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Simplifying the Solution of Ljunggren's Equation $X^2 + 1 = 2Y^4$

RAY STEINER

Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, Ohio 43403-0221

AND

NIKOS TZANAKIS

Department of Mathematics, University of Crete, Iraklion, Crete, Greece

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In 1942 Ljunggren gave a very complicated proof of the fact that the only positive integer solutions of the equation $X^2+1=2Y^4$ are $(X, Y)=(1, 1)$ and (239,13). In the present paper we give a simpler solution of Ljunggren's problem. This is accomplished by reducing the problem to a Thue equation and then solving it by using a deep result of Mignotte and Waldschmidt on linear forms in logarithms and continued fractions.

I. INTRODUCTION

In 1942 Ljunggren [4] gave a very complicated proof of the following

THEOREM 1. *The only positive integer solutions of the diophantine equation*

$$X^2+1=2Y^4 \tag{1.1}$$

are $(X, Y)=(1, 1)$ *and* (239,13).

Ljunggren's proof depends upon the study of units of relative norm -1 in a quadratic extension of a quartic field and Skolem's p-adic method and is very difficult to follow. Indeed, the late Professor L. J. Mordell used to say: "One cannot imagine a more involved solution (of Eq. (1)). One could only wish for a simpler proof."

The purpose of this paper is to fulfill Mordell's desire by giving a simpler

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solution of (1.1). This is accomplished by reducing it to a Thue equation and then solving the latter by using some elementary results of Tzanakis and de Weger [6], a deep but easily applicable result of Mignotte and Waldschmidt [5] on lower bounds for linear forms in logarithms of algebraic numbers and the theory of continued fractions. In fact, our solution is conceptually quite simple; anyway, far simpler than Ljunggren's solution. As in any case in which the theory of linear forms in logarithms of algebraic numbers is applied to the solution of a specific Diophantine equation, high precision calculations are required. A remarkable fact in our solution is that, thanks to Mignotte and Waldschmidt's theorem, the decimal digits required in our computations are "very few" compared to analogous situations: 30 decimal digits suffice!

II. Derivation of the Thue Equation

Factorization of Eq. (1.1) over the Gaussian field yields

$$(X+i)(X-i)=2Y^4,$$

and we have $2=-i(1-i)^2$. Clearly, both $X+i$ and $X-i$ must be divisible by $1+i$ and none of them by $(1+i)^2$. Therefore, we have the ideal equation

$$\left(\frac{X+i}{1+i}\right)\left(\frac{X-i}{1+i}\right)=(Y)^4,$$

in which the two ideals in the left-hand side are relatively prime. It follows then that

$$(X+i)=i^s(1+i)(a+bi)^4, \qquad s\in\{0,1,2,3\}, \tag{2.1}$$

where $a, b\in\mathbb{Z}$ and $Y=\mathrm{Norm}(a+bi)=a^2+b^2$. Consider now (2.1). If $s=0$ or 2 then $\mathrm{Im}\{(1+i)(a+bi)^4\}=1$ or -1, respectively. If $s=1$ then $(X+i)=-(1-i)(a+bi)^4$. Replacing b by $-b$ (this does not affect Y) and taking conjugates gives $\mathrm{Im}\{(1+i)(a+bi)^4\}=1$. Finally, if $s=3$ then in a completely analogous way we obtain a similar equation with -1 in the right-hand side. We conclude therefore that, in any case, (2.1) implies

$$\pm 1=\mathrm{Im}\{(1+i)(a+bi)^4\}=a^4+4a^3b-6a^2b^2-4ab^3+b^4.$$

To simplify the last equation a bit we make the substitution $a=x-y$, $b=y$ and we obtain the Thue equation

$$x^4-12x^2y^2+16xy^3-4y^4=\pm 1.$$

Note that Y is related to x, y by

$$Y=(x-y)^2+y^2. \tag{2.2}$$

III. Solution of the Thue Equation

$$x^4-12x^2y^2+16xy^3-4y^4=\pm 1. \tag{3.1}$$

In this section we will prove the following:

THEOREM 2. *The only solutions of* (3.1) *are given by* $\pm(x, y)=(1, 3)$, $(1, 0)$, $(1, 1)$, $(5, 2)$.

