elementary equivalence or its inverse, it is directly to construct a path homotopy from λ_α to λ_β. Since β is obtained from α by a finite sequence of the elementary equivalences, we have $\lambda_\alpha \sim \lambda_\beta$.

$\Longleftarrow$ Let $\alpha = (v^0, v^1, \ldots, v^n)$ and $\beta = (w^0, w^1, \ldots, w^m)$ with $v^0 = w^0$ and $v^n = w^m$. Suppose that $\lambda_\alpha \simeq \lambda_\beta$. Then there is a homotopy

$$F \colon I \times I \to |K|$$

such that $F(s, 0) = \lambda_\alpha(s)$, $F(s, 1) = \lambda_\beta(s)$, $F(0, t) = v^0$ and $F(1, t) = v^n$ for $0 \leq s, t \leq 1$. Let $I \times I$ be triangulated by the simplicial complex T given by joining the line segments from $(1/2, 1/2)$ to the points given by $(i/n, 0)$ and $(j/m, 1)$ for $0 \leq i \leq n$ and $0 \leq j \leq m$. Let A be the simplicial subcomplex of T with $|A|$ given by the boundary of $I \times I$. The picture is as follows:



Then $F|_{|A|}$ is a simplicial map. By the Relative Simplicial Approximation Theorem, there exists N and a simplicial map

$$g\colon \operatorname{sd}^N(T,A) \to K$$

such that $g|_{|A|} = F|_{|A|}$ and $g \simeq F \mathrm{rel}\ |A|$.

Let θ_1 be the edge path in T given by

$$\theta_1 = ((0,0),(0,1),(1/m,1),(2/m,1),\ldots,(1,1),(1,0)),$$

that is the edge path goes through the left edge, top edge and right edge of $I \times I$. Let θ_2 be the edge path in T given by

$$\theta_2 = ((0,0),(1/n,0),(2/n,0),\ldots,(1,0)),$$

that is θ_2 is the bottom edge of $I \times I$. Then, in the simplicial complex $\operatorname{sd}^N(T,A)$, we have

$$\theta_1 \sim \theta_2$$

because θ_2 can be obtained from θ_1 by a finite sequence of elementary equivalences or their inverses. (**Note.** The iterated relative barycentric subdivisions of (T,A) is a triangulation of $I \times I$ without changing the simplicial structure on the boundary of $I \times I$. The equivalence between θ_1 and θ_2 can be obtained by moving down the edge path θ_1 to θ_2 through 2-simplices in $\operatorname{sd}^N(T,A)$.)

Now we have the following:

$$\begin{aligned}
\alpha &= (v^0,v^1,\ldots,v^n)\\
&= (g(0,0),g(1/n,0),g(2/n,0),\ldots,g(1,0))\\
&\quad\ \text{because } g|_{|A|} = F|_{|A|}\\
&= g(\theta_2)\\
&\sim g(\theta_1)\\
&\quad\ \text{because } g \text{ is simplicial}\\
&= (g(0,0),g(0,1),g(1/m,1),g(2/m,1),\ldots,g(1,1),g(1,0)\\
&= (w^0,w^0,w^1,w^2,\ldots,w^m,v^n)\\
&\sim (w^0,w^1,w^2,\ldots,w^m)\\
&\quad\ \text{because } w^m = v^n\\
&= \beta.
\end{aligned}$$

The proof is finished. □

THEOREM 4.6.7. *Let K be a simplicial complex. Then $\mathcal{C}^\pi(K)$ is a full subcategory of $\mathcal{C}^{\pi_1}(|K|)$.*

PROOF. The objects in $\mathcal{C}^\pi(K)$ are the vertices of K and so we can identify the objects of $\mathcal{C}^\pi(K)$ as the objects of $\mathcal{C}^{\pi_1}(|K|)$. Consider the function $\alpha \mapsto \lambda_\alpha$ from edge paths in K to paths in $|K|$. By the definition of path product, we have

$$\lambda_{\alpha\star\beta} \simeq \lambda_\alpha * \lambda_\beta.$$

By Lemma 4.6.6, if $\alpha \sim \beta$, then $\lambda_\alpha \simeq \lambda_\beta$. Thus the function $\alpha \mapsto \lambda_\alpha$ from edge paths in K to paths in $|K|$ induces a functor

$$\lambda\colon \mathcal{C}^\pi(K) \to \mathcal{C}^{\pi_1}(|K|).$$

We show that

$$\lambda\colon \operatorname{Hom}_{\mathcal{C}^\pi(K)}(v,w) \to \operatorname{Hom}_{\mathcal{C}^{\pi_1}(|K|)}(v,w)$$

is an isomorphism for any vertices v and w.

By Lemma 4.6.6, $\alpha \sim \beta$ if $\lambda_\alpha \simeq \lambda_\beta$. Thus $\lambda\colon \operatorname{Hom}_{\mathcal{C}^{\pi}(K)}(v,w) \to \operatorname{Hom}_{\mathcal{C}^{\pi_1}(|K|)}(v,w)$ is one-to-one.

Let μ be a path in $|K|$ from v to w. By the Relative Simplicial Approximation Theorem, there exists N and a simplicial map

$$g\colon \operatorname{sd}^N(I,\{0,1\}) \to K$$

such that $g \simeq \mu$ relative to $\{0,1\}$, that is g is path homotopic to μ. The simplicial map $g\colon \operatorname{sd}^N(I,\{0,1\}) \to K$ defines an edge path α of K with

$$\lambda_\alpha \simeq g$$

under path homotopy. It follows that $\lambda\colon \operatorname{Hom}_{\mathcal{C}^{\pi}(K)}(v,w) \to \operatorname{Hom}_{\mathcal{C}^{\pi_1}(|K|)}(v,w)$ is onto and hence the result. □

COROLLARY 4.6.8. *Let K be a simplicial complex and let v be a vertex of K. Then $\pi(K,v) \cong \pi_1(|K|,v)$.*

PROOF. By above theorem,

$$\pi(K,v) = \operatorname{Hom}_{\mathcal{C}^{\pi}(K)}(v,v) \xrightarrow[\cong]{\lambda} \pi_1(|K|,v) = \operatorname{Hom}_{\mathcal{C}^{\pi_1}(|K|)}(v,v).$$

□

EXAMPLE 4.6.1. Let K be the boundary of a 2-simplex. The vertices of K has v^0, v^1 and v^2 with 1-simplices v^0v^1, v^1v^2 and v^0v^2. The edge loops at v^0 are given by (v^0,v^1,v^2,v^0), (v^0,v^2,v^1,v^0), $(v^0,v^1,v^2,v^0,v^1,v^2,v^0)$ and etc. From this information, we can see that $\pi(K,v^0)=\mathbb{Z}$ and so $\pi_1(S^1)=\mathbb{Z}$. □

A simplicial complex K is called *path-connected* if the polyhedron $|K|$ is path-connected. A simplicial complex K is called *simply connected* if $|K|$ is path-connected and $\pi_1(|K|,x_0)=\{1\}$ for some $x_0 \in |K|$.

COROLLARY 4.6.9. *Let K be a simply connected simplicial complex and let v and w be two vertices of K. Let α and β be two edge paths from v to w. Then $\alpha \sim \beta$.*

PROOF. Since $|K|$ is path-connected and $\pi_1(|K|,x_0)=\{1\}$ for some $x_0 \in |K|$, $\pi_1(|K|,x)=\{1\}$ for any point $x \in |K|$. Thus

$$\pi(K,v) \cong \pi_1(|K|,v) = \{1\}.$$

It follows that $\alpha \star \beta^{-1} \sim (v)$ and so

$$\alpha \sim \alpha \star \beta^{-1} \star \beta \sim (v) \star \beta \sim \beta$$

and hence the result. □

A 1-dimensional simplicial complex A of K is called a *tree* in K if $|A|$ is simply connected. A tree A of K is called *maximal* if, for any tree B of K with $A \subseteq B$, we have $A = B$.

LEMMA 4.6.10. *Let K be a path-connected simplicial complex. Then*

(1). *There exists a maximal tree of K.*
(2). *Any path-connected simplicial subcomplex of a tree is also a tree.*
(3). *A tree A of K is maximal if and only if A contains all vertices of K.*

PROOF. (1) We use Zorn Lemma to show the existence of maximal trees. The trees in K are partially ordered by the inclusions. By applying Zorn Lemma, it suffices to show that every chain of the trees has an upper bound of a tree. Let $\{A_\alpha \mid \alpha \in J\}$ be a chain of trees, that is the index set J is a well-ordered set and $A_\alpha \subseteq A_\beta$ for $\alpha < \beta$. Let

$$A = \bigcup_{\alpha \in J} A_\alpha.$$

To see that A is path-connected, let v and w be two vertices in A. Then there exists α such that $v, w \in A_\alpha$. Since A_α is a tree, A_α is path-connected and so there is an edge path in A_α (and so in A) joining v and w. Now let $\gamma = (v^0, v^1, \ldots, v^n)$ be an edge loop in A. Since γ is a finite union of 1-simplices and $\{A_\alpha\}$ is a chain, there exists α such that $\gamma \in A_\alpha$. Since A_α is simply connected, $\gamma \sim (v_0)$ in A_α (and so in A) by Corollary 4.6.9. Thus A is simply connected and so A is a tree.

(2) Let A be a tree and let A_0 be a path-connected simplicial subcomplex of A. Let v^0 be a vertex of A_0. It suffices to show that $\pi(A_0, v^0) = \{1\}$.

Let α and β be two edge paths in A_0. Denote $\alpha \sim_A \beta$ if $\alpha \sim \beta$ in A (that is there is a finite sequence of elementary equivalences in the simplicial complex A that moves α to β), and $\alpha \sim_{A_0} \beta$ if $\alpha \sim \beta$ in A_0.

Let us make an observation on the equivalence relation in A. The first type elementary relation is just given by repeating a vertex in an edge path. The second type elementary relation is given as follows: Let v, v', w be vertices of A such that v, v', w spans a simplex of A. Since A has no 2-simplices, at least two of v, v', w are the same. Thus the second type elementary equivalence becomes:

a) $(\cdots, v^{i-1}, v^{i-1}, v^{i-1}, \cdots) \sim (\cdots, v^{i-1}, v^{i-1}, \cdots)$.
b) $(\cdots, v^{i-1}, v^{i-1}, v^{i+1}, \cdots) \sim (\cdots, v^{i-1}, v^{i+1}, \cdots)$.
c) $(\cdots, v^{i-1}, v^{i}, v^{i}, \cdots) \sim (\cdots, v^{i-1}, v^{i}, \cdots)$.
d) $(\cdots, v^{i-1}, v^{i}, v^{i-1}, \cdots) \sim (\cdots, v^{i-1}, v^{i-1}, \cdots)$.

Together with the first type elementary relation, the equivalence relation in A is given by finite sequences of the first type elementary relation and Relation (d) above.

Given two edge paths α, β in A_0 such that $\alpha \sim_A \beta$, we may assume that β is obtained by a single elementary relation from α. Thus α and β are given either in the form $(\cdots, v^i, v^i, \cdots)$ and $(\cdots, v^i, \cdots)$ or $(\cdots, v^{i-1}, v^i, v^{i-1}, \cdots)$ and $(\cdots, v^{i-1}, v^{i-1}, \cdots)$. In both cases, $\alpha \sim_{A_0} \beta$. It follows that

$$\alpha \sim_A \beta \quad \Longrightarrow \alpha \sim_{A_0} \beta.$$

Now let α be an edge loop at v^0 in A_0. By considering α as an edge loop in A, since A is simply connected, $\alpha \sim_A (v^0)$ in A. From the above, $\alpha \sim_{A_0} (v^0)$. It follows that $\pi(A_0, v^0) = \{1\}$ and so A_0 is a tree.

(3) Suppose that A is a tree in K such that A contains all vertices of K. Then clearly A is a maximal tree in K. Now suppose that there exists a vertex w of K such that $w \notin A$. Choose v to be a vertex in A. By the assumption, $|K|$ is path-connected and so there is a path from v to w. By Theorem 4.6.7, there is an edge path $\alpha = (v^0, v^1, \ldots, v^n)$ with $v^0 = v$ and $v^n = w$. Let

$$r = \max\{i \mid v^0, v^1, \ldots, v^i \in A\}.$$

Then $r < n$, $v^r \in A$, $v^{r+1} \notin A$ and $v^r v^{r+1}$ spans a 1-simplex of K because $v^r \neq v^{r+1}$. Now let B be the simplicial subcomplex of K given by adding the

1-simplex $v^r v^{r+1}$ and the vertex v^{r+1} to A. Then $|B| \simeq |A|$ because $|A|$ is obtained from $|B|$ by collapsing the 1-simplex $v^r v^{r+1}$. It follows that B is a tree in K with A a proper simplicial subcomplex of B. Thus A is not maximal. □

For a path-connected simplicial complex K, there is a combinatorial methods for computing the fundamental group described as follows. Let A be a fixed maximal tree in K and let the set of vertices of K be totally ordered. Define a group $\mathrm{Gp}(K)$ combinatorially as follows: The generators of $\mathrm{Gp}_A(K)$ are given by the letters

$$g_{vw},$$

where vw is a 1-simplex of K with $v < w$ with defining relations given by

(1). $g_{vw} = 1$ if the 1-simplex vw lies in A and

(2). $g_{vw} = g_{vv'} g_{v'w}$ if $v < v' < w$ and $vv'w$ is a 2-simplex of K.

Equivalently $\mathrm{Gp}_A(K)$ is generated by g_{vw}, where vw is a 1-simplex in $K \smallsetminus A$ with $v < w$, subject to the defining relation given in (2).

Theorem 4.6.11. *Let K be a path-connected simplicial complex and let v^0 be a vertex of K. Then* $\mathrm{Gp}_A(K) \cong \pi(K, v^0)$.

Proof. Let v be any vertex of K. By Lemma 4.6.10, $v \in A$. Since $|A|$ is contractible, $|A|$ is path-connected and so there is an edge path $\alpha_{v^0,v}$ in A from v^0 to v. Let $\alpha'_{v^0,v}$ be another edge path in A from v^0 to v. Then

$$\alpha_{v^0,v} \sim \alpha'_{v^0,v}$$

because $|A|$ is contractible. Thus the edge path $\alpha_{v^0,v}$ is unique up to equivalence.

For any 1-simplex vw of K with $v < w$, let

$$\theta(g_{vw}) = [\alpha_{v^0,v} \star (v, w) \star \alpha^{-1}_{v^0,w}]$$

be the equivalence class of the edge path $\alpha_{v^0,v} \star (v, w) \star \alpha^{-1}_{v^0,w}$. If $vv'w$ spans a 2-simplex of K with $v < v' < w$, then

$$\begin{aligned}
\theta(g_{vv'})\theta(g_{v'w}) &= [\alpha_{v^0,v} \star (v, v') \star \alpha^{-1}_{v^0,v'} \star \alpha_{v^0,v'} \star (v', w) \star \alpha^{-1}_{v^0,w}] \\
&= [\alpha_{v^0,v} \star (v, v') \star (v', w) \star \alpha^{-1}_{v^0,w}] \\
&= [\alpha_{v^0,v} \star (v, w) \star \alpha^{-1}_{v^0,w}] \\
&= \theta(g_{vw}).
\end{aligned}$$

If vw is a 1-simplex of A, then $\theta(g_{vw}) = [\alpha_{v^0,v} \star (v, w) \star \alpha^{-1}_{v^0,w}]$ is a loop in A. Since $|A|$ is contractible, $\theta(g_{vw}) = 1$. Thus the function $g_{vw} \mapsto \theta(g_{vw})$ defines a group homomorphism

$$\theta\colon \mathrm{Gp}_A(K) \longrightarrow \pi(K, v^0).$$

Now we construct a group homomorphism $\phi\colon \pi(K, v^0) \to \mathrm{Gp}_A(K)$ as follows: Let

$$\alpha = (v^0, v^1, \ldots, v^n)$$

be an edge path in K with $v^n = v^0$. Write it as a product of edge paths

$$\alpha = (v^0, v^1) \star (v^1, v^2) \star \cdots \star (v^{n-2}, v^{n-1}) \star (v^{n-1}, v^0).$$

For each edge path (v^i, v^{i+1}), define $\phi(v^i, v^{i+1}) \in \mathrm{Gp}_A(K)$ by setting

$$\phi(v^i, v^{i+1}) = \begin{cases} 1 & \text{if} \quad v^i = v^{i+1} \\ g_{v^i v^{i+1}} & \text{if} \quad v^i < v^{i+1} \\ g^{-1}_{v^i v^{i+1}} & \text{if} \quad v^i > v^{i+1} \end{cases}$$

and define

$$\phi(\alpha) = \phi(v^0, v^1)\phi(v^1, v^2)\cdots\phi(v^{n-2}, v^{n-1})\phi(v^{n-1}, v^0) \in \mathrm{Gp}(K).$$

Observe that if v^{i-1}, v^i, v^{i+1} spans a simplex of K, then

$$\phi(v^{i-1}, v^i)\phi(v^i, v^{i+1}) = \phi(v^{i-1}, v^{i+1})$$

in $\mathrm{Gp}_A(K)$. Thus if $\alpha \sim \beta$, then $\phi(\alpha) \sim \phi(\beta)$. Clearly $\phi(\alpha \star \beta) = \phi(\alpha)\phi(\beta)$. Thus the function ϕ defines a group homomorphism

$$\phi\colon \pi(K, v^0) \longrightarrow \mathrm{Gp}_A(K).$$

We check that ϕ is the inverse of θ. Let vw be a 1-simplex of K with $v < w$. Then

$$\begin{aligned} \phi \circ \theta(g_{vw}) &= \phi[\alpha_{v^0,v} \star (v, w) \star \alpha^{-1}_{v^0,w}] \\ &= \phi(\alpha_{v^0,v)}\phi(v, w)\phi(\alpha_{v^0,w})^{-1} \\ &= \phi(\alpha_{v^0,v)} g_{vw} \phi(\alpha_{v^0,w})^{-1} \end{aligned}$$