In view of (2.2), Theorem 2 immediately implies Theorem 1.

3.1. *Preliminaries*

Let θ be defined by

$$\theta^4-12\theta^2+16\theta-4=0.$$

It is easy to check that $\mathbb{Q}(\theta)=\mathbb{Q}(\rho)$, where

$$\rho=\sqrt{4+2\sqrt{2}},$$

and this is a totally real normal (Galois) field, since the four conjugates of ρ are: $\pm\rho$ and $\pm(-3\rho+\frac{1}{2}\rho^3)=\pm\sqrt{4-2\sqrt{2}}$. Put

$$\mathbb{K}=\mathbb{Q}(\rho) \qquad \text{and} \qquad R=\mathbb{Z}[1, \rho, \tfrac{1}{2}\rho^2, \tfrac{1}{2}\rho^3].$$

The four conjugates of θ are

$$\theta^{(1)}=2+\rho-\tfrac{1}{2}\rho^2, \qquad \theta^{(2)}=2-\rho-\tfrac{1}{2}\rho^2$$
$$\theta^{(3)}=-2-3\rho+\tfrac{1}{2}\rho^2+\tfrac{1}{2}\rho^3, \qquad \theta^{(4)}=-2+3\rho+\tfrac{1}{2}\rho^2-\tfrac{1}{2}\rho.$$

In view of (3.1), $x-y\theta$ is a unit of the order R. Applying Billevic's method [1] (see [6, Appendix I]) we computed the following triad of fundamental units of R:

$$\varepsilon_1=-1-\rho+\rho^2+\tfrac{1}{2}\rho^3=-6+21\theta-\tfrac{5}{2}\theta^2-2\theta^3$$
$$\varepsilon_2=-5-2\rho+4\rho^2+\tfrac{3}{2}\rho^3=-25+79\theta-9\theta^2-\tfrac{15}{2}\theta^3$$
$$\varepsilon_3=-7-2\rho+\tfrac{11}{2}\rho^2+2\rho^3=-36+111\theta-\tfrac{25}{2}\theta^2-\tfrac{21}{2}\theta^3$$

(ε_1, ε_2, $\varepsilon_3 > 0$). Thus we obtain

$$x - y\theta = \pm\varepsilon_1^{a_1}\varepsilon_2^{a_2}\varepsilon_3^{a_3}, \qquad (a_1, a_2, a_3) \in \mathbb{Z}^3 \tag{3.2}$$

and we put

$$A = \max\{|a_1|, |a_2|, |a_3|\}.$$

3.2. *Searching for Solution with Small* $|y|$

A direct search shows that the only solutions (x, y) of (3.1) with $|y| \leqslant 5$ are those listed in the following table, in which the corresponding values of the a_i's in (3.2) are also shown.

a_1	a_2	a_3	$\pm(x, y)$
-1	2	1	(1, 3)
0	0	0	(1, 0)
1	0	-1	(1, 1)
10	-2	-4	(5, 2)

Now let (x, y) be a solution of (3.1). In view of the above table we may assume that $|y| \geqslant 6$. We put

$$\beta = x - y\theta.$$

According to a simple lemma (see [6, Chap. II, Lemma 1.1]), if $|y| > Y_1$, then there exists an index $i_0 \in \{1, 2, 3, 4\}$ such that

$$|\beta^{(i_0)}| \leqslant C_1 |y|^{-3}. \tag{3.3}$$

The formulas of Y_1 and C_1 give in our case

$$Y_1 = 3, \qquad C_1 = 1.3604.$$

Let $d_0, d_1, d_2, \ldots$ be the partial quotients and $p_1/q_1, p_2/q_2, \ldots$ the convergents in the continued fraction expansion of $\theta^{(i_0)}$ (for the actual computation of the continued fraction of a real algebraic number see [3] or [7, Chap. 4]). Put in view of the above mentioned lemma, $x/y = p_n/q_n$ for some $n = 1, 2, \ldots$. By a well-known result on continued fractions, we have

$$\frac{1}{(d_{n+1}+2)q_n^2} < \left|\theta^{(i_0)} - \frac{p_n}{q_n}\right|.$$

Combine this with the first relation (3.3) and the fact that $|q_n| = |y|$ to obtain

$$d_{n+1} > \frac{|q_n|^2}{C_1} - 2 \tag{3.4}$$

(note that $|q_n| = |y| \geqslant 6$; on the other hand, since $|q_n|$ grows very fast with n, we expect that (3.4) can be true for only a very few values of n).

We now want to search for solutions of (3.1) in the range $6 \leqslant |y| \leqslant 10^{30}$. For every $i_0 \in \{1, 2, 3, 4\}$ we check which convergents satisfy (3.4). If some p_n/q_n is such a convergent, then we check whether $(x, y) = (p_n, q_n)$ is a solution of (3.1).

In this way we checked that no solution exists in the range $6 \leqslant |y| \leqslant 10^{30}$. Therefore, from now on we suppose that

$$|y| > 10^{30} \tag{3.5}$$

and we will prove that (3.1) has no solutions in this range. This will imply that the only solutions of (3.1) are $\pm(x, y) = (1, 3), (1, 0), (1, 1), (5, 2)$.

We note now that from (3.6) we can easily find a useful lower bound for A as follows (this idea is due to A. Pethö): For every $(i, j) \in \{1, 2, 3\} \times \{1, , 3, 4\}$ put

$$v_{ij} = \begin{cases} 1 & \text{if } |\varepsilon_i^{(j)}| > 1 \\ -1 & \text{if } |\varepsilon_i^{(j)}| < 1 \end{cases} \quad \text{and} \quad E_j = \prod_{i=1}^{3} |\varepsilon_i^{(j)}|^{v_{ij}}.$$

Then, for every $j \in \{1, 2, 3, 4\}$,

$$|\beta^{(j)}| = \prod_{i=1}^{3} |\varepsilon_i^{(j)}|^{a_i} \leqslant E_j^A$$

and hence, from any pair j_1, j_2 ($j_1 \neq j_2$) we have

$$|y| = \frac{|\beta^{(j_1)} - \beta^{(j_2)}|}{|\theta^{(j_1)} - \theta^{(j_2)}|} \leqslant \frac{E_{j_1}^A + E_{j_2}^A}{|\theta^{(j_1)} - \theta^{(j_2)}|}. \tag{3.6}$$

Therefore, if we know a lower bound for $|y|$ (such as in (3.5), for example), then we can find a lower bound for A. Note that j_1 and j_2 can be chosen in such a way that the resulting lower bound for A can be the best possible. For example, in our case an easy computation shows that

$$E_1 < 32476.1, \qquad E_2 < 28.1422, \qquad E_3 < 33.9, \qquad E_4 < 34.1$$

and if we choose $j_1 = 2, j_2 = 4$ ($|\theta^{(2)} - \theta^{(4)}| > 2.16478$) and take into account (3.5), then we easily see from (3.6) that

$$A \geqslant 20. \tag{3.7}$$

3.3. *From (3.2) to an Inequality Involving a Linear Form in Logarithms*

Let $i_0 \in \{1, 2, 3, 4\}$ be as before (we have to check four possibilities). Take any pair (j, k) of indices from the set $\{1, 2, 3, 4\}$ such that the three indices i_0, j, and k be distinct. Consider the i_0, j, k-conjugates of the relation $\beta = x - y\theta$ and eliminate x and y to obtain

$$\frac{\theta^{(i_0)} - \theta^{(j)}}{\theta^{(i_0)} - \theta^{(k)}} \cdot \frac{\beta^{(k)}}{\beta^{(j)}} - 1 = -\frac{\theta^{(k)} - \theta^{(j)}}{\theta^{(k)} - \theta^{(i_0)}} \cdot \frac{\beta^{(i_0)}}{\beta^{(j)}}. \tag{3.8}$$

For simplicity in our notation we put

$$\delta_0 = \frac{\theta^{(i_0)} - \theta^{(j)}}{\theta^{(i_0)} - \theta^{(k)}}, \qquad \delta_i = \frac{\varepsilon_i^{(k)}}{\varepsilon_i^{(j)}} \qquad (i = 1, 2, 3).$$