Since $\alpha_{v^0,v}$ is an edge path in A, $\phi(\alpha_{v^0,v})$ is a product of $g^{\pm}_{v',w'}$ with the property that $v'w'$ spans a 1-simplex of A with $v' < w'$. Thus $\phi(\alpha_{v^0,v}) = 1$. Similarly $\phi(\alpha_{v^0,w}) = 1$. Thus $\phi \circ \theta = \mathrm{id}$. Now let

$$\alpha = (v^0, v^1, \ldots, v^n)$$

be an edge path of K with $v^n = v^0$. Then

$$\begin{aligned} \theta \circ \phi(\alpha) &= \theta(\phi(v^0, v^1))\theta(\phi(v^1, v^2))\cdots\theta(\phi(v^{n-2}, v^{n-1}))\theta(\phi(v^{n-1}, v^0)) \\ &= [\alpha_{v^0,v^0} \star (v^0, v^1) \star \alpha^{-1}_{v^0,v^1}][\alpha_{v^0,v^1} \star (v^1, v^2) \star \alpha^{-1}_{v^0,v^2}] \\ &\quad \cdots [\alpha_{v^0,v^{n-1}} \star (v^{n-1}, v^0) \star \alpha^{-1}_{v^0,v^0}] \\ &= [\alpha_{v^0,v^0} \star (v^0, v^1) \star \alpha^{-1}_{v^0,v^1} \star \alpha_{v^0,v^1} \star (v^1, v^2) \star \alpha^{-1}_{v^0,v^2} \star \\ &\quad \cdots \star \alpha_{v^0,v^{n-1}} \star (v^{n-1}, v^0) \star \alpha^{-1}_{v^0,v^0}] \\ &= [\alpha], \end{aligned}$$

where we use the formula

$$\theta(\phi(v^i, v^{i+1})) = \begin{cases} 1 = [\alpha_{v^0,v^i} \star (v^i, v^{i+1}) \star \alpha^{-1}_{v^0,v^i}] & \text{if} \quad v^i = v^{i+1} \\ [\alpha_{v^0,v^i} \star (v^i, v^{i+1}) \star \alpha^{-1}_{v^0,v^{i+1}}] & \text{if} \quad v^i < v^{i+1} \\ \theta(g^{-1}_{v^{i+1}v^i}) = \theta(g_{v^{i+1}v^i})^{-1} & \\ = [\alpha_{v^0,v^i} \star (v^i, v^{i+1}) \star \alpha^{-1}_{v^0,v^{i+1}}] & \text{if} \quad v^i > v^{i+1}. \end{cases}$$

Thus $\theta \circ \phi = \mathrm{id}$ and hence the result. □

COROLLARY 4.6.12. *Let K be a 1-dimensional simplicial complex. Then*

(1). *$\pi_1(|K|, v)$ is a free group for any vertex v.*

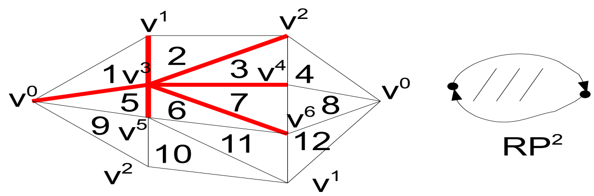
(2). *Let L be a simplicial subcomplex of K and let v be a vertex of L. Then $\pi(L, v)$ is a subgroup of $\pi(K, v)$ with* $\mathrm{rank}(\pi(L, v)) \leq \mathrm{rank}(\pi(K, v))$.

PROOF. (1) Let A be a maximal tree of K. By Theorem 4.6.11, $\pi(K, v)$ is the group generated by g_{vw} with $v < w$. Since K has no 2-simplices, the defining relations are given by $g_{vw} = 1$ for vw a 1-simplex in A. Thus $\pi(K, v)$ is the free group with a basis given by g_{vw} with $v < w$ and vw is not a 1-simplex in A.

(2) Let $A_0 = A \cap L$. Then, by (2) of Lemma 4.6.10, A_0 is also a tree. By (3) of Lemma 4.6.10, A contains all vertices of K and so A_0 contains all vertices of L. By applying (3) of Lemma 4.6.10 again, A_0 is a maximal tree of L. From Part (1), $\pi(L, v)$ is the free group with a basis given by g_{vw} with $v < w$, vw is a 1-simplex in $L \smallsetminus A_0$. The assertion follows.

□

EXAMPLE 4.6.2. Let K be a simplicial complex with twelve 2-simplices and their faces as in the following picture, where the maximal tree A is given by the red colored 1-simplices.



We compute $\pi(K, v^0)$ using the above theorem. Write g_{ij} for $g_{v^i v^j}$. If $v^i v^j$ is a 1-simplex of A, then $g_{ij} = 1$. So $\pi(K, v^0)$ is generated by g_{ij} for those 1-simplices $v^i v^j$ not in A. Thus there are 12 generators

$$\{g_{01}, g_{02}, g_{04}, g_{05}, g_{06}, g_{12}, g_{15}, g_{16}, g_{24}, g_{25}, g_{46}, g_{56}\}.$$

Now the twelve 2-simplices give 12 relations:

$$\begin{cases} g_{01} & = & 1 \\ g_{12} & = & 1 \\ g_{24} & = & 1 \\ g_{04} & = & g_{02}g_{24} = g_{02} \\ g_{05} & = & 1 \\ g_{56} & = & 1 \\ g_{46} & = & 1 \\ g_{06} & = & g_{04}g_{46} = g_{04} = g_{02} \\ g_{05} & = & g_{02}g_{25} \\ \Rightarrow & & g_{25} = g_{02}^{-1} \\ g_{15} & = & g_{12}g_{25} = g_{25} \\ g_{16} & = & g_{15}g_{56} = g_{15} = g_{25} \\ g_{06} & = & g_{01}g_{16} = g_{16} = g_{25} \\ \Rightarrow & & g_{25} = g_{02} \end{cases}$$

Thus $\pi(K, v^0)$ is generated by g_{02} with $g_{02} = g_{02}^{-1}$ and so $\pi(K, v^0) = \mathbb{Z}/2$. $\square$

As an application of Theorem 4.6.11, we have the Seifert-van Kampen Theorem for simplicial complexes.

THEOREM 4.6.13 (Seifert-van Kampen Theorem). *Let K be a simplicial complex and let K_1 and K_2 be simplicial subcomplexes of K such that $K = K_1 \cup K_2$ and $K_1 \cap K_2 \neq \emptyset$. Let v^0 be a vertex of $K_1 \cap K_2$. Suppose that $K_1 \cap K_2$, K_1 and K_2 are path-connected. Then*

$$\pi(K, v^0) \cong \pi(K_1, v^0) \coprod_{\pi(K_1 \cap K_2, v^0)} \pi(K_2, v^0)$$

the free product with amalgamation.

PROOF. Since K_1 and K_2 are path-connected with $K_1 \cap K_2 \neq \emptyset$, $K = K_1 \cup K_2$ is path-connected.

Let A_0 be a maximal tree in $K_1 \cap K_2$. Let A_1 be a maximal tree in K_1 contains A_0 and let A_2 be a maximal tree in K_2 contains A_0. Then $A_0 = A_1 \cap K_1 \cap K_2$ because $A_0 \subseteq A_1 \cap K_1 \cap K_2$ and A_0 is a maximal tree in $K_1 \cap K_2$. Similarly $A_0 = A_2 \cap K_1 \cap K_2$. Thus $A_0 = A_1 \cap A_2$. Let $A = A_1 \cup A_2$. We check that A is a maximal tree in K. Since A_1 and A_2 are path-connected and $A_1 \cap A_2 = A_0 \neq \emptyset$, $A = A_1 \cup A_2$ is path-connected. Note that A contains all vertices of K as A_1 contains all vertices of K_1 and A_2 contains all vertices of K_2. It suffices to show that $\pi(A, v^0) = \{1\}$. Let

$$\alpha = (v^0, v^1, \ldots, v^n)$$

be an edge path in A with $v^n = v^0$. We show that $\alpha \sim (v^0)$ by induction on the number $k(\alpha)$ of vertices v^i in α with $v^i \notin A_0$. If all $v^i \in A_0$, then $\alpha \sim (v^0)$ because $\pi(A_0, v^0) = \{1\}$. Thus the statement holds for $k(\alpha) = 0$. Suppose that the statement holds for edge paths β with $k(\beta) < k(\alpha)$. Note that $\alpha \sim (v^0)$ if all $v^i \in A_1$ and $\alpha \sim (v^0)$ if all $v^i \in A_2$. We may assume that there exist $v^i A_1 \smallsetminus A_2$ and $v^j \in A_2 \smallsetminus A_1$ for some i, j. We may assume that i is the smallest integer with $v^i \in A_1 \smallsetminus A_2$ and j is the smallest integer with $v^j \in A_2 \smallsetminus A_1$ with $i < j$. Then there exists $1 \leq r \leq n-2$ such that $v^1, \ldots, v^r \in A_1$ and $v^{r+1} \in A_2 \smallsetminus A_1$ and $v^t \in A_1 \smallsetminus A_2$ for some $1 \leq t \leq r$. Consider the 1-simplex $v^r v^{r+1}$ of A. Then $v^r v^{r+1} \in A_1$ if the barycenter $\widehat{v^r v^{r+1}} \in |A_1|$ and $v^r v^{r+1} \in A_2$ if the barycenter $\widehat{v^r v^{r+1}} \in |A_2|$. Since $v^{r+1} \notin A_1$, $v^r v^{r+1}$ is a 1-simplex of A_2. Thus $v^r \in A_2$ and so $v^r \in A_0 = A_1 \cap A_2$. Since A_0 is path-connected, there is an edge path $\alpha' = (v^0, w^1, \ldots, w^q, v^r)$ in A_0. Since A_1 is simply connected,

$$(v^0, v^1, \ldots, v^r) \sim (v^0, w^1, \ldots, w^q, v^r)$$

by Corollary 4.6.9. If follows that

$$\begin{aligned}\alpha = (v^0, v^1, \ldots, v^r, v^{r+1}, \ldots, \ldots, \ldots, v^n) &\sim \beta = (v^0, w^1, \ldots, w^q, v^r, v^{r+1}, \ldots, v^n)\\ &\sim (v^0)\\ &\quad\ \text{because } k(\beta) < k(\alpha).\end{aligned}$$

The induction is finished and so A is a maximal tree in K.

The group $\mathrm{Gp}_A(K)$ is generated by

$$g_{vw}$$

for 1-simplices vw of K with $v < w$, with the defining relations given by

(a) $g_{vw} = 1$ for 1-simplices $vw \in A$ with $v < w$ and
(b) $g_{vw} = g_{vv'} g_{v'w}$ for 2-simplex $vv'w$ of K with $v < v' < w$.

Now the $\mathrm{Gp}_{A_1}(K_1) \coprod_{\mathrm{Gp}_{A_1 \cap A_2}(K_1 \cap K_2)} \mathrm{Gp}_{A_2}(K_2)$ is generated by g_{vw} for 1-simplices vw of K_1 with $v < w$, and $g_{v',w'}$ for 1-simplices $v'w'$ with $v' < w'$. Note that the group $\mathrm{Gp}_{A_1 \cap A_2}(K_1 \cap K_2)$ is generated by $g_{vw} = g_{v'w'}$ for 1-simplices $vw = v'w'$ of $K_1 \cap K_2$. Thus the defining relations for the group

$$\mathrm{Gp}_{A_1}(K_1) \coprod_{\mathrm{Gp}_{A_1 \cap A_2}(K_1 \cap K_2)} \mathrm{Gp}_{A_2}(K_2)$$

can be given by

(1). $g_{vw} = g_{v'w'}$ for 1-simplices $vw = v'w'$ of $K_1 \cap K_2$;
(2). $g_{vw} = 1$ for 1-simplices vw of A_1;
(3). $g_{v'w'} = 1$ for 1-simplices $v'w'$ of A_2;
(4). $g_{v_1 v_3} = g_{v_1 v_2} g_{v_2 v_3}$ for 2-simplices (v_1, v_2, v_3) of K_1 with $v_1 < v_2 < v_3$, and
(5). $g_{v_1' v_3'} = g_{v_1' v_2'} g_{v_2' v_3'}$ for 2-simplices (v_1', v_2', v_3') of K_1 with $v_1' < v_2' < v_3'$.

By Relation (1), the group $\mathrm{Gp}_{A_1}(K_1) \coprod_{\mathrm{Gp}_{A_1 \cap A_2}} \mathrm{Gp}_{A_2}(K_2)$ is also generated by g_{vw} for 1-simplices vw of K with $v < w$.

Let $\sigma = vw$ be a 1-simplex of A. Since $|A| = |A_1| \cup |A_2|$, $\sigma \in A_1$ if the barycenter $\hat{\sigma} \in |A_1|$ and $\sigma \in A_2$ if $\hat{\sigma} \in |A_2|$. This σ is either in A_1 or A_2. It follows that Relations (2) and (3) are the same as Relation (a). Similarly Relations (4) and (5) are the same as Relation (b). The assertion follows by Theorem 4.6.11 now. □

EXERCISE 4.1. Determine all possible simplicial maps from $K \to S^1$, where K is a simplicial complex such that $|K| \cong S^2$.

Projects

Triangulations of Surfaces. A Hausdorff space M is called an *n-manifold* if each point of M has a neighborhood homeomorphic to an open set in $\mathbb{R}^n$. In this proposed project, one can work out a classification of triangulable 2-manifolds, namely 2-manifolds given by the polyhedron of a finite simplicial complex.

CHAPTER 2

Abstract Simplicial Complexes, Δ-sets and Simplicial Homology

Week 7 September 30 (T) October 3 (F): Sections 1.1, 1.2, 1.3
Week 8 October 7 (T) October 10 (F): Sections 2.1, 2.2, 2.3
Week 9 October 14 (T) October 17 (F): Sections 3.1, 3.2

1. Abstract Simplicial Complexes

1.1. Definition and Geometric Realizations of Abstract Simplicial Complexes.

Definition 1.1.1. An *abstract simplicial complex* $\mathcal{K}$ is a collection of finite nonempty sets, such that if A is an element in $\mathcal{K}$, so is every nonempty subset of A. The element A of $\mathcal{K}$ is called a *simplex* of $\mathcal{K}$; its *dimension* is one less than the number of its elements. Each nonempty subset of A is called a *face* of A. The *dimension* of $\mathcal{K}$ is the supremum of the dimensions of its simplices. The vertex set $V(\mathcal{K})$ is the union of the one-point elements of $\mathcal{K}$; we shall make no distinction between the vertex v and the 0-simplex $\{v\}$. A sub collection of $\mathcal{K}$ that is itself a complex is called a subcomplex of $\mathcal{K}$. Two abstract simplicial complexes $\mathcal{K}$ and $\mathcal{K}'$ are called to be *isomorphic* if there exists a bijective correspondence f mapping the vertex set of $\mathcal{K}$ to the vertex set of $\mathcal{K}'$ such that $\{a_0, a_1, \ldots, a_n\} \in \mathcal{K}$ if and only if $\{f(a_0), f(a_1), \ldots, f(a_n)\} \in \mathcal{K}'$.

Remark 1.1.2. *In some references, the empty set may be allowed in the definition of abstract simplicial complex. For such a case, we call augmented abstract simplicial complex. In other words, an augmented abstract simplicial complex $\mathcal{K}$ means a collection of finite sets such that if A is an element in $\mathcal{K}$, so is every subset of A.* □

Definition 1.1.3. Let K be a geometric simplicial complex. Let V be the vertex set of K. Let $\mathcal{K}$ be the collection of all subsets $\{a_0, a_1, \ldots, a_n\}$ of V such that $a_0, a_1, \ldots, a_n$ span a simplex of K. The collection $\mathcal{K}$ is called the *vertex scheme* of K, or *abstraction* of K. The geometric simplicial complex K is called a *geometric realization* of the abstract simplicial complex $\mathcal{K}$.

Theorem 1.1.4. *A relation between abstract simplicial complexes and geometric simplicial complexes is as follows:*

(a) *Every abstract simplicial complex $\mathcal{K}$ is isomorphic to the vertex scheme of some geometric simplicial complex.*
(b) *Two geometric simplicial complexes are linearly isomorphic if and only if their vertex schemes are isomorphic as abstract simplicial complexes.*

PROOF. We leave (b) as an exercise. To prove (a), we proceed as follows: Given an index set J, let Δ^J be the collection of all simplices in $\mathbb{E}^J$ spanned by finite subsets of the standard basis $\{e_\alpha\}_{\alpha\in J}$ for $\mathbb{E}^J$. It is easy to see that Δ^J is a simplicial complex. Moreover if σ and τ are two simplices of Δ^J, then their combined vertex set is geometrically independent and spans a simplex of Δ^J.

Now let $\mathcal{K}$ be an abstract simplicial complex with vertex set V. Choose the index set $J = V$. We specify a subcomplex K of Δ^J by the condition that for each abstract simplex $\{a_0, \ldots, a_n\} \in \mathcal{K}$, the geometric simplex spanned by $e_{a_0}, e_{a_1}, \ldots, e_{a_n}$ is to be in K. It is immediate that K is a geometric simplicial complex and $\mathcal{K}$ is isomorphic to the vertex scheme of K. □

By the above theorem, up to isomorphisms, geometric simplicial complexes are one-to-one correspondent to abstract simplicial complexes. As a tool, abstract simplicial complexes help to give some canonical constructions on simplicial complexes as well as to set up mathematical models in applications.

EXAMPLE 1.1.1. Let $V = \{a^0, a^1, \ldots, a^n\}$ be a nonempty finite set. Let $\mathcal{K}$ be the abstract simplicial complex given by all nonempty subsets of V. Then the geometric realization of $\mathcal{K}$ is an n-simplex. □

PROPOSITION 1.1.5. *Any finite geometric simplicial complex is linearly isomorphic to a simplicial subcomplex of a simplex.*

PROOF. Let K be a finite geometric simplicial complex and let $\mathcal{K}$ be its abstraction. Let $V = \{a^0, a^1, \ldots, a^n\}$ be the vertex set. Let $\mathcal{K}'$ be the abstract simplicial complex given by all nonempty subsets of V. Let K' be a geometric realization of $\mathcal{K}'$. Then K' is an n-simplex. Note that $\mathcal{K}$ is a simplicial subcomplex of $\mathcal{K}'$. By taking geometric realization, K is linearly isomorphic to a simplicial subcomplex of K'. □

1.2. Face Posets and Subdivision of Abstract Simplicial Complexes. Recall that a *partially ordered set*, or simply *poset*, V is a set together with a binary relation $\leq$ such that

(1). *idempotency*: $x \leq x$ for any $x \in V$;
(2). *antisymmetry*: for any $x, y \in V$, if $x \leq y$ and $y \leq x$, then $x = y$;
(3). *transitivity*: for any $x, y, z \in V$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

Given a poset V, let

$$\mathcal{S}(V) = \{\{a_0, a_1, \ldots, a_n\} \mid a_i \in V \ a_0 < a_1 < \cdots < a_n\}.$$

Then $\mathcal{S}(V)$ is an abstract simplicial complex with V as its vertex set because if $\{a_0, a_1, \ldots, a_n\}$ is well-ordered, then any nonempty subset of $\{a_0, a_1, \ldots, a_n\}$ is well-ordered. Roughly speaking $\mathcal{S}(V)$ is the set of all finite well-ordered sequences.

DEFINITION 1.2.1. Let $\mathcal{K}$ be an arbitrary abstract simplicial complex. A *face poset* of $\mathcal{K}$, denoted by $\mathrm{FP}(\mathcal{K})$, is the poset whose elements are given by all simplices of $\mathcal{K}$ with partial order relation given by the inclusions. In other words, $\mathrm{FP}(\mathcal{K})$ is the same set as $\mathcal{K}$ with partial order given by

$$A \leq B$$

if A is a face of B.

The subdivision of abstract simplicial complexes follows the ideas of barycentric subdivision of geometric simplicial complexes. Let $\mathcal{K}$ be an abstract simplicial complex. Then we have the face poset $\mathrm{FP}(\mathcal{K})$. Define

$$\mathrm{sd}(\mathcal{K}) = \mathcal{S}(\mathrm{FP}(\mathcal{K})),$$

which is called the *subdivision* of $\mathcal{K}$. The simplices of $\mathrm{sd}(\mathcal{K})$ are given by the sequences

$$\{\sigma_1, \sigma_2, \ldots, \sigma_q\}$$

with $\sigma_1 < \sigma_2 < \cdots < \sigma_q$. If K is the geometric realization of $\mathcal{K}$, then, by definition, the geometric realization of $\mathrm{sd}(\mathcal{K})$ is $\mathrm{sd}(K)$.

The subdivision of (abstract) simplicial complexes can be described by the following commutative diagram

$$\begin{tikzcd}
\{\text{(abstract) simplicial complexes}\} \arrow[r, "\mathrm{FP}"] \arrow[d, "\mathrm{sd}"'] & \{\text{posets}\} \arrow[dl, "\mathcal{S}"] \arrow[d] \\
\{\text{(abstract) simplicial complexes}\} \arrow[r, "\mathrm{FP}"] & \{\text{posets}\},
\end{tikzcd}$$

where the right column is the functor that sends a poset to the poset of its finite well-ordered sequences with new partial order given by the inclusions.

1.3. Simplicial Joins. The notion of the cone $K * w$ of a simplicial complex K and a point w can be generalized to have a notion of join. As we see from the definition of cone, we need to have the point w in a *good position* that is each ray from w intersecting with $|K|$ at most one point. By using abstract simplicial complexes, we do not have to be worried about the technical assumption on *good position* of w. Namely, we can start with two abstract simplicial complexes and then construct a new abstract simplicial complex. Its geometric realization gives a construction in geometry. The precise definition of join is as follows.

DEFINITION 1.3.1. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be abstract simplicial complexes. The *join* $\mathcal{K}_1 * \mathcal{K}_2$ of $\mathcal{K}_1$ and $\mathcal{K}_2$ is the abstract simplicial complex with the vertex set given by the disjoint union of the vertices of $\mathcal{K}_1$ and that of $\mathcal{K}_2$, and its simplices given by the disjoin unions of the simplices of $\mathcal{K}_1$ and that of $\mathcal{K}_2$ along all faces of such simplices. More precisely,

$$V(\mathcal{K}_1 * \mathcal{K}_2) = V(\mathcal{K}_1) \coprod V(\mathcal{K}_2)$$

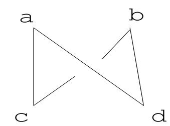
and a nonempty subset $A \coprod B$ of $V(\mathcal{K}_1 * \mathcal{K}_2)$ with $A \subseteq V(\mathcal{K}_1)$ and $B \subseteq V(\mathcal{K}_2)$ is an simplex of $\mathcal{K}_1 * \mathcal{K}_2$ if and only if A is an empty set or a simplex of $\mathcal{K}_1$ and B is an empty set or a simplex of $\mathcal{K}_2$.

EXAMPLE 1.3.1. Let $\mathcal{K}_1$ be an abstract simplicial complex and let $\mathcal{K}_2 = \{w\}$ be an (abstract) 0-simplex. Then, from the definition of cone, the geometric realization of $\mathcal{K}_1 * \mathcal{K}_2$ is a cone of the geometric realization of $\mathcal{K}_1$. $\square$

EXAMPLE 1.3.2. Let $\mathcal{K}_1 = \{a, b\}$ and let $\mathcal{K}_2 = \{c, d\}$. Then $\mathcal{K}_1 * \mathcal{K}_2$ has the simplices:

$$\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a\}, \{b\}, \{c\}, \{d\}.$$

Its geometric realization is a circle S^1 as one can see from the following picture:



Definition 1.3.2. Let K_1 be a geometric simplicial complex with its abstraction $\mathcal{K}_1$ and let K_2 be a geometric simplicial complex with its abstraction $\mathcal{K}_2$. The *join* $K_1 * K_2$ of K_1 and K_2 is defined to be the geometric realization of $\mathcal{K}_1$ and $\mathcal{K}_2$.

Proposition 1.3.3. *Let $K_1 = \sigma^n$ be an n-simplex and let $K_2 = \tau^m$ be an m-simplex. Then $K_1 * K_2$ is an $(m+n+1)$-simplex.*

Proof. Let $K_1 = a^0a^1\cdots a^n$ and let $K_2 = b^0b^1\cdots b^m$. Then the abstraction $\mathcal{K}_1$ of K_1 is given by all nonempty subsets of $\{a^0, a^1, \ldots, a^n\}$, and the abstraction $\mathcal{K}_2$ of K_2 is given by all nonempty subsets of $\{b^0, b^1, \ldots, b^m\}$. Thus $\mathcal{K}_1 * \mathcal{K}_2$ is given by all nonempty subsets of $\{a^0, a^1, \ldots, a^n, b^0, b^1, \ldots, b^m\}$ and hence the result. □

Proposition 1.3.4. *Let K_1, K_2 and K_3 be geometric simplicial complexes. Then*

(1). *$K_1 * K_2$ is linearly isomorphic to $K_2 * K_1$.*
(2). *$(K_1 * K_2) * K_3$ is linearly isomorphic to $K_1 * (K_2 * K_3)$.*

Proof. Let $\mathcal{K}_i$ be the abstraction of K_i. The assertion follows from that $\mathcal{K}_1 * \mathcal{K}_2 \cong \mathcal{K}_2 * \mathcal{K}_1$ and $(\mathcal{K}_1 * \mathcal{K}_2) * \mathcal{K}_3 \cong \mathcal{K}_1 * (\mathcal{K}_2 * \mathcal{K}_3)$. □

Proposition 1.3.5. *Let K and L be geometric simplicial complexes and let $\{K_\alpha \mid \alpha \in J\}$ be a collection of simplicial subcomplex of K such that $K = \bigcup_{\alpha\in J} K_\alpha$. Then there is a linear isomorphism*

$$K * L \cong \bigcup_{\alpha\in J} (K_\alpha * L).$$

Proof. Let $\mathcal{K}$ and $\mathcal{L}$ be the abstraction of K and L, respectively. Let $\mathcal{K}_\alpha$ be the abstraction of K_α. Then

$$\mathcal{K} = \bigcup_{\alpha\in J} \mathcal{K}_\alpha.$$

From the definition of the join of abstract simplicial complexes, we have

$$\mathcal{K} * \mathcal{L} = \bigcup_{\alpha \in J} (\mathcal{K}_\alpha * \mathcal{L}.$$

The assertion follows by taking geometric realization. □

PROPOSITION 1.3.6. *Let K be a geometric simplicial complex and let v be a vertex of K. Then*

$$\overline{\mathrm{St}}(v) = v * \mathrm{Lk}(v).$$

PROOF. Recall $\overline{\mathrm{St}}(v) = \bigcup_{v\in\sigma} \sigma$. Given a simplex σ having vertices $v, a^0, a^1, \ldots, a^n$. Let $\tau_\sigma = a^0 a^1 \cdots a^n$ be the face of σ opposite to v. (If $\sigma = v$, we allow $\tau = \emptyset$ in this proof.) From the definition of the join,

$$\sigma = v * \tau_\sigma.$$

It follows

$$\begin{aligned}\overline{\mathrm{St}}(v) &= \bigcup_{v\in\sigma} \sigma \\ &= \bigcup_{v\in\sigma} v * \tau_\sigma \\ &= v * \left(\textstyle\bigcup_{v\in\sigma} \tau_\sigma\right).\end{aligned}$$

Thus it suffices to show that

$$\mathrm{Lk}(v) = \left(\bigcup_{v\in\sigma} \tau_\sigma\right).$$

Let t_v be the barycentric coordinate function with respect to v. Recall that $x \in \mathrm{St}(v)$ if and only if $t_v(x) > 0$. Thus

$$\mathrm{Lk}(v) = \overline{\mathrm{St}}(v) \cap t_v^{-1}(0).$$

For any simplex $\sigma = va^0a^1 \cdots a^n$ of K having v as one of its vertices, observe that a point $x \in \sigma$ lies in its face $\tau_\sigma = a^0 a^1 \cdots a^n$ if and only if $t_v(x) = 0$. It follows that $\tau_\sigma \subseteq \mathrm{Lk}(v)$ and so

$$\bigcup_{v\in\sigma} \tau_\sigma \subseteq \mathrm{Lk}(v).$$

Now let $x \in \mathrm{Lk}(v) = \overline{\mathrm{St}}(v) \smallsetminus \mathrm{St}(v)$. There exists a simplex σ of K having v as one of its vertices such that $x \in \sigma$. Since $t_v(x) = 0$ as $x \in \mathrm{Lk}(v)$, we have $t_v \in \tau_\sigma$. Thus

$$\mathrm{Lk}(v) \subseteq \bigcup_{v\in\sigma} \tau_\sigma$$

and hence the result. □

Let X and Y be topological spaces. The *join* of X and Y, denoted by $X\#Y$, is defined to be the quotient space of $X \times Y \times I$ by the equivalence relation generated by:

(1). $(x, y_1, 0) \sim (x, y_2, 0)$ for $x \in X, y_1, y_2 \in Y$ and
(2). $(x_1, y, 1) \sim (x_2, y, 1)$ for $x_1, x_2 \in X$ and $y \in Y$.

LEMMA 1.3.7. *Let σ^n be an n-simplex and let τ^m be an m-simplex. Then $\sigma^n \# \tau^m \cong \sigma^n * \tau^m$ is an $(n+m+1)$-simplex .*

PROOF. Let $\sigma = a^0a^1\cdots a^n \subseteq \mathbb{R}^{N_1}$ and let $\tau = b^0b^1\cdots b^m \subseteq \mathbb{R}^{N_2}$. For having the join $\sigma * \tau$ as a geometric realization of an $(n+m+1)$-simplex, we may put the vertices $a^0, a^1, \ldots, a^n, b^0, b^1, \ldots, b^m$ in $\mathbb{R}^{N_1+N_2}$ such that $a^0, a^1, \ldots, a^n \in \mathbb{R}^{N_1} = \mathbb{R}^{N_1}\times 0 \subseteq \mathbb{R}^{N_1+N_2}$ and $b^0, b^1, \ldots, b^m \in \mathbb{R}^{N_2} = 0\times\mathbb{R}^{N_2} \subseteq \mathbb{R}^{N_1+N_2}$. Then $a^0, a^1, \ldots, a^n, b^0, b^1, \ldots, b^m$ are geometrically independent and the simplex $\sigma*\tau$ is given by

$$\sigma * \tau = a^0a^1\cdots a^nb^0b^1\cdots b^m$$

as an $(n+m+1)$-simplex. Now define a map $\phi\colon \sigma\times\tau\times I \longrightarrow \sigma*\tau$ by setting

$$\phi(\sum_{i=0}^n t_ia^i, \sum_{j=0}^m s_jb^j, t) = \sum_{i=0}^n (1-t)t_ia_i + \sum_{j=0}^m ts_jb_j,$$

where the right-hand side is also the barycentric coordinates because $(1-t)t_i \geq 0$, $ts_j \geq 0$ and

$$\sum_{i=0}^n tt_i + \sum_{j=0}^m (1-t)s_j = t\sum_{i=0}^n t_i + (1-t)\sum_{j=0}^m s_j = t + (1-t) = 1.$$

Note that

$$\phi(\sum_{i=0}^n t_ia^i, \sum_{j=0}^m s_jb^j, 0) = \sum_{i=0}^n t_ia_i$$

and

$$\phi(\sum_{i=0}^n t_ia^i, \sum_{j=0}^m s_jb^j, 1) = \sum_{j=0}^m s_jb_j.$$

Thus the map ϕ factors through the quotient space $\sigma\#\tau$ because

$$\phi(\sum_{i=0}^n t_ia^i, \sum_{j=0}^m s_jb^j, 0) = \phi(\sum_{i=0}^n t_ia^i, \sum_{j=0}^m s'_jb^j, 0)$$

for any $\sum_{j=0}^m s_jb^j, \sum_{j=0}^m s'_jb^j \in \tau$ and

$$\phi(\sum_{i=0}^n t_ia^i, \sum_{j=0}^m s_jb^j, 1) = \phi(\sum_{i=0}^n t'_ia^i, \sum_{j=0}^m s_jb^j, 1)$$

for any $\sum_{i=0}^n t_ja^j, \sum_{i=0}^n t'_ia^i \in \sigma$. Let

$$\bar{\phi}\colon \sigma\#\tau \longrightarrow \sigma*\tau \tag{1.3.1}$$

be the map induced by ϕ. Then $\bar{\phi}$ is one-to-one and onto. Since $\sigma\#\tau$ has quotient topology, $\bar{\phi}$ is continuous.

To see $\bar{\phi}^{-1}$ is continuous, let $q\colon \sigma\times\tau\times I \to \sigma\#\tau$ be the quotient map and let A be closed subset of $\sigma\#\tau$. Then $q^{-1}(A)$ is a closed subset of $\sigma\times\times I$. Since $\sigma\times\tau\times I$ is compact, $q^{-1}(A)$ is compact. It follows that

$$\bar{\phi}(A) = \phi(q^{-1}(A))$$

is compact subset of $\sigma*\tau$. Thus $\bar{\phi}(A)$ is closed and so $\bar{\phi}^{-1}$ is continuous. The proof is finished. □

Let K and L be geometric simplicial complexes. Then

$$|K|\#|L| = \bigcup_{\substack{\sigma \in K \\ \tau \in L}} \sigma\#\tau.$$

The *weak topology* on $|K|\#|L|$ is defined by requiring that a subset A to be closed if and only if $A \cap (\sigma\#\tau)$ is closed in $\sigma\#\tau$ for any $\sigma \in K$ and $\tau \in L$.

THEOREM 1.3.8. *Let K and L be geometric simplicial complexes and let $|K|\#|L|$ have weak topology. Then there is a homeomorphism*

$$|K|\#|L| \cong |K * L|.$$

PROOF. Let $|K| \subseteq \mathbb{R}^J$ and let $|L| \subseteq \mathbb{R}^{J'}$. Consider J and J' as disjoint index sets and regard $|K|$ and $|L|$ as subsets of $\mathbb{R}^{J \sqcup J'} \subseteq \mathbb{R}^J \times \mathbb{R}^{J'}$. By the proof of above lemma, for $\sigma \in K$ and $\tau \in L$, the join $\sigma * \tau$ is the simplex spanned by the vertices of σ and τ. Let

$$\bar{\phi}_{\sigma,\tau} \colon \sigma\#\tau \longrightarrow \sigma * \tau$$

be the homeomorphism defined in (1.3.1). Then we obtain the function

$$\bar{\phi} = \bigcup_{\sigma,\tau} \phi_{\sigma,\tau} \colon |K|\#|L| = \bigcup_{\substack{\sigma \in K \\ \tau \in L}} \sigma\#\tau \longrightarrow |K * L| = \bigcup_{\substack{\sigma \in K \\ \tau \in L}} \sigma * \tau$$

which is one-to-one and onto. The function $\bar{\phi}$ is continuous because the restriction

$$\bar{\phi}|_{\sigma\#\tau} \colon \sigma\#\tau \xrightarrow{\bar{\phi}_{\sigma,\tau}} \sigma * \tau \subseteq |K * L|$$

is continuous for each $\sigma \in K$ and $\tau \in L$. Also the inverse $\bar{\phi}^{-1}$ is continuous because its restriction

$$\bar{\phi}^{-1}|_{\sigma * \tau} \colon \sigma * \tau \xrightarrow{\bar{\phi}^{-1}_{\sigma,\tau}} \sigma\#\tau \subseteq |K|\#|L|$$

is continuous for every $\sigma \in K$ and $\tau \in L$. (Note that all simplices in $|K * L|$ are given in the form $\sigma * \tau$, σ, τ for $\sigma \in K$ and $\tau \in L$. The function $\bar{\phi}^{-1}$ restricted to the last two cases is also continuous because σ and τ are subspaces of $\sigma\#\tau$.) The proof is finished. □

2. Δ-sets

2.1. Definition of Δ-sets. A generalization of abstract simplicial complex is so-called Δ-set. The definition of Δ-set is as follows.

DEFINITION 2.1.1. A Δ-*set* means a sequence of sets $X = \{X_n\}_{n \geq 0}$ with *faces* $d_i \colon X_n \to X_{n-1}$, $0 \leq i \leq n$, such that

$$d_i d_j = d_j d_{i+1} \tag{2.1.1}$$

for $i \geq j$, which is called the Δ-*identity*.

In this definition, a Δ-set refers to a sequence of sets together with structured functions called faces rather than a single set. For helping to remember the Δ-identity, one can look at the coordinate projections

$$d_i \colon (x_0, \ldots, x_n) \longrightarrow (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

to catch the identities $d_i d_j = d_j d_{i+1}$ for $i \geq j$, where x_i are letters.