In view of (3.2), (3.8) becomes

$$\delta_0 \delta_1^{a_1} \delta_2^{a_2} \delta_3^{a_3} - 1 = -\frac{\theta^{(k)} - \theta^{(j)}}{\theta^{(k)} - \theta^{(i_0)}} \cdot \frac{\beta^{(i_0)}}{\beta^{(j)}}. \tag{3.9}$$

If we put

$$\Lambda = \log |\delta_0 \delta_1^{a_1} \delta_2^{a_2} \delta_3^{a_3}|$$

and estimate the right-hand side of (3.9) with the aid of (3.3) we can prove easily (see [6, Chap. II, Lemma 1.2]) that, if $|y| > Y_2^*$ then $0 < |\Lambda| < 1.39 C_1 C_3 / C_2 \, |y|^{-4}$. The formulas of Y_2^* and C_3 in our case give

$$Y_2^* = 3 \qquad \text{and} \qquad C_3 = 6.02734$$

and therefore

$$0 < |\Lambda| < 13.146 \, |y|^{-4}. \tag{3.10}$$

We would like now, to replace the right-hand side of (3.10) by an expression containing Λ but not $|y|$. We first need some notations. Consider the 4×3 matrix

$$\mathscr{E} = (\log |\varepsilon_h^{(i)}|)_{1 \leqslant h \leqslant 3, 1 \leqslant i \leqslant 4}.$$

For every $j \in \{1, 2, 3, 4\}$ let $\mathscr{E}_j$ be the matrix which results from $\mathscr{E}$ if we omit the jth row. Then $|\det(\mathscr{E}_j)|$ is equal to the regulator of the order R (in our case this is equal to 4.8835898...). Let

$$N_0 = \min\{3 \cdot \min_{1 \leqslant j \leqslant 4} N[\mathscr{E}_j^{-1}], \max_{1 \leqslant j \leqslant 4} N[\mathscr{E}_j^{-1}]\},$$

where, in general, for an $m \times n$ matrix (a_{ij}), $N[(a_{ij})]$ is the row-norm of the matrix defined by

$$N[(a_{ij})] = \max_{1 \leqslant i \leqslant m} \left(\sum_{j=1}^{n} |a_{ij}| \right).$$

Define also

$$|\bar{\theta}| = \max_{1 \leqslant i \leqslant 4} |\theta^{(i)}|.$$

Then, for a solution satisfying $|y| > 10^S$ we can easily show (see [2, relation (3)] that

$$A \leqslant C_5 \log |y|, \qquad C_5 = N_0 \left(1 + \frac{1}{S} \log_{10} |\bar{\theta}|\right). \tag{3.11}$$

Combine now (3.10) and (3.11) to obtain

$$0 < |\Lambda| < 13.146 \cdot e^{-4A/c_5}. \tag{3.12}$$

In our case $S = 30$ and we computed that $N_0 < 5.475513$, so that

$$C_5 < 5.58594.$$

Then, in view also of (3.7), (3.12) implies

$$0 < |\Lambda| < e^{-0.5872777A}, \tag{3.13}$$

and this is the required inequality. Note that (3.13) combined with (3.7) implies, in particular

$$|\Lambda| < 7.93 \cdot 10^{-6}. \tag{3.14}$$

3.4. *Explicit Computation of Λ*

As already noted, once i_0 is chosen we can choose j and k arbitrarily ($i_0 \neq j \neq k \neq i_0$). So, we make the following choices:

If $i_0 = 3$ or 4 we take $k = 1$ and $j = 2$. In both cases it is a routine matter to compute that

$$|\delta_1| = \varepsilon_1^{-2} \varepsilon_3^2, \qquad |\delta_2| = \varepsilon_1^{-8} \varepsilon_2^2 \varepsilon_3^4, \qquad |\delta_3| = \varepsilon_1^{-4} \varepsilon_3^4.$$

Also, if $i_0 = 3$ then

$$\delta_0 = \frac{\theta^{(3)} - \theta^{(2)}}{\theta^{(3)} - \theta^{(1)}} = \frac{-4 - 2\rho + \rho^2 + \frac{1}{2}\rho^3}{-4 - 4\rho + \rho^2 + \frac{1}{2}\rho^3} = -1 + \rho + \tfrac{1}{2}\rho^2 = \varepsilon_1^{-1} \varepsilon_3$$

and, analogously, if $i_0 = 4$ then $\delta_0 = -\varepsilon_1^{-1} \varepsilon_3$. Thus, if $i_0 = 3$ or 4 then

$$\begin{aligned} \Lambda &= \log(\varepsilon_1^{-1} \varepsilon_3) + a_1 \log(\varepsilon_1^{-2} \varepsilon_3^2) + a_2 \log(\varepsilon_1^{-8} \varepsilon_2^2 \varepsilon_3^4) + a_3 \log(\varepsilon_1^{-4} \varepsilon_3^4) \\ &= (1 + 2a_1 + 4a_3) \log(\varepsilon_1^{-1} \varepsilon_3) + 2a_2 \log(\varepsilon_1^{-4} \varepsilon_2 \varepsilon_3^2) \\ &= (1 + 2a_1 + 2a_2 + 4a_3) \log(\varepsilon_1^{-1} \varepsilon_3) - 2a_2 \log(\varepsilon_1^3 \varepsilon_2^{-1} \varepsilon_3^{-1}). \end{aligned}$$

In an analogous way we find that if $i_0 = 1$ or 2 then

$$\Lambda = (1 + 2a_1 + 4a_3)\log(\varepsilon_1^3\varepsilon_2^{-1}\varepsilon_3^{-1}) + 2a_2\log(\varepsilon_1^2\varepsilon_2^{-1})$$
$$= 2a^2\log(\varepsilon_1^{-1}\varepsilon_3) + (1 + 2a_1 + 2a_2 + 4a_3)\log(\varepsilon_1^3\varepsilon_2^{-1}\varepsilon_3^{-1}).$$

Thus

$$\Lambda = b_1 \log \gamma_1 + b_2 \log \gamma_2,$$

where

$$\gamma_1 = \varepsilon_1^{-1}\varepsilon_3 = -1 + \rho + \tfrac{1}{2}\rho^2, \qquad \gamma_2 = \varepsilon_1^3\varepsilon_2^{-1}\varepsilon_3^{-1} = 3 - 3\rho - \tfrac{1}{2}\rho^2 + \tfrac{1}{2}\rho^3$$

and

$$(b_1, b_2) = (1 + 2a_1 + 2a_2 + 4a_3, -2a_2) \text{ or } (2a_2, 1 + 2a_1 + 2a_2 + 4a_3). \qquad (3.15)$$

We now put

$$B = \max\{|b_1|, |b_2|\},$$

so that $B \leqslant 8.05A$ and then, by (3.13),

$$0 < |\Lambda| < e^{-C_6 B}, \qquad C_6 = 0.072954. \qquad (3.16)$$

3.5. *An Upper Bound for B*

Up to now, the results and arguments were elementary. At this point we use a really deep theorem of Mignotte and Waldschmidt.

THEOREM [5, Corollary 1.1]. *Let α_1, α_2 be two multiplicatively independent algebraic numbers and b_1, b_2 two positive rational integers such that $b_1 \log \alpha_1 \neq b_2 \log \alpha_2$ (where $\log \alpha_i$ $(i = 1, 2)$ is an arbitrary but fixed determination of the logarithm). Define $D = D[\mathbb{Q}(\alpha_1, \alpha_2): \mathbb{Q}]$, $B = \max\{b_1, b_2\}$ and choose two positive real numbers a_1, a_2 satisfying*

$$a_j = \max\left\{1, h(\alpha_j) + \log 2, \frac{2e\,|\log \alpha_j|}{D}\right\} \qquad (j = 1, 2)$$

(*where, as usual, $h(\cdot)$ denotes the absolute logarithmic height*). *Then,*

$$|b_1 \log \alpha_1 - b_2 \log \alpha_2| \geqslant \exp\{-500D^4 a_1 a_2 (7.5 + \log B)^2\}.$$