EXAMPLE 2.1.1. Let $\mathcal{K}$ be an abstract simplicial complex. Let the vertices of $\mathcal{K}$ be well-ordered. Let $\mathcal{K}_n$ be the set of n-simplices of $\mathcal{K}$. Define the faces

$$d_i \colon \mathcal{K}_n \to \mathcal{K}_{n-1}, \quad 0 \le i \le n,$$

by

$$d_i\{a^0, a^1, \ldots, a^n\} = \{a^0, a^1, \ldots, a^{i-1}, a^{i+1}, \ldots, a^n\}$$

for any n-simplex $\{a^0, a^1, \ldots, a^n\}$ with $a^0 < a^1 < \cdots < a^n$. Then the sequence $\mathcal{K}^\Delta = \{\mathcal{K}_n\}_{n\ge 0}$ forms a Δ-set. Note that the faces depend on the choice of well-order on vertices. □

There is a categorical way to describe Δ-set. Namely we can describe Δ-sets as functors from a category $\mathcal{O}^+$ below to the category of sets.

DEFINITION 2.1.2. Let $\mathcal{O}^+$ be the category whose objects are finite well-ordered sets and whose morphisms are functions $f \colon X \to Y$ such that $f(x) < f(y)$ if $x < y$.

Note that the objects in $\mathcal{O}^+$ are given by $[n] = \{0, 1, \ldots, n\}$ for $n \ge 0$ and the morphisms in $\mathcal{O}^+$ are generated by $d^i \colon [n-1] \longrightarrow [n]$ with

$$d^i(j) = \begin{cases} j & \text{if} \quad j < i \\ j+1 & \text{if} \quad j \ge i \end{cases}$$

for $0 \le i \le n$, that is d^i is the ordered embedding missing i. We may write the function d^i in matrix form:

$$d^i = \begin{pmatrix} 0 & 1 & \cdots & i-1 & i & i+1 & \cdots & n-1 \\ 0 & 1 & \cdots & i-1 & i+1 & i+2 & \cdots & n. \end{pmatrix}.$$

The morphisms d^i satisfy the following identity:

$$d^j d^i = d^{i+1} d^j$$

for $i \ge j$.

REMARK 2.1.3. *For seeing that morphisms in $\mathcal{O}^+$ are generated by d^i, observe that any morphism in $\mathcal{O}^+$ means an ordered embedding, which can be written as the compositions of d^i's.*

Let $\mathcal{S}$ denote the category of sets.

PROPOSITION 2.1.4. *Δ-sets are one-to-one correspondent to contravariant functors from $\mathcal{O}^+$ to $\mathcal{S}$.*

PROOF. Let $F \colon \mathcal{O}^+ \to \mathcal{S}$ be a contravariant functor. Define $X_n = F([n])$ and

$$d_i = F(d^i) \colon X_n = F([n]) \to X_{n-1} = F([n-1]).$$

Then X is a Δ-set.

Conversely suppose that X is a Δ-set. Define the $F \colon \mathcal{O}^+ \to \mathcal{S}$ by setting $F([n]) = X_n$ and $F(d^i) = d_i$. Then F is a contravariant functor. □

DEFINITION 2.1.5. A Δ-set $G = \{G_n\}_{n\ge 0}$ is called a Δ-*group* if each G_n is a group, and each face d_i is a group homomorphism. In other words, a Δ-group means a contravariant functor from $\mathcal{O}^+$ to the category of groups. More abstractly, for any category $\mathcal{C}$, a Δ-object over $\mathcal{C}$ means a contravariant functor from $\mathcal{O}^+$ to $\mathcal{C}$. In other words, a Δ-object over $\mathcal{C}$ means a sequence of objects over $\mathcal{C}$, $X = \{X_n\}_{n\ge 0}$ with faces $d_i \colon X_n \to X_{n-1}$ as morphisms in $\mathcal{C}$.

EXAMPLE 2.1.2 (n-simplex). The n-simplex $\Delta^+[n]$, as a Δ-set, is as follows:

$$\Delta^+[n]_k = \{(i_0, i_1, \ldots, i_k) \mid 0 \leq i_0 < i_1 < \cdots < i_k \leq n\}$$

for $k \leq n$ and $\Delta^+[n]_k = \emptyset$ for $k > n$. Namely, $\Delta^+[n]_k$ is the collection of the subsets of cardinal $(k+1)$ of the set $(0, 1, 2, \ldots, n)$.

The face $d_j \colon \Delta^+[n]_k \to \Delta^+[n]_{k-1}$ is given by

$$d_j(i_0, i_1, \ldots, i_k) = (i_0, i_1, \ldots, \hat{i}_j, \ldots, i_k),$$

that is deleting i_j. Let $\sigma_n = (0, 1, \ldots, n)$. Then

$$(i_0, i_1, \ldots, i_k) = d_{j_1} d_{j_2} \cdots d_{j_{n-k}} \sigma_n,$$

where $j_1 < j_2 < \cdots < j_{n-k}$ with $\{j_1, \ldots, j_k\} = \{0, 1, \ldots, n\} \smallsetminus \{i_0, i_1, \ldots, i_k\}$. In other words, any elements in $\Delta^{[n]}$ can be written an iterated face of σ_n. Note that $\Delta^+[n]$ is the abstraction of an n-simplex by considering $\Delta^+[n]$ as an abstract simplicial complex. □

DEFINITION 2.1.6. A Δ-*map* $f \colon X \to Y$ means a sequence of functions

$$f \colon X_n \to Y_n$$

for each $n \geq 0$ such that $f \circ d_i = d_i \circ f$, that is the diagram

$$\begin{tikzcd} X_n \arrow[r, "f"] \arrow[d, "d_i"] & Y_n \arrow[d, "d_i"] \\ X_{n-1} \arrow[r, "f"] & Y_{n-1} \end{tikzcd}$$

commutes. A Δ-*subset* A of a Δ-set X means a sequence of subsets $A_n \subseteq X_n$ such that

$$d_i(A_n) \subseteq A_{n-1}$$

for all $0 \leq i \leq n < \infty$. A Δ-set X is called to be *isomorphic* to a Δ-set Y, denoted by $X \cong Y$, if there is a bijective Δ-map $f \colon X \to Y$.

Let X be a Δ-set and let A be a Δ-subset. Clearly the inclusion $A \subseteq X$, that is, $A_n \subseteq X_n$ for each $n \geq 0$, is a Δ-map.

PROPOSITION 2.1.7. *Let X be a Δ-set and let $x \in X_n$ be an element. Then there exists a unique Δ-map*

$$f_x \colon \Delta^+[n] \to X$$

such that $f_x(\sigma_n) = x$.

PROOF. Suppose that $f \colon \Delta^+[n] \to X$ is a Δ-map such that $f(\sigma_n) = x$. From the assumption $f(\sigma_n) = x$, we have

$$\begin{aligned} f(i_0, i_1, \ldots, i_k) &= f(d_{j_1} d_{j_2} \cdots d_{j_{n-k}} \sigma_n) \\ &= d_{j_1} d_{j_2} \cdots d_{j_{n-k}} f_x(\sigma_n) \\ &= d_{j_1} d_{j_2} \cdots d_{j_{n-k}} x. \end{aligned}$$

This proves the uniqueness because the value $f(i_0, i_1, \ldots, i_{k+1})$ must be given by $d_{j_1} d_{j_2} \cdots d_{j_{n-k}} x$. Now define the functions

$$(f_x)_k \colon \Delta^+[n]_k \longrightarrow X_k$$

by setting $(f_x)_k(i_0, i_1, \ldots, i_{k+1}) = d_{j_1} d_{j_2} \cdots d_{j_{n-k}} x$. Then

$$f_x = \{(f_x)_k\} \colon \Delta^+[n] \longrightarrow X$$

is a Δ-map. The proof is finished. □

The simplicial map $f_x \colon \Delta^+[n] \to X$ is called *representing map* of x.

DEFINITION 2.1.8. Let X be a Δ-set and let $S \subseteq \bigcup_{n=0}^{\infty} X_n$. The Δ-*subset* generated by S is defined by

$$\langle S \rangle^{\Delta} = \bigcap \{A \subseteq X \mid S \subseteq \bigcup_{n=0}^{\infty} A_n \; A = \{A_n\} \text{ is a } \Delta - \text{subset of } X\}.$$

For $x \in X_n$, $\langle \{x\} \rangle^{\Delta}$ is simply denoted by $\langle x \rangle^{\Delta}$. A Δ-set X is called *monogenic* if it is generated by a single element.

PROPOSITION 2.1.9. *Let X be a Δ-set and let $S \subseteq \bigcup_{n=0}^{\infty} X_n$. Then*

$$\langle S \rangle^{\Delta}_n = (S \cap X_n) \cup \bigcup_{\substack{0 \leq j_1 < j_2 < \cdots < j_k \leq n+k \\ 1 \leq k < \infty}} d_{j_1} d_{j_2} \cdots d_{j_k} (S \cap X_{n+k})$$

for each $n \geq 0$.

PROOF. Let

$$K_n = (S \cap X_n) \cup \bigcup_{\substack{0 \leq j_1 < j_2 < \cdots < j_k \leq n+k \\ 1 \leq k < \infty}} d_{j_1} d_{j_2} \cdots d_{j_k} (S \cap X_{n+k})$$

Then $d_i(K_n) \subseteq K_{n-1}$ by the Δ-identity (2.1.1) for $0 \leq i \leq n$. Thus $K = \{K_n\}_{n \geq 0}$ is a Δ-subset of X. If A is a Δ-subset of X with $S \subseteq \bigcup_{n=0}^{\infty} A_n$, then clearly $K_n \subseteq A_n$ for each $n \geq 0$. Thus $K = \langle S \rangle^{\Delta}$ and hence the assertion. □

2.2. Polyhedral Δ-sets. Given an abstract simplicial complex $\mathcal{K}$ (with a well-order on the vertices), as in Example 2.1.1, we obtain a Δ-set $\mathcal{K}^{\Delta}$.

DEFINITION 2.2.1. A Δ-set X is called *polyhedral* if there exists an abstract simplicial complex $\mathcal{K}$ such that $X \cong \mathcal{K}^{\Delta}$.

In general, a Δ-set may not be polyhedral.

EXAMPLE 2.2.1. Let $X = \Delta^+[1] \cup_{\Delta^+[1]_0} \Delta^+[1]$ be the union of two copies of $\Delta^+[1]$ by identifying the vertices. We show that X is not polyhedral. Note that X_0 has two vertices 0 and 1 and X_1 has two elements $\sigma_1 = (0,1)$ and $\bar{\sigma}_1 = \overline{(0,1)}$, where $\overline{(0,1)}$ is a copy of σ_1. Assume that there is an abstract simplicial complex $\mathcal{K}$ such that $\mathcal{K}^{\Delta} = X$. Then $\mathcal{K}$ has only two vertices that is impossible to create two 1-simplices σ_1 and $\bar{\sigma}_1$. □

Now let X be a Δ-set and let 2^{X_0} be the set of all subsets of X_0. Define

$$\phi \colon \coprod_{n \geq 0} X_n \longrightarrow 2^{X_0}$$

by setting $\phi(x) = \{f_x(0), f_x(1), \ldots, f_x(n)\}$ for $x \in X_n$.

THEOREM 2.2.2. *Let X be a Δ-set. Then X is polyhedral if and only if the following holds:*

(1). *There exists an order of* X_0 *such that, for each* $x \in X_n$,

$$f_x(0) \leq f_x(1) \leq \cdots \leq f_x(n).$$

(2). *The function* $\phi\colon \bigcup_{n\geq 0} X_n \longrightarrow 2^{X_0}$ *is one-to-one.*

Proof. If $X \cong \mathcal{K}^\Delta$ for some abstract simplicial complex $\mathcal{K}$, then each n-simplex is uniquely determined by its vertices. Thus ϕ is one-to-one. From the construction, we have $f_x(0) < f_x(1) < \cdots < f_x(n)$.

Conversely suppose that X is a Δ set satisfies the two conditions in the statement. First we show that, for $x \in X_n$, the cardinal of $\{f_x(0), f_x(1), \ldots, f_x(n)\}$ is $n+1$. Otherwise there exists $0 \leq i < j \leq n$ such that $f_x(i) = f_x(j)$. Then

$$d_i(x) = f_x d_i(0, 1, \ldots, n) = f_x(0, 1, \ldots, i-1, i+1, \ldots, n)$$

and so

$$\begin{array}{rcl}\phi(d_i(x)) & = & \{f_x(0), f_x(1), \ldots, f_x(i-1), f_x(i+1), \ldots, f_x(n)\} \\ & = & \{f_x(0), f_x(1), \ldots, f_x(i-1), f_x(i), f_x(i+1), \ldots, f_x(n)\} \\ & = & \phi(x).\end{array}$$

Thus $d_i(x) = x$, which contradicts to that $X_n \cap X_{n-1} = \emptyset$.

Let the vertices of $\mathcal{K}$ be the elements of X_0. Define a subset

$$\{a^0, a^1, \ldots, a^n\} \subseteq X_0$$

to be an n-simplex of $\mathcal{K}$ if and only if $\{a^0, a^1, \ldots, a^n\} \in \mathrm{Im}(\phi)$, where the elements are given in the order that $a^0 < a^1 < \cdots < a^n$.

For checking that $\mathcal{K}$ is an abstract simplicial complex, let $\{a^0, a^1, \ldots, a^n\}$ be an n-simplex of $\mathcal{K}$ such that $\{a^0, a^1, \ldots, a^n\} = \phi(x)$. From the above arguments, $x \in X_n$ and so we may assume that $a^i = f_x(i)$ for $0 \leq i \leq n$. Let $\{a^{i_0}, a^{i_1}, \ldots, a^{i_p}\}$ be a subset of $\{a^0, a^1, \ldots, a^n\}$ with $0 \leq i_0 < i_1 < \cdots < i_p \leq n$. Let

$$\{j_0, j_1, \ldots, j_{n-p-1}\} = \{0, 1, \ldots, n\} \smallsetminus \{i_0, i_1, \ldots, i_p\}$$

with $0 \leq j_0 < j_1 < \cdots < j_{n-p-1}$. Then

$$\begin{array}{rcl}d_{j_0} d_{j_1} \cdots d_{j_{n-p-1}} x & = & f_x d_{j_0} d_{j_1} \cdots d_{j_{n-p-1}}(0, 1, \ldots, n) \\ & = & f_x(i_0, i_1, \ldots, i_p).\end{array}$$

Thus

$$\phi(d_{j_0} d_{j_1} \cdots d_{j_{n-p-1}} x) = \{a^{i_0}, a^{i_1}, \ldots, a^{i_p}\}$$

and so $\{a^{i_0}, a^{i_1}, \ldots, a^{i_p}\}$ is a p-simplex of $\mathcal{K}$.

Finally the function

$$g\colon X \longrightarrow \mathcal{K}^\Delta$$

with $g(x) = \{f_x(0), \ldots, f_x(n)\}$ for $x \in X_n$ is a bijective Δ-map. This proves that $X \cong \mathcal{K}^\Delta$. □

2.3. Δ-complexes and the geometric realization of Δ-sets. Recall that the standard *geometric* n*-simplex* Δ^n is defined by

$$\Delta^n = \{(t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0 \text{ and } \sum_{i=0}^{n} t_i = 1\}.$$

Let $e^0 = (1,0,\ldots,0), e^1 = (0,1,0,\ldots,0),\ldots,e^n = (0,\ldots,0,1)$ be the vertices of Δ^n. For $0 \le i \le n$, the i-th face of Δ^n is given by the image of the map $d^i\colon \Delta^{n-1} \to \Delta^n$ defined by

$$d^i(t_0,t_1,\ldots,t_{n-1}) = (t_0,\ldots,t_{i-1},0,t_i,\ldots,t_{n-1}).$$

Namely $d^i\colon \Delta^{n-1} \to \Delta^n$ is a simplicial map with

$$d^i(e^j) = \begin{cases} e^j & \text{if} \quad j < i \\ e^{j+1} & \text{if} \quad j \ge i. \end{cases}$$

It is straightforward to check that the maps d^i satisfy the identity:

$$d^j d^i = d^{i+1} d^j \tag{2.3.1}$$

for $i \ge j$. This identity is dual to the Δ-identity. In such a sense, the map $d^i\colon \Delta^{n-1} \to \Delta^n$ is called *i-th coface* map and its image $d^i(\Delta^{n-1}) \subseteq \Delta^n$ is called *i-th face* of Δ^n. Recall that the *boundary* $\partial(\Delta^n) = \bigcup_{i=0}^{n} d^i(\Delta[n-1])$ is the union of all faces of Δ^n and the *interior* of Δ^n is given by $\mathrm{Int}(\Delta^n) = \Delta^n \smallsetminus \partial\Delta^n$.

DEFINITION 2.3.1. A *Δ-complex structure* on a space X is a collection of maps

$$\mathcal{C}(X) = \{\sigma_\alpha\colon \Delta^n \to X \mid \alpha \in J_n\ n \ge 0\}.$$

such that

(1). $\sigma_\alpha|_{\mathrm{Int}(\Delta^n)}\colon \mathrm{Int}(\Delta^n) \to X$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha|_{\mathrm{Int}(\Delta^n)}$.

(2). For each $\sigma_\alpha \in \mathcal{C}(X)$, each face

$$\sigma_\alpha \circ d^i \in \mathcal{C}(X).$$

(3). A set $A \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each $\sigma_\alpha \in \mathcal{C}(X)$.

The last condition says that the topology on X is given by the *weak* topology with respect to the structure maps $\sigma_\alpha\colon \Delta^n \to X$. Define

$$C^\Delta_n(X) = \{\sigma_\alpha\colon \Delta^n \to X \mid \alpha \in J_n\} \subseteq \mathcal{C}(X)$$

with $d_i\colon C^\Delta_n(X) \to C^\Delta_{n-1}(X)$ given by

$$d_i(\sigma_\alpha) = \sigma_\alpha \circ d^i\colon \Delta^{n-1} \longrightarrow X$$

for $0 \le i \le n$.