It is easy to check that in our case the above theorem implies

$$|\Lambda| > \exp\{-500 \cdot 4^4 \cdot 2.63 \cdot (7.5 + \log B)^2\}$$

and this inequality combined with (3.16) gives

$$B < 4.05 \cdot 10^9.$$

3.6. *Reducing the Upper Bound of B*

Equation (3.16) is equivalent to

$$\left|\delta - \frac{b_1}{b_2}\right| < \frac{1}{|b_2|} \cdot \frac{1}{|\log \gamma_1|}\, e^{-C_6 B}, \tag{3.17}$$

where $\delta = -\log \gamma_2 / \log \gamma_1$ and $B < C = 4.05 \cdot 10^9$. We have

$$\frac{1}{|b_2|\, |\log \gamma_1|}\, e^{-c_6 B} < \frac{1}{1.61489\, |b_2|}\, 1.075681^{-B} < \frac{1}{2.1\, |b_2|^2},$$

provided that $B \geqslant 60$. Now let $\tilde{\delta}$ be a rational approximation of δ such that

$$|\tilde{\delta} - \delta| < \frac{1}{1000 C^2}. \tag{3.18}$$

Then,

$$\left|\tilde{\delta} - \frac{b_1}{b_2}\right| \leqslant |\tilde{\delta} - \delta| + \left|\delta - \frac{b_1}{b_2}\right| < \frac{1}{1000 C^2} + \frac{1}{2.1\, |b_2|^2}$$
$$< \frac{1}{1000\, |b_2|^2} + \frac{1}{2.1\, |b_2|^2} < \frac{1}{2\, |b_2|^2},$$

which implies that b_1/b_2 is a convergent of the continued fraction expansion of $\tilde{\delta}$. Denote by d_0, d_1, d_2, ... the partial quotients and by p_1/q_1, p_2/q_2, ... the convergents in the continued fraction expansion of $\tilde{\delta}$. Suppose that $b_1/b_2 = p_n/q_n$. Then,

$$\frac{1}{(d_{n+1}+2)\, |b_2|^2} \leqslant \frac{1}{(d_{n+1}+2)\, |q_n|^2} < \left|\tilde{\delta} - \frac{p_n}{q_n}\right| = \left|\tilde{\delta} - \frac{b_1}{b_2}\right|$$
$$\leqslant |\tilde{\delta} - \delta| + \left|\delta - \frac{b_1}{b_2}\right|$$
$$< \frac{1}{1000 C^2} + \frac{1}{1.61489\, |b_2|}\, 1.075681^{-B},$$

from which

$$d_{n+1} + 2 > \left(10^{-3} + \frac{B}{1.61489} \cdot 1.076581^{-B}\right)^{-1} > 29$$

provided that $B \geqslant 104$. We computed a rational approximation $\tilde{\delta}$ of δ up to 30 decimal digits (so that (3.18) is satisfied) and we looked for all convergents p_n/q_n of $\tilde{\delta}$ with $\max\{p_n, q_n\} \geqslant 104$ and such that $d_{n+1} \geqslant 28$. It turned out that no such convergent exists and consequently there are no solutions of (3.17) with $B \geqslant 104$. If $60 \leqslant B < 104$ then, by our previous arguments, b_1/b_2 is a convergent in the continued fraction expansion of $\tilde{\delta}$, but it is straightforward to check that no convergent p_i/q_i satisfies $60 \leqslant \max\{|p_i|, |q_i|\} < 104$.

Therefore we are left with the case $B \leqslant 59$. From (3.17) we see that $b_2/b_1 > 1$; i.e., $B = |b_2|$, and by (3.15) b_1, b_2 have opposite parities. Since they must satisfy (3.17), we have $B \geqslant 4$ and then (3.17) implies in particular that

$$0.140343\ |b_2| < |b_1| < 0.359009\ |b_2|. \tag{3.19}$$

We have determined all pairs $(|b_1|, |b_2|)$, satisfying $4 \leqslant |b_2| \leqslant 59$ and (3.19), and for each such pair we calculated the corresponding value of Λ. In all cases it turned out that $|\Lambda| > 0.00209$, which contradicts (3.14). This contradiction completes the proof of Theorem 2.

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