PROPOSITION 2.3.2. *$C^\Delta(X) = \{C^\Delta_n(X)\}_{n\ge 0}$ is a Δ-set.*

PROOF. The assertion follows from Identity (2.3.1) for the cofaces maps d^i. □

DEFINITION 2.3.3. Let K be a Δ-set. The *geometric realization* $|K|$ of K is defined to be

$$|K| = \coprod_{\substack{x \in K_n \\ n \ge 0}} (\Delta^n, x)/\sim = \coprod_{n=0}^{\infty} \Delta^n \times K_n/\sim,$$

where (Δ^n, x) is Δ^n labeled by $x \in K_n$ and $\sim$ is generated by

$$(z, d_i x) \sim (d^i z, x) \tag{2.3.2}$$

for any $x \in K_n$ and $z \in \Delta^{n-1}$ labeled by $d_i x$. For any $x \in K_n$, let $\sigma_x \colon \Delta^n = (\Delta^n, x) \to |K|$ be the canonical *characteristic map*. The topology on K is defined by $U \subseteq |K|$ is open if and only if the pre-image $\sigma_x^{-1}(U)$ is open in Δ^n for any $x \in K_n$ and $n \geq 0$. Equivalently the topology on $|K|$ is given by the quotient topology under the quotient map

$$q\colon \coprod_{\substack{x \in K_n \\ n \geq 0}} (\Delta^n, x) \longrightarrow |K|.$$

The geometric realization of a Δ-set K can be intuitively described as follows: Consider every element $x \in K_n$ as a n-simplex. We assign a copy of Δ^n to each n-simplex x. Then we obtain a collection of geometric simplices and make them as a disjoint union. The *gluing procedure* is then given in the way that the simplex Δ^{n-1} labeled by $d_i x$ is identified with the i-th face of the simplex Δ^n labeled by x under the coface map $d^i \colon \Delta^{n-1} \to \Delta^n$. From this, we obtain the set $|K|$ and then put the weak topology on $|K|$.

Example 2.3.1. We show that the geometric realization $|\Delta^+[n]| \cong \Delta^n$. Note that the elements in $\Delta^+[n]$ are given by

$$(i_0, i_1, \ldots, i_k)$$

with $0 \leq i_0 < \cdots < i_k \leq n$. Let $\alpha = (0, 1, \ldots, n)$. By definition, $|\Delta^+[n]|$ is the quotient of the disjoint union of

$$|\Delta^+[n]| = \coprod_{0 \leq i_0 < i_1 < \ldots < i_k \leq n} (\Delta^k, (i_0, \ldots, i_k)) / \sim .$$

Let $\phi = \sigma_\alpha \colon \Delta^n = (\Delta^n, \alpha) \to |\Delta^+[n]|$ be the characteristic map. Then ϕ is continuous because, for any open subset U of $|\Delta^+[n]|$, $\phi^{-1}(U) = \sigma_\alpha^{-1}(U)$ is open in Δ^n.

Now we define a continuous map

$$\psi\colon \coprod_{0 \leq i_0 < i_1 < \ldots < i_k \leq n} (\Delta^k, (i_0, \ldots, i_k)) \longrightarrow \Delta^n$$

as follows. Let

$$\psi|_{(\Delta^n, \alpha)} \colon (\Delta^n, \alpha) \longrightarrow \Delta^n$$

be the identity map. For each $(i_0, i_1, \ldots, i_k)$ with $k < n$, let

$$\{0, 1, \ldots, n\} \smallsetminus \{i_0, i_1, \ldots, i_k\} = \{j_1, j_2, \ldots, j_{n-k}\}$$

with $j_1 < j_2 < \ldots < j_{n-k}$. Let

$$\psi|_{(\Delta^k, (i_0, i_1, \ldots, i_k))} = d^{j_{n-k}} d^{j_{n-k-1}} \cdots d^{j_1} \colon (\Delta^k, (i_0, i_1, \ldots, i_k)) \longrightarrow \Delta^n$$

be the iterated face. This defines a continuous map ψ.

Let

$$q\colon \coprod_{0 \leq i_0 < i_1 < \ldots < i_k \leq n} (\Delta^k, (i_0, \ldots, i_k)) \twoheadrightarrow |\Delta^+[n]|$$

be the quotient map. Then the map ψ factors through the quotient map q because, for any $z \in \Delta^k$,

$$\begin{array}{rcl} (z, d_{j_1} d_{j_2} \cdots d_{j_{n-k}} \alpha) & \sim & (d^{j_1} z, d_{j_2} \cdots d_{j_{n-k}} \alpha) \\ & \sim & (d^{j_2} d^{j_1} z, d_{j_3} \cdots d_{j_{n-k}} \alpha) \\ & \cdots & \\ & \sim & (d^{j_{n-k}} d^{j_{n-k-1}} \cdots d^{j_1} z, \alpha). \end{array}$$

Let $\bar{\psi}\colon |\Delta^+[n]| \to \Delta^n$ be the function with $\psi = \bar{\psi} \circ q$. Then $\bar{\psi}$ is continuous because $|\Delta^+[n]|$ has the quotient topology.

From the definition of ϕ and ψ, the composite $\bar{\psi} \circ \phi\colon \Delta^n \to \Delta^n$ is the identity map. This forces that $\phi\colon \Delta^n \to \Delta^+[n]$ is one-to-one. Since each simplex Δ^k labeled by $(i_0, i_1, \ldots, i_k)$ is identified with a face of Δ^n, the map $\phi\colon \Delta^n \to |\Delta^+[n]|$ is onto. Thus ϕ is bijective and $\bar{\psi} = \phi^{-1}$. Hence $|\Delta^+[n]| \cong \Delta^n$. ☐

Similarly we have $|\partial\Delta^+[n]| \cong \partial\Delta^n$.

Proposition 2.3.4. *Let K be a Δ-set. Then $|K|$ is Δ-complex.*

Proof. Let $\mathrm{sk}_n K = \{K_j\}_{0 \le j \le n}$ be the n-skeleton of K. Then $\mathrm{sk}_n K$ is a Δ-subset of K. Let $\partial\Delta^+[n] = \mathrm{sk}_{n-1} \Delta^+[n]$ be the boundary of $\Delta^+[n]$. Note that

$$|\Delta^+[n]| \cong \Delta^n \text{ and } |\partial\Delta^+[n]| \cong \partial\Delta^n.$$

From the push-out diagram

$$\begin{tikzcd} \coprod_{x \in K_n} \partial\Delta^+[n] \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\text{push}"] & \mathrm{sk}_{n-1} K \arrow[d, hook] \\ \coprod_{x \in K_n} \Delta^+[n] \arrow[r] & \mathrm{sk}_n K, \end{tikzcd}$$

there is a push-out diagram

$$\begin{tikzcd} \coprod_{x \in K_n} \partial\Delta^n \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\text{push}"] & |\mathrm{sk}_{n-1} K| \arrow[d, hook] \\ \coprod_{x \in K_n} \Delta^n \arrow[r] & |\mathrm{sk}_n K|. \end{tikzcd}$$

Thus $|\mathrm{sk}_n K|$ is obtained from $\mathrm{sk}_{n-1} K|$ by attaching cells with labels in K_n. By induction, it follows that $|K|$ is a Δ-complex. ☐

Proposition 2.3.5. *Let $\mathcal{K}$ be an abstract simplicial complex. Then $|\mathcal{K}^{\Delta}|$ is a geometric simplicial complex whose abstraction is $\mathcal{K}$.*

Proof. Let K be a geometric realization of $\mathcal{K}$. We identify the vertices of $\mathcal{K}$ with the vertices of K. For each copy of the simplex (Δ^n, x) with $x \in \mathcal{K}_n$, the element x is determined by its vertices $\{a^0, a^1, \ldots, a^n\}$ with $a^0 < a^1 < \cdots < a^n$. Since $\mathcal{K}$ is the abstraction of K, we have the simplex $\sigma_x = a^0 a^1 \cdots a^n$ in K. Define the linear map

$$\phi\colon (\Delta^n, x) \longrightarrow \sigma_x$$

by

$$\phi(t_0,t_1,\ldots,t_n,x)=\sum_{i=0}^{n} t_i a^i.$$

Note that $d_i x=\{a^0,a^1,\ldots,a^{i-1},a^{i+1},\ldots,a^n\}$. We have

$$\begin{aligned}\phi(t_0,t_1,\ldots,t_{n-1},d_ix) &= t_0a^0+t_1a^1+\cdots+t_{i-1}a^{i-1}+t_ia^{i+1}+\cdots+t_{n-1}a^n\\ &= \phi(t_0,t_1,\ldots,t_{i-1},0,t_i,\ldots,t_{n-1},x)\\ &= \phi(d^i(t_0,t_1,\ldots,t_{n-1}),x)).\end{aligned}$$

Namely $\phi(z,d_ix)=\phi(d^iz,x)$ for $(z,d_ix)\in(\Delta^{n-1},d_ix)$ with $x\in\mathcal{K}_n$. Thus the map ϕ induces a (continuous) map

$$\bar{\phi}\colon |\mathcal{K}^{\Delta}| \longrightarrow |K|$$

that is one-to-one and onto. By the definition of the weak topology on $|K|$, $\bar{\phi}$ is a homeomorphism and hence the result. □

3. Homology

3.1. Homology of Δ-sets. Recall that a chain complex of groups means a sequence $C=\{C_n\}$ of groups with differential $\partial_n\colon C_n\to C_{n-1}$ such that $\partial_n\circ\partial_{n+1}$ is trivial, that is $\mathrm{Im}(\partial_{n+1})\subseteq\mathrm{Ker}(\partial_n)$ and so the homology is defined by

$$H_n(C)=\mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n+1}),$$

which is a coset in general. A chain complex C is called *normal* if $\mathrm{Im}(\partial_{n+1})$ is a normal subgroup of $\mathrm{Ker}(\partial_n)$ for each n. In this case $H_n(C)$ is a group for each n.

Proposition 3.1.1. *Let G be a Δ-abelian group. Define*

$$\partial_n=\sum_{i=0}^{n}(-1)^i d_i\colon G_n\to G_{n-1}.$$

Then $\partial_{n-1}\circ\partial_n=0$, that is, G is a chain complex under ∂_*.

Proof.

$$\begin{aligned}\partial_{n-1}\circ\partial_n &= \sum_{i=0}^{n-1}(-1)^i d_i\sum_{j=0}^{n}(-1)^j d_j\\ &= \sum_{0\le i<j\le n}(-1)^{i+j}d_id_j+\sum_{0\le j\le i\le n-1}(-1)^{i+j}d_id_j\\ &= \sum_{0\le i<j\le n}(-1)^{i+j}d_id_j+\sum_{0\le j<i+1\le n}(-1)^{i+j}d_jd_{i+1}\\ &= \sum_{0\le i<j\le n}(-1)^{i+j}d_id_j+\sum_{0\le j<i\le n}(-1)^{i+j-1}d_jd_i\\ &= 0.\end{aligned}$$

□

Let X be a Δ-set. The homology $H_*(X;G)$ of X with coefficients in an abelian group G is defined by

$$H_*(X;G)=H_*(\mathbb{Z}(X)\otimes G,\partial_*),$$

where $\mathbb{Z}(X)=\{\mathbb{Z}(X_n)\}_{n\ge 0}$ and $\mathbb{Z}(X_n)$ is the free abelian group generated by X_n.

PROPOSITION 3.1.2. *Let*

$$1 \longrightarrow C' \xrightarrow{i} C \xrightarrow{p} C'' \longrightarrow 1$$

be any short exact sequence of chain complexes of (possibly non-commutative) groups. Then there is a long exact sequence

$$\cdots \longrightarrow H_{k+1}(C'') \xrightarrow{\partial_{k+1}} H_k(C') \xrightarrow{i_*} H_k(C) \xrightarrow{p_*} H_k(C'') \longrightarrow \cdots .$$

Moreover if C' and C'' are normal chain complexes, then ∂_{k+1} is a group homomorphism for each k.

PROOF. Consider the commutative diagram

$$\begin{tikzcd}
C'_{k+2} \arrow[r, hook, "i"] \arrow[d, "\partial'"] & C_{k+2} \arrow[r, two heads, "p"] \arrow[d, "\partial"] & C''_{k+2} \arrow[d, "\partial''"] \\
C'_{k+1} \arrow[r, hook, "i"] \arrow[d, "\partial'"] & C_{k+1} \arrow[r, two heads, "p"] \arrow[d, "\partial"] & C''_{k+1} \arrow[d, "\partial''"] \\
C'_{k} \arrow[r, hook, "i"] \arrow[d, "\partial'"] & C_{k} \arrow[r, two heads, "p"] \arrow[d, "\partial"] & C''_{k} \arrow[d, "\partial''"] \\
C'_{k-1} \arrow[r, hook, "i"] & C_{k-1} \arrow[r, two heads, "p"] & C''_{k-1}.
\end{tikzcd}$$

Let $x \in C''_{k+1}$ with $\partial''(x) = 1$. There exists $\tilde{x} \in C_{k+1}$ such that $p(\tilde{x}) = x$. Since

$$p(\partial(\tilde{x})) = \partial''(p(\tilde{x})) = \partial''(x) = 1,$$

there exists $\bar{x} \in C'_k$ such that $i(\bar{x}) = \partial(\tilde{x})$. Now

$$i(\partial'(\bar{x})) = \partial(i(\bar{x})) = \partial \circ \partial(\tilde{x}) = 1.$$

Thus $\bar{x}$ is a circle in C' and so $\{\bar{x}\}$ defines an element in $H_k(C')$.

Let $\hat{x}$ be another element in C_{k+1} such that $p(\hat{x}) = x$. Then

$$p(\tilde{x}\hat{x}^{-1}) = 1$$

and so there exists an element $z \in C'_{k+1}$ such that $i(z) = \tilde{x}^{-1}\hat{x}$. Now

$$i(\bar{x}\partial'(z)) = \partial(\tilde{x})(\partial(\tilde{x}))^{-1}\partial(\hat{x}) = \partial(\hat{x}).$$

Thus $\{\bar{x}\} \in H_k(C')$ is independent on the choice of the pre-image of x in C_{k+1}.

Suppose that $x' = x\partial''(y)$ with $\partial''(x) = 1$ for some $y \in C''_{k+2}$. There exists $\tilde{y} \in C_{k+2}$ such that $p(\tilde{y}) = y$. Then

$$x' = p(\tilde{x}\partial(\tilde{y}))$$

with

$$\bar{x}' = \partial(\tilde{x}\partial(\tilde{y})) = \partial(\tilde{x}) = \bar{x}.$$

This shows that

$$\partial_{k+1} \colon H_{k+1}(C'') \to H_k(C') \quad \{x\} \mapsto \{\bar{x}\}$$

is well-defined. Assume that C' and C'' are normal chain complexes. For $x, x' \in C''_{k+1}$ with $\partial''(x) = \partial''(x') = 1$. Then $p(\tilde{x}\tilde{x}') = xx'$ and so

$$\partial_{k+1}(\{x\}\{x'\}) = \partial_{k+1}(\{x\})\partial_{k+1}(\{x'\})$$

provided that C' and C'' are normal.

The composite $i_* \circ \partial_{k+1}$ is trivial because $i(\bar{x}) = \partial(\tilde{x})$. Let $y \in C'_k$ with $\partial'(y) = 1$ and $i_*(y)$ is trivial in $H_k(C)$. Then there exists $\tilde{y} \in C_{k+1}$ such that

$$i(y) = \partial(\tilde{y}).$$

By the construction of ∂_{k+1}, $\partial_{k+1}(p(\tilde{y})) = y$. This shows that

$$H_{k+1}(C'') \xrightarrow{\partial_{k+1}} H_k(C') \xrightarrow{i_*} H_k(C)$$

is exact.

Now we show that

$$H_{k+1}(C) \xrightarrow{p_*} H_{k+1}(C'') \xrightarrow{\partial_{k+1}} H_k(C')$$

is exact. Let $y \in C_{k+1}$ such that $\partial(y) = 1$. Then by the construction of ∂_{k+1}, $\partial_{k+1}(p(y)) = 1$. Thus the composite $\partial_{k+1} \circ p_*$ is trivial. Suppose that $x \in C''_{k+1}$ with $\partial''(x) = 1$ and $\bar{x} = \partial_{k+1}(x)$ is trivial in $H_k(C')$. There exists an element $z \in C'_{k+1}$ such that

$$\partial'(z) = \bar{x}.$$

Let $\hat{x} = i(z)^{-1}\tilde{x}$. Then

$$p(\hat{x}) = p(i(z)^{-1}\tilde{x}) = p(\tilde{x}) = x$$

with

$$\partial(\hat{x}) = \partial(i(z)^{-1}\tilde{x}) = i(\partial'(z)^{-1}\bar{x}) = 1.$$

Thus $\hat{x}$ defines an elements in $H_{k+1}(C)$ with $p_*(\{\hat{x}\}) = \{x\}$.

Finally we show that

$$H_k(C') \xrightarrow{i_*} H_k(C) \xrightarrow{p_*} H_k(C'')$$

is exact. Since $p \circ i$ is trivial, so is $p_* \circ i_*$. Let $y \in C_k$ with $\partial(y) = 1$ and $p_*(y)$ is trivial in $H_k(C'')$. There exists an element $z \in C''_{k+1}$ such that

$$p(y) = \partial''(z).$$

Let $\tilde{z} \in C_{k+1}$ such that $p(\tilde{z}) = z$. Then

$$\begin{aligned} p(y\partial(\tilde{z}^{-1})) &= \partial''(z)p(\partial(\tilde{z}^{-1})) \\ &= \partial''(z)\partial''(p(\tilde{z})^{-1}) \\ &= \partial''(z)\partial''(z^{-1}) \\ &= 1. \end{aligned}$$

Thus there exists $w \in C'_k$ such that $i(w) = y\partial(\tilde{z}^{-1})$ with

$$i(\partial'(w)) = \partial(i(w)) = \partial(y\partial(\tilde{z}^{-1})) = 1.$$

and so $\partial'(w) = 1$. Hence $i_*(\{w\}) = \{y\}$. The proof is finished now. □

Let X' be a Δ-subset of X. The relative homology $H_*(X, X'; G)$ is defined by

$$H_*(X, X'; G) = H_*((\mathbb{Z}(X)/\mathbb{Z}(X') \otimes G, \partial_*).$$

Corollary 3.1.3. *Let X' be a Δ-subset of X. Then there is a long exact sequence*

$$\cdots \longrightarrow H_{k+1}(X,X';G) \xrightarrow{\partial_{k+1}} H_k(X';G) \xrightarrow{i_*} H_k(X;G) \xrightarrow{p_*} H_k(X,X';G) \longrightarrow \cdots$$

for abelian group G. □

3.2. Simplicial and Singular Homology.

Definition 3.2.1. Let X be a Δ-complex. Then *simplicial homology* of X with coefficients in an abelian group G is defined by

$$H^{\Delta}_*(X;G) = H_*(C^{\Delta}_*(X);G).$$

For any space X, define

$$S_n(X) = \mathrm{Map}(\Delta^n, X)$$

be the set of all continuous maps from Δ^n to X with

$$d_i = d^{i*}\colon S_n(X) = \mathrm{Map}(\Delta^n, X) \longrightarrow S_{n-1}(X) = \mathrm{Map}(\Delta^{n-1}, X).$$

for $0 \leq i \leq n$. Then $S_*(X) = \{S_n(X)\}_{n\geq 0}$ is a Δ-set. This allows us to define singular homology:

Definition 3.2.2. For a pair of spaces (X,A), the *singular homology* $H_*(X,A;G)$ with coefficients in an abelian group G is defined by

$$H_*(X,A;G) = H_*(S_*(X), S_*(A);G).$$

For any continuous map $f\colon X \to Y$, there is an induced Δ-map If

Let (X,A) and (Y,B) be pairs of spaces and let $f\colon X \to Y$ be a continuous map such that $f(A) \subseteq B$. Then the map f induces a Δ-map

$$f_{\#}\colon S_*(X) \longrightarrow S_*(Y)$$

given by $f_{\#}(\lambda) = f \circ \lambda$ for any $\lambda\colon \Delta^n \to X$ with

$$f_{\#}(S_*(A)) \subseteq S_*(B)$$

and so it induces a group homomorphism

$$f_*\colon H_n(X,A;G) = H_*(S_*(X), S_*(A);G) \longrightarrow H_*(Y,B;G) = H_n(S_*(Y), S_*(B);G)$$

for each n. Thus the homology is a functor from the category of (pairs) of spaces to the category of graded abelian groups. See Hatcher's book [**7**] for further properties. We only list few properties without proofs here.

Proposition 3.2.3. *If $X \simeq Y$, then $H_*(X;G) \cong H_*(Y;G)$. Thus the homology only depends on the homotopy type of spaces.* □

Proposition 3.2.4. *Let X be a Δ-complex and let G be an abelian group. Then there is a natural isomorphism*

$$H^{\Delta}_*(X;G) \cong H_*(X;G).$$

Thus the simplicial homology is the same as the singular homology.

EXAMPLE 3.2.1. If $X = \{x_0\}$ be the space of a single point, then $H_0(X;G) = G$ and $H_q(X;G) = 0$ for $q > 0$. Let $X = \Delta^n$. Since $X \simeq \{x_0\}$ a single point, $H_*(\Delta^n;G) \cong H_*(\{x_0\};G)$.

Now we compute the homology of a sphere S^n. If $n = 0$, then $H_0(S^0;G) = G \oplus G$ and $H_q(S^0;G) = 0$ for $q > 0$. We assume that $n > 0$. Since $S^n \cong \partial(\Delta^{n+1})$, we have

$$H_*(S^n;G) \cong H_*(\partial(\Delta^{n+1});G) \cong H_*^{\Delta}(\partial(\Delta^{n+1});G).$$

Let

$$D = (\mathbb{Z}(\Delta^+[n+1]) \otimes G)/(\mathbb{Z}(\partial(\Delta^+[n+1])) \otimes G).$$

Since

$$\partial(\Delta^+[n+1])_k = \Delta^+[n+1]_k$$

for $k \leq n$ and $\partial(\Delta^+[n+1])_{n+1} = \emptyset$, we have

$$D_k = 0$$

for $k \leq n$ and

$$D_{n+1} = \mathbb{Z}(\Delta^+[n+1]_{n+1}) \otimes G = G$$

as $\Delta^+[n+1]$ has only one element in dimension $n+1$. Thus

$$H_q(D) = \begin{cases} G & \text{for} \quad q = n+1 \\ 0 & \text{for} \quad q \neq n+1. \end{cases}$$

From the short exact sequence of chain complex

$$\mathbb{Z}(\partial(\Delta^+[n])) \otimes G \hookrightarrow \mathbb{Z}(\Delta^+[n]) \otimes G \twoheadrightarrow D,$$

there is a long exact sequence

$$\cdots \to H_q(\partial(\Delta^+[n+1])) \to H_q(\Delta^+[n+1]) \to H_q(D) \to H_{q-1}^{\Delta}(\partial(\Delta^+[n+1])) \to \cdots.$$

Since $|\Delta^+[n+1]| \cong \Delta^{n+1}$,

$$H_q(\Delta^+[n+1];G) = \begin{cases} G & \text{for} \quad q = 0 \\ 0 & \text{for} \quad q \neq 0. \end{cases}$$

Thus

$$H_q(S^n;G) \cong H_q^{\Delta}(\partial\Delta^n;G) = H_q(\partial(\Delta^+[n+1]);G) = \begin{cases} G & \text{if} \quad q = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 3.2.5 (Brouwer Fixed-point Theorem). *Any continuous map $f\colon D^n \to D^n$ has a fixed point.*

PROOF. Suppose that $f\colon D^n \to D^n$ has no fixed point. For any $x \in D^n$, let $r(x)$ be the intersecting point of the line segment from $\phi(x)$ in the direction $\overrightarrow{\phi(x)x}$. This defines a continuous map $r\colon D^n \to S^{n-1}$ such that

$$r|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}.$$

In other words, there is a commutative diagram

$$\begin{tikzcd} D^n \arrow[r, "r"] & S^{n-1} \\ S^{n-1}. \arrow[u] \arrow[ur, equal] & \end{tikzcd}$$

By applying the functor $H_{n-1}(-;\mathbb{Z})$ to this diagram, we have the commutative diagram

$$\begin{tikzcd} H_{n-1}(D^n;\mathbb{Z}) \arrow[r, "r_*"] & H_{n-1}(S^{n-1};\mathbb{Z}) \\ H_{n-1}(S^{n-1};\mathbb{Z}). \arrow[u] \arrow[ur, equal] & \end{tikzcd}$$

If $n = 1$, this gives the commutative diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \\ \mathbb{Z}\oplus\mathbb{Z}, \arrow[u] \arrow[ur, equal] & \end{tikzcd}$$

which is impossible. For $n > 1$, the above diagram gives a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \\ \mathbb{Z}, \arrow[u] \arrow[ur, equal] & \end{tikzcd}$$

which is impossible and hence the result. □

Note. A generalization of Brouwer Fixed-point Theorem is Lefschetz Fixed-point Theorem [**?**, Section 22].

CHAPTER 3

Covering Spaces

Week 10 October 21 (T) October 24 (F): Sections 1.1, 1.2
Week 12 November 4 (T) November 7 (F): Sections: Section 1.3

1. Covering Spaces

1.1. Covering Spaces.

Definition 1.1.1. A map $p\colon \tilde{X} \to X$ is a *covering projection* and $\tilde{X}$ (or $(\tilde{X}, p)$ is a *covering space* of X if

1) p is onto, and
2) for any $x \in X$ there is an open neighborhood U (called an *elementary neighborhood*) of x such that
$$p^{-1}(U) = \coprod_{\alpha \in J} U_\alpha$$
is a topological disjoint union of open sets (called *sheets*), each U_α is mapped homeomorphically onto U by p. So $p^{-1}(U) \cong U \times($ discrete space.)

Roughly speaking covering space just means that 'locally' the pre-image $p^{-1}(U)$ is disjoint union of copies of U.

Example 1.1.1. (1). Any homeomorphism $p\colon \tilde{X} \to X$ is a one-sheeted covering projection.
(2). Let F be a discrete space and $\tilde{X} = X \times F$. Then the coordinate projection $p\colon \tilde{X} \to X$ is a covering projection.
(3). The projection $p\colon S^n \to \mathbb{R}P^n$ is a two-sheeted covering projection.
(4). $p\colon S^1 \to S^1$, $z \mapsto z^n$, is an n-sheeted covering.
(5). The exponential map $\mathrm{e}\colon \mathbb{R} \to S^1$ is a covering with infinite sheets.

Exercise 1.1. Let $p\colon \tilde{X} \to X$ and $q\colon \tilde{Y} \to Y$ be covering projections. Show that $p \times q\colon \tilde{X} \times \tilde{Y} \to X \times Y$ is also a covering projection.

Let G be a group and let Y be a G-space. For $g \in G$ and a subset $S \subseteq Y$, let $g \cdot S$ denote the set $\{g \cdot x | x \in S\}$.

Definition 1.1.2. Let G be a (discrete) group and let Y be a G-space. A G-action on Y is called *properly discontinuous* if

for any $y \in Y$ there exists a neighborhood W_y such that
$$g_1 \neq g_2 \quad \Rightarrow \quad g_1 \cdot W_y \cap g_2 \cdot W_y = \emptyset$$
(or, equivalently, $g \neq 1 \quad \Rightarrow \quad g \cdot W_y \cap W_y = \emptyset$).

Theorem 1.1.3. *Let X be a G-space. If the G-action on X is properly discontinuous, then $X \to X/G$ is a covering.*

PROOF. Let $p\colon X \to X/G$ be the quotient map. By Theorem **??**, p is an open map. For any $x \in X$, let W be an open neighborhood satisfying the condition of proper discontinuity. Then $p(U)$ is an open neighborhood of $p(x)$ and

$$p^{-1}(W) = \coprod_{g\in G} g \cdot W$$

is a disjoint union of open subsets of X. Furthermore $p|_{g\cdot W} : g \cdot W \to p(W)$ is a continuous open bijective map and hence a homeomorphism. □

EXERCISE 1.2. Let X be a G-space. Suppose that $X \to X/G$ is a covering. Show that the G-action on X is properly discontinuous.

Now the next question is how can we know a group-action is properly discontinuous. Recall that a group G acts freely on X if $g \cdot x \neq x$ for all $x \in X$ and $g \in G$ with $g \neq 1$.

EXERCISE 1.3. Let X be a G-space. Suppose that the G-action on X is properly discontinuous. Then G acts freely on X.

THEOREM 1.1.4. *Let G be a finite group and let X be a Hausdorff G-space. Then the G-action on X is properly discontinuous if and only if G acts freely on X.*

PROOF. $\Rightarrow$ is obvious (see Exercise 1.3).
$\Leftarrow$ Let $G = \{g_0 = 1, g_1, \cdots, g_n\}$. Since X is Hausdorff, there exist open neighborhoods $U_0, \cdots U_n$ of $g_0 \cdot x, \cdots, g_n \cdot x$, respectively such that $U_0 \cap U_j = \emptyset$ for $1 \le j \le n$. Let $U = \cap_{j=0}^n g_j^{-1} \cdot U_j$. Then U is an open neighborhood of x with $g_j \cdot U \cap U = \emptyset$ for each $1 \le j \le n$ because

$$\begin{aligned} g_j \cdot U = g_j \cdot \bigcap_{i=0}^n g_i^{-1} U_i &= \bigcap_{i=0}^n g_j (g_i^{-1} \cdot U_i) \\ = \bigcap_{i=0}^n (g_j g_i^{-1}) \cdot U_i &\subseteq (g_j g_j^{-1}) \cdot U_j = U_j. \end{aligned}$$

Thus the G-action on X is properly discontinuous. □

Note: If G has infinite elements, a free G-action may or may not be properly discontinuous. In other words, the quotient $X \to X/G$ may or may not be a covering even if G acts freely on X and X is Hausdorff.

Now we have more examples of covering spaces.

EXAMPLE 1.1.2.
1) Let $\mathbb{Z}$ act on $\mathbb{R}$ by $x \mapsto x + n$. Then this action is discontinuous and so $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$ is a covering.
2) Let $\mathbb{Z}^n = \mathbb{Z}^{\oplus n}$ act on $\mathbb{R}^n$ by $(x_1, \cdots, x_n) \mapsto (x_1 + l_1, \cdots, x_n + l_n)$ for $x_j \in \mathbb{R}$ and $l_j \in \mathbb{Z}$. Then this action is discontinuous and so $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n = S^1 \times \cdots \times S^1$ is a covering. In particular, when $n = 2$, we have the covering projection : $\mathbb{R}^2 \to T = S^1 \times S^1$.
3) Let p be a prime integer and let $q_1, \cdots q_n$ be integers prime to p. We define a $\mathbb{Z}/p$-action on
$$S^{2n+1} = \{(z_0, \cdots, z_n) \in \mathbb{C}^n | |z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 1\}$$
by
$$l \cdot (z_0, \cdots, z_n) = (\mathrm{e}^{2\pi i l/p}\, z_0, \mathrm{e}^{2\pi i l q_1/p}\, z_1, \cdots, \mathrm{e}^{2\pi i l q_n/p}\, z_n).$$

We show that this action is free. Suppose that

$$l \cdot (z_0, \cdots, z_n) = (z_0, \cdots, z_n).$$

Then

$$\mathrm{e}^{2\pi i l q_j/p}\, z_j = z_j$$

for each $0 \leq j \leq n$, where $q_0 = 1$. Since $(z_0, \cdots z_n) \in S^{2n+1}$, there exists $z_{j_0} \neq 0$ for some j_0. It follows that

$$\mathrm{e}^{2\pi i l q_{j_0}/p} = 1$$

and so $lq_{j_0} \equiv 0 \bmod p$. Since $q_{j_0} \not\equiv 0 \bmod p$ and p is a prime, $l \equiv 0 \bmod p$, that is l is the identity in $\mathbb{Z}/p$. Thus this action is free.

Since S^{2n+1} is Hausdorff, $S^{2n+1} \to S^{2n+1}/(\mathbb{Z}/p)$ is a covering. The quotient $S^{2n+1}/(\mathbb{Z}/p)$, denoted by $L^n(p, q_1, \cdots, q_n)$, is called a *lens space*. Note that $L^n(2) = \mathbb{R}P^{2n+1}$.

4) Let p be any non-zero integer. We define a $\mathbb{Z}/p$-action on

$$S^{2n+1} = \{(z_0, \cdots, z_n) \in \mathbb{C}^n \,||z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 1\}$$

by

$$l \cdot (z_0, \cdots, z_n) = (\mathrm{e}^{2\pi i l/p}\, z_0, \mathrm{e}^{2\pi i l/p}\, z_1, \cdots, \mathrm{e}^{2\pi i l/p}\, z_n).$$

The argument above show that this action is free. (**Note:** in this case, we do not need to assume that p is a prime.) The quotient $S^{2n+1}/(\mathbb{Z}/p)$ is denoted by $L^n(p)$. Again we have a covering projection $S^{2n+1} \to L^n(p)$.

5) Let M be a manifold and let

$$F(M, n) = \{(x_1, \cdots, x_n) \in M^n | x_i \neq x_j \text{ for } i \neq j\}$$

be a ordered configuration space. Let the symmetric group Σ_n act on $F(M, n)$ by permuting positions. Then $F(M, n) \to F(M, n)/\Sigma_n$ is a covering. The quotient $F(M, n)/\Sigma_n$, denoted by $B(M, n)$, is called the space of *unordered configurations*.

6) Let G be a (Hausdorff) topological group and let H be a finite subgroup of G. Let G/H be the set of left cosets with quotient topology. Then $G \to G/H$ is a covering. (**Note:** One can directly show that $G \to G/H$ is a covering if H is a discrete subgroup of G (without assuming that H is finite).

1.2. The Lifting Theorem For Covering Spaces. If $p\colon \tilde{X} \to X$ is a covering and $f\colon Y \to X$ is a map, then a *lifting* of f is a continuous map $\tilde{f}\colon Y \to \tilde{X}$ such that $f = p \circ \tilde{f}$.

The lifting problem is: Given a map $f\colon Y \to X$.

i) When does there exist a lifting of f?
ii) Must such a lifting be unique?

The 'uniqueness' can be answered as follows.

Lemma 1.2.1. *Let $p\colon \tilde{X} \to X$ be a covering and let $\tilde{f}, \bar{f}\colon Y \to \tilde{X}$ be two lifting of $f\colon Y \to X$. Suppose that Y is connected and $\tilde{f}(y_0) = \bar{f}(y_0)$ for some $y_0 \in Y$. Then $\tilde{f} = \bar{f}$.*

Proof. Let $Y' = \{y \in Y | \tilde{f}(y) = \bar{f}(y)\}$. Then $y_0 \in Z$. It suffices to show that Z is open and closed. (**Note:** A space Y is connected if and only if Y and $\emptyset$ are only open and closed subsets of Y (or, equivalently, Y is not disjoint union of two open subsets). A path-connected space is connected, but a connected space may not be path connected in general.)

First we show that Y' is an open subset of Y. Let $y \in Y'$ and let U be an elementary neighborhood of $f(y)$ in X. There is a (unique) sheet U_α of $p^{-1}(U)$ such that $\tilde{f}(y) = \bar{f}(y) \in U_\alpha$. Then $\tilde{f}^{-1}(U_\alpha) \cap \bar{f}^{-1}(U_\alpha)$ is an open neighborhood of y. Since $p|_{U_\alpha} \colon U_\alpha \to U$ is a homeomorphism,

$$\tilde{f}|_{\tilde{f}^{-1}(U_\alpha) \cap \bar{f}^{-1}(U_\alpha)} = \bar{f}|_{\tilde{f}^{-1}(U_\alpha) \cap \bar{f}^{-1}(U_\alpha)}.$$

Thus

$$\tilde{f}^{-1}(U_\alpha) \cap \bar{f}^{-1}(U_\alpha) \subseteq Y'$$

and so Y' is open.

Now we show that $Y \setminus Y'$ is open. Let $y \in Y \setminus Y'$ and let U be an elementary neighborhood of $f(y)$ in X. Since $\tilde{f}(y) \neq \bar{f}(y)$, there are two different sheets U_α and U_β of $p^{-1}(U)$ such that $\tilde{f}(y) \in U_\alpha$ and $\bar{f}(y) \in U_\beta$. ($\alpha \neq \beta$ because p restricted to each sheet is a homeomorphism.) Now $\tilde{f}^{-1}(U_\alpha) \cap \bar{f}^{-1}(U_\beta)$ is an open neighborhood of y. Since $U_\alpha \cap U_\beta = \emptyset$, $\tilde{f}(z) \neq \bar{f}(z)$ for any $z \in f^{-1}(U_\alpha) \cap \bar{f}^{-1}(U_\beta)$ and so

$$\tilde{f}^{-1}(U_\alpha \cap \bar{f}^{-1}(U_\beta) \subseteq Y \setminus Y'.$$

Thus $Y \setminus Y'$ is open and hence the result. □

Corollary 1.2.2. *Suppose that $\tilde{X}$ is connected and $\phi \colon \tilde{X} \to \tilde{X}$ is a map such that $p \circ \phi = p$. If $\phi(x_1) = x_1$ for some $x_i \in \tilde{X}$, then ϕ is the identity map.*

Proof. Both ϕ and the identity map $\mathrm{id}_{\tilde{X}}$ are liftings of the map $p \colon \tilde{X} \to X$. Since $\phi(x_1) = \mathrm{id}_{\tilde{X}}(x_1)$, the assertion follows from Lemma 1.2.1. □

Let X be a pointed space with a base-point x_0 and $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = x_0$.

Theorem 1.2.3 (Path-lifting Theorem). *Let $p \colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering. Then*

i) *Every path $\lambda \colon (I, 0) \to (X, x_0)$ has a unique lifting $\tilde{\lambda} \colon (I, 0) \to (\tilde{X}, \tilde{x}_0)$.*
ii) *Every map $F \colon (I \times I, (0,0)) \to (X, x_0)$ has a unique lifting $\tilde{F} \colon (I \times I, (0,0)) \to (\tilde{X}, \tilde{x}_0)$.*

Proof. We already prove the uniqueness of a lifting. So we only need to prove the existence.

i) There exist $0 = t_0 < t_1 < \cdots t_m = 1$ such that $\lambda([t_i, t_{i+1}])$ is contained in some elementary neighborhood of each i. We show that there is a lifting $\tilde{\lambda}_i \colon [0, t_i] \to \tilde{X}$ of $\lambda|_{[0,t_i]}$ by induction on i. When $i = 0$, $\tilde{\lambda}_0 \colon [0, 0] \to \tilde{X}$ is given by $\tilde{\lambda}(0) = \tilde{x}_0$. Suppose that there is a lifting $\tilde{\lambda}_i \colon [0, t_i] \to \tilde{X}$. Since $\lambda([t_i, t_{i+1}])$ lies in an elementary neighborhood. There is a unique lifting $\mu \colon [t_i, t_{i+1}] \to \tilde{X}$ of $\lambda|_{[t_i, t_{i+1}]}$ such that $\mu(t_i) = \tilde{\lambda}_i(t_i)$ (The map μ is obtained by composing $\lambda|_{[t_i, t_{i+1}]}$ with the inverse homeomorphism to p-restricted-to-the-sheet-containing-$\tilde{\lambda}_i(t_i)$. Let

$$\tilde{\lambda}_{i+1} = \tilde{\lambda}_i \cup \mu \colon [0, t_{i+1}] \to \tilde{X}.$$

Then $\tilde{\lambda}_{i+1}$ is a lifting of $\lambda|_{[0,t_{i+1}]}$. This gives a construction of $\tilde{\lambda}$ by induction.

ii) The proof essentially follows from the same idea, that is there are sequence $0 = s_0 < s_1 < \cdots < s_m = 1$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ such that F maps each small rectangle $R_{i,j} = [s_i, s_{i+1}] \times [t_j, t_{j+1}]$ into an elementary neighborhood and then defined $\tilde{F}$ inductively over the rectangles

$$R_{0,0}, R_{0,1}, \cdots, R_{0,n}, R_{1,0}, \cdots.$$

$\square$

Corollary 1.2.4 (Monodromy Lemma). *Let $\tilde{\lambda}_0, \tilde{\lambda}_1 \colon (I, 0) \to (\tilde{X}, \tilde{x}_0)$ be paths with $p \circ \tilde{\lambda}_0 \simeq p \circ \tilde{\lambda}_1$. Then $\tilde{\lambda}_0 \simeq \tilde{\lambda}_1$. In particular, $\tilde{\lambda}_0(1) = \tilde{\lambda}_1(1)$.*

Proof. Let $\lambda_0 = p \circ \tilde{\lambda}_0$ and $\lambda_1 = p \circ \lambda_1$. Let $F \colon I \times I \to X$ be a homotopy relative to $\{0, 1\}$ from λ_0 to λ_1. Then there is a unique lifting $\tilde{F} \colon I \times I \to \tilde{X}$ of F with $\tilde{F}(0, 0) = \tilde{\lambda}_0(0) = \tilde{\lambda}_1(0)$. Then

1) $\tilde{F}(t, 0) = \tilde{\lambda}_0(t)$ for any t because both of them are lifting of λ_0 with $\tilde{F}(0, 0) = \tilde{\lambda}_0(0)$. And $\tilde{F}(1, 0) = \tilde{\lambda}_0(1)$.
2) $\tilde{F}(0, s) = \epsilon_{\tilde{\lambda}_0(0)}$ because both of them are liftings of $F(0, s) = \epsilon_{\lambda(0)}$ with $\tilde{F}(0, 0) = \lambda_0(0)$. And $\tilde{F}(0, 1) = \tilde{\lambda}_0(0) = \tilde{\lambda}_1(0)$.
3) $\tilde{F}(t, 1) = \tilde{\lambda}_1(t)$ because $\tilde{F}(0, 1) = \tilde{\lambda}_1(0)$ and both of them are liftings of λ_1. In particular, $\tilde{F}(1, 1) = \tilde{\lambda}_1(1)$.
4) $\tilde{F}(1, s) = \epsilon_{\tilde{\lambda}_0(1)}$ because $\tilde{F}(1, 0) = \tilde{\lambda}_0(1)$ and both of them are liftings of $\epsilon_{\lambda_0(1)}$.

This show that $\tilde{F}$ is a path homotopy from $\tilde{\lambda}_0$ to $\tilde{\lambda}_1$. $\square$

If in Corollary 1.2.4 we consider only loops, then we immediately have

Theorem 1.2.5. *If $p \colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering, then $p_* \colon \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is a monomorphism.* $\square$

Let $p \colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering projection. The function $\psi \colon \pi_1(X, x_0) \to p^{-1}(x_0)$ is defined by $[\alpha] \mapsto \tilde{\alpha}(1)$, where $\tilde{\alpha} \colon (I, 0, 1) \to (\tilde{X}, \tilde{x}_0, \tilde{\alpha}(1))$ is the unique lifting of α as in Theorem 1.2.3. The function ψ is well-defined by the Monodromy Lemma (Corollary 1.2.4).

Exercise 1.4. Suppose that $\tilde{X}$ is path-connected. Show that the function $\psi \colon \pi_1(X, x_0) \to p^{-1}(x_0)$ is onto.

Hint: Let $y \in p^{-1}(x_0)$. There is a path β from $\tilde{x}_0$ to y. Let $\alpha = p \circ \beta$. Then $\beta = \tilde{\alpha}$ by the uniqueness of the lifting and so $\psi([\alpha]) = \tilde{\alpha}(1) = \beta(1) = y$.

Theorem 1.2.6. *If $\tilde{X}$ is simply connected, then ψ is a bijection.*

Proof. By Exercise 1.4, it suffices to show that ψ is one-to-one.

Suppose that $[\alpha], [\beta] \in \pi_1(X, x_0)$ with $\psi([\alpha]) = \psi([\beta]) = y \in p^{-1}(x_0)$, that is $\tilde{\alpha}(1) = \tilde{\beta}(1) = y$, where $\tilde{\alpha}$ and $\tilde{\beta}$ are the liftings of $[\alpha]$ and β, respectively. Since $\tilde{X}$ is simply connected, $[\tilde{\alpha} * \tilde{\beta}^{-1}] = 1 \in \pi_1(\tilde{X}, \tilde{x}_0)$. Thus

$$[\alpha][\beta]^{-1} = [(p \circ \tilde{\alpha}) * (p \circ \tilde{\beta}^{-1}] = [p \circ (\tilde{\alpha} * \tilde{\beta}^{-1})] = p_*([\tilde{\alpha} * \tilde{\beta}^{-1}]) = p_*(1) = 1.$$

Hence $[\alpha] = [\beta] \in \pi_1(X, x_0)$. $\square$

Now suppose that the quotient $p \colon \tilde{X} \to \tilde{X}/G$, $\tilde{x} \mapsto [\tilde{x}]$, is a covering space arising from a properly discontinuous group action. Here we can do much better.

Since $p^{-1}([\tilde{x}_0]) = G\cdot\tilde{x} = \{g\cdot\tilde{x}_0|g\in G\}$, we can identify $p^{-1}([\tilde{x}_0])$ with G by $g\cdot\tilde{x}_0 \leftrightarrow g$. (Recall: $g\cdot\tilde{x}_0 = g'\cdot\tilde{x}_0 \Rightarrow g = g'$ by the properly discontinuous property.)

THEOREM 1.2.7. *If $\tilde{X}$ is path-connected, then $\psi\colon \pi_1(\tilde{X}/G,[\tilde{x}_0]) \to G$ is a group epimorphism with kernel $p_*\pi_1(\tilde{X},\tilde{x}_0)$.*

PROOF. (i) By Exercise 1.4, the function ψ is onto.
(ii) To see that it's a homomorphism, recall that the lifting $\tilde{\alpha}\colon (I,0,1)\to(\tilde{X},\tilde{x}_0,\tilde{\alpha}(1))$ of a loop α representing $[\alpha]\in\pi_1(\tilde{X}/G,[\tilde{x}_0])$ has $\alpha(1) = g_\alpha\cdot\tilde{x}_0$ for some unique $g_\alpha\in G$ (independent of choice of $\alpha\in[\alpha]$.)

Given $[\alpha],[\beta]\in\pi_1(\tilde{X}/G,[\tilde{x}])$, with α,β lifting to $\tilde{\alpha}\colon (I,0,1)\to(\tilde{X},\tilde{x}_0,g_\alpha\cdot\tilde{x}_0)$, $\tilde{\beta}\colon (I,0,1)\to(\tilde{X},\tilde{x}_0,g_\beta\cdot\tilde{x}_0)$, note that in general $\tilde{\alpha}*\tilde{\beta}$ is not defined (since $g_\alpha\cdot\tilde{x}_0\neq\tilde{x}_0$). However the map $g_\alpha\cdot\colon\tilde{X}\to\tilde{X}$ composes with $\tilde{\beta}$ to give

$$g_\alpha\cdot\tilde{\beta}\colon (I,0,1)\to(\tilde{X},g_\alpha\cdot\tilde{x}_0,g_\alpha\cdot(g_\beta\cdot\tilde{x}_0))$$

which lifts β (Note $g_\alpha\cdot\tilde{\beta}$ is from $g_\alpha\cdot\tilde{x}_0$ to $g_\alpha\cdot(g_\beta\cdot\tilde{x}_0)$). Thus

$$\tilde{\alpha}*(g_\alpha\cdot\tilde{\beta})\colon (I,0,1)\to(\tilde{X},\tilde{x}_0,g_\alpha g_\beta\cdot\tilde{x}_0)$$

is well-defined and lifts $\alpha*\beta$. Since this lifting of $\alpha*\beta$ has final point $g_\alpha g_\beta\cdot\tilde{x}_0$, we have $\psi([\alpha*\beta]) = g_\alpha g_\beta$ and hence

$$\psi([\alpha][\beta]) = \psi([\alpha*\beta]) = g_\alpha g_\beta = \psi([\alpha])\psi([\beta]).$$

(iii) If $\psi([\alpha]) = e\in G$, then $\tilde{\alpha}(1) = e\cdot\tilde{x}_0 = \tilde{x}_0$, making $\tilde{\alpha}$ a loop. Hence

$$[\alpha] = [p\circ\tilde{\alpha}] = p_*([\tilde{\alpha}])\in p_*\pi_1(\tilde{X},\tilde{x}_0).$$

Conversely, for any $\tilde{\alpha}\colon (I,\partial I)\to(\tilde{X},\tilde{x}_0)$, $p\circ\tilde{\alpha}$ has lifting $\tilde{\alpha}$ with $\tilde{\alpha}(1) = e\cdot\tilde{x}_0$, and so $\psi(p_*([\tilde{\alpha}])) = \psi([p\circ\tilde{\alpha}]) = e\in G$. $\square$

COROLLARY 1.2.8. *Suppose that $\tilde{X}$ is path-connected space on which the group G acts properly discontinuously. Then*

$$\psi\colon \pi_1(\tilde{X}/G,\tilde{x}_0)\longrightarrow G$$

is an isomorphism if and only if $\tilde{X}$ is simply-connected.

EXAMPLE 1.2.1. 1) Since S^n is simply connected for $n\geq 2$, we have

$$\pi_1(\mathbb{R}P^n) = \pi_1(S^n/\mathbb{Z}/2) = \mathbb{Z}/2$$

for $n\geq 2$.

2) $\pi_1(L^n(p)) = \pi_1(S^{2n+1}/\mathbb{Z}/p) = \mathbb{Z}/p$ $(n\geq 1)$.
3) $\pi_1(S^1) = \pi_1(\mathbb{R}/\mathbb{Z}) = \mathbb{Z}$.

A space X is called to be *locally path-connected* if for each point $x\in X$ and **any** neighborhood U of x there exists a path-connected open neighborhood V of x with $V\subseteq U$. (**Note.** In Spanier's book [**21**], the definition of locally path-connected is as follows: A space X is said to be *locally path-connected* if, for each $x\in X$ and any neighborhood U of x, there is an open neighborhood V of x such that $x\in V\subseteq U$ and any two points in V can be connected by a path in U. Thus our definition of locally path-connected is stronger than Spanier's definition. Our definition follows from Hatcher's book [**7**, pp.62]. This more restrictive definition simplifies the proofs. But keep in mind that the following statements hold for locally path-connected in the sense of Spanier's book.)

THEOREM 1.2.9 (Lifting Theorem). *Let* $p\colon (\tilde{X},\tilde{x}_0) \to (X,x_0)$ *is a covering space. Let* $f\colon (Y,y_0) \to (X,x_0)$ *be a map. Suppose that* Y *is path-connected and locally path-connected. Then* $f\colon (Y,y_0) \to (X,x_0)$ *admits a unique lifting* $\tilde{f}\colon (Y,y_0) \to (\tilde{X},\tilde{x}_0)$ *if and only if*

$$f_*(\pi_1(Y,y_0)) \subseteq p_*\pi_1(\tilde{X},\tilde{x}_0).$$

PROOF. $\Rightarrow$ is obvious.

$\Leftarrow$ By Lemma 1.2.1, if f admits a lifting, then the lifting is unique. Thus it suffices to prove the existence of the lifting. The construction of $\tilde{f}$ is as follows:

> For each $y \in Y$, since Y is path-connected, there is a path $\lambda\colon (I,0,1) \to (Y,y_0,y)$. So lift $f\circ\lambda\colon (I,0) \to (X,x_0)$ uniquely (by Theorem 1.2.3) to $\widetilde{f\circ\lambda}\colon (I,0) \to (\tilde{X},\tilde{x}_0)$. Let
> $$\tilde{f}(y) = \widetilde{f\circ\lambda}(1).$$
> Then $p\circ\tilde{f} = f$.

We must prove that

i) $\tilde{f}(y)$ is independent of choice of $\lambda\colon (I,0,1) \to (Y,y_0,y)$, that is $\tilde{f}$ is well-defined as a function, and
ii) $\tilde{f}$ is continuous.

To see (i), let λ and λ' be two paths in Y from y_0 to y. Then the path product $\lambda * \lambda'^{-1}$ is a loop in Y from y_0 to y_0. By the assumption

$$[(f\circ\lambda) * f(\lambda'^{-1})] \in f_*(\pi_1(Y,y_0)) \subseteq p_*\pi_1(\tilde{X},\tilde{x}_0).$$

Thus the loop $(f\circ\lambda)*f(\lambda'^{-1})$ admits a lifting in $(\tilde{X},\tilde{x}_0)$ as a **loop**. By uniqueness of lifting, the first half lifting of this loop is given by $\widetilde{f\circ\lambda}$ and second half lifting is given $\widetilde{f\circ\lambda'}$. In particular, $\widetilde{f\circ\lambda}(1) = \widetilde{f\circ\lambda'}(1)$ because they form a loop.

For showing (ii), let W be an open neighborhood of $\tilde{f}(y)$. Choose a small open neighborhood U of $f(y)$ such that $p^{-1}(U)$ is disjoint union of open sets in $\tilde{X}$ with one piece $\tilde{f}(y) \in \tilde{U} \cong U$ and $\tilde{U} \subseteq W$. By the assumption of locally path-connected, there exists a path open neighborhood V of y with $V \subseteq f^{-1}(U)$. Fix a path λ from y_0 to y. For any $y' \in V$, there is a path η from y to y'. Then the path product $\lambda * \eta$ is a path from y_0 to y'. Since

$$p\colon \tilde{U} \to U$$

is a homeomorphism, $p|_{\tilde{U}}^{-1}(f\circ\eta)$ is a path in $\tilde{U}$ from $\tilde{f}(y)$ from a point in $\tilde{U}$. Now the path product $\widetilde{f\circ\lambda} * p|_{\tilde{U}}^{-1}(f\circ\eta)$ is a lifting of $f(\lambda*\eta)$. By the uniqueness of lifting,

$$\widetilde{f(\lambda*\eta)} = \widetilde{f\circ\lambda} * p|_{\tilde{U}}^{-1}(f\circ\eta).$$

In particular

$$\tilde{f}(y') = \widetilde{f(\lambda*\eta)}(1) \in \tilde{U}.$$

It follows that

$$V \subseteq \tilde{f}^{-1}(\tilde{U}) \subseteq \tilde{f}^{-1}(W)$$

and so f is continuous. This finishes the proof.

$\square$

COROLLARY 1.2.10. *Any maps from a simply-connected locally path-connected* (Y,y_0) *lifts (uniquely).*

Corollary 1.2.11. *Any map from* $(S^n, (1, 0, \cdots, 0))$ *lifts uniquely* $(n \geq 2)$.

Corollary 1.2.12. *For* $n \geq 2$, $p_*\colon \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ *is an isomorphism.*

Proof. By Corollary 1.2.11, p_* is onto. By Corollary 1.2.10, p_* is one-to-one because $S^n \times I$ is simply connected for $n \geq 2$. □

Theorem 1.2.13 (Borsuk-Ulam). *There exists no map* $f\colon S^2 \to S^1$ *such that* $f(-x) = -f(x)$ *for any* x.

Proof. Let $q\colon S^2 \to \mathbb{R}P^2$ be the covering projection, and suppose that for all $x \in S^2$

$$f(-x) = -f(x).$$

Then we can define $g\colon \mathbb{R}P^2 \to S^1$ by $g(\pm x) = (f(x))^2$, making $g \circ q = p \circ f$, where $p\colon S^1 \to S^1$ is defined by $z \mapsto z^2$.

$$\begin{tikzcd} S^2 \arrow[r, "f"] \arrow[d, "q"] & S^1 \arrow[d, "p"] \\ \mathbb{R}P^2 \arrow[r, "g"] & S^1 \end{tikzcd}$$

Since $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2$, $g_*\pi_1(\mathbb{R}P^2)$ is a torsion subgroup of $\pi_1(S^1) = \mathbb{Z}$ and hence $g_*\pi_1(\mathbb{R}P^2) = 0$. Thus, by Theorem 1.2.9, there is a lifting $\tilde{g}\colon \mathbb{R}P^2 \to S^1$ such that $g = p \circ \tilde{g}$. (Note the map p is a covering.) Since $\tilde{g} \circ q$ and f are two liftings of $g \circ q$, we have

$$\tilde{g} \circ q = f.$$

It follows that

$$f(x) = \tilde{g} \circ q(x) = \tilde{g} \circ q(-x) = f(-x) = -f(x),$$

a contradiction. □

Corollary 1.2.14. *If* $g\colon S^2 \to \mathbb{R}^2$ *is an antipode-preserving map, that is* $g(-x) = -g(x)$, *then some* $x \in S^2$ *has* $g(x) = 0$.

Proof. Otherwise $f\colon S^2 \to S^1 \quad x \to \frac{g(x)}{\|g(x\|}$ contradicts Theorem 1.2.13. □

Corollary 1.2.15. *If* $h\colon S^2 \to \mathbb{R}^2$, *then some* $x \in S^2$ *has* $h(x) = h(-x)$; *so* h *is not injective.*

Proof. If this were not the case, then $g\colon S^2 \to \mathbb{R}^2 \quad x \mapsto h(x) - h(-x)$ would contradicts Corollary 1.2.14. □

Corollary 1.2.16. *No subspace of* $\mathbb{R}^2$ *is homeomorphic to* S^2.

Example 1.2.2. Regard the Earth as S^2 and the functions

$P\colon S^2 \to \mathbb{R}$, $x \mapsto$ barometric pressure at x,
$T\colon S^2 \to \mathbb{R}$, $x \mapsto$ temperature at x

as continuous. Then Corollary 1.2.15 says that

$$h\colon S^2 \to \mathbb{R}^2 \quad h(x) = (P(x), T(x))$$

has $h(-x) = h(x)$ for some $x \in S^2$, in other words, there are always two antipodal places on Earth with the same temperature and pressure.

1.3. Universal Covering. Let X be a path-connected space. A covering $p\colon \tilde{X} \to X$ is called *universal* if $\tilde{X}$ is simply connected.

PROPOSITION 1.3.1. *Let X be path-connected and locally path-connected. Then the universal covering over X is unique provided that it exists.*

PROOF. Suppose that $p\colon \tilde{X} \to X$ and $p'\colon \tilde{X}' \to X$ be two universal coverings over X. By the definition, both $\tilde{X}$ and $\tilde{X}'$ are simply connected. In particular, both $\tilde{X}$ and $\tilde{X}'$ are path-connected. Since X is locally path-connected, so are $\tilde{X}$ and $\tilde{X}'$. Let x_0 be a basepoint of X. Choose basepoints $\tilde{x}_0 \in p^{-1}(x_0)$ and $\tilde{x}'_0 \in p'^{-1}(x_0)$. By the lifting theorem, there exist liftings

$$\begin{tikzcd}
\tilde{X} \arrow[r, "f"] \arrow[d, "p"] & \tilde{X}' \arrow[r, "g"] \arrow[d, "p'"] & \tilde{X} \arrow[d, "p"] \\
X \arrow[r, equal] & X \arrow[r, equal] & X
\end{tikzcd}$$

with $f(\tilde{x}_0) = \tilde{x}'_0$ and $g(\tilde{x}'_0) = \tilde{x}_0$ because $\pi_1(\tilde{X})$ and $\pi_1(\tilde{X}')$ are trivial. By the uniqueness of the lifting, $g \circ f = \mathrm{id}_{\tilde{X}}$ and $f \circ g = \mathrm{id}_{\tilde{X}'}$ and hence the result. $\square$

1.3.1. *Existence of Universal Covering Space.* A space X is called *semi-locally simply-connected* if for each point $x \in X$ there exists a neighborhood U of x such that $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.

THEOREM 1.3.2. *Let X be path-connected, locally path-connected, and semi-locally simply connected. Then there exists the universal covering $\tilde{X} \to X$.*

PROOF. The proof is given by constructing a universal covering over X. Let x_0 be a basepoint of X.
Construction: Define

$$\tilde{X} = \{[\lambda] \mid \lambda(0) = x_0\},$$

where $[\lambda]$ is the homotopy class relative to the ending points, that is, with respect to the homotopies that fix the endpoints. Define $p\colon \tilde{X} \to X$ by $p([\lambda]) = \lambda(1)$.
Topology on $\tilde{X}$. Let

$$\mathcal{U} = \{U \subseteq X \mid U \text{ path-connected open } \pi_1(U) \to \pi_1(X) \text{ is trivial }\}.$$

By the assumption of semi-locally simply connected, for any $x \in X$, there exists a neighborhood U of x such that $\pi_1(U) \to \pi_1(X)$ is trivial. By the assumption of locally path-connected, there exists a path-connected open neighborhood V of x such that $V \subseteq U$ with $\pi_1(V) \to \pi_1(X)$ is trivial as it is the composite $\pi_1(V) \to \pi_1(U) \to \pi_1(X)$. Thus $\mathcal{U}$ is a basis for the topology on X. (**Note.** We use the assumptions that X locally path-connected and semi-locally simply connected.)

For $U \in \mathcal{U}$ and a path λ from x_0 to a point in U, define

$$U_{[\lambda]} = \{[\lambda * \eta] \mid \eta \text{ path in } U \text{ with } \eta(0) = \lambda(1)\}.$$

1) $U_{[\lambda]}$ depends only on the path homotopy class of λ, that is, if $\lambda \simeq \lambda' \mathrm{rel} 0, 1$, then $U_{[\lambda]} = U_{[\lambda']}$.
2) $p\colon U_{[\lambda]} \to U$ is onto because U is path-connected.

3) $p\colon U_{[\lambda]} \to U$ is one-to-one. Let η and η' be two paths in U such that $\eta(0) = \eta'(0) = \lambda(1)$ and $\eta(1) = \eta'(1)$. Then $\eta * \eta'^{-1}$ form a loop in U. Since $\pi_1(U) \to \pi_1(X)$ is trivial, $[\eta] * [\eta'^{-1}]$ is trivial in X and so the loop
$$[(\lambda * \eta) * (\lambda * \eta')^{-1}] = 1.$$
It follows that the path homotopy class $[\lambda * \eta] = [\lambda * \eta']$.

4) If $[\lambda'] \in U_{[\lambda]}$, then $U_{[\lambda']} = U_{[\lambda]}$. For seeing this, let $\lambda' = \lambda * \eta$. For any $[\lambda * \eta'] \in U_{[\lambda]}$, let μ be a path in U from $\eta(1)$ to $\eta'(1)$. Then $\eta * \mu * \eta'^{-1}$ is a loop in U from $\lambda(1)$ to $\lambda(1)$. By using the assumption that $\pi_1(U) \to \pi_1(X)$ is trivial, $[\eta * \mu * \eta'^{-1}]$ is trivial in $\pi_1(X, \lambda(1))$. Thus
$$[(\lambda * \eta * \mu) * (\lambda * \eta')^{-1}] = 1$$
in $\pi_1(X, x_0)$ and so
$$[\lambda * \eta'] = [(\lambda * \eta) * \mu],$$
that is $[\lambda * \eta'] \in U_{[\lambda']}$. Or $U_{[\lambda]} \subseteq U_{[\lambda']}$. Similarly, $U_{[\lambda']} \subseteq U_{[\lambda]}$. Thus $U_{[\lambda']} = U_{[\lambda]}$.

Now we show that
$$\tilde{\mathcal{U}} = \{U_{[\lambda]} \mid U \in \mathcal{U}, \lambda \text{ path from } x_0 \text{ a point in } U\}$$
forms a basis for a topology on $\tilde{X}$. Let $U_{[\lambda]}, V_{[\lambda']} \in \tilde{\mathcal{U}}$ with $[\lambda''] \in U_{[\lambda]} \cap V_{[\lambda']}$. Then
$$U_{[\lambda'']} = U_{[\lambda]} \quad V_{[\lambda'']} = V_{[\lambda']}$$
by assertion (4) above. Let W be in $\mathcal{U}$ with $\lambda''(1) \in W \subseteq U \cap V$. Then
$$[\lambda''] \in W_{[\lambda'']} \subseteq U_{[\lambda'']} \cap V_{[\lambda'']} = U_{[\lambda]} \cap V_{[\lambda']}$$
and so $\tilde{\mathcal{U}}$ forms a basis for a topology on $\tilde{X}$.

$p\colon U_{[\lambda]} \to U$ **is a homeomorphism:** Recall that the open subsets of $U_{[\lambda]}$ is given by $U_{[\lambda]} \cap W$ for open sets W in $\tilde{X}$.

First we show that p is continuous. Let $[\lambda * \eta]$ be any element in $U_{[\lambda]}$ and let V be an open subset of U with $x = \lambda * \eta(1) \in V$. There exists $V' \in \mathcal{U}$ such that $x \in V' \subseteq V$.
$$p^{-1}(V') = \{[\lambda * \eta'] \mid \eta'(0) = \lambda(1)\ \eta'(1) \in V'\}.$$
Note that
$$[\lambda * \eta] \in V'_{[\lambda * \eta]} = \{[\lambda * \eta * \eta''] \mid \eta''(0) = x,\ \eta'(1) \in V'\} \subseteq p^{-1}(V') \subseteq p^{-1}(V).$$
Thus p is continuous.

Next we show that $p|^{-1}_{U_{[\lambda]}}$ is continuous. Let x be any point in U. Fix a path η in U from $\lambda(1)$ to x. For any open subset W in $\tilde{X}$ with $p|^{-1}_{U_{[\lambda]}}(x) = [\lambda * \eta] \in W \cap U_{[\lambda]}$ ✓ since $\tilde{\mathcal{U}}$ is basis, there is $V_{[\lambda']}$ such that
$$[\lambda * \eta] \in V_{[\lambda']} \subseteq W \cap U_{[\lambda]}.$$
Then
$$V = p(V_{[\lambda']}) = (p|^{-1}_{U_{[\lambda]}})^{-1}(V_{[\lambda']})$$
is open. Thus $p|^{-1}_{U_{[\lambda]}}$ is continuous.

$p\colon \tilde{X} \to X$ **is a covering:** For $U \in \mathcal{U}$,
$$p^{-1}(U) = \bigcup \{U_{[\lambda]} \mid \lambda(1) \in U\}.$$

Assume that $U_{[\lambda]} \cap U_{[\lambda']} \neq \emptyset$. Let $[\lambda''] \in U_{[\lambda]} \cap U_{[\lambda']}$. By assertion (4) above, $U_{[\lambda'']} = U_{[\lambda]} = U_{[\lambda']}$. Thus $p^{-1}(U)$ is a disjoint union of $U_{[\lambda]}$ and so p is a covering map.

The space $\tilde{X}$ is simply connected. First we show that $\tilde{X}$ is path-connected. Given two points $[\lambda], [\lambda'] \in \tilde{X}$. Since X is path connected, there is a path η in X from $\lambda(1)$ to $\lambda'(1)$. Let η_t be part of the path η from $\eta(0)$ to $\eta(t)$. Then $[\lambda * \eta_t]$ gives a path in $\tilde{X}$ from $[\lambda]$ to $[\lambda']$. Thus $\tilde{X}$ is path-connected.

Finally we show that $\pi_1(\tilde{X}, [x_0])$ is trivial. Since p is a covering map, $p_*\colon \pi_1(\tilde{X}, [x_0]) \to \pi_1(X, x_0)$ is a monomorphism. Let ω be a loop in $\tilde{X}$, which means that $\omega(t)$ is the path homotopy class of a from x_0 to $\omega(t)(1)$ with $\omega(0) = \omega(1) = [x_0]$ the constant path. Consider the homotopy $F_s := \omega(t)(s)$, namely the path $\omega(t)$ evaluating at s. Then $F_s(0) = \omega(0)(s) = x_0$, $F_s(1) = \omega(1)(s) = x_0$, $F_0(t) = \omega(t)(0) = x_0$ and $F_1(t) = \omega(t)(1) = (p \circ \omega)(t)$. Thus F_s defines a null homotopy for $p \circ \omega$. Thus $[p \circ \omega] = 1$ and so $\pi_1(\tilde{X})$ is trivial. □

Note. In Spanier's book [**21**], the construction of the universal covering space is given as the quotient space of the path space $PX = \{\lambda\colon I \to X \mid \lambda(0)\}$ with compact-open topology. In his book, the arguments use the knowledge of compact-open topology.

COROLLARY 1.3.3. *Let X be path-connected, locally path-connected, and semi-locally simply connected. Then for every subgroup $H \subseteq \pi_1(X, x_0)$ there exists a covering space $p\colon \tilde{X}_H \to X$ such that $p_*(\pi_1(\tilde{X}_H, \tilde{x}_0)) = H$ for a suitably chosen basepoint $\tilde{x}_0 \in \tilde{X}_H$.*

PROOF. In the universal covering space $\tilde{X}$, define the equivalence relation

$$[\lambda] \sim [\lambda']$$

if $\lambda(1) = \lambda'(1)$ and $[\lambda * \lambda'^{-1}] \in H$. Define $\tilde{X}_H = \tilde{X}/\sim$. Then the resulting covering $\tilde{X}_H \to X$ is the desired coving. See Hatcher's book [**7**] for details. □

An application to combinatorial group theory is to give a geometric proof of the following theorem:

THEOREM 1.3.4. *Any subgroup of a free group is free.*

PROOF. Let F be a free group. Then we can choose X a connected 1-dimensional cell complex such that $\pi_1(X) = F$. Let H be a subgroup of F. Then there is a covering $p\colon \tilde{X}_H \to X$ such that $p_*(\pi_1(\tilde{X}_H)) = H$. Since any covering over X is still a 1-dimensional cell-complex, $\tilde{X}_H$ is homotopy to a wedge of circles. It follows that $H \cong \pi_1(\tilde{X}_H)$ is a free group. □

EXERCISE 1.5. Let $p\colon \tilde{X} \to X$ be a covering and let B be a subspace of X. Let $\tilde{B} = p^{-1}(B)$ with projection

$$p' = p|_{\tilde{B}}\colon \tilde{B} \to B$$

be the induced covering. Suppose that

(1). $\tilde{X}$, X and B are path-connected;
(2). $\pi_1(B) \to \pi_(X)$ is onto.

Show that $\tilde{B}$ is path-connected.
[Hint: Try a proof by the following steps:
Step 1. By using the Changing-base theorem, show that $\pi_1(B, b) \to \pi_1(X, b)$ is onto for any $b \in B$.
Step 2. Let $x, y \in \tilde{B}$. Show that there is a path λ in $\tilde{B}$ such that $\lambda(0) = x$ and $p'(\lambda(1)) = p'(y)$. (Thus it suffices to show that there is a path in $\tilde{B}$ joint any two points in the fibre.)
Step 3. Let $x, y \in \tilde{B}$ such that $p'(x) = p'(y)$. Let $b = p'(x)$. Since $\tilde{X}$ is path-connected, there is a path λ in $\tilde{X}$ from x to y. Then $p \circ \lambda$ is a loop in X from b to b. By using the statement in Step 1, there is a loop ω in B from b to b such that $\omega \simeq p \circ \lambda$. Let λ' be a path lifting of ω with $\lambda'(0) = x$. By using Monodromy Theorem, $\lambda' \simeq \lambda \mathrm{rel} 0, 1$. In particular, λ' is a path from x to y. Since λ' is a lifting of a loop ω in B, λ' is a path in $\tilde{B}$ joint x and y.]

PROPOSITION 1.3.5. *Let X be path-connected and let Y be a simply connected. Suppose that*

(1). *There exist small contractible open neighborhoods of x_0 and y_0, respectively.*
(2). *$p\colon \tilde{X} \to X$ is the universal covering over X.*

Then
$$\widetilde{X \vee Y} = \{(x, y) \in \tilde{X} \times Y \mid (p(x), y) \in X \vee Y\}$$
with $p' = (p \times \mathrm{id}_Y)|_{\widetilde{X \vee Y}}\colon \widetilde{X \vee Y} \to X \vee Y$ is the universal covering over $X \vee Y$.

PROOF. Since $p\colon \tilde{X} \to X$ is a covering, so is $p \times \mathrm{id}_Y\colon \tilde{X} \times Y \to X \times Y$. Thus
$$p' = (p \times \mathrm{id}_Y)|_{\widetilde{X \vee Y}}\colon \widetilde{X \vee Y} \to X \vee Y$$
is a covering because it is induced from $p \times \mathrm{id}_Y$. By the above exercise, $\widetilde{X \vee Y}$ is path-connected. From the commutative diagram

$$\begin{tikzcd}
\pi_1(\widetilde{X \vee Y}) \arrow[r] \arrow[d, hook] & \pi_1(\tilde{X} \times Y) = \pi_1(\tilde{X}) \times \pi_1(Y) = \{1\} \arrow[d, hook] \\
\pi_1(X \vee Y) = \pi_1(X) \coprod \{1\} = \pi_1(X) \arrow[r, "\cong"] & \pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y) = \pi_1(X),
\end{tikzcd}$$

$\pi_1(\widetilde{X \vee Y}) = \{1\}$ and hence the result. $\square$

